Popular Discontent, Recall Elections, and Ranked-Choice Voting

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Abstract

Consider a scenario in which voters select a candidate through a two-stage process: an intra-party stage in which only party members vote, followed by an inter-party stage in which all voters participate. In such a setting, the elected candidate may not be a Condorcet winner, potentially leading to significant voter dissatisfaction. Recall elections, which allow the public to remove a politician from office before the end of her term, can serve as a partial remedy for this discontent. This study examines the effectiveness of recall elections in selecting a Condorcet winner as a replacement candidate using two voting methods: the current first-past-the-post (FPP) system and an alternative, ranked-choice voting (RCV). Our results indicate that RCV generally outperforms FPP in selecting the Condorcet winner, especially in ideologically polarized societies. While challenges remain when the Condorcet winner is a compromise candidate for a majority, slight modifications to RCV improve its performance over FPP.

Keywords: First-past-the-post, Polarization, Ranked-Choice Voting, Recall Election, Single-Peakedness.

Journal of Economic Literature Classification Numbers: D71, D72.

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1. Introduction

The recall election is a political procedure that allows citizens to remove an elected official from office through a referendum before the end of the official's term, serving as a mechanism for direct democracy. Recall election proponents, citing the National Conference of State Legislatures (NCSL), argue that it enables voters to exercise control over elected officials who do not represent their constituents' best interests, are unresponsive, incompetent, or fail to fulfill their duties in a manner aligned with the public's interest. By allowing constituents to hold their elected representatives accountable outside the regular election cycle, the recall election acts as a crucial tool for maintaining a responsive and responsible government.

Recall elections have been used in the United States to remove statewide elected officials from office, although they tend to be more prevalent in certain regions, particularly in some states, than in others. In the history of the United States, there have been approximately 30 recall elections targeting state legislators (National Conference of State Legislatures, 2021), with the state of California being a significant contributor to this total, accounting for more than a third of these efforts at recall (California Secretary of State's office, 2024). The rules and procedures governing recall elections, including the specific grounds on which an official may be recalled, are outlined in state constitutions or statutes, reflecting some variation in their application across jurisdictions. While the process for initiating a recall election exhibits differences from one state to another, it typically follows a sequence of established steps.

The first step in triggering a recall election involves gathering a petition, requiring a certain number of signatures from registered voters. This number is often linked to a percentage of the votes from the last election for that office. Once this petition is submitted, election officials verify the signatures to confirm they meet the necessary criteria. If the petition has the required number of valid signatures, the process moves to the next phase: organizing a recall election. In this election, voters decide whether the official should be removed from office. The rules of the jurisdiction may also allow for a list of candidates to be included on the ballot to replace the official if the recall is successful. Voters are presented with the option to vote "yes" to remove the official and choose a replacement, or "no" to keep the current official in office. If the majority votes for the recall, the official is removed, and the candidate with the most votes fills the position for the remainder of the term. If the recall effort fails, the official continues in his or her position.

This research is propelled by an essential insight: within the current mechanics of recall elections, an elected official may be supplanted by a candidate who commands less support from the electorate than the incumbent who was displaced. This issue is far from trivial, especially considering that recall elections are fundamentally designed to remedy potential dissatisfaction among voters with the political class, specifically when elected representatives no longer reflect the populace's values and/or interests.

One of the most notable instances of recall occurred in California during 2018. During his term, Democratic Senator Josh Newman faced a recall election. Among the participants in the recall referendum, 58.1% voted in favor of the recall, while 41.9% opposed it, supporting the senator's continued tenure in his position. Ultimately, with a majority of the votes, the recall option prevailed. This led to the next phase; determining Newman's replacement. A total of six candidates contested to fill the senator's position in the event of his recall. Among these candidates, Republican Ling-Ling Chang emerged as the frontrunner, securing the highest percentage of votes at 33.8%. It is not unlikely that the 33.8% support Chang received came from the 58.1% of the electorate that voted to recall the senator. Accepting this assumption would imply that Ling-Ling Chang actually garnered the support of just 19.65% of the referendum participants. When compared to the 41.9% who supported Democrat Newman, this discrepancy significantly challenges the efficiency of the recall election process.

In this paper, we evaluate the performance of the current first-past-the-post (FPP) system and an alternative voting mechanism that has been increasingly recognized and discussed in recent times, ranked-choice voting (RCV), in electing a replacement candidate for the recalled incumbent.[1] Ranked-choice voting is an electoral system that aims to capture voters' prefer-

[1]Note that neither first-past-the-post nor ranked-choice voting is strategy-proof over the full preference domain,

ences more accurately. This method asks voters to articulate their preferences in a hierarchical manner, designating their most favored candidate as the primary choice and sequentially ranking subsequent candidates—second choice, third choice, and so forth—in accordance with their individual preferences. Voters are not required to rank all candidates and can rank as many or as few as they wish.[2] The counting process begins by tallying only the first-choice votes. If a candidate secures a majority of these votes (more than 50%), she is declared the winner. However, if no candidate achieves a majority from the first-choice votes, the system triggers a sequence of elimination and reallocation rounds. The candidate with the fewest first-choice votes is eliminated, and votes for this candidate transfer directly to each voter's next-highest choice. This procedure for eliminating the candidate receiving the least first-choice support and reallocating their votes is repeated in successive rounds until a candidate gathers a majority of first-choice votes in a given round.

We propose a closed primary election model that allows for a recall election. Closed primaries are a two-stage process: an intra-party stage in which only party members vote, followed by an inter-party stage in which all voters participate. The reason we have chosen to model closed primaries is due to their significance in the context of recall elections. More than 50% of all recall elections for state officials in the US occurred in states that, at the time of the recall election, operated with either closed primaries or partially closed primaries (National Conference of State Legislatures, 2021).[3] Today, out of the 50 American states, more than half operate under a primary election system linked to closed primaries: 10 use a closed primary system for state elections, 9 use a partially closed primary system, and 8 use a primary system open to unaffiliated voters (National Conference of State Legislatures, 2024).[4] In closed

as they are special cases of the Gibbard-Satterthwaite theorem. For a simple proof, see Benoıt (2000). Dellis and Kröger (2024) experimentally investigate the bias induced by strategic behavior in ranked-choice voting.

[2]Our model assumes that voters rank all candidates based on their order of preferences. This approach aims to address one of the recognized issues of ranked-choice voting: ballot exhaustion. Ballot exhaustion occurs when all of a voter's ranked candidates are eliminated before the final round, rendering their ballot "exhausted" and thus excluded from the final count. This situation may arise if voters do not rank all candidates or if their chosen candidates are eliminated in the early rounds.

[3]In partially closed primaries, parties may allow unaffiliated voters to participate in their nominating contests, but these voters must affiliate with the party before casting their ballot.

[4]In primaries open to unaffiliated voters, these voters can participate in any party primary without officially affiliating, though their choice is public information, which may result in voters participating in only one party's primary.

primaries, the inter-party stage, commonly referred to as the general election, may produce a winner who is not the Condorcet winner. This occurs when there is support for extreme views within a party, while socially, its moderate candidate is more desirable.

If electing a candidate that is not the Condorcet winner triggers a recall election, we study the effectiveness of first-past-the-post and ranked-choice voting systems in selecting a Condorcet winner candidate to replace an ousted incumbent. We show that in recall elections, ranked-choice voting has a superior ability to select the Condorcet winner compared to first-past-the-post, especially in societies showing signs of ideological polarization where the ideological center is weakening. However, we may still fail to select the Condorcet winner if she is a compromise candidate —one acceptable to the majority but only passionately supported by a minority. Nevertheless, slight modifications to ranked-choice voting can improve its performance over first-past-the-post. Specifically, implementing an instant runoff between the two candidates with the fewest first-choice votes, rather than eliminating the candidate with the fewest first-choice votes outright, allows ranked-choice voting to select the Condorcet winner even in such scenarios. Our results suggest that integrating ranked-choice voting into the process of selecting a replacement candidate after a successful recall election would offer several potential benefits. By enabling voters to rank their preferred candidates, RCV ensures that the newly elected official reflects a wider consensus among the electorate. This method would therefore enhance the democratic process by producing outcomes that more accurately reflect the preferences of voters, thereby making the election of a replacement more equitable and in line with the collective will of the electorate. Such a fusion of recall elections and RCV would underscore a commitment to refining democratic mechanisms to be more inclusive and representative.

1.1. Related Literature

Vandamme (2020) investigates the use of recall mechanisms, analyzing theoretical arguments and empirical evidence to assess their potential in enhancing representative democracy.

He specifically explores whether recalls increase citizens' support for political systems and improve the quality of democratic decisions. The author acknowledges that the existing empirical evidence regarding the use of recall elections does not support the hypothesis that this mechanism radically transforms representative institutions. Given this backdrop, conducting a theoretical analysis to identify the challenges posed by recall elections and how their design might be refined becomes even more crucial. This would aim to ensure that the mechanism effectively achieves its primary goal of fostering a political class that truly represents the will of the citizens.

Maskin (2022) underscores the benefits of ranked-choice voting over the traditional first-past-the-post system in the context of the US general election. The paper details how ranked-choice voting yields election outcomes that more accurately reflect voter preferences compared to first-past-the-post. Furthermore, it presents the argument that majority-rule voting represents an even more superior system. The Arrow Impossibility Theorem (Arrow, 1951) posits that it is impossible for any voting rule to fulfill all five of the following desirable principles simultaneously: consensus, anonymity, neutrality, independence of irrelevant alternatives, and decisiveness. Dasgupta and Maskin (2008) have shown that majority rule adheres to these five principles more consistently than other voting systems do. Maskin (2022) shows that ranked-choice voting does not satisfy independence of irrelevant alternatives, thereby making it evidently less optimal than majority rule. But the author claims that combining ranked-choice voting with an instant runoff between the two candidates who receive the least first-place votes would lead to the results achieved by majority rule, making ranked-choice voting a clear step forward in comparison to the current first-past-the-post system.

Holden and Quiggin (2022) assess the effectiveness of ranked-choice voting versus first-past-the-post voting in tripartite elections, featuring two major parties and a third, minor party. Operating under the premise that the minor party is never the Condorcet winner, they conclude that ranked-choice voting consistently selects the Condorcet winner, in scenarios of sincere voting, in contrast to first-past-the-post voting, which does not. In contrast to their work, we evaluate the performance of ranked-choice voting and first-past-the-post in three candidate scenarios where the minority-supported candidate could actually be the Condorcet winner. This allows us to uncover instances where ranked-choice voting also fails to select the Condorcet winner candidate.

Our basic model has similarities to the one proposed in Amorós et al. (2016) but it addresses a different problem. The authors find that closed primaries could prevent the median voter's preferred candidate from securing victory. In particular, they uncover that in circumstances where the median voter is moderate, yet the median partisan voter of her party is extreme, closed primaries can result in a winner with extreme positions. Our paper demonstrates that within this scenario, a recall election—whether executed via first-past-the-post or ranked-choice voting—enables the election of the candidate preferred by the median voter. Furthermore, we identify a scenario not possible in Amorós et al. (2016) where the candidate preferred by the median voter also fails to win under closed primaries.[5] This occurs when a moderate candidate wins the election, yet the candidate preferred by the median voter, albeit also moderate, was a different one. In this scenario, we reveal that the electoral system employed in the recall election significantly impacts outcomes: ranked-choice voting facilitates the selection of the median voter's preferred candidate in more instances compared to first-past-the-post.

While the ranked-choice voting system is celebrated for its potential to encourage a more moderate and inclusive political environment, emerging research also brings to light nuanced challenges associated with its implementation. Buisseret and Prato (2024)'s analysis suggests that, under certain conditions, ranked-choice voting may inadvertently amplify candidates' focus on their core base at the expense of broader electoral appeal, and potentially enable a candidate less favored in head-to-head comparisons to secure multi-candidate victories. McCarty (2024)'s investigation into ranked-choice voting's implications on minority representation further illuminates complex dynamics, highlighting that, despite its intentions, ranked-choice voting's majoritarian nature might limit minority groups' electoral influence,

[5]This new case, in contrast to Amorós et al. (2016), emerges because while their model considers a preliminary stage where candidates may or may not be incentivized to enter the race, our study assumes that candidates invariably choose to run.

particularly where ballot exhaustion rates are high among these communities.

The remainder of the paper is organized as follows. In Section 2 we set out the model. In Section 3, we delineate the circumstances in which the candidate who is the Condorcet winner is not ultimately chosen, a scenario we describe as majority dissatisfaction. In Section 4, we compare the performance of two different voting systems —first-past-the-post and ranked-choice voting— applied to recall elections. Specifically, we study the conditions under which each of these systems would select the Condorcet winner to replace the candidate responsible for the majority dissatisfaction. In Section 5 we offer some final remarks. In the Appendix we provide the proofs of the results presented in Sections 3 and 4.

2. The Model

Let $\mathcal{L}$ and $\mathcal{R}$ denote the *left* and *right* political parties, respectively. Each political party consists of two candidates: one extreme and one moderate. Let $\mathcal{C}_{\mathcal{L}} = \{L^+, L^-\}$ be the set of candidates of the left party. Let $\mathcal{C}_{\mathcal{R}} = \{R^+, R^-\}$ be the set of candidates of the right party. The superscripts $+$ and $-$ denote *extreme* and *moderate* candidate. Let then $\mathcal{C} = \{L^+, L^-, R^+, R^-\}$ be the set of candidates. The elements in $\mathcal{C}$ can be linearly ordered as follows: $L^+ < L^- < R^- < R^+$. This order represents the relative positions of candidates on the political spectrum from left to right.

Let $\mathcal{N} = \{1, ..., n\}$ be the set of voters, each of whom votes sincerely.[6] Each voter $i \in \mathcal{N}$ has a strict single-peaked preference over the set of candidates $\mathcal{C}$, that we denote by P_i. Each voter can be either a *left-partisan* or a *right-partisan* depending on whether the voter's most preferred candidate is a left-wing candidate (L^+ or L^-) or a right-wing candidate (R^+ or R^-),

[6]Adams and Merrill (2014) conclude that whether voters cast their ballots sincerely or strategically in the primary election is irrelevant to the candidates' chances of winning the general election within a closed primary model. This conclusion stems from their finding that the optimal policy strategies of candidates remain unchanged regardless of whether voters behave sincerely or strategically when selecting a candidate in closed primary systems. Candidates tend to position themselves closer to the center of the general electorate rather than their primary electorate, thereby homogenizing the electability of all candidates and nullifying the potential effects of strategic behavior where voters invariably support electable primary candidates over unelectable ones.

respectively. We suppose that left-partisans always prefer the extreme left-wing candidate over the extreme right-wing candidate, while right-partisans always prefer the extreme right-wing candidate over the extreme left-wing candidate. Then, the admissible preferences for each voter $i \in \mathcal{N}$, which we display in Table 1, are a particular subset of all the strict single-peaked preferences over $\mathcal{C}$.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^+	L^-	L^-	R^-	R^-	R^+
L^-	L^+	R^-	L^-	R^+	R^-
R^-	R^-	L^+	R^+	L^-	L^-
R^+	R^+	R^+	L^+	L^+	L^+

Table 1: Admissible preferences

Let $\mathcal{P} = \{P_{L^+}, P^1_{L^-}, P^2_{L^-}, P^2_{R^-}, P^1_{R^-}, P_{R^+}\}$ be the set of admissible preference relations and let $P = (P_i)_{i \in \mathcal{N}} \in \mathcal{P}^n$ be a voters' preference profile. We say that a candidate is the Condorcet winner relative to the voters' preference profile when no other candidate defeats her in pairwise majority comparisons. Formally, candidate $c \in \mathcal{C}$ is the Condorcet winner at P if there is no $c' \in \mathcal{C}$ such that $\# \{i \in \mathcal{N} \; / \; c' \; P_i \; c\} > \frac{n}{2}$. From Black (1948) we know that the existence of a Condorcet winner candidate is guaranteed under our assumptions. Let $CW(P)$ be the candidate who is the Condorcet winner, given the voters' preference profile P.

Let $\mathcal{N}_L = \{i \in \mathcal{N} : P_i \in \{P_{L^+}, P^1_{L^-}, P^2_{L^-}\}\}$ and $\mathcal{N}_R = \{i \in \mathcal{N} : P_i \in \{P_{R^+}, P^1_{R^-}, P^2_{R^-}\}\}$ be the sets of left-partisans and right-partisans, respectively. Let n_L and n_R denote the cardinality of the sets $\mathcal{N}_L$ and $\mathcal{N}_R$, respectively, where $n_L + n_R = n$. Let α_{L^+} be the proportion of left-partisans whose preferences are represented by P_{L^+}. Formally, $\alpha_{L^+} = \frac{\#\{i \in \mathcal{N} : P_i = P_{L^+}\}}{n}$. Proportions of all other types of voters $\alpha^1_{L^-}, \alpha^2_{L^-}, \alpha^2_{R^-}, \alpha^1_{R^-}, \alpha_{R^+}$ are defined in a similar way. Note that $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+} = 1$.

Let $P_{L^+} < P^1_{L^-} < P^2_{L^-} < P^2_{R^-} < P^1_{R^-} < P_{R^+}$ be the order for the elements of $\mathcal{P}$. Given this order, and for each $P \in \mathcal{P}^n$, let P^m be the median of the elements of $\mathcal{P}$ at P, i.e., $P^m \in \mathcal{P}$ is such that $\#\{i \in \mathcal{N} : P_i \leq P^m\} \geq \frac{n}{2}$ and $\#\{i \in \mathcal{N} : P_i \geq P^m\} \geq \frac{n}{2}$. Suppose, for simplicity, that P^m is unique. We call P^m the *median voter's preferences*. Notice that, for all $c, c' \in \mathcal{C}$

such that cP^mc', either (1) cP_ic' for all $i \in \mathcal{N}$ such that $P_i \leq P^m$, or (2) cP_ic' for all $i \in \mathcal{N}$ such that $P_i \geq P^m$. Hence, when comparing any two candidates c and c', if the median voter prefers c to c', then a majority of voters also prefer c to c'.

For each $P \in \mathcal{P}^n$, let P_L^m be the median of the elements of the set $\{P_{L^+}, P_{L^-}^1, P_{L^-}^2\}$, i.e., $P_L^m \in \{P_{L^+}, P_{L^-}^1, P_{L^-}^2\}$ is such that $\#\{i \in \mathcal{N}_L : P_i \leq P_L^m\} \geq \frac{n_L}{2}$ and $\#\{i \in \mathcal{N}_L : P_i \geq P_L^m\} \geq \frac{n_L}{2}$. We call P_L^m the preferences of the *partisan median voter of the left party.* The preferences of the *partisan median voter of the right party,* $P_R^m \in \{P_{R^-}^2, P_{R^-}^1, P_{R^+}\}$, are defined in a similar way. Suppose, for simplicity, that P_L^m and P_R^m are unique.

Candidates have preferences about being in office. We use w_c where $c \in \mathcal{C}$ to denote that candidate c is in office. Let $\mathcal{O} = \{w_{L^+}, w_{L^-}, w_{R^+}, w_{R^-}\}$ be the set of office-outcomes. We make the following assumptions about candidates' preferences:

(i) For each candidate $c \in \mathcal{C}$, the most preferred office-outcome is that in which she is in office, that is, w_c.

(ii) Each candidate $c \in \mathcal{C}$ is indifferent among the office-outcomes in which she is not in office, that is, w_{-c}.

We suppose that there are no costs associated with running for office. This, along with the preferences considered for candidates, means that all candidates always run for office.

The timing of the game is described below.

1. *Intra-party election stage.* At the party level, left-partisans elect one of the two left-wing candidates and right-partisans elect one of the two right-wing candidates, each using a first-past-the-post voting system. Let $x^L \in \mathcal{C}_\mathcal{L}$ be the candidate chosen by the left-partisans. Let $x^R \in \mathcal{C}_\mathcal{R}$ be the candidate chosen by the right-partisans.

2. *Inter-party election stage.* All voters (both groups, left-partisans and right-partisans) elect one of the two winning candidates from the previous stage by first-past-the-post voting. Let $x^w \in \{x^L, x^R\}$ be the candidate chosen by all voters at the inter-party

election stage. If x^w is a Condorcet winner at P the game ends, otherwise we move to the next stage.

3. *Recall election stage.* All voters (both groups, left-partisans and right-partisans) elect one of the three remaining candidates other than the winner of the inter-party election stage, by either first-past-the-post voting or ranked-choice voting. Let $\mathcal{C}_0 \subset \mathcal{C}$ be the set of candidates running in the recall election. Let $x^w_{FPP}, x^w_{RCV} \in \mathcal{C}_0$ denote the candidates chosen by all voters at the recall election stage by first-past-the-post voting and ranked-choice voting, respectively.

3. Existence of Majority Dissatisfaction

For centuries, representative democracy has been understood as the delegation of decision making power by the electorate to a political class that must ensure the fulfillment of the electorate's interests. It would only be wise to entrust such an important task to a political class that is popular with the people. Thus, it is quite reasonable that if the incumbent politician, for whatever reason, does not have sufficient popular support, voices will emerge calling for her removal. This is where the recall election would come into play, as a replacement for the removed incumbent would have to be appointed.

Since the goal of this paper is to analyze the performance of two voting systems applied to recall elections, we need to ensure that such a recall referendum will take place. Therefore, we introduce a sufficient condition that guarantees the incumbent politician will be recalled from office. We call this condition majority dissatisfaction.

Definition 1. *There is majority dissatisfaction among voters if the Condorcet winner candidate relative to the voters' preference profile is not chosen in the inter-party election stage.*

Note that the Condorcet winner candidate is the most preferred candidate of the (overall) median voter. If a candidate other than the Condorcet winner is elected in the inter-party election stage, this means that there is a majority of voters who would prefer another candidate to hold the office. Under sincere voting, this majority of voters will vote "yes" to remove the official and elect a replacement. That is why we argue that majority dissatisfaction is a sufficient condition for incumbent recall to occur.

Depending on the moderate or extreme nature of the candidates competing in the inter-party election stage, majority dissatisfaction among voters may or may not emerge. Remember that the candidates who advance to the inter-party election are determined by the results of the intra-party election at the left and right political parties. Proposition 1 presents a sufficient condition to prevent majority dissatisfaction in society. All proofs can be found in the Appendix.

Proposition 1. *If the inter-party election takes place between two moderate candidates, then there is no majority dissatisfaction among voters.*

We now offer an intuitive understanding of this result. L^- and R^- are the preferred candidates of the partisan median voters of the left and right parties, respectively. Therefore, they are chosen at each intra-party election and reach the inter-party election. Formally, we have that $P_L^m \in \{P_{L^-}^1, P_{L^-}^2\}$ and $P_R^m \in \{P_{R^-}^2, P_{R^-}^1\}$. Such configuration of voters' preferences is only compatible with the existence of a median voter whose most preferred candidate is either L^- or R^-. That is, $P^m \in \{P_{L^-}^1, P_{L^-}^2, P_{R^-}^2, P_{R^-}^1\}$. Hence, the candidate chosen in the inter-party election will be the Condorcet winner, meaning there will be no room for majority dissatisfaction.

Proposition 1 teaches us a key lesson: If an extreme candidate reaches the inter-party election stage, majority dissatisfaction may occur. Proposition 2 presents the necessary and sufficient condition for the existence of majority dissatisfaction.

Proposition 2. *There is majority dissatisfaction if and only if the Condorcet winner candidate relative to the voters' preference profile is a moderate candidate who was not selected by the* *partisans of her party at the intra-party election stage.*

Majority dissatisfaction arises because partisans of the Condorcet winner candidate's party select the extreme candidate at the intra-party election. Therefore, the Condorcet winner cannot advance to the inter-party election. In other words, the partisan median voter of the Condorcet winner candidate's party is more extreme than the *overall* median voter.

A corollary of Proposition 2 is that at least one extreme candidate reaching the inter-party election is a necessary condition for majority dissatisfaction to exist. There are two types of candidate configurations that could lead to majority dissatisfaction. The first is when the two candidates competing in the inter-party election are both extreme. The second is when one candidate is extreme and the other is moderate. Note that it is only under these two settings that the (moderate) Condorcet winner candidate is not present at the inter-party election stage. Still, the existence of at least one extreme candidate at the inter-party election is not a sufficient condition for majority dissatisfaction. It is possible that, for example, such extreme candidate participating in the inter-party election is the Condorcet winner candidate at P and is elected, resulting in no majority dissatisfaction.

4. Majority Dissatisfaction and the Recall Election

We aim to evaluate the performance of the current first-past-the-post system and the alternative ranked-choice voting system in a recall election. We have argued that under sincere voting, majority dissatisfaction is a sufficient condition for the incumbent to be recalled through a referendum recall. Majority dissatisfaction guarantees that more than 50% of the participants in the referendum vote to remove the incumbent from office and thus the challenger with the most votes replaces her. Specifically, we are interested in knowing whether majority dissatisfaction disappears after the replacement of the incumbent through a recall election.

Definition 2. *Let Y represent a voting method. Voting method Y is beneficial for the recall election if the Condorcet winner candidate relative to the voters' preference profile is selected as the replacement candidate for the incumbent.*

We have previously noted that the configuration of candidates in the inter-party election, in instances of widespread dissatisfaction, can manifest in one of two forms: (i) an extreme candidate versus another extreme candidate, and (ii) an extreme candidate versus a moderate candidate. This majority dissatisfaction stems from the exclusion of a moderate candidate —who was the Condorcet winner— from the inter-party election. Consequently, irrespective of whether the ultimate victor of the inter-party election is an extreme or a moderate candidate, majority dissatisfaction will persist and a recall election will be initiated against that incumbent. In presenting our findings, it is particularly beneficial to differentiate between scenarios wherein an extreme candidate is removed or a moderate candidate experiences removal through a recall election. Propositions 3 and 4 present our findings for each of these cases, respectively.

Proposition 3. *Suppose there is majority dissatisfaction and an extreme candidate is recalled. Then both first-past-the-post (FPP) and ranked-choice voting (RCV) systems are beneficial for the recall election.*

An intuitive interpretation of this proposition is the following. When there is majority dissatisfaction and an extreme candidate is recalled, the pool of replacement candidates will be composed of the two moderate candidates (including the Condorcet winner) and the other extreme candidate. All voters for whom the removed extreme candidate was their preferred candidate, now support their party's moderate in the recall election. Because of our preference domain, such moderate candidate is the Condorcet winner. Thus, this candidate becomes preferred by more than half of all voters, leading to her selection in the recall referendum, regardless of whether the system employed is first-past-the-post or ranked-choice voting.

Proposition 4 studies the case where there is majority dissatisfaction and the recall election was initiated against a moderate candidate.

Proposition 4. *Suppose there is majority dissatisfaction and a moderate candidate is recalled. Then:*

(i) The first-past-the-post (FPP) system is beneficial for the recall election if and only if more voters rank the Condorcet winner candidate first than they do any other replacement candidate.

(ii) The ranked-choice voting (RCV) system is beneficial for the recall election if and only if the voters who rank the Condorcet winner candidate first than they do any other replacement candidate are not a minority.

When there is majority dissatisfaction and a moderate candidate is recalled, the pool of replacement candidates will be composed of the two extreme candidates and the other moderate candidate. It follows from Proposition 2 that the latter candidate is necessarily the Condorcet winner. Proposition 4 states that our findings depend on the distribution of voter preferences for the most preferred candidate within the replacement set. Below we provide an intuition for the results. We distinguish three cases, depending on the size of the fraction of voters for whom the Condorcet winner is the most preferred candidate within the replacement set.

***Case 1:* $CW(P)$ is the preferred replacement candidate for the largest number of voters.**

Suppose that, among the group of candidates competing to replace the recalled incumbent, the moderate candidate —who is the Condorcet winner— is the one whom more voters rank first. Then, both FPP and RCV systems are beneficial for the recall election, i.e., selecting the Condorcet winner as the replacement candidate.

Figures 1 and 2 represent the two possible distributions of voter preferences for the most preferred replacement candidate that we may find in Case 1. Note that in both, the moderate candidate is the preferred one for the largest number of voters, while the distinction arises in the identity of the second most preferred candidate: in Figure 1, it is the extreme left candidate, whereas in Figure 2, it is the extreme right candidate.

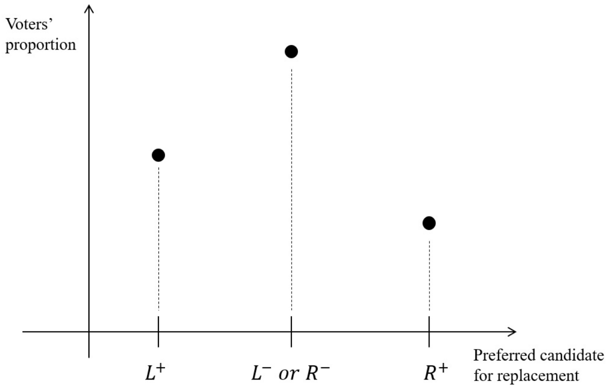

Figure 1: Center-left electorate

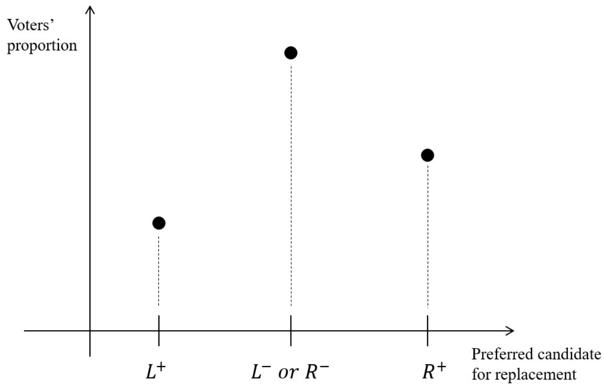

Figure 2: Center-right electorate

In the FPP system, $CW(P)$ is elected because she receives more votes than any other candidate. Under the RCV system, there are two possible situations. The first is when the (largest) number of voters who prefer the Condorcet winner as the replacement candidate already constitutes a majority by itself, in which case RCV directly elects her. The second situation is when these voters still do not form a majority. Then, at the first stage of the RCV, one of the extreme candidates —the one ranked first by the fewest voters— is removed. Subsequently, the Condorcet winner secures a majority because voters who preferred the eliminated extreme candidate now transfer their support to the moderate candidate. Thus, RCV chooses $CW(P)$ at its second stage.

***Case 2:* $\boldsymbol{CW(P)}$ is the preferred replacement candidate for a number of voters that is not the largest but not the smallest.**

Suppose that the number of voters who have the Condorcet winner as their preferred replacement candidate is neither a relative majority nor a minority. Then, RCV is beneficial for the recall election, while FPP is not. That is, while RCV chooses the Condorcet winner as the replacement candidate, FPP chooses one of the extreme candidates.

Figures 3 and 4 represent the two possible distributions of voter preferences for the most preferred replacement candidate that we may find in Case 2. Note that in both, the moderate candidate is not the preferred candidate for either the largest or smallest number of voters, while the distinction arises in the identity of the preferred candidate: in Figure 3 it is the extreme left candidate, whereas in Figure 4 it is the extreme right candidate.

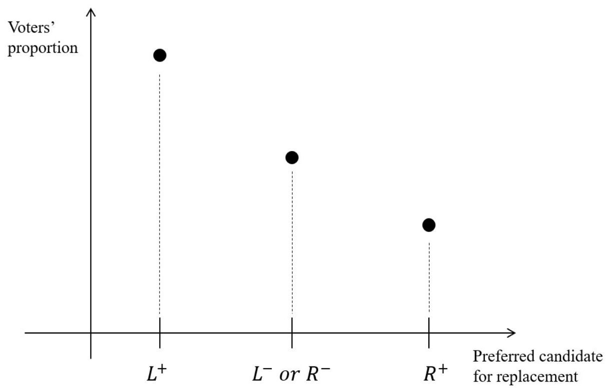

Figure 3: Left-leaning electorate

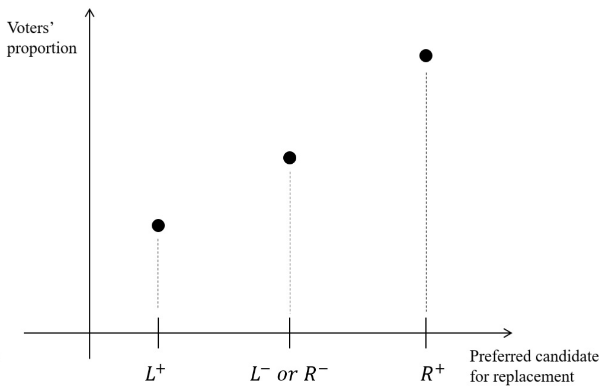

Figure 4: Right-leaning electorate

In the FPP system, an extreme candidate is elected because she receives more votes than the Condorcet winner. Under the RCV system, if it happens that the voters who prefer one of the extreme candidates to replace the incumbent constitute a majority, then this candidate is elected. Otherwise, note that the other extreme candidate is necessarily the one ranked first by the fewest voters. At the first stage of the RCV, such extreme candidate is removed. After this elimination, the Condorcet winner secures a majority —as supporters of the eliminated candidate pivot to the moderate candidate— and is subsequently chosen. Thus, $CW(P)$ is elected at the second stage of the RCV.

***Case 3:* $\boldsymbol{CW(P)}$ is the preferred replacement candidate for the smallest number of voters.**

Suppose that the number of voters with the Condorcet winner as their preferred replacement candidate is the smallest. Then neither FPP nor RCV systems are beneficial for the recall election, i.e., both select an extreme candidate to replace the recalled incumbent.

Figures 5 and 6 represent the two possible distributions of voter preferences for the most preferred replacement candidate that we may find in Case 3. Note that in both, the moderate candidate is the preferred candidate for a minority of voters, while the distinction arises in the identity of the preferred candidate: in Figure 5, it is the extreme left candidate, whereas in Figure 6, it is the extreme right candidate.

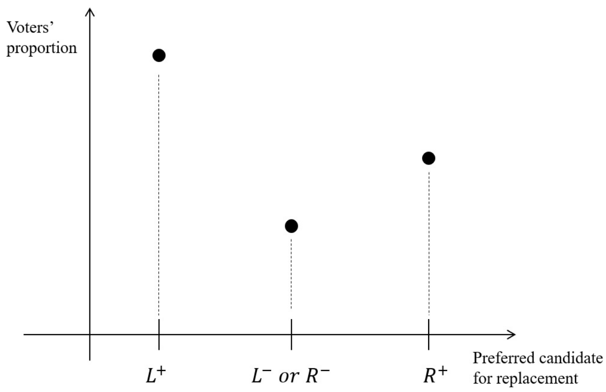

Figure 5: Polarized electorate (I)

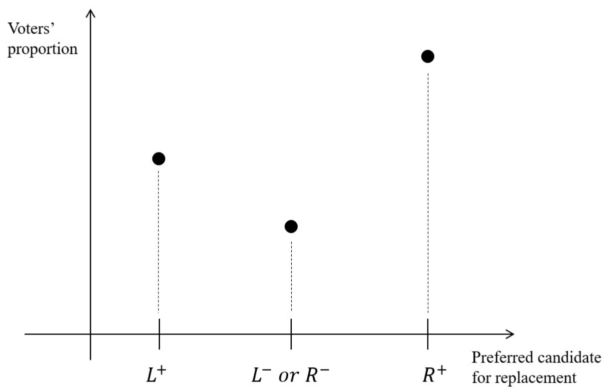

Figure 6: Polarized electorate (II)

As in Case 2, the FPP system chooses an extreme candidate because she receives more votes than the Condorcet winner. Regarding the RCV system, again as in Case 2, if it happens that the voters who prefer one of the extreme candidates to replace the incumbent constitute a majority, then this candidate is elected. Otherwise, note that the moderate candidate —the Condorcet winner— is the one ranked first by the fewest voters. Thus, she is eliminated in the first stage of the RCV, thereby precluding its selection. As a result, an extreme candidate is also ultimately elected under this system.

Cases 1 and 2 above exemplify circumstances wherein RCV for the recall election performs at least as well as FPP. In Case 3, however, we find that RCV is not beneficial for the recall election. Therefore, the advantage over the traditional FPP system is not clear in that specific setting. At this point, it is appropriate to revisit the modification of the RCV mechanism suggested by Maskin (2022), which could lead to outcomes even closer to those obtained by majority rule. He proposes that rather than dismissing the candidate who has received the least number of first-place rankings from voters, an instant runoff should be conducted between the two candidates who have garnered the fewest first-place rankings. Proposition 5 below shows the advantage of this modified RCV over traditional FPP in situations where the Condorcet winner is the candidate preferred by only a minority of voters to replace the recalled incumbent.

Proposition 5. *Suppose there is majority dissatisfaction, a moderate candidate is recalled, and voters who rank the Condorcet winner candidate first than they do any other replacement candidate are a minority. Then the ranked-choice voting (RCV) system with instant runoff between the two candidates who have been ranked first the least frequently is beneficial for the recall election.*

The inclusion of this instant runoff voting prevents the elimination of the Condorcet winner in the first stage of RCV. Since the moderate candidate is the Condorcet winner, she wins over every single candidate in pairwise majority comparisons, especially the extreme candidate with the second-fewest first-place votes. Thus, she moves on to the second stage of the RCV where, faced against the other extreme candidate, she is chosen because she secures a majority.

5. Concluding Remarks

Our study reveals that in the context of social dissatisfaction with the political class in power, ranked-choice voting has an advantage over the traditional first-past-the-post system in replacing the incumbent candidate through a recall election. Particularly, ranked-choice voting chooses the Condorcet winner candidate under more configurations of voter preferences than first-past-the-post. Once the fraction of voters who rank the Condorcet winner candidate first is not the largest, first-past-the-post stops the election of that candidate in a recall election. On the contrary, ranked-choice voting will continue to elect the Condorcet winner as long as that fraction of voters is not the smallest. Still, even in the case where the fraction of voters who rank the Condorcet winner first is the smallest, we find that with a slight modification in the design of this voting system —the inclusion of an instant runoff— it also elects the Condorcet winner candidate in a recall election.

When examining voters' preferences among the available candidates in the recall election, the scenario in which ranked-choice voting outperforms first-past-the-post can be compared to an ideologically polarized electorate. Draca and Schwarz (2024) find that the US stands out as experiencing the sharpest increase in polarization and identify that the ideological polarization of US citizens is due to a "disappearing center" driven by the growth of low trust in institutions or "anarchist" types. As long as the fraction of voters who rank the (moderate) Condorcet winner candidate first is not the largest, we can conclude that ranked-choice voting has an advantage over first-past-the-post in recall elections in societies where the ideological center is declining.

We acknowledge that this paper represents only an initial exploration of the recall election process and the advantages of using ranked-choice voting rather than traditional first-past-the-post voting in this political procedure. Moving forward, our findings provide a solid foundation for exploring promising avenues for future research.

In our model, the trigger for a recall election was exogenous: a candidate who is not the Condorcet winner emerges from a particular electoral process given certain voter preferences. However, one could think of an alternative model in which the triggering of the recall election is endogenous. Imagine that there is an incumbent who is a Condorcet winner, regardless of the electoral system by which she was elected. Now suppose that some event occurs that causes voter preferences to shift to another candidate, so that the incumbent is no longer a Condorcet winner. Our findings constitute a partial solution to understanding the effectiveness of an endogenously-triggered recall election, as they specifically address the scenario where a shift in voter preferences results in a moderate candidate becoming the new Condorcet winner, regardless of whether the incumbent was extreme or moderate. However, our study does not explore the scenario where an extreme candidate becomes the new Condorcet winner, leaving this area for future research.

While this paper has focused on a closed primary model because of its relevance to recall elections, examining how different electoral systems, such as open primaries and multi-party primaries, influence the occurrence of recall elections could provide valuable insights. At this point, it is worth revisiting one of the assumptions in this paper. In our closed primary model, we assume sincere voting by voters, i.e., voters who do not weigh the general election prospects of candidates when casting their votes in the primary. This assumption was motivated by the findings in Adams and Merrill (2014), which establish an equivalence in candidates' general election prospects whether voters behave sincerely or strategically when choosing a candidate in a closed primary system. However, in contexts such as open primaries, the consideration of sincere or strategic voting by voters could indeed lead to significant differences (see Chen and Yang, 2002; Cho and Kang, 2015). Therefore, including this factor would enrich the analysis of the recall elections under these alternative electoral systems.

We have considered office-motivated candidates who consistently seek election. Another possibility, however, might be to consider policy-motivated candidates who have the option of deciding whether or not to run for office. In this regard, it is worth referring to Amorós et al. (2016) to understand the implications. For policy-oriented politicians, the authors show that when closed primaries produce a winning candidate who is not the Condorcet winner, that candidate is extreme. In such a case, only our Proposition 3 would apply, suggesting that both FPP and RCV systems are beneficial for the recall election. However, the case in which RCV has an advantage over FPP in the recall election –which occurs when the incumbent who is not the Condorcet winner is moderate– (Proposition 4) arises precisely when we consider office-oriented candidates. An interesting avenue for future research would be to explore the case where candidate preferences exhibit a mixture of both policy-driven and office-seeking motivations.

Another promising avenue for future research is to determine whether our results hold when there are more than three candidates in the recall election. While this certainly warrants further investigation, our approach remains significant. This is supported by the historical data on candidate affiliations in replacement candidacies during recall elections. For instance, in California, recognized as one of the most active states in such elections, nearly all candidates were affiliated with either the Democratic or Republican Party (see Vassar and Meyers, 2024). The remaining candidates, whether independents, with no party preference, or affiliated with other parties such as Libertarian or Green, individually had a relatively marginal presence compared to the aforementioned groups. Our results would hold in a context where each of the three candidates considered in our model represents a Democratic, Republican, and minor party ideology, respectively.

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Appendix

Proof of Proposition 1. We distinguish 4 cases regarding different combinations of values for P_L^m and P_R^m:

Case 1. If $P_L^m = P_{L^-}^2$ and $P_R^m = P_{R^-}^2$, then $x^L = L^-$ and $x^R = R^-$.

Candidates L^- and R^- are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2$, while R^- receives the votes from $\alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 1.1. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 > \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = L^-$. Note that necessarily $P^m = P_{L^-}^2$, so candidate L^- is a Condorcet winner at $\mathcal{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 1.2. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 < \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = R^-$. Note that necessarily $P^m = P_{R^-}^2$, so candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 2. If $P_L^m = P_{L^-}^2$ and $P_R^m = P_{R^-}^1$, then $x^L = L^-$ and $x^R = R^-$.

Candidates L^- and R^- are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2$, while R^- receives the votes from $\alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 2.1. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 > \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = L^-$. Note that necessarily $P^m = P_{L^-}^2$, so candidate L^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 2.2. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 < \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = R^-$. Note that $P^m \in \{P_{R^-}^2, P_{R^-}^1\}$, so candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 3. If $P_L^m = P_{L^-}^1$ and $P_R^m = P_{R^-}^2$, then $x^L = L^-$ and $x^R = R^-$.

Candidates L^- and R^- are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2$, while R^- receives the votes from $\alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 3.1. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 > \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = L^-$. Note that $P^m \in \{P_{L^-}^1, P_{L^-}^2\}$, so candidate L^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 3.2. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 < \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = R^-$. Note that necessarily $P^m = P_{R^-}^2$, so candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 4. If $P_L^m = P_{L^-}^1$ and $P_R^m = P_{R^-}^1$, then $x^L = L^-$ and $x^R = R^-$.

Candidates L^- and R^- are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2$, while R^- receives the votes from $\alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 4.1. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 > \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = L^-$. Note that $P^m \in \{P_{L^-}^1, P_{L^-}^2\}$, so candidate L^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 4.2. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 < \alpha_{R^-}^2 + \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = R^-$. Note that $P^m \in \{P_{R^-}^2, P_{R^-}^1\}$, so candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

■

Proof of Proposition 2. The statement of this proposition implies that the inter-party election necessarily takes place between either (i) a moderate and an extreme candidate, or (ii) two extreme candidates. We begin by considering scenario (i), that is, when a moderate and an extreme candidate reach the inter-party election. In order to do that, we distinguish 4 cases regarding different combinations of values for P_L^m and P_R^m:

Case 1. If $P_L^m = P_{L^-}^2$ and $P_R^m = P_{R^+}$, then $x^L = L^-$ and $x^R = R^+$.

Candidates L^- and R^+ are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 + \alpha_{R^-}^2$, while R^+ receives the votes from $\alpha_{R^-}^1 + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 1.1. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 + \alpha_{R^-}^2 > \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = L^-$. Note that $P^m \in \{P_{L^-}^2, P_{R^-}^2\}$. We distinguish two subcases:

Case 1.1.1. Let $P^m = P_{L^-}^2$. Then, candidate L^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 1.1.2. Let $P^m = P_{R^-}^2$. Then, candidate L^- is not a Condorcet winner at $\mathbb{P}$. Candidate R^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 1.2. If $\alpha_{L^+} + \alpha_{L^-}^1 + \alpha_{L^-}^2 + \alpha_{R^-}^2 < \alpha_{R^-}^1 + \alpha_{R^+}$, then $x^w = R^+$. Note that $P^m \in \{P_{R^-}^1, P_{R^+}\}$. We distinguish two subcases:

Case 1.2.1. Let $P^m = P_{R^+}$. Then, candidate R^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 1.2.2. Let $P^m = P_{R^-}^1$. Then, candidate R^+ is not a Condorcet winner at $\mathbb{P}$. Candidate R^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 2. If $P^m_L = P^1_{L^-}$ and $P^m_R = P_{R^+}$, then $x^L = L^-$ and $x^R = R^+$.

Candidates L^- and R^+ are running in the inter-party election. L^- receives the votes from $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-}$, while R^+ receives the votes from $\alpha^1_{R^-} + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 2.1. If $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} > \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = L^-$. Note that $P^m \in \{P^1_{L^-}, P^2_{L^-}, P^2_{R^-}\}$. We distinguish two subcases:

Case 2.1.1. Let $P^m \in \{P^1_{L^-}, P^2_{L^-}\}$. Then, candidate L^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 2.1.2. Let $P^m = P^2_{R^-}$. Then, candidate L^- is not a Condorcet winner at $\mathbb{P}$. Candidate R^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 2.2. If $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} < \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = R^+$. Note that $P^m \in \{P^1_{R^-}, P_{R^+}\}$. We distinguish two subcases:

Case 2.2.1. Let $P^m = P_{R^+}$. Then, candidate R^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 2.2.2. Let $P^m = P^1_{R^-}$. Then, candidate R^+ is not a Condorcet winner at $\mathbb{P}$. Candidate R^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 3. If $P^m_L = P_{L^+}$ and $P^m_R = P^2_{R^-}$, then $x^L = L^+$ and $x^R = R^-$.

Candidates L^+ and R^- are running in the inter-party election. L^+ receives the votes from $\alpha_{L^+} + \alpha^1_{L^-}$, while R^- receives the votes from $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 3.1. If $\alpha_{L^+} + \alpha^1_{L^-} > \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = L^+$. Note that $P^m \in \{P_{L^+}, P^1_{L^-}\}$. We distinguish two subcases:

Case 3.1.1. Let $P^m = P_{L^+}$. Then, candidate L^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 3.1.2. Let $P^m = P^1_{L^-}$. Then, candidate L^+ is not a Condorcet winner at $\mathbb{P}$. Candidate L^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 3.2. If $\alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = R^-$. Note that $P^m \in \{P^2_{L^-}, P^2_{R^-}\}$. We distinguish two subcases:

Case 3.2.1. Let $P^m = P^2_{L^-}$. Then, candidate R^- is not a Condorcet winner at $\mathbb{P}$. Candidate L^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 3.2.2. Let $P^m = P^2_{R^-}$. Then, candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 4. If $P^m_L = P_{L^+}$ and $P^m_R = P^1_{R^-}$, then $x^L = L^+$ and $x^R = R^-$.

Candidates L^+ and R^- are running in the inter-party election. L^+ receives the votes from $\alpha_{L^+} + \alpha^1_{L^-}$, while R^- receives the votes from $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 4.1. If $\alpha_{L^+} + \alpha^1_{L^-} > \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = L^+$. Note that $P^m \in \{P_{L^+}, P^1_{L^-}\}$. We distinguish two subcases:

Case 4.1.1. Let $P^m = P_{L^+}$. Then, candidate L^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 4.1.2. Let $P^m = P^1_{L^-}$. Then, candidate L^+ is not a Condorcet winner at $\mathbb{P}$. Candidate L^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 4.2. If $\alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = R^-$. Note that $P^m \in \{P^2_{L^-}, P^2_{R^-}, P^1_{R^-}\}$. We distinguish two subcases:

Case 4.2.1. Let $P^m = P^2_{L^-}$. Then, candidate R^- is not a Condorcet winner at $\mathbb{P}$. Candidate L^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 4.2.2. Let $P^m \in \{P^2_{R^-}, P^1_{R^-}\}$. Then, candidate R^- is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

We now consider scenario (ii), that is, when two extreme candidates reach the inter-party election. In order to do that, we consider the following case regarding values for P^m_L and P^m_R:

Case 5. If $P^m_L = P_{L^+}$ and $P^m_R = P_{R^+}$, then $x^L = L^+$ and $x^R = R^+$.

Candidates L^+ and R^+ are running in the inter-party election. L^+ receives the votes from $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-}$, while R^+ receives the votes from $\alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$. We distinguish two cases for determining the winner of the inter-party election:

Case 5.1. If $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} > \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = L^+$. Note that $P^m \in \{P_{L^+}, P^1_{L^-}, P^2_{L^-}\}$. We distinguish two subcases:

Case 5.1.1. Let $P^m = P_{L^+}$. Then, candidate L^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 5.1.2. Let $P^m \in \{P^1_{L^-}, P^2_{L^-}\}$. Then, candidate L^+ is not a Condorcet winner at $\mathbb{P}$. Candidate L^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

Case 5.2. If $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} < \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, then $x^w = R^+$. Note that $P^m \in \{P^2_{R^-}, P^1_{R^-}, P_{R^+}\}$. We distinguish two subcases:

Case 5.2.1. Let $P^m = P_{R^+}$. Then, candidate R^+ is a Condorcet winner at $\mathbb{P}$. Since the winner at the inter-party election is the preferred candidate in pairwise comparisons for a majority of voters **there is no majority dissatisfaction** among voters.

Case 5.2.2. Let $P^m \in \{P^2_{R^-}, P^1_{R^-}\}$. Then, candidate R^+ is not a Condorcet winner at $\mathbb{P}$. Candidate R^- is a Condorcet winner at $\mathbb{P}$ so she defeats the winner of the inter-party election in pairwise majority comparisons. Thus, **there is majority dissatisfaction** among voters.

■

Proof of Proposition 3. We focus on those cases from the proof of Proposition 2 in which an extreme candidate wins the inter-party election and there is majority dissatisfaction: Cases 1.2.2., 2.2.2., 3.1.2., 4.1.2., 5.1.2., and 5.2.2. Table 2 summarizes the preferences of the partisan median voter of the left party; the preferences of the partisan median voter of the right party; the preferences of the median voter; and the winner at the inter-party election stage, for each of the six cases listed above.

CASE	$\mathbf{P^m_L}$	$\mathbf{P^m_R}$	$\mathbf{P^m}$	$\mathbf{x^w}$
1.2.2.	$P^2_{L^-}$	P_{R^+}	$P^1_{R^-}$	R^+
2.2.2.	$P^1_{L^-}$	P_{R^+}	$P^1_{R^-}$	R^+
3.1.2.	P_{L^+}	$P^2_{R^-}$	$P^1_{L^-}$	L^+
4.1.2.	P_{L^+}	$P^1_{R^-}$	$P^1_{L^-}$	L^+
5.1.2.	P_{L^+}	P_{R^+}	$\{P^1_{L^-}, P^2_{L^-}\}$	L^+
5.2.2.	P_{L^+}	P_{R^+}	$\{P^2_{R^-}, P^1_{R^-}\}$	R^+

Table 2: Cases of majority dissatisfaction and recall of an extreme candidate

Note that cases 3.1.2. and 4.1.2. are symmetric to cases 1.2.2. and 2.2.2., respectively. Moreover, in cases 1.2.2. and 2.2.2., the only difference lies in the preferences of the partisan median voter of the left party, which are $P^2_{L^-}$ and $P^1_{L^-}$, respectively. Note, however, that in both cases the most preferred candidate of this partisan median voter is the same: L^-.

Therefore, we can analyze both cases together. Let $P^m = P^1_{R^-}$ so R^- is the Condorcet winner at P. Let $P^m_L \in \{P^1_{L^-}, P^2_{L^-}\}$ and $P^m_R = P_{R^+}$. Let $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} < \alpha^1_{R^-} + \alpha_{R^+}$ so $x^w = R^+$. Since the Condorcet winner at P is not the Condorcet winner at its voters' partition (candidate R^- was discarded at the right party's intra-party election), from Proposition 2 we know that there is majority dissatisfaction, which is a sufficient condition for the recall of the incumbent. We now determine and compare the candidates that would be elected to replace the recalled incumbent in the recall election under both FPP and RCV. Let $\mathcal{C}_0 = \{L^+, L^-, R^-\}$ be the set of candidates running in the recall election.

- First-past-the-post voting

  In the recall election under first-past-the-post voting:

  - L^+ receives the votes from α_{L^+},
  - L^- receives the votes from $\alpha^1_{L^-} + \alpha^2_{L^-}$, while
  - R^- receives the votes from $\alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$.

  Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} < \alpha^1_{R^-} + \alpha_{R^+}$, candidate R^- is elected in the recall election.

- Ranked-choice voting

  The voters' admissible preferences restricted to the set $\mathcal{C}_0$ are given by Table 3.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^+	L^-	L^-	R^-	R^-	R^-
L^-	L^+	R^-	L^-	L^-	L^-
R^-	R^-	L^+	L^+	L^+	L^+

Table 3: Admissible preferences defined over $\mathcal{C}_0$

Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} < \alpha^1_{R^-} + \alpha_{R^+}$, R^- is the candidate who is ranked first by a majority of voters, so she is not removed. Then, we focus on L^+ and L^-. Recall that $P^m_L \in \{P^1_{L^-}, P^2_{L^-}\}$ which implies that $\alpha^1_{L^-} + \alpha^2_{L^-} > \alpha_{L^+}$. Hence, L^+ is the candidate that has been ranked first by a minority of voters, so she is removed. Let $\mathcal{C}_1 = \{L^-, R^-\}$ be the set of candidates running in the recall election after the first iteration of the ranked-choice voting method. The voters' admissible preferences restricted to the set $\mathcal{C}_1$ are given by Table 4.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^-	L^-	L^-	R^-	R^-	R^-
R^-	R^-	R^-	L^-	L^-	L^-

Table 4: Admissible preferences defined over $\mathcal{C}_1$

Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} < \alpha^1_{R^-} + \alpha_{R^+}$, candidate R^- is elected in the recall election because she has been ranked first by more than 50% of the voters.

Going back to Table 2, note that case 5.2.2. is symmetric to case 5.1.2., so we will focus on analyzing the latter case. Let $P^m \in \{P^1_{L^-}, P^2_{L^-}\}$ so L^- is the Condorcet winner at P. Let $P^m_L = P_{L^+}$ and $P^m_R = P_{R^+}$. Let $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} > \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$ so $x^w = L^+$. Since the Condorcet winner at P is not the Condorcet winner at its voters' partition (candidate L^- was discarded at the left party's intra-party election), from Proposition 2 we know that there is majority dissatisfaction, which is a sufficient condition for the recall of the incumbent. We now determine and compare the candidates that would be elected to replace the recalled incumbent in the recall election under both FPP and RCV. Let $\mathcal{C}_0 = \{L^-, R^-, R^+\}$ be the set of candidates running in the recall election.

- First-past-the-post voting

  In the recall election under first-past-the-post voting:

  - L^- receives the votes from $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-}$,
  - R^- receives the votes from $\alpha^2_{R^-} + \alpha^1_{R^-}$, while
  - R^+ receives the votes from α_{R^+}.

  Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} > \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, candidate L^- is elected in the recall election.

- Ranked-choice voting

  The voters' admissible preferences restricted to the set $\mathcal{C}_0$ are given by Table 5.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^-	L^-	L^-	R^-	R^-	R^+
R^-	R^-	R^-	L^-	R^+	R^-
R^+	R^+	R^+	R^+	L^-	L^-

Table 5: Admissible preferences defined over $\mathcal{C}_0$

Note that:

- L^- has been ranked first by $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-}$,
- R^- has been ranked first by $\alpha^2_{R^-} + \alpha^1_{R^-}$, while
- R^+ has been ranked first by α_{R^+}.

Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} > \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, L^- is the candidate who is ranked first by a majority of voters, so she is not removed. Then, we focus on R^- and R^+. Recall that $P^m_R = P_{R^+}$ which implies that $\alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{R^+}$. Hence, R^- is the candidate that has been ranked first by a minority of voters, so she is removed. Let $\mathcal{C}_1 = \{L^-, R^+\}$ be the set of candidates running in the recall election after the first iteration of the ranked-choice voting method. The voters' admissible preferences restricted to the set $\mathcal{C}_1$ are given by Table 6.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^-	L^-	L^-	L^-	R^+	R^+
R^+	R^+	R^+	R^+	L^-	L^-

Table 6: Admissible preferences defined over $\mathcal{C}_1$

Note that:

- L^- has been ranked first by $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-}$, while
- R^+ has been ranked first by $\alpha^1_{R^-} + \alpha_{R^+}$.

Since $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} > \alpha^2_{R^-} + \alpha^1_{R^-} + \alpha_{R^+}$, candidate L^- is elected in the recall election because she has been ranked first by more than 50% of the voters.

From the analysis of the cases in Table 2, we conclude that when there is majority dissatisfaction and an extreme candidate is recalled both first-past-the-post voting and ranked-choice voting elect the Condorcet winner candidate relative to the voters' preference profile. Therefore, both FPP and RCV systems are beneficial for the recall election.

■

Proof of Proposition 4. We focus on those cases from the proof of Proposition 2 in which a moderate candidate wins the inter-party election and there is majority dissatisfaction: Cases 1.1.2., 2.1.2., 3.2.1., and 4.2.1. Table 7 summarizes the preferences of the partisan median voter of the left party; the preferences of the partisan median voter of the right party; the preferences of the median voter; and the winner at the inter-party election stage, for each of the four cases listed above.

CASE	$\mathbf{P^m_L}$	$\mathbf{P^m_R}$	$\mathbf{P^m}$	$\mathbf{x^w}$
1.1.2.	$P^2_{L^-}$	P_{R^+}	$P^2_{R^-}$	L^-
2.1.2.	$P^1_{L^-}$	P_{R^+}	$P^2_{R^-}$	L^-
3.2.1.	P_{L^+}	$P^2_{R^-}$	$P^2_{L^-}$	R^-
4.2.1.	P_{L^+}	$P^1_{R^-}$	$P^2_{L^-}$	R^-

Table 7: Cases of majority dissatisfaction and recall of a moderate candidate

Note that cases 3.2.1. and 4.2.1. are symmetric to cases 1.1.2. and 2.1.2., respectively. Moreover, in cases 1.1.2. and 2.1.2., the only difference lies in the preferences of the partisan median voter of the left party, which are $P^2_{L^-}$ and $P^1_{L^-}$, respectively. Note, however, that in both cases the most preferred candidate of this partisan median voter is the same: L^-. Therefore, we can analyze both cases together. Let $P^m = P^2_{R^-}$ so R^- is the Condorcet winner at P. Let $P^m_L \in \{P^1_{L^-}, P^2_{L^-}\}$ and $P^m_R = P_{R^+}$. Let $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} > \alpha^1_{R^-} + \alpha_{R^+}$ so $x^w = L^-$. Since the Condorcet winner at P is not the Condorcet winner at its voters' partition (candidate R^- was discarded at the right party's intra-party election), from Proposition 2 we know that there is majority dissatisfaction, which is a sufficient condition for the recall of the incumbent. We now determine and compare the candidates that would be elected to replace the recalled incumbent in the recall election under both FPP and RCV. Let $\mathcal{C}_0 = \{L^+, R^-, R^+\}$ be the set of candidates running in the recall election.

- <u>First-past-the-post voting</u>

  In the recall election under first-past-the-post voting:

  - L^+ receives the votes from $\alpha_{L^+} + \alpha^1_{L^-}$,
  - R^- receives the votes from $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$, while
  - R^+ receives the votes from α_{R^+}.

  The relationship among the shares of voters above can be any of the following six:

  (i) $\alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{R^+}$

  (ii) $\alpha_{L^+} + \alpha^1_{L^-} < \alpha_{R^+} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$

  (iii) $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{L^+} + \alpha^1_{L^-} < \alpha_{R^+}$

  (iv) $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{R^+} < \alpha_{L^+} + \alpha^1_{L^-}$

  (v) $\alpha_{R^+} < \alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$

  (vi) $\alpha_{R^+} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{L^+} + \alpha^1_{L^-}$

  Note that all of them are compatible with $\alpha_{L^+} + \alpha^1_{L^-} + \alpha^2_{L^-} + \alpha^2_{R^-} > \alpha^1_{R^-} + \alpha_{R^+}$. The candidate that is elected by first-past-the-post voting clearly depends on the case in question. In particular, candidate R^+ is elected in the recall election in cases (i) and (iii); candidate R^- is elected in cases (ii) and (v); while candidate L^+ is elected in cases (iv) and (vi).

- <u>Ranked-choice voting</u>

  The voters' admissible preferences restricted to the set $\mathcal{C}_0$ are given by Table 8.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^+	L^+	R^-	R^-	R^-	R^+
R^-	R^-	L^+	R^+	R^+	R^-
R^+	R^+	R^+	L^+	L^+	L^+

Table 8: Admissible preferences defined over $\mathcal{C}_0$

Note that the six cases (i) to (vi) about the relationship among the shares of voters specified above are also helpful in order to determine which candidate is elected in the recall election by ranked-choice voting.

Suppose first that we are either in case (i) or (ii). Then, L^+ is the candidate that has been ranked first by a minority of voters, so she is removed. Let $\mathcal{C}_1 = \{R^-, R^+\}$ be the set of candidates running in the recall election after the first iteration of the ranked-choice voting method. The voters' admissible preferences restricted to the set $\mathcal{C}_1$ are given by Table 9.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
R^-	R^-	R^-	R^-	R^-	R^+
R^+	R^+	R^+	R^+	R^+	R^-

Table 9: Admissible preferences defined over $\mathcal{C}_1$

Note that the fraction of voters $\alpha_{L^+} + \alpha^1_{L^-}$ that were voting for candidate L^+ at the first round of the recall election now votes for candidate R^-. Since all voters except those whose preferences are of type P_{R^+} vote for R^- and $P^m \neq P_{R^+}$, then candidate R^- is elected in the recall election because she has been ranked first by more than 50% of the voters.

Suppose next that we are either in case (iii) or (iv). Then, R^- is the candidate that has been ranked first by a minority of voters, so she is removed. Let $\mathcal{C}_1 = \{L^+, R^+\}$ be the set of candidates running in the recall election after the first iteration of the ranked-choice voting method. The voters' admissible preferences restricted to the set $\mathcal{C}_1$ are given by Table 10.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^+	L^+	L^+	R^+	R^+	R^+
R^+	R^+	R^+	L^+	L^+	L^+

Table 10: Admissible preferences defined over $\mathcal{C}_1$

Note that among voters that were voting for candidate R^- at the first round of the recall election (the fraction equal to $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$), the fraction $\alpha^2_{L^-}$ now votes for candidate L^+ while the remaining fraction $\alpha^2_{R^-} + \alpha^1_{R^-}$ now votes for candidate R^+. Since $P^m = P^2_{R^-}$, then candidate R^+ is elected in the recall election because she has been ranked first by more than 50% of the voters.

Suppose finally that we are either in case (v) or (vi). Then, R^+ is the candidate that has been ranked first by a minority of voters, so she is removed. Let $\mathcal{C}_1 = \{L^+, R^-\}$ be the set of candidates running in the recall election after the first iteration of the ranked-choice voting method. The voters' admissible preferences restricted to the set $\mathcal{C}_1$ are given by Table 11.

P_{L^+}	$P^1_{L^-}$	$P^2_{L^-}$	$P^2_{R^-}$	$P^1_{R^-}$	P_{R^+}
L^+	L^+	R^-	R^-	R^-	R^-
R^-	R^-	L^+	L^+	L^+	L^+

Table 11: Admissible preferences defined over $\mathcal{C}_1$

Note that the fraction of voters α_{R^+} that were voting for candidate R^+ at the first round of the recall election now vote for candidate R^-. Since $P^m = P^2_{R^-}$, then candidate R^- is elected in the recall election because she has been ranked first by more than 50% of the voters.

Table 12 summarizes the candidates elected in the recall election by first-past-the-post voting and ranked-choice voting depending on the specific configuration of voters' share.

CASES	$\mathbf{x^w_{FPP}}$	$\mathbf{x^w_{RCV}}$
(i) $\alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{R^+}$	R^+	R^-
(ii) $\alpha_{L^+} + \alpha^1_{L^-} < \alpha_{R^+} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$	R^-	R^-
(iii) $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{L^+} + \alpha^1_{L^-} < \alpha_{R^+}$	R^+	R^+
(iv) $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{R^+} < \alpha_{L^+} + \alpha^1_{L^-}$	L^+	R^+
(v) $\alpha_{R^+} < \alpha_{L^+} + \alpha^1_{L^-} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$	R^-	R^-
(vi) $\alpha_{R^+} < \alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{L^+} + \alpha^1_{L^-}$	L^+	R^-

Table 12: FPP vs. RCV elected candidates in the recall election

Recall that $P^m = P^2_{R^-}$. Thus, first-past-the-post voting elects the Condorcet winner candidate relative to the voters' preference profile (candidate R^-) only in cases (ii) and (v), while ranked-choice voting elects her in all cases except (iii) and (iv). Note that the fraction of voters who rank the Condorcet winner candidate first (i.e., $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-}$) is the largest in cases (ii) and (v), while it is the smallest in cases (iii) and (iv). Hence, from the analysis of the cases in Table 7, we conclude that when there is majority dissatisfaction and a moderate candidate is recalled (I) the first-past-the-post system is beneficial for the recall election if and only if more voters rank the Condorcet winner candidate first than they do any other replacement candidate, while (II) the ranked-choice voting system is beneficial for the recall election if and only if the voters who rank the Condorcet winner candidate first than they do any other replacement candidate are not a minority.

■

Proof of Proposition 5. We focus on those cases from Table 12 in the proof of Proposition 4 in which the fraction of voters who rank the Condorcet winner candidate first than they do any other replacement candidate are a minority: Cases (iii) and (iv). Case (iii) is such that $\alpha^2_{L^-} + \alpha^2_{R^-} + \alpha^1_{R^-} < \alpha_{L^+} + \alpha^1_{L^-} < \alpha_{R^+}$ which, from Table 8, implies that candidate R^- –the Condorcet winner– is ranked first by a minority of voters, and candidate L^+ is ranked first by the second smallest fraction of voters. Let an instant runoff voting between candidates R^- and L^+ takes place. Since candidate R^- is the Condorcet winner candidate relative to the voters' preference profile, she is preferred to any other candidate —in particular to candidate L^+— in pairwise majority comparisons. Thus, L^+ is removed. The next stage of the ranked-choice voting system would therefore pit candidates R^- and R^+ against each other. Again, since R^- is the Condorcet winner, she is elected in the recall election because she is ranked first by more than 50% of the voters. Case (iv) is proven in an analogous way.

■