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Periodic solutions of second order equations via rotation numbers

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Abstract

We consider the problem of the existence and multiplicity of periodic solutions associated to a class of scalar equations of the form $x'' + f(t, x) = 0$. The class considered is such that the behaviour of its solutions near zero and infinity can be compared two suitable piecewise linear systems. We show how a rotation number approach, together with the Poincaré–Birkhoff theorem and the phase-plane analysis of the spiral properties, allows to obtain multiplicity results in terms of the gap between the rotation numbers of the referred piecewise linear systems at zero and at infinity. These systems may also be resonant. In particular, our approach can be used to deal with the problems without both global existence of the Cauchy problems associated to the equation and the sign assumption on f. The typical example is a partially superlinear second order equation. Our main result generalizes some classical results of Jacobowitz and Hartman, among others.

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Keywords: Periodic solution; Second order equation; Poincaré–Birkhoff theorem; Superlinear growth; Rotation number

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1. Introduction

We are interested in the existence and multiplicity of periodic solutions of the equation

$$x'' + f(t,x) = 0. \tag{1.1}$$

We assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic with respect to first variable and locally Lipschitz-continuous with respect to the second variable, in order that uniqueness for the associated Cauchy problems is guaranteed. Weaker conditions could be assumed as well, see [9, 11].

The second order differential equation (1.1) is the canonical model for the motion of a periodically forced nonlinear oscillator without friction effects. In this context, the study of the existence and multiplicity of periodic solutions has a long history, with methods ranging from variational methods [1,2], Poincaré–Birkhoff twist theorem [9,19,20], to Leray–Schauder continuation method [5,6] or many other techniques based on fixed point theory and topological degree.

As a result of the absence of friction and the underlying hamiltonian structure, the Poincaré map of (1.1) is an area-preserving homeomorphism. Following a suggestion by Moser, Jacobowitz [20] first applied the celebrated Poincaré–Birkhoff twist theorem to prove the existence of infinitely many periodic solutions for superlinear second order equations. His result was refined one year later by Hartman [19]. To apply the Poincaré–Birkhoff theorem one needs to estimate the rotations of solutions of (1.1) around the origin. If a solution $x(t)$ exists in $[t_0, t_0 + 2\pi]$ and the solution curve $(x(t), x'(t))$ in the phase plane never passes through the origin, the associated rotation number can be defined as

$$\operatorname{rot}_f(z_0) = \frac{\theta(2\pi; z_0) - \theta(0; z_0)}{2\pi},$$

where $\theta(t; z_0)$ is the polar angle function of the solution curve $z(t; z_0) = (x(t), x'(t))$ with the initial value condition $z(t_0; z_0) = z_0$. In the cited classical references, the following sign condition is assumed to estimate the rotation number

(f_0) $\operatorname{sgn}(x) f(t,x) > 0$ for $|x| \gg 1$ and $t \in \mathbb{R}$.

On the other hand, to assure that Poincaré map is well-defined, it is commonly assumed conditions for the global existence of the solutions.

When (1.1) is of Duffing type,

$$x'' + g(x) = p(t), \tag{1.2}$$

the sign condition reads as

(g_0) $\operatorname{sgn}(x) g(x) > \max_{t\in\mathbb{R}} |p(t)|$ for $|x| \gg 1$.

From (g_0) it follows that all the “large” solutions x of the autonomous equation $x'' + g(x) = 0$ are periodic and the orbits of the equivalent system define a global center in the phase plane $\mathbb{R}^2$ (except for a compact neighbourhood of the origin). Thus, (1.1) is a case of perturbation of a center and the global existence of the solutions of (1.1) is guaranteed by using the Grownwall inequality. Moreover, one can use the fundamental periods $\tau_g(M)$ of solutions satisfying $\max x = M$ for $M \to +\infty$ to estimate the rotations behaviour of solutions of (1.1). Using this approach, Ding and Zanolin [9] applied a generalized version of the Poincaré–Birkhoff theorem by Ding [8] to prove the existence of infinitely many periodic solutions for the Duffing equations with superquadratic potential. We also refer to [13,23–26,28] and the references therein for other related papers on this line.

When (1.1) is not of Duffing type, the case is relatively more complex. The continuity of f and the sign condition are not enough to guarantee the global existence of solutions of (1.1). In fact, it was shown in [7] that there are positive continuous 2π-periodic functions $q(t)$ such that the differential equation $x'' + q(t)x^3 = 0$ has a solution which does not exist on $[0, 2\pi]$. Thus the Poincaré map may not be well defined. To overcome this difficulty, besides assuming the uniqueness for the initial value problems, Jacobowitz [20] considered the successor map, instead of the Poincaré map. However, in order to have this map well defined, he needed to assume a stronger sign condition on the nonlinearity, namely (f_0) for any $x \neq 0$. One year later, Hartman [19] was able to avoid this additional condition. His approach consists in modifying the nonlinearities and making use of some a priori estimates for the solutions with a prescribed number of rotations in the phase plane. Recently, Fonda and Sfecci [15] develop this idea and use the so-called admissible spiral method, a tool introduced by the same authors in [14], to prove the existence of infinitely many periodic solutions for weakly coupled superlinear second order systems by using a higher dimensional version of the Poincaré–Birkhoff theorem recently obtained by Fonda and Ureña [16]. In [15], a sign condition similar to (f_0) is also used to construct admissible spiral curves.

In what concerns the use of Poincaré–Birkhoff theorem without the sign condition, one can mention the paper of Boscaggin, Ortega and Zanolin [4] for multiplicity of subharmonic solutions of the forced pendulum equation. We also refer to [14,22,29] for recent results for planar systems. A different line of research considers (1.1) under asymptotically linear conditions. For instance, Zanini [30] analyses a relation obtained in [18] between rotation numbers and eigenvalues of Hill's equation and multiplicity results of periodic solutions are obtained using this relation. More recently, Margheri, Rebelo and one of the authors [21] consider the equation (1.1) with an asymptotically linear property at the origin and at infinity. Then, a rotation number approach together with the Poincaré–Birkhoff theorem and a recent variant of it [22], allows to obtain multiplicity results in terms of the gap between the Morse indexes of the referred linear systems at zero and at infinity. More precisely, consider for $i = 0, \infty$ the following assumptions

(H_0) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic with respect to the first variable and locally Lipschitz-continuous with respect to the second variable, $f(t, 0) \equiv 0$.

(H_i^l) there exists a function $a_i \in L^1([0, T])$ such that

$$a_i(t) \le \liminf_{x \to i} \frac{f(t, x)}{x} \quad \text{uniformly a.e. in } t \in [0, T].$$

(H_i^r) there exists a function $b_i \in L^1([0, T])$ such that

$$b_i(t) \ge \limsup_{x \to i} \frac{f(t, x)}{x} \quad \text{uniformly a.e. in } t \in [0, T].$$

The main result of [21] gives multiplicity of periodic solutions if either (H_0^l) or (H_0^r) holds and both (H_∞^l) and (H_∞^r) hold. In fact, although for each i only one of the two assumptions (H_i^l), (H_i^r) is used at a time to estimate the rotations of the solutions of (1.1) at zero and at infinity, assumptions (H_∞^l) and (H_∞^r) together are needed to guarantee that all solutions of Cauchy problems associated to (1.1) are defined globally on $[0, 2\pi]$. So, it is natural to ask if (H_∞^r) can be removed.

The main aim of this paper is to provide a rotation number approach to deal with the problem without both global existence and sign assumption. The model example is a partially superlinear second order equation, that is for (1.1) we assume that

(f_1) $f(t,x)/x \geq l(t)$ for $|x| \gg 1$ and $t \in [0, 2\pi]$, moreover,

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = +\infty, \quad \text{for } t \in I \subset [0, 2\pi],$$

where $l(t) \in L^1$ and I is a set of positive measure. A similar condition was used in [3] (see condition (f_∞^2) therein).

If compared to [21], we will consider a more general piecewise linear setting, that is we assume that

$(H_\infty^l)'$ there exists functions $a_\pm \in L^1([0,T])$ such that

$$a_\pm(t) \leq \liminf_{x\to\pm\infty} \frac{f(t,x)}{x} \quad \text{uniformly a.e. in } t \in [0,T],$$

$(H_0^r)'$ there exists a function $b_\pm \in L^1([0,T])$ such that

$$b_\pm(t) \geq \limsup_{x\to 0^\pm} \frac{f(t,x)}{x} \quad \text{uniformly a.e. in } t \in [0,T].$$

The following is the main result of this paper. The usual notation $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ for the positive and negative part of a number is used.

Theorem 1.1. *Suppose that (1.1) satisfies* (H_0), $(H_0^r)'$ *and* $(H_\infty^l)'$. *Let us define* ρ_∞, ρ_0 *the rotation numbers of the piecewise linear equations* $x'' + a_+(t)x^+ - a_-(t)x^- = 0$ *and* $x'' + b_+(t)x^+ - b_-(t)x^- = 0$, *respectively. If* $\rho_\infty > \rho_0$, *then for any* $n/m \in (\rho_0, \rho_\infty)$, *equation (1.1) has two* $2m\pi$*-periodic solutions. Moreover, such* $2m\pi$*-periodic solutions make exactly n turns around the origin in the time* $[0, 2m\pi]$.

This theorem improves the main result of [21]. As a consequence, it is not difficult to prove the following result, that generalizes the classical results of Jacobowitz and Hartman, among others.

Corollary 1.1. *Suppose that (1.1) satisfies* (H_0) *and* (f_1), *then for any* $m \in \mathbb{N}$, *there exist infinitely many* $2m\pi$*-periodic solutions for equation (1.1).*

The rest of the paper is organized as follows. In Section 2, the fundamental notion of rotation number is introduced together with some auxiliary lemmas. Section 3 is devoted to the proof of Theorem 1.1. Finally, some applications and examples are exposed on Section 4, including the proof of Corollary 1.1, which is presented as a particular case of a more general result (see Corollary 4.1).

2. Definition and properties of the rotation number

Consider the first order planar system

$$x' = -y, \quad y' = f(t, x), \tag{2.1}$$

associated to (1.1). Let $z = (x, y) \in \mathbb{R}^2$ and the solution $z(t; t_0, z_0)$ of (2.1) such that $z(t_0; t_0, z_0) = z_0$. We will write it simply as $z(t; z_0)$ when no confusion can arise. Under condition (H_0), from the uniqueness for the associated Cauchy problems we know that $z(t; z_0) \neq 0$ if $z_0 \neq 0$. Passing to polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have

$$\begin{cases} \theta' = \sin^2\theta + \frac{f(t,x)}{r}\cos\theta, \\ r' = -r\sin\theta\cos\theta + \frac{f(t,x)}{r}\sin\theta. \end{cases} \tag{2.2}$$

If $z(t; z_0)$ exists in $[0, 2\pi]$, we can define the 2π-rotation number associated to $z(t; z_0)$ as

$$\mathrm{rot}_f(z_0) = \frac{\theta(2\pi; z_0) - \theta(0; z_0)}{2\pi},$$

where $\theta(t; z_0)$ is the argument function of $z(t; z_0)$. Accordingly, $\mathrm{rot}_f(z_0)$ represents the total algebraic count of the counterclockwise rotations of the solution $z(t; z_0)$ around the origin during the time interval $[0, 2\pi]$. When (2.1) is a piecewise linear system

$$x' = -y, \quad y' = q_+(t)x^+ - q_-(t)x^-, \tag{2.3}$$

the argument function $\theta(t; z_0)$ satisfies

$$\theta'(t) = q_+(t)((\cos\theta)^+)^2 + q_-(t)((\cos\theta)^-)^2 + \sin^2\theta. \tag{2.4}$$

Thus, $\theta(t; z_0)$ only depends on the initial time t_0 and the argument value $\theta_0 \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. In this case we can write the 2π-rotation number of $z(t; z_0)$ as $\mathrm{rot}_q(w_0)$, where $w_0 = z_0/|z_0|$. Moreover, the function

$$q_+(t)((\cos\theta)^+)^2 + q_-(t)((\cos\theta)^-)^2 + \sin^2\theta$$

is 2π-periodic in t and 2π-periodic in θ. In other words, (2.4) is a differential equation on a torus. Therefore the rotation number of (2.4)

$$\rho(q) = \lim_{t\to\infty} \frac{\theta(t;\theta_0)-\theta_0}{t} \tag{2.5}$$

exists and it is independent of (t_0, θ_0).

In analogy to Propositions 2.1–2.3 in [18], we have the following relations for the rotation number $\rho(q)$ and 2π-rotation number $\mathrm{rot}_q(w_0)$ of system (2.3).

Lemma 2.1. *Let $n \in \mathbb{Z}$, then*

(i) $\rho(q) > n \Leftrightarrow \mathrm{rot}_q(w_0) > n,\ \forall\ w_0 \in \mathbb{S}^1$;
(ii) $\rho(q) < n \Leftrightarrow \mathrm{rot}_q(w_0) < n,\ \forall\ w_0 \in \mathbb{S}^1$;
(iii) $\rho(q) = n$ *if and only if there is at least one nontrivial 2π-periodic solution $\theta(t;\theta_0)$ of (2.4) with $\theta(2\pi;\theta_0) - \theta_0 = 2n\pi$.*

Similarly, for $m \in \mathbb{N}$, we can define

$$\mathrm{rot}_q^m(w_0) = \frac{\theta(2m\pi;\theta_0)-\theta_0}{2\pi}.$$

Then, the following lemma holds.

Lemma 2.2. *Let $n, m \in \mathbb{Z}$, then*

(i) $m\rho(q) > n \Leftrightarrow \mathrm{rot}_q^m(w_0) > n,\ \forall\ w_0 \in \mathbb{S}^1$;
(ii) $m\rho(q) < n \Leftrightarrow \mathrm{rot}_q^m(w_0) < n,\ \forall\ w_0 \in \mathbb{S}^1$.

Next, we discuss a comparison result associated to the 2π-rotation number for the nonlinear system (2.1). Then the behaviour of its solutions near zero and near infinity can be compared with two suitable piecewise linear systems, one considered near zero and the other near infinity. Similar to that in [11], we can prove the following lemma.

Lemma 2.3. *Let $f : [0,2\pi]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and let $q_\pm \in L^1([0,2\pi],\mathbb{R})$ be such that*

$$\liminf_{x\to\pm\infty} \frac{f(t,x)}{x} \ge q_\pm(t) \quad \textit{uniformly a.e. in } t\in[0,2\pi].$$

Then, for each $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that, for each solution $z(t)$ of (2.1) with $|z(t)| \ge R_\varepsilon$, $\forall t \in [0,2\pi]$, it follows that

$$\mathrm{rot}_f(z_0) \ge \mathrm{rot}_q(w_0) - \varepsilon, \quad \textit{for } t\in[0,2\pi], \textit{ with } w_0 = \frac{z_0}{|z_0|}.$$

Respectively, if

$$\limsup_{x\to\pm\infty} \frac{f(t,x)}{x} \le q_\pm(t) \quad \textit{uniformly a.e. in } t\in[0,2\pi],$$

then, for each $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that, for each solution $z(t)$ of (2.1) with $|z(t)| \ge R_\varepsilon$, $\forall t \in [0,2\pi]$, it follows that

$$\mathrm{rot}_f(z_0) \le \mathrm{rot}_q(w_0) + \varepsilon, \quad \textit{for } t \in [0, 2\pi], \textit{ with } w_0 = \frac{z_0}{|z_0|}.$$

Furthermore, if we assume that

$$\liminf_{x \to 0^{\pm}} \frac{f(t,x)}{x} \ge q_{\pm}(t) \quad \textit{uniformly a.e. in } t \in [0, 2\pi],$$

then, for each $\varepsilon > 0$*, there is* $r_\varepsilon > 0$ *such that, for each solution* $z(t)$ *of (2.1) with* $0 < |z(t)| \le r_\varepsilon$*,* $\forall t \in [0, 2\pi]$*, it follows that*

$$\mathrm{rot}_f(z_0) \ge \mathrm{rot}_q(w_0) - \varepsilon, \quad \textit{for } t \in [0, 2\pi], \textit{ with } w_0 = \frac{z_0}{|z_0|}.$$

Respectively, if

$$\limsup_{x \to 0^{\pm}} \frac{f(t,x)}{x} \le q_{\pm}(t) \quad \textit{uniformly a.e. in } t \in [0, 2\pi],$$

then, for each $\varepsilon > 0$*, there is* $r_\varepsilon > 0$ *such that, for each solution* $z(t)$ *of (2.1) with* $0 < |z(t)| \le r_\varepsilon$*,* $\forall t \in [0, 2\pi]$*, it follows that*

$$\mathrm{rot}_f(z_0) \le \mathrm{rot}_q(w_0) + \varepsilon, \quad \textit{for } t \in [0, 2\pi], \textit{ with } w_0 = \frac{z_0}{|z_0|}.$$

Remark 2.1. As remarked in [11], Lemma 2.3 does not require the global continuability of the solutions on $[0, 2\pi]$ (which may fail, for example, if $f(t,x)$ has superlinear growth in x). Hence, the claims of this lemma have to be considered only in regard to those solutions $z(t)$ of (2.1) defined on $[0, 2\pi]$ and such that $z(t) \ne 0$ for all $t \in [0, 2\pi]$.

3. Spiral property, modified Hamiltonian systems and the existence of periodic solutions

If there is no global existence of the solutions $z(t; z_0)$ in $[0, 2\pi]$ for the nonlinear system (2.1), the 2π-rotation number $\mathrm{rot}_f(z_0)$ is not well-defined. Thus the global existence is a crucial requirement for applying Poincaré–Birkhoff twist theorem. Although the assumptions $(H_0), (H_\infty^l), (H_0^r)$ alone are not enough to guarantee the global existence, we will find that the solutions of nonlinear system (2.1) have a spiral property under $(H_0), (H_\infty^l)$ and (H_0^r), that is if $z(t; t_0, z_0)$ is defined in $[t_0, t_1]$, and $\Delta r = |r(t_1; t_0, \theta_0, r_0) - r_0| \gg 1$ then $\Delta\theta = |\theta(t_1; t_0, \theta_0, r_0) - \theta_0| \gg 1$, where $(\theta(t; t_0, \theta_0, r_0), r(t; t_0, \theta_0, r_0))$ are the polar coordinates of $z(t; t_0, z_0)$. This idea is similar to that used in [10,12,15], but in our case a more delicate phase plane analysis is needed. More precisely, the following lemma is crucial.

Lemma 3.1. *Let* $f(t,x)$ *satisfy* (H_0) *and* $(H_\infty^l)'$*. Then, for any fixed* m*,* $N_0 \in \mathbb{N}$ *and sufficiently large* r_**, there are two strictly monotonically increasing functions* $\xi_{N_0}^-$*,* $\xi_{N_0}^+ : [r_*, +\infty) \to \mathbb{R}$*, such that*

$$\xi_{N_0}^{\pm}(r) \to +\infty \iff r \to +\infty. \tag{3.1}$$

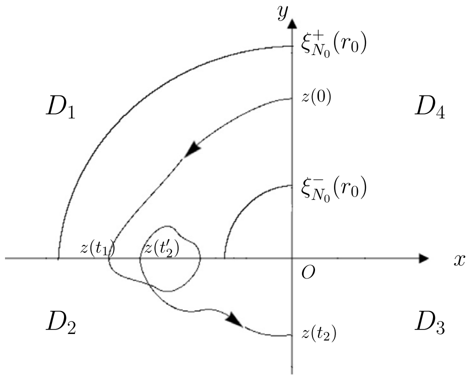

Fig. 1. The spiral property.

Moreover, for any $r_0 \geq r_*$, *the solution* $(\theta(t;\theta_0,r_0), r(t;\theta_0,r_0))$ *of* (2.2) *satisfies that either*

$$\xi^-_{N_0}(r_0) \leq r(t) \leq \xi^+_{N_0}(r_0), \quad t \in [t_0, t_0 + 2m\pi],$$

or there exists $\widehat{t}_{N_0} \in (t_0, t_0 + 2m\pi)$ *such that*

$$\theta(\widehat{t}_{N_0};\theta_0,r_0) - \theta_0 = 2N_0\pi.$$

Proof. For simplicity, we assume $m = 1$, $N_0 = 1$ and $t_0 = 0$, but in the same manner the conclusion can be proved for any $t_0 \in [0, 2\pi]$, $N_0 > 1$ and $m > 1$. The first, second, third and fourth quadrant are respectively denoted by D_4, D_1, D_2 and D_3 (see Fig. 1).

Let $z(t) = (x(t), y(t))$ be the solution of (2.1) satisfying $(x(0), y(0)) = (0, y_0)$ with $y_0 = r_0$ large enough. We will describe the behaviour of the solution in the following steps.

Step 1. We will prove that there exists $\xi^\pm_{N_0(2)}(r_0)$, with

$$\xi^\pm_{N_0(2)}(r_0) \to +\infty \iff r_0 \to +\infty,$$

such that either

$$z(t) \in D_1 \cup D_2, \quad \xi^-_{N_0(2)}(r_0) \leq r(t) \leq \xi^+_{N_0(2)}(r_0)$$

for $t \in [0, 2\pi]$, or there exists $t_2 \in (0, 2\pi)$ such that the latter inequality holds for $t \in [0, t_2)$ and, moreover,

$$x(t_2) = 0,\ y(t_2) < 0,\ x'(t_2) > 0 \quad \text{and} \quad x(t) > 0 \text{ for } t > t_2 \text{ and } t \text{ near } t_2,$$

see Fig. 1.

We discuss the estimates of $z(t)$ in the following cases.

Case 1. Let $z(t) \in D_1$ for $t \in (0, t_1)$, where $t_1 \leq 2\pi$. Define an energy function $v(x, y) = \frac{y^2}{2} + G_M(x)$, where

$$G_M(x) = \int_0^x g_M(s)ds, \;\; g_M(x) = \mathrm{sgn}(x) \max\{|x|, \max_{t\in[0,2\pi]} |f(t, x)| + 1\}.$$

Then,

$$G_M(x) \to +\infty \quad \text{as} \quad |x| \to \infty.$$

Moreover,

$$\frac{d}{dt} v(x(t), y(t)) = yy' + g_M(x)x' = (f(t, x) - g_M(x))y \geq 0.$$

It follows that

$$v(x(t), y(t)) \geq v(x(0), y(0)) = \frac{r_0^2}{2}, \quad \text{for} \quad t \in (0, t_1].$$

Denote by Γ_{M_0} the curve $v(x, y) = \frac{r_0^2}{2}$, then

$$r(t) \geq \xi^-_{N_0(1)}(r_0), \quad \text{for} \quad t \in (0, t_1], \tag{3.2}$$

where $\xi^-_{N_0(1)}(r_0) = \min\{\sqrt{x^2 + y^2}\,|\,(x, y) \in D_1 \cap \Gamma_{M_0}\} > 0$ for r_0 sufficiently large.

On the other hand, we have

$$\frac{d}{dt} r^2 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = -2xy + 2yf(t, x). \tag{3.3}$$

From $(H^l_\infty)'$, there exists $\varepsilon_0 \leq 1$ and M_{ϵ_0}, such that $f(t, x) \leq (a_-(t) - \varepsilon_0)x$ for $x < -M_{\epsilon_0}$, where $a_-(t) \in L^1([0, 2\pi])$. Using (3.2), for a sufficiently large r_0, there exists $t_1' \in (0, t_1)$ such that $x(t_1') = -M_{\varepsilon_0}$. Then $x(t) \in (0, -M_{\varepsilon_0})$ for $t \in (0, t_1')$ and $x(t) \leq -M_{\varepsilon_0}$ for $t \in [t_1', t_1)$. When $t \in (0, t_1')$, we have

$$|y'(t)| \leq K_{\varepsilon_0} = \max\{|f(t, x)| : t \in [0, 2\pi], x \in [-M_{\varepsilon_0}, 0]\}.$$

Thus $y(t) \leq y_0 + 2K_{\varepsilon_0}\pi$, which implies $r(t) \leq r_0 + 2K_{\varepsilon_0}\pi + M_{\varepsilon_0}$. When $t \in [t_1', t_1)$, from (3.3), we have

$$\frac{d}{dt} r^2 \leq -2xy + 2xy(a_-(t) - \varepsilon_0) \leq r^2(2 + |a_-(t)|).$$

Taking integrals on both sides of the above inequality gives

$$\int_{t_1'}^{t} \frac{1}{r^2} d(r^2) \le \int_{t_1'}^{t} (2 + |a_-(t)|) dt.$$

It follows that

$$r(t) \le r(t_1') e^{\int_0^t (1+|a_-(t)|/2)dt} \le (r_0 + 2K_{\varepsilon_0}\pi + M_{\varepsilon_0}) e^{\int_0^t (1+|a_-(t)|/2)dt}, \quad \text{for} \quad t \in [t_1', t_1). \quad (3.4)$$

Denote by $\xi^+_{N_0(1)}(r_0) = (r_0 + 2K_{\varepsilon_0}\pi + M_{\varepsilon_0}) e^{\int_0^t (1+|a_-(t)|/2)dt}$. Then, combining with (3.2) we find

$$\xi^-_{N_0(1)}(r_0) \le r(t) \le \xi^+_{N_0(1)}(r_0), \quad \text{for} \quad t \in [0, t_1]. \quad (3.5)$$

If $t_1 = 2\pi$, the discussion is completed. If $t_1 < 2\pi$, $z(t)$ will meet $y = 0$ entering into D_2 at $t = t_1$.

Note that now we are not assuming the sign condition $y' = f(t, x) < 0$ for $-x$ large enough. Thus $z(t)$ could eventually return back to D_1 again. Without loss of generality, we assume that there are $t_2', t_2 \in (t_1, 2\pi]$ such that:

(1) $z(t) \in D_1$ for $t \in [0, t_1)$, $y(t_1) = 0$;
(2) $z(t) \in D_1 \cup D_2$ for $t \in [t_1, t_2']$, $y(t_2') = 0$
(3) $z(t) \in D_2$ for $t \in [t_2', t_2)$, $t_2 = 2\pi$ or $x(t_2) = 0$.

The estimate of (1) has been given in Case 1. Since we can obtain an estimate of $z(t)$ for $t \in [t_2', t_2]$ only dependent on $r(t_2')$, we will discuss the estimate of (3) firstly, then give the estimates of (2).

Case 2. Let $z(t) \in D_2$, $t \in (t_2', t_2)$, then $x(t) < 0$, $y(t) < 0$. In this case,

$$\frac{d}{dt} v(x(t), y(t)) = yy' + g_M(x)x' = (f(t, x) - g_M(x))y \le 0.$$

It follows that

$$v(x(t), y(t)) \le v(x(t_2'), y(t_2')) = G_M(x(t_2')), \quad \text{for} \quad t \in [t_2', t_2).$$

Denote by $\Gamma_{M_2'}$ the curve $v(x, y) = G_M(x(t_2'))$, then

$$r(t) \le \xi^+_{t_2'}, \quad \text{for} \quad t \in [t_2', t_2), \quad (3.6)$$

where $\xi^+_{t_2'} = \max\{\sqrt{x^2 + y^2} \mid (x, y) \in D_2 \cap \Gamma_{M_2'}\}$ is dependent on $r(t_2')$.

On the other hand, we have

$$\frac{d}{dt} r^2 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = -2xy + 2yf(t, x).$$

From $(H^l_\infty)'$, there exist positive constants $\varepsilon_0 \leq$ and M_{ϵ_0}, such that $f(t,x) \leq (a_-(t) - \varepsilon_0)x$ for $x \leq M_{\epsilon_0}$. Then

$$\frac{dr^2}{dt} \geq -2xy + 2xy(a_-(t) - \varepsilon_0) \geq -r^2(2 + |a_-(t)|).$$

Using the Gronwall inequality we have

$$r(t) \geq r(t_2')\mathrm{e}^{-\int_{t_2'}^{t}(1+|a_-(t)|/2)dt}, \quad \text{for} \quad t \in [t_2', t_2). \tag{3.7}$$

If there is $t_2'' > t_2'$ such that $x(t_2'') = -M_{\varepsilon_0}$, then $y'(t_2'') \leq -r(t_2'') + x(t_2'') = -r(t_2'') + M_{\varepsilon_0}$. We can prove that if $r(t_2'')$ is sufficiently large then there is $t_2''' > t_2''$ such that $x(t_2''') = 0$ and $x'(t) = -y(t) < 0$ for $t \in [t_2'', t_2''']$. In this case $z(t)$ could not return back to D_1 again. So we just discuss the case when $x(t) < -M_{\varepsilon_0}$ for $t \in [t_2', t_2)$.

We note that (3.6) and (3.7) give estimates of $z(t)$ for $t \in [t_2', t_2]$ dependent on $r(t_2')$.

Case 3. We proceed to the estimation of $z(t)$ for $t \in [t_1, t_2']$. We will prove that

$$\xi^-_{N_0(2)}(r_0) \leq r(t) \leq \xi^+_{N_0(2)}(r_0), \quad \text{for} \quad t \in [t_1, t_2'], \tag{3.8}$$

where

$$\xi^-_{N_0(2)}(r_0) = \xi^-_{N_0(1)}(r_0)\mathrm{e}^{-(2\pi+M)}$$

and

$$\xi^+_{N_0(2)}(r_0) = \max\{\sqrt{x^2+y^2} | (x,y) \in D_2 \cap \Gamma_{M_2}\},$$

where Γ_{M_2} is the curve $v(x,y) = G_M(\xi^+_{N_0(1)}(r_0))$.

From the uniqueness for the associated Cauchy problems and the vector field constraint condition $x' = -y$, the nonzero solutions can never perform clockwise rotations at y-axis. Then there is $d_0 > 0$, such that $x(t) \leq -d_0$ for $t \in [t_1, t_2']$. Let $f_n(t,x)$, $n = 1, 2, \cdots$, be analytic functions satisfying

$$\lim_{n\to\infty} f_n(t,x) = f(t,x) \quad \text{uniformly for} \quad x \in [-\xi^+_{N_0(2)}(r_0), -d_0/2],\ t \in [0, 2\pi].$$

Let $x_n(t)$ be the solution of $x'' + f_n(t,x) = 0$ satisfying the initial conditions $x_n(t_1) = x(t_1)$, $x_n'(t_1) = 0$. Then, by using the theorem for the dependence on initial conditions and parameters we know that $z_n(t) = (x_n(t), -x_n'(t))$ exists for $t \in [t_1, t_2']$. Moreover,

$$\lim_{n\to\infty} z_n(t) = z(t) \quad \text{uniformly for} \quad t \in [t_1, t_2']. \tag{3.9}$$

Without loss of generality, we let $x_n(t) \leq -d_0/2$ for $t \in [t_1, t_2']$.

Next, we will prove that

$$\xi^-_{N_0(2)}(r_0) \leq r_n(t) = |z_n(t)| \leq \xi^+_{N_0(2)}(r_0), \quad \text{for} \quad t \in [t_1, t_2']. \tag{3.10}$$

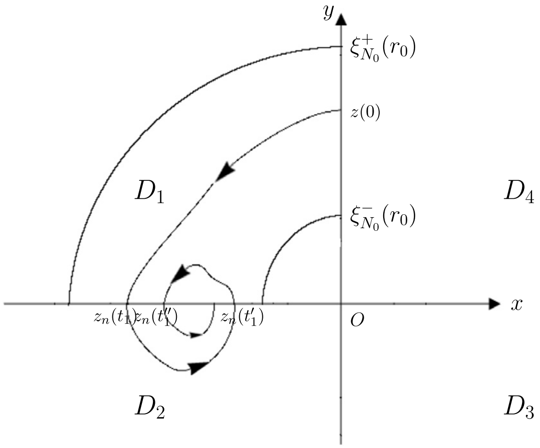

Fig. 2. A solution turning back to region D_1.

Therefore, (3.10) and (3.9) implies (3.8).

Since $f_n(t,x)$ is analytic, so is $z_n(t)$. Thus $y_n(t)$ has only a finite number of zeros in $[t_1, t_2']$. Without loss of generality, we let $t_1', t_1'' \in (t_1, t_2')$ such that $y_n(t_1') = y_n(t_1'') = 0$ and $z_n(t) \in D_2$ for $t \in [t_1, t_1']$, $z_n(t) \in D_1$ for $t \in (t_1', t_1'']$ (see Fig. 2).

When $z_n(t) \in D_2$, we use the similar argument as in (3.6) and (3.7) to obtain

$$r_n(t_1)\mathrm{e}^{-\int_{t_1}^{t}(1+|a_-(t)|/2)dt} \le r_n(t) \le \xi^+_{t_1}, \quad \text{for} \quad t \in [t_1, t_1'], \tag{3.11}$$

where $\xi^+_{t_1} = \max\{\sqrt{x^2+y^2}\,|\,(x,y) \in D_2 \cap \Gamma_{M_1}\}$ and Γ_{M_1} is the curve $v(x,y) = G_M(x(t_1))$. Moreover, since $x_n'(t) = -y_n(t) > 0$, we have

$$-x_n(t_1') < -x_n(t_1) = r_n(t_1). \tag{3.12}$$

When $z_n(t) \in D_1$, using the argument in Case 1, and from the analogous estimation of (3.4), we find

$$r_n(t) \le r_n(t_1')\mathrm{e}^{\int_{t_1'}^{t}(1+|a_-(t)|/2)dt}, \quad \text{for} \quad t \in [t_1', t_1'']. \tag{3.13}$$

Moreover, since $x_n'(t) = -y_n(t) < 0$, we have $-x_n(t) > -x_n(t_1') = r_n(t_1')$. Thus,

$$r_n(t) \ge -x_n(t) \ge r_n(t_1'), \quad \text{for} \quad t \in [t_1', t_1'']. \tag{3.14}$$

In particular, we get the following estimates related to the zeros t_1', t_1'' of $y_n(t)$:

$$r_n(t_1)\mathrm{e}^{-\int_{t_1}^{t_1'}(1+|a_-(t)|/2)dt} \le |x_n(t_1')| \le r_n(t_1),$$

and

$$r_n(t_1)\mathrm{e}^{-\int_{t_1}^{t_1''}(1+|a_-(t)|/2)dt} \le |x_n(t_1'')| \le r_n(t_1)\mathrm{e}^{\int_{t_1}^{t_1''}(1+|a_-(t)|/2)dt}.$$

In general, assume that $t_1^{(k)}$ is the k-th zero of $y_n(t)$, then it satisfies

$$r_n(t_1)e^{-\int_{t_1}^{t_1^{(k)}}(1+|a_-(t)|/2)dt} \leq |x_n(t^{(k)})| \leq r_n(t_1)e^{\int_{t_1}^{t_1^{(k)}}(1+|a_-(t)|/2)dt}.$$

Then for the $k+1$-th zero $t_1^{(k+1)}$, if it exists, we have two possibilities. The first one is $z_n(t) \in D_1$ for $t \in (t_1^{(k)}, t_1^{(k+1)})$, then using the similar argument for (3.13) and (3.14) we have

$$|x_n(t_1^{(k)})| < |x_n(t_1^{(k+1)})| \leq r_n(t_1^{(k)})e^{\int_{t_1^{(k)}}^{t_1^{(k+1)}}(1+|a_-(t)|/2)dt}. \tag{3.15}$$

The second option is $z_n(t) \in D_2$ for $t \in (t_1^{(k)}, t_1^{(k+1)})$. In this case, using the similar argument for (3.11) and (3.12), we have

$$r_n(t_1^{(k)})e^{-\int_{t_1^{(k)}}^{t_1^{(k+1)}}(1+|a_-(t)|/2)dt} \leq |x_n(t_1^{(k+1)})| < |x_n(t_1^{(k)})|. \tag{3.16}$$

Inequalities (3.15) and (3.16) imply that

$$r_n(t_1)e^{-\int_{t_1}^{t_1^{(k+1)}}(1+|a_-(t)|/2)dt} \leq |x_n(t_1^{(k+1)})| \leq r_n(t_1)e^{\int_{t_1}^{t_1^{(k+1)}}(1+|a_-(t)|/2)dt}.$$

Therefore

$$r_n(t_1)e^{-\int_{t_1}^{\widetilde{t}}(1+|a_-(t)|/2)dt} \leq |x_n(\widetilde{t})| \leq r_n(t_1)e^{\int_{t_1}^{\widetilde{t}}(1+|a_-(t)|/2)dt} \tag{3.17}$$

holds for any $\widetilde{t} \in (t_1, t_2')$ such that $y_n(\widetilde{t}) = 0$.

Now we can give the estimate of $z_n(t)$ for $t \in (t_1, t_2')$. If $z_n(t) \in D_1$, then there exists $\widetilde{t_1} \in (t_1, t_2')$ with $y_n(\widetilde{t_1}) = 0$, such that $z_n(s) \in D_1$ for $s \in (\widetilde{t_1}, t]$. Using the similar argument for (3.13) and (3.14) we have

$$|x_n(\widetilde{t_1})| = r_n(\widetilde{t_1}) \leq r_n(t) \leq r_n(\widetilde{t_1})e^{\int_{\widetilde{t_1}}^{t}(1+|a_-(t)|/2)dt}$$

which recalls (3.17) and (3.4) to get

$$\xi_{N_0(1)}^-(r_0)e^{-\int_0^{2\pi}(1+|a_-(t)|/2)dt} \leq r_n(t) \leq r_n(t_1)e^{\int_{t_1}^{t}(1+|a_-(t)|/2)dt} \leq \xi_{N_0(1)}^+(r_0). \tag{3.18}$$

If $z_n(t) \in D_2$, then there exists $\widetilde{t_2} \in (t_1, t_2')$ with $y_n(\widetilde{t_2}) = 0$, such that $z_n(s) \in D_2$ for $s \in (\widetilde{t_2}, t]$. Then, reasoning as in (3.11), we have

$$r_n(\widetilde{t_2})e^{-\int_{\widetilde{t_2}}^{t}(1+|a_-(t)|/2)dt} \leq r_n(t) \leq \xi_{\widetilde{t_2}}^+, \tag{3.19}$$

where $\xi_{\widetilde{t_2}}^+ = \max\{\sqrt{x^2+y^2} \,|\, (x,y) \in D_2 \cap \Gamma_{\widetilde{M}_2}\}$ and $\Gamma_{\widetilde{M}_2}$ is the curve $v(x,y) = G_M(x(\widetilde{t_2}))$. Note that from (3.4) we have

$$|x_n(\widetilde{t_2})| \le r_n(t_1)e^{\int_{t_1}^{\widetilde{t}}(1+|a_-(t)|/2)dt} \le r(0)e^{\int_0^{\widetilde{t}}(1+|a_-(t)|/2)dt} \le \xi^+_{N_0(1)}(r_0).$$

Then $\xi^+_{\widetilde{t_2}} \le \xi^+_{N_0(2)}(r_0) = \max\{\sqrt{x^2+y^2}\,|\,(x,y)\in D_2\cap\Gamma_{M_2}\}$, where Γ_{M_2} is the curve $v(x,y) = G_M(\xi^+_{N_0(1)}(r_0))$. Therefore (3.18) and (3.19) implies (3.10).

For $t > t_2'$, there are two possibilities. The first one is $z(t)\in D_2$ for $t\in[t_2', 2\pi]$. In this case, the estimate (3.8) holds for $t\in[t_2', 2\pi]$ by using (3.11). Thus, in view of (3.5), (3.11), estimate (3.8) holds for $t\in[0,2\pi]$. Otherwise, there exists $t_2\in(t_1,2\pi)$ such that $x(t_2)=0$ and $z(t)\in D_3$ for $t\ge t_2$. In this case, the estimate (3.8) also holds for $t\in[0,t_2]$.

Step 2. Similar arguments apply to the cases of $z(t)\in D_3$ and D_4. For example, when $z(t)\in D_3$, we have $x>0$ and $y<0$, then we have

$$\frac{d}{dt}v(x(t),y(t)) = yy' + g_M(x)x' = (f(t,x)-g_M(x))y \ge 0.$$

On the other hand, there exists ε_0, such that $f(t,x)\ge(a_+(t)-\varepsilon_0)x$. Then

$$\frac{d}{dt}r^2 \le -2xy + 2xy(a_+(t)-\varepsilon_0) \le r^2(2+|a_+(t)|).$$

Thus the argument for $z(t)\in D_3$ is similar to that for $z(t)\in D_1$.

In conclusion, we can find $\xi^\pm_{N_0(4)}(r_0)$ with

$$\xi^\pm_{N_0(4)}(r_0)\to+\infty \iff r_0\to+\infty,$$

such that

$$\xi^-_{N_0(4)}(r_0)\le r(t)\le \xi^+_{N_0(4)}(r_0), \quad \text{for} \quad z(t)\in D_3\cup D_4.$$

Let

$$\xi^-_{N_0}(r_0) = \min\{\xi^-_{N_0(i)}(r_0), i=2,4\}, \quad \xi^+_{N_0}(r_0) = \max\{\xi^+_{N_0(i)}(r_0), i=2,4\}.$$

Then either

$$\xi^-_{N_0}(r_0)\le r(t)\le \xi^+_{N_0}(r_0), \quad \text{for} \quad t\in[0,2\pi],$$

or there exists $\widehat{t_1}\in[0,2\pi)$, such that $z(t)$ intersects $x=0$ at $t=\widehat{t_1}$ and $z(t)$ completes one clockwise turn around the origin when $t\in[0,\widehat{t_1}]$. Moreover,

$$\xi^-_{N_0}(r_0)\le r(t)\le \xi^+_{N_0}(r_0), \quad \text{for} \quad t\in[0,\widehat{t_1}].$$

Finally, it is clear that $\xi^\pm_{N_0}(r_0)$ can be chosen as strictly increasing functions that satisfy (3.1). □

The geometric meaning of Lemma 3.1 is that there is an annulus depending on initial values such that the solution of (2.1) either is located in this annulus for $t \in [t_0, t_0 + 2m\pi]$ or completes N_0 turns around the origin when $t \in [0, \widehat{t}_{N_0}]$ with $\widehat{t}_{N_0} \in (t_0, t_0 + 2m\pi)$. In other words, there are two spiral curves guiding the solutions of (2.1) in the phase plane (see Figs. 1, 2), forcing them to rotate around the origin as they increase in norm.

Next, let us define a modified Hamiltonian system. Consider a Hamiltonian function

$$H = \frac{y^2}{2} + k(x^2 + y^2)G(t, x) + [1 - k(x^2 + y^2)]x^2,$$

where $G(t, x) = \int_0^x f(t, s)ds$, $k(x^2 + y^2) \in C^\infty(\mathbb{R}^+, \mathbb{R})$ is a truncating function satisfying that

$$k(x^2 + y^2) = \begin{cases} 0, & x^2 + y^2 \geq r_2; \\ \text{smooth connection}, & r_1 < x^2 + y^2 < r_2; \\ 1, & x^2 + y^2 \leq r_1, \end{cases}$$

where r_1, r_2 are positive parameters, whose specific value of will be given in the proof of Theorem 1.1.

For the associated Hamiltonian system

$$\begin{cases} x' = -\dfrac{\partial H}{\partial y} = -y - 2y\dfrac{dk}{dr^2}[G(t, x) - x^2]; \\ y' = \dfrac{\partial H}{\partial x} = 2x\dfrac{dk}{dr^2}[G(t, x) - x^2] + k(x^2 + y^2)[f(t, x) - 2x] + 2x, \end{cases} \tag{3.20}$$

it is very easy to check the following lemma.

Lemma 3.2. *Assume* (H_0). *The initial value problem associated to (3.20) has a unique solution. Moreover, every solution* $z(t; z_0) = (x(t; x_0, y_0), y(t; x_0, y_0))$ *exists for any* $t \in \mathbb{R}$. *If* $|z(t; z_0)| \leq r_1$, $z(t; z_0)$ *is also a solution of (2.1).*

If $z_0 \neq (0, 0)$, *then* $z(t; z_0) \neq (0, 0)$ *for every* $t \in \mathbb{R}$. *Thus, polar coordinates*

$$x(t; x_0, y_0) = r(t; z_0)\cos\theta(t; z_0), \quad y(t; x_0, y_0) = r(t; z_0)\sin\theta(t; z_0),$$

are well-defined. Moreover, when $x(t; x_0, y_0) = 0$ *then* $x'(t; x_0, y_0) = -y(t; x_0, y_0)$, *the solutions can never perform clockwise rotations at y-axis. More precisely, for any* $t_2 > t_1 > 0$ *and any* $k \in \mathbb{Z}$, *if* $\theta(t_1; z_0) > k\pi + \pi/2$, *then*

$$\theta(t_2; z_0) > k\pi + \pi/2.$$

Proof of Theorem 1.1. We will divide the proof into the following steps.

Step 1. By the continuity of the solutions with respect to the initial value and the fact that $z = (0, 0)$ is a solution of (3.20), we can find $\Gamma_- = \{z : |z| = \widetilde{r}_\varepsilon\}$, such that if $z_0 \in \Gamma_-$ then $|r(t)| = |z(t; z_0)| \leq r_\varepsilon$, for $t \in [0, 2m\pi]$, where $\varepsilon < \min\{n/m - \rho_0, \rho_\infty - n/m\}$. Then, by Lemma 2.3 we have

$$\theta(2m\pi, z_0) - \theta(0, z_0) < 2n\pi, \quad \text{for} \quad z_0 \in \Gamma_-. \tag{3.21}$$

Step 2. Let $\Gamma_+ = \{z : |z| = R_\infty\}$ and choose $r_1 = R'_\infty$, where

$$R_\infty > (\xi^-_{n+1})^{-1}(R_\varepsilon), \quad R'_\infty > \xi^+_{n+1}(R_\infty)$$

and R_ε is defined in Lemma 2.3. Then, system (3.20) is equivalent to the original system (2.1) for $|z| \le r_1$.

Now, consider the solution of (3.20) starting from $z_0 \in \Gamma_+ = \{z : |z| = R_\infty\}$. If $R_\varepsilon \le r(t) = |z(t; z_0)| \le R'_\infty$ for all $t \in [0, 2m\pi]$, using Lemma 2.3, we have

$$\theta(2m\pi, z_0) - \theta(0, z_0) > 2n\pi, \quad \text{for} \quad z_0 \in \Gamma_+. \tag{3.22}$$

If there exists $t_\star \in (0, 2m\pi)$ such that $r(t_\star) \ge R'_\infty > \xi^+_{n+1}(R_\infty)$, then the inequality

$$\xi^-_{n+1}(|z_0|) \le r(t) \le \xi^+_{n+1}(|z_0|)$$

doesn't satisfy for all $t \in [0, 2m\pi]$. Therefore, by Lemma 3.1, we know that there exists $t'_\star \in (0, t_1]$ such that

$$\theta(t'_\star; z_0) - \theta(0; z_0) = 2(n+1)\pi.$$

Moreover, by using Lemma 3.2, we have

$$\begin{aligned}\theta(2m\pi, z_0) - \theta(0, z_0) &= (\theta(2m\pi, z_0) - \theta(t'_\star; z_0)) + (\theta(t'_\star; z_0) - \theta(0, z_0)) \\ &\ge -\pi + 2(n+1)\pi > 2n\pi.\end{aligned}$$

Finally, if there exists $t''_\star \in (0, 2m\pi)$ such that $r(t''_\star) \le R_\varepsilon < \xi^-_{n+1}(R_\infty)$, then the inequality

$$\xi^-_{n+1}(|z_0|) \le r(t) \le \xi^+_{n+1}(|z_0|)$$

doesn't satisfy for all $t \in [0, 2m\pi]$, the same argument as above proves that (3.22) holds.

Step 3. Denote by $\mathcal{A}$ the annulus bounded by Γ_- and Γ_+. Consider the Poincaré map

$$\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2, \quad z_0 \mapsto z(2m\pi; z_0).$$

Using Lemma 3.2, we know that $\mathcal{P}$ is a homeomorphism. In fact, since (3.20) is a Hamiltonian system $\mathcal{P}$ is area-preserving homeomorphism. Moreover, (3.21) and (3.22) imply that $\mathcal{P}$ satisfies the boundary twist condition on $\mathcal{A}$. Applying the Poincaré–Birkhoff twist theorem (see the generalized versions exposed in [27], [8], [17] and [26]), we conclude that $\mathcal{P}$ has at least two geometrically distinct fixed points $z_i,\ i = 1, 2$, which correspond to two $2m\pi$-periodic solutions $z(t; z_i),\ i = 1, 2$ of system (3.20) with

$$\theta(2m\pi; z_i) - \theta(0; z_i) = 2n\pi, \quad i = 1, 2. \tag{3.23}$$

Step 4. We will show that $z(t; z_i)$, $i = 1, 2$ are in fact $2m\pi$-periodic solutions of (2.1). Note that $0 < |z_1|, |z_2| < R_\infty$. If there exists $t_1 \in (0, 2m\pi)$ such that $|z(t_1; z_1)| \geq R'_\infty$ and $|z(t; z_1)| \leq R'_\infty$ for $t \in [0, t_1]$, then by using Lemma 3.1 we have

$$\theta(t_1; z_1) - \theta(0; z_1) = 2(n+1)\pi.$$

Moreover, by means of Lemma 3.2, we have

$$\begin{aligned}\theta(2m\pi, z_1) - \theta(0, z_1) &= (\theta(2m\pi, z_1) - \theta(t_1; z_1)) + (\theta(t_1; z_1) - \theta(0, z_1)) \\ &\geq -\pi + 2(n+1)\pi > 2n\pi,\end{aligned}$$

which contradicts (3.23). Therefore, $|z(t; z_1)| \leq R'_\infty$ for $t \in [0, 2m\pi]$, and $z(t; z_1)$ is a $2m\pi$-periodic solution of (2.1). The same arguments are valid for $z(t; z_2)$. Besides, (3.23) and Lemma 3.2 show that $x(t; , z_i)$, $i = 1, 2$ are $2m\pi$-periodic solutions of (1.1) with $2n$ zeros in $[0, 2m\pi)$. Theorem 1.1 is thus proved. □

4. Some examples

We begin with some discussions for the estimation of the rotation number of the asymmetric oscillator

$$x'' + q_+(t)x^+ - q_-(t)x^- = 0. \tag{4.1}$$

By the arguments of Section 2, the rotation number $\rho(q)$ exists with the definition (2.5), and it is sufficient to consider to just consider the limit

$$\rho(q) = \lim_{n\to\infty} \frac{\theta(2n\pi; t_0, \theta_0)}{2n\pi},$$

where t_0, θ_0 can be selected arbitrarily.

The following simple lemmas will be useful.

Lemma 4.1. *Let $\theta(t; t_0, \theta_0)$ a nonzero solution of (2.4) and assume that $\theta(t_1; t_0, \theta_0) \geq k\pi + \pi/2$ for some $k \in \mathbb{Z}$. Then $\theta(t; t_0, \theta_0) > k\pi + \pi/2$ for any $t > t_1$.*

Lemma 4.2. *Let $\theta(t; t_0, \theta_0)$ and $\theta(t; t_0, \theta_1)$ be two solutions of (2.4) with the initial values $\theta(t_0; t_0, \theta_0) = \theta_0 > \theta_1 = \theta(t_0; t_0, \theta_1)$. Then*

$$\theta(t; t_0, \theta_0) > \theta(t; t_0, \theta_1), \quad \textit{for} \quad t > t_0.$$

Lemma 4.3. *Let $\theta(t; t_0, \theta_0)$ be the solutions of (2.4) with the initial values $\theta(t_0; t_0, \theta_0) = \theta_0$ and $\rho(q)$ the rotation number of (2.4). Then*

$$|\theta(2m\pi + t_0; t_0, \theta_0) - \theta_0 - \rho(q)m| < 2\pi, \quad \textit{for} \quad m \in \mathbb{Z}.$$

The proof of Lemma 4.1 relies on the fact that a nonzero solution of (2.3) can never perform clockwise rotations at y-axis. The proof of Lemma 4.2 uses the uniqueness of the solution of (2.4) with respect to the initial value, then since the equation is scalar the solutions are ordered. Finally, the proof of Lemma 4.3 uses Lemma 4.2 and the definition of rotation number.

With these basic tools, the rotation number can be estimated in some particular cases.

Proposition 1. *The following statements hold*

(i) *If* $q_\pm(t) \geq a^2$ *for all* $t \in [0, 2\pi]$, *then* $\rho(q) \geq a$

(ii) *Assume that* $q_\pm(t) \geq a^2 > 0$ *for* $t \in [t_0, t_0 + \alpha]$ *and* $q_\pm(t)$ *takes other values for* $t \in (t_0 + \alpha, t_0 + 2\pi]$. *Then,*

$$\rho(q) \geq \left[\frac{a\alpha}{2\pi}\right].$$

(iii) $\rho(q) > 0$ *if and only if there exists a solution* $x(t; t_0, x_0, x_0')$ *of (4.1) such that* $x(t; t_0, x_0, x_0')$ *has at least three zeros. In particular, if there exists a solution* $x(t; t_0, x_0, x_0')$ *of (4.1) such that* $x(t; t_0, x_0, x_0')$ *has at least three zeros in* $[0, 2\pi)$ *then* $\rho(q) \geq 1$.

Proof. Let us begin by proving (i). By using the hypothesis,

$$\int_0^{2k\pi} \frac{d\theta}{q_+(t)((\cos\theta)^+)^2 + q_-(t)((\cos\theta)^-)^2 + \sin^2\theta} \leq \frac{2k\pi}{a}.$$

Then $\Delta\theta[t_0, t] = \theta(t, t_0, \theta_0) - \theta_0 \geq 2k\pi$ for $\Delta t = t - t_0 = 2k\pi/a$. From (2.5) we have

$$\rho(q) \geq \lim_{t\to+\infty} \frac{\Delta\theta[t_0, t]}{\Delta t} \geq a.$$

To prove (ii), we consider the solution $\theta(t; t_0, \pi/2)$. Let $k \in \mathbb{Z}^+$ such that $2k\pi \leq a\alpha < (2k + 1)\pi$. Then $\theta(t_0 + \alpha; t_0, \pi/2) \geq 2k\pi + \pi/2$. Moreover,

$$\theta(t; t_0, \pi/2) > 2k\pi + \pi/2, \quad \text{for} \quad t \in (t_0 + \alpha, t_0 + 2\pi]$$

by using Lemma 4.1.

Next, denote by $\theta_1 = \theta(t_0 + 2\pi; t_0, \pi/2)$. From Lemma 4.2 we have

$$\theta(t_0 + 4\pi; t_0 + 2\pi, \theta_1) > \theta(t_0 + 4\pi; t_0 + 2\pi, 2k\pi + \pi/2) = \theta(t_0 + 4\pi; t_0 + 2\pi, \pi/2) + 2k\pi.$$

Thus $\theta(t_0 + 4\pi; t_0, \pi/2) = \theta(t_0 + 4\pi; t_0 + 2\pi, \theta_1) > \theta(t_0 + 2\pi; t_0, \pi/2) + 2k\pi > 4k\pi + \pi/2$.

Similarly, we have

$$\theta(t_0 + 2m\pi; t_0, \pi/2) > 2mk\pi + \pi/2, \quad \text{for} \quad m \in \mathbb{N}.$$

Therefore, we have

$$\rho(q) = \lim_{m\to\infty} \frac{\theta(t_0+2m\pi; t_0, \pi/2)}{2m\pi} \geq k = \left[\frac{a\alpha}{2\pi}\right].$$

Finally, we proceed to prove (iii). Indeed, let us assume that $x(t) = x(t; t_0, x_0, x_0')$ has three adjacent zeros

$$x(t_1) = x(t_2) = x(t_3) = 0.$$

Denote by $\theta(t)$ the argument value of $(x(t), x'(t))$, then

$$\cos\theta(t_i) = 0, \quad |\sin\theta(t_i)| = 1, \quad i = 1, 2, 3,$$

which implies that $\theta'(t_i) > 0,\ i = 1, 2, 3$, so in consequence $\theta(t_3) - \theta(t_1) = 2\pi$. Suppose that $t_3 - t_1 \leq 2m\pi$ and $\theta(t_1) = k\pi + \pi/2$ for some $k \in \mathbb{Z}$, then $\theta(t_1 + 2m\pi) \geq (2 + k)\pi + \pi/2$ by using Lemma 4.1, that is

$$|\theta(t_1 + 2m\pi) - \theta(t_1)| \geq 2\pi.$$

As a result of Lemma 4.2, we conclude that $\rho(q) > 0$.

On the other hand, if $\rho(q) > 0$ we have a sufficiently large $m \in \mathbb{N}$ such that

$$|\theta(t_0 + 2m\pi) - \theta(t_0)| \geq m\rho(q) - 2\pi \geq 4\pi.$$

Then, there are at least three times $t_1, t_2, t_3 \in [t_0, t_0 + 2m\pi]$ such that $\theta(t_i) = (k + i)\pi + \pi/2$ for $k \in \mathbb{Z}$, which follows that

$$x(t_1) = x(t_2) = x(t_3) = 0.$$

Moreover, if $x(t) = x(t; t_0, x_0, x_0')$ has three adjacent zeros in $[t_0, t_0 + 2\pi)$, that is, there are three times $t_1, t_2, t_3 \in [t_0, t_0 + 2\pi)$ such that $\theta(t_i) = (k + i)\pi + \pi/2$ for some $k \in \mathbb{Z}$, then we find $\theta(t_1 + 2\pi) - \theta(t_1) > 2\pi$ by using Lemma 4.1, and $\theta(t_1 + 2n\pi) - \theta(t_1) > 2n\pi$ for $n \in \mathbb{Z}$ by Lemma 4.2. Therefore, we obtain $\rho(q) \geq 1$. □

Now, we can prove the following corollary for a partially superlinear differential equation.

Corollary 4.1. *Suppose that* (H_0) *holds and*

(f_2) *There are* r_k *and* $a_k(t) \in L^1,\ k = 1, 2, \cdots,$ *such that*

$$\frac{f(t,x)}{x} \geq a_k(t)x, \quad \textit{for} \quad |x| \geq r_k \quad \textit{and} \quad t \in [0, 2\pi],$$

where the rotation number $\rho(a_k)$ *of* $x'' + a_k(t)x = 0$ *satisfies that* $\rho(a_k) \to +\infty$ *as* $k \to \infty$.

Then for any $m \in \mathbb{N}$, *there exists* $n_m\mathbb{N}$ *such that equation (1.1) has two* $2m\pi$*-periodic solutions* $x_{i,m,n}(t),\ i = 1, 2,$ *for any* $n > n_m$, *such that* $x_{i,m,n}(t),\ i = 1, 2$ *makes exactly* n *turns around the origin in the interval time* $[0, 2m\pi]$.

Proof. From (H_0), we have $L, \delta > 0$ such that $|f(t,x) - f(t,0)| \leq L|x|$ for $|x| \leq \delta$, which implies (H_0^r) if we let $b_0(t) = L$. Moreover, $\rho(b_0) = \sqrt{L}$. On the other hand, taking $a_\infty(t) = a_k(t)$, condition (H_∞^l) is satisfied. Thus Theorem 1.1 shows that, for any $n/m \in (\rho(b_0), \rho(a_k))$, equation (1.1) has two $2m\pi$-periodic solutions $x_{i,m,n}(t)$, $i = 1, 2$. Moreover, these $2m\pi$-periodic solutions make exactly n turns around the origin in the time $[0, 2m\pi]$. □

As it is seen below, it is not difficult to check that Corollary 1.1 is a particular case of the latter result.

Proof of Corollary 1.1. It is enough to prove that (f_1) implies (f_2) and apply Corollary 4.1. If (f_1) holds, then for any $k > 0$, we have $r_k > 0$ such that $f(t,x)/x \geq k$ for $|x| \geq r_k$ and $t \in I$. Denote by

$$a_k(t) = k, \quad \text{for } t \in I \quad \text{and} \quad a_k(t) = l(t), \quad \text{for } t \in [0, 2\pi]\backslash I.$$

Then from the argument used in the proof of Proposition 1 (ii), one has $\rho(a_k) \geq \left[\frac{k\alpha}{2\pi}\right]$ where $\alpha = \text{mes}(I)$, which implies $\rho(a_k) \to +\infty$ as $k \to \infty$. The conclusion of Corollary 1.1 is thus proved. □

Example 4.1. The typical example of a differential equation with a partially superlinear nonlinearity, where Corollary 1.1 applies, is

$$x'' + l(t)x^3 = 0,$$

with $l(t) \geq 0$ for any $t \in [0, 2\pi]$ and $\int_0^{2\pi} l(t)dt > 0$.

Nevertheless, it is possible to deal with other examples. To this purpose, it is sometimes helpful to use the general polar coordinates

$$x = u\cos\varphi, \quad x' = \beta u \sin\varphi, \tag{4.2}$$

where $\beta > 0$. Then

$$\varphi_\beta'(t) = \frac{1}{\beta}(q_+(t)((\cos\varphi_\beta)^+)^2 + q_-(t)((\cos\varphi_\beta)^-)^2 + \beta^2 \sin^2\varphi_\beta).$$

Moreover,

$$|\varphi_\beta(t; t_0, \theta_0) - \theta(t; t_0, \theta_0)| < \pi/2, \quad \text{for} \quad t \in \mathbb{R},$$

which follows that

$$\rho(q) = \lim_{n\to\infty} \frac{\varphi_\beta(2n\pi; t_0, \varphi_\beta(t_0))}{2n\pi},$$

where $t_0, \varphi_\beta(t_0)$ can be selected arbitrarily. In this way, we can estimate $\rho(q)$ via φ_β. This approach is particularly appropriate in the following example.

Example 4.2. For any sufficiently large $m \in \mathbb{N}$, the equation

$$x'' + 4\sin t \frac{(x^+)^3}{1+x^2} + 4\cos t x^- = 0 \tag{4.3}$$

has two $2m\pi$-periodic solutions.

Proof. The key point is to compute the rotation numbers $\rho_\infty,\ \rho_0$ of the equations

$$x'' + a_+(t)x^+ - a_-(t)x^- = 0$$

and

$$x'' + b_+(t)x^+ - b_-(t)x^- = 0,$$

respectively, where $a_+(t) = 4\sin t,\ a_-(t) = -4\cos t,\ b_+(t) = 0,\ b_-(t) = -4\cos t$. Considering the argument function $\varphi_2(t; 0, \pi/2)$ via the general polar coordinates (4.2), we have

$$\varphi_2'(t; 0, \pi/2) = \frac{1}{2}(4\sin t((\cos\varphi_2)^+)^2 - 4\cos t((\cos\varphi_2)^-)^2 + 4\sin^2\varphi_2),$$

from where

$$\varphi_2'(t; 0, \pi/2) \geq -2\cos t, \quad \text{for} \quad \varphi_2(t; 0, \pi/2) \in [\pi/2, \pi + \pi/2],$$
$$\varphi_2'(t; 0, \pi/2) \geq 2\sin t, \quad \text{for} \quad \varphi_2(t; 0, \pi/2) \in [\pi + \pi/2, 2\pi + \pi/2].$$

This implies that

$$\varphi_2(\pi + \pi/2; 0, \pi/2) - \pi/2 > \int_{\pi/2}^{\pi+\pi/2} -2\cos t dt = 4 > \pi.$$

Moreover,

$$\varphi_2(t; 0, \pi/2) > \pi + \pi/2, \quad \text{for} \quad t > \pi + \pi/2.$$

If $\varphi_2(3\pi; 0, \pi/2) > 2\pi + \pi/2$ then $x(t) = u(t)\cos\varphi_2(t; 0, \pi/2)$ has three zeros in $[0, 3\pi)$, which implies that $\rho(a_\infty) > 0$ by Proposition 1 (iii). Otherwise, $\varphi_2(3\pi; 0, \pi/2) \leq 2\pi + \pi/2$. Then,

$$\pi + \pi/2 < \varphi_2(t; 0, \pi/2) \leq 2\pi + \pi/2, \quad \text{for} \quad t \in [2\pi, 3\pi],$$

which implies that $\varphi_2'(t; 0, \pi/2) \geq 2\sin t$ for $t \in [2\pi, 3\pi]$. Then,

$$\varphi_2(3\pi; 0, \pi/2) - \varphi_2(2\pi; 0, \pi/2) \geq \int_{2\pi}^{3\pi} 2\sin t dt = 4 > \pi.$$

This is a contradiction. Therefore, $\rho_\infty > 0$.

Using Lemma 4.3, it is easy to show that if $\rho_0 > 0$ then every solution $x(t; 0, x_0, x_0')$ of $x'' + 4\cos t x^- = 0$ has infinitely many zeros in $(0, +\infty)$. Now we observe that $x(t) = t$ satisfies that $x(0) = 0,\ x'(0) = 1$ and $x(t) > 0,\ x''(t) = 0$ for $t > 0$. Thus $x(t) = t$ is the solution of $x'' + 4\cos t x^- = 0$ for $t \in (0 + \infty)$. But $x(t) = t$ does not vanish on $(0 + \infty)$. Therefore we have proved that $\rho_0 = 0$.

From Theorem 1.1, for any $n/m \in (0, \rho_\infty)$, the equation (4.3) has infinitely many $2m\pi$-periodic solutions $x_{i,m,n}(t),\ i = 1, 2$, such that $x_{i,m,n}(t),\ i = 1, 2$ make exactly n turns around the origin in the time $[0, 2m\pi]$. □

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