Moroccan Journal of Algebra and Geometry with Applications
x (xx) (2024), 1–17

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco ISSN: 2820-7114

Associated ideals to totally noetherian modules

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Communicated by Najib Mahdou

(Received 24 January 2024, Revised 23 April 2024, Accepted 17 May 2024)

Abstract. One problem in the study of the decomposition of modules is to choose the simple pieces to build such decompositions. In the noetherian case these simple pieces are the coprimary modules; therefore, each noetherian module is a subdirect product of finitely many coprimary modules, and each coprimary module has associated a unique prime ideal. A relative notion of noetherian modules was introduced by Anderson and Dumitrescu as S–noetherian modules, relative to a multiplicative subset $S \subseteq A$ of a commutative ring, in [1]. Since then many authors have worked on this notion introducing prime and primary ideal and submodules relative to S. We have chosen a more general point of view, and work on a hereditary torsion theory σ in **Mod**–A and extend S–noetherian to totally σ–noetherian, recovering earlier notions when we take $\sigma = \sigma_S$. Since we first show that σ is of finite type whenever A is totally σ–noetherian, hence our theory can be regarded as an extension of the Anderson–Dumitrescu's theory taking a multiplicative subset of finitely generated ideals instead of a multiplicative subset of elements. In this context we establish new results on prime and primary ideals and submodules, provide a primary decomposition of totally σ–noetherian modules, and show some applications of this primary decomposition, in particular, to totally σ–artinian modules.

Key Words: primary submodule, S–finite module, noetherian ring, hereditary torsion theory, totally torsion.

2020 MSC: Primary 13E05, 13E10; Secondary 13C12.

1 Introduction

The study of noetherian rings and modules, and their decompositions, has been essential in the develop of Commutative Algebra. The main idea under these decompositions is to study the module relative to the prime ideals of the ring, and grouping all the elements that each prime ideal determines. If A is a noetherian commutative ring, and M a noetherian module, we have a decomposition of the injective hull $E(M)$ of M, as direct sum of finitely many indecomposable injective modules, hence we obtain a decomposition of M: the primary decomposition induced by the decomposition of its injective hull.

Since there are many generalizations of noetherian rings and modules, if we study one of them, the natural question if: when there exists a primary decomposition of noetherian modules?

In this paper we are interested in S–noetherian modules, for a multiplicative set $S \subseteq A$, in the sense of Anderson–Dumitrescu, see [1], or in totally σ–noetherian modules, in the sense of Jara, see [5, 6]; the main aim of this paper is to develop the primary decomposition of totally σ–noetherian modules.

S–finite and S–noetherian modules were introduced by Anderson and Dumitrescu, and studied by many others authors, giving the essential of their structure, see [10]. Even some work are realized to define prime and primary submodules in this context, see [2, 4, 11]; a theory of primary decomposition of ideals is formulated in [14]. Later, in [6, 5], Jara abstracts these notions to consider a hereditary torsion theory σ in **Mod**$-A$, in such a way that the S–finite and S–noetherian modules are particular cases of totally σ–finitely generated, and totally σ–noetherian modules, respectively. In addition, he proves that σ is defined by a multiplicative set of finitely generated ideals whenever A is totally σ–noetherian: that is, the study by means hereditary torsion theories is the natural extension of multiplicative sets of elements to multiplicative sets of (finitely generated) ideals.

With this background, we proceed to study the primary decomposition of totally noetherian modules. To do that, given a totally σ–noetherian module we study the totally σ–prime modules which are submodules of M, with the goal of establishing a set of prime ideals of A that control this type of submodules. After that, we find that a finite set of prime ideals parameterizes these totally σ–prime modules: the associated totally σ–prime ideals. This set of prime ideals is denoted by $\mathrm{Ass}_{t\sigma}(M)$, and it has the important property that for any submodule $N \subseteq M$ the inclusion

$$\mathrm{Ass}_{t\sigma}(N) \subseteq \mathrm{Ass}_{t\sigma}(M) \subseteq \mathrm{Ass}_{t\sigma}(N) \cup \mathrm{Ass}_{t\sigma}(M/N) \quad (1)$$

holds; it is well known that the second inclusion can be strict.

Nevertheless, for some special submodules $N \subseteq M$ the equality holds; this is the case of submodules $N \subseteq M$ such that: (1) $\mathrm{Ass}_{t\sigma}(N) = \{\mathfrak{p}\}$, is unitary set, being $\mathfrak{p} \in \mathrm{Ass}(M)$ a maximal element, and (2) there is not any prime module H such that $H \subseteq M$ and $N \subseteq H$; i.e., N is a maximal totally prime extension, in M, of $0 \subseteq M$. In this case we can pass from M to M/N without the addition of any new prime ideals in $\mathrm{Ass}_{t\sigma}$.

To define a primary decomposition of a totally noetherian module first we introduce totally primary submodules and define when an module has a totally primary decomposition, and tackling the problem of show the existence of totally primary decomposition, that finally is proven in Theorem (6.14).

This paper is divide in sections; this is the first section. In section 2 we give a brief introduction to the basic notion on hereditary torsion theories. Since a few results on totally noetherian modules are necessary, we collect them in section 3, and use as a reference the paper [6]. The second necessary tool are totally prime modules and submodules; in section 4 we collect some result that show their behaviour and properties; in particular, we show that the σ–torsion submodule of any totally prime module is totally σ–torsion; and that for any submodule $N \subseteq M$ is M which is a totally σ–prime module, with companion ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$, we have associated a prime ideal: $(\sigma N : N)$; we represent by $\mathrm{Ass}_{t\sigma}(M)$ the set constituted by this prime ideal. Properties of the sets $\mathrm{Ass}_{t\sigma}(M)$ are studied to finally show that $\mathrm{Ass}_{t\sigma}(M)$ is non–empty and has finitely many elements whenever M is a totally σ–noetherian non σ–torsion module. To work with associated totally prime ideals and find the existence of primary decompositions we need to study chain of submodules whose factors are totally prime modules; in order to check that we do not increase the amount of totally associated prime ideals in (1), in section 5 we introduce totally prime extensions and maximal prime extensions submodules, proving Theorem (5.2) that show one case in which the equality $\mathrm{Ass}_{t\sigma}(M) = \mathrm{Ass}_{t\sigma}(N) \cup \mathrm{Ass}_{t\sigma}(M/N)$ which will be used to prove the existence of primary decompositions.

In section 6 we introduce totally primary submodules, coprimary modules and the primary decomposition of totally noetherian modules for non σ–torsion modules. Using associated prime ideals it is possible to characterize totally finitely generated modules which are totally coprimary as those with a unitary set of totally associated ideals. Therefore, we introduce a new type of modules: the totally tertiary modules as those such that $\mathrm{Ass}_{t\sigma}(M)$ is unitary. In this meanwhile we show that each associated totally prime ideal is σ–closed, and this fact together with Theorem (6.11) gives that the injective hull of $M/\sigma M$ has a decomposition as a direct sum of indecomposable injective modules that finally in Theorem (6.14) shows the existence of primary decomposition for totally noetherian modules. In section 7 we include some examples of application of the theory; in particular, we study the associated totally prime ideals of a totally artinian module.

2 Background on hereditary torsion theories

Let us introduce hereditary torsion theories and the elements of the theory we shall use in this work.

Recall that a hereditary torsion theory can be described by means of

- A class $\mathcal{T}_\sigma$ of modules which is closed under submodules, homomorphic images, direct sums and group–extensions: the **torsion class**.
- A class $\mathcal{F}_\sigma$ of modules which is closed under submodules, essential extensions, direct products and group–extensions: the **torsionfree class**.
- A filter of ideals $\mathcal{L}(\sigma)$ satisfying the following property: for any ideal $\mathfrak{a} \subseteq A$, and any ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$, if $(\mathfrak{a} : h) \in \mathcal{L}(\sigma)$, for every $h \in \mathfrak{h}$, then $\mathfrak{a} \in \mathcal{L}(\sigma)$. The **Gabriel filter**.
- A **left–exact functor** σ satisfying $\sigma M \subseteq M$, and $\sigma(M/\sigma M) = 0$, for every A–module M, and $\sigma N = N \cap \sigma M$ for any submodule $N \subseteq M$.

The relationship between them is given by the following:

- $\mathcal{T}_\sigma = \{M \in \mathbf{Mod}\text{–}A \mid \sigma M = M\}$, the class of σ**–torsion modules**.
- $\mathcal{F}_\sigma = \{M \in \mathbf{Mod}\text{–}A \mid \sigma M = 0\}$, the class of σ**–torsionfree modules**.
- $\mathcal{L}(\sigma) = \{\mathfrak{a} \subseteq A \mid A/\mathfrak{a} \in \mathcal{T}_\sigma\}$.
- $\sigma M = \{m \in M \mid (0 : m) \in \mathcal{L}(\sigma)\} = \sum\{N \subseteq M \mid N \in \mathcal{T}_\sigma\} = \cap\{N \subseteq M \mid M/N \in \mathcal{F}_\sigma\}$, the σ**–torsion submodule** of M.

Every hereditary torsion theory produces a partition of the prime spectrum of A in two subsets:

- $\mathcal{Z}(\sigma) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A/\mathfrak{p} \in \mathcal{T}_\sigma\}$, and
- $\mathcal{K}(\sigma) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A/\mathfrak{p} \in \mathcal{F}_\sigma\}$.

The set of all maximal elements of $\mathcal{K}(\sigma)$ is denoted by $\mathcal{C}(\sigma)$.

For any ideal $\mathfrak{a} \subseteq A$ we define a new ideal as the solution of the equation: $\sigma(A/\mathfrak{a}) = \mathrm{Cl}^A_\sigma(\mathfrak{a})/\mathfrak{a}$; it satisfies $\mathfrak{a} \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$, and it is the smallest ideal $\mathfrak{b}$ satisfying $\mathfrak{a} \subseteq \mathfrak{b}$ and $A/\mathfrak{b} \in \mathcal{F}_\sigma$. It is called the σ**–closure** of $\mathfrak{a}$ because $\mathrm{Cl}^A_\sigma(-)$ is a closure operator in the set of all ideal of A. An ideal $\mathfrak{a} \subseteq A$ is called σ**–closed** whenever $\mathfrak{a} = \mathrm{Cl}^A_\sigma(\mathfrak{a})$, and σ**–dense** if $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = A$.

The set of all σ–closed ideals of the ring A is denoted by $C(A,\sigma)$, and is a lattice with the operations:

- $\mathfrak{a}_1 \wedge \mathfrak{a}_2 = \mathfrak{a}_1 \cap \mathfrak{a}_2$ and
- $\mathfrak{a}_1 \vee \mathfrak{a}_2 = \mathrm{Cl}^A_\sigma(\mathfrak{a}_1 + \mathfrak{a}_2)$.

The ring A is σ**–noetherian** (resp. σ**–artinian**) whenever the lattice $C(A,\sigma)$ is noetherian (resp. artinian).

In the same way, for any A–module M, and any submodule $N \subseteq M$, we have a closure operator $\mathrm{Cl}^M_\sigma(N) = \{m \in M \mid (N : m) \in \mathcal{L}(\sigma)\}$, and the lattice $C(M,\sigma)$ of all σ–closed submodules. An A–module M is σ**–noetherian** (resp. σ**–artinian**) whenever the lattice $C(M,\sigma)$ is noetherian (resp. artinian).

A submodule $N \subseteq M$ is σ**–finitely generated** whenever there exists a finitely generated submodule $H \subseteq M$ such that $\mathrm{Cl}^M_\sigma(N) = \mathrm{Cl}^M_\sigma(H)$, and σ**–cyclic** whenever H is a cyclic submodule. It turns out that M is σ–noetherian if, and only if, every submodule is σ–finitely generated.

A **simple** module is a non–zero A–module M such that the only submodules are 0 and M. In the same way, an A–module M is σ**–simple** whenever $M \neq \sigma M$ and $C(M,\sigma) = \{\sigma M, M\}$. Using the lattice $C(M,\sigma)$ we can also define σ**–essential submodules**, σ**–uniform modules**, and so on.

A stronger condition, relative to chain conditions, was introduced by Anderson and Dumitrescu in [1] in the particular case where σ is the hereditary torsion theory defined by a multiplicative set $S \subseteq A$. This torsion theory is denoted by σ_S, and is defined by $\mathcal{L}(\sigma_S) = \{\mathfrak{h} \subseteq A \mid \mathfrak{h} \cap A \neq \varnothing\}$. They define that an ideal $\mathfrak{a} \subseteq A$ is S**–finite** if there exists a finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$ and an element $s \in S$ such that $\mathfrak{a}s \subseteq \mathfrak{b} \subseteq \mathfrak{a}$; and S**–principal** whenever $\mathfrak{b}$ is a principal ideal. In a straightforward way we can consider σ–finite and σ–principal ideals with respect to a hereditary torsion theory σ and call them **totally σ–finitely generated** and **totally σ–principal**, respectively.

The two notions are related in the following sense: $\mathfrak{a} \subseteq A$ is σ–finitely generated if, and only if, $\mathrm{Cl}_\sigma^A(\mathfrak{a})$ is σ–finitely generated, i.e., there exists a finitely generated ideal $\mathfrak{b} \subseteq \mathrm{Cl}_\sigma^A(\sigma)$ such that $\mathrm{Cl}_\sigma^A(\mathfrak{a})/\mathfrak{b}$ is σ–torsion. On the other hand, $\mathfrak{a} \subseteq A$ is totally σ–finitely generated whenever there exists a finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$ and an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $(\mathfrak{a}/\mathfrak{b})\mathfrak{h} = 0$; i.e., $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{b}$. In this sense σ–finitely generated is a generalization of totally σ–finitely generated, as totally σ–finitely generated is a generalization of finitely generated.

A hereditary torsion theory σ is of **finite type** whenever $\mathcal{L}(\sigma)$ has a cofinal set of finitely generated ideals; in fact, it is an extension of the hereditary torsion theories σ_S, for a multiplicative set $S \subseteq A$, which are **principal**; that is, they have a cofinal set of principal ideals. Both finite type and principal hereditary torsion theories are in bijective correspondence with multiplicative sets of finitely generated ideals and multiplicative set of elements, respectively.

The necessary background on hereditary torsion theories can be found in [6], and references therein.

3 Totally noetherian modules

In the study of noetherian modules the primary decomposition allows to reduce the study of modules to other with a more simple arithmetical structure. We'll follow this strategy to give a simple description of noetherian modules with respect to a multiplicative set or a hereditary torsion theory.

First of all we situate the problem in an easy and confortable situation. Since we are interested in totally σ–noetherian modules, with respect to a hereditary torsion theory σ, the next result allows to change the ground ring to another ring satisfying extra properties, which say us that our hereditary torsion theory has some restrictions from its origin.

Proposition 3.1. *Let M be an A–module, and $B = A/\mathrm{Ann}(M)$. If for any hereditary torsion theory σ in **Mod**–A we denote by $\overline{\sigma}$ the hereditary torsion theory defined in **Mod**–B, the following statements hold:*

(1) If M is totally σ–noetherian, then it is a totally $\overline{\sigma}$–noetherian and faithful B–module.

(2) If M is a totally σ–noetherian and faithful A–module, then A is a totally σ–noetherian ring.

Proof. (1) is immediate.

(2). Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ be a companion ideal of M, and $m_1,\ldots,m_t \in M$ be elements such that $M\mathfrak{h} \subseteq (m_1,\ldots,m_t) \subseteq M$. Hence we have

$$\mathrm{Ann}(M) \subseteq \mathrm{Ann}(m_1,\ldots,m_t) \subseteq \mathrm{Ann}(M\mathfrak{h}) = (\mathrm{Ann}(M) : \mathfrak{h}).$$

Therefore, $\mathrm{Ann}(M\mathfrak{h})\mathfrak{h} \subseteq \mathrm{Ann}(M)$. In particular, we have $\mathrm{Ann}(m_1,\ldots,m_t)\mathfrak{h} \subseteq \mathrm{Ann}(M) = 0$, so the ideal $\mathrm{Ann}(m_1,\ldots,m_t)$ is totally σ–torsion.

If we define $N = (m_1,\ldots,m_t)$, and a map $f : A \longrightarrow N^t$ as $f(a) = (m_1a,\ldots,m_ta)$, then $\mathrm{Ker}(f) = \mathrm{Ann}(m_1,\ldots,m_t)$ is totally σ–torsion, hence totally σ–noetherian; since $A/\mathrm{Ker}(f) \subseteq N^t$ is totally σ–torsion, then A is totally σ–noetherian. □

Consequently, we have found that the hereditary torsion theory $\overline{\sigma}$ must be of finite type; that is, the Gabriel's filter has a cofinal set of finitely generated ideals; therefore, it is a generalization of the principal hereditary torsion theories that appear when we consider multiplicative sets.

Let M be a totally σ–noetherian module with companion ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$, we may assume A is also totally σ–noetherian.

Lemma 3.2. *Let M be a module, the following statements hold:*

(1) If M is totally σ–noetherian there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$ and $m_1,\dots,m_t \in \sigma M$ such that $\sigma M\mathfrak{h} \subseteq (m_1,\dots,m_t)$. In addition, σM is totally σ–torsion and satisfies $\sigma M = (0 : \mathfrak{h})$.
(2) If σM is totally σ–torsion, and $\mathfrak{a} = \mathrm{Ann}(\sigma M)$, then for any $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $\mathfrak{h} \subseteq \mathfrak{a}$ we have $\sigma M = (0 : \mathfrak{h})$.

Proof. (1). Since M is totally σ–noetherian then σM is totally σ–noetherian, and there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$, $m_1,\dots,m_t \in M$ such that $\sigma M\mathfrak{h} \subseteq (m_1,\dots,m_t) \subseteq \sigma M$. Hence we have the inclusions

$$\mathrm{Ann}(\sigma M) \subseteq \mathrm{Ann}(m_1,\dots,m_t) = \cap_{i=1}^t \mathrm{Ann}(m_i) \subseteq \mathrm{Ann}(M\mathfrak{h}) = (\mathrm{Ann}(\sigma M) : \mathfrak{h}).$$

Since $m_i \in \sigma M$, then $\mathrm{Ann}(m_i) \in \mathcal{L}(\sigma)$, and $(\mathrm{Ann}(\sigma M) : \mathfrak{h}) \in \mathcal{L}(\sigma)$, which is a Gabriel filter, hence $\mathrm{Ann}(\sigma M) \in \mathcal{L}(\sigma)$, and σM is totally σ–torsion.

(2). It is clear that $\mathfrak{a} \in \mathcal{L}(\sigma)$, then the equality is obvious. □

Corollary 3.3. *Let M be a totally σ–noetherian module, for any element $m \in M \setminus \sigma M$ there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$ and $m_1,\dots,m_t \in \sigma(mA)$ such that $\sigma(mA)\mathfrak{h} \subseteq (m_1,\dots,m_t) \subseteq \sigma(mA)$, $\sigma(mA) = (0 :_{mA} \mathfrak{h})$ and $\mathfrak{h} = \mathrm{Ann}(\sigma(mA))$.*

4 Totally prime submodules

Our aim is to deep into the structure of totally σ–noetherian rings and modules using prime submodules as a tool. To do that we recall the notion of totally σ–prime module; an A–module M is **totally σ–prime** whenever M is not totally σ–torsion, and there exists an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that for every $m \in M$ and any $a \in A$, if $ma = 0$, then either $m\mathfrak{h} = 0$ or $Ma\mathfrak{h} = 0$. The ideal $\mathfrak{h}$ is named a companion ideal of the totally σ–prime module M.

First we address the reader to the basic results on totally prime modules and submodules developed in [8, Theorem 4.4], and recall that if M is a totally σ–prime module, the following statements hold:
(1) σM is totally σ–torsion;
(2) $\mathfrak{h} = (0 : \sigma M) \in \mathcal{L}(\sigma)$ can be taken as a companion ideal to M;
(3) $\sigma M = (0 : \mathfrak{h})$, and
(4) $(0 : M\mathfrak{h}) = (\sigma M : M) \in \mathcal{K}(\sigma)$ is a prime ideal.

In particular, if M is totally σ–prime, then M is not σ–torsion.

A submodule $N \subseteq M$ is a **totally σ–prime submodule**, whenever M/N is a totally σ–prime module.

Now we study the relationship between totally σ–prime ideals and totally σ–noetherian modules. We begin proving the existence of totally σ–prime ideals and totally σ–prime submodules related with a totally σ–noetherian module.

Theorem 4.1. If M is a totally σ–noetherian module, non totally σ–torsion, there exists totally σ–prime submodules of M.

Proof. Since M is totally σ–noetherian, then $(0 : \sigma M) \in \mathcal{L}(\sigma)$ is a companion ideal of M.

If we consider the family $\Gamma = \{N \subseteq M \mid M/N \text{ is not totally } \sigma\text{–torsion}\}$, there are σ–maximal elements; that is, there exists an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ and elements $N \in \Gamma$ such that for any submodule $H \supseteq N$, with $H \in \Gamma$, we have $H\mathfrak{h} \subseteq N$.

Let $N \in \Gamma$ be a σ–maximal element. Since M/N is totally σ–noetherian, there exists a companion ideal $\mathfrak{h} = (0 : \sigma(M/N)) = (N : \mathrm{Cl}_\sigma^M(N)) \in \mathcal{L}(\sigma)$. For any $m \in M$, $a \in A$ such that $ma \in N$, if $m\mathfrak{h} \not\subseteq N$, then $N + mA \notin \Gamma$; hence $M/(N + mA)$ is totally σ–torsion.

Consider the short exact sequence $0 \to (N+mA)/N \to M/N \to M/(N+mA) \to 0$; since σ is a left exact functor, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Cl}_\sigma^{N+mA}(N)/N & \rightarrowtail & \mathrm{Cl}_\sigma^{M}(N)/N & \twoheadrightarrow & \mathrm{Cl}_\sigma^{M}(N)/\mathrm{Cl}_\sigma^{N+mA}(N) \\
\| & & \| & & \| \\
\sigma((N+mA)/N) & \rightarrowtail & \sigma(M/N) & \twoheadrightarrow & \sigma(M/(N+mA)) \\
\downarrow & & \downarrow & & \| \\
(N+mA)/N & \rightarrowtail & M/N & \twoheadrightarrow & M/(N+ma)
\end{array}$$

Therefore, $(M/(N+mA))\mathfrak{h} = 0$, and we have: $M\mathfrak{h} \subseteq (N+mA)$, so $Ma\mathfrak{h} \subseteq N$. □

Once we have shown the existence of totally σ–prime submodules of a non totally σ–torsion and totally σ–noetherian module, we shall check the existence of submodules which are totally σ–prime modules; i.e., of associated totallt prime ideals with respect to a hereditary torsion theory. First we prove the existence of totally σ–prime ideals which are annihilators of elements of M.

Theorem 4.2. Let M be a totally σ–noetherian module (we assume A is totally σ–noetherian), the σ–maximal elements of

$$\Gamma = \{\mathrm{Ann}(m) \mid m \in M \setminus \sigma M\}$$

are totally σ–prime ideals.

Proof. If M is σ–torsion then $\Gamma = \emptyset$, and the result holds. Otherwise, if M is not σ–torsion, then $\Gamma \neq \emptyset$; since A is totally σ–noetherian there are σ–maximal elements in Γ; i.e., there exists $\mathfrak{k}$ and elements $\mathrm{Ann}(m) \in \Gamma$ such that if $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(x)$, for some $\mathrm{Ann}(x) \in \Gamma$, then $\mathrm{Ann}(x)\mathfrak{k} \subseteq \mathrm{Ann}(m)$.

Let $\mathrm{Ann}(m) \in \Gamma$ be a σ–maximal element; by Corollary (3.3) there exists $\mathfrak{h} = \mathrm{Ann}(\sigma(mA)) \in \mathcal{L}(\sigma)$. We can take $\mathfrak{h} \subseteq \mathfrak{k}$ (simply take the product $\mathfrak{h}\mathfrak{k}$).

If $a, b \in A$ satisfy $ab \in \mathrm{Ann}(m)$ we have the two following options:

(1) $a\mathfrak{h} \subseteq \mathrm{Ann}(m)$, hence $ma\mathfrak{h} = 0$, and $ma \in \sigma M$; conversely, if $ma \in \sigma M = (0 : \mathfrak{h})$, then $ma\mathfrak{h} = 0$, and $a\mathfrak{h} \subseteq \mathrm{Ann}(m)$.

(2) $a\mathfrak{h} \nsubseteq \mathrm{Ann}(m)$, hence $ma\mathfrak{h} \neq 0$, and $ma \notin \sigma(mA) = (0 : \mathfrak{h})$. Therefore we have an inclusion $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(ma)$, hence $\mathrm{Ann}(ma)\mathfrak{h} \subseteq \mathrm{Ann}(m)$, and $b\mathfrak{h} \subseteq \mathrm{Ann}(ma)\mathfrak{h} \subseteq \mathrm{Ann}(m)$.

In consequence, $\mathrm{Ann}(m) \subseteq A$ is totally σ–prime with companion ideal $\mathfrak{h}$. □

Lemma 4.3. *Let M be a module, $m \in M \setminus \sigma M$, and $\mathfrak{h} \in \mathcal{L}(\sigma)$; the following statements are equivalent:*

(a) mA is totally σ–prime and $\mathfrak{h} = \mathrm{Ann}(\sigma(mA))$.

(b) $\mathrm{Ann}(m) \subseteq A$ is totally σ–prime with companion ideal $\mathfrak{h}$.

(c) $\mathfrak{p} = \mathrm{Ann}(m\mathfrak{h}) \subseteq A \in \mathcal{K}(\sigma)$ and $mA/(0 : \mathfrak{h})$ is $\sigma_{A\setminus\mathfrak{p}}$–torsionfree.

Proof. (a) ⇒ (b), (c). It is a consequence of [8, Theorem 4.4].

(c) ⇒ (a). It is a consequence of [8, Proposition 6.1].

(b) ⇒ (c). It is a consequence of [8, Proposition 4.3]. □

Corollary 4.4. *Let $\mathfrak{a} \subseteq A$ be an ideal, and $\mathfrak{h} \in \mathcal{L}(\sigma)$, the following statements are equivalent:*

(a) $\mathfrak{a} \subseteq A$ is a totally σ–prime ideal with companion ideal $\mathfrak{h} = \mathrm{Ann}(\sigma(A/\mathfrak{a}))$.

(b) $\mathfrak{p} = \mathrm{Ann}(\mathfrak{h}/\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h}) \subseteq A$ is a prime ideal, and $A/(\mathfrak{a} : \mathfrak{h})$ is $\sigma_{A\setminus\mathfrak{p}}$–torsionfree.

The last results establish a bijective correspondence between a subset of totally σ–prime ideals and a subset of prime ideals of A. Therefore, the next definition can be realized either for totally σ–prime ideals or for prime ideals, we prefer the latest option.

Let M be a non σ–torsion module, a prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ is a **associated totally σ–prime ideal** to M whenever there exists an element $m \in M \setminus \sigma M$ such that $\mathfrak{p} = \operatorname{Ann}(m\mathfrak{h})$, being $\mathfrak{h} = \operatorname{Ann}(\sigma(mA))$. In other words, there exists $m \in M \setminus \sigma M$ such that $\mathfrak{p} = (\sigma(mA) : m) = (\sigma M : m)$. We represent by $\operatorname{Ass}_{t\sigma}(M)$ the set of all associated totally σ–prime ideals of M.

As a consequence of Theorem (4.2) we have:

Proposition 4.5. *If M is a totally σ–noetherian module which is not σ–torsion, then* $\operatorname{Ass}_{t\sigma}(M) \neq \varnothing$.

Remark 4.6. Note that another case in which we can assure that the set of associated totally σ–prime ideals is non–empty is when M is a totally σ–prime module. Indeed, if $\mathfrak{p} = \operatorname{Cl}^A_\sigma(\operatorname{Ann}(M)) \subseteq A$, then $\operatorname{Ass}_{t\sigma}(M) = \{\mathfrak{p}\}$.

Lemma 4.7. *Let M be a module, the following statements hold:*
(1) If M is totally σ–noetherian, then σM is totally σ–torsion.
(2) If M is totally σ–noetherian and σ–torsion, then it is totally σ–torsion.
(3) M is totally σ–noetherian if, and only if, σM is totally σ–torsion and $M/\sigma M$ is totally σ–noetherian.

Proof. (1). The case when M is totally σ–torsion is obvious. Otherwise, there exist $\mathfrak{h}$ and $m_1,\ldots,m_t \in M$ such that $M\mathfrak{h} \subseteq (m_1,\ldots,m_t)$. If M is σ torsion then $\mathfrak{h}' = \operatorname{Ann}(m_1,\ldots,m_t) \in \mathcal{L}(\sigma)$, and $M\mathfrak{h}\mathfrak{h}' = 0$, which is a contradiction.

(2). It is obvious because $M = \sigma M$ is totally σ–torsion.

(3). If σM is totally σ–torsion, then both σM and $M/\sigma M$ are totally σ–noetherians, hence M is. □

A similar result can be established for totally σ–artinian modules.

Lemma 4.8. *Let M be a module, the following hold:*
(1) If M is totally σ–artinian then σM is totally σ–torsion.
(2) If M is totally σ–artinian and σ–torsion, then it is totally σ–torsion.
(3) M is totally σ–artinian if, and only if, σM is totally σ–torsion and $M/\sigma M$ is totally σ–artinian.

Proof. (1). We may assume M is σ–torsion. Consider the family $\Gamma = \{mA \mid 0 \neq m \in M\}$, since M is totally σ–artinian, there exist an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ and a σ–minimal element in Γ, say mA, that satisfies $xA\mathfrak{h} \subseteq mA$ for every $0 \neq x \in M$; in particular we have $M\mathfrak{h} \subseteq mA$, and $\mathfrak{h}\operatorname{Ann}(m) \in \mathcal{L}(\sigma)$ is an annihilator of M, so M totally σ–torsion.

(2) is obvious.

(3) is a consequence of (1) and [5, Lemma 3.3]. □

Proposition 4.9. *If N is a totally σ–prime module, $\mathfrak{t} \in \mathcal{L}(\sigma)$, and $T = \operatorname{Ann}_N(\mathfrak{t}) = \{n \in N \mid n\mathfrak{t} = 0\}$, then N/T is totally σ–prime and N and N/T have the same associated prime ideals.*

Proof. Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ the companion ideal of N. It is clear that N/T is not totally σ–torsion. Let $\overline{n} = n + T \in N/T$ such that $\overline{n}a = 0$, then $na \in T$, and $na\mathfrak{t} = 0$; hence either $n\mathfrak{t}\mathfrak{h} = 0$, and $\overline{n}\mathfrak{h} = 0$, or $Na\mathfrak{h} = 0$, and $(N/T)a\mathfrak{h} = 0$.

Observe that the ideal $\mathfrak{h}$ is companion with both N and N/T, hence we have two prime ideals: $\mathfrak{p} = \operatorname{Ann}(N\mathfrak{h}) = (0 : N\mathfrak{h})$ and $\mathfrak{p}' = \operatorname{Ann}((N/T)\mathfrak{h}) = (T : N\mathfrak{h})$. The inclusion $\mathfrak{p} \subseteq \mathfrak{p}'$ is clear; otherwise, if $y \in \mathfrak{p}'$ then $N\mathfrak{h}y \subseteq T$, and $N\mathfrak{h}y\mathfrak{t} = 0$, hence $y\mathfrak{t} \subseteq \mathfrak{p}$, and we obtain that $y \in \mathfrak{p}$. □

Proposition 4.10. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of modules, the following statements hold:*
(1) $\operatorname{Ass}_{t\sigma}(M_1) \subseteq \operatorname{Ass}_{t\sigma}(M_2)$.

(2) $\mathrm{Ass}_{t\sigma}(M_2) \subseteq \mathrm{Ass}_{t\sigma}(M_1) \cup \mathrm{Ass}_{t\sigma}(M_3)$.

Proof. (1) is obvious.

(2). Let $\mathfrak{p} \in \mathrm{Ass}_{t\sigma}(M_2)$, there exists $m \in M_2 \setminus \sigma M_2$ such that mA is totally σ–prime, $\mathfrak{h} = \mathrm{Ann}(\sigma(mA))$, and $\mathfrak{p} = \mathrm{Ann}(m\mathfrak{h})$. If $\mathfrak{p} \notin \mathrm{Ass}_{t\sigma}(M_1)$, then $mA \cap N$ is totally σ–torsion. Since we have $\frac{mA+M_1}{M_1} \cong \frac{mA}{mA\cap M_1}$, then $\frac{mA+M_1}{M_1} = (m+M_1)A$ is a totally σ–prime module with associated prime ideal $\mathfrak{p}$. □

We can paraphrase [8, Proposition 4.3] as: for any non totally σ–torsion A–module M the following statements are equivalent:
(a) M is a totally σ–prime module.
(b) $\mathfrak{h} = (0 : \sigma M) \in \mathcal{L}(\sigma)$ and $\sigma M \subseteq M$ is a prime submodule.

Hence we can apply it to study the behaviour of totally prime modules, in the following sense.

Proposition 4.11. *Given a module M such that σM is totally σ–torsion the following statements hold:*
(1) For any submodule $N \subseteq M$ we have N is a totally σ–prime module if, and only if, $(N+\sigma M)/\sigma M$ is a prime module.
(2) If $N \subseteq M$ is a totally σ–prime module, then $N+\sigma M$ is a totally σ–prime module.
(3) There exists a bijective correspondence between submodules of $M/\sigma M$ which are prime modules and submodules of M containing σM which are totally σ–prime modules.
(4) There is an equality $\mathrm{Ass}_{t\sigma}(M) = \mathrm{Ass}(M/\sigma M)$.

Proof. (1). By the hypothesis $N/\sigma N$ is prime; since $N/\sigma N \cong N/(N\cap\sigma M) \cong (N+\sigma M)/\sigma M$, then $(N+\sigma M)/\sigma M$ is prime. On the other hand, since σM is totally σ–torsion, then σN is totally σ–torsion.

(2), (3) and (4) are obvious. □

Theorem 4.12. Let M be a totally σ–noetherian non σ–torsion module, then $\mathrm{Ass}_{t\sigma}(M)$ is finite.

Proof. Since M a totally σ–noetherian module then $\mathrm{Ass}_{t\sigma}(M) \neq \emptyset$. Given $\mathfrak{p}_1 \in \mathrm{Ass}_{t\sigma}(M)$, there exists $m_1 \in M \setminus \sigma M$ such that m_1A is a totally σ–prime module, $\mathfrak{h}_1 = \mathrm{Ann}(\sigma(m_1A))$, and $\mathfrak{p}_1 = \mathrm{Ann}(m_1\mathfrak{h}_1)$.

Say $M_1 = m_1A$. We consider M/M_1; if it is not totally σ–torsion, then it is not σ–torsion; since it is totally σ–noetherian, then $\mathrm{Ass}_{t\sigma}(M/M_1) \neq \emptyset$, and there exists $\mathfrak{p}_2 \in \mathrm{Ass}_{t\sigma}(M/M_1)$; hence there exists $\overline{m_2} = m_2 + M_1 \in M/M_1 \setminus \sigma(M/M_1)$ such that $\overline{m_2}A$ is a totally σ–prime module, $\mathfrak{h}_2 = \mathrm{Ann}(\sigma(\overline{m_2}A))$, and $\mathfrak{p}_2 = \mathrm{Ann}(\overline{m_2}\mathfrak{h}_2)$. Now we put $M_2 = (m_1, m_2)$, and consider M/M_2; if it is not totally σ–torsion we build M_3 and so on.

If for some M_s we have M/M_s is totally σ–torsion we can not continue with this procedure, and we have a chain

$$0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots \subsetneqq M_s \subseteq M$$

Being each quotient M_i/M_{i-1} a totally σ–prime modules with associated prime ideal $\mathfrak{p}_i$, for $i = 1, \ldots, s$. In consequence, $\mathrm{Ass}_{t\sigma}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ has finitely many elements. □

5 Filtration of prime submodules

The set $\mathrm{Ass}_{t\sigma}(M)$ has a lot of information on M itself, as we'll show in the following. First we point out that we can distinguish two kinds of elements in $\mathrm{Ass}_{t\sigma}(M)$. On one hand the minimal elements, and, on the other hand, the rest of elements called embedded associated totally prime ideals. A particular and interesting situation appears when there is no embedded elements; that is, when $\mathrm{Min}(\mathrm{Ass}_{t\sigma}(M)) = \mathrm{Ass}_{t\sigma}(M)$, the unmixed modules, we shall show that this is the case, among others, of totally artinian modules.

Since we shall work with submodules of M which are totally prime modules, we introduce new notation to manipulate them.

Let M be a module and $N \subseteq H \subseteq M$ submodules:

- If $N \subseteq H$ is a totally σ–prime submodule of H, we say H is a **totally σ–prime extension** of N.
- If the associated prime ideal to $N \subseteq H$ is $\mathfrak{p}$, we call it a **totally $(\sigma,\mathfrak{p})$–prime submodule** of H, and H a **totally $(\sigma,\mathfrak{p})$–prime extension** of N.
- A totally $(\sigma,\mathfrak{p})$–prime extension $N \subseteq H \subseteq M$ is **maximal** whenever, for any totally $(\sigma,\mathfrak{p})$–prime extension $N \subseteq L \subseteq M$ with $H \subseteq L$, we have $H = L$.

These definitions are adapted from [3].

Lemma 5.1. *Let $\mathfrak{p},\mathfrak{q} \subseteq A$ be prime ideals. If an A–module M is $\mathfrak{p}$–prime and $\mathfrak{q}$–prime, then $\mathfrak{p} = \mathfrak{q}$.*

Proof. If M is $\mathfrak{p}$–prime for every $0 \neq x \in M$ we have $\operatorname{Ann}(x) = \mathfrak{p}$, and the same for $\mathfrak{q}$; therefore, $\mathfrak{p} = \mathfrak{q}$. □

In the following we shall show that maximal elements of $\operatorname{Ass}_{t\sigma}(M)$ allow to construct chains of totally prime submodules; we start with the next theorem.

Theorem 5.2. Given a totally σ–noetherian module M and a prime ideal $\mathfrak{p} \in \operatorname{Ass}_{t\sigma}(M)$, which is maximal, there exists a submodule $N \subseteq M$ which is a maximal totally $(\sigma,\mathfrak{p})$–extension of σM.

Proof. We may assume M is σ–torsionfree, hence $\mathfrak{p} \in \operatorname{Ass}(M)$. Since $\mathfrak{p} \in \operatorname{Ass}(M)$, there exists $0 \neq x \in M$ such that $(0:x) = \mathfrak{p}$. Let $N = \{x \in M \mid \mathfrak{p} \subseteq (0:x)\}$; it is clear that $N \neq 0$. We define $\Gamma = \{(0:x) \mid 0 \neq x \in N\}$; this set is non–empty, and since M is totally σ–noetherian, there exists $x \in M$ such that $(0:x) \in \Gamma$ is σ–maximal; i.e., there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that, for any $(0:m) \in \Gamma$ satisfying $(0:x) \subseteq (0:m)$ we have $(0:m)\mathfrak{h} \subseteq (0:x)$. We claim $(0:x) \subseteq A$ is prime; indeed, if $ab \in (0:x)$ and $a \notin (0:x)$, then $xa \neq 0$, and $b \in (0:xa)$; but $(0:x) \subseteq (0:xa)$, and by the maximality we have $(0:xa)\mathfrak{h} = (0:x)$, hence $b\mathfrak{h} \subseteq (0:x)$, and $xb\mathfrak{h} = 0$, which implies that $xb \in \sigma M = 0$; i.e., $b \in (0:x)$. In consequence, $(0:x) \in \operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$ is prime. Since $\mathfrak{p} \subseteq (0:x)$, by the maximality of $\mathfrak{p}$ we have $\mathfrak{p} = (0:x)$, and N is a $\mathfrak{p}$–prime module. □

Remark 5.3. Observe that, in this theorem, N is a σ–closed submodule of M; that is, $\operatorname{Cl}_\sigma^M(N) = N$.

The maximal totally prime extensions have the property if limit the size of the set $\operatorname{Ass}_{t\sigma}(M/N)$, as the following theorem shows.

Theorem 5.4. Given a totally σ–noetherian module, a prime ideal $\mathfrak{p} \in \operatorname{Ass}_{t\sigma}(M)$, and a submodule N which is a maximal totally $(\sigma,\mathfrak{p})$–extension of 0, we have $\operatorname{Ass}_{t\sigma}(M/N) \subseteq \operatorname{Ass}_{t\sigma}(M)$. As a consequence we have the equality:

$$\operatorname{Ass}_{t\sigma}(M) = \operatorname{Ass}_{t\sigma}(N) \cup \operatorname{Ass}_{t\sigma}(M/N).$$

Proof. We may assume M is σ–torsionfree, hence $\mathfrak{p} \in \operatorname{Ass}(M)$, and N is a maximal $\mathfrak{p}$–prime extension of 0. Let $\mathfrak{q} \in \operatorname{Ass}_{t\sigma}(M/N) = \operatorname{Ass}\left(\frac{M/N}{\sigma(M/N)}\right) = \operatorname{Ass}\left(M/\operatorname{Cl}_\sigma^M(N)\right)$. Since $\sigma(M/N)$ is totally σ–torsion, there exists $\mathfrak{k} \in \mathcal{L}(\sigma)$ such that $\operatorname{Cl}_\sigma^M(N)\mathfrak{k} \subseteq N$. Hence $\operatorname{Cl}_\sigma^M(N)$ is $\mathfrak{p}$–prime, and, by the maximality of N, we have $\operatorname{Cl}_\sigma^M(N) = N$. There exists $m \in M \setminus N$ such that $\mathfrak{q} = (N:m)$; if we define $H = N + mA$, then $N \subseteq H$ is a $\mathfrak{q}$–prime submodule.

We continue analyzing the relationship of $\mathfrak{p}$ and $\mathfrak{q}$.

Case 1. If $\mathfrak{p} \not\subseteq \mathfrak{q}$, for any $x \in H \setminus N$ we have $\mathfrak{p} \not\subseteq (0:x) \subseteq (N:x) = \mathfrak{q}$, hence there exists $y \in \mathfrak{p} \setminus \mathfrak{q}$ such that $xy \neq 0$ and $xy \notin N$. Since $xy\mathfrak{q} \subseteq N$, and since $x\mathfrak{q} \subseteq N$, then $xy\mathfrak{q} = 0$ because $y \in \mathfrak{p}$. In consequence, $\mathfrak{q} \subseteq (0:xy) \subseteq (N:xy) = \mathfrak{q}$, and $\mathfrak{q} \in \operatorname{Ass}(M)$.

Case 2. If $\mathfrak{p} \subseteq \mathfrak{q}$; we claim $H\mathfrak{p} = N$. There are two possibilities:

- First: $H\mathfrak{p} \neq 0$, there exist $x \in H \setminus N$, and $y \in \mathfrak{p}$ such that $xy \neq 0$. Since $x\mathfrak{q} \subseteq N$ and $y \in \mathfrak{p}$, then $xy\mathfrak{q} = 0$, hence $\mathfrak{q} \subseteq (0:xy) = \mathfrak{p}$ because $xy \in N$. In consequence, $\mathfrak{p} = \mathfrak{q}$, and $\mathfrak{q} \in \operatorname{Ass}(M)$.

- Second: $H\mathfrak{p} = 0$ (in this case we have a contradiction). For any $x \in H \setminus N$ we have: $\mathfrak{p} \subseteq (0 : x) \subseteq (N : x) = \mathfrak{q}$. If $\mathfrak{p} \subseteq (0 : x) \subsetneq \mathfrak{q}$, for any $y \in \mathfrak{q} \setminus (0 : x)$ we have $0 \neq xy \in N$, hence $(0 : x) \subseteq (0 : xy) = \mathfrak{p}$, and $(0 : x) = \mathfrak{p}$. In consequence, $0 \subseteq H$ is a $\mathfrak{p}$–prime extension, which is a contradiction with the maximality of N.

□

In conclusion, the existence of maximal prime extensions, N of 0, allows us to control the set $\mathrm{Ass}_{t\sigma}$ of the quotient module M/N, they allow to count the associated totally prime ideals.

6 Primary decomposition

Let $\mathfrak{p} \subseteq A$ be a prime ideal.

- An A–module M is **totally σ–coirreducible** (= **totally σ–uniform**) whenever, M is not σ–torsion, σM is totally σ–torsion and for any non totally σ–torsion submodules $N_1, N_2 \subseteq M$ we have $N_1 \cap N_2 \nsubseteq \sigma M$. See [8].
- A submodule $N \subseteq M$ of an A–module M is a **totally σ–irreducible submodule** whenever M/N is totally σ–coirreducible:

Lemma 6.1 ([8, Proposition 5.6])**.** *Every totally σ–prime ideal $\mathfrak{a} \subseteq A$ is a totally σ–irreducible ideal.*

The following definition of totally σ–primary are then natural extensions of the classical ones.

- An M–module M is a **totally σ–coprimary module** whenever M is not totally σ–torsion, and, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that for any $m \in M$, $a \in A$, if $ma = 0$ either $m\mathfrak{h} = 0$ or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} = 0$.
- A submodule $N \subseteq M$ is a **totally σ–primary submodule** whenever the quotient M/N is a totally σ–coprimary module.

The ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$, in the above definition is called a companion ideal of M. For any totally σ–coprimary module M there are evident consequences that should be emphasized.

Lemma 6.2. *For any A–module M the following statements hold:*
*(1) Let σ be a finite type hereditary torsion theory in **Mod**–A, and M is totally σ–coprimary, then $(0 : \sigma M) \in \mathcal{L}(\sigma)$, and it is a companion ideal of M.*
(2) If $N \subseteq M$ and M/N is not totally σ–torsion, then $N \subseteq M$ is a totally σ–primary submodule if, and only if, $(N : \mathrm{Cl}^M_\sigma(N)) \in \mathcal{L}(\sigma)$ and $\mathrm{Cl}^M_\sigma(N) \subseteq M$ is a primary submodule.
(3) If $N \subseteq M$ is a totally σ–primary submodule, then $\mathrm{Cl}^M_\sigma(N) \subseteq M$ is a primary submodule.

Proof. (1). If $\mathfrak{h} \in \mathcal{L}(\sigma)$ is a companion ideal of M, for any $m \in \sigma M$ there exists $\mathfrak{k} \subseteq \mathcal{L}(\sigma)$, finitely generated, such that $m\mathfrak{k} = 0$. If $m\mathfrak{h} \neq 0$, for any $k \in \mathfrak{k}$ there exists $n_k \in \mathbb{N}$ such that $Mk^{n_k}\mathfrak{h} = 0$; hence, there exists $n \in \mathbb{N}$ such that $M\mathfrak{k}^n\mathfrak{h} = 0$, which is a contradiction because M is not totally σ–torsion. Consequently, $m\mathfrak{h} = 0$, and σ is totally σ–torsion. We can take $\mathfrak{h} = \mathrm{Ann}(\sigma M)$ as companion ideal of M.

(2). Necessary condition. Since $\sigma(M/N)$ is totally σ–torsion, then $\mathfrak{h} = (N : \mathrm{Cl}^M_\sigma(N)) \in \mathcal{L}(\sigma)$. Otherwise, if $m \in M$ and $a \in A$ satisfy $ma \in \mathrm{Cl}^M_\sigma(N)$, then $ma\mathfrak{h} \subseteq N$; then either $m\mathfrak{h}\mathfrak{h} \nsubseteq N$, hence $m \in \mathrm{Cl}^M_\sigma(N)$, or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} \subseteq N$, and $Ma^n \subseteq \mathrm{Cl}^M_\sigma(N)$.

Sufficient condition. Since $(N : \mathrm{Cl}^M_\sigma(N)) \in \mathcal{L}(\sigma)$, we take $\mathfrak{h} = (N : \mathrm{Cl}^M_\sigma(N))$. If $m \in M$ and $a \in A$ satisfy $ma \in N \subseteq \mathrm{Cl}^M_\sigma(N)$, then either $m \in \mathrm{Cl}^M_\sigma(N)$, hence $m\mathfrak{h} \subseteq N$, or there exists $n \in \mathbb{N}$ such that $Ma^n \subseteq \mathrm{Cl}^M_\sigma(N)$, and $Ma^n\mathfrak{h} \subseteq N$.

(3). As a consequence of (2) we have $\mathrm{Cl}^M_\sigma(N) \neq M$. Otherwise, let $\mathfrak{h} \in \mathcal{L}(\sigma)$ be a companion ideal of N, for any $m \in M$ and $a \in A$, if $ma \in \mathrm{Cl}^M_\sigma(N)$, then either $m\mathfrak{h} \subseteq \mathrm{Cl}^M_\sigma(N)$, hence $m \in \mathrm{Cl}^M_\sigma(N)$; or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} \subseteq \mathrm{Cl}^M_\sigma(N)$, hence $Ma^n \subseteq \mathrm{Cl}^M_\sigma(N)$. □

Lemma 6.3. *Let M be a totally σ–coprimary module, the following statements hold:*
(1) $M/\sigma M$ is totally σ–coprimary.
(2) If $\mathfrak{t} \in \mathcal{L}(\sigma)$, and $T = \mathrm{Ann}_M(\mathfrak{t})$, then M/T is totally σ–coprimary.

Proof. Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ be a companion ideal of M.

(1). For any $m \in M$ and $a \in M$ let $\overline{m} = m + \sigma M$. If $\overline{m}a = 0$, then $ma \in \sigma M$, and there exists $\mathfrak{k} \in \mathcal{L}(\sigma)$ such that $ma\mathfrak{k} = 0$; therefore, $m\mathfrak{k}\mathfrak{h} = 0$, hence $\overline{m} = 0$, or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} = 0$, and $(M/\sigma M)a^n = 0$.

(2). For any $m \in M$ and $a \in M$ let $\overline{m} = m + T$. If $\overline{m}a = 0$, then $ma \in T = \mathrm{Ann}_M(\mathfrak{t})$, hence $ma\mathfrak{t} = 0$, and we can continue as before. □

This result allows us to construct new totally coprimary modules, we shall give two different examples.

Corollary 6.4. *Let $N \subseteq M$ be a submodule, and $\mathfrak{t} \in \mathcal{L}(\sigma)$ be an ideal, then the following statements are equivalent:*
(a) $N \subseteq M$ is totally σ–primary.
(b) $(N : \mathfrak{t}) \subseteq M$ is totally σ–primary.

Proof. Note that there is a short exact sequence $0 \to (N : \mathfrak{t})/N \to M/N \to M/(N : \mathfrak{t}) \to 0$, where $(N : \mathfrak{t})$ is totally σ–torsion, hence M/N is totally σ–torsion if, and only if, $M/(N : \mathfrak{t})$ is.

(a) $\Rightarrow$ (b). By Lemma (6.3), since M/N is totally σ–coprimary, then $\frac{M/N}{\mathrm{Ann}_{M/N}(\mathfrak{t})} \cong \frac{M}{(N:\mathfrak{t})}$ is totally σ–coprimary.

(b) $\Rightarrow$ (a). Let $m \in M$ and $a \in A$ such that $ma \in N \subseteq (N : \mathfrak{t})$, and let $\mathfrak{h} \in \mathcal{L}(\sigma)$ be a companion ideal of $M/(N : \mathfrak{t})$, then either $m\mathfrak{h} \subseteq (N : \mathfrak{t})$, hence $m\mathfrak{h}\mathfrak{t} \subseteq N$, or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} \subseteq (N : \mathfrak{t})$, hence $Ma^n\mathfrak{h}\mathfrak{t} \subseteq N$. In conclusion, M/N is totally σ–coprimary with companion ideal $\mathfrak{h}\mathfrak{t}$. □

Corollary 6.5. *Let $N \subseteq M$ be a submodule, and $\mathfrak{t} \in \mathcal{L}(\sigma)$ be an ideal, then the following statements are equivalent:*
(a) $N \subseteq M$ is totally σ–primary.
(b) $N\mathfrak{t} \subseteq M$ is totally σ–primary.

Proof. First note that there is a short exact sequence $0 \to N/N\mathfrak{t} \to M/N\mathfrak{t} \to M/N \to 0$, where $N/N\mathfrak{t}$ is totally σ–torsion, hence $M/N\mathfrak{t}$ is totally σ–torsion if, and only if, M/N is.

(a) $\Rightarrow$ (b). Let $\mathfrak{h}$ a companion ideal of M/N, for any $m \in M$ and $a \in A$, if $ma \in N\mathfrak{t} \subseteq N$, then either $m\mathfrak{h} \subseteq N$, hence $m\mathfrak{h}\mathfrak{t} \subseteq N\mathfrak{t}$; or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} \subseteq N$, hence $Ma^n\mathfrak{h}\mathfrak{t} \subseteq N\mathfrak{t}$. Thus $N\mathfrak{t} \subseteq M$ is a totally σ–primary submodule with companion ideal $\mathfrak{h}\mathfrak{t}$.

(b) $\Rightarrow$ (a). Let $\mathfrak{h}$ a companion ideal of $N\mathfrak{t} \subseteq M$, for any $m \in M$ and $a \in A$ such that $ma \in N$ we have $ma\mathfrak{t} \subseteq N\mathfrak{t}$, then either $m\mathfrak{t}\mathfrak{h} \subseteq N\mathfrak{t} \subseteq N$, or there exists $n \in \mathbb{N}$ such that $Ma^n\mathfrak{h} \subseteq N\mathfrak{t} \subseteq N$. Consequently, $N \subseteq M$ is a totally σ–primary submodule with companion ideal $\mathfrak{h}\mathfrak{t}$. □

We rewrite the definition of totally σ–primary ideal as follows: an ideal $\mathfrak{q} \subseteq A$ is a **totally σ–primary ideal** whenever $\mathfrak{q} \notin \mathcal{L}(\sigma)$ and if $\mathfrak{h} = (\mathfrak{q} : \mathrm{Cl}^A_\sigma(\mathfrak{q}))$, for any $a, b \in A$ is $ab \in \mathfrak{q}$, then either $a\mathfrak{h} \in \mathfrak{q}$ or there exists $n \in \mathbb{N}$ such that $b^n\mathfrak{h} \in \mathfrak{q}$.

Note that the annihilator of a totally σ–coprimary A–module is a totally σ–primary ideal.

Proposition 6.6. *If M is a totally σ–coprimary module, then $\mathrm{Ann}(M) \subseteq A$ is a totally σ–primary ideal.*

Proof. If $A/\mathrm{Ann}(M)$ is not totally σ–torsion because $\mathrm{Ann}(M) \notin \mathcal{L}(\sigma)$. Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ the companion ideal of M; for any $a, b \in A$, if $ab \in \mathrm{Ann}(M)$ and $a\mathfrak{h} \nsubseteq \mathrm{Ann}(M)$, then since $mab = 0$ there exists $n \in \mathbb{N}$ such that $Mb^n\mathfrak{h} = 0$; therefore, $b^n\mathfrak{h} \subseteq \mathrm{Ann}(M)$, and $\mathrm{Ann}(M) \subseteq A$ is a totally σ–primary ideal. □

To well understand the structure of totally σ–primary modules we check that the σ–radical of the annihilator of any totally σ–coprimary A–module is a prime ideal in $\mathcal{K}(\sigma)$. In general, the **σ–radical** of a submodule $N \subseteq M$ is $\operatorname{rad}_\sigma(N) = \cap\{K \subseteq M \mid N \subseteq K, K \subseteq M \text{ is } \sigma\text{–prime}\}$ is always σ–closed. See [7].

Proposition 6.7. *Let σ be a finite type hereditary torsion theory in* ***Mod**–A*. *If $\mathfrak{q} \subseteq A$ is a totally σ–primary ideal, then $\operatorname{rad}_\sigma(\mathfrak{q}) \subseteq A$ is a prime ideal.*

Proof. If $\mathfrak{q} \subseteq A$ is a totally σ–primary ideal, then $\mathrm{Cl}_\sigma^A(\mathfrak{q}) \subseteq A$ is totally σ–primary; therefore, $\operatorname{rad}_\sigma(\mathfrak{q}) = \operatorname{rad}_\sigma(\mathrm{Cl}_\sigma^A(\mathfrak{q}))$. In addition, $\mathrm{Cl}_\sigma^A(\mathfrak{q}) \subseteq A$ is a primary ideal, hence $\mathfrak{p} = \operatorname{rad}(\mathrm{Cl}_\sigma^A(\mathfrak{q})) \subseteq A$ is prime. If $\mathfrak{p} \in \mathcal{Z}(\sigma)$, there exists $\mathfrak{h} \subseteq \mathcal{L}(\sigma)$, finitely generated, such that $\mathfrak{h} \subseteq \mathfrak{p}$; hence, there exists $n \in \mathbb{N}$ such that $\mathfrak{h}^n \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{q})$, which is a contradiction. Consequently, $\mathfrak{p} = \operatorname{rad}(\mathrm{Cl}_\sigma^A(\mathfrak{q})) = \operatorname{rad}_\sigma(\mathrm{Cl}_\sigma^A(\mathfrak{q})) \subseteq A$ is a prime ideal. □

For any A–module M and any ideal $\mathfrak{q} \subseteq A$ we have:

- If M is a totally σ–prime module and $\mathfrak{p} = \mathrm{Cl}_\sigma^A(\mathrm{Ann}(M))$, we call it a **totally $\mathfrak{p}$–prime module**.
- If M is a totally σ–coprimary module and $\mathfrak{p} = \operatorname{rad}(\mathrm{Cl}_\sigma^A(\mathrm{Ann}(M)))$, we call it a **totally $\mathfrak{p}$–coprimary module**.
- If $N \subseteq M$ is a totally σ–primary submodule and $\mathfrak{p} = \operatorname{rad}(\mathrm{Cl}_\sigma^A((N : M)))$, we call it a **totally $\mathfrak{p}$–primary submodule**.
- A family of totally σ–primary submodules $\{Q_i \mid i = 1, \ldots, s\}$ gives a **totally σ–primary decomposition** of M whenever if $\mathrm{Ass}_{t\sigma}(Q_i) = \{\mathfrak{p}_i\}$, then (1) $\mathfrak{p}_i \neq \mathfrak{p}_j$ whenever $i \neq j$, and (2) $\sigma M = \cap_{i=1}^s Q_i$.

We proceed now to characterize totally coprimary modules through associated totally prime ideals. First we extend Remark (4.6) to totally coprimary modules.

Proposition 6.8. *If A is a totally σ–noetherian ring and M is a totally σ–coprimary module, then we have $\mathrm{Ass}_{t\sigma}(M) = \{\mathfrak{p}\}$ whenever $\mathfrak{p} = \operatorname{rad}_\sigma(\mathrm{Ann}(M))$.*

Proof. We may translate the problem to study M modulo σM. Thus, by Propositions (4.5) and (4.11), we have $\varnothing \neq \mathrm{Ass}_{t\sigma}(M) = \mathrm{Ass}_{t\sigma}(M/\sigma M) = \mathrm{Ass}(M/\sigma M)$. In addition $A/\sigma A$ is totally $\overline{\sigma}$–noetherian. We can assume that A is a torsionfree totally σ–noetherian ring, and M is a torsionfree totally σ–coprimary module. Denote by $\mathfrak{p} = \operatorname{rad}(\mathrm{Ann}(M))$, and let $q \in \mathrm{Ass}(M)$; there exists $m \in M$ such that $\mathfrak{q} = \mathrm{Ann}(m)$; hence $m\mathfrak{q} = 0$, then, for any $y \in \mathfrak{q}$ there exists $n_y \in \mathbb{N}$ such that $My^{n_y} = 0$; therefore $y \in \operatorname{rad}(\mathrm{Ann}(M))$, and $\mathfrak{q} \subseteq \operatorname{rad}(\mathrm{Ann}(M)) = \mathfrak{p}$; since $\mathrm{Ann}(M) \subseteq \mathfrak{q}$ we have the equality.

Alternatively we can use that $\mathfrak{q}$ is totally σ–finitely generated; indeed, there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$ and $y_1, \ldots, y_t \in \mathfrak{q}$ such that $\mathfrak{q}\mathfrak{h} \subseteq \langle y_1, \ldots, y_t \rangle$; therefore, there exists $n \in \mathbb{N}$ such that $(\mathfrak{q}\mathfrak{h})^n \subseteq \langle y_1, \ldots, y_t \rangle^t \subseteq \mathrm{Ann}(M)$, and $\mathfrak{q} \subseteq \mathfrak{p}$. □

The converse of this result also holds whenever we add an extra condition to M. Thus we have:

Theorem 6.9. Let A be a totally σ-noetherian ring and M a totally σ–finitely generated A–module which is not totally σ–torsion. If $\mathrm{Ass}_{t\sigma}(M) = \{\mathfrak{p}\}$, then $\mathfrak{p} \in \mathcal{K}(\sigma)$ and M is totally $\mathfrak{p}$–coprimary.

Proof. Since M is totally σ–finitely generated then, $\mathfrak{h} = Ann(\sigma M) \in \mathcal{L}(\sigma)$. On the other hand, since $\mathrm{Ass}_{t\sigma}(M) = \{\mathfrak{p}\}$, there exists $x \in M \setminus \sigma M$ such that $\mathfrak{p} = (\sigma M : x)$.

Let $m \in M$, and $a \in A$ such that $ma = 0$. If $m \in \sigma M$, then $m\mathfrak{h} = 0$. If $m \in M \setminus \sigma M$ we proceed as follow. Define $\Gamma = \{(\sigma M : z) \mid z \in M \setminus \sigma M \text{ and } za = 0\}$. This family is non–empty because $(\sigma M : x) \in \Gamma$. Since A is totally σ–noetherian, there is a companion ideal $\mathfrak{k} \in \mathcal{L}(\sigma)$, and there are σ–maximal elements in Γ. Say $(\sigma M : z) \in \Gamma$ an element σ–maximal, we show that $(\sigma M : z) \subseteq A$ is a prime ideal. Let $u, v \in A$ such that $uv \in (\sigma M : z)$ and $u \notin (\sigma M : z)$, then $zu \in M \setminus \sigma M$, and $(\sigma M : zu) \in \Gamma$; therefore, $(\sigma M : zu)\mathfrak{k} \subseteq (\sigma M : z)$; that is, $v\mathfrak{k} \subseteq (\sigma M : z)$; hence $zv\mathfrak{k} \subseteq \sigma M$, and $zv \in \sigma M$. This means that $(\sigma M : z) \in \mathrm{Ass}_{t\sigma}(M)$, so $a \in (\sigma M : z) = \mathfrak{p}$. In conclusion, $\mathrm{Ann}(m) \subseteq \mathfrak{p}$ for every $m \in M \setminus \sigma M$.

Since $\text{Ass}(M) = \{\mathfrak{p}\}$, by Theorem (5.2), there is a maximal $\mathfrak{p}$–extension of σM, say M_1, that, by Remark (4.6) is a σ–closed submodule.

Now using Theorem (5.4) we have $\text{Ass}_{t\sigma}(M) = \text{Ass}_{t\sigma}(M_1) \cup \text{Ass}_{t\sigma}(M/M_1)$; consequently, we have $\text{Ass}(M/M_1) = \{\mathfrak{p}\}$. In this way, if $M_1 \neq M$ we can construct M_2, and so on. Since M is noetherian, there exists an index n such that $M_n = M$. This means that $p^n \subseteq (\sigma M : M)$, and $\text{rad}_\sigma(\text{Ann}(M)) = \text{rad}(\sigma M : M) = \mathfrak{p}$. Therefore, $a^n \in (\sigma M : M)$, and $a^n\mathfrak{h} = 0$. □

In view of this result, we will give the following definition: a non totally σ–torsion A–module M is a **totally $\mathfrak{p}$–cotertiary module** whenever $\text{Ass}_{t\sigma}(M) = \{\mathfrak{p}\}$, for some $\mathfrak{p} \in \mathcal{K}(\sigma)$. Consequently any totally cotertiary A–module is a totally coprimary A–module. See [9] and [13] for the earliest definition of tertiary modules.

At this point we may prove that every totally noetherian, non totally torsion, module has a totally primary decomposition.

Theorem 6.10. If M is a totally σ–noetherian non totally σ–torsion A–module, there exists a totally primary decomposition of M.

Proof. Since M is totally σ–noetherian we know that for every non totally σ–torsion submodule or homomorphic image X of M we have $\text{Ass}_{t\sigma}(X) \neq \varnothing$. In addition, σM is totally σ–torsion, hence we shall assume that M is σ–torsionfree. Let us consider $\mathfrak{p} \in \text{Ass}_{t\sigma}(M)$, and define $\Gamma = \{N \subseteq M \mid \mathfrak{p} \notin \text{Ass}_{t\sigma}(N)\}$. Since $0 \in \Gamma$, then $\Gamma \neq \varnothing$, and there are σ–maximal elements in Γ. Let $Q \in \Gamma$ be a σ–maximal element, hence $Q \subsetneq M$ is σ–closed; in the contrary there exist $y \in \text{Cl}_\sigma^M(Q)$ and $\mathfrak{k} \in \mathcal{L}(\sigma)$ such that $\mathfrak{p} = (0 : y$ and $0 \neq y\mathfrak{k} \subseteq Q$, hence $\mathfrak{q} \in \text{Ass}_{t\sigma}(Q)$, which is a contradiction. On the other hand, if $Q \subseteq M$ is not $\mathfrak{p}$–tertiary, there exists $\mathfrak{q} \in \text{Ass}_{t\sigma}(M/Q)$, $\mathfrak{q} \neq \mathfrak{p}$, and there exists $H/Q \subseteq M/Q$ such that $A/\mathfrak{q} \cong H/Q$, hence $\text{Ass}_{t\sigma}(H/Q) = \{\mathfrak{q}\}$. Therefore, we have $\text{Ass}_{t\sigma}(H) \subseteq \text{Ass}_{t\sigma}(Q) \cup \text{Ass}_{t\sigma}(H/Q) \subseteq \text{Ass}_{t\sigma}(Q) \cup \{\mathfrak{q}\}$; hence, $H \in \Gamma$, since $Q \subseteq H$, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $H\mathfrak{h} \subseteq Q$, which is a contradiction. □

Injective hulls of modules

Now we study the relationship between coprimary modules and their injective hull. If M is an A–module, we denote by $E(M)$ its injective hull.

First we recall the theory in the absolute case, that is, when $\mathcal{L}(\sigma) = \{A\}$, then totally σ–noetherian is exactly noetherian, totally σ–prime is prime, and so on. Let M be a $\mathfrak{p}$–coprimary A–module satisfying $\text{Ass}(M) = \{\mathfrak{p}\}$, hence there exists $0 \neq m \in M$ such that $\mathfrak{p} = \text{Ann}(m)$; therefore, $A/\mathfrak{p} \cong mA \subseteq M$, and there exists monomorphism $E(A/\mathfrak{p}) \subseteq E(M)$. Consequently, if M is coirreducible, there is an isomorphism $E(A/\mathfrak{p}) \cong E(M)$. Since for any essential extension $N \subseteq^e M$ we have $\text{Ass}(N) = \text{Ass}(M)$, then for any coirreducible $\mathfrak{p}$–primary A–module M we have $\text{Ass}(E(M)) = \text{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$, which means that $E(M)$ is $\mathfrak{p}$–cotertiary, but non–necessarily $\mathfrak{p}$–coprimary.

For any hereditary torsion theory σ, the use of the lattices $C(M,\sigma)$, for any A–module M allows the definitions of σ–noetherian, σ–prime and so on. This case is well understood, and decomposition properties of modules relative to σ had been established. We are interested in the decomposition of σ–torsionfree injective modules; we cite the following Matlis–like's theorem.

Theorem 6.11 (Matlis–like's theorem, [12, Propositions XIII.2.4 and 2.6])**.** Let σ be a hereditary torsion theory in **Mod**–A, if A is a σ–noetherian ring the following statements hold:

(1) Every direct sum of σ–torsionfree injective modules is injective.
(2) Every σ–torsionfree injective module has a decomposition is a direct sum of indecomposable injective modules.
(3) There is a bijective correspondence between the set of prime ideal $\mathcal{K}(\sigma)$, and the set of isomorphism classes of a σ–torsionfree indecomposable injective modules.

We are interested in totally σ–noetherian rings and modules, and we know that totally σ–noetherian implies σ–noetherian, and we can apply the Matlis–like's theorem in the totally σ–noetherian case. In the following, in this section, the ring A will be a totally σ–noetherian ring.

Since the ring A is totally σ–noetherian then non totally σ–torsion submodules as well as direct sum of $\mathfrak{p}$–cotertiary modules are totally σ–cotertiary.

By the Matlis–like's correspondence if $\mathfrak{p} \in \mathcal{K}$ is a prime ideal its image is $E(A/\mathfrak{p})$, and if E is a σ–torsionfree indecomposable injective A–module, it is cotertiary and there exists a prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$ such that $\mathrm{Ass}(E) = \{\mathfrak{p}\}$, and define the image of E as $\mathfrak{p}$.

For any A–module M the uniform dimension of M is defined as the cardinal of an independent family $\{U_i \mid i \in I\}$, of uniform submodules, such that $\oplus_i U_i \subseteq^e M$.

Proposition 6.12. *If M is a σ–torsionfree and totally σ–noetherian, the uniform dimension of M is finite.*

Proof. Let $\{U_i \mid i \in I\}$ be an independent family of submodules of M; we consider $\{V_i = \mathrm{Cl}_\sigma^M(U_i) \mid i \in I\}$, which also is an independent family. Indeed, if $x \in V_j \cap \oplus_{i\neq j} V_i$, for some index $j \in I$, there exists an ideal $\mathfrak{k} \in \mathcal{L}(\sigma)$ such that $x\mathfrak{k} \in U_j \cap \oplus_{i\neq j} U_i = 0$; hence $x \in \sigma M = 0$.

Consider the family $\Gamma = \{\oplus_{i\in F} V_i \mid F \subseteq I \text{ is finite}\}$. Since M is totally σ–noetherian, there exists a companion ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ and σ–maximal elements in Γ. Let $\oplus_{i\in F} V_i$ be a σ–maximal element; if there exists $h \in I \setminus F$, then $\oplus_{i\in F\cup\{h\}} V_i \in \Gamma$, and $\oplus_{i\in F} V_i \subseteq \oplus_{i\in F\cup\{h\}} V_i$, so $(\oplus_{i\in F\cup\{h\}} V_i)\mathfrak{h} \subseteq \oplus_{i\in F} V_i$; therefore $V_j\mathfrak{h} = 0$, which is a contradiction. Consequently, I must be a finite set. □

The proof of the next result is straightforward.

Lemma 6.13. *Let M be a σ–torsionfree module, for any uniform submodule $U \subseteq M$ the σ–closure $\mathrm{Cl}_\sigma^M(U) \subseteq M$ is also uniform.*

Let $\{U_i \mid i = 1, \ldots, t\}$ an independent family of σ–closed submodules of M such that $\oplus_{i=1}^t U_i \subseteq^e M$, there exists an isomorphism

$$E(M) \cong E(U_1 \oplus \cdots \oplus U_t) \cong E(U_1) \oplus \cdots \oplus E(U_t).$$

Let $\mathfrak{p}_i \subseteq A$ be the prime ideal such that $\mathrm{Ass}(E(U_i)) = \{\mathfrak{p}_i\}$, then $\mathrm{Ass}(M) = \mathrm{Ass}(E(M)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$, where in this family may be $\mathfrak{p}_i = \mathfrak{p}_j$ even if $i \neq j$. By eliminating repetitions we have a set, say $\{p_1, \ldots, p_s\}$, and putting together, in the same summand, all modules $E(U)$, with the same associated prime ideal, we obtain a decomposition of $E(M)$ as a direct sum of cotertiary injective modules:

$$E(M) \cong E_1 \oplus \cdots \oplus E_s,$$

being $\mathrm{Ass}(E_j) = \{\mathfrak{p}_j\}$ for $i = 1, \ldots, s$.

For any index $j \in \{1, \ldots, s\}$ we have a map

$$f_j : M \longrightarrow E(M) \cong E_1 \oplus \cdots \oplus E_s \longrightarrow E_j,$$

if we call $Q_j = \mathrm{Ker}(f_j)$, then $M/Q_j \cong \mathrm{Im}(f_j)$ is a totally σ–noetherian totally $\mathfrak{p}_j$–cotertiary module, hence $Q_j \subseteq M$ is a totally $\mathfrak{p}$–primary submodule. In addition, we have the following intersection $\cap_{j=1}^s Q_j = 0$.

Given a totally σ–noetherian A–module M a **totally σ–primary decomposition** of M is a finite family of totally σ–primary submodules $\{Q_i \mid i = 1, \ldots, s\}$ such that $0 = Q_1 \cap \ldots \cap Q_s$. The above exposition gives the proof of the following theorem.

Theorem 6.14. Any non totally σ–torsion totally σ–noetherian module has a totally σ–primary decomposition.

7 Applications

Let us consider a totally σ–artinian module M; by Lemma (4.8) we have that σM is totally σ–torsion. Therefore, a module M is totally σ–artinian if, and only if $M/\sigma M$ and σM is totally σ–artinian and totally σ–torsion.

Our aim is to check the associated totally σ–prime ideals to M, and, in particular, the set $\mathrm{Ass}_{t\sigma}(M)$.

An A–module M is **totally σ–simple** whenever it satisfies:

(1) M is not σ–torsion.
(2) σM is totally σ–torsion.
(3) There exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that, for every non σ–submodule $N \subseteq M$ we have $S\mathfrak{h} \subseteq N$.

A submodule $N \subseteq M$ is a σ**–minimal submodule** whenever

(1) N in not totally σ–torsion.
(2) There exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that for any non totally σ–torsion submodule $H \subseteq N$, we have $N\mathfrak{h} \subseteq H$.

Note that a submodule $S \subseteq M$ of a module M, which is totally σ–simple, is a σ–minimal submodule; the converse non necessarily holds.

Remark 7.1. We have that a σ–minimal submodule $N \subseteq M$ is not totally σ–torsion, and satisfies property (3) in the definition of totally σ–simple module, but we have no information on σN. Indeed, is σN is not totally σ–torsion, then $N\mathfrak{h} \subseteq \sigma N$; therefore, N is σ–torsion.

In consequence, for any non σ–torsion module M, with σM totally σ–torsion, the two notions coincide: each σ–minimal submodule is a totally σ–simple module. This is the case of totally σ–artinian modules.

Dually we may define σ**–maximal submodules** of a module M. A submodule $N \subseteq M$ is a σ–maximal submodule whenever

(1) M/N is not totally σ–torsion.
(2) There exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that for any submodule $N \subseteq H \subseteq M$, with M/H non totally σ–torsion, we have $H\mathfrak{h} \subseteq N$.

Remark 7.2. For a σ–maximal submodule $N \subseteq M$ we can consider the quotient M/N, which is not totally σ–torsion, and satisfies (3) in the definition of totally σ–simple module. But, in general, $\sigma(M/N)$ is not totally σ–torsion. As in Remark (7.1) the two notion coincide whenever M is non σ–torsion and σM is totally σ–torsion. This is the case of totally σ–noetherian modules.

In this situation we have the following result.

Proposition 7.3. *If A is not σ–torsion and σA is totally σ–torsion, for every σ–maximal ideal $\mathfrak{a} \subseteq A$ we have that A/σ is totally σ–simple.*

The reciprocal of this result is the following one:

Proposition 7.4. *Given a totally σ–simple module M we have that* $\mathrm{Ann}(M) \subseteq A$ *is σ–maximal.*

Proof. Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ the companion ideal of M. Since M is totally σ–simple, for any $m \in M \setminus \sigma M$ we have $M\mathfrak{h} \subseteq mA$, then we have $M\mathfrak{h}\mathrm{Ann}(m) \subseteq m\mathrm{Ann}(m) = 0$, and $\mathrm{Ann}(M)\mathfrak{h} \subseteq \mathrm{Ann}(m) \subseteq \mathrm{Ann}(M)$.

For any ideal $\mathfrak{a} \notin \mathcal{L}(\sigma)$ such that $\mathrm{Ann}(m) \subseteq \mathfrak{a} \subseteq A$ we have $\mathfrak{a}/\mathrm{Ann}(m) \subseteq A/\mathrm{Ann}(m) = mA$. If $\mathfrak{a}/\mathrm{Ann}(m)$ is not totally σ–torsion, we have $(A/\mathrm{Ann}(m))\mathfrak{h} \subseteq \mathfrak{a}/\mathrm{Ann}(m)$, hence $\mathfrak{h} \subseteq \mathfrak{a}$, which is a contradiction. If $\mathfrak{a}/\mathrm{Ann}(m)$ is totally σ–torsion, there exists $\mathfrak{k} = (0 : \sigma M) \in \mathcal{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{k} \subseteq \mathrm{Ann}(m)$. Consequently, $\mathrm{Ann}(m) \subseteq A$ is σ–maximal. □

Proposition 7.5. *If M is a non totally σ–torsion and totally σ–artinian module, the following statements are equivalent:*

(a) M is totally σ–prime module.
(b) The ideal $\operatorname{Ann}(M) \subseteq A$ *is σ–maximal ideal.*

Proof. (a) ⇒ (b) If M is totally σ–prime consider the family

$$\Gamma = \{N \subseteq M \mid N \text{ is not totally } \sigma\text{–torsion}\}.$$

Since $M \in \Gamma$, there exists a σ–minimal element $N \in \Gamma$. Let $\mathfrak{h} \in \mathcal{L}(\sigma)$ be the companion ideal of N. For any $n \in N \setminus \sigma N$ we have $N\mathfrak{h} \subseteq nA$. Therefore, $\operatorname{Ann}(N) = \operatorname{Ann}(m) \subseteq A$ is a σ–maximal ideal.

(b) ⇒ (a). Let $\mathfrak{h} = \operatorname{Ann}(\sigma M)$. If $\operatorname{Ann}(M) \subseteq A$ is σ–maximal, since for any $m \in M \setminus \sigma M$ we have $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(m)$, then $\operatorname{Ann}(m)\mathfrak{h} \subseteq \operatorname{Ann}(M)$.

For any $x \in M$ and $a \in A$ such that $xa = 0$, if $x \in \sigma M$, then $x\mathfrak{h} = 0$; if $x \in M \setminus \sigma M$, then $\operatorname{Ann}(x)\mathfrak{h} \subseteq \operatorname{Ann}(M)$; therefore, $a\mathfrak{h} \subseteq \operatorname{Ann}(M)$. In conclusion, M is totally σ–prime. □

Corollary 7.6. *Every associated totally σ–prime ideal to a totally σ–artinian module is σ–maximal.*

Proof. For any $\mathfrak{p} \in \operatorname{Ass}_{t\sigma}(M)$ there exists a totally σ–prime module $N \subseteq M$ such that $\mathfrak{p} = (\sigma N : N)$, or equivalently, $\mathfrak{p} = (\sigma M : m)$ for some $m \in M \setminus \sigma M$. □

Corollary 7.7. *For any totally σ–artinian module M the set* $\operatorname{Ass}_{t\sigma}(M)$ *is finite.*

References

[1] D. D. Anderson and T. Dumitrescu, *S-noetherian rings*, Comm. Algebra **30(9)** (2002), 4407–4416.

[2] H. Ansari-Toroghy and S. S. Pourmortazavi, *On S–primary submodules*, Int. Electron. J. Algebra **31** (2022), 74–89.

[3] T. Duraivel, S. Mangayarcarassy, and K. Premkumar, *Prime extension filtration of modules*, Int. J. Pure Appl. Mathematics **98 (2)** (2015), 211–220.

[4] A. Hamed and A. Malek, *S–prime ideals of a commutative ring*, Beitr. Algebra Geom. **61** (2020), 533–542.

[5] P. Jara, *An extension of S–artinian rings and modules to a hereditary torsion thery setting*, Comm. Algebra **49** (2021), 1583–1599.

[6] ______, *An extension of S–noetherian rings and modules*, Int. Electron. J. Algebra **34** (2023), 1–20, arXiv: 2011.03008.

[7] P. Jara, F. Omar, and E. Santos, *An extension of S–noetherian spectrum property*, Journal Algebra Appl. **.** (2023), 17 pp., Granada.

[8] ______, *Totally simple modules*, Granada, 2023.

[9] L. Lesieur and R. Croisot, *Algèbre noethérienne non commutative*, Gautier–Villars, 1963.

[10] J. W. Lim, *A note on S–noetherian domains*, Kyungpook Math. J. **55** (2015), 507–514.

[11] E. S. Sevim, T. Arabaci, U. Tekir, and S. Koc, *On S–prime submodules*, Turkish J. Math. **43** (2019), 1036–1046.

[12] B. Stenström, *Rings of quotients*, Springer–Verlag, Berlin, 1975.

[13] A. Verschoren, *Tertiary decomposition in Grothendieck categories*, Czechoslovak Mathematical Journal **30** (1980), 661–672.

[14] S. Visweswaran, *Some results on S–primary ideals of a commutative ring*, Beitr. Algebra Geom. **63** (2022), 247–266.