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# A variational method for solving two-dimensional Bratu's problem

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**Abstract** In this paper we propose a variational method in order to solve Bratu Problem for two dimension in an adequate space of biquadratic spline functions. The solution is obtained by resolving a sequence of boundary value problems. We study some characterizations of the functions of such sequence and we express them as some linear combination of biquadratic spline bases functions. We finish by showing some numerical and graphical examples in order to prove the validity and the effectiveness of our method.

**Keywords** Bratu's problem · PDE · variational method · bi-quadratic spline
**Mathematical Subject Classification** [2008] 65D05 · 65D07 · 65D10 · 65D17

## 1 Introduction

The non-linear boundary problems occur in engineering and science, including of diffusion processes of the chemical reactions and heat transfers. Particularly, Bratu Problem appears in a large variety of applications areas such as the fuel ignition model of thermal combustion, radiative heat transfer, the Chandrasekhar model of the expansion of the universe, chemical reactor theory

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and nanotechnology [17], [20], [10] and [11]. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction [2]. An interested application of nonlinear Bratus equation in two and three dimensions to electrostatics are studied and solved in [8]. In [10] a summary of the history such kind of problem is given.

The two-dimensional Bratu Problem is an elliptic partial differential equation with homogeneous Dirichlet boundary conditions. This problem is given by

$$\begin{cases} \Delta u + \lambda e^u = 0, & \text{in } \Omega = (0,1) \times (0,1), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ and $\Omega$ is a bounded domain with boundary $\partial\Omega$. The problem is a nonlinear eigenvalue problem that is commonly used as a test problem for many numerical methods.

In the planar 1D case the problem is reduced to the expression

$$\begin{cases} u_{xx} + \lambda e^u = 0 \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$

The exact solution of (2) is given in [3] and [7] as

$$u(x) = -2 \ln \left( \frac{\cosh((x - \frac{1}{2})\frac{\theta}{2})}{\cosh \frac{\theta}{4}} \right), \tag{3}$$

where $\theta$ solves

$$\theta = \sqrt{2\lambda} \cosh \left( \frac{\theta}{4} \right). \tag{4}$$

Let $\lambda_c = 8(\alpha^2 - 1)$, where $\alpha$ is the fixed point of the hyperbolic cotangent function ($\lambda_c \simeq 3.513830719$). Then, for $0 < \lambda < \lambda_c$ the equation (4) has two solutions, for $\lambda = \lambda_c$ it has only unique solution and for $\lambda > \lambda_c$ there are no solutions.

Many analytical and numerical methods have been applied to solve Problem (1) and (2).

For problem (2), the authors in [5] propose a B-spline method for solving the one-dimensional Bratu's problem. They computed the numerical approximations to the exact solution and then compared with other existing methods. They verified the B-spline method for different values of the parameter, below its critical value, where two solutions occur.

For Problem (2) some variational iteration methods are applied to obtain approximate analytical solution without any discretization [4,6]. In [1] the authors transform this problem into a non-linear initial value problem and then they solve it by the Lie-group shooting method. In [21] the author present a framework to determine exact solutions of (2) by the Adomian decomposition method.

A Laplace transform decomposition numerical algorithm is introduced in [12] and a numerical algorithm based on the decomposition technique is presented in [7] for solving a class on non-linear boundary values problem, including Troesch and Bratu one-dimensional problems.

In other hand, in [9] the authors use the Reproducing Kernel Hilbert space method for solving Problem (2) and the obtained numerical approximations to the exact solution are computed and compared with other existing methods.

Recently, the authors in [18] have applied a numerical scheme based on differential quadrature methods to solve nonlinear Bratu problem. The unknown field quantity and their derivatives are approximated using differential quadrature approximations.

In the 2D case the solution of problem (1) is not known. Thus an approximation method of a solution is necessary.

For example, in [19] the authors present a method of construction of a near exact solution based in the expression of the solution of the 1D case.

A function of the form

$$u(x,y) = 2\ln\left(\frac{\cosh\left(\frac{\theta}{4}\right)\cosh\left(\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)\theta\right)}{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)\cosh\left(\left(y-\frac{1}{2}\right)\frac{\theta}{2}\right)}\right), \ \forall (x,y)\in\Omega. \quad (5)$$

where $\theta$ is a constant to be determined, is carefully chosen and assumed to be the solution of Problem (1). Substituting (5) in (1), simplifying and collocating at the point $x=\frac{1}{2}$ and $y=\frac{1}{2}$ we have

$$\theta^2 = \lambda\cosh\left(\frac{\theta}{4}\right)^2. \quad (6)$$

Obtaining $\frac{d\lambda}{d\theta}$ from equation (6) and equating to zero the critical value $\lambda_c$ satisfies

$$\theta = \frac{\lambda_c}{4}\sinh(\frac{\theta}{4})\cosh(\frac{\theta}{4}). \quad (7)$$

By eliminating $\lambda$ from equations (6) and (7) we have the value of $\theta_c$ for the critical $\lambda_c$ satisfying

$$\frac{\theta_c}{4} = \coth(\frac{\theta_c}{4}) \quad (8)$$

and $\theta_c = \pm 4.798714561$. From equation (7) $\lambda_c = 7.027661438$. For $\lambda > \lambda_c$ the equation (6) has not solution, for $\lambda = \lambda_c$ has a unique solution and for $\lambda < \lambda_c$ has two solutions $\theta_1 < \theta_2$, which determine two functions $u_1(x,y)$ and $u_2(x,y)$, given by (5), which are called the lower branch solution and the upper branch solution near exact solution.

Different techniques for the construction of a surface have been developed in recent years, for example, interpolation by spline functions, based on the minimization of a certain functional in an adequate Sobolev space [13–15]. Such a functional may represent a energy measure, or geometric considerations of a surface as a air measure or a curvature or curvature variation measure or this surface [13]. These techniques have many applications in CAD, CAGD and Earth Sciences.

In [16] the authors present a variational approximation method for solving Troesch Problem. They show the existence and uniqueness of the solution of the problem and construct a sequence of approximate solutions of such problem. Under adequate conditions, such sequence converges to the exact solution of Troesch Problem.

In this paper, we propose a variational method in order to solve Bratu Problem (1) in a space of biquadratic spline functions. The solution is obtained by resolving a sequence of boundary value problems. We study some characterizations of the functions of such sequence and we express them as some linear combination of biquadratic spline bases functions.

The remainder of the manuscript is organized as follows. In section 2 we formulate the problem. In section 3 we present the numerical solution of the problem. Section 4 is devoted to study how to compute each approximate solution of the sequence. The last section is devoted to show some graphical and numerical examples in order to prove the effectiveness and the useful of our methods.

## 2 Formulating the problem

For any $n \in \mathbb{N}$, $n \geq 1$, let $T_n = \{t_0, \ldots, t_n\}$ be a subset of $n+1$ distinct points of $[0,1]$ such that $t_i = \dfrac{i}{n}$, for all $i = 0, \ldots, n$. We denote by $S_2^1(T_n)$ the space of spline functions of degree 2 and class $C^1$ constructed over the partition $T_n$.

Let $W_n = S_2^1(T_n) \otimes S_2^1(T_n)$ the space of bi-quadratic spline functions of class $C^1$ constructed over the partition $T_n \times T_n$ of $\Omega = [0,1] \times [0,1]$.

Now, Let $\{B_i^n,\ i = 1, \ldots n+2\}$ be the B-spline basis functions of $S_2^1(T_n)$ and, for $i = 1, \ldots, n+2$ and $j = 1, \ldots, n+2$, we define

$$B^n_{(n+2)(i-1)+j}(x,y) = B_i^n(x) B_j^n(y), \quad \forall\, (x,y) \in \Omega.$$

Then, denoted $N = (n+2)^2$, the set $\{B_1^n, \ldots, B_N^n\}$ is a bivariate B-spline function basis of $W_n$. Moreover, it is verified that $W_n$ is continuously embedded in the Sobolev space $H^1(\Omega)$.

Let $F : L^2(\Omega) \to L^2(\Omega)$ be the functional application defined by

$$Fu(x,y) = \lambda \mathrm{e}^{u(x,y)}, \quad \forall\, (x,y) \in \Omega, \quad \forall\, u \in L^2(\Omega).$$

We consider the problem: Find $\sigma \in C^1(\overline{\Omega})$ such that

$$\begin{cases} \Delta\sigma = -F\sigma & \text{in } \ \Omega, \\ \sigma = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{9}$$

Then, Problem (9) is Problem (1). Let $H = \{u \in H^1(\Omega) :\ u = 0 \text{ on } \partial\Omega\}$.

**Proposition 1** *Let $\sigma$ be a solution of Problem (9). Then, $\sigma$ is a solution of the following variational problem: Find $\sigma \in H$ such that*

$$(\sigma, u)_1 = (F\sigma, u)_0, \; \forall\, u \in H, \tag{10}$$

*where $(\,\cdot\,,\,\cdot\,)_\ell$, for $\ell = 0, 1$, are the usual semi–inner products defined in $H^1(\Omega)$ by*

$$(u, v)_0 = \textstyle\int_\Omega uvdxdy,$$

$$(u, v)_1 = \textstyle\int_\Omega \langle \nabla u, \nabla v\rangle_2 dxdy,$$

*being $\langle\,\cdot\,,\,\cdot\,\rangle_2$ the Euclidean inner product in $\mathbb{R}^2$.*

*Proof* From the Green's first identity we have

$$\int_\Omega \Delta\sigma u dxdy = -\int_\Omega \langle \nabla\sigma, \nabla u\rangle_2 dxdy + \int_{\partial\Omega} u\frac{\partial\sigma}{\partial n} d\mu, \tag{11}$$

where $\dfrac{\partial\sigma}{\partial n}$ indicates the normal derivative and $d\mu$ indicates the line differential element on $\partial\Omega$.

From (9) we have

$$\Delta\sigma u = -F\sigma u, \quad \forall\, u \in H. \tag{12}$$

From here, by integration, from (11) and taking into account that $u = 0$ on $\partial\Omega$, we obtain

$$-\int_\Omega \langle \nabla\sigma, \nabla u\rangle_2 dxdy = -\int_\Omega F\sigma u dxdy$$

and thus (10) holds. □

**Definition 1** A solution of Problem (10) is called a weak solution of Bratu Problem.

*Remark 1* Obviously, if Problem (10) has a unique solution $\overline{\sigma}$ and $\sigma$ is a solution of (9) then $\sigma$ is unique and $\sigma = \overline{\sigma}$.

## 3 Numerical solution of the Problem

Now, we are going to construct a sequence of approximate solutions of Problem (10).

For this, we consider the near exact solution $\sigma_0 \in C^2(\overline{\Omega})$ (see [19]) given by (5) for $\lambda <= \lambda_c$ and $\theta$ a solution of the equation (6).

We denote

$$a_{i,n} = \begin{cases} (t_i, 0), & i = 1, \dots, n-1, \\ (t_{i-n+1}, 1), & i = n, \dots, 2n-2, \\ (0, t_{i-2n+1}), & i = 2n-1, \dots, 3n-1, \\ (1, t_{i-3n}), & i = 3n, \dots, 4n, \\ (\dfrac{t_{i-4n-1} + t_{i-4n}}{2}, 0), & i = 4n+1, \dots, 5n, \\ (\dfrac{t_{i-5n-1} + 1 + t_{i-5n}}{2}, 1), & i = 5n+1, \dots, 6n, \\ (0, \dfrac{t_{i-6n-1} + t_{i-6n}}{2}), & i = 6n+1, \dots, 7n, \\ (1, \dfrac{t_{i-7n-1} + t_{i-7n}}{2}), & i = 7n+1, \dots, 8n. \end{cases}$$

For any $n \in \mathbb{N}$, $n \geq 1$, we define

$$H_n = \{u \in W_n \ : \ u(a_i) = 0, \ i = 1, \dots, 8n\}.$$

From $\sigma_0$ we are going to construct a sequence of approximate solutions of Problem (10) by induction.

Suppose that, for some $n \in \mathbb{N}$, $n \geq 1$, we have constructed an approximation function $\sigma_{n-1} \in H$ of the solution of Problem (10). We are going to construct a new approximation function $\sigma_n \in H_n$.

**Proposition 2** *We assume that the approximation $\sigma_{n-1} \in H$ is constructed. Then there exists a unique $\sigma_n \in H_n \subset H$ such that*

$$J(\sigma_n) \leq J_n(v), \quad \forall\, v \in H_n, \tag{13}$$

*being $J_n : H^1(\Omega) \to \mathbb{R}$ the functional defined by*

$$J_n(v) = |v|_1^2 - 2(F\sigma_{n-1}, v)_0, \tag{14}$$

*where $|\cdot|_1$ is the semi-norm in $H^1(\Omega)$ given by $|v|_1 = (v, v)_1^{\frac{1}{2}}$.*

*Moreover $\sigma_n$ verifies*

$$(\sigma_n, v)_1 = (F\sigma_{n-1}, v)_0, \quad \forall\, v \in H_n. \tag{15}$$

*Proof* Let consider the bilinear form $\tilde{a} : H \times H \to \mathbb{R}$ given by

$$\tilde{a}(u, v) = 2(u, v)_1.$$

It is obvious that $\tilde{a}$ is symmetric, continuous and it endows the space $H$ with a norm defined by $\tilde{a}(v, v)^{\frac{1}{2}} = |v|_1$, which is equivalent to the Sobolev norm $\|v\| = (|v|_0^2 + |v|_1^2)^{\frac{1}{2}}$ of $H \subset H^1(\Omega)$. Hence, $\tilde{a}$ is $H$-elliptic. Moreover, $H_n$ is a convex non empty subset of $H^1(\Omega)$ since $\omega(x, y) = (x - x^2)(y - y^2) \in H_n$.

Now, the application defined by

$$\varphi(v) = 2(F\sigma_{n-1}, v)_0, \quad \forall\, v \in H,$$

is linear and continuous. So, by applying the Lax-Milgram Lemma, we deduce that there exists a unique $\sigma_n \in H_n$ such that $\tilde{a}(\sigma_n, v) = \varphi(v)$, for all $v \in H_n$. Then (15) holds.

Furthermore, $\sigma_n$ is the function of $H_n$ where the functional $\Phi(v) = \frac{1}{2}\tilde{a}(v,v) - \varphi(v)$ is minimum which, in turn, is equivalent to minimize in $H_n$ the functional $J_n$ since $J_n(v) = \Phi(v)$, for all $v \in H_n$. □

**Proposition 3** *It exists a unique* $(\sigma_n, \lambda_1, \ldots, \lambda_{8n}) \in H_n \times \mathbb{R}^{8n}$ *such that*

$$(\sigma_n, u)_1 + \sum_{i=1}^{8n} \lambda_i u(a_i) = (F\sigma_{n-1}, u)_0, \quad \forall\, u \in W_n, \tag{16}$$

*where* $\sigma_n$ *is the unique solution of (15).*

*Proof* Let $u \in W_n$ and let $\{\varphi_1, \ldots, \varphi_{8n}\}$ be the Lagrange basis functions of $W_n$ associates with the functionals $\{\phi_1, \ldots, \phi_{8n}\}$ given by $\phi_i(v) = v(a_i)$, for $i = 1, \ldots, 8n$ and for any $v \in W_n$. Then it is verified that $\phi_i(a_j) = \delta_{ij}$, for all $i, j = 1, \ldots, 8n$.

Then, the function $\omega = u - \sum_{i=1}^{8n} u(a_i)\varphi_i$ verifies

$$\omega(a_j) = u(a_j) - \sum_{i=1}^{8n} u(a_i)\varphi_i(a_j) = u(a_j) - \sum_{i=1}^{8n} u(a_i)\delta_{ij} = 0,\ j = 1, \ldots, 8n.$$

Thus $\omega \in H_n$ and from (15) we obtain that

$$(\sigma_n, \omega)_1 = (F\sigma_{n-1}, \omega)_0$$

and, from here,

$$(\sigma_n, u)_1 - \sum_{i=1}^{8n} (\sigma_n, \varphi_i)_1 u(a_i) = (F\sigma_{n-1}, u)_0 - \sum_{i=1}^{8n} (F\sigma_{n-1}, \varphi_i)_0 u(a_i). \tag{17}$$

Now, let

$$\lambda_i = -(\sigma_n, \varphi_i)_1 + (F\sigma_{n-1}, \varphi_i)_0, \quad i = 1, \ldots, 8n.$$

Then, from (17), we obtain (16). The uniqueness is immediate. □

## 4 Computation

Now, we are going to show how to calculate in practice the sequence $(\sigma_n)$.

By induction, a sequence $\{\sigma_n\}_{n\geq 1} \subset H^1(\Omega)$ has been obtained such that $\sigma_n \in W_n$, for all $n \geq 1$.

Then, for any $n \geq 1$, there exist $c_1, \ldots, c_N \in \mathbb{R}$ such that

$$\begin{aligned} &\sigma_n(x,y) = \sum_{i=1}^{N} c_i B_i(x,y), \quad && \forall\,(x,y) \in \Omega, \\ &\sigma_n(a_i) = 0, && i = 1, \ldots, 8n. \end{aligned} \tag{18}$$

Thus, by linearity, from (16) and (17) we obtain a linear system with unknowns $c_1, \ldots, c_N, \lambda_1, \ldots \lambda_{8n}$, that, in matrix form, it can be expressed by

$$\begin{pmatrix} \mathcal{A}_n & \mathcal{D}_n^t \\ \mathcal{D}_n & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C}^t \\ \Lambda^t \end{pmatrix} = \begin{pmatrix} \mathcal{B}_n \\ 0 \end{pmatrix},$$

being

$$\mathcal{A}_n = ((B_i, B_j)_1)_{1\leq i,j\leq N},$$

$$\mathcal{D}_n = (B_j(a_i))_{\substack{1\leq i\leq 8n, \\ 1\leq j\leq N}},$$

$$\mathcal{C} = (c_1, \ldots, c_N),$$

$$\Lambda = (\lambda_1, \ldots, \lambda_{8n})$$

and

$$\mathcal{B}_n = ((F\sigma_{n-1}, B_i)_0)_{1\leq i\leq N}.$$

## 5 Numerical and graphical examples

The objective of this section is the application of our method in order to compute some approximation of the solution of Bratu's problem. To this end, we consider the Bratu's problem:

$$\Delta u + \lambda e^u = 0, \quad \text{in } \Omega = (0,1) \times (0,1),$$

with boundary conditions

$$u(x,0) = u(x,1) = 0, \forall x \in [0,1],$$

$$u(0,y) = u(1,y) = 0, \forall y \in [0,1].$$

By applying the method studied in section 3, we construct a sequence of biquadratic spline functions $\sigma_n$, with $n \in \mathbb{N}$, depending on the number of the knots $n \times n$. We take $\lambda = 1 < \lambda_c$.

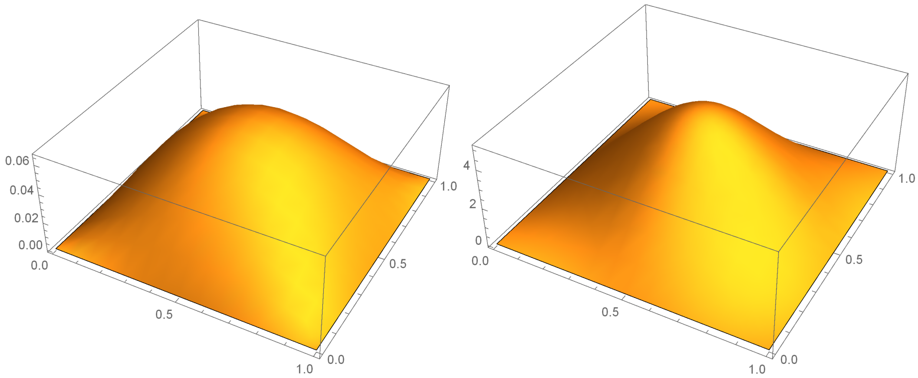


**Fig. 1** Graphs of the approximate solutions $\sigma_0$ defined in (10) with $\lambda = 1$ and $\theta$ the solutions of the equation (6), that is the lower branch solution, on the left, and the upper branch solution, on the right.

Furthermore, to test the goodness of the approximation $\sigma_n$, we compute an error estimate by the expression

$$Error_n = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left(\Delta\sigma_n(a_i) + \lambda e^{\sigma_n(a_i)}\right)^2}, \tag{19}$$

where $\{a_1, \ldots, a_{5000}\}$ is a set of scattered points of $[0,1] \times [0,1]$.

Observe that this error estimate measures how close the function $\sigma_n$ is to verifying the differential equation of Problem (1). In fact, if this error estimate tends to zero, the function $\sigma_n$ tends to be a solution of the differential equation.

For $\lambda = 1$, the solutions of the equation (6) has two solution $\theta_1 = 1.03356946$ and $\theta_2 = 13.038239297$ and thus there exists two near exact solution $\sigma_0$, the lower branch solution and the upper branch solution. The error estimates of these functions are $3.31045 \times 10^{-1}$ and 15.2532 and its maximum values are 0.066036 and 5.13577, respectively. To start with a good approximation we take $\sigma_0$ as the lower branch solution.

Figure 1 shows the graphs of the lower and the upper branch solutions, from left to right.

Taken $\sigma_0$ equals the lower branch solution, we apply the method and calculate the sequence of the weak solutions of Bratu problem.

Figures 2 and 3 show the graphs of the functions $\sigma_n$, for $n = 3, 5, 9, 17$, and Table 1 shows the error estimates $Error_n$ and the maximum values of the functions $\sigma_n$, $(\sigma_n)_{max}$, for $n = 3, 5, 9, 17, 33$.

Observe that this error estimate decreases when the number of the knots increases.

**Conclusion:** From Figures 2 and 3 and Table 1, one can observe that the numerical results are compatible with the theory presented in this work, since the computation of the error estimate diminishes when the number of the knots

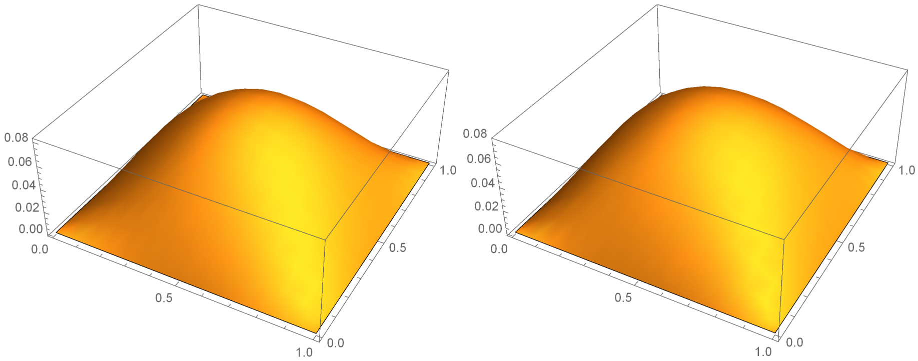


**Fig. 2** Graphs of the weak solutions $\sigma_3$ from $3 \times 3$ equidistant knots, on the left, and $\sigma_5$ from $5 \times 5$ equidistant knots, on the right.

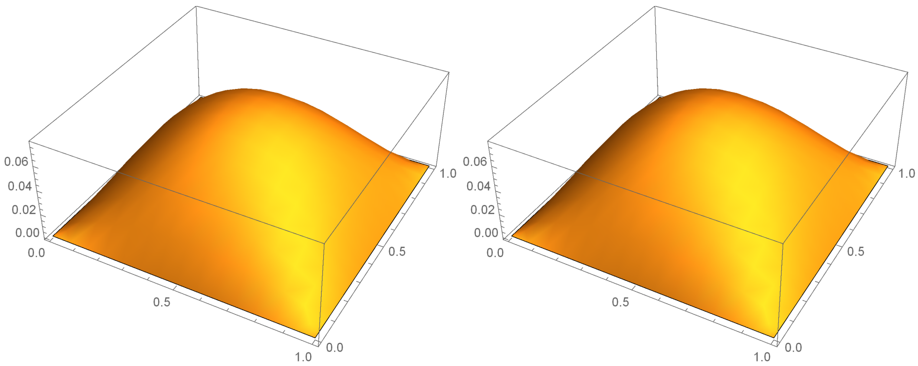


**Fig. 3** Graphs of the weak solutions $\sigma_9$ from $9 \times 9$ equidistant knots, on the left, and $\sigma_17$ from $17 \times 17$ equidistant knots, on the right.

| $n \times n$ | $Error_n$ | $(\sigma_n)_{max}$ |
|---|---|---|
| $3 \times 3$ | $2.3150 \times 10^{-1}$ | 0.0811496 |
| $5 \times 5$ | $5.7886 \times 10^{-2}$ | 0.0783703 |
| $9 \times 9$ | $1.4579 \times 10^{-2}$ | 0.0781169 |
| $17 \times 17$ | $3.6283 \times 10^{-3}$ | 0.0781021 |
| $33 \times 33$ | $9.1233 \times 10^{-4}$ | 0.0781012 |

**Table 1** Table of some values of the error estimate $Error_n$ and the maximum value of $\sigma_n$ ($(\sigma_n)_{max}$) from $n \times n$ equidistant knots.

increases (meaning $n$ tends to the infinite). Furthermore, with only a very small data set provides improvement in terms of the degree of approximation. Hence, we can conclude that our proposal is valid as a numerical method to solve the Bratu problem.

Most of the articles that have studied the Bratu problem only do so by calculating the solutions numerically, while in this paper we present not only the numerical calculation but also we add mathematical techniques, see Propositions 1, 2 and 3. Also, in our opinion, we believe that for the first time, we can highlight our article with respect to others, by the study of the theory of variational approximation for the obtention of the approximate solutions of Bratu problem for dimension two.

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