Journal name manuscript No.
(will be inserted by the editor)

Radial discrete PDE splines on Lipschitz domains

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Received: date / Accepted: date

Abstract In the framework of an elliptic partial differential equation (PDE), certain boundary conditions and a set of points to approximate in a Lipschitz domain and arbitrary dimension, we use radial basis function (RBF) techniques for the construction and characterization of discrete PDE splines. We also show convergence and derive error estimates.

Keywords Approximation · interpolation · radial basis functions · PDE

Mathematics Subject Classification (2000) 41A25 · 41A30 · 41A63 · 65D10 · 65N15

1 Introduction

Radial basis function (RBF) methods have emerged as an important and effective tool for the numerical solution of partial differential equations (PDE) in any number of dimensions and for the approximation of an unknown multivariate function by interpolation at scattered sites [7,10,16,17,18], e.g. see also [8], entering in a field traditionally tackled by finite element methods (FEM) [9]. They have been used in many applications in engineering, medical imaging, computer aided geometric design (CAGD), neural networks, and economics.

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Also, PDE surfaces, which are surfaces whose behaviour is governed by PDEs [4], have been shown to possess many modelling advantages in a wide range of fields and they frequently appear in a lot of physical problems. A combination of conditions of interpolation and approximation can be used for the PDE method of surface design, as it appears in [1,13,14,15]: on one hand, the surface has to approximate a given data set, and on the other hand, it has to be modelled by a partial differential equation. In addition, the surface has to satisfy some boundary conditions that are included along with the equation as a boundary value problem. This technique can be applied mainly to CAGD and modeling fields as we can observe in [5]. Although the method is not confined to any particular type of PDE, mainly elliptic PDEs have been considered as they produce smooth surfaces for boundary value problems. Moreover, this 2-dimensional approximation problem may be generalized to the d-dimensional case, for any positive integer d.

By using RBF techniques we will study in this paper the existence and the uniqueness of the solution of the generalized problem. Moreover, we will show the convergence of this solution to a function from which the data values are obtained. More specifically, we will extend an approach of approximation method for multivariate functions from data constituted by a given data point set and a PDE. In [13] this problem was formulated and solved by variational methods. In [14], the authors discretized the problem by using techniques in $\mathbb{R}^2$ based on FEM. However, several disadvantages appeared for the PDE splines used in this approach: the domain was polygonal, there was need for mesh generation and it was not useful for higher dimensions.

In this paper we improve the previous results. We solve the problem of obtaining our discrete PDE spline for some more general domains, as Lipschitz domains are, and for arbitrary dimensions, by using RBF techniques. We formulate our variational problem in an adequate function space, the native space, that can be the whole Sobolev space or a subset thereof. We discretize the solution of the problem in terms of RBF and we establish some estimations of the error.

The outline of the paper is as follows: In Section 2, we briefly recall some preliminary notations and results. In Section 3, the notion of a discrete PDE spline is defined. Theorem 1 is stated, which will be crucial to the rest of the paper. For the sake of clarity, the convergence results and estimates of the approximation error are also presented in this section. In Section 4, a characterization of the discrete PDE splines is obtained. Section 5 is dedicated to prove the convergence results and error estimates stated in Section 3, after establishing some relevant results on the interpolant to the given function. Details of the computation are given in Section 6. The proof of Theorem 1 will appear in the Appendix.

2 Preliminaries

Before we shall outline our problem at the beginning of the next section and state it in more detail later-on, let us give some necessary definitions and other preliminaries.

We use $\mathbb{N}$ (resp. $\mathbb{N}^*$) for the set of non-negative integers (resp. for the set of positive integers). Let $d \in \mathbb{N}^*, n \in \mathbb{N}$. Let Ω be a nonempty open set in $\mathbb{R}^d$. We will use the following notations:

- For $p \in [0,\infty)$, $L^p(\Omega)$ stands for the linear space of real Lebesgue measurable functions u such that $\int_\Omega |u(x)|^p dx < \infty$;
- $H^n(\Omega)$ represents the usual Sobolev space of order n of (classes of) functions $u \in L^2(\Omega)$, having all their partial derivatives $\partial^i u$, in the distribution sense, of order $|i| \leq n$, in $L^2(\Omega)$, where for all $i = (i_1,\dots,i_d) \in \mathbb{N}^d$,
  $|i| = i_1 + \cdots + i_d$ and $\partial^i u(x) = \dfrac{\partial^{|i|} u(x)}{\partial x_1^{i_1} \dots \partial x_d^{i_d}}$, for any $x = (x_1,\dots,x_d) \in \Omega$.

The linear space $L^2(\Omega)$ is equipped with the inner product

$$(u,v)_{0,\Omega} = \int_\Omega u(x)v(x)dx$$

and the corresponding norm $\|u\|_{0,\Omega} = (u,u)_{0,\Omega}^{\frac{1}{2}}$.

Analogously, the Sobolev space $H^n(\Omega)$ is equipped with the norm

$$\|u\|_{n,\Omega} = \left(\sum_{|i|\leq n} \|\partial^i u\|_{0,\Omega}^2 \right)^{1/2},$$

and the semi-norms

$$|u|_{k,\Omega} = \left(\sum_{|i|=k} \|\partial^i u\|_{0,\Omega}^2 \right)^{1/2}, \ 0 \leq k \leq n.$$

We clarify that, for any compact set $K \subset \mathbb{R}^d$ whose interior $\mathring{K}$ is nonempty, for the sake of simplicity we shall write $H^n(K)$ instead of $H^n(\mathring{K})$, $\|\cdot\|_{n,K}$ instead of $\|\cdot\|_{n,\mathring{K}}$, etc.

We shall also use the following notations:

- $\dfrac{\partial^j v}{\partial \mathrm{n}^j}(x)$, $j \in \mathbb{N}$, will indicate the j-th derivative of v with respect to n at $x \in \partial\Omega$, where $\mathrm{n}(x)$ is the unit outer normal vector at x. For $j = 0$, $\dfrac{\partial^0 v}{\partial \mathrm{n}^0}(x)$ indicates $v(x)$.
- For $f \in L^2(\mathbb{R}^d)$, the Fourier transform of f is defined as

$$\hat{f}(y) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}x^T y} f(x)dx.$$

We also use the generalized Fourier transform as it is defined in [11].

– Finally, with the same letter C we will denote various strictly positive constants.

We will use the theory of RBF. Denote by $\mathbb{P}_{m-1}$, $m \in \mathbb{N}$, the space of real-valued d-variate polynomials of degree at most $m-1$ ($\mathbb{P}_{-1} := \{0\}$), and by Δ_m its dimension. We suppose that Ω contains at least one $\mathbb{P}_{m-1}$-unisolvent subset.

Definition 1 A continuous symmetric function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called a conditionally positive definite kernel on Ω with respect to $\mathbb{P}_{m-1}$ if, for all $M \in \mathbb{N}^*$, all pairwise distinct centers $x_1, \dots, x_M \in \Omega$, and all $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{R}^M \setminus \{0\}$ with

$$\sum_{j=1}^{M} \alpha_j p(x_j) = 0$$

for all $p \in \mathbb{P}_{m-1}$, the inequality

$$\sum_{j=1}^{M} \sum_{k=1}^{M} \alpha_j \alpha_k \Phi(x_j, x_k) > 0$$

is valid. A conditionally positive definite kernel on Ω with respect to $\{0\}$ is called a positive definite kernel on Ω.

We now introduce the native space, $\mathcal{N}_\Phi(\Omega)$, for such a kernel Φ. Its definition will be based on finitely supported linear functionals on the space of real-valued continuous functions on Ω, $C(\Omega)$, which vanish on $\mathbb{P}_{m-1}$. Consider the set

$$L_{\mathbb{P}_{m-1}}(\Omega) := \Bigg\{ \lambda_{M,\alpha,\{x_1,\dots,x_M\}} = \sum_{j=1}^{M} \alpha_j \delta_{x_j} : M \in \mathbb{N}^*, \alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{R}^M, \\ \{x_1, \dots, x_M\} \subset \Omega, \text{ with } \lambda_{M,\alpha,\{x_1,\dots,x_M\}}(p) = 0, \text{ for all } p \in \mathbb{P}_{m-1} \Bigg\},$$

where δ_{x_j} is the evaluation functional at x_j, and equip $L_{\mathbb{P}_{m-1}}(\Omega)$ with the inner product

$$(\lambda_{M,\alpha,\{x_1,\dots,x_M\}}, \lambda_{N,\beta,\{y_1,\dots,y_N\}})_\Phi := \sum_{j=1}^{M} \sum_{k=1}^{N} \alpha_j \beta_k \Phi(x_j, y_k).$$

Definition 2 The native space corresponding to a symmetric kernel Φ that is conditionally positive definite on Ω with respect to $\mathbb{P}_{m-1}$ is defined by

$$\mathcal{N}_\Phi(\Omega) = \{f \in C(\Omega) : |\lambda(f)| \le C_f \|\lambda\|_\Phi \text{ for all } \lambda \in L_{\mathbb{P}_{m-1}}(\Omega)\}.$$

This space carries the semi-norm

$$|f|_{\mathcal{N}_\Phi(\Omega)} = \sup_{\substack{\lambda \in L_{\mathbb{P}_{m-1}}(\Omega) \\ \lambda \neq 0}} \frac{|\lambda(f)|}{\|\lambda\|_\Phi}. \tag{1}$$

The notion of a conditionally positive definite kernel generalizes the one of a conditionally positive definite function.

Definition 3 A continuous, even function $\Phi_m : \mathbb{R}^d \to \mathbb{R}$ is said to be conditionally positive definite of order m if the kernel $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by $\Phi(x, y) := \Phi_m(x-y)$ is conditionally positive definite on $\mathbb{R}^d$ with respect to $\mathbb{P}_{m-1}$. A conditionally positive definite function of order 0 is called a positive definite function.

We will usually work in the context of conditionally positive definite functions that will also be radial functions.

Definition 4 A function $\Phi_m : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a function $\phi : [0,\infty) \to \mathbb{R}$ such that $\Phi_m(x) = \phi(\|x\|_2)$ for all $x \in \mathbb{R}^d$.

3 Formulation of the problem, convergence and error estimates

In the sequel, we assume that $n > \dfrac{d}{2}$ and that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, i.e. a bounded, connected, nonempty, open subset of $\mathbb{R}^d$ with a Lipschitz-continuous boundary in the J. Necas [12] sense.

Our objective is to find a function in a finite-dimensional space spanned by radial basis function kernels (shifts in the easiest of all cases) satisfying certain conditions regarding the boundary of Ω, while simultaneously the function has to approximate a data point set in the interior of the same domain. This could take place either by interpolation or by smoothing, i.e. by minimising a certain functional.

Namely, we study the following problem: finding the (unique) minimizer σ_θ of the cost functional J in the set H^X, where (details will be given on the following pages) J is the functional defined on $H^n(\Omega)$ by

$$J(v) = \|\rho v - \beta\|_2^2 + \theta\left(|v|_{\mathcal{L}}^2 - 2(f, v)_{0,\Omega}\right).$$

This functional J includes a term for the approximation to a data vector β from the vector of pointwise values of v (through the evaluation operator ρ) on a set $\mathcal{A}$ of interior points of Ω, together with the energy of a strongly elliptic differential operator $\mathcal{L}$ in weak form, linked by a positive parameter θ. H^X will be a set of elements of the native space of a certain conditionally positive definite kernel which interpolate the values of zero- to possibly high-order normal derivatives on a finite set $\mathcal{B}$ of points on the boundary of Ω.

Let us now present all the details. Suppose we are given

- a non-negative integer m;
- an ordered set $\mathcal{A} = \{a_1, \ldots, a_M\}$ of $M \in \mathbb{N}^*$ distinct points of Ω such that

$$ker\rho \cap \mathbb{P}_{\max\{m,n\}-1} = \{0\}, \tag{2}$$

where the operator $\rho : H^n(\Omega) \longrightarrow \mathbb{R}^M$, is given by $\rho(v) = (v(a_i))_{i=1,\ldots,M}$;

– a data vector $\beta = (\beta_1, \ldots, \beta_M) \in \mathbb{R}^M$;
– an ordered set $\mathcal{B} = \{b_1, \ldots, b_N\}$ of $N \in \mathbb{N}^*$ distinct points of $\partial\Omega$, such that there exists the unit outer normal vector $\mathrm{n}(b_\ell)$, for each $\ell = 1, \ldots, N$;

For each $\ell = 1, \ldots, N$ and $j = 0, \ldots, n-1$, we form the functionals $\lambda_{jN+\ell} = \delta_{b_\ell} \circ \dfrac{\partial^j}{\partial \mathrm{n}^j}$, where δ_{b_ℓ} is the evaluation functional at b_ℓ, i.e. $\lambda_{jN+\ell}(v) = \dfrac{\partial^j v}{\partial \mathrm{n}^j}(b_\ell)$ for each $v \in H^n(\Omega)$. We will also suppose that

$$\text{if } v \in \mathbb{P}_{n-1} \text{ and } \lambda_i(v) = 0, \text{ for all } i = 1, \ldots, Nn, \text{ then } v = 0.$$

Suppose we are also given $\Phi_m \in C^{2n}(\mathbb{R}^d)$ a conditionally positive definite function of order m, such that it has a generalized Fourier transform $\widehat{\Phi_m}$ of order m that is continuous on $\mathbb{R}^d \setminus \{0\}$. Let $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\Phi(x,y) := \Phi_m(x-y)$, be the corresponding conditionally positive definite kernel on $\mathbb{R}^d$ with respect to $\mathbb{P}_{m-1}$.

From (2) we may choose a $\mathbb{P}_{m-1}$-unisolvent set $\Xi = \{\xi_1, \ldots, \xi_{\Delta_m}\} \subset \Omega$. Let $\{p_1, \ldots, p_{\Delta_m}\}$ be a Lagrange basis of $\mathbb{P}_{m-1}$ with respect to this set Ξ. Define the kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\begin{aligned} K(x,y) :=& \Phi(x,y) - \sum_{k=1}^{\Delta_m} p_k(y)\Phi(x,\xi_k) - \sum_{k=1}^{\Delta_m} p_k(x)\Phi(\xi_k,y) \\ &+ \sum_{k=1}^{\Delta_m}\sum_{\ell=1}^{\Delta_m} p_k(x)p_\ell(y)\Phi(\xi_k,\xi_\ell) + \sum_{k=1}^{\Delta_m} p_k(x)p_k(y). \end{aligned} \tag{3}$$

From [18, Th. 10.20], the native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ carries an inner product $(\cdot,\cdot)_{\mathcal{N}_\Phi(\mathbb{R}^d)}$, with which $\mathcal{N}_\Phi(\mathbb{R}^d)$ becomes a reproducing-kernel Hilbert space with reproducing kernel $K(\cdot,\cdot)$. Moreover, $K(\cdot,\cdot)$ is a symmetric positive definite kernel on $\mathbb{R}^d$ by [18, Th. 12.9]. Hence, the native space associated to Φ, $\mathcal{N}_\Phi(\mathbb{R}^d)$, coincides with the native space associated to K, $\mathcal{N}_K(\mathbb{R}^d)$, and the inner products are the same, by [18, Th. 10.11].

We consider

– an ordered set $\{c_1, \ldots, c_{I_0}\}$ of $I_0 \in \mathbb{N}^*$ distinct points of Ω, and the corresponding evaluation functionals $\lambda_{Nn+i} := \delta_{c_i}$, $i = 1, \ldots, I_0$,
– and X will denote the finite-dimensional subspace of $\mathcal{N}_\Phi(\mathbb{R}^d)$, with $\dim X = I = Nn + I_0$, given by

$$X := \operatorname{span}\{\lambda_i^y K(\cdot,y) : i = 1, \ldots, I\}, \tag{4}$$

where the notation λ_i^y indicates that the functional λ_i acts on K viewed as a function of its second argument.

Remark 1 By restricting to our Lipschitz domain Ω, Φ may also be considered as a conditionally positive definite kernel on Ω with respect to $\mathbb{P}_{m-1}$ and, as a consequence, X may also be seen as a finite-dimensional subspace of the native space $\mathcal{N}_\Phi(\Omega)$.

Since each element of $\mathcal{N}_\Phi(\Omega)$ can be extended to $\mathcal{N}_\Phi(\mathbb{R}^d)$, and $\mathcal{N}_\Phi(\mathbb{R}^d) \subseteq C^n(\mathbb{R}^d)$ by [18, Th. 10.45], we derive that $\mathcal{N}_\Phi(\Omega) \subseteq H^n(\Omega)$. Therefore, the whole Sobolev machinery can be used for the treatment of the elements of X. □

On the one hand, the function we are looking for as our objective will have to approximate a certain data set point, and on the other hand, it will have to be modelled by an elliptic partial differential equation. Moreover, the function will have to satisfy some boundary conditions that are included along with the equations as a boundary value problem.

We are obtaining this function by minimising a functional in a subset of our finite-dimensional space X. The functional will contain the information concerning the desired conditions.

Let us detail all of this. Let $\mathcal{L} : H^{2n}(\Omega) \to L^2(\Omega)$ be a differential operator given by

$$\mathcal{L}u(x) = \sum_{|i|,|j|\le n} (-1)^{|j|} \partial^j (p_{ij}(x)\partial^i u(x)), \quad x \in \Omega, \tag{5}$$

where $p_{ij} \in C^{|j|}(\overline{\Omega})$ and $p_{ij} = p_{ji}$ for all $|i|,|j| \le n$. We now consider the symmetric bilinear form associated with $\mathcal{L}$ defined on $H^n(\Omega)\times H^n(\Omega)$ by $(u,v)_{\mathcal{L}} = \sum_{|i|,|j|\le n}(p_{ij}\partial^i u, \partial^j v)_{0,\Omega}$. In addition, we suppose that $\mathcal{L}$ is strongly elliptic on Ω, i.e.

$$\sum_{|i|,|j|\le n-1} \alpha_i p_{ij}(x)\alpha_j \ge 0, \quad \forall\, x \in \Omega, \tag{6}$$

and that there exists $\nu > 0$ such that

$$\sum_{|i|,|j|=n} \alpha_i p_{ij}(x)\alpha_j \ge \nu \sum_{|i|=n} \alpha_i^2, \quad \forall\, x \in \Omega, \tag{7}$$

for all $\alpha_i \in \mathbb{R}$, where $i \in \mathbb{N}^d$.

According to the hypotheses (6)–(7) the bilinear form $(\cdot,\cdot)_{\mathcal{L}}$ defines a semi–inner product on $H^n(\Omega)$ whose associated semi-norm is denoted by $|u|_{\mathcal{L}} = (u,u)_{\mathcal{L}}^{\frac{1}{2}}$.

Suppose we are given

- the functions $f \in L^2(\Omega)$ and $\eta_j \in C(\overline{\Omega})$, for $j = 0,\dots,n-1$;
- a vector $z = (z_i)_{i=1\dots,Nn}$ with $z_{jN+\ell} = \eta_j(b_\ell)$, for $\ell = 1,\dots,N$, $j = 0,\dots,n-1$.

We define the vector space

$$H_0^X = \{u \in X : \lambda_i(u) = 0, \quad 1 \le i \le Nn\}$$

and the set

$$H^X = \{u \in X : \lambda_i(u) = z_i, \quad 1 \le i \le Nn\}.$$

Let $\mathcal{L}$ be the operator given in (5) and let us consider the problem

$$\begin{cases} \mathcal{L}u(x) = f(x), & x \in \Omega, \\ \dfrac{\partial^j u}{\partial \mathrm{n}^j}(x) = \eta_j(x), & x \in \partial\Omega,\ 0 \le j \le n-1. \end{cases}$$

Definition 5 We say that σ_θ is a discrete PDE spline in X associated to $\mathcal{L}, \mathcal{B}, z$, $\mathcal{A}, \beta$ and $\theta > 0$, if σ_θ is a solution of the problem

$$\begin{cases} \sigma_\theta \in H^X, \\ \forall v \in H^X, \quad J(\sigma_\theta) \le J(v), \end{cases} \tag{8}$$

where J is the functional defined on $H^n(\Omega)$ by

$$J(v) = \|\rho v - \beta\|_2^2 + \theta \left(|v|_{\mathcal{L}}^2 - 2(f, v)_{0,\Omega} \right).$$

We will prove the existence and uniqueness of a solution for Problem (8) and we will state an equivalent formulation of it in the next section, characterizing the notion of a discrete PDE spline. An essential result which will play a key rôle in our paper is contained in the following theorem whose proof will be given in the Appendix:

Theorem 1 *For any $Q \in \mathbb{N}^*$, let $\alpha^{(1)}, \dots, \alpha^{(Q)} \in \mathbb{N}^d$ and $x_1, \dots, x_Q \in \mathbb{R}^d$. If the functionals $\tilde{\lambda}_i := \delta_{x_i} \circ \partial^{\alpha^{(i)}}$, $1 \le i \le Q$, with $|\alpha^{(i)}| \le n$, are pairwise distinct, meaning that $\alpha^{(i)} \ne \alpha^{(j)}$ if $x_i = x_j$ for two different $i \ne j$, then they are also linearly independent over $\mathcal{N}_\Phi(\mathbb{R}^d)$.*

To state the notion of a discrete PDE spline in Definition 5, the user provides, in the following order, the data: $d, n, \Omega, m, M, \mathcal{A}, \beta, \mathcal{B}, \Phi_m$ and the corresponding native space, the set of interior points $\{c_1, \dots, c_{I_0}\}$, and $\mathcal{L}, f, \eta_j$, for $j = 1, \dots, n-1$, to formulate the boundary value problem (z is then obtained by evaluation of the functions η_j at $\mathcal{B}$).

To establish our convergence results and error estimates, we will consider a function $g \in \mathcal{N}_\Phi(\Omega)$. This function g will be our approximand, and the convergence of the discrete PDE spline to g will be obtained by using, in terms of g, some of the elements given for the definition of the discrete PDE spline.

Firstly, the vector z for the discrete PDE spline will be taken as $z = (z_i)_{i=1\dots,Nn}$ with $z_i = \lambda_i(g)$, for $i = 1, \dots, Nn$.

Suppose we are also given a subset $\mathcal{H}$ of $(0, +\infty)$ such that $0 \in \overline{\mathcal{H}}$ and, for each $h \in \mathcal{H}$, an ordered set $\mathcal{A}^h = \{a_1, \dots, a_M\}$ of $M = M(h) \in \mathbb{N}^*$ distinct points of Ω such that the following condition holds

$$\sup_{x \in \Omega} \min_{a \in \mathcal{A}^h} \|x - a\| = h. \tag{9}$$

For each $h \in \mathcal{H}$, we define $\rho^h : H^n(\Omega) \longrightarrow \mathbb{R}^M$ as $\rho^h(v) = (v(a))_{a \in \mathcal{A}^h}$.

Finally, let us consider a data vector $\beta^h = (g(a))_{a \in \mathcal{A}^h} \in \mathbb{R}^M$ and X as the finite-dimensional subspace of $\mathcal{N}_\Phi(\Omega)$ given by (4), where we have taken $\{c_1, \dots, c_{I_0}\} := \mathcal{A}^h$. We define the nonempty convex closed set

$$H^X = \{u \in X : \lambda_i(u) = z_i, \quad 1 \le i \le Nn\}$$

and H_0^X the same with zero boundary conditions.

For most basis functions (as it is referred in page 14 previously to the statement of Corollary 3) and denoting by σ_θ^h the discrete PDE spline in X associated to $\mathcal{L},\mathcal{B},z,\mathcal{A}^h,\beta^h$ and $\theta > 0$, we will prove in Section 5 the convergence of σ_θ^h to g as the fill distance h tends to 0 and we will get estimates of the corresponding approximation error in terms of h. Specifically, we will prove the following results:

Theorem 2 *Suppose that* $\theta : \mathcal{H} \to (0,+\infty)$ *satisfies*

$$\theta = o(h^{-d}), \quad h \to 0. \tag{10}$$

Then

$$\lim_{h\to 0} \|\sigma_\theta^h - g\|_{n,\Omega} = 0.$$

Theorem 3 *For each* $\theta > 0$ *and* $k = 0,\dots,n-1$, *we have*

$$|g - \sigma_\theta^h|_{k,\Omega} = O\left(|\sigma_\theta^h - g|_{n,\Omega} h^{n-k} + h^{\frac{d}{2}-k}\theta^{\frac{1}{2}}\right), \quad h \to 0.$$

Corollary 1 *Let the conditions of Theorem 2 hold. Then we have the upper bounds*

$$\forall k = 0,\dots,n-1, \quad |g - \sigma_\theta^h|_{k,\Omega} = o(h^{n-k}) + O(h^{\frac{d}{2}-k}\theta^{\frac{1}{2}}),\ h \to 0$$

and

$$|g - \sigma_\theta^h|_{n,\Omega} = o(1),\ h \to 0.$$

Remark 2 If, additionally, $\theta = o(h^{-d+2(n-1)})$, $h \to 0$, then Corollary 1 implies for all $k = 0,\dots,n-1$, $|g - \sigma_\theta^h|_{k,\Omega} = o(h^{(n-1)-k})$, $h \to 0$, which converges to 0. □

We note that the stated results require extensions of some earlier theorems which make them applicable in our general framework of a Lipschitz domain and arbitrary dimension. We do this by using the more general case of conditionally positive definite basis functions in place of just positive definite ones (see, for instance, the weaker version of Theorem 1 in [18, Th. 16.4] for the positive definite case).

4 Characterization of discrete PDE splines

Once Problem (8) has been stated, we are giving an equivalent formulation of it. Our analysis aims now to address our problem into a symmetric collocation problem. Theorem 1 will be crucial for our purpose.

Let us consider the assumptions and notations before Definition 5. We firstly need to show that H^X is a non-empty set.

The following result is a consequence of Theorem 1:

Corollary 2 *The functionals* $\lambda_{jN+\ell}$, $1 \le \ell \le N$, $j < n$, *are linearly independent over* $\mathcal{N}_\Phi(\mathbb{R}^d)$.

Proof First, we point out that

$$\lambda_{jN+\ell} = \sum_{k=1}^{d} (\mathrm{n}^{je_k}) \tilde{\lambda}_{b_\ell, je_k} + \sum_{\substack{\alpha \in \mathbb{N}^d \setminus \{je_1, \ldots, je_d\} \\ |\alpha| = j}} a_\alpha^{(j)} \tilde{\lambda}_{b_\ell, \alpha} \tag{11}$$

with $a_\alpha^{(j)} \in \mathbb{R}$ and $\tilde{\lambda}_{b_\ell,\alpha}$ the functionals given by

$$\tilde{\lambda}_{b_\ell,\alpha} = \delta_{b_\ell} \circ \partial^\alpha.$$

Consider

$$\sum_{\ell=1}^{N} \sum_{j=0}^{n-1} c_{\ell j} \lambda_{jN+\ell} = 0, \quad c_{\ell j} \in \mathbb{R}.$$

We know that $\{\tilde{\lambda}_{b_\ell, \alpha^{(j)}}\}$, for all ℓ and j, with $\alpha^{(j)} \in \mathbb{N}^d$, $|\alpha^{(j)}| = j$, are linearly independent by Theorem 1. Applying this result and (11) we have that for each ℓ and j,

$$c_{\ell j} \mathrm{n}^{je_k} = 0, \ \forall k = 1, \ldots, d,$$

and hence $c_{\ell j} = 0$. □

Consider now $\{\lambda_1, \ldots, \lambda_{Nn}\}$ and the finite-dimensional subspace of the native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ defined by

$$\mathrm{span}\{\lambda_i^y K(\cdot, y) : i = 1, \ldots, Nn\}.$$

Proposition 1 *There exists a (unique) element $u \in span\{\lambda_i^y K(\cdot, y) : i = 1, \ldots, Nn\}$ such that, for all $i = 1, \ldots, Nn$,*

$$\lambda_i(u) = z_i.$$

Proof Let u be the element of $\mathrm{span}\{\lambda_i^y K(\cdot, y) : i = 1, \ldots, Nn\}$ given by

$$u(\cdot) = \sum_{j=0}^{Nn} \alpha_j \lambda_j^y K(\cdot, y).$$

Then, the system to be solved is

$$A\alpha = z,$$

where $A = (a_{ij})_{i,j=1,\ldots,Nn}$ with

$$a_{ij} = \lambda_i^x \lambda_j^y K(x, y), \quad \alpha = (\alpha_j)_{j=1,\ldots,Nn}.$$

If we show that A is a positive definite and symmetric matrix, then the system has a unique solution.

– A is symmetric. For any $i, j = 1, \ldots, Nn$, and applying [18, Th. 16.7], we have

$$a_{ij} = \lambda_i^x \lambda_j^y K(x,y) = (\lambda_i, \lambda_j)_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} = (\lambda_j, \lambda_i)_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} = \lambda_j^x \lambda_i^y K(x,y) \\ = a_{ji}.$$

– A is positive definite. For each $\gamma = (\gamma_i)_{i=1,\ldots,Nn} \in \mathbb{R}^{Nn}$,

$$\gamma^T A \gamma = \sum_{i=1}^{Nn} \sum_{j=1}^{Nn} \gamma_i \gamma_j \lambda_i^x \lambda_j^y K(x,y) = \sum_{i=1}^{Nn} \sum_{j=1}^{Nn} \gamma_i \gamma_j (\lambda_i, \lambda_j)_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} \\ = \left(\sum_{i=1}^{Nn} \gamma_i \lambda_i, \sum_{j=1}^{Nn} \gamma_j \lambda_j \right)_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} = \left\| \sum_{i=1}^{Nn} \gamma_i \lambda_i \right\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)^*}.$$

Hence, $\gamma^T A \gamma \geq 0$ and if $\gamma^T A \gamma = 0$ then

$$\left\| \sum_{i=1}^{Nn} \gamma_i \lambda_i \right\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} = 0 \implies \sum_{i=1}^{Nn} \gamma_i \lambda_i = 0.$$

Now, the linear independence of $\{\lambda_i\}_{i=1,\ldots,Nn}$ over $\mathcal{N}_\Phi(\mathbb{R}^d)$ implies that $\gamma_i = 0$, for all $i = 1, \ldots, Nn$.

□

As a consequence, we obtain that H^X is a non-empty set.

We also need this preparatory lemma:

Lemma 1 *The map* $(\cdot,\cdot)_{(\theta)} : H^n(\Omega) \times H^n(\Omega) \longrightarrow \mathbb{R}$ *defined by*

$$(u,v)_{(\theta)} = (\rho u, \rho v) + \theta (u,v)_{\mathcal{L}}$$

is an inner product on $H^n(\Omega)$ *and its associated norm* $\|\cdot\|_{(\theta)}$*, given by*

$$\forall v \in H^n(\Omega), \qquad \|v\|_{(\theta)} = (v,v)_{(\theta)}^{1/2},$$

is equivalent to the Sobolev norm $\|\cdot\|_{n,\Omega}$.

Proof This result is derived from our assumptions and [3, Th. 7.3.12]. □

We are ready to give the following equivalent formulation of Problem (8) which allows us to consider it in symmetric collocation:

Theorem 4 *Problem (8) admits a unique solution which is also the unique solution of the variational problem: find* $\sigma_\theta \in H^X$ *such that*

$$\forall\, v \in H_0^X,\ (\rho \sigma_\theta, \rho v) + \theta (\sigma_\theta, v)_{\mathcal{L}} = (\beta, \rho v) + \theta (f, v)_{0,\Omega}.$$

Proof We consider the map $a : H^n(\Omega) \times H^n(\Omega) \to \mathbb{R}$, given by

$$a(u,v) = 2\Big((\rho u, \rho v) + \theta(u,v)_{\mathcal{L}}\Big).$$

From Lemma 1, $a(\cdot,\cdot)$ is a continuous coercive symmetric bilinear form on $H^n(\Omega)$.

Let $F(v) = 2((\beta, \rho v) + \theta(f,v)_{0,\Omega})$, which is clearly linear and continuous on $H^n(\Omega)$. As H^X is a non-empty convex closed set we conclude, by applying Stampacchia's Theorem (see [6]), that there exists a unique $\sigma_\theta \in H^X$ such that $a(\sigma_\theta, w-\sigma_\theta) \geq F(w-\sigma_\theta)$, for all $w \in H^X$, which implies that $a(\sigma_\theta, v) \geq F(v)$ for all $v \in H_0^X$. As H_0^X is a vector subspace, then if $v \in H_0^X$ hence $-v \in H_0^X$, and it follows that $a(\sigma_\theta, -v) \geq F(-v)$, for any $v \in H_0^X$.

From this we obtain that $a(\sigma_\theta, v) = F(v)$ for any $v \in H_0^X$. Furthermore, σ_θ is the minimum in H^X of the functional $\Theta(v) = \frac{1}{2}a(v,v) - F(v)$, which is the minimum of J since $\Theta(v) = J(v) - \|\beta\|_2^2$. Hence we conclude the result. □

We shall now prove a result that is very useful if we want to obtain an expression of the discrete PDE spline.

Theorem 5 *There exists a unique* $(\sigma_\theta, \alpha) \in H^X \times \mathbb{R}^{Nn}$ *such that*

$$(\rho\sigma_\theta, \rho v) + \theta(\sigma_\theta, v)_{\mathcal{L}} + \sum_{i=1}^{Nn} \alpha_i \lambda_i(v) = (\beta, \rho v) + \theta(f,v)_{0,\Omega}, \tag{12}$$

for all $v \in X$*, where* σ_θ *is the unique solution of Problem (8) and* $\alpha = (\alpha_1, \ldots, \alpha_{Nn})$.

Proof By Proposition 1 we may consider $\{\psi_1, \ldots, \psi_{Nn}\}$ functions of X such that $\lambda_i(\psi_j) = \delta_{ij}$, for all $i,j = 1,\ldots,Nn$.

For each $v \in X$, let $\psi = v - \sum_{j=1}^{Nn} \lambda_j(v)\psi_j$.

Then, $\psi \in X$ and for each $i = 1,\ldots,Nn$, we have

$$\lambda_i(\psi) = \lambda_i(v) - \sum_{j=1}^{Nn} \lambda_j(v)\lambda_i(\psi_j) = 0$$

and consequently, $\psi \in H_0^X$. Let σ_θ be the (unique) solution of (8). Then, by Theorem 4, we have $\sigma_\theta \in H^X$ and

$$(\rho\sigma_\theta, \rho\psi) + \theta(\sigma_\theta, \psi)_{\mathcal{L}} = (\beta, \rho\psi) + \theta(f,\psi)_{0,\Omega},$$

that is,

$$(\rho\sigma_\theta, \rho(v - \sum_{j=1}^{Nn} \lambda_j(v)\psi_j)) + \theta(\sigma_\theta, v - \sum_{j=1}^{Nn} \lambda_j(v)\psi_j)_{\mathcal{L}}$$
$$= (\beta, \rho(v - \sum_{j=1}^{Nn} \lambda_j(v)\psi_j)) + \theta(f, v - \sum_{j=1}^{Nn} \lambda_j(v)\psi_j)_{0,\Omega},$$

and by linearity, we obtain

$$(\rho\sigma_\theta, \rho v) + \sum_{j=1}^{Nn} \Big((\beta - \rho\sigma_\theta, \rho\psi_j) - \theta(\sigma_\theta, \psi_j)_{\mathcal{L}} + \theta(f, \psi_j)_{0,\Omega} \Big) \lambda_j(v)$$
$$+ \theta(\sigma_\theta, v)_{\mathcal{L}} = (\beta, \rho v) + \theta(f, v)_{0,\Omega}.$$

If we denote $\alpha_j = (\beta - \rho\sigma_\theta, \rho\psi_j) - \theta(\sigma_\theta, \psi_j)_{\mathcal{L}} + \theta(f, \psi_j)_{0,\Omega}$, then we conclude that for $\alpha = \big(\alpha_j\big)_{j=1,\ldots,Nn}$,

$$(\rho\sigma_\theta, \rho v) + \theta(\sigma_\theta, v)_{\mathcal{L}} + \sum_{j=1}^{Nn} \alpha_j \lambda_j(v) = (\beta, \rho v) + \theta(f, v)_{0,\Omega},$$

and (12) holds. Now, we suppose that there exists $\alpha, \overline{\alpha} \in \mathbb{R}^{Nn}$ such that (σ_θ, α) and $(\sigma_\theta, \overline{\alpha})$ satisfy (12). Then, for each $v \in X$, we have that

$$(\rho\sigma_\theta, \rho v) + \theta(\sigma_\theta, v)_{\mathcal{L}} + \sum_{i=1}^{Nn} \alpha_i \lambda_i(v) = (\beta, \rho v) + \theta(f, v)_{0,\Omega},$$

$$(\rho\sigma_\theta, \rho v) + \theta(\sigma_\theta, v)_{\mathcal{L}} + \sum_{i=1}^{Nn} \overline{\alpha}_i \lambda_i(v) = (\beta, \rho v) + \theta(f, v)_{0,\Omega},$$

and, by subtracting, we have $\sum_{i=1}^{Nn} (\alpha_i - \overline{\alpha}_i)\lambda_i(v) = 0$. In particular, if we take for each $j = 1, \ldots, Nn$, $v = \psi_j$, we have

$$0 = \sum_{i=1}^{Nn} (\alpha_i - \overline{\alpha}_i)\lambda_i(\psi_j) = \sum_{i=1}^{Nn} (\alpha_i - \overline{\alpha}_i)\delta_{ij} = \alpha_j - \overline{\alpha}_j$$

from which we derive $\alpha = \overline{\alpha}$ and, hence, the uniqueness of (σ_θ, α). □

Remark 3 It is easy to check that $\{\psi_1, \ldots, \psi_{Nn}\}$ considered above are linearly independent functions of X. □

5 Proof of the convergence results and estimates of the approximation error

Consider the assumptions and notations before Theorem 2.

Remark 4 Define the non-empty convex closed set

$$\check{H} = \Big\{ u \in H^n(\Omega) \cap C^{n-1}(\overline{\Omega}) : \lambda_i(u) = z_i, \quad 1 \le i \le Nn \Big\}$$

and $\check{H}_0$ the same with zero boundary conditions. Let $\widetilde{J}$ be the functional defined on $H^n(\Omega)$ by $\widetilde{J}(v) = |v|^2_{\mathcal{L}} - 2(f, v)_{0,\Omega}$. Reasoning in a similar way as in the proof of Theorem 4, we deduce that there exists a unique $\widetilde{\sigma} \in \check{H}$ such that

$\widetilde{J}(\widetilde{\sigma}) \le \widetilde{J}(v)$, for all $v \in \check{H}$, which is characterized as the unique solution of the variational problem: Find $\widetilde{\sigma} \in \check{H}$ such that

$$(\widetilde{\sigma}, v)_{\mathcal{L}} = (f, v)_{0,\Omega}, \ \forall v \in \check{H}_0.$$

□

Remark 5 Our assumptions imply that $\mathcal{H}$ is bounded and that the Hausdorff distance between $\mathcal{A}^h$ and $\overline{\Omega}$ tends to 0 as h does, i.e., the fact that $0 \in \overline{\mathcal{H}}$ and (9) imply the weaker condition

$$\lim_{h\to 0} \sup_{x\in\Omega} \min_{a\in\mathcal{A}^h} \|x-a\| = 0.$$

As $\mathcal{H}$ is bounded we may suppose that for each $h \in \mathcal{H}$, $\mathcal{A}^h$ contains a $\mathbb{P}_{\max\{m,n\}-1}$-unisolvent set and, in particular, $\ker\rho^h \cap \mathbb{P}_{\max\{m,n\}-1} = \{0\}$. □

5.1 First main results

A similar proof to the Proposition 1 gives us that there exists a (unique) element $s_g^h \in X$ such that $\lambda_i(s_g^h) = z_i$, for all $i = 1,\ldots,Nn$, and $s_g^h(a) = g(a)$, for all $a \in \mathcal{A}^h$.

Fix $x \in \Omega$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \le n$. The following error estimates for the interpolant s_g^h state our first main result:

Theorem 6 *There exists a positive constant $c_2^{(\alpha)}$ such that*

$$|\partial^\alpha g(x) - \partial^\alpha s_g^h(x)| \le C C_\Phi(x)^{1/2} h^{n-|\alpha|} \|g\|_{\mathcal{N}_\Phi(\Omega)},$$

for small enough h. The function $C_\Phi(x)$ is defined by

$$C_\Phi(x) := \max_{\substack{\beta,\nu\in\mathbb{N}^d \\ |\beta|+|\nu|=2n}} \max_{z,w\in\Omega\cap B(x,c_2^{(\alpha)}h)} |\partial_1^\beta \partial_2^\nu \Phi(z,w)|,$$

where the constant C is independent of x, g and Φ, and where ∂_1^β (or ∂_2^ν, respectively) denotes the derivative with respect to the first argument (or with respect to the second argument, respectively).

As all derivatives of Φ of order $2n$ are continuous on $\overline{\Omega} \times \overline{\Omega}$, $C_\Phi(x)$ is uniformly bounded on Ω. Moreover, the former approximation order can be improved for most basis functions (because of their infinite smoothness or by squeezing out additional powers of h through the function $C_\Phi(x)$) (see [18, Ch. 11]). For these basis functions, which will be considered for the rest of the subsections of Section 5, we have the following corollary:

Corollary 3 *For all $k = 0,\ldots,n$,*

$$|g - s_g^h|_{k,\Omega} = o(h^{n-k}), \ h \to 0. \tag{13}$$

Before proving Theorem 6, we state two preparatory propositions. Here, we need to make a suitable adaptation of several results of [18].

Consider the linear and continuous functional $\tilde{\lambda}_{x,\alpha} : \mathcal{N}_\Phi(\Omega) \to \mathbb{R}$ given by $\tilde{\lambda}_{x,\alpha} := \delta_x \circ \partial^\alpha$.

Proposition 2 *Set $\Lambda_2 := span\{\delta_{a_i}\}_{i=1,\dots,M} = span\{\lambda_{Nn+i}\}_{i=1,\dots,M}$. Denote by $P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha})$ the generalized power function given by*

$$P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha}) = \inf_{\mu\in\Lambda_2} \|\tilde{\lambda}_{x,\alpha} - \mu\|_{\mathcal{N}_\Phi(\Omega)^*}.$$

Then $P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha}) \le P^{(\alpha)}_{\Phi,\mathcal{A}^h}(x)$, where the power function $P^{(\alpha)}_{\Phi,\mathcal{A}^h}(x)$ is defined as the minimum of the square root of the quadratic form $Q : \mathbb{R}^M \to \mathbb{R}$

$$Q(c) := \partial_1^\alpha \partial_2^\alpha \Phi(x,x) - 2\sum_{i=1}^M c_i \partial_1^\alpha \Phi(x,a_i) + \sum_{j=1}^M \sum_{i=1}^M c_j c_i \Phi(a_j,a_i),$$

on the set

$$\mathbb{M}_\alpha = \{c = (c_1,\dots,c_M) \in \mathbb{R}^M : \sum_{i=1}^M c_i p(a_i) = \partial^\alpha p(x) \textit{ for all } p \in \mathbb{P}_{m-1}\}.$$

Proof Let $\Xi = \{\xi_1,\dots,\xi_{\Delta_m}\}$ be a $\mathbb{P}_{m-1}$-unisolvent subset of $\mathcal{A}^h$ and $\{p_1,\dots,p_{\Delta_m}\}$ a Lagrange basis of $\mathbb{P}_{m-1}$ with respect to this set Ξ. Consider the kernel $K : \Omega \times \Omega \to \mathbb{R}$ given by (3). As the Riesz representer of $\tilde{\lambda}_{x,\alpha}$ (or of δ_{a_i}, $i = 1,\dots,M$, respectively) on the native Hilbert space $\mathcal{N}_\Phi(\Omega)$ is given by $\partial_2^\alpha K(\cdot,x)$ (or $K(\cdot,a_i)$, $i = 1,\dots,M$, respectively), we thus have

$$\begin{aligned} P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha}) &= \inf_{\mu\in\Lambda_2} \|\tilde{\lambda}_{x,\alpha} - \mu\|_{\mathcal{N}_\Phi(\Omega)^*} = \inf_{c=(c_1,\dots,c_M)\in\mathbb{R}^M} \|\tilde{\lambda}_{x,\alpha} - \sum_{i=1}^M c_i \delta_{a_i}\|_{\mathcal{N}_\Phi(\Omega)^*} \\ &= \inf_{c=(c_1,\dots,c_M)\in\mathbb{R}^M} \|\partial_2^\alpha K(\cdot,x) - \sum_{i=1}^M c_i K(\cdot,a_i)\|_{\mathcal{N}_\Phi(\Omega)} \qquad (14) \\ &\le \inf_{c=(c_1,\dots,c_M)\in\mathbb{M}_\alpha} \|\partial_2^\alpha K(\cdot,x) - \sum_{i=1}^M c_i K(\cdot,a_i)\|_{\mathcal{N}_\Phi(\Omega)}. \end{aligned}$$

Taking into account that for each $c = (c_1,\dots,c_M) \in \mathbb{M}_\alpha$ and $k = 1,\dots,\Delta_m$,

$$\partial_2^\alpha K(\xi_k,x) = \partial^\alpha p_k(x) = \sum_{i=1}^M c_i p_k(a_i) = \sum_{i=1}^M c_i K(\xi_k,a_i),$$

we obtain from (14) that

$$P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha}) \le \inf_{c=(c_1,\dots,c_M)\in\mathbb{M}_\alpha} |\partial_2^\alpha \Phi(\cdot,x) - \sum_{i=1}^M c_i \Phi(\cdot,a_i)|_{\mathcal{N}_\Phi(\Omega)}. \qquad (15)$$

By applying [18, Lemma 11.3], the right-hand side of (15) is the infimum of the square root of Q on $\mathbb{M}_\alpha$, which in fact is the minimum by [18, Th. 11.5]. □

The following proposition can be directly derived from [18, Cor. 16.2].

Proposition 3 *The interpolant s_g^h is the best approximation from X to g in the native space norm $\|\cdot\|_{\mathcal{N}_\Phi(\Omega)}$, i.e.*

$$\|g - s_g^h\|_{\mathcal{N}_\Phi(\Omega)} = \min\{\|g - v\|_{\mathcal{N}_\Phi(\Omega)} : v \in X\}.$$

In particular, $\|g - s_g^h\|_{\mathcal{N}_\Phi(\Omega)} \leq \|g\|_{\mathcal{N}_\Phi(\Omega)}$.

We are ready to prove Theorem 6:

Proof With the notation of Proposition 2, we have that, for each $\mu \in \Lambda_2$,

$$\begin{aligned}|\partial^\alpha g(x) - \partial^\alpha s_g^h(x)| &= |\tilde{\lambda}_{x,\alpha}(g - s_g^h)| = |(\tilde{\lambda}_{x,\alpha} - \mu)(g - s_g^h)| \\ &\leq \|\tilde{\lambda}_{x,\alpha} - \mu\|_{\mathcal{N}_\Phi(\Omega)^*} \|g - s_g^h\|_{\mathcal{N}_\Phi(\Omega)},\end{aligned}$$

which implies

$$|\partial^\alpha g(x) - \partial^\alpha s_g^h(x)| \leq P_{\Lambda_2}(\tilde{\lambda}_{x,\alpha}) \|g - s_g^h\|_{\mathcal{N}_\Phi(\Omega)}.$$

By applying now Proposition 2, [18, Th. 11.5], the proof of [18, Th. 11.13], and Proposition 3, we obtain the desired result. □

5.2 Convergence

Here, and in the following subsection, we reformulate and adapt some results in [2] to the theorems in our context. Their proofs are very closely related, but we state and we detail them nonetheless for the convenience of the reader.

We will need the following result whose proof is similar to that of Proposition V.1.2 in [2].

Proposition 4 *Let $T_0 = \{t_{01}, \ldots, t_{0\Delta_n}\}$ be a $\mathbb{P}_{n-1}$-unisolvent set of points of $\overline{\Omega}$. For any $r > 0$, we denote by $\mathcal{T}_r$ the family of all subsets $T = \{t_1, \ldots, t_{\Delta_n}\}$ of $\overline{\Omega}$ that satisfy the condition*

$$\forall j = 1, \ldots, \Delta_n,\ \|t_j - t_{0j}\|_2 \leq r.$$

Then, there exists $r_0 > 0$ such that the family $\mathcal{T}_{r_0}$ is formed by $\mathbb{P}_{n-1}$-unisolvent subsets and the mapping defined for any subset $T = \{t_1, \ldots, t_{\Delta_n}\}$ of $\overline{\Omega}$ by

$$\forall v \in H^n(\Omega), \quad v \mapsto \left(\sum_{j=1}^{\Delta_n} |v(t_j)|^2 + |v|_{\mathcal{L}}^2 \right)^{\frac{1}{2}},$$

is, for every $T \in \mathcal{T}_{r_0}$, a norm on $H^n(\Omega)$ uniformly equivalent on $\mathcal{T}_{r_0}$ to the usual norm $\|\cdot\|_{n,\Omega}$.

We are now ready to prove Theorem 2:

Proof Step 1: Let us prove that the family $(\sigma_\theta^h)_{h\in\mathcal{H}\cap(0,h^*]}$ is bounded in $H^n(\Omega)$ for a certain $h^* > 0$:

By using Theorem 6, it is satisfied that

$$\|g - s_g^h\|_{n,\Omega} = O(1), \quad h \to 0. \tag{16}$$

Let $h \in \mathcal{H}$. Since $s_g^h \in H^X$, and σ_θ^h is the minimum of the functional J in H^X, we have that

$$J(\sigma_\theta^h) \le J(s_g^h)$$

that is,

$$\|\rho^h(\sigma_\theta^h) - \beta^h\|_2^2 + \theta\left(|\sigma_\theta^h|_{\mathcal{L}}^2 - 2(f,\sigma_\theta^h)_{0,\Omega}\right) \le \theta\left(|s_g^h|_{\mathcal{L}}^2 - 2(f,s_g^h)_{0,\Omega}\right), \tag{17}$$

which implies

$$|\sigma_\theta^h|_{\mathcal{L}}^2 \le |s_g^h|_{\mathcal{L}}^2 + 2(f,\sigma_\theta^h - s_g^h)_{0,\Omega}. \tag{18}$$

By Remark 4 there exists a unique $\widetilde{\sigma} \in \check{H}$ such that

$$\widetilde{J}(\widetilde{\sigma}) \le \widetilde{J}(v), \qquad \forall v \in \check{H}.$$

Moreover, this element is characterized as the unique solution of the following variational problem: Find $\widetilde{\sigma} \in \check{H}$ such that

$$(\widetilde{\sigma}, v)_{\mathcal{L}} = (f, v)_{0,\Omega},\ \forall v \in \check{H}_0.$$

As $\sigma_\theta^h - s_g^h \in \check{H}_0$, we have

$$(f, \sigma_\theta^h - s_g^h)_{0,\Omega} = (\widetilde{\sigma}, \sigma_\theta^h - s_g^h)_{\mathcal{L}}. \tag{19}$$

By using (19) on the right-hand side of (18), we obtain

$$|\sigma_\theta^h|_{\mathcal{L}}^2 \le |s_g^h|_{\mathcal{L}}^2 + 2(\widetilde{\sigma}, \sigma_\theta^h - s_g^h)_{\mathcal{L}}.$$

Now, by adding the term $(\widetilde{\sigma},\widetilde{\sigma})_{\mathcal{L}}$ on both sides of the inequality and by rearranging we obtain

$$|\sigma_\theta^h - \widetilde{\sigma}|_{\mathcal{L}}^2 \le |s_g^h - \widetilde{\sigma}|_{\mathcal{L}}^2$$

and

$$|\sigma_\theta^h|_{\mathcal{L}} \le |s_g^h - g|_{\mathcal{L}} + |g - \widetilde{\sigma}|_{\mathcal{L}} + |\widetilde{\sigma}|_{\mathcal{L}}.$$

From Lemma 1

$$|s_g^h - g|_{\mathcal{L}} = \|s_g^h - g\|_{(1)} = O(\|s_g^h - g\|_{n,\Omega}), \quad h \to 0$$

and then, by using (16)

$$|\sigma_\theta^h|_{\mathcal{L}} - |g - \widetilde{\sigma}|_{\mathcal{L}} - |\widetilde{\sigma}|_{\mathcal{L}} = O(1), \quad h \to 0. \tag{20}$$

Analogously, by (17)

$$\|\rho^h \sigma_\theta^h - \beta^h\|_2^2 \le \theta\left(|s_g^h|_{\mathcal{L}}^2 + 2(f, \sigma_\theta^h - s_g^h)_{0,\Omega}\right) = \theta\left(|s_g^h|_{\mathcal{L}}^2 + 2(\widetilde{\sigma}, \sigma_\theta^h - s_g^h)_{\mathcal{L}}\right)$$
$$\le \theta\left(|s_g^h - g|_{\mathcal{L}}^2 + |g|_{\mathcal{L}}^2 + 2|s_g^h - g|_{\mathcal{L}}|g|_{\mathcal{L}} + 2|\widetilde{\sigma}|_{\mathcal{L}}|s_g^h - g|_{\mathcal{L}} + |\widetilde{\sigma}|_{\mathcal{L}}^2 + |\sigma_\theta^h|_{\mathcal{L}}^2 - 2(\widetilde{\sigma}, g)_{\mathcal{L}}\right). \tag{21}$$

Therefore, from (21) and (20), we obtain

$$\|\rho^h \sigma_\theta^h - \beta^h\|_2^2 = O(\theta), \quad h \to 0. \tag{22}$$

Let $T_0 = \{t_{01}, \ldots, t_{0\Delta_n}\}$ be a $\mathbb{P}_{n-1}$-unisolvent set of points of Ω. Then $T_0 \subset \Omega$ and we can find R_0 such that $\overline{B}(t_{0j}, R_0) \subset \Omega$, for all $j = 1, \ldots, \Delta_n$.

Let r_0 be the constant given in Proposition 4 for the set T_0 and let us consider

$$r_0' = \min\{R_0, r_0\}.$$

By using (9) we obtain that

$$\forall h \in \mathcal{H} \cap (0, r_0'),\ \forall j = 1, \ldots, \Delta_n,\ \overline{B}(t_{0j}, r_0' - h) \subset \bigcup_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} \overline{B}(a, h). \tag{23}$$

Set $N_j = \mathrm{card}(\mathcal{A}^h \cap \overline{B}(t_{0j}, r_0'))$. From (23) we have that

$$\forall h \in \mathcal{H} \cap (0, r_0'),\ \forall j = 1, \ldots, \Delta_n,\ (r_0' - h)^d \le N_j h^d. \tag{24}$$

Let us consider $h_0 \in (0, r_0')$. Then, by (24), for all $h \in \mathcal{H} \cap (0, h_0)$,

$$N_j \ge (r_0' - h)^d h^{-d} \ge (r_0' - h_0)^d h^{-d}. \tag{25}$$

On the other hand, for all $j = 1, \ldots, \Delta_n$, we have

$$\sum_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} |\sigma_\theta^h(a) - g(a)|^2 \le \sum_{a \in \mathcal{A}^h} |\sigma_\theta^h(a) - g(a)|^2 = \|\rho^h \sigma_\theta^h - \beta^h\|_2^2.$$

Therefore, by using (22) and (10), we obtain

$$\sum_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} |\sigma_\theta^h(a) - g(a)|^2 = o(h^{-d}), \quad h \to 0. \tag{26}$$

For each $h \in \mathcal{H} \cap (0, r_0')$ and $j = 1, \ldots, \Delta_n$ there exists at least one point $t_j^h \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')$ such that

$$|\sigma_\theta^h(t_j^h) - g(t_j^h)| = \min_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} |\sigma_\theta^h(a) - g(a)|.$$

Now, if we choose h_0 as in (25), then, for each $h \in \mathcal{H} \cap (0, h_0)$ and $j = 1, \ldots, \Delta_n$, we have

$$\sum_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} |\sigma_\theta^h(a) - g(a)|^2 \geq N_j \min_{a \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0')} |\sigma_\theta^h(a) - g(a)|^2$$
$$= N_j |\sigma_\theta^h(t_j^h) - g(t_j^h)|^2 \geq (r_0' - h_0)^d h^{-d} |\sigma_\theta^h(t_j^h) - g(t_j^h)|^2 \geq 0,$$

and taking into account (26) we deduce that the nonnegative sequence $(r_0' - h_0)^d |\sigma_\theta^h(t_j^h) - g(t_j^h)|^2 \to 0$, as $h \to 0$.

This implies that for each $h \in \mathcal{H} \cap (0, h_0)$ and $j = 1, \ldots, \Delta_n$,

$$|\sigma_\theta^h(t_j^h) - g(t_j^h)| = o(1), \quad h \to 0. \tag{27}$$

Now, given $h \in \mathcal{H} \cap (0, h_0)$, we take $T^h = \{t_1^h, \ldots, t_{\Delta_n}^h\}$. Since $h \leq h_0 \leq r_0'$, for each $j = 1, \ldots, \Delta_n$

$$t_j^h \in \mathcal{A}^h \cap \overline{B}(t_{0j}, r_0') \Rightarrow \|t_j^h - t_{0j}\|_2 \leq r_0' \Rightarrow T^h \in \mathcal{T}_{r_0'} \subset \mathcal{T}_{r_0}$$

and by Proposition 4

$$\forall v \in H^n(\Omega), \quad v \mapsto \left(\sum_{j=1}^{\Delta_n} |v(t_j^h)|^2 + |v|_{\mathcal{L}}^2 \right)^{\frac{1}{2}},$$

is a norm on $H^n(\Omega)$ uniformly equivalent on $\mathcal{T}_{r_0}$ to the usual norm $\|\cdot\|_{n,\Omega}$.

Now, we have that for our element $h \in \mathcal{H} \cap (0, h_0)$ and for each $j = 1, \ldots, \Delta_n$

$$|\sigma_\theta^h(t_j^h)| \leq |\sigma_\theta^h(t_j^h) - g(t_j^h)| + |g(t_j^h)|. \tag{28}$$

Since $n > \dfrac{d}{2}$, it is satisfied that $H^n(\Omega) \hookrightarrow C^0(\overline{\Omega})$. As $g \in H^n(\Omega)$, we thus have that g is bounded on $\overline{\Omega}$. Combining this with (27), and taking into account (28), we deduce that

$$|\sigma_\theta^h(t_j^h)| = O(1), \quad h \to 0.$$

This fact, together with (20), allows us to obtain

$$\left(\sum_{j=1}^{\Delta_n} |\sigma_\theta^h(t_j^h)|^2 + |\sigma_\theta^h|_{\mathcal{L}}^2 \right)^{\frac{1}{2}} = O(1), \quad h \to 0.$$

Therefore, by applying Proposition 4, we conclude that there exist $C > 0$ and $h^* > 0$ such that for each $h \in \mathcal{H} \cap (0, h^*]$,

$$\|\sigma_\theta^h\|_{n,\Omega} \leq C,$$

i.e., the family $(\sigma_\theta^h)_{h \in \mathcal{H} \cap (0, h^*]}$ contains bounded elements in $H^n(\Omega)$.

As a consequence, there exists a subsequence $(\sigma_{\theta_\ell}^{h_\ell})_{\ell\in\mathbb{N}}$ extracted from the family $(\sigma_\theta^h)_{h\in\mathcal{H}\cap(0,h^*]}$ such that, for each $\ell\in\mathbb{N}$, $\theta_\ell=\theta(h_\ell)$ and

$$\lim_{\ell\to+\infty} h_\ell = \lim_{\ell\to+\infty} h_\ell^d\theta_\ell = 0,$$

and there exists an element g^* of $H^n(\Omega)$ such that

$$g^* = \lim_{\ell\to+\infty} \sigma_{\theta_\ell}^{h_\ell} \quad \text{weakly in } H^n(\Omega).$$

Step 2: In this step we prove that $g^*=g$ by contradiction. We suppose, that $g^*\neq g$, then by the continuous injection of $H^n(\Omega)$ into $C^0(\overline{\Omega})$, there exist $\gamma>0$ and a nonempty open set $\mathcal{O}\subset\Omega$ such that

$$\forall x\in\mathcal{O}, \quad |g^*(x)-g(x)|>\gamma.$$

This injection is in fact compact. Hence, $(\sigma_{\theta_\ell}^{h_\ell})_{\ell\in\mathbb{N}}$ converges strongly to g^* in $C^0(\overline{\Omega})$ and, as a consequence,

$$\exists\,\ell_0\in\mathbb{N},\ \forall\,\ell\geq\ell_0,\ \forall\,x\in\mathcal{O},\ |\sigma_{\theta_\ell}^{h_\ell}(x)-g^*(x)|\leq\frac{\gamma}{2}.$$

Therefore,

$$\forall\,\ell\geq\ell_0,\ \forall x\in\mathcal{O}, |\sigma_{\theta_\ell}^{h_\ell}(x)-g(x)|\geq|g^*(x)-g(x)|-|\sigma_{\theta_\ell}^{h_\ell}(x)-g^*(x)|>\frac{\gamma}{2}. \tag{29}$$

Now, by reasoning as in Step 1, we can prove that, for sufficiently large $\ell\in\mathbb{N}$, there exists $t^{h_\ell}\in\mathcal{A}^{h_\ell}\cap\mathcal{O}$ such that

$$|\sigma_{\theta_\ell}^{h_\ell}(t^{h_\ell})-g(t^{h_\ell})|_n=o(1),\ \ell\to+\infty,$$

in contradiction to (29). Hence, $g^*=g$.

Step 3: Since $g^*=g$ and the space $H^n(\Omega)$ is compactly imbedded in $H^{n-1}(\Omega)$, we have that $(\sigma_{\theta_\ell}^{h_\ell})_{\ell\in\mathbb{N}}$ strongly converges to g in $H^{n-1}(\Omega)$, i.e.

$$g=\lim_{\ell\to+\infty}\sigma_{\theta_\ell}^{h_\ell} \quad \text{in } H^{n-1}(\Omega). \tag{30}$$

From the definitions of the seminorm $|\cdot|_{n,\Omega}$, the semi-inner product $(\cdot,\cdot)_{\mathcal{L}}$ and the strongly ellipticity of the operator $\mathcal{L}$ on Ω, we can prove that there exists $\nu>0$ such that

$$\begin{aligned}|\sigma_{\theta_\ell}^{h_\ell}-g|_{n,\Omega}^2 \leq \frac{1}{\nu}\Big[&2\left(|g|_{\mathcal{L}}^2-(\sigma_{\theta_\ell}^{h_\ell},g)_{\mathcal{L}}\right)+2\left((f,\sigma_{\theta_\ell}^{h_\ell})_{0,\Omega}-(f,g)_{0,\Omega}\right)\\ &+|s_g^{h_\ell}-g|_{\mathcal{L}}^2+2|s_g^{h_\ell}-g|_{\mathcal{L}}|g|_{\mathcal{L}}+2(f,g-s_g^{h_\ell})_{0,\Omega}\Big].\end{aligned} \tag{31}$$

From the weak convergence of $\sigma_{\theta_\ell}^{h_\ell}$ to g in $H^n(\Omega)$ we obtain

$$\lim_{\ell\to+\infty}(\sigma_{\theta_\ell}^{h_\ell}, g)_{\mathcal{L}} = (g,g)_{\mathcal{L}} = |g|_{\mathcal{L}}^2. \tag{32}$$

Analogously, from the compact and continuous injection of $H^n(\Omega)$ into $L^2(\Omega)$,

$$\lim_{\ell\to+\infty}(f, \sigma_{\theta_\ell}^{h_\ell})_{0,\Omega} = \lim_{\ell\to+\infty}(f,g)_{0,\Omega} \tag{33}$$

and from Corollary 3,

$$\lim_{\ell\to+\infty}\left[|s_g^{h_\ell} - g|_{\mathcal{L}}^2 + 2|s_g^{h_\ell} - g|_{\mathcal{L}}|g|_{\mathcal{L}} + 2(f, g - s_g^{h_\ell})_{0,\Omega}\right] = 0. \tag{34}$$

Hence, by using (32), (33) and (34) in (31) we have that

$$\lim_{\ell\to+\infty}|\sigma_{\theta_\ell}^{h_\ell} - g|_{n,\Omega}^2 = 0,$$

that, jointly with (30), implies that

$$\lim_{\ell\to+\infty}\|\sigma_{\theta_\ell}^{h_\ell} - g\|_{n,\Omega}^2 = 0.$$

Step 4: To conclude the proof, we argue by contradiction. Assume that $\|\sigma_\theta^h - g\|_{n,\Omega}$ does not tend to 0 when $h \to 0$. Then there exists a real number $r > 0$ and two sequences $(h'_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{H}$ and $(\theta'_\ell)_{\ell\in\mathbb{N}} \subset (0,+\infty)$ with, for each $\ell \in \mathbb{N}$, $\theta'_\ell = \theta(h'_\ell)$ and

$$\lim_{\ell\to+\infty} h'_\ell = \lim_{\ell\to+\infty}(h'_\ell)^d\theta'_\ell = 0,$$

satisfying

$$\forall\,\ell\in\mathbb{N},\quad \|\sigma_{\theta'_\ell}^{h'_\ell} - g\|_{n,\Omega} \geq r. \tag{35}$$

Now, the sequence $(\sigma_{\theta'_\ell}^{h'_\ell})_{\ell\in\mathbb{N}}$ is bounded in $H^n(\Omega)$. A similar argument to that of steps 1, 2, 3, proves that there exists a subsequence of $(\sigma_{\theta'_\ell}^{h'_\ell})_{\ell\in\mathbb{N}}$ which converges to g in $H^n(\Omega)$, in contradiction to (35). This observation completes the proof. □

5.3 Estimates of the approximation error

Let us remark some facts we will need:

- By [2, Prop. II-6.1], there exist constants $\vartheta > 1$ and $\kappa_0 > 0$ such that, for all $\kappa \in (0,\kappa_0]$, there exists a $T_\kappa \subset \Omega$ satisfying:

$$\forall t \in T_\kappa,\ \overline{B}(t,\kappa) \subset \Omega \ \text{ and } \ \Omega \subset \bigcup_{t\in T_\kappa}\overline{B}(t,\vartheta\kappa).$$

– By [2, Prop. II-6.6], there exist constants $R>1$ and $C>0$ and, for each $h\in \mathcal{H}\cap(0,\kappa_0/R]$ and $t\in T_{Rh}$, Δ_n closed balls $\mathcal{B}_1,\dots,\mathcal{B}_{\Delta_n}$ of radius h, contained in the ball $\overline{B}(t,Rh)$ such that, for any $v\in H^n(\overline{B}(t,\vartheta Rh))$ which is zero at one point or more of each of the balls $\mathcal{B}_1,\dots,\mathcal{B}_{\Delta_n}$, we have

$$\forall k=0,\dots,n-1,\quad |v|_{k,\overline{B}(t,\vartheta Rh)}\le Ch^{n-k}|v|_{n,\overline{B}(t,\vartheta Rh)}.$$

Moreover, any Δ_n-tuple belonging to $\Pi_{j=1}^{\Delta_n}\mathcal{B}_j$ is $\mathbb{P}_{n-1}$-unisolvent.

From (9), there exists a Δ_n-tuple $a_t^h=(a_{1t}^h,\dots,a_{\Delta_n t}^h)\in\Pi_{j=1}^{\Delta_n}(\mathcal{B}_j\cap\mathcal{A}^h)$. Since a_t^h is $\mathbb{P}_{n-1}$-unisolvent, it is possible to define on $H^n(\overline{B}(t,\vartheta Rh))$ the Lagrange $\mathbb{P}_{n-1}$-interpolating operator Π_t^h given by

$$\Pi_t^h v\in\mathbb{P}_{n-1}\text{ and, for } j=1,\dots,\Delta_n,\quad \Pi_t^h v(a_{jt}^h)=v(a_{jt}^h).$$

For each $h\in\mathcal{H}$, we let $u_\theta^h=\sigma_\theta^h-g$ and $\tilde u_\theta^h=\tilde P u_\theta^h$, where $\tilde P=P(I-\Pi_{n-1})+E\Pi_{n-1}$ is the prolongation operator defined over $H^n(\Omega)$ in [2, II-6.2].

Proposition 5 *For any $\theta>0$ and $k=0,\dots,n-1$, we have*

$$\sum_{t\in T_{Rh}}|\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)}=O(h^{2(n-k)}|u_\theta^h|^2_{n,\Omega}),\quad h\to 0.$$

Proof Let $h\in\mathcal{H}\cap(0,\kappa_0/R]$, $\theta>0$ and $t\in T_{Rh}$.

First, it is clear that $\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h\in H^n(\overline{B}(t,\vartheta Rh))$. For each $j=1,\dots,\Delta_n$, $(\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h)(a_{jt}^h)=0$, i.e. $(\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h)$ vanishes at least in a point of every ball $\mathcal{B}_1,\dots,\mathcal{B}_{\Delta_n}$. By applying [2, Prop. II-6.6], we have

$$\forall k=0,\dots,n-1,\ |\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|_{k,\overline{B}(t,\vartheta Rh)}=O(h^{n-k}|\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|_{n,\overline{B}(t,\vartheta Rh)}),\quad h\to 0.$$

As a consequence, for each $k=0,\dots,n-1$,

$$\sum_{t\in T_{Rh}}|\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)}=O(h^{2(n-k)}\sum_{t\in T_{Rh}}|\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|^2_{n,\overline{B}(t,\vartheta Rh)}),\quad h\to 0.\tag{36}$$

By [2, Prop. II-6.1], the sum

$$\sum_{t\in T_{Rh}}\chi_{\overline{B}(t,\vartheta Rh)}\qquad\text{is uniformly bounded,}\tag{37}$$

where $\chi_{\overline{B}(t,\vartheta Rh)}$ denotes the characteristic function of the set $\overline{B}(t,\vartheta Rh)$. Hence

$$\begin{aligned}\sum_{t\in T_{Rh}}|\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h|^2_{n,\overline{B}(t,\vartheta Rh)}&=\sum_{t\in T_{Rh}}\int_{\mathbb{R}^d}\chi_{\overline{B}(t,\vartheta Rh)}\left(\sum_{|\alpha|=n}|\partial^\alpha(\tilde u_\theta^h-\Pi_t^h\tilde u_\theta^h)(x)|^2\right)dx\\&=\sum_{t\in T_{Rh}}\int_{\mathbb{R}^d}\chi_{\overline{B}(t,\vartheta Rh)}\left(\sum_{|\alpha|=n}|\partial^\alpha\tilde u_\theta^h(x)|^2\right)dx=O(|\tilde u_\theta^h|^2_{n,\mathbb{R}^d}),\quad h\to 0.\end{aligned}\tag{38}$$

By [2, Prop. II-6.3]

$$|\tilde{u}_\theta^h|_{n,\mathbb{R}^d} = O(|u_\theta^h|_{n,\Omega}), \quad h \to 0.$$

By substituting in the last term of (38) we obtain

$$\sum_{t \in T_{Rh}} |\tilde{u}_\theta^h - \Pi_t^h \tilde{u}_\theta^h|^2_{n,\overline{B}(t,\vartheta Rh)} = O(|u_\theta^h|^2_{n,\Omega}), \quad h \to 0$$

and, now replacing it in (36), we deduce that

$$\sum_{t \in T_{Rh}} |\tilde{u}_\theta^h - \Pi_t^h \tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} = O(h^{2(n-k)} |u_\theta^h|^2_{n,\Omega}), \quad h \to 0. \tag{39}$$

This completes the proof. □

Proposition 6 *For any $\theta > 0$, we have*

$$\forall k = 0, \ldots, n-1, \quad \sum_{t \in T_{Rh}} |\Pi_t^h \tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} = O(h^{d-2k}\theta), \quad h \to 0. \tag{40}$$

Proof Let $h \in \mathcal{H} \cap (0, \kappa_0/R]$, $\theta > 0$ and $t \in T_{Rh}$.

Step 1:

From the proof of [2, Prop. II.6.6] we know that:

- There exists a set $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_{\Delta_n}\} \subset \mathbb{R}^d$ such that $\Pi_{j=1}^{\Delta_n} \overline{B}(\hat{\alpha}_j, 1)$ is a compact subset of $(\mathbb{R}^d)^{\Delta_n}$ formed by $\mathbb{P}_{n-1}$-unisolvent Δ_n-tuples; the constant R is chosen, in fact, so that, for some $\hat{a} \in \mathbb{R}^d$, $\displaystyle\bigcup_{j=1}^{\Delta_n} \overline{B}(\hat{\alpha}_j, 1) \subset \overline{B}(\hat{a}, R)$.
- The closed balls of radius h, $\mathcal{B}_j \subset \overline{B}(t, Rh), j = 1, \ldots, \Delta_n$ are defined as follows

  $$\mathcal{B}_j = F_t^h(\overline{B}(\hat{\alpha}_j, 1)), \forall j = 1, \ldots, \Delta_n,$$

  where $F_t^h : \mathbb{R}^d \to \mathbb{R}^d$ is the invertible affine map given by

  $$F_t^h(x) = t + h(x - \hat{a}), \ \forall x \in \mathbb{R}^d.$$

- Each Δ_n-tuple belonging to $\Pi_{j=1}^{\Delta_n} \mathcal{B}_j$ is $\mathbb{P}_{n-1}$-unisolvent.

Step 2:

The balls $B(\hat{a}, \vartheta R)$ and $B(t, \vartheta Rh)$ are affine-equivalent because the affine bijection F_t^h transforms the former ball into the latter.

Now, applying [2, Prop. II-6.4, (ii)] with $F = F_t^h, \|L^{-1}\| = h^{-1}$ and $\det L = h^d$, we obtain

$$|\Pi_t^h \tilde{u}_\theta^h|_{k,\overline{B}(t,\vartheta Rh)} = O(h^{-k} h^{\frac{d}{2}} |\Pi_t^h \tilde{u}_\theta^h \circ F_t^h|_{k,\overline{B}(\hat{a},\vartheta R)}), \quad h \to 0. \tag{41}$$

Now, we can check that the application $v \mapsto \left(\sum_{j=1}^{\Delta_n} |v(\hat{\alpha}_j)|^2 + |v|^2_{n,\overline{B}(\hat{a},\vartheta R)} \right)^{\frac{1}{2}}$ is a norm over $H^n(\overline{B}(\hat{a},\vartheta R))$ which is equivalent to the norm $\|\cdot\|_{n,\overline{B}(\hat{a},\vartheta R)}$. Therefore,

$$\|\Pi_t^h \tilde{u}_\theta^h \circ F_t^h\|_{n,\overline{B}(\hat{a},\vartheta R)} = O\big(\left(\sum_{j=1}^{\Delta_n} |\Pi_t^h \tilde{u}_\theta^h \circ F_t^h(\hat{\alpha}_j)|^2 + |\Pi_t^h \tilde{u}_\theta^h \circ F_t^h|^2_{n,\overline{B}(\hat{a},\vartheta R)} \right)^{\frac{1}{2}} \big),\ h \to 0. \tag{42}$$

Since $\Pi_t^h \tilde{u}_\theta^h \in \mathbb{P}_{n-1}$, it follows that $|\Pi_t^h \tilde{u}_\theta^h \circ F_t^h|_{n,\overline{B}(\hat{a},\vartheta R)} = 0$. So, from (41)–(42) and the fact $|v|_{k,\overline{B}(\hat{a},\vartheta R)} \le \|v\|_{n,\overline{B}(\hat{a},\vartheta R)}$ we deduce

$$|\Pi_t^h \tilde{u}_\theta^h|_{k,\overline{B}(t,\vartheta Rh)} = O\big(h^{\frac{d}{2}-k} \left(\sum_{j=1}^{\Delta_n} |\Pi_t^h \tilde{u}_\theta^h \circ F_t^h(\hat{\alpha}_j)|^2 \right)^{\frac{1}{2}} \big), \quad h \to 0. \tag{43}$$

Step 3:

Let $\hat{B} = \Pi_{j=1}^{\Delta_n} \overline{B}(\hat{\alpha}_j, 1)$. Let $\{\tilde{p}_1, \dots, \tilde{p}_{\Delta_n}\}$ be a basis of $\mathbb{P}_{n-1}$ and, for any $\hat{b} = (\hat{b}_1, \dots, \hat{b}_{\Delta_n}) \in \hat{B}$, let $M(\hat{b}) = \left(\tilde{p}_i(\hat{b}_j) \right)_{1 \le i,j \le \Delta_n}$.

Taking into account that $\hat{b}$ is $\mathbb{P}_{n-1}$-unisolvent we deduce that $M(\hat{b})$ is regular. So, if we denote $M(\hat{b})^{-1} = \left(m'_{ij}(\hat{b}) \right)_{1 \le i,j \le \Delta_n}$ we have that

$$\forall \Psi \in \mathbb{P}_{n-1}, \quad \Psi = \sum_{i,j=1}^{\Delta_n} \Psi(\hat{b}_i) m'_{ij}(\hat{b}) \tilde{p}_j.$$

Now, as the matrix inversion is a continuous operation, each function $m'_{ij}(\hat{b})$ is bounded over the compact set $\hat{B}$. If we denote

$$\gamma_0 = \max_{1 \le i,j \le \Delta_n} \sup_{\hat{b} \in \hat{B}} |m'_{ij}(\hat{b})| \quad \text{and} \quad \gamma_1 = \max_{1 \le i,j \le \Delta_n} |\tilde{p}_j(\hat{\alpha}_i)|,$$

we have that for all $\Psi \in \mathbb{P}_{n-1}$, $\hat{b} = (\hat{b}_1, \dots, \hat{b}_{\Delta_n}) \in \hat{B}$, for all $j = 1, \dots, \Delta_n$,

$$|\Psi(\hat{\alpha}_j)| \le \gamma_0 \gamma_1 \left| \sum_{i=1}^{\Delta_n} \Psi(\hat{b}_i) \right|.$$

Therefore

$$|\Psi(\hat{\alpha}_j)|^2 \le \gamma_0^2 \gamma_1^2 |\sum_{i=1}^{\Delta_n} \Psi(\hat{b}_i)|^2 \le \gamma_0^2 \gamma_1^2 2^{\Delta_n - 1} \sum_{i=1}^{\Delta_n} |\Psi(\hat{b}_i)|^2$$

and we have obtained that for all $\Psi \in \mathbb{P}_{n-1}$, for all $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_{\Delta_n}) \in \hat{B}$, there exists a constant $C > 0$ such that

$$\sum_{j=1}^{\Delta_n} |\Psi(\hat{\alpha}_j)|^2 \leq C \sum_{i=1}^{\Delta_n} |\Psi(\hat{b}_i)|^2. \tag{44}$$

Now, for each $j = 1, \ldots, \Delta_n$, let $\hat{a}_{jt}^h = (F_t^h)^{-1}(a_{jt}^h)$. By the definition of F_t^h and a_t^h, the Δ_n-tuple $\hat{a}_t^h = (\hat{a}_{1t}^h, \ldots, \hat{a}_{\Delta_n t}^h)$ belongs to $\hat{B}$. By using (44) with $\Psi = (\Pi_t^h \tilde{u}_\theta^h) \circ F_t^h$ and $\hat{b} = \hat{a}_t^h$, and taking into account that

$$\begin{aligned}
\Psi(\hat{b}_j) &= \Big((\Pi_t^h \tilde{u}_\theta^h) \circ F_t^h\Big)(\hat{b}_j) = \Big((\Pi_t^h \tilde{u}_\theta^h) \circ F_t^h\Big)(\hat{a}_{jt}^h) \\
&= \Big((\Pi_t^h \tilde{u}_\theta^h) \circ F_t^h\Big)\Big((F_t^h)^{-1} a_{jt}^h\Big) = (\Pi_t^h \tilde{u}_\theta^h)(a_{jt}^h) \\
&= \tilde{u}_\theta^h(a_{jt}^h),
\end{aligned}$$

we have that

$$\sum_{j=1}^{\Delta_n} |(\Pi_t^h \tilde{u}_\theta^h) \circ F_t^h(\hat{\alpha}_j)|^2 \leq C \sum_{j=1}^{\Delta_n} |\tilde{u}_\theta^h(a_{jt}^h)|^2.$$

By replacing in (43), we obtain

$$|\Pi_t^h \tilde{u}_\theta^h|_{k,\overline{B}(t,\vartheta Rh)} = O\left(h^{\frac{d}{2}-k}\left(\sum_{j=1}^{\Delta_n} |\tilde{u}_\theta^h(a_{jt}^h)|^2\right)^{\frac{1}{2}}\right), \quad h \to 0. \tag{45}$$

Step 4:

From (45) we have that

$$|\Pi_t^h \tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} = O\left(h^{d-2k}\left(\sum_{j=1}^{\Delta_n} |\tilde{u}_\theta^h(a_{jt}^h)|^2\right)\right), \quad h \to 0,$$

and

$$\begin{aligned}
\sum_{t \in T_{Rh}} |\Pi_t^h \tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} &= O\left(h^{d-2k} \sum_{t \in T_{Rh}} \left(\sum_{j=1}^{\Delta_n} |\tilde{u}_\theta^h(a_{jt}^h)|^2\right)\right) \\
&= O\left(h^{d-2k} \sum_{t \in T_{Rh}} \left(\sum_{a \in \mathcal{A}^h \cap \overline{B}(t,\vartheta Rh)} |\tilde{u}_\theta^h(a)|^2\right)\right) \\
&= O\left(h^{d-2k} \sum_{t \in T_{Rh}} \left(\sum_{a \in \mathcal{A}^h} \chi_{\overline{B}(t,\vartheta Rh)}(a) |\tilde{u}_\theta^h(a)|^2\right)\right) \\
&= O\left(h^{d-2k} \sum_{a \in \mathcal{A}^h} |\tilde{u}_\theta^h(a)|^2 \left(\sum_{t \in T_{Rh}} \chi_{\overline{B}(t,\vartheta Rh)}(a)\right)\right), \quad h \to 0.
\end{aligned}$$

Hence, by using (37), we obtain

$$\sum_{t\in T_{Rh}} |\Pi_t^h \tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} = O\left(h^{d-2k}\sum_{a\in\mathcal{A}^h}|\tilde{u}_\theta^h(a)|^2\right), \quad h\to 0. \tag{46}$$

Now, since $\mathcal{A}^h\subset\Omega$ and $(\tilde{u}_\theta^h)|_\Omega = (\tilde{P}u_\theta^h)|_\Omega = u_\theta^h$, we have that

$$\begin{aligned}\sum_{a\in\mathcal{A}^h}|\tilde{u}_\theta^h(a)|^2 &= \sum_{a\in\mathcal{A}^h}|u_\theta^h(a)|^2 = \|\rho^h u_\theta^h\|_2^2 = \|\rho^h(\sigma_\theta^h - g)\|_2^2\\ &= \|\rho^h(\sigma_\theta^h)-\beta^h\|_2^2 = O(\theta),\ h\to 0,\end{aligned} \tag{47}$$

where (22) has been used. From (46) and (47) we derive (40) . □

We are now ready to prove Theorem 3:

Proof Let k be an integer with $0\le k\le n-1$. Let κ_0 be the constant given by [2, Prop. II-6.1] and $R>1$ be the constant of [2, Prop. II-6.6].

For each $h\in\mathcal{H}\cap(0,\kappa_0/R]$ and $\theta>0$, we have that $hR\in(0,\kappa_0]$, and applying [2, Prop. II-6.1] we have

$$\Omega\subset\bigcup_{t\in T_{Rh}}\overline{B}(t,\vartheta Rh).$$

Since $\tilde{P}$ is a prolongation operator from $H^n(\Omega)$ to $H^n(\mathbb{R}^d)$ we have that

$$|u_\theta^h|^2_{k,\Omega} \le |\tilde{u}_\theta^h|^2_{k,\bigcup_{t\in T_{Rh}}\overline{B}(t,\vartheta Rh)} \le \sum_{t\in T_{Rh}}|\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)}. \tag{48}$$

Now, it is clear that

$$|\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} \le 2\left(|\tilde{u}_\theta^h - \Pi_t^h\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} + |\Pi_t^h\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)}\right).$$

So, (48) can be expressed as

$$|u_\theta^h|^2_{k,\Omega} \le 2\left(\sum_{t\in T_{Rh}}|\tilde{u}_\theta^h - \Pi_t^h\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)} + \sum_{t\in T_{Rh}}|\Pi_t^h\tilde{u}_\theta^h|^2_{k,\overline{B}(t,\vartheta Rh)}\right). \tag{49}$$

By introducing (39) and (40) in (49) we have that

$$|u_\theta^h|^2_{k,\Omega} = O(|u_\theta^h|^2_{n,\Omega}h^{2(n-k)} + h^{d-2k}\theta), \quad h\to 0,$$

i.e.

$$|\sigma_\theta^h - g|^2_{k,\Omega} = O(|\sigma_\theta^h - g|^2_{n,\Omega}h^{2(n-k)} + h^{d-2k}\theta), \quad h\to 0,$$

from which we deduce the result. □

Corollary 1 is now derived as a consequence of Theorem 2 and Theorem 3: *Proof of Corollary 1* By Theorem 2, $\|g - \sigma_\theta^h\|_{n,\Omega} = o(1),\ h \to 0$. Therefore, since $|g - \sigma_\theta^h|_{n,\Omega} \le \|g - \sigma_\theta^h\|_{n,\Omega}$, we have that

$$|g - \sigma_\theta^h|_{n,\Omega} = o(1),\ h \to 0. \tag{50}$$

From (50) we also have that, for any $k = 0,\ldots,n-1$,

$$|g - \sigma_\theta^h|_{n,\Omega} \cdot h^{n-k} = o(h^{n-k}),\ h \to 0.$$

and applying Theorem 3,

$$|g - \sigma_\theta^h|_{k,\Omega} = o(h^{n-k}) + O(h^{\frac{d}{2}-k}\theta^{\frac{1}{2}}),\ h \to 0.$$

□

6 Computation

We are now going to obtain the expression of the discrete PDE spline σ_θ.

We number the basis functions of the space X, $\omega_1,\ldots,\omega_I$. We can then express σ_θ as the linear combination $\sigma_\theta(x) = \sum_{i=1}^I \gamma_i \omega_i(x)$, and if we calculate the unknown coefficients γ_i, we then have the expression of σ_θ.

By substituting in (12), we obtain, for all $v \in X$,

$$\sum_{i=1}^{I} \gamma_i \left((\rho\omega_i, \rho v) + \theta(\omega_i, v)_{\mathcal{L}}\right) + \sum_{k=1}^{Nn} \alpha_k \lambda_k(v) = (\beta, \rho v) + \theta\,(f, v)_{0,\Omega}\,,$$

subject to the restrictions $\lambda_j\left(\sum_{i=1}^{I} \gamma_i \omega_i\right) = z_j,\ 0 \le j \le Nn$, which are equivalent to

$$\begin{cases} \displaystyle\sum_{i=1}^{I} \gamma_i \Big((\rho\omega_i, \rho\omega_j) + \theta(\omega_i, \omega_j)_{\mathcal{L}}\Big) + \sum_{k=1}^{Nn} \alpha_k \lambda_k(\omega_j) \\ = (\beta, \rho\omega_j) + \theta(f, \omega_j)_{0,\Omega},\ 1 \le j \le I, \\ \displaystyle\sum_{i=1}^{I} \gamma_i \lambda_j(\omega_i) = z_j,\ 0 \le j \le Nn, \end{cases}$$

that is, a linear system with $I + Nn$ equations and the unknowns

$$\gamma_1,\ldots,\gamma_I,\alpha_1,\ldots,\alpha_{Nn}.$$

Its matricial form is

$$\begin{pmatrix} G & D \\ D^T & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \end{pmatrix} = \begin{pmatrix} \check{f} \\ z \end{pmatrix},$$

where $G = (g_{ij})_{1\le i,j\le I}$, with $g_{ij} = (\rho\omega_i, \rho\omega_j) + \theta\left(\omega_i, \omega_j\right)_{\mathcal{L}}$, $D = (d_{ij})_{\substack{1\le i\le I\\ 1\le j\le Nn}}$, with $d_{ij} = \lambda_j(\omega_i)$, $\gamma = (\gamma_1, \dots, \gamma_I)^T$, $\alpha = (\alpha_1, \dots, \alpha_{Nn})^T$, $\check{f} = ((\beta, \rho\omega_1) + \theta(f, \omega_1)_{0,\Omega}, \dots, (\beta, \rho\omega_I) + \theta(f, \omega_I)_{0,\Omega})^T$, $z = (z_1, \dots, z_{Nn})^T$.

If we call

$$A = \left(\omega_j(a_i)\right)_{\substack{1\le i\le M\\ 1\le j\le I}}, R = \left((\omega_i, \omega_j)_{\mathcal{L}}\right)_{1\le i,j\le I}, \text{and } \widetilde{f} = ((\omega_1, f)_{0,\Omega}, \dots, (\omega_I, f)_{0,\Omega})^T,$$

then $G = A^T A + \theta R$ and $\check{f} = A^T\beta + \theta\widetilde{f}$.

Appendix

It is known that every conditionally positive definite function of order m grows at most like a polynomial of degree $2m$, i.e. it is a slowly increasing function. Moreover, its corresponding native space admits the following characterization:

Theorem 7 *Suppose that $\Phi_m : \mathbb{R}^d \to \mathbb{R}$ is a conditionally positive definite function of order m. Suppose further that Φ_m has a generalized Fourier transform $\widehat{\Phi_m}$ of order m that is continuous on $\mathbb{R}^d \setminus \{0\}$. Let $\mathcal{G}$ be the real vector space consisting of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ that are slowly increasing and have a generalized Fourier transform $\widehat{f}$ of order $m/2$ that satisfies $\widehat{f}\Big/\sqrt{\widehat{\Phi_m}} \in L^2(\mathbb{R}^d)$. Then $\mathcal{G}$ is the native space corresponding to $\Phi(x,y) := \Phi_m(x-y)$, i.e. $\mathcal{G} = \mathcal{N}_\Phi(\mathbb{R}^d)$, and the semi-norm defined on $\mathcal{G}$ by the square root of*

$$(2\pi)^{-d/2}\int_{\mathbb{R}^d} \frac{|\widehat{f}(\omega)|^2}{\widehat{\Phi_m}(\omega)}d\omega,$$

coincides with the semi-norm given by (1).

Within the assumptions and notations before Theorem 1, let us now prove it:

Proof of Theorem 1

Let $(c_i)_{i=1}^Q$ be a set of real coefficients such that

$$\sum_{i=1}^{Q} c_i\tilde{\lambda}_i = 0 \tag{51}$$

on $\mathcal{N}_\Phi(\mathbb{R}^d)$. Then, as the native space norm, $\|\cdot\|_{\mathcal{N}_\Phi(\mathbb{R}^d)}$, is given, for each $v \in \mathcal{N}_\Phi(\mathbb{R}^d)$, by

$$\|v\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = |v|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} + \sum_{k=1}^{\Delta_m}|v(\xi_k)|^2,$$

where $|\cdot|_{\mathcal{N}_\Phi(\mathbb{R}^d)}$ is the native space semi-norm defined in (1), and taking into account that, for every $i = 1,\dots,Q$, the Riesz representer of $\tilde{\lambda}_i$ on the native Hilbert space $\mathcal{N}_\Phi(\mathbb{R}^d)$ is $\tilde{\lambda}_i^y K(\cdot,y)$, by [18, Th. 16.7], we have that:

$$
\begin{aligned}
0 = \left\| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i \right\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)^*} &= \left\| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y K(\cdot,y) \right\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} \\
&= \left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y K(\cdot,y) \right|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} + \sum_{k=1}^{\Delta_m} \left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y K(\xi_k,y) \right|^2
\end{aligned}
$$

and, as a consequence, in the native space semi-norm $|\cdot|_{\mathcal{N}_\Phi(\mathbb{R}^d)}$,

$$
\left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y K(\cdot,y) \right|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = 0. \tag{52}
$$

From (51) we obtain that

$$
\sum_{i=1}^{Q} c_i \tilde{\lambda}_i(p) = 0 \tag{53}
$$

for all $p \in \mathbb{P}_{m-1}$. Thus

$$
\begin{aligned}
&\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y \Phi(\cdot,y) - \sum_{i=1}^{Q} c_i \sum_{k=1}^{\Delta_m} \Phi(\cdot,\xi_k)\tilde{\lambda}_i(p_k) \\
&= \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y \Phi(\cdot,y) - \sum_{k=1}^{\Delta_m} \Phi(\cdot,\xi_k) \sum_{i=1}^{Q} c_i \tilde{\lambda}_i(p_k) = \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y \Phi(\cdot,y)
\end{aligned}
$$

and taking into account that $\tilde{\lambda}_i^y \Phi(\cdot,y) - \sum_{k=1}^{\Delta_m} \Phi(\cdot,\xi_k)\tilde{\lambda}_i(p_k)$ and $\tilde{\lambda}_i^y K(\cdot,y)$ differ only by a polynomial in $\mathbb{P}_{m-1}$, we obtain from (52) that

$$
\left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y \Phi(\cdot,y) \right|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = 0. \tag{54}
$$

For each $i = 1,\dots,Q$, it is satisfied that

$$
\tilde{\lambda}_i^y \Phi(\cdot,y) = \tilde{\lambda}_i^y \Phi_m(\cdot - y) = (-1)^{|\alpha^{(i)}|} (\partial^{\alpha^{(i)}} \Phi_m)(\cdot - x_i)
$$

and

$$
(\tilde{\lambda}_i^y \Phi(\cdot,y))^\wedge(\omega) = (-\mathrm{i}\omega)^{\alpha^{(i)}} \mathrm{e}^{-\mathrm{i}\omega^T x_i} \widehat{\Phi_m}(\omega) = \tilde{\lambda}_i^y (\mathrm{e}^{-\mathrm{i}\omega^T y}) \widehat{\Phi_m}(\omega). \tag{55}
$$

On the other hand, for fixed $\omega \in \mathbb{R}^d$, the expansion of the exponential function gives us that

$$
\mathrm{e}^{-\mathrm{i}\omega^T y} = \sum_{k=0}^{\infty} \frac{(-\mathrm{i})^k}{k!} (\omega^T y)^k
$$

and then

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{-i\omega^T y} = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y (\omega^T y)^k.$$

By applying (53) we have that, for each $k = 0, \ldots, m-1$,

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y (\omega^T y)^k = 0$$

and therefore

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{-i\omega^T y} = \sum_{k=m}^{\infty} \frac{(-i)^k}{k!} \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y (\omega^T y)^k,$$

which leads to

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{-i\omega^T y} = O(\|\omega\|_2^m), \qquad \|\omega\|_2 \to 0. \tag{56}$$

Therefore, taking into account (54) and (55), and by applying Theorem 7, we obtain that

$$0 = \left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y \Phi(\cdot, y) \right|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left| \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{-i\omega^T y} \right|^2 \widehat{\Phi_m}(\omega) d\omega, \tag{57}$$

where this integral is well defined by (56).

Since Φ_m is a conditionally positive definite function of order m that has a nonnegative and nonvanishing generalised Fourier transform $\widehat{\Phi_m}$ of order m which is continuous on $\mathbb{R}^d \setminus \{0\}$, there exists an open set $U \subseteq \mathbb{R}^d$ where $\widehat{\Phi_m}(\omega) > 0$. Hence, from (57), we obtain that

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{i\omega^T y} = 0$$

for all ω with $-\omega \in U$ and then, by analytic continuation, we derive that

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{i\omega^T y} = 0$$

for all $\omega \in \mathbb{R}^d$.

Therefore, for the Fourier transform $\hat{u}$ of any test function $u \in \mathcal{S}$ (being $\mathcal{S}$ the Schwartz space of test functions of rapid decay [11], i.e., the space of all functions $u \in C^\infty(\mathbb{R}^d)$ which satisfy that $x^\alpha \partial^\beta u$ is uniformly bounded on $\mathbb{R}^d$, for all multi-indices $\alpha, \beta \in \mathbb{N}^d$), we have the identity

$$0 = \sum_{i=1}^{Q} c_i \tilde{\lambda}_i^y e^{i\omega^T y} \hat{u}(\omega) = \sum_{i=1}^{Q} c_i (i\omega)^{\alpha^{(i)}} e^{i\omega^T x_i} \hat{u}(\omega) = \left(\sum_{i=1}^{Q} c_i (\partial^{\alpha^{(i)}} u)(\cdot + x_i) \right)^{\wedge} (\omega)$$

for all $\omega \in \mathbb{R}^d$, which implies

$$\sum_{i=1}^{Q} c_i (\partial^{\alpha^{(i)}} u)(x + x_i) = 0, \qquad x \in \mathbb{R}^d,$$

and in particular, taking $x = 0$,

$$\sum_{i=1}^{Q} c_i \tilde{\lambda}_i(u) = 0. \tag{58}$$

Let us find now Q test functions $u_j \in \mathcal{S}$, $j = 1, \ldots, Q$, such that

$$\tilde{\lambda}_i(u_j) = \delta_{ij}, \qquad i, j \in \{1, \ldots, Q\}.$$

Let $u_0 \in C^\infty(\mathbb{R}^d)$ be a compactly supported function with support contained in the ball around zero with radius $0 < \varepsilon < \min_{j \neq k} \|x_j - x_k\|_2$ and $u_0(x) = 1$ if $\|x\|_2 < \varepsilon/2$. For $j = 1, \ldots, Q$ we define

$$u_j(x) := \frac{(x - x_j)^{\alpha^{(j)}}}{\alpha^{(j)}!} u_0(x - x_j), \qquad x \in \mathbb{R}^d.$$

We then have that $u_j \in \mathcal{S}$ and $\tilde{\lambda}_i(u_j) = \delta_{ij}$, for all $i, j \in \{1, \ldots, Q\}$. Now, by applying (58), we get, for each $j = 1, \ldots, Q$,

$$0 = \sum_{i=1}^{Q} c_i \tilde{\lambda}_i(u_j) = c_j$$

and hence our result. □

Acknowledgements This work has been supported by Junta de Andalucía (Research groups FQM178 and FQM191). The authors are grateful for useful advices and suggestions from Professor Dr. M. Pasadas which have enhanced this work.

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