Instability of closed orbits obtained by minimization*

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ABSTRACT. We study the dynamics around closed orbits of autonomous Lagrangian systems. When the configuration space is two-dimensional and orientable we show that every closed orbit minimizing the free-period action functional is orbitally unstable. This result applies even when the minimizers are degenerate or nonisolated, but a particularly strong form of instability holds in the isolated case. Under some symmetry assumptions, free-period action minimizers are unstable also in the higher-dimensional case. Applications to geodesics and Celestial Mechanics are given.

1 Introduction

Many connections between the dynamical and variational properties of solutions of various classes of dynamical systems are known. The purpose of this paper is to analyze the extent to which *dynamical instability* holds for *action-minimizing closed orbits* of autonomous Lagrangian systems.

The history of this problem is long and fruitful, starting with a classical 1887 result of Poincaré [43, §358] on the hyperbolicity of nondegenerate length-minimizing closed geodesics on orientable surfaces. Forty-five years later Hedlung [22, Theorem XVI] used variational arguments due to Morse [32] to show the existence of heteroclinics connecting (possibly degenerate) adjacent minimal closed geodesics on two-dimensional tori. In particular, it follows from these classical results that degenerate minimal closed geodesics are also unstable *provided that they are isolated*. The isolatedness assumption will be removed in Corollary 3.2, which also contains a reinforced instability result in the isolated case.

Poincaré's instability result for nondegenerate minimal geodesics was extended to even-dimensional orientable manifolds by Bolotin [7] and Treschev [49] (see also [11]). This result does not hold in the odd-dimensional case; in fact, there is a 1935 example by Carathéodory [13, §411] of a Riemannian metric on $\mathbb{R}^3$ and a closed, length-minimizing geodesic which is orbitally stable. However, under some

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symmetry conditions, Bolotin and Rabinowitz [9], [10] have extended Hedlung's theorem to all dimensions $d \geq 3$, showing that heteroclinics connecting isolated closed minimizing geodesics continue to exist in the higher-dimensional setting. In particular, these results imply that symmetric closed minimizing geodesics are unstable (in all space dimensions) *provided that they are isolated.* We shall prove instability without the isolatedness assumption in Corollary 3.3.

The dynamics of minimal closed geodesics have also been explored by means of Maslov index techniques. In this line, [28, Theorem 1.5] states that in a quite general framework (without symmetry assumptions) nondegenerate local minima of the periodic action are hyperbolic; [36] studies hyperbolicity in a symmetric framework, and [24] considers the possible extension of the Bolotin-Treschev instability results to the degenerate (parabolic) case[1].

The discussion above concerning with geodesic flows, a second question arises: *to what extent the principle of instability of minimizing geodesics can be extended to more general (autonomous) Lagrangian systems?* At a linear level, this problem has been treated by several authors, mainly motivated by the dynamics of the so-called Aubry-Mather-Mañé sets (see, e.g. [29], [16], [33, Section 2.6], [4]), and problems from Celestial Mechanics ([26], [23]). Maslov-index arguments have also been used in this setting, see, e.g., [45]. There is, however, an important obstruction to this general idea, and it is provided by Gordon's characterization of the (dynamically stable) elliptic orbits of the Kepler problem as minimizers of the fixed period action functional ([20]; further examples can be found in [3]). It points to the fact that some extra assumptions must be added to minimizing the *fixed period* action functional if one wants to predict dynamical instability.

In view of this situation we deal instead with (local) minimizers of the *free period action functional*

$$A_h(T,x) = \int_0^T \left[h + L(x(t),\dot{x}(t)) \right] dt \,, \tag{1}$$

where $L = L(x,\dot{x})$ is the Lagrangian function and $h \in \mathbb{R}$ is a parameter. The new domain is the set of couples (T,x) where $T > 0$ and $x = x(t)$ is a closed loop of period T. This functional has been previously considered in the literature (see, e.g. [15], [1]), and its critical points are the periodic solutions of any period having energy[2] h. While it is clear that any local minimizer (T_*, x_*) of A_h locally minimizes the fixed-period action functional $A_h(T_*,\cdot)$, the converse holds for geodesic flows (Lemma 3.1) but not in general (Example *4.* in Section 12; see also [30, Theorem 1.3] and [2, Proposition 2.1]). Theorems 2.1 and 2.3, which will

[1] Extra assumptions should be added in [28, Theorem 1.5] or [36, Theorem 3.1], see Example *3* in Section 12. On the other hand, [24, Theorem 1.1] uses in an essential way the fact that linear instability implies nonlinear instability, which is not always true in the parabolic situation, see Example *1* in Section 12.

[2] In fact, vanishing first-order variation with respect to x is equivalent to the Euler-Lagrange equations and zero first-order variation with respect to T is another way to say that the energy of the orbit is h. See also Lemma 9.2*(iii)*; part of this result was already present in [15, Proposition 3-3.1].

be the main results of this paper, roughly state that if the configuration space is either 2-dimensional and orientable, or symmetric of any dimension, then *all nonconstant minimizers of A_h are unstable*. This result does not require any nondegeneracy/isolatedness assumptions.

The main idea in our proofs consists in combining the classical principle of Jacobi-Maupertuis with an argument of reduction of order suggested by Carathéodory in [13, Chapter 17]. It can be seen that in the vicinity of a closed orbit with nowhere-vanishing velocity, the dynamics of an autonomous Lagrangian system inside a given energy level can be represented as the dynamics of a nonautonomous, time-periodic Lagrangian system in one degree less of freedom. This procedure, which is classical in the context of Hamiltonian systems (see, e.g., [5, §45-B]), reduces the problem of the instability of free-period minimizers to the Lyapunov-instability of action minimizers in time-periodic problems, an issue which has been the object of separate attention in the literature. In the nonautonomous framework we refer to the pioneering work of Carathéodory [13, §412-§413] on the linear approximation as well as to the more recent works [18], [38], [39], [50], [51] (for the one-dimensional problem), or [8], [35],[44], [46], [47], [52], [53] for the higher-dimensional situation.

2 Precise statement of the main results

Let the configuration space $M = M^d$ be a differentiable manifold of class C^3 without boundary (assumed connected and metrizable but not neccessarily orientable or compact); $2 \leq d := \dim M < +\infty$. Denote by TM its tangent bundle, let $\mathcal{O} \subset TM$ be an open set, and let the Lagrangian $L : \mathcal{O} \to \mathbb{R}$, $L = L(x, \dot{x})$ be a C^2 function satisfying the *Legendre convexity condition* on fibers, i.e.

$$\partial^2_{\dot{x}\dot{x}} L(x, \dot{x}) \text{ is positive definite for every } (x, \dot{x}) \in \mathcal{O}\,. \tag{2}$$

The associated Euler-Lagrange differential equations constitute a first-order system on $\mathcal{O}$ that in local tangent-bundle coordinates adopts the well-known form

$$\frac{d}{dt}\partial_{\dot{x}} L\big(x(t), \dot{x}(t)\big) = \partial_x L\big(x(t), \dot{x}(t)\big)\,. \tag{3}$$

We shall denote by

$$H : \mathcal{O} \to \mathbb{R}, \qquad (x, \dot{x}) \mapsto \partial_{\dot{x}} L(x, \dot{x})\dot{x} - L(x, \dot{x}), \tag{4}$$

the energy function associated to our Lagrangian L. Notice that it has class C^1 on $\mathcal{O}$. It is well-known that H is a first integral (i.e., constant along solutions) of our Euler-Lagrange system (3). In addition, one checks that

$$\frac{d}{d\mu} H(x, \mu\dot{x}) = \partial_{\dot{x}} H(x, \mu\dot{x})\dot{x} = \mu\, \partial^2_{\dot{x}\dot{x}} L(x, \mu\dot{x})(\dot{x}, \dot{x})\,, \qquad (x, \mu\dot{x}) \in \mathcal{O}. \tag{5}$$

The free-period action functional A_h has already appeared in (1). It is time to be precise concerning its domain, and with this purpose we consider the fiber bundle Λ of free closed loops with arbitrary periods. With formulas,

$$\Lambda := \bigsqcup_{T>0} C^1(\mathbb{R}/T\mathbb{Z}, M) = \big\{(T,x) : T>0,\ x\in C^1(\mathbb{R}/T\mathbb{Z}, M)\big\}\,.$$

The 'linear reparametrization map' $(T,x) \longleftrightarrow (T,\hat{x})$ where $\hat{x}(\theta) := x(T\theta)$, naturally identifies Λ with $]0,+\infty[\times C^1(\mathbb{R}/\mathbb{Z}, M)$, and consequently[3] Λ is endowed with a differentiable (of class C^2) structure. The (metrizable) topology of Λ is the natural one: a sequence $\{(T_n,x_n)\}_n \subset \Lambda$ converges to (T,x) if and only if $T_n \to T$ and $x_n(sT_n) \to x(sT)$ in the $C^1([0,1], M)$-topology. In this way the set

$$\Lambda_{\mathcal{O}} := \{(T,x)\in\Lambda : (x(t),\dot{x}(t))\in\mathcal{O}\ \forall t\in[0,T]\} \tag{6}$$

is open. We are interested in $\Lambda_{\mathcal{O}}$ because it is the natural domain of A_h; in fact, $A_h : \Lambda_{\mathcal{O}} \to \mathbb{R}$ has class C^2. As discussed in the Introduction, its critical points turn out to be the closed orbits (T,x) of (3) having energy $H(x(t),\dot{x}(t)) \equiv h$, see also Lemma 9.2*(iii)* below.

We remind that the nonconstant closed orbit (T_*,x_*) is called *isolated inside its energy level* provided that the corresponding fixed point of the Poincaré return map associated to a transversal section of the energy level is isolated. Equivalently, if for any sequence of closed orbits (T_n,x_n), all of them having the same energy as x_* and such that $T_n \to T_*$ and $x_n(sT_n) \to x_*(sT_*)$ in the $C^1([0,1],M)$ sense, one has that $T_n = T_*$ and $x_n(t) = x_*(t+\theta_n)$ for some sequence $\theta_n \to 0$ and sufficiently big n.

Similarly, *orbital instability* (also referred to as *dynamical instability* in the Introduction) is understood as the logical negation of the Lyapunov stability of the corresponding fixed point of the Poincaré section, and therefore, means both past and future instability, which are equivalent concepts in the measure-preserving context[4]. More explicitly, the nonconstant closed orbit x_* is called *orbitally unstable* (inside its energy level) provided that there exists an open neighborhood $N_0 \subset TM$ of the trajectory $\mathscr{T} := \{(x_*(t),\dot{x}_*(t)) : t\in\mathbb{R}\}$ such that, for any other open neighborhood N of $\mathscr{T}$ there is some solution $x = x(t)$ of (3) having the same energy as x_* and with $(x(0),\dot{x}(0))\in N$ but $(x(\tau),\dot{x}(\tau)) \notin N_0$ for some time τ. Roughly speaking, it means that there are solutions having the same energy of x_* and starting nearby, which get away from it at some (past or future) time.

We shall also consider a stronger notion of instability, referred to as *orbital instability in the sense of Siegel and Moser*, which was introduced by these authors in [48, §25]: the closed orbit x_* is unstable in this sense if there is an open neighborhood $N_0 \subset TM$ of the trajectory $\mathscr{T}$ such that *every* solution $x \not\equiv x_*$ having the same energy as x_* and satisfying $(x(0),\dot{x}(0))\in N_0$ leaves N_0 either in

[3]The Banach manifold structure of $C^1(\mathbb{R}/\mathbb{Z}, M)$ was first studied by Eells [19] and has been considered by many authors since. Under our assumption that M has class C^3 then $C^1(\mathbb{R}/\mathbb{Z}, M)$ has a natural C^2 structure [42], [25].

[4]See [40, p. 114-115]; the statements there are made for the 2-dimensional case but the argument keeps its validity for higher dimensions.

the past or in the future. In broad terms, it means that the only solution having the same energy as x_* and remaining close to it for all (past and future) time is x_* itself. Notice that it can happen only when the closed orbit x_* is isolated inside its energy level; on the other hand, hyperbolic closed orbits are always orbitally unstable in this reinforced sense.

Our first result, which extends Poincaré-Hedlung's theorem on the instability of isolated geodesics, is valid only in the two-dimensional orientable case. We emphasize that the concept of *local minimizer* is referred to the topology of $\Lambda_{\mathcal{O}}$ (or Λ); thus, the action on the closed loop x_* is compared to closed loops which have a similar period and, after linear reparametrization, are near in the C^1 topology.

Theorem 2.1. *Let M be an orientable surface. If $(T_*, x_*) \in \Lambda_{\mathcal{O}}$ is a nonconstant local minimizer of A_h, then x_* is orbitally unstable. If, moreover, (T_*, x_*) is isolated inside its energy level (but possibly degenerate), then it is orbitally unstable in the sense of Siegel and Moser.*

For higher-dimensional configuration manifolds ($d \geq 3$) we need to introduce some symmetry assumptions. Thus, let the C^3 diffeomorphism $\mathcal{S} : M \to M$ be involutive, i.e. $\mathcal{S}^2 = \mathrm{Id}_M$. By a theorem due to Bochner ([6, Theorem 1], see also [31, Lemma 2]), each connected component of the set $\mathrm{Fix}(\mathcal{S})$ of fixed points of $\mathcal{S}$ must be a C^3 submanifold of M. We shall assume that they are all hypersurfaces[5], i.e.

$$\dim \mathrm{Fix}(\mathcal{S}) = d - 1\,. \tag{7}$$

Moreover, let the Lagrangian L be $\mathcal{S}$-symmetric, meaning that

$$(x, \dot{x}) \in \mathcal{O} \implies \big(\mathcal{S}x, -(d\mathcal{S})_x\dot{x}\big) \in \mathcal{O} \text{ and } L(x, \dot{x}) = L\big(\mathcal{S}x, -(d\mathcal{S})_x\dot{x}\big)\,. \tag{8}$$

Equivalently, this condition can be restated as follows: $\mathcal{S}x(-t)$ is an extremal of (3) for any extremal $x = x(t)$. The pair $(T, x) \in \Lambda$ will be called *symmetric* provided that $x(t) = \mathcal{S}x(-t)$ for any t; we denote by $\Lambda_{\mathcal{O}}^{\mathcal{S}}$ the set of symmetric elements $(T, x) \in \Lambda_{\mathcal{O}}$. Being the set of fixed points of the involutive map $(T, x(t)) \mapsto (T, \mathcal{S}x(-t))$, the infinite-dimensional version of Bochner's theorem ([41, Theorem 5.3]) states that each connected component of $\Lambda_{\mathcal{O}}^{\mathcal{S}}$ is a C^2-submanifold of Λ. In addition, Palais' principle of symmetric criticality ([41, Theorem 5.4]) ensures that the critical points of $A_h\big|_{\Lambda_{\mathcal{O}}^{\mathcal{S}}}$ coincide with the $\mathcal{S}$-symmetric critical points of A_h. This equivalence can be extended to minimizers, as the following result shows:

Lemma 2.2. *Let $(T_*, x_*) \in \Lambda_{\mathcal{O}}^{\mathcal{S}}$ be given. Then (T_*, x_*) is a local minimizer of A_h on $\Lambda_{\mathcal{O}}$ if and only if (T_*, x_*) is a local minimizer of $A_h\big|_{\Lambda_{\mathcal{O}}^{\mathcal{S}}}$.*

In order to avoid breaking the pace of the presentation, the proof of this lemma is posponed to the first subsection of the Appendix, at the end of the paper. It allows us to use the expression *symmetric local minimizer* with any of these equivalent meanings. In this paper we also study the dynamics around symmetric local minimizers.

[5] This assumption has been previously considered by Arnaud [4].

Theorem 2.3. *Every nonconstant symmetric local minimizer* $(T_*, x_*) \in \Lambda_{\mathcal{O}}^{\mathcal{S}}$ *is orbitally unstable. Moreover, the associated Floquet multipliers are real and positive.*

We recall that the Floquet multipliers associated to the closed orbit $x_* \in \Lambda_{\mathcal{O}}^{\mathcal{S}}$ are the eigenvalues of the linearized Poincaré map associated to a transversal section inside the energy level. It follows that *if the minimizing closed orbit* x_* *is nondegenerate, then it is hyperbolic.*

Many classical variational methods are designed to find minimizers of the action functional when the period is fixed. Motivated by this fact let us assume next that, either,

(a) M is orientable and 2-dimensional, and Ξ is a nontrivial[6] connected component of $C^1(\mathbb{R}/\mathbb{Z}, M)$, or

(b) there exists an involutive C^3-map $\mathcal{S} : M \to M$ with (7) and (8), and Ξ is a nontrivial connected component of $\{x \in C^1(\mathbb{R}/\mathbb{Z}, M) : x \text{ is } \mathcal{S}\text{-symmetric}\}$.

For any $T > 0$ we consider the open set Ξ_T (assumed nonempty) of closed loops $x \in C^1(\mathbb{R}/T\mathbb{Z}, M)$ such that $(T, x) \in \Lambda_{\mathcal{O}}$ and $\hat{x}(\theta) := x(\theta T)$ belongs to Ξ.

Corollary 2.4. *Assume that* $m(T) := \min_{x \in \Xi_T} A_h(T, x)$ *exists for all* $T > 0$*, and* $\lim_{T \to 0} m(T) = +\infty = \lim_{T \to +\infty} m(T)$*. Then (3) has an orbitally unstable closed orbit* $x \in \Xi_T$*, for some* $T > 0$*.*

Admitting the validity of Theorems 2.1-2.3, the proof of this result follows from the fact that the continuous and coercive function $m :]0, +\infty[\to \mathbb{R}$ must attains its global minimum, which is the minimum of A_h.

The proof of Theorems 2.1 and 2.3 will be divided in several steps, which will be distributed along Sections 5-11. Before closing this section we announce that the question concerning to the necessity of the various assumptions of Theorems 2.1 and 2.3 will be addressed by means of several examples in Section 12. For instance, the orientability of M (required in Theorem 2.1) is shown to be necessary in Example *2*. The symmetry conditions (7)-(8) of Theorem 2.3 cannot be skipped; this is shown in Example *3*. The need of considering the free-period action functional instead of the more classical fixed-period action functional is clear from Gordon's characterization of the elliptic orbits of the Kepler problem as minimizers of the action (Example *4*). The assumption on x_* being not an equilibrium is also necessary[7], as it will be checked in Example *5*. When comparing Theorems 2.1 and 2.3, it seems reasonable to conjecture the orbital instability in the sense of Siegel and Moser of higher-dimensional isolated symmetric minimizers: in general this is not true, as it will be shown in Example *6*.

[6] We call *trivial* to the connected component of $C^1(\mathbb{R}/\mathbb{Z}, M)$ containing the constant loops. When the fundamental group of M is nontrivial, then $C^1(\mathbb{R}/\mathbb{Z}, M)$ is disconnected ([21, Section 1.1, Problem 6]).

[7] However, if the Lagrangian L is assumed reversible in time, i.e., $L(x, \dot{x}) = L(x, -\dot{x})$, then the equilibria which minimize the (fixed-period) action functional are Lyapunov-unstable [52, Corollary 2.3], and the corresponding Floquet multipliers are real and positive [53, Corollary 1.2].

3 Length-minimizing geodesics

Throughout this section $M = M^d$ stands for a Riemannian manifold of class C^3, with associated Riemannian norm $\|\cdot\|$. We set $\mathcal{O} := TM$ and assume that $\|\cdot\|^2 : TM \to \mathbb{R}$ has class C^2. In addition to the free-period action functional

$$A_h : \Lambda \to \mathbb{R}, \qquad (T, x) \mapsto \int_0^T \left(h + \frac{1}{2}\|\dot{x}(t)\|^2 \right) dt$$

associated to each energy level h, the length functional

$$\mathcal{A} : C^1(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}, \qquad x \mapsto \int_0^1 \|\dot{x}(\theta)\| d\theta$$

has an obvious geometrical significance. In this context, it is well-known that a closed loop $x_* : \mathbb{R}/\mathbb{Z} \to M$ having constant speed $\|\dot{x}(\theta)\| \equiv v > 0$ is a geodesic if and only if any of the following equivalent conditions holds: *(a):* $(1, x_*)$ is a critical point of the free-period action functional $A_{v^2/2}$; *(b):* x_* is a critical point of the fixed-period action functional $A_0(1, \cdot) : C^1(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$; *(c):* x_* is a critical point of the length functional $\mathcal{A}$. The following elementary lemma extends this equivalence to action minimizers.

Lemma 3.1. *Let M be a Riemannian manifold and let $x_* : \mathbb{R}/\mathbb{Z} \to M$ be a closed geodesic with (constant) speed $\|\dot{x}_*(\theta)\| \equiv v > 0$. Then, the following statements are equivalent:*

(i) $(1, x_*)$ *is a local minimizer of the free-period action functional* $A_{v^2/2} : \Lambda \to \mathbb{R}$,

(ii) x_* *is a local minimizer of the fixed-period energy action functional:* $A_0(1, \cdot) : C^1(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$,

(iii) x_* *is a local minimizer of the length action functional* $\mathcal{A} : C^1(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$.

In order to keep the rhythm of the exposition, the proof is postponed to the second subsection of the Appendix, at the end of the paper. Closed geodesics satisfying any of these equivalent statements will be called *length-minimizing geodesics* in what follows. By assuming the validity of Theorem 2.1 we immediately obtain the following refinement of Poincaré-Hedlung's planar instability theorem:

Corollary 3.2. *If M is 2-dimensional and orientable, length-minimizing geodesics in M are always unstable. Moreover, if a length-minimizing geodesic is isolated, then it unstable in the sense of Siegel and Moser.*

Assume now that the (possibly higher-dimensional) Riemannian manifold M is endowed with an isometry $\mathcal{S} : M \to M$ with (7) and such that $\mathcal{S}^2 = \mathrm{Id}_M$. In this setting, Theorem 2.3 gives rise to the following

Corollary 3.3. *Length-minimizing symmetric geodesics in M are always unstable. In addition, the associated Floquet multipliers are real and positive.*

Under some topological conditions, the existence of length-minimizing geodesics can be ensured. In the following result we assume that the Riemannian manifold M is compact and one of the two following possibilities holds:

(a') M is orientable and 2-dimensional, and Ξ is a nontrivial connected component of $C^1(\mathbb{R}/\mathbb{Z}, M)$, or

(b') There exists an isometry $\mathcal{S} : M \to M$ with (7) and such that $\mathcal{S}^2 = \mathrm{Id}_M$, and Ξ is a nontrivial connected component of the set $\{x \in C^1(\mathbb{R}/\mathbb{Z}, M) : x \text{ is } \mathcal{S}\text{-symmetric}\}$.

It is well-known that either *(a')* or *(b')* implies the existence of a length-minimizing geodesic in Ξ. We arrive to the following

Corollary 3.4. *If either (a') or (b') holds, then M has a closed geodesic $x \in \Xi$ which is unstable and whose Floquet multipliers are real and positive.*

For mechanical Lagrangians $L(x, \dot{x}) := \|\dot{x}\|^2/2 - V(x)$ one obtains a instability result by applying the previous corollary to the Jacobi metric

$$\| \cdot \|_h := \sqrt{h - V(x)}\, \| \cdot \| \,. \tag{9}$$

In the last result of this section we keep the framework and assumptions of Corollary 3.4 to which we add a new one: the potential $V : M \to \mathbb{R}$ has class C^3 and, in case *(b')*, satisfies $V \circ \mathcal{S} = V$. Adapting the terminology of the previous section for any $T > 0$ we set

$$\Xi_T := \{x \in C^1(\mathbb{R}/T\mathbb{Z}, M) \text{ such that } \hat{x}(\theta) := x(\theta T) \text{ belongs to } \Xi\} \,.$$

Corollary 3.5. *Under the above, for any energy level $h > \max_M V$ there exists some $T > 0$ such that the Lagrangian system (3) has a a closed orbit $x_* \in \Xi_T$ which is orbitally unstable and whose Floquet multipliers are real and positive.*

4 Unstable periodic orbits in Celestial Mechanics

We illustrate Corollary 2.4 with an application concerning the instability of minimizing orbits in some problems from Celestial Mechanics. We consider the restricted planar problem defined by a given number N of primaries with masses $m_1, m_2, \ldots, m_N > 0$ rotating solidly around the origin at angular speed $\omega = 1$. Using complex notation, the positions of these primaries are $e^{it}q_1, \ldots, e^{it}q_N$, where $q_1, \ldots, q_N$ are arbitrary different points of the complex plane $\mathbb{C}$ (identified with $\mathbb{R}^2$), which may or may not conform a central configuration.

We study the motion of a massless particle $z = z(t)$ which is attracted by the primaries through the inverse $p > 3$ (strong force) law, and our interest lies on trajectories which look periodic from a reference system rotating solidly with the primaries. Using this rotating reference system, and after some normalization, the Lagrangian governing the motion of the massless particle is given by

$$L(z,\dot z) = \frac{1}{2}|\dot z + iz|^2 + \sum_{k=1}^{N} \frac{m_k}{|z-q_k|^{p-1}}\,, \qquad z \in M := \mathbb{C}\backslash\{q_1,\dots,q_N\},\ \dot z \in \mathbb{C}\,.$$

The free loop space $C^1(\mathbb{R}/\mathbb{Z}, M)$ is divided into infinitely many connected components (or, what is the same, homotopy classes). We denote by Ξ^j the homotopy class of $\xi^j(\theta) := re^{-2\pi j\theta i}$ for $r > \max_{1\le k\le N}|q_k|$, i.e. Ξ^0 stands for the homotopy class of the constant loops while Ξ^j denotes the homotopy class of closed loops rotating around the convex envelope of the primaries j times in the clockwise sense. Given a connected component $\Xi \subset C^1(\mathbb{R}/\mathbb{Z}, M)$ and $T>0$ we denote by Ξ_T the set of closed loops $z \in C^1(\mathbb{R}/T\mathbb{Z}, M)$ such that $\hat z(\theta) := z(\theta T)$ belongs to Ξ. For simplicity we set $\Xi^j_T := (\Xi^j)_T$.

Finally, we recall that the Jacobi constant

$$\mathscr{J} = \frac{1}{2}|z|^2 + \sum_{k=1}^{N} \frac{m_k}{|z-q_k|^{p-1}} - \frac{1}{2}|\dot z|^2\,,$$

is a first integral of the motion.

Theorem 4.1. *Under the above, for each value $\mathscr{J} < 0$ of the Jacobi constant and for any homotopy class $\Xi \subset C^1(\mathbb{R}/\mathbb{Z}, M)$ with $\Xi \ne \Xi^j$ for every $j \ge 0$, there is some $T>0$ and some orbitally unstable closed orbit $z \in \Xi_T$ having this value of the Jacobi constant.*

Proof. We start by noticing that the energy associated to an extremal is its Jacobi constant with the reversed sign; thus, fixing $\mathscr{J} < 0$ is equivalent to fixing the energy $h = -\mathscr{J} > 0$. The free-period action functional is given by

$$A_h(T,z) := \frac{1}{2}\int_0^T |\dot z(t) + iz(t)|^2 dt + \sum_{k=1}^{N}\int_0^T \frac{m_k}{|z(t)-q_k|^{p-1}}dt - T\mathscr{J}\,, \quad (T,z) \in \Lambda\,.$$

In order to apply Corollary 2.4 in the planar, non-symmetric case *(a)*, two assumptions must be checked:

[1.] $A_h(T,\cdot)$ attains its minimum on Ξ_T for every $T>0$. This fact is essentially due to Poincaré, see also [14, Section 3]. Notice that this part of the argument actually works for $p \ge 3$.

Fix $T>0$ and let $\{z_n\}_n \subset \Xi_T$ be a minimizing sequence, i.e., $A_h(T,z_n) \to \inf_{z\in\Xi_T} A_h(T,z)$.

We start by noticing that $\{z_n\}_n$ is bounded on the Sobolev space $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{C})$; this assertion follows from the inequality

$$A_h(T,z_n) \ge \frac{1}{2}\int_0^T |\dot z_n(t) + iz_n(t)|^2 dt\,, \qquad n \in \mathbb{N}\,.$$

Indeed, if T is not an integer multiple of 2π, then the usual Wirtinger inequalities imply that the right hand side is bounded from below by $c\|z_n\|_{H^1}$, where $c > 0$ is some constant not depending on n, and the result follows. If, on the other hand, $T = 2\pi j$ is an integer multiple of 2π, we can write $z_n(t) = \ell_n e^{-it} + \eta_n(t)$, where $\ell_n \in \mathbb{C}$ and $\{\eta_n\}$ is bounded on $H^1(\mathbb{R}/2\pi j\mathbb{Z}, \mathbb{C})$. Then, if z_n were not bounded, after possibly passing to a subsequence one may assume that $|\ell_n| \to +\infty$. It follows that $z_n \in \Xi^j_{2\pi j}$ for n big enough, and consequently, $\Xi_{2\pi j} = \Xi^j_{2\pi j}$, implying that $\Xi = \Xi^j$, which contradicts the assumptions.

As a consequence, after possibly passing to a subsequence, one may assume that $\{z_n\} \to z_*$ weakly on $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$ (and in particular, uniformly on $\mathbb{R}$). We claim that $z_*(t) \in M$ for any $t \in \mathbb{R}$, i.e., z_* does not have collisions with the primaries. We check it by a contradiction argument and assume instead the existence of collisions, say, with the first mass q_1. If we assume that, actually, $z_*(t) \equiv q_1$, then the inequality $A_h(T, z_n) \geq m_1 \int_0^T \frac{1}{|z_n(t)-q_1|^{p-1}} dt$ implies that $A_h(T, z_n) \to +\infty$, a contradiction. Thus, there must be numbers $t_1 < t_2 < t_1 + T$ such that $z_*(t_1) = q_1$, $z_*(t_2) \neq q_1$ and $|z_*(t) - q_1| < 1$ for any $t \in [t_1, t_2]$. For n big enough, $|z_n(t) - q_1| < 1$ for every $t \in [t_1, t_2]$, and one has

$$A_h(T, z_n) \geq \frac{1}{2} \int_{t_1}^{t_2} |\dot{z}_n(t) + i z_n(t)|^2 dt + \int_{t_1}^{t_2} \frac{m_1}{|z_n(t) - q_1|^2} dt \,, \tag{10}$$

where we have used the assumption $p \geq 3$. We set $w_n(t) := e^{it}(z_n(t) - q_1)$; the parallelogram identity $|u - iv|^2 = |u|^2 + |v|^2 - 2\,\mathrm{im}(u\overline{v})$ leads us to

$$|\dot{w}_n(t)|^2 = |\dot{z}_n(t) + i z_n(t)|^2 + |q_1|^2 - 2\,\mathrm{im}\big[(\dot{z}_n(t) + i z_n(t))\overline{q_1}\big] \,,$$

and using the inequality $|z_n(t)| \leq |q_1| + 1$ we see that, for every $t \in [t_1, t_2]$,

$$|\dot{z}_n(t) + i z_n(t)|^2 \geq |\dot{w}_n(t)|^2 + 2\,\mathrm{im}(\dot{z}_n(t)\,\overline{q_1}) - 3|q_1|^2 - 2|q_1| \,.$$

On the other hand,

$$\int_{t_1}^{t_2} \mathrm{im}(\dot{z}_n(t)\,\overline{q_1}) dt = \mathrm{im}(z_n(t_2)\overline{q_1}) - \mathrm{im}(z_n(t_1)\overline{q_1}) \geq -2|q_1|(|q_1| + 1) \,,$$

and (10) implies that

$$A_h(T, z_n) \geq \int_{t_1}^{t_2} \left[\frac{1}{2}|\dot{w}_n(t)|^2 + \frac{m_1}{|w_n(t)|^2}\right] dt - (T + 1)(5|q_1|^2 + 4|q_1|) \,.$$

The last summand on the right term does not depend on n. On the other hand, using the inequality $a^2 + b^2 \geq 2ab$, we see that the integral I_n in the inequality above can be estimated as follows:

$$I_n \geq \sqrt{2m_1} \int_{t_1}^{t_2} \frac{|\dot{w}_n(t)|}{|w_n(t)|} dt \geq \sqrt{2m_1} \int_{t_1}^{t_2} \left| \frac{d}{dt}\big(\log|w_n(t)|\big) \right| dt \geq$$
$$\geq \sqrt{2m_1} \Big| \log|w_n(t_2)| - \log|w_n(t_1)| \Big| \to +\infty \quad \text{as } n \to +\infty \,.$$

Consequently, $A_h(T,z_n)\to+\infty$ as $n\to+\infty$, which is not possible since z_n was a minimizing sequence. It follows from here that z_* does not have collisions, and we deduce, on the first hand, that $z_*\in\Xi_T$, and on the second, that $A_h(T,\cdot)$ (which is lower-semicontinuous with respect to the H^1 topology) attains on z_* its minimum over Ξ_T.

[2.] $a(T):=\min_{\Xi_T}A_h(T,\cdot)$ diverges at $T=0$ and at $T=+\infty$. Since $a(T)\geq -T\mathscr{J}$ and we assumed $\mathscr{J}<0$, the fact that $a(T)\to+\infty$ as $T\to+\infty$ is immediate. Thus, let us assume, by a contradiction argument, that there is a sequence $T_n\to 0$ and a constant $K>0$ such that $a(T_n)\leq K$ for any $n\in\mathbb{N}$. Set $a(T_n)=A_h(T_n,z_n)$ for some $z_n\in\Xi_{T_n}$, define $u_n:[0,T_n]\to\mathbb{C}$ by $t\mapsto e^{it}z_n(t)$, and observe that

$$\int_0^{T_n}|\dot u_n(t)|^2dt=\int_0^{T_n}|\dot z_n(t)+iz_n(t)|^2dt\leq 2A_h(T_n,z_n)\leq 2K\,,\qquad n\in\mathbb{N}\,,$$

so that, by the Cauchy-Schwarz inequality,

$$|u_n(t)-u_n(0)|\leq\int_0^{T_n}|\dot u_n(t)|dt\leq\sqrt{2K}\sqrt{T_n}\,,\qquad t\in[0,T_n],\ n\in\mathbb{N}\,,$$

and, using the inequality

$$|z_n(t)-z_n(0)|=|e^{it}z_n(t)-e^{it}z_n(0)|\leq|u_n(t)-u_n(0)|+|1-e^{it}||z_n(0)|\,,$$

we obtain that $|z_n(t)-z_n(0)|\leq\sqrt{2K}\sqrt{T_n}+|z_n(0)|\,T_n$ for any $t\in\mathbb{R}/T_n\mathbb{Z}$ and any $n\in\mathbb{N}$.

If we assume now that $|z_n(0)|\to+\infty$, we obtain that $z_n\in\Xi^0_{T_n}$ for n big enough, a contradiction. Thus, after possibly passing to a subsequence, we may suppose that $z_n(0)$ is bounded, and this implies the existence of some constant $R>0$ such that

$$|z_n(t)-z_n(0)|\leq R\sqrt{T_n}\,,\qquad t\in\mathbb{R}/T_n\mathbb{Z}\,,\ n\in\mathbb{N}\,. \tag{11}$$

Now, if one assumes that, for some natural index n, the inequality $|z_n(0)-q_k|>R\sqrt{T_n}$ holds for every $1\leq k\leq N$, we see that $z_n\in\Xi^0_{T_n}$, which is a contradiction. It follows that, after possibly passing to a subsequence, we may assume that $|z_n(0)-q_1|\leq R\sqrt{T_n}$ for every n. Thus, (11) gives

$$|z_n(t)-q_1|\leq 2R\sqrt{T_n}\,,\qquad t\in\mathbb{R}/T_n\mathbb{Z}\,,\ n\in\mathbb{N}\,,$$

and the inequality $A_h(T_n,z_n)\geq m_1\int_0^{T_n}\frac{1}{|z_n(t)-q_1|^{p-1}}dt$, together with the fact that $p>3$, implies now that $a(T_n)=A_h(T_n,z_n)\to+\infty$ as $n\to+\infty$. This is a contradiction and concludes the proof. □

In connection with Theorem 4.1, Llibre and Stoica [27] have used continuation methods to study the dynamics around both *comet orbits*, i.e., the almost-circular closed orbits of large amplitude, and *Hill's orbits*, the almost-circular closed orbits of small amplitude rotating a primary; they all are shown to be unstable for $p>3$. In comparison, we deal with more types of orbits and our result is not of perturbative nature.

5 Free-period minimizers do not have cusp points

Let us go back to the general framework set in Section 2 and let M be a differentiable manifold, let $\mathcal{O} \subset TM$ be an open set, and let the C^2 Lagrangian $L : \mathcal{O} \to \mathbb{R}$ satisfy the Legendre convexity condition (2). The energy function $H : \mathcal{O} \to \mathbb{R}$ is defined as in (4). In this section we prepare the proof of Theorems 2.1-2.3 by observing that the trajectories of free-period minimizers are regular curves in the geometrical sense. More precisely:

Proposition 5.1. *Let $x_* \in \Lambda_{\mathcal{O}}$ be a local minimizer of the free-period action functional A_h. If x_* is not an equilibrium, then*

$$\dot{x}_*(t) \neq 0 \qquad \textit{for any } \; t \in \mathbb{R}. \tag{12}$$

In the reversible case $L(x, -\dot{x}) = L(x, \dot{x})$, periodic solutions which travel back and forth over some set $\mathscr{C} \cong [0, 1]$ are sometimes called *brake orbits*. Such orbits cannot be free-period minimizers, in view of the above. It must be noticed, however, that Proposition 5.1 holds also in a non-reversible setting. Observe also that it does not require symmetry assumptions, or restrictions on the dimension of M.

An observation before the proof of Proposition 5.1. Let $x_* \in \Lambda_{\mathcal{O}}$ and $t_0 \in \mathbb{R}$ be given, and assume that we are given families of positive numbers $\{a_\rho\}$, $\{b_\rho\}$, $\{\tau_\rho\}$, all three defined for small $\rho > 0$ and converging to zero as $\rho \to 0_+$. Assume also that for small $\rho > 0$ there exists some curve $\gamma_\rho \in C^1([0, \tau_\rho], M)$ (not necessarily extremal) with

(i) $\gamma_\rho(0) = x_*(t_0 - a_\rho)$, $\;\gamma_\rho(\tau_\rho) = x_*(t_0 + b_\rho)$, for each ρ,

(ii) $\max_{0 \le t \le \tau_\rho} |\gamma_\rho(t) - x_*(t_0)| + \max_{0 \le t \le \tau_\rho} |\dot{\gamma}_\rho(t) - \dot{x}_*(t_0)| \to 0$ as $\rho \to 0_+$,

(iii) $\int_0^{\tau_\rho} (h + L(\gamma_\rho(t), \dot{\gamma}_\rho(t)))dt < \int_{t_0 - a_\rho}^{t_0 + b_\rho} (h + L(x_*(t), \dot{x}_*(t)))dt$ for $\rho > 0$ small.

Then x_* is not a local minimizer of the free-period action functional A_h.

The proof of this statement follows easily from a regularization argument, but it has the following important consequence: in order to prove Proposition 5.1 we only have to check that families $a_\rho, b_\rho, \tau_\rho, \gamma_\rho$ as above do exist if one assumes that the nonconstant extremal $x_* = x_*(t)$ satisfies $\dot{x}_*(t_0) = 0$ and has energy h (this latter condition being necessary for local minimizers; see Lemma 9.2*(iii)* below). It will be our task next.

Proof of Proposition 5.1. In view of the observation above, which reduces the problem to a local question, there is no loss of generality in assuming that $M = \mathbb{B}_1^d \subset \mathbb{R}^d$ is the unit ball and $\mathcal{O} = \mathbb{B}_1^d \times \mathbb{B}_1^d \subset \mathbb{R}^d \times \mathbb{R}^d$. In order to simplify the notation we shall further assume that $x_* = x_*(t)$ is a nonconstant extremal with energy $h = 0$ and $x_*(0) = 0 = \dot{x}_*(0)$. In particular, $L(0, 0) = 0$. Additionally, we shall assume that

$$\partial_{\dot{x}} L(0, 0) = 0 \,, \tag{13}$$

since otherwise it suffices to replace $L(x,\dot{x})$ with $\tilde{L}(x,\dot{x}) := L(x,\dot{x}) - \partial_{\dot{x}}L(0,0)\dot{x}$, which has the same Euler-Lagrange equations, the same free-period action functional and the same free-period minimizers.

We consider the C^2 function $V : \mathbb{B}_1^d \to \mathbb{R}$ defined by $V(x) := H(x,0) = -L(x,0)$. Uniqueness implies that

$$\nabla V(0) \neq 0\,, \tag{14}$$

and the Euler-Lagrange equations (3) give $\ddot{x}_*(0) \neq 0$; in fact, (2) leads to $\langle \nabla V(0), \ddot{x}_*(0)\rangle < 0$. Thus,

$$V(x_*(0)) = 0\,, \qquad \frac{d}{dt}\Big|_{t=0} V(x_*(t)) = 0\,, \qquad \frac{d^2}{dt^2}\Big|_{t=0} V(x_*(t)) < 0\,.$$

Therefore, there exists some $\rho_0 > 0$ and strictly-increasing functions $a, b : [0,\rho_0[\to [0,+\infty[$ of class C^1 such that $a(0) = b(0) = 0$, $a'(0) = b'(0) > 0$, and $V(x_*(-a(\rho))) = V(x_*(b(\rho))) = -\rho^2$ for any $\rho \in [0,\rho_0[$. By combining l'Hôpital's rule with Lemma 13.1 in the Appendix we see that there exists some constant $0 < \kappa < +\infty$ such that

$$\lim_{\rho\to 0_+} \frac{1}{\rho^3}\int_{-a(\rho)}^{b(\rho)} L(x_*(t),\dot{x}_*(t))dt = \kappa\,. \tag{15}$$

We also observe that

$$\begin{aligned}\lim_{\rho\to 0_+} \frac{x_*(b(\rho)) - x_*(-a(\rho))}{\rho^2} &= \lim_{\rho\to 0_+} \left[\left(\frac{b(\rho)}{\rho}\right)^2 \frac{x_*(b(\rho))}{b(\rho)^2} - \left(\frac{a(\rho)}{\rho}\right)^2 \frac{x_*(-a(\rho))}{(-a(\rho))^2}\right] \\ &= (b'(0)^2 - a'(0)^2)\ddot{x}_*(0)/2 = 0\,,\end{aligned}$$

since $a'(0) = b'(0) > 0$. Furthermore, the points $x_*(-a(\rho))$ and $x_*(b(\rho))$ both belong to the C^2-hypersurface $V_\rho := \{x \in \mathbb{B}_1^d : V(x) = -\rho^2\}$, and, in view of (14), these hypersurfaces continue from V_0 in a C^2-fashion. Thus, for small $\rho > 0$ it is possible to find $\tau_\rho > 0$ and a C^1 curve $\gamma_\rho : [0,\tau_\rho] \to \mathbb{B}_1^d$ with $\gamma_\rho(0) = x_*(-a(\rho))$, $\gamma_\rho(\tau_\rho) = x_*(b(\rho))$, $\dot{\gamma}_\rho(t) \neq 0$ for every $t \in [0,\tau_\rho]$,

$$H(\gamma_\rho(t),0) = V(\gamma_\rho(t)) = -\rho^2 \text{ for any } t \in [0,\tau_\rho]\,, \tag{16}$$

and such that

$$\int_0^{\tau_\rho} |\dot{\gamma}_\rho(t)|dt = o(\rho^2), \qquad \text{as } \rho \to 0_+\,. \tag{17}$$

In this way *(i)* holds trivially. In addition, $\gamma_\rho(0) = x_*(-a(\rho)) = o(\rho)$ as $\rho \to 0_+$, and we deduce that

$$\max_{0\le t\le\tau_\rho} |\gamma_\rho(t)| = o(\rho) \text{ as } \rho \to 0_+\,. \tag{18}$$

Remembering (2) and (5) we deduce that for any $x, \dot{x} \in \mathbb{B}_1$ with $\dot{x} \neq 0$, the function $\mu \mapsto H(x,\mu\dot{x})$ is strictly increasing on $[0,1]$. Expressions (2) and (5) also imply the existence of constants $0 < \epsilon_1 < 1$, $0 < k < K$ such that

$$k|\dot{x}|^2 \le H(x,\dot{x}) - V(x) \le K|\dot{x}|^2\,, \qquad |x| \le \epsilon_1,\ |\dot{x}| < 1\,. \tag{19}$$

Thus, after possibly replacing ϵ_1 by an smaller constant there exists some $\epsilon_2 \in]\epsilon_1, 1[$ such that $H(x,\dot{x}) > 0$ if $|x| \le \epsilon_1$ and $\epsilon_2 \le |\dot{x}| < 1$. By (18), $\max_{0\le t\le\tau_\rho} |\gamma_\rho(t)| < \epsilon_1$ for small $\rho > 0$, and by (16) we see that

$$H(\gamma_\rho(t), 0) < 0 < H(\gamma_\rho(t), \dot{x}), \text{ for all } t \in [0,\tau_\rho], \text{ all } \epsilon_2 \le |\dot{x}| < 1 \text{ and small } \rho > 0.$$

Therefore, after reparametrization we may assume that

$$H(\gamma_\rho(t), \dot{\gamma}_\rho(t)) = 0, \ |\dot{\gamma}_\rho(t)| < \epsilon_2\,, \qquad \text{for all } t \in [0,\tau_\rho] \text{ and small } \rho > 0. \tag{20}$$

The inequality in the previous line concerning $|\dot{\gamma}_\rho(t)|$ can be improved, since the combination of (16), (19) and (20) shows the existence of constants $0 < \kappa_1 < \kappa_2$ such that

$$\kappa_1\rho \le |\dot{\gamma}_\rho(t)| \le \kappa_2\rho \text{ for every } t \in [0,\tau_\rho] \text{ and small } \rho > 0\,. \tag{21}$$

Remembering (18) we see that *(ii)* holds. Moreover, by (17) we obtain that $\tau_\rho = o(\rho)$ as $\rho \to 0_+$. On the other hand, by the first part of (20) and (13) we see that

$$L(\gamma_\rho(t), \dot{\gamma}_\rho(t)) = \partial_{\dot{x}} L(\gamma_\rho(t), \dot{\gamma}_\rho(t))\dot{\gamma}_\rho(t) =$$
$$= \int_0^1 \partial_{x\dot{x}} L(s\gamma_\rho(t), s\dot{\gamma}_\rho(t))(\gamma_\rho(t), \dot{\gamma}_\rho(t))ds + \int_0^1 \partial_{\dot{x}\dot{x}} L(s\gamma_\rho(t), s\dot{\gamma}_\rho(t))(\dot{\gamma}_\rho(t), \dot{\gamma}_\rho(t))ds,$$

and the estimates in (18) and (21) imply the existence of some constant $c > 0$ such that $|L(\gamma_\rho(t), \dot{\gamma}_\rho(t))| \le c\rho^2$ for every $t \in [0,\tau_\rho]$ and sufficiently small $\rho > 0$. Thus,

$$\left|\int_0^{\tau_\rho} L(\gamma_\rho(t), \dot{\gamma}_\rho(t))dt\right| \le c\rho^2\tau_\rho = o(\rho^3) \text{ as } \rho \to 0_+\,.$$

When combined with (15), it shows *(iii)* and completes the proof. □

6 The Jacobi-Maupertuis Lagrangian

For any $(x,\dot{x}) \in \mathcal{O}$ with $\dot{x} \ne 0$, the set

$$\mathcal{I}(x,\dot{x}) := \Big\{\mu > 0 : \big(x, (1 + s(\mu-1))\dot{x}\big) \in \mathcal{O} \ \forall s \in [0,1]\Big\}\,,$$

is an open interval containing $\mu = 1$. Moreover, in view of (5) the energy function $H : \mathcal{O} \to \mathbb{R}$ satisfies

$$\frac{d}{d\mu} H(x, \mu\dot{x}) > 0\,, \qquad (x,\dot{x}) \in \mathcal{O}, \ \dot{x} \ne 0, \ \mu \in \mathcal{I}(x,\dot{x})\,. \tag{22}$$

We fix some energy level $h \in \mathbb{R}$ and consider the sets $\mathcal{N}_h \subset \mathcal{O}_h \subset \mathcal{O}$ defined by

$$\mathcal{N}_h := \{(x,\dot{x}) \in \mathcal{O} : \dot{x} \ne 0 \text{ and } H(x,\dot{x}) = h\}\,,$$
$$\mathcal{O}_h := \Big\{(x,\dot{x}) \in \mathcal{O} : \dot{x} \ne 0 \text{ and } H(x,\mu\dot{x}) = h \text{ for some } \mu \in \mathcal{I}(x,\dot{x})\Big\}\,.$$

It follows from (22) that $\mathcal{O}_h$ is open in TM. Let the maps $\lambda_h : \mathcal{O}_h \to \mathbb{R}$ and $\Pi_h : \mathcal{O}_h \to \mathcal{N}_h$ be defined by

$$\lambda_h(x,\dot{x}) \in \mathcal{I}(x,\dot{x}) \text{ and } H(x,\lambda_h(x,\dot{x})\dot{x}) = h, \qquad \Pi_h(x,\dot{x}) := (x,\lambda_h(x,\dot{x})\dot{x})\,.$$

They are both of class C^1, and Π_h is a retraction of $\mathcal{O}_h$ into the energy level set $\mathcal{N}_h$. Some further properties of the function λ_h, the intervals $\mathcal{I}(x,\dot{x})$ and the set $\mathcal{O}_h$ are collected below:

Lemma 6.1. *Given $(x,\dot{x}) \in \mathcal{O}_h$ and $\mu \in \mathcal{I}(x,\dot{x})$ one has*

$$\begin{cases} (x,\mu\dot{x}) \in \mathcal{O}_h, \quad \mathcal{I}(x,\mu\dot{x}) = \dfrac{1}{\mu}\mathcal{I}(x,\dot{x}), \quad \lambda_h(x,\mu\dot{x}) = \dfrac{\lambda_h(x,\dot{x})}{\mu}, \\ \dfrac{h + L(x,\mu\dot{x})}{\mu} \geq \dfrac{h + L(x,\lambda_h(x,\dot{x})\dot{x})}{\lambda_h(x,\dot{x})}\,. \end{cases}$$

Proof. The three assertions in the first line are easy to check. Concerning the last inequality we define $\varphi : \mathcal{I}(x,\dot{x}) \to \mathbb{R}$ by $\varphi(\mu) := (L(x,\mu\dot{x}) + h)/\mu$ and observe that $\varphi'(\mu) = (H(x,\mu\dot{x}) - h)/\mu^2$, so that, by (22),

$$\varphi'(\mu) = 0 \Leftrightarrow \mu = \lambda_h(x,\dot{x})\,, \qquad \varphi''(\lambda_h(x,\dot{x})) > 0\,.$$

We deduce that $\varphi'(\mu) > 0$ if $\mu > \lambda_h(x,\dot{x})$ and $\varphi'(\mu) < 0$ if $\mu < \lambda_h(x,\dot{x})$, which implies the result. $\square$

The *Jacobi-Maupertuis* Lagrangian $\mathscr{L}_h : \mathcal{O}_h \to \mathbb{R}$ is defined as follows:

$$\mathscr{L}_h(x,\dot{x}) := \frac{h + L(x,\lambda_h(x,\dot{x})\dot{x})}{\lambda_h(x,\dot{x})} = \partial_{\dot{x}} L(x,\lambda_h(x,\dot{x})\dot{x})\dot{x}\,, \qquad (x,\dot{x}) \in \mathcal{O}_h\,. \quad (23)$$

It follows from Lemma 6.1 that $\mathscr{L}_h$ is *locally homogeneous of degree* 1 *on* $\dot{x}$ in the sense that

$$\mathscr{L}_h(x,\mu\dot{x}) = \mu\mathscr{L}_h(x,\dot{x}) \text{ for all } (x,\dot{x}) \in \mathcal{O}_h \text{ and } \mu \in \mathcal{I}(x,\dot{x})\,. \quad (24)$$

In addition, it is clear from (23) that $\mathscr{L}_h$ has class C^1. In the result below we show that $\mathscr{L}_h$ has some additional unsuspected regularity.

Lemma 6.2. *$\mathscr{L}_h \in C^2(\mathcal{O}_h)$. Moreover, $\partial^2_{\dot{x}\dot{x}}\mathscr{L}_h(x,\dot{x})(v,v) > 0$ if $v \notin \mathbb{R}\dot{x}$.*

Proof. Assuming, with no loss of generality, that $M = \mathbb{B}^d_1$ is the unit ball in $\mathbb{R}^d$ and $\mathcal{O} = \mathbb{B}^d_1 \times \mathbb{B}^d_1$, some computations give

$$\begin{cases} \partial_x \lambda_h(x,\dot{x}) = \dfrac{\partial_x L(x,\lambda_h\dot{x}) - \lambda_h\, \partial^2_{x\dot{x}} L(x,\lambda_h\dot{x})\dot{x}}{\lambda_h\, \partial^2_{\dot{x}\dot{x}} L(x,\lambda_h\dot{x})(\dot{x},\dot{x})}, \\ \partial_{\dot{x}} \lambda_h(x,\dot{x}) = -\dfrac{\lambda_h\, \partial^2_{\dot{x}\dot{x}} L(x,\lambda_h\dot{x})\dot{x}}{\partial^2_{\dot{x}\dot{x}} L(x,\lambda_h\dot{x})(\dot{x},\dot{x})}\,, \end{cases}$$

(for the sake of simplicity, in the right hand sides we write λ_h instead of $\lambda_h(x,\dot{x})$). From here one checks the equalities

$$\partial_x \mathscr{L}_h(x,\dot{x}) = \frac{1}{\lambda_h(x,\dot{x})}\partial_x L(x,\lambda_h(x,\dot{x})\dot{x})\,, \qquad \partial_{\dot{x}} \mathscr{L}_h(x,\dot{x}) = \partial_{\dot{x}} L(x,\lambda_h(x,\dot{x})\dot{x})\,,$$

so that $\mathscr{L}_h$ has class C^2, as claimed. Now, we observe that

$$\partial^2_{\dot{x}\dot{x}} \mathscr{L}_h(x,\dot{x})(v,v) = \frac{\lambda_h(x,\dot{x})}{(\dot{x}|\dot{x})}\Big[(v|v)(\dot{x}|\dot{x}) - (v|\dot{x})^2\Big]\,,$$

where we denote $(v|w) := \partial^2_{\dot{x}\dot{x}} L(x,\lambda_h(x,\dot{x})\dot{x})(v,w)$; for $(x,\dot{x}) \in \mathcal{O}_h$ fixed this is a scalar product on T_xM. The result follows from the Cauchy-Schwarz inequality. □

In many physical applications the Lagrangian L has the form

$$L(x,\dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + E(x)\dot{x} - V(x)\,, \qquad (x,\dot{x}) \in \mathcal{O} = TM\,,$$

where $\|\cdot\|^2$ stands for a Riemannian quadratic form, $E : M \to T^*M$ is a linear form, and $V : M \to \mathbb{R}$ is the potential, all three assumed of class C^2. Then, $H(x,\dot{x}) = \|\dot{x}\|^2/2 + V(x)$, and, setting $M_h := \{x \in M : V(x) < h\}$ one has

$$\mathscr{L}_h(x,\dot{x}) = \sqrt{2(h - V(x))}\,\|\dot{x}\| + E(x)\dot{x}\,, \qquad (x,\dot{x}) \in \mathcal{O}_h = TM_h\,.$$

Notice in particular that in the special (mechanical) case in which E vanishes, $\mathscr{L}_h$ is ($\sqrt{2}$ times) the length associated to the classical Jacobi metric (9).

7 Local structure of the phase space around a closed extremal

Let $x_* \in C^2(\mathbb{R}/\mathbb{Z}, M)$ with (12) be a closed orbit of the Lagrangian system (3) having energy h. We assume, with no loss of generality, that $T_* = 1$ is the minimal period of x_*. In addition, let one of the following assumptions (corresponding respectively to Theorems 2.1 and 2.3) hold:

($\overline{a}$): M is orientable and 2-dimensional,

($\overline{b}$): There exists a C^3 involution $\mathcal{S} : M \to M$ satisfying (7)-(8) and such that x_* is $\mathcal{S}$-symmetric.

Under any of these conditions, it is possible to find some small radius $0 < \epsilon < 1$ and a C^2 map $\psi : (\mathbb{R}/\mathbb{Z}) \times \mathbb{B}^{d-1}_\epsilon \to M$, $\psi = \psi(\theta, y)$ satisfying[8]:

[ψ_1] $\psi(\theta, 0) = x_*(\theta)$ for every $\theta \in \mathbb{R}/\mathbb{Z}$.

[8]We denote by $\mathbb{B}^{d-1}_\epsilon$ the ball of radius ϵ in $\mathbb{R}^{d-1}$.

[ψ_2] The restriction of ψ to $]\theta-\epsilon/2,\theta+\epsilon/2[\times\mathbb{B}_\epsilon^{d-1}$ is a diffeomorphism into its open image, for all $\theta\in\mathbb{R}/\mathbb{Z}$.

[ψ_3] $\partial_\theta\psi,\partial_{y_i}\psi:(\mathbb{R}/\mathbb{Z})\times\mathbb{B}_\epsilon^{d-1}\to TM$ (for $1\le i\le d-1$), are $C^{0,2}$ maps.

Moreover, if $(\bar{b})$ holds then the map ψ can be chosen with the additional property

[ψ_4] $\psi(-\theta,y)=\mathcal{S}(\psi(\theta,y))$ for any $(\theta,y)\in(\mathbb{R}/\mathbb{Z})\times\mathbb{B}_\epsilon^{d-1}$.

Even though the closed loop x_* might have self-intersections, we shall see in the following section that curves which are close to x_* in the C^1-sense can be lifted via ψ to curves in the (θ,y) space. The differential of ψ is defined on $(\mathbb{R}/\mathbb{Z})\times\mathbb{B}_\epsilon^{d-1}\times\mathbb{R}\times\mathbb{R}^{d-1}$ as follows:

$$d\psi(\theta,y,\dot\theta,\dot y):=\Big(\psi(\theta,y),\partial_\theta\psi(\theta,y)\dot\theta+\partial_y\psi(\theta,y)\dot y\Big)\,.$$

Since the closed loop $\mathbb{R}/\mathbb{Z}\to\mathcal{O}$, $\theta\mapsto d\psi(\theta,0,1,0)=(x_*(\theta),\dot x_*(\theta))$ is simple (by uniqueness), and the differential of $d\psi$ at each point $(\theta,0,1,0)$ is a linear isomorphism from $\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}^{d-1}$ into $T_{(x_*(\theta),\dot x_*(\theta))}(TM)$, the semilocal version of the inverse function theorem implies that, after possibly replacing ϵ with some smaller number, there exists some $0<\epsilon'<1$ such that $d\psi$ establishes a diffeomorphism from

$$\Omega_*:=\Big\{(\theta,y,\dot\theta,\dot y)\in(\mathbb{R}/\mathbb{Z})\times\mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}^{d-1}:|\dot\theta-1|<\epsilon',\ |y|<\epsilon,\ |\dot y|<\epsilon\dot\theta\Big\}\quad(25)$$

into its image $\mathcal{O}_*:=d\psi(\Omega_*)$, assumed to be an open subset of $\mathcal{O}_h$. Alternatively, (setting $\dot y=\dot\theta\zeta$) one can reformulate this condition by considering the set $\mathbb{O}:=(\mathbb{R}/\mathbb{Z})\times\mathbb{B}_\epsilon^{d-1}\times\mathbb{B}_\epsilon^{d-1}$ and the map $\Psi:\mathbb{O}\times]1-\epsilon',1+\epsilon'[\to\mathcal{O}_*$ defined by

$$\Psi(\theta,y,\zeta;\lambda):=\Big(\psi(\theta,y),\lambda(\partial_\theta\psi(\theta,y)+\partial_y\psi(\theta,y)\zeta)\Big)\,.$$

Then, the requirement that $d\psi:\Omega_*\to\mathcal{O}_*$ is a diffeomorphism means that Ψ is a C^1-diffeomorphism. We may think of $\mathcal{O}_*\subset TM$ as a solid tube around the orbit, and Ψ as providing, in some way, 'cylindrical coordinates' for $\mathcal{O}_*$. Replacing again $\epsilon>0$ with some smaller quantity we may further assume that

$$1-\epsilon'<\lambda_h\big(\Psi(\theta,y,\zeta;1)\big)<1+\epsilon',\qquad(\theta,y,\zeta)\in\mathbb{O}\,.\tag{26}$$

Lemma 7.1. *The set $\mathcal{N}_*:=\mathcal{N}_h\cap\mathcal{O}_*=\{(x,\dot x)\in\mathcal{O}_*:H(x,\dot x)=h\}$ is a C^1-hypersurface of TM, and the map*

$$\Xi:\mathbb{O}\to\mathcal{N}_*,\qquad(\theta,y,\zeta)\mapsto(\Pi_h\circ\Psi)\big(\theta,y,\zeta;1\big)\,,$$

is a C^1-diffeomorphism.

Proof. We start by observing that, as a consequence of (26), the image of Ξ is contained in $\mathcal{O}_*$. The result follows immediately. $\square$

Before concluding this section we consider the C^1 submanifolds $\mathcal{N}_0 \subset \mathcal{O}_0$ of TM defined as follows:

$$\mathcal{O}_0 := \Psi\Big(\{0\} \times \mathbb{B}_\epsilon^{d-1} \times \mathbb{B}_\epsilon^{d-1} \times]1-\epsilon', 1+\epsilon'[\Big)\,, \qquad \mathcal{N}_0 := \mathcal{O}_0 \cap \mathcal{N}_*\,.$$

They can be regarded as C^1 'slices' of $\mathcal{O}_*$ and $\mathcal{N}_*$ respectively. Moreover, the maps $\Psi_0 : \mathbb{B}_\epsilon^{d-1} \times \mathbb{B}_\epsilon^{d-1} \times]1-\epsilon', 1+\epsilon'[\to \mathcal{O}_0$ and $\Xi_0 : \mathbb{B}_\epsilon^{d-1} \times \mathbb{B}_\epsilon^{d-1} \to \mathcal{N}_0$ defined by

$$\Psi_0(y, \zeta; \lambda) := \Psi(0, y, \zeta; \lambda)\,, \qquad \Xi_0(y, \zeta) := \Xi(0, y, \zeta)\,, \tag{27}$$

are C^1-diffeomorphisms. The fact that (by (25)),

$$\dot{\theta} > 0 \text{ for all } (\theta, y, \dot{\theta}, \dot{y}) \in \Omega_*\,, \tag{28}$$

implies that $\mathcal{O}_0$ and $\mathcal{N}_0$ are transversal surfaces of sections for the flow of the Lagrangian system (3) restricted to $\mathcal{O}_*$ and $\mathcal{N}_*$, respectively.

8 A Banach fiber bundle of curves

In order to study dynamics one has to keep track of *all* solutions near a given one, and not only the periodic ones. For this reason we consider, for each $T > 0$, the Banach manifold $C^1([0, T], M)$ of (not necessarily closed) curves. These manifolds are pasted together to yield the fiber bundle

$$\widehat{\Gamma} := \bigsqcup_{T>0} C^1([0,T], M) := \big\{(T, x) : T > 0,\ x \in C^1([0,T], M)\big\}\,.$$

Exploiting some ideas already presented in Section 2, $\widehat{\Gamma}$ may be identified with $]0, +\infty[\times C^1([0,1], M)$ by means of the transformation

$$(T, x) \in \widehat{\Gamma} \longleftrightarrow (T, \hat{x}) \in]0, +\infty[\times C^1([0,1], M), \quad \text{where } \hat{x}(\theta) := x(\theta T)\,.$$

In particular, $\widehat{\Gamma}$ is naturally endowed with a C^2-differentiable structure modeled on the Banach space $\mathbb{R} \times C^1([0,1], \mathbb{R}^d)$. For the purposes of this paper we shall be interested in the subset

$$\Gamma := \left\{(T, x) \in \widehat{\Gamma} : \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathcal{O}_0 \text{ for } t \in \{0, T\},\ \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathcal{O}_* \backslash \mathcal{O}_0 \ \forall t \in]0, T[\right\}\,. \tag{29}$$

In addition to being a (C^1, codimension 2)-submanifold of $\widehat{\Gamma}$, Γ can be covered by a single (infinite-dimensional) coordinate patch. To check this out we consider the affine space

$$\mathfrak{F} := \Big\{(T, \varkappa) \in \mathbb{R} \times C^1([0,1], \mathbb{R}^d) : \varkappa(0) \in \{0\} \times \mathbb{R}^{d-1},\ \varkappa(1) \in \{1\} \times \mathbb{R}^{d-1}\Big\},$$

endowed with the topological and differentiable structures inherited from the parallel Banach space $\mathbb{R} \times \{\tilde{\varkappa} \in C^1([0,1], \mathbb{R}^d) : \tilde{\varkappa}(\theta) \in \{0\} \times \mathbb{R}^{d-1} \text{ for } \theta = 0, 1\}$ (we use the identification $\mathbb{R}^d \equiv \mathbb{R} \times \mathbb{R}^{d-1}$). The set

$$\mathfrak{G} := \left\{(T, \varkappa) \in \mathfrak{F} : T > 0,\ \Big(\varkappa(\theta), \frac{1}{T}\dot{\varkappa}(\theta)\Big) \in \Omega_* \ \ \forall \theta \in [0,1]\right\}\,,$$

is open in $\mathfrak{F}$ and contains the point $(1, \varkappa_*)$ where $\varkappa_*(\theta) := (\theta, 0_{\mathbb{R}^{d-1}})$. Let $\Phi : \mathfrak{G} \to \widehat{\Gamma}$ be defined by

$$\Phi(T, \varkappa) := (T, x), \qquad \text{where } x(t) := (\psi \circ \varkappa)\left(\frac{t}{T}\right), \ t \in [0, T].$$

Lemma 8.1. *$\Phi(\mathfrak{G}) = \Gamma$, and $\Phi : \mathfrak{G} \to \Gamma$ is a C^1-diffeomorphism. Moreover, a point $(T, \varkappa) \in \mathfrak{G}$ belongs to $]0, +\infty[\times C^2([0,1], \mathbb{R}^d)$ if and only if $\Phi(T, x) \in]0, +\infty[\times C^2([0,T], M)$.*

Proof. We start by observing that Φ has class C^1 because ψ has class C^2 (see [25, Property (B), p. 74]). Let $(T, \varkappa) \in \mathfrak{G}$ be given and write $\varkappa := (\tau, y)$; since τ is strictly increasing on $[0,1]$ (by (28)), $\tau(0) = 0$ and $\tau(1) = 1$, we deduce that $\tau(\theta) \in]0, 1[$ for all $\theta \in]0, 1[$. It follows that $\Phi(T, \varkappa) \in \Gamma$.

In order to check that $\Phi(\mathfrak{G}) = \Gamma$ and construct Φ^{-1} choose $(T, x) \in \Gamma$ and set $(\mathfrak{t}, y, \dot{\mathfrak{t}}, \dot{y}) := (d\psi)^{-1} \circ (x, \dot{x}) : [0, T] \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1}$. Observe that, as the notation suggests, $\dot{\mathfrak{t}}$ and $\dot{y}$ are the time derivatives of $\mathfrak{t}$ and y respectively. The C^1-function $\mathfrak{t} : [0, T] \to \mathbb{R}/\mathbb{Z}$ satisfies that $\mathfrak{t}(0) = 0 + \mathbb{Z} = \mathfrak{t}(T)$ and therefore its C^1-lifting $\tau : [0, T] \to \mathbb{R}$ with $\tau(0) = 0$ satisfies $\tau(T) \in \mathbb{Z}$. But τ is strictly increasing (since $0 < 1 - \epsilon' < \dot{\tau}(t) = \dot{\mathfrak{t}}(t)$), and $\tau(t) \notin \mathbb{Z}\ \forall t \in]0, T[$ (by (29)), and we deduce that $\tau(T) = 1$. Thus, setting

$$\varkappa(\theta) := \big(\tau(\theta T), y(\theta T)\big), \qquad \theta \in [0, 1],$$

we see that $(T, \varkappa) \in \mathfrak{G}$ and the C^1-map $\Gamma \to \mathfrak{G}$ defined by $(T, x) \mapsto (T, \varkappa)$ is the inverse of Φ. It proves the first part of the result.

Concerning the 'Moreover...' statement we observe that in case that $\varkappa \in C^2([0,1], \mathbb{R}^d)$ then $x \in C^2([0,T], M)$ since ψ has class C^2. Conversely, if $x \in C^2([0,T], M)$ then both $\varkappa$ and $\dot{\varkappa}$ are C^1 maps (since $d\psi$ has class C^1). It completes the proof. □

We conclude this section by pointing out a couple of facts related to this result:

(i) Lemma 8.1 can be thought of as a 'lifting lemma': given $(T, x) \in \Gamma$ there exists an unique C^1 function $\varkappa : [0,1] \to \mathbb{R} \times \mathbb{R}^{d-1}$, $\varkappa(s) := (\mathfrak{t}(s), y(s))$, with $\mathfrak{t}(0) = 0,\ \mathfrak{t}(1) = 1$, and such that

$$\left(\varkappa(t/T), \frac{\dot{\varkappa}(t/T)}{T}\right) \in \Omega_*, \quad x(t) = (\psi \circ \varkappa)(t/T), \qquad t \in [0, T].$$

(ii) Readers unfamiliar with the differentiable structure of the set of paths on a manifold may simply think of Γ as endowed with the pull-forward differentiable structure associated to the bijective map Φ. In this paper we shall not essentially use the fact that this differentiable structure coincides with the usual one.

9 Reparametrizations

We shall begin with a definition. Two elements $(T_a, x_a), (T_b, x_b) \in \Gamma$ will be called *reparametrizations* of each other provided that there exists a C^1-diffeomorphism $\sigma : [0, T_b] \to [0, T_a]$ with

$$\sigma(0) = 0, \quad \sigma(T_b) = T_a, \qquad \dot{\sigma}(t) \in \mathcal{I}\Big(x_a\big(\sigma(t)\big), \dot{x}_a\big(\sigma(t)\big)\Big) \text{ for any } t \in [0, T_b]\,, \tag{30}$$

and such that $x_b = x_a \circ \sigma$. It follows from Lemma 6.1 that this is an equivalence relation on Γ. In the following result we check that every element of Γ can be uniquely reparametriced to constant energy h:

Lemma 9.1. *The following hold:*

(i) Every element $(T_a, x_a) \in \Gamma$ admits a unique reparametrization $(T_b, x_b) = \mathcal{R}(T_a, x_a)$ such that $H(x_b(t), \dot{x}_b(t)) = h$ for all $t \in [0, T_b]$.

(ii) The map $\mathcal{R} : \Gamma \to \Gamma$ defined in this way is a continuous retraction into

$$\Gamma(h) := \big\{(T, x) \in \Gamma : H(x(t), \dot{x}(t)) = h \ \forall t \in [0, T]\big\}\,.$$

(iii) If $(T_a, x_a) \in \Gamma$ is such that $x_a \in C^2([0, T_a], M)$, then $\mathcal{R}$ is differentiable at (T_a, x_a), and $(T_b, x_b) := \mathcal{R}(T_a, x_a)$ satisfies that $x_b \in C^2([0, T_b], M)$.

Proof. Writing $x_b := x_a \circ \sigma$ where $\sigma : [0, T_b] \to \mathbb{R}$ has class C^1, we see that (30) holds and $(T_b, x_b) \in \Gamma(h)$, if and only if σ solves the boundary value problem

$$\dot{\sigma} = \lambda_h(x_a(\sigma), \dot{x}_a(\sigma))\,, \qquad \sigma(0) = 0,\ \sigma(T_b) = T_a\,. \tag{31}$$

The right hand side of this separable equation is continuous and positive, implying uniqueness for initial value problems (see, e.g., [17, Theorem 1.2]). The above looks like an overdetermined problem, but the point is that $T_b > 0$ is one of the unknowns of the problem. Thus, there is a unique solution (T_b, σ) to (31). In addition, $(T_b, x_a \circ \sigma) \in \Gamma$ by Lemma 7.1, meaning that $\mathcal{R} : \Gamma \to \Gamma$ is well-defined. It proves *(i)*.

It is now clear that all points $(T, x) \in \Gamma$ having constant energy h are fixed by $\mathcal{R}$. In order to check the claimed continuity of $\mathcal{R}$ as well as *(c)*, we use the chart $\Phi : \mathfrak{G} \to \Gamma$. Some elementary calculus show that $(T_b, \varkappa_b) := (\Phi^{-1} \circ \mathcal{R} \circ \Phi)(T_a, \varkappa_a)$ is defined by:

- $T_b = T_a \displaystyle\int_0^1 \frac{1}{(\lambda_h \circ d\psi)\big(\varkappa_a(s), \dot{\varkappa}_a(s)/T_a\big)} ds$, and
- $\varkappa_b = \varkappa_a \circ \mathfrak{s}$, where the diffeomorphism $\mathfrak{s} : [0, 1] \to [0, 1]$ is defined by

$$\dot{\mathfrak{s}} = \frac{T_b}{T_a} (\lambda_h \circ d\psi)\big(\varkappa_a(\mathfrak{s}), \dot{\varkappa}_a(\mathfrak{s})/T_a\big)\,, \qquad \mathfrak{s}(0) = 0\,.$$

The result follows. $\square$

The free-period action functional $A_h : \Lambda_{\mathcal{O}} \to \mathbb{R}$ associated to the Lagrangian function L and the energy level h (defined by (1)) can be naturally extended to nonperiodic curves; in this way we may think of it as defined on Γ. We shall be particularly interested in the action of A_h on the set Γ_p of periodic curves

$$\Gamma_p := \big\{(T,x) \in \Gamma : x(0) = x(T),\ \dot{x}(0) = \dot{x}(T)\big\}$$

and on the sets $\Gamma(x_0,x_1)$ of Dirichlet curves

$$\Gamma(x_0,x_1) := \Big\{(T,x) \in \Gamma : x(0) = x_0, \quad x(1) = x_a\Big\}\,,$$

where $x_0 = \psi(0,y_0)$ and $x_1 = \psi(0,y_1) \in \psi(\{0\} \times \mathbb{B}^{d-1}_\epsilon)$ are arbitrary.

The usual identification $x \mapsto x\big|_{[0,T]}$ between closed loops $x \in C^1(\mathbb{R}/T\mathbb{Z}, M)$ and curves $x \in C^1([0,T], M)$ with $x(0) = x(T)$ and $\dot{x}(0) = \dot{x}(T)$ makes it possible to see Γ_p as an open subset of $\Lambda_{\mathcal{O}}$, the set of closed loops considered in (6). Notice that under this identification, the new definition of the action functional A_h is coherent with its previous meaning. The result below extends [15, Proposition 3-3.1].

Lemma 9.2. *For a fixed energy level $h \in \mathbb{R}$ the following hold:*

(i) $A_h \in C^1(\Gamma)$.

(ii) Γ_p *and* $\Gamma(x_0,x_1)$ *are* C^1*-submanifolds of* Γ.

(iii) The element $(T,x) \in \Gamma_p$ is a critical point of $A_h\big|_{\Gamma_p}$ (or the element $(T,x) \in \Gamma(x_0,x_1)$ is a critical point of $A_h\big|_{\Gamma(x_0,x_1)}$), if and only if $x \in \Gamma(h)$ and solves the Euler-Lagrange equations (3).

Proof. Straightforward computations show that

$$(A_h \circ \Phi)(T,\varkappa) = Th + T\int_0^1 L\left(\psi\big(\varkappa(s)\big), \frac{1}{T}\psi'\big(\varkappa(s)\big)\dot{\varkappa}(s)\right) ds\,, \quad (T,\varkappa) \in \mathfrak{G}\,. \tag{32}$$

In this way *(i)* follows from Lemma 8.1.

Concerning *(ii)* and in view of Lemma 8.1 we can equivalently check that $\mathfrak{G}_p := \Phi^{-1}(\Gamma_p)$ and $\mathfrak{G}(y_0,y_1) := \Phi^{-1}(\Gamma(x_0,x_1))$ are C^1-submanifolds of $\mathfrak{G}$. One checks that

$$\mathfrak{G}_p = \big\{(T,\varkappa) \in \mathfrak{G} : \varkappa(1) - \varkappa(0) \in \{1\} \times \{0_{\mathbb{R}^{d-1}}\},\ \dot{\varkappa}(0) = \dot{\varkappa}(1)\big\}\,,$$
$$\mathfrak{G}(y_0,y_1) = \big\{(T,\varkappa) \in \mathfrak{G} : \varkappa(0) = (0,y_0),\ \varkappa(1) = (1,y_1)\big\}\,,$$

and the result follows.

Regarding *(iii)* we set $(T,\varkappa) := \Phi^{-1}(T,x)$, so that $(T,x) \in \Gamma_p$ is a critical point of $A_h\big|_{\Gamma_p}$ (respectively, $(T,x) \in \Gamma(x_0,x_1)$ is a critical point of $A_h\big|_{\Gamma(x_0,x_1)}$) if and only if $(T,\varkappa)\in\mathfrak{G}_p$ is a critical point of $(A_h\circ\Phi)\big|_{\mathfrak{G}_p}$ (resp., $(T,\varkappa)\in\mathfrak{G}(y_0,y_1)$ is a critical point of $(A_h\circ\Phi)\big|_{\mathfrak{G}(y_0,y_1)}$). Next, it follows from (32) that

$$\frac{\partial (A_h\circ\Phi)|_{\mathfrak{G}_p}}{\partial\varkappa}(T,\varkappa)=0 \qquad \Big(\text{resp., } \frac{\partial (A_h\circ\Phi)|_{\mathfrak{G}(y_0,y_1)}}{\partial\varkappa}(T,\varkappa)=0\Big),$$

if and only if $x=x(t)$ is a solution of the Euler-Lagrange equations (3). On the other hand, the partial derivative with respect to T of $(A_h\circ\Phi)|_{\mathfrak{G}_p}$ (resp., $(A_h\circ\Phi)|_{\mathfrak{G}(y_0,y_1)}$) is given by

$$h-\frac{1}{T}\int_0^T H(x(t),\dot{x}(t))dt,$$

and the result follows from the fact that the energy is a first integral for the Euler-Lagrange flow.

□

We consider now the (free-period) action functional $\mathscr{A}_h\in C^2(\Gamma)$, defined by

$$\mathscr{A}_h(T,x):=\int_0^T \mathscr{L}_h(x(t),\dot{x}(t))dt\,,\qquad (T,x)\in\Gamma\,.$$

where $\mathscr{L}_h:\mathcal{O}_h\to\mathbb{R}$ is the Jacobi-Maupertuis Lagrangian, already considered in (23). To conclude this section we shall check that the retraction $\mathcal{R}$ lowers the value of A_h; on the other hand $\mathscr{A}_h$ encapsulates the action of A_h on curves with constant energy h. More formally, one has the following

Lemma 9.3. $\mathscr{A}_h(T,x)=A_h\big(\mathcal{R}(T,x)\big)\le A_h(T,x)\,,\qquad (T,x)\in\Gamma.$

Proof. It follows from (24) that $\mathscr{A}_h$ is invariant by reparametrizations. Thus,

$$\mathscr{A}_h(T,x)=\mathscr{A}_h(\mathcal{R}(T,x))=A_h\big(\mathcal{R}(T,x)\big)\,,\qquad (T,x)\in\Gamma\,.$$

Set now $(T_h,x_h):=\mathcal{R}(T,x)$ and write $x_h=x\circ\sigma$, where $\sigma:[0,T_h]\to[0,T]$ is an increasing diffeomorphism. Then, changing variables in the integral we get:

$$\begin{aligned}A_h(T,x)&=\int_0^T\big(h+L(x(t),\dot{x}(t))\big)dt=\int_0^{T_h}\Big(h+L\big(x(\sigma(t)),\dot{x}(\sigma(t))\big)\Big)\dot{\sigma}(t)dt=\\&=\int_0^{T_h}\left(h+L\left(x_h(t),\frac{\dot{x}_h(t)}{\dot{\sigma}(t)}\right)\right)\dot{\sigma}(t)dt\ge A_h(T_h,x_h)\,,\end{aligned}$$

where we have used the inequality in the second line of Lemma 6.1. It completes the proof. □

10 Carathéodory's reduction of order

Following some ideas already appearing in [13, §404], the (nonautonomous) *Carathéodory's reduced Lagrangian* associated to the energy level h is constructed from the Jacobi-Maupertuis Lagrangian $\mathscr{L}_h$ of Section 6 as follows:

$$\mathbb{L}_h : \mathbb{O} \to \mathbb{R}, \qquad (\theta, y, \dot{y}) \mapsto \mathscr{L}_h\big(\psi(\theta, y), \partial_\theta \psi(\theta, y) + \partial_y \psi(\theta, y)\dot{y}\big)\,.$$

Here, $\mathbb{O} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{B}^{d-1}_\epsilon \times \mathbb{B}^{d-1}_\epsilon$ is the open set considered in Section 7. One has the following

Lemma 10.1. *Seeing $\mathbb{O}$ as an open subset of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^{2(d-1)}$, the Lagrangian $\mathbb{L}_h : \mathbb{O} \to \mathbb{R}$ is a $C^{0,2}$ function satisfying the Legendre convexity condition:*

$$\partial^2_{\dot{y}\dot{y}} \mathbb{L}_h(\theta, y, \dot{y}) > 0\,, \qquad (\theta, y, \dot{y}) \in \mathbb{O}\,.$$

Moreover, if assumptions $(\bar{b})$ and $[\psi_4]$ of Section 7 hold, then $\mathbb{L}_h$ is time-reversible, i.e.

$$\mathbb{L}_h(-\theta, y, -\dot{y}) = \mathbb{L}_h(\theta, y, \dot{y})\,, \qquad (\theta, y, \dot{y}) \in \mathbb{O}\,.$$

Proof. The combination of Lemma 6.2 and assumption $[\psi_3]$ implies that $\mathbb{L}_h$ has class $C^{0,2}$ on $\mathbb{O}$ and satisfies the Legendre convexity condition. On the other hand, the 'moreover...' statement is easy to prove. $\square$

The Euler-Lagrange equations associated to $\mathbb{L}_h$ are given by

$$\frac{d}{ds} \partial_{\dot{y}} \mathbb{L}_h(\theta, y, \dot{y}) = \partial_y \mathbb{L}_h(\theta, y, \dot{y})\,, \qquad (\theta, y, \dot{y}) \in \mathbb{O}\,, \tag{33}$$

and the corresponding (fixed-period) action functional is defined as follows:

$$\mathbb{A}_h[y] := \int_0^1 \mathbb{L}_h(\theta, y(\theta), \dot{y}(\theta))\, ds\,.$$

The natural domain of $\mathbb{A}_h$ is the set

$$\mathbb{G} := \{y \in C^1([0,1], \mathbb{R}^{d-1}) : (\theta, y(\theta), \dot{y}(\theta)) \in \mathbb{O}\ \forall \theta \in [0,1]\},$$

which is open in $C^1([0,1], \mathbb{R}^{d-1})$. Consider now the map $\Upsilon : \mathbb{G} \to \Gamma(h)$ defined by

$$\Upsilon[y] := (\mathcal{R} \circ \Phi)(1, \varkappa_y)\,, \qquad \text{where } \varkappa_y(\theta) := (\theta, y(\theta)),\ 0 \le \theta \le 1\,. \tag{34}$$

We shall also introduce the set of periodic functions

$$\mathbb{G}_p := \big\{y \in \mathbb{G} : y(0) = y(1),\ \dot{y}(0) = \dot{y}(1)\big\}\,.$$

Notice that $\Upsilon(\mathbb{G}_p) \subset \Gamma_p$, the function $y_* \equiv 0$ is an element of $\mathbb{G}_p$ and $(1, x_*)$ is an element of $\Gamma_p \cap \Gamma(h)$. Moreover, $\Upsilon[y_*] = x_*$. The following lemma collects some further properties of Υ.

Lemma 10.2. *The following hold:*

(i) $\Upsilon(\mathbb{G}) = \Gamma(h)$ *and* $\Upsilon : \mathbb{G} \to \Gamma(h)$ *is a homeomorphism.*

(ii) *A function* $y \in \mathbb{G}$ *has class* C^2 *on* $[0,1]$ *if and only if* $\Upsilon[y]$ *has class* C^2*. Moreover, in this case* $\Upsilon : \mathbb{G} \to \Gamma$ *is differentiable at* y*.*

(iii) $\mathbb{A}_h[y] = A_h[\Upsilon[y]]$ *for every* $y \in \mathbb{G}$*.*

(iv) *If* $(T,x) \in \Gamma(h)$ *is an extremal of (3) then* $y := \Upsilon^{-1}(T,x)$ *is an extremal of (33).*

(v) *There exists some* $0 < \epsilon_1 < \epsilon$ *with the following property: for any* $y \in \mathbb{G}$ *extremal of (33) with* $y(0), \dot{y}(0) \in \mathbb{B}^{d-1}_{\epsilon_1}$*, one has that* $(T,x) := \Upsilon[y]$ *is an extremal of (3).*

Proof. Let us start by constructing the inverse of Υ. Given $(T,x) \in \Gamma(h)$ we set $\Upsilon^{-1}(T,x) := y \circ \theta^{-1}$, where $(T,\theta,y) := \Phi^{-1}(T,x) \in \mathfrak{G}$. The map $\Upsilon^{-1} : \Gamma(h) \to \mathbb{G}$ constructed in this way is well-defined (by Lemma 7.1) and continuous, and it is clear that it is the inverse of $\Upsilon : \mathbb{G} \to \Gamma(h)$. Statements *(i)*-*(ii)* follow.

Concerning *(iii)* we simply observe that, by (34) and Lemma 9.3,

$$\mathbb{A}_h[y] = \mathscr{A}_h[\Phi(1,\varkappa_y)] = A_h[\mathcal{R}(\Phi(1,\varkappa_y))] = A_h[\Upsilon[y]]\,, \qquad y \in \mathbb{G}\,.$$

In order to check *(iv)*, let $y \in \mathbb{G}$ be such that $\Upsilon[y]$ is a solution of the Euler-Lagrange equations (33). By *(ii)* we know that $y \in C^2([0,1],\mathbb{R}^{d-1})$ and $\Upsilon : \mathbb{G} \to \Gamma$ is differentiable at y. In particular, setting $y_0 := y(0)$, $y_1 := y(1)$, $x_0 := \psi(0,y_0)$, $x_1 := \psi(0,y_1)$ and letting $\mathbb{G}(y_0,y_1)$ be the set of functions in $\mathbb{G}$ satisfying the same Dirichlet boundary conditions as y (which is a C^1-submanifold of $\mathbb{G}$), we see that the restriction $\Upsilon\big|_{\mathbb{G}(y_0,y_1)} : \mathbb{G}(y_0,y_1) \to \Gamma(x_0,x_1)$ is well-defined and differentiable at y. In view of Lemma 9.2*(iii)*, $\Upsilon[y]$ is a critical point of $A_h\big|_{\Gamma(x_0,x_1)}$; on the other hand, by part *(iii)* of this lemma,

$$\mathbb{A}_h\big|_{\mathbb{G}(y_0,y_1)} = \left(A_h\big|_{\Gamma(x_0,x_1)}\right) \circ \left(\Upsilon\big|_{\mathbb{G}(y_0,y_1)}\right)\,,$$

so that, by the chain rule, y is a critical point of $\mathbb{A}_h\big|_{\mathbb{G}(y_0,y_1)}$. It proves *(iv)*.

Finally, in order to check *(v)* choose some sequence $\{\bar{y}_n\}_n \subset \mathbb{G}$ with $|\bar{y}_n(0)|+|\dot{\bar{y}}_n(0)| \to 0$ and such that all functions $\bar{y}_n$ are solutions of (33). It will be shown that $(\bar{T}_n,\bar{x}_n) := \Upsilon[\bar{y}_n]$ is an extremal of (3) for big n. By continuous dependence, $\{\bar{y}_n\} \to y_* \equiv 0$ in the $C^1([0,1],\mathbb{R}^{d-1})$-topology, and the continuity of Υ implies that $(\bar{T}_n,\bar{x}_n) \to (1,x_*)$ in Γ. In particular, $(\bar{x}_n(0),\dot{\bar{x}}_n(0)) \to (x_*(0),\dot{x}_*(0))$. Continuous dependence applies again to show that, for n big enough, the extremal x_n of (3) with $x_n(0) = \bar{x}_n(0)$ and $\dot{x}_n(0) = \dot{\bar{x}}_n(0)$ satisfies that $(T_n,x_n) \in \Gamma$ for some $T_n > 0$. Setting $y_n := \Upsilon^{-1}(T_n,x_n) \in \mathbb{G}$ (which is an extremal of (33) by assertion *(iv)* above), we see that $y_n(0) = \bar{y}_n(0)$ and $\dot{y}_n(0) = \dot{\bar{y}}_n(0)$, and uniqueness implies that $\bar{y}_n \equiv y_n$ for big n. Then also $(\bar{T}_n,\bar{x}_n) = (T_n,x_n)$, so that $(\bar{T}_n,\bar{x}_n)$ is an extremal of (3) for big n, as claimed. It completes the proof. $\square$

Lemma 10.3. *The following hold:*

(i) $y_* \equiv 0$ *is a local minimizer of* $\mathbb{A}_h\big|_{\mathbb{G}_p}$ *if and only if* $(1, x_*)$ *is a local minimizer of* $A_h\big|_{\Gamma_p}$.

(ii) $y_* \equiv 0$ *is isolated in* $\mathbb{G}_p$ *as a critical point of* $\mathbb{A}_h\big|_{\mathbb{G}_p}$ *if and only if* $(1, x_*)$ *is isolated in* Γ_p *as a critical point of* $A_h\big|_{\Gamma_p}$.

Proof. In view of Lemma 10.2*(iii)* one has:

$$\mathbb{A}_h\big|_{\mathbb{G}_p} = \Big(A_h\big|_{\Gamma_p \cap \Gamma(h)}\Big) \circ \Big(\Upsilon\big|_{\mathbb{G}_p}\Big)\ .$$

On the other hand, it follows from Lemma 10.2*(i)* that $\Upsilon\big|_{\mathbb{G}_p} : \mathbb{G}_p \to \Gamma_p \cap \Gamma(h)$ is a homeomorphism. Thus, $y_* \equiv 0$ is a local minimizer of $\mathbb{A}_h\big|_{\mathbb{G}_p}$ if and only if $(1, x_*)$ is a local minimizer of $A_h\big|_{\Gamma_p \cap \Gamma(h)}$. On the other hand, by combining Lemma 9.1*(ii)* and Lemma 9.3 we see that $\mathcal{R} : \Gamma_p \to \Gamma_p \cap \Gamma(h)$ is a continuous retraction lowering the value of A_h. It implies the first part of the statement.

On the other hand, by Lemma 10.2*(iv)*-*(v)* there are open neighborhoods of $y_* \equiv 0$ and $(1, x_*)$, in $\mathbb{G}_p$ and $\Gamma_p \cap \Gamma(h)$ respectively, such that Υ carries bijectively the critical points of $\mathbb{A}_h\big|_{\mathbb{G}_p}$ in the first open set into the critical points of $A_h\big|_{\Gamma_p}$ in the second open set. In view of Lemma 9.2*(iii)* the critical points of $A_h\big|_{\Gamma_p}$ must belong to $\Gamma(h)$ and the result follows. □

Notice now that, for any $y \in \mathbb{G}$, the function $(T, x) := \Upsilon[y]$ satisfies

$$(x(0), \dot{x}(0)) = \Xi_0(y(0), \dot{y}(0))\,, \qquad (x(T), \dot{x}(T)) = \Xi_0(y(1), \dot{y}(1))\,, \tag{35}$$

where $\Xi_0 : \mathbb{B}^{d-1}_\epsilon \times \mathbb{B}^{d-1}_\epsilon \to \mathcal{N}_0$ is the C^1-diffeomorphism defined in (27). We are led to consider the sets

$$\begin{cases} \mathbb{O}_{00} := \Big\{(y(0), \dot{y}(0)) : y \in \mathbb{G} \text{ is a solution of (33)}\Big\}\,, \\ \mathcal{N}_{00} := \Big\{\big(x(0), \dot{x}(0)\big) : (T, x) \in \Gamma \text{ is a solution of (3) with energy } h\Big\}\,. \end{cases}$$

By continuous dependence on initial conditions, $\mathbb{O}_{00}$ is open in $\mathbb{B}^{d-1}_\epsilon \times \mathbb{B}^{d-1}_\epsilon$, and $\mathcal{N}_{00}$ is open in $\mathcal{N}_0$. These sets are the natural domains of the Poincaré maps $\mathbb{P} : \mathbb{O}_{00} \to \mathbb{B}^{d-1}_\epsilon \times \mathbb{B}^{d-1}_\epsilon$ and $\mathcal{P} : \mathcal{N}_{00} \to \mathcal{N}_0$, associated respectively to the flows of (33) (for the period 1), and (3) inside the energy level h (for the transversal section $\mathcal{N}_0$). Notice also that $(0,0) \in \mathbb{O}_{00}$ and $(x_*(0), \dot{x}_*(0)) \in \mathcal{N}_{00}$. In the following result we see that $\mathbb{P}$ and $\mathcal{P}$ are conjugate around these points.

Lemma 10.4. $\Xi_0 \circ \mathbb{P} = \mathcal{P} \circ \Xi_0$ *on* $(\mathbb{B}^{d-1}_{\epsilon_1} \times \mathbb{B}^{d-1}_{\epsilon_1}) \cap \mathbb{O}_{00}$. *Here,* $0 < \epsilon_1 < \epsilon$ *is the number appearing in Lemma 10.2(v).*

Proof. Choose some initial condition $(y_0, \dot{y}_0) \in (\mathbb{B}^{d-1}_{\epsilon_1} \times \mathbb{B}^{d-1}_{\epsilon_1}) \cap \mathbb{O}_{00}$. Then, the associated solution $y = y(\theta)$ of (33) is defined for all $\theta \in [0,1]$ and belongs to $\mathbb{G}$. We set $(y_1, \dot{y}_1) := (y(1), \dot{y}(1)) = \mathbb{P}(y_0, \dot{y}_0)$. By Lemma 10.2*(v)* we see that $(T, x) := \Upsilon[y]$ is an extremal of (3), and by (35) we obtain

$$\mathcal{P}\big(\Xi_0(y_0, \dot{y}_0)\big) = \mathcal{P}\big(x(0), \dot{x}(0)\big) = \big(x(T), \dot{x}(T)\big) = \Xi_0(y_1, \dot{y}_1) = \Xi_0\big(\mathbb{P}(y_0, \dot{y}_0)\big)\,,$$

proving the result. □

11 From the autonomous framework to a time-dependent problem

In this section we complete the proofs of the main results of this paper, namely Theorems 2.1 and 2.3. Actually, almost all the work has already been done now; the only piece of the jigsaw puzzle which waits to be put on the table is a theorem concerning time-periodic Lagrangian systems of the form (33). Here, the Lagrangian $\mathbb{L} = \mathbb{L}(\theta, y, \dot{y})$ is assumed to be a $C^{0,2}$ function, defined on an open set $\mathbb{O} \subset (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ for some $d \geq 2$. It will be assumed that $\mathbb{L}$ satisfies the Legendre convexity condition $\partial^2_{\dot{y}\dot{y}}\mathbb{L}(\theta, y, \dot{y}) > 0$ on $\mathbb{O}$. Moreover, let us further assume that, either *(a"): $d = 2$*, or *(b"):* $\mathbb{L}$ is time-reversible, i.e., $(\theta, y, \dot{y}) \in \mathbb{O} \Rightarrow (-\theta, y, -\dot{y}) \in \mathbb{O}$ and $\mathbb{L}(-\theta, y, -\dot{y}) = \mathbb{L}(\theta, y, \dot{y})$. We consider the (fixed period)-action functional

$$\mathbb{A}[y] := \int_0^1 \mathbb{L}(\theta, y(\theta), \dot{y}(\theta))d\theta\,, \qquad y \in C^1_{\mathbb{O}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{d-1})\,,$$

the domain $C^1_{\mathbb{O}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{d-1})$ being the set of C^1 functions $y : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{d-1}$ such that $(\theta, y(\theta), \dot{y}(\theta)) \in \mathbb{O}$ for every θ. As mentioned at the end of the Introduction, the instability of the *minimizers* of $\mathbb{A}$ has been treated in several works, and we collect some of these results below. It will be convenient to adapt the notion of instability in the sense of Siegel and Moser for nonautonomous systems; thus, we shall say that the 1-periodic solution y_* of (33) is *unstable in the sense of Siegel and Moser* provided that there exists some $\epsilon_0 > 0$ such that the unique globally-defined solution $y = y(\theta)$ of (33) with $|y(\theta) - y_*(\theta)| + |\dot{y}(\theta) - \dot{y}_*(\theta)| < \epsilon_0\ \forall \theta \in \mathbb{R}$ is $y \equiv y_*$.

Theorem 11.1 ([13, 50, 51, 52, 53]). *Let $y_* \in C^1_{\mathbb{O}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{d-1})$ be a local minimizer of $\mathbb{A}$. In situation (b") assume further that y_* is even, i.e. $y_*(-\theta) = y_*(\theta)$ for any θ. Then:*

- *(i) y_* is unstable in the Lyapunov sense. Moreover, the associated Floquet multipliers are real and positive.*
- *(ii) If (a") holds and y_* is isolated as a 1-periodic solution of (33), then it is unstable in the sense of Siegel and Moser.*

The instability assertion in *(i)* was shown in [51] in situation *(a")*, and in [52, Lemma 2.3] if *(b")* holds. The statement on the Floquet multipliers was shown by Carathéodory [13, §413] in situation *(a")*, and more recently, by the author [53] for the case *(b")*. Finally, assertion *(ii)* has been proved in [50].

Proof of Theorems 2.1 and 2.3. Let $x_* \in \Lambda_{\mathcal{O}}$ be a nonconstant local minimizer of A_h; then Proposition 5.1 implies (12). Let h be the energy level of x_* and let $\mathbb{L} = \mathbb{L}_h$ be the associated Carathéodory Lagrangian, constructed as in Section 6. Now, the result follows from the combination of Lemma 10.3, which establishes that $y_* \equiv 0$ is a local minimizer of Carathéodory's reduced action; Lemma 10.4, which states that the associated Poincaré maps are conjugate and so have the same dynamics; and Theorem 11.1, depicting the dynamics around the minimizer $y_* \equiv 0$ for Carathéodory's reduced system. $\square$

12 A list of examples

We devote this section to develop some insight about the assumptions of Theorems 2.1 and 2.3 by presenting several counterexamples when some of these assumptions fail.

1. Degenerate closed geodesics which are not minimal may be stable even with Morse index zero. Consider the revolution surfaces

$$\mathfrak{R}_\pm : \qquad\qquad r = 1 \pm z^4\,, \quad -1 < z < 1\,,$$

written here in cylindrical coordinates (r, θ, z). Clairaut's relation[9] implies that, for $\mathfrak{R}_+$, the closed geodesic $z = 0$ is unstable, while for $\mathfrak{R}_-$, the closed geodesic $z = 0$ is stable. Moreover, this geodesic is length-minimizing in the first case but not in the second, and consequently, in view of Lemma 3.1, it is energy-minimizing in $\mathfrak{R}_+$ but not in $\mathfrak{R}_-$. Nonetheless, the first-order linear approximation is the same and so, the associated Morse index is zero in both cases. Notice that this example contradicts [24, Theorem 1.1].

2. Nonorientable surfaces: The 2-dimensional, symmetry-free Theorem 2.1 loses its validity without the orientability assumption. The following counterexample is inspired by [13, §411]. We start from the two-dimensional sphere $\mathbb{S}^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$, endowed with the usual Riemannian metric inherited from $\mathbb{R}^3$. The antipodal map $-I : \mathbb{S}^2 \to \mathbb{S}^2,\ (u, v, w) \mapsto (-u, -v, -w)$ is an isometry; hence, the (nonorientable) projective plane $\mathbb{RP}^2 := \mathbb{S}^2/\{I, -I\}$ is naturally endowed with the quotient Riemannian metric. The corresponding geodesics are the projections of geodesics on the sphere; thus, they are all closed loops (with the minimal period divided by 2) and dynamically stable. Moreover, they are length-minimizing, as this is the case with the geodesics of the sphere till the moment they arrive to the antipodal point.

Observe that the reflection map $\mathbb{S}^2 \to \mathbb{S}^2,\ (u, v, w) \mapsto (u, -v, w)$ induces an involutive isometry $\mathcal{S} : \mathbb{RP}^2 \to \mathbb{RP}^2$, and the closed geodesic $x_* : \mathbb{R}/\pi\mathbb{Z} \to \mathbb{RP}^2,\ t \mapsto (\cos t, \sin t, 0)$, is $\mathcal{S}$-symmetric. However, the set of fixed points of $\mathcal{S}$ is composed of a topological circle and an isolated point; it is therefore not a hypersurface of $\mathbb{RP}^2$.

3. An elliptic countexample with bad symmetries: The previous example is *degenerate*; indeed, the closed geodesic x_* lies inside a continuous family of closed geodesics. We are now going to build a new example where the configuration manifold will be the open solid torus

$$M := (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D} = \{(\theta, z) : \theta \in \mathbb{R}/2\pi\mathbb{Z},\ z \in \mathbb{C},\ |z| < 1\}\,,$$

[9] According to Clairaut's equality, the quantity $r(t)\cos\tau(t)$ is constant for each geodesic. Here, $\tau = \tau(t)$ stands for the angle between the geodesic and the parallels $z = \mathit{const}$ of the revolution surface.

endowed with some convenient Riemannian metric. Our example, which is a variation of a (parabolic) construction due to Carathéodory [13, §411], will satisfy:

(i) $x_*(t) := (t, 0_{\mathbb{C}})$, $t \in \mathbb{R}/2\pi\mathbb{Z}$, is a length-minimizing geodesic;

(ii) x_* is $\mathcal{S}$-symmetric for some involutive isometry $\mathcal{S} : M \to M$;

(iii) x_* is elliptic (and in particular, nondegenerate), and orbitally stable.

It will not disprove Theorem 2.3 because $\mathrm{Fix}(\mathcal{S})$ will include isolated points and/or connected components of codimension 2. Notice also that such an example contradicts previous attempts to show the hyperbolicity of minimizers in more general frameworks, such as [28, Theorem 1.5] or [36, Theorem 3.1] (concerning this last paper, take as antisymplectic involution $R : TM \to TM$, $(x, v) \mapsto (\mathcal{S}(x), -\mathcal{S}'(x)v)$).

We begin with the 3-dimensional sphere $\mathbb{S}^3 := \{(p_1, p_2) \in \mathbb{C} \times \mathbb{C} : |p_1|^2 + |p_2|^2 = 1\}$, which we write with the help of complex notation but consider as a hypersurface of $\mathbb{R}^4$. It is naturally endowed with the Riemannian metric inherited from the scalar product of $\mathbb{R}^4$, and the map $\eta : \mathbb{S}^3 \to \mathbb{S}^3$, $(p_1, p_2) \mapsto (ip_1, -p_2)$ is an isometry. Moreover, $\eta^4 = \mathrm{Id}_{\mathbb{S}^3}$, and η takes the circumference $\mathcal{N} := \{0_{\mathbb{C}}\} \times \mathbb{S}^1$ into itself. We can now see η as a map from $\mathbb{S}^3 \backslash \mathcal{N}$ into itself; it generates the isometry subgroup $G := \{\mathrm{Id}_{\mathbb{S}^3 \backslash \mathcal{N}}, \eta, \eta^2, \eta^3\}$, and this group acts in a properly discontinuous way on $\mathbb{S}^3 \backslash \mathcal{N}$ (see, e.g., [12, Chapter III.7]). The quotient space $\mathfrak{T}^3 := (\mathbb{S}^3 \backslash \mathcal{N})/G$ is a 3-dimensional manifold diffeomorphic to the open solid torus; moreover, the maps $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D} \to \mathfrak{T}^3$, $(\theta, z) \mapsto (\sqrt{1 - |z|^2}\, e^{i\theta/4}, e^{i\theta/2} z)$ and $\mathfrak{T}^3 \to (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D}$, $(p_1, p_2) \mapsto (4 \arg(p_1), \overline{p_1} p_2 / p_1)$ are mutually inverse diffeomorphisms. This fact makes it equivalent to construct the announced pathological Riemannian metric on $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D}$ or on $\mathfrak{T}^3$; we opt for this second possibility and consider the quotient Riemannian metric on $\mathfrak{T}^3$ inherited from $\mathbb{S}^3 \backslash \mathcal{N}$.

Notice that the parameterized curve $x_* : \mathbb{R}/(\pi/2)\mathbb{Z} \to \mathfrak{T}^3$ given by $t \mapsto (e^{it}, 0_{\mathbb{C}})$ is a closed geodesic; we claim that it is length-minimizing. Indeed, if $x : \mathbb{R}/(\pi/2)\mathbb{Z} \to \mathfrak{T}$ is a closed loop near x_*, then it can be lifted to a closed loop $\hat{x} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{S}^3 \backslash \mathcal{N}$ satisfying $\hat{x}(t + \pi/2) = \eta(\hat{x}(t))$ for any t; moreover, the lengths of x and $\hat{x}$ on the time interval $[0, \pi/2]$ coincide. This length is therefore not smaller than the distance on $\mathbb{S}^3$ between $\hat{x}(0)$ and $\hat{x}(\pi/2) = \eta(\hat{x}(0))$. The distance between two points on the sphere coincides with the angle between them; on the other hand, writting $\hat{x}(0) = (p_1, p_2)$, $\hat{x}(\pi/2) = (ip_1, -p_2)$, we see that this angle is $\arccos(-|p_2|^2) \geq \pi/2$. This shows that x_* is length-minimizing.

It is clear from the construction that x_* is orbitally stable; in principle it could be parabolic, but we claim that it is elliptic. To see this we consider the Poincaré map $P : T\mathfrak{T}^3 \to T\mathfrak{T}^3$ associated to the time period $\pi/2$. Then, $\xi_* := (x_*(0), \dot{x}_*(0)) = ((1, 0_{\mathbb{C}}), (i, 0_{\mathbb{C}}))$ is a fixed point of P, and $P'(\xi_*) : T_{\xi_*}(T\mathfrak{T}^3) \to T_{\xi_*}(T\mathfrak{T}^3)$ is a linear isomorphism. It has $\lambda = 1$ as a double eigenvalue (see, e.g. [34, Lemma 2.3 (p. 90)]), and in order to check the ellipticity of x_* we shall show that the remaining eigenvalues are $\pm i$ (also double).

With this goal, we denote by $\mathscr{P} : T\mathbb{S}^3 \to T\mathbb{S}^3$ the Poincaré map associated to the geodesic flow on $\mathbb{S}^3$ and the time period $\pi/2$. Then, $(d\eta)^{-1} \circ \mathscr{P} : T\mathbb{S}^3 \to T\mathbb{S}^3$

also has ξ_* as a fixed point, and is in fact locally conjugate around ξ_* to P (via the differential of the canonical projection $\mathbb{S}^3\backslash\mathcal{N} \to \mathfrak{T}^3$). Correspondingly, it suffices to check that $\pm i$ are double eigenvalues of $\zeta := d_{\xi_*}[(d\eta)^{-1} \circ \mathscr{P}] : T_{\xi_*}(T\mathbb{S}^3) \to T_{\xi_*}(T\mathbb{S}^3)$.

We use the identification $T\mathbb{S}^3 \equiv \{(p,v) \in \mathbb{C}^2 \times \mathbb{C}^2 : |p| = 1, \mathrm{re}\,\langle p,v\rangle = 0\}$. (Here, $\langle p,q\rangle := p_1\overline{q_1} + p_2\overline{q_2}0$ stands for the canonical Hermitian form on $\mathbb{C}^2$). Then, $\mathcal{M} := \{(p,v) \in T\mathbb{S}^3 : |v| = 1\}$ is a 5-dimensional submanifold of $T\mathbb{S}^3$. This manifold is invariant both for $\mathscr{P}$ and $d\eta$; indeed,

$$\mathscr{P}(p,v) = (v,-p); \qquad (d\eta)(p,v) = (ip_1, -p_2, iv_1, -v_2)\,,$$

for any $(p,v) = (p_1,p_2,v_1,v_2) \in \mathcal{M}$. Consequently,

$$\zeta(u,w) = (-iw_1, -w_2, iu_1, u_2), \qquad (u,w) \in T_{\xi_*}\mathcal{M}\,.$$

Notice now that $(\{0_{\mathbb{C}}\}\times\mathbb{C})\times(\{0_{\mathbb{C}}\}\times\mathbb{C}) \equiv \mathbb{R}^4$ is a ζ-invariant subspace of $T_{\xi_*}\mathcal{M}$ and $\zeta(0_{\mathbb{C}}, u_2, 0_{\mathbb{C}}, w_2) = (0_{\mathbb{C}}, -w_2, 0_{\mathbb{C}}, u_2)$ for any $u_2, w_2 \in \mathbb{C}$. The eigenvalues of this linear map are $\pm i$ (both repeated twice and semisimple). This completes the proof of the announced ellipticity of x_*.

Finally, we notice that there are several involutive isometries $\mathcal{S} : \mathfrak{T}^3 \to \mathfrak{T}^3$ such that x_* is $\mathcal{S}$-symmetric. For instance, one can take $\mathcal{S}(p_1,p_2) := (\overline{p_1}, p_2)$; in principle $\mathcal{S}$ is an isometry on $\mathbb{S}^3$, but one checks that $\mathcal{S} \circ \eta = \eta^3 \circ \mathcal{S}$, and so, $\mathcal{S}$ induces an isometry (which we may still call $\mathcal{S}$) on $\mathfrak{T}^3$. The previously-considered diffeomorphisms between $\mathfrak{T}^3$ and $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D}$ allow us to see it as a map from $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{D}$ to itself; it is then given by $\mathcal{S}(\theta,z) = (-\theta, e^{i\theta}z)$. In particular, we observe that the set of fixed points is made by the 2-cell $\theta = 0$ and the 0-cell $(\theta,z) = (\pi,0)$.

One can repeat the procedure for the choice $\mathcal{S}(p_1,p_2) := (\overline{p_1}, -p_2)$, and also $\mathcal{S}_\tau(p_1,p_2) := (\overline{p_1}, e^{i\tau}\overline{p_2})$ for some fixed $\tau \in \mathbb{R}$. Arguing as before, one checks that they give rise to involutive isometries in $\mathfrak{T}$. In the first case the set of fixed points is again composed by a 2-cell and a 0-cell, while in the second its is made of two 1-cells.

4. A counterexample for the fixed period action functional: Despite the geodesics example, instability does not hold in general for (symmetric) minimizers of the *fixed-period* action functional if one deals with more general Lagrangians. As an example, we recall that Gordon [20] showed that the, say, $2\pi-$periodic orbits of the Kepler problem

$$\ddot{x} = -\frac{x}{|x|^3}\,, \qquad x \in \mathbb{R}^2\backslash\{0\}\,,$$

minimize the fixed-period action functional $x \mapsto \displaystyle\int_0^{2\pi} \left(\frac{|\dot{x}|^2}{2} + \frac{1}{|x|}\right) dt$. Moreover, the circular one $x_*(t) := (\cos t, \sin t)$ is symmetric with respect to the orthogonal reflection over the abscissa axis. However, x_* is orbitally-stable. We observe that the energy of x_* is $h = -1/2$; on the other hand, setting $x_c(t) := \frac{1}{c^{2/3}}(\cos ct, \sin ct)$, simple computations show that the function $c > 0 \mapsto$

$A_{-1/2}(2\pi/c, x_c) = \int_0^{2\pi/c}(-1/2 + |\dot{x}_c|^2/2 + 1/|x_c|)\, dt = 3\pi/\sqrt[3]{c} - \pi/c$ has a *maximum* at $c = 1$. Thus, the free-period action functional $A_{-1/2}$ has in fact a *saddle point* at x_*.

Other examples of elliptic orbits which minimize the fixed-period action functional have been given in [3].

5. On x_* being not an equilibrium. The condition that x_* is not an equilibrium cannot be eliminated, either. Here is a counterexample, constructed by combining some ideas of Moser [33, p. 78] and Ortega [37]. We consider, for $M = \mathbb{R}^2$ and $\mathcal{O} = TM = \mathbb{R}^2 \times \mathbb{R}^2$, the Lagrangian

$$L_\epsilon(x, \dot{x}) := \frac{1}{2}\,|\dot{x}|^2 + \frac{1}{2}|x|^2 + (1+\epsilon)\langle \dot{x}, Jx\rangle\,, \qquad x, \dot{x} \in \mathbb{R}^2.$$

Here, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ stands for the 90° rotation in the positive sense, and $\epsilon > 0$ is a small parameter. Setting $\mathcal{S}(x_1, x_2) := (x_1, -x_2)$ one checks that (8) holds, so that L_ϵ is $\mathcal{S}$-symmetric. The associated Euler-Lagrange equations are linear, and given by:

$$\ddot{x} + 2(1+\epsilon)J\dot{x} - x = 0\,, \qquad x \in \mathbb{R}^2\,. \tag{36}$$

This equation can be solved explicitly, and the trivial solution $x_* \equiv 0$ is stable. What is more, if one fixes the period $T_* := 1$ and chooses $\epsilon > 0$ sufficiently small, then x_* is of elliptic-elliptic type with four different Floquet multipliers on the unit circle. This is easy to confirm: denoting by $R[-(1+\epsilon)t]$ the rotation of angle $-(1+\epsilon)t$ in the anticlockwise sense, the change of variables $x(t) = R[-(1+\epsilon)t]u(t)$ transforms (36) into $\ddot{u} + (\epsilon^2 + 2\epsilon)u = 0$.

Yet, setting $x_* \equiv 0$, for small $\epsilon > 0$ the extremal $(1, x_*)$ is also a (strict) local minimizer of the free-period action functional

$$A_0 : \Lambda \to \mathbb{R}\,, \qquad (T, x) \mapsto \int_0^T L_\epsilon(x(t), \dot{x}(t))dt\,.$$

This can be equivalently reformulated by saying that the fixed-period action functional $A_0(T, \cdot)$ is positive definite on $C^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$ provided that $|T-1|$ and $\epsilon > 0$ are small. To check this statement we first notice that $A_0(T, \cdot)$ is positive definite on the set of constant functions $\bar{x} \in \mathbb{R}$; on the other hand, setting $\widetilde{C^1}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2) := \{\tilde{x} \in C^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2) : \int_0^T \tilde{x}(s)ds = 0\}$, one has:

$$A_0(T, \bar{x} + \tilde{x}) \geq A_0(T, \tilde{x}) \qquad \forall \bar{x} \in \mathbb{R}\,,\ \forall \tilde{x} \in \widetilde{C^1}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)\,.$$

Thus, it suffices to see that $A_0(T, \cdot)$ is positive-definite on $\widetilde{C^1}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$ whenever $\epsilon > 0$ is small and T is close to 1. By combining Cauchy-Schwarz's and Wirtinger's inequalities we see that, letting $p(r, s) := r^2/2 + s^2/2 - (1+\epsilon)rs$, for every $\tilde{x} \in \widetilde{C^1}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$ one has:

$$A_0(T, \tilde{x}) \geq \min\left\{p(r, \|\dot{\tilde{x}}\|_2) : 0 \leq r \leq (T/2\pi)\|\dot{\tilde{x}}\|_2\right\} = p\big((T/2\pi)\|\dot{\tilde{x}}\|_2, \|\dot{\tilde{x}}\|_2\big) = c\|\dot{\tilde{x}}\|_2^2\,,$$

where $c = (4\pi^2 - 4\pi(1+\epsilon)T + T^2)/8\pi^2$, which is positive for $\epsilon > 0$ small and T close to 1. It completes the argument.

6. Instability vs. instability in the sense of Siegel and Moser. In view of Theorems 2.1 and 2.3, another question appears: is it true that, when $d \geq 3$, symmetric closed minimizing orbits which are isolated are unstable in the sense of Siegel and Moser?

The answer turns out to be negative. Set $M := (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$, with coordinates $x = (\theta, y)$; there exists a C^∞ Lagrangian function

$$L : TM \equiv M \times \mathbb{R}^3 \to \mathbb{R}, \qquad L = L(\theta, y, \dot\theta, \dot y)$$

such that, defining $\mathcal{S} : M \to M$ by $S(\theta, y) := (-\theta, y)$ and $x_* : \mathbb{R}/\mathbb{Z} \to M$ by $t \mapsto (t, 0)$, the following hold:

(a) L is $\mathcal{S}$-symmetric and satisfies $\partial^2_{\dot x \dot x} L(x_*(t), \dot x_*(t)) > 0$ for any time $t \in \mathbb{R}/\mathbb{Z}$,

(b) $(1, x_*)$ is a local minimizer of the free-period action functional A_h corresponding to some energy value h. Moreover, it is isolated as a periodic orbit inside its energy level,

but x_* is *not* orbitally unstable in the sense of Siegel and Moser (it must be orbitally unstable by Theorem 2.3).

In fact, our Lagrangian L will have the form

$$L(\theta, y, \dot\theta, \dot y) = \frac{1}{2}|\dot y|^2 - V(y) + \frac{1}{2}\left(1 - V(y) - \frac{1}{2}|\dot y|^2\right)(\dot\theta^2 - 1)\,,$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is some C^∞ function with

$$V(0) = 0, \qquad \nabla V(0) = 0, \qquad \mathrm{Hess} V(0) = 0\,, \tag{37}$$

which will be made precise later. Now, *(a)* follows immediately. One checks that the energy function

$$H(\theta, y, \dot\theta, \dot y) = \partial_{\dot\theta} L(\theta, y, \dot\theta, \dot y)\dot\theta + \partial_{\dot y} L(\theta, y, \dot\theta, \dot y)\dot y - L(\theta, y, \dot\theta, \dot y)\,,$$

satisfies the identity $H(\theta, y, 1, \dot y) \equiv 1$; in particular, $h := 1$ is the energy level of the extremal $x_*(t) := (t, 0)$. Carrying out the scheme described in Section 6, on some open neighborhood $\mathcal{U}$ of the closed loop $(\mathbb{R}/\mathbb{Z}) \times \{(0,1,0)\}$ one has $\lambda_1(\theta, y, \dot\theta, \dot y) = 1/\dot\theta$ and the associated Jacobi-Maupertuis Lagrangian is given by:

$$\mathscr{L}_1(\theta, y, \dot\theta, \dot y) = \big(1 + L(\theta, y, 1, \dot y/\dot\theta)\big)\dot\theta = \frac{|\dot y|^2}{2\dot\theta} + \big(1 - V(y)\big)\dot\theta\,, \qquad (\theta, y, \dot\theta, \dot y) \in \mathcal{U}\,.$$

Further performing Carathéodory's reduction of order procedure (as described in Section 10) with $\psi(t, y) := (t, y)$, we get the Lagrangian

$$\mathbb{L}_1(t, y, \dot y) = \mathscr{L}_1(t, y, 1, \dot y) = \frac{|\dot y|^2}{2} + 1 - V(y)\,, \qquad t \in \mathbb{R}/\mathbb{Z},\ |y| < \epsilon,\ |\dot y| < \epsilon,$$

with associated Euler-Lagrange equations

$$\ddot{y} = -\nabla V(y)\,. \tag{38}$$

In order to fix V we shall use the following result, which was recently obtained in [54]: there exists a C^∞ function $V : \mathbb{R}^2 \to \mathbb{R}$ with $V(0) = 0 > V(y)$ for any $y \neq 0$, $\nabla V(y) \neq 0$ for $y \neq 0$, and there exists a sequence of nontrivial periodic orbits $y_n : \mathbb{R}/T_n\mathbb{Z} \to \mathbb{R}^2$ of (38) with $\max_t |y_n(t)| \to 0$ as $n \to +\infty$.

It follows now from Lemma 10.3*(i)* that $(1, x_*)$ is a local minimum of the free-period action functional A_1 associated to L, whereas Lemma 10.4 implies that x_* is not unstable inside its energy level in the sense of Siegel and Moser. It only remains to see that z_* is isolated, and, by Lemma 10.3*(ii)* it is equivalent to show that $y_* \equiv 0$ is isolated as a 2π-periodic solution of the Newtonian equation $\ddot{y} = -\nabla V(y)$. We use a contradiction argument and assume instead the existence of a sequence $\{y_n\}_n \to 0$ of nontrivial 1-periodic solutions; they cannot be constant, and so, it makes sense to define $z_n := \dot{y}_n/\|y_n\|_\infty$, for each $n \in \mathbb{N}$. Differentiation in the equation $\ddot{y}_n = -\nabla V(y_n)$ leads to $\dddot{z}_n = -\mathrm{Hess}V(y_n(t))z_n$, and since $\{z_n\}$ is bounded in the L^∞ norm, we deduce that also $\ddot{z}_n$ is bounded in the L^∞ norm. We deduce from here that the sequence z_n is actually bounded in the C^2 norm, and, by the Ascoli-Arzela lemma, after possibly passing to a subsequence, we may assume $z_n \to z_*$ in the C^1 norm. Passing to the limit and remembering (37) we see that z_* must solve the equation

$$\ddot{z}_* = -\mathrm{Hess}V(0)z_* = 0\,,$$

and, since z_* must be periodic, it must be constant. On the other hand, all functions z_n have zero mean value, and we conclude that also z_* has zero mean value, so that $z_* \equiv 0$. But $\|z_n\|_\infty = 1$ for every n, and passing to the limit we find that $\|z_*\|_\infty = 1$. This is a contradiction and concludes the proof.

13 Appendix

13.1 Minimizing on loops vs. minimizing on symmetric loops

We open this section with a general result which applies to Lagrangian systems (3) under the sole assumptions that the Lagrangian function $L : \mathcal{O} \to \mathbb{R}$ has class C^2 and satisfies the Legendre convexity condition (2). Our starting point will be the following observation: if $x : I \to M$ is a solution of the Euler-Lagrange equations (3) with energy h and $\dot{x}(t_0) = 0$ for some $t_0 \in I$, then

$$h + L(x(t_0), \dot{x}(t_0)) = \partial_{\dot{x}} L(x(t_0), \dot{x}(t_0))\dot{x}(t_0) = 0.$$

The lemma below, which was used in the proof of Proposition 5.1, gives a deeper insight in the behavior of the function $t \mapsto h + L(x(t), \dot{x}(t))$ near $t = t_0$:

Lemma 13.1. *Let $x : I \to M$ be a solution of (3) with energy h and $\dot{x}(t_0) = 0$ for some $t_0 \in I$. If x is not an equilibrium, then*

$$\lim_{t\to t_0} \frac{h + L(x(t), \dot{x}(t))}{(t-t_0)^2} > 0\,.$$

.

Proof. Since the statement is local, there is no loss of generality in assuming $M = \mathbb{R}^d$ and $x(t_0) = 0$. Since x is not an equilibrium, $\partial_x L(0, 0) \neq 0$, and the Euler-Lagrange equations (3) imply that $\ddot{x}(t_0) \neq 0$.

Now, for $t \in I \backslash \{t_0\}$ one has:

$$\begin{aligned}\frac{h + L(x(t), \dot{x}(t))}{(t-t_0)^2} &= \frac{\partial_{\dot{x}} L(x(t), \dot{x}(t))\dot{x}(t)}{(t-t_0)^2} = \\ &= (t-t_0)\left(\int_0^1 \partial^2_{x\dot{x}} L(sx(t), s\dot{x}(t))\, ds\right)\left(\frac{x(t)}{(t-t_0)^2}, \frac{\dot{x}(t)}{(t-t_0)}\right) + \\ &\quad + \left(\int_0^1 \partial^2_{\dot{x}\dot{x}} L(sx(t), s\dot{x}(t)) ds\right)\left(\frac{\dot{x}(t)}{t-t_0}, \frac{\dot{x}(t)}{t-t_0}\right)\,.\end{aligned}$$

Since $x(t)/(t-t_0)^2 \to \ddot{x}(0)/2$ and $\dot{x}(t)/(t-t_0) \to \ddot{x}(0)$ as $t \to t_0$ we see that the first term of the last side of the equality converges to 0, and the second one to $\partial^2_{\dot{x}\dot{x}} L(0,0)\,(\ddot{x}_*(0), \ddot{x}_*(0)) > 0$. The result follows. □

The remaining of this subsection is devoted to prove Lemma 2.2. Thus, we further assume now that the Lagrangian $L : \mathcal{O} \to \mathbb{R}$ satisfies the symmetry condition (8) for some involutive C^3-map $\mathcal{S} : M \to M$ with (7). Let $(T_*, x_*) \in \Lambda^{\mathcal{S}}_{\mathcal{O}}$ be given; by symmetry, $x_*(0)$ and $x_*(T_*/2)$ must be fixed points of $\mathcal{S}$. The following lemma gives some additional information on the local behavior of x_* near $t = 0$ when (T_*, x_*) is a local minimizer of $A_h\big|_{\Lambda^{\mathcal{S}}_{\mathcal{O}}}$.

Lemma 13.2. *Let $(T_*, x_*) \in \Lambda^{\mathcal{S}}_{\mathcal{O}}$ be a local minimizer of $A_h\big|_{\Lambda^{\mathcal{S}}_{\mathcal{O}}}$. Then,*

$$x_*(]-\varepsilon, \varepsilon[) \backslash \mathrm{Fix}(\mathcal{S}) \neq \emptyset \quad \textit{for all } \varepsilon > 0\,.$$

Proof. We argue by a contradiction argument and assume instead the existence of some $0 < \varepsilon_0 < T_*/2$ such that $x_*\big(]-\varepsilon_0, \varepsilon_0[\big) \subset \mathrm{Fix}(\mathcal{S})$. Then, by symmetry, $x_*(-t) = x_*(t)$ for if $|t| < \varepsilon_0$, and in particular, $\dot{x}_*(0) = 0$. Lemma 13.1 implies the existence of some $0 < \varepsilon_1 < \varepsilon_0$ such that $h + L(x_*(t), \dot{x}_*(t)) > 0$ if $0 < |t| < \varepsilon_1$, and we deduce that, for any $n \in \mathbb{N}$ one has

$$\int_{\varepsilon_1/n}^{T_*/2} \big(h + L(x_*(t), \dot{x}_*(t))\big) dt < \int_0^{T_*/2} \big(h + L(x_*(t), \dot{x}_*(t))\big) dt.$$

By slightly modifying $x_*\big|_{[\varepsilon_1/n, T_*/2]}$ near ε_1/n (so that its derivative there vanishes) and extending it by symmetry and periodicity, we can now construct a sequence $\{(T_* - 2\varepsilon_1/n, x_n)\}_n \subset \Lambda^{\mathcal{S}}_{\mathcal{O}}$ converging to (T_*, x_*) and with $A_h(T_* - 2\varepsilon_1/n, x_n) < A_h(T_*, x_*)$ for all n. This is a contradiction with the fact that x_* was a local minimizer in $\Lambda^{\mathcal{S}}_{\mathcal{O}}$ and concludes the proof. □

Proof of Lemma 2.2. It is clear from the definitions that any $\mathcal{S}$-symmetric local minimizer of A_h is a local minimizer of $A_h\big|_{\Lambda_{\mathcal{O}}^{\mathcal{S}}}$. We check the converse by a contradiction argument and assume that (T_*, x_*) is a local minimizer of $A_h\big|_{\Lambda_{\mathcal{O}}^{\mathcal{S}}}$ for which there exists a sequence $(T_n, x_n) \to (T_*, x_*)$ with $(T_n, x_n) \in \Lambda_{\mathcal{O}}$ and $A_h(T_n, x_n) < A_h(t_*, x_*)$ $\forall n$.

By Lemma 13.2 there is a sequence $\{\tau_n\} \to 0$ with $x_*(\tau_n) \notin \mathrm{Fix}(\mathcal{S})$ for every $n \in \mathbb{N}$. Then, by symmetry, $x_*(-\tau_n) \notin \mathrm{Fix}(\mathcal{S})$, and in fact, for n big enough $x_*(\pm\tau_n)$ belong to the two different connected components in which $\mathrm{Fix}(\mathcal{S})$ divides M near $x_*(0)$. Consequently, after possibly replacing x_n by a subsequence we may assume that $x_n(\pm\tau_n)$ also belong to the two local connected components of $M\backslash\mathrm{Fix}(\mathcal{S})$ near $x_*(0)$, and therefore there exists some sequence $\{\alpha_n\} \to 0$ such that $x_n(\alpha_n) \in \mathrm{Fix}(\mathcal{S})$ for every n. Similarly, after possibly passing to a second subsequence we may assume the existence of $\{\beta_n\} \to T_*/2$ such that $x_n(\beta_n) \in \mathrm{Fix}(\mathcal{S})$ for every n. Finally, after possibly replacing $x_n = x_n(t)$ with $\bar{x}_n(t) := \mathcal{S}(x_n(-t))$ and α_n, β_n by $-\alpha_n, T_n - \beta_n$ respectively, we may additionally assume that

$$\int_{\alpha_n}^{\beta_n} \big(h + L(x_n(t), \dot{x}_n(t)\big)dt < \frac{1}{2}A_h(T_*, x_*) \text{ for all } n \in \mathbb{N}.$$

Thus, after slightly modifying each function $x_n\big|_{[\alpha_n,\beta_n]}$ near α_n, β_n (so that its derivative at these points vanishes), and extending it to the real line by symmetry and periodicity we obtain a sequence $\{(2(\beta_n - \alpha_n), \tilde{x}_n)\}_n \subset \Lambda_{\mathcal{O}}^{\mathcal{S}}$ converging to (T_*, x_*) and with $A_h(2(\beta_n - \alpha_n), \tilde{x}_n) < A_h(T_*, x_*)$ for all n. It contradicts the fact that x_* was a local minimizer in $\Lambda_{\mathcal{O}}^{\mathcal{S}}$ and concludes the proof. $\square$

13.2 Minimizing length vs. minimizing energy

Proof of Lemma 3.1. The implication *(i)* $\Longrightarrow$ *(ii)* is immediate. Concerning *(ii)* $\Longrightarrow$ *(iii)* we notice that every $x \in C^1(\mathbb{R}/\mathbb{Z}, M)$ with $\dot{x}(t) \neq 0$ $\forall t$ can be reparameterized to constant speed inside $C^1(\mathbb{R}/\mathbb{Z}, M)$, i.e., there exists a C^1 diffeomorphism $\sigma : \mathbb{R} \to \mathbb{R}$ with $\sigma(t+1) = \sigma(t) + 1$ for every $t \in \mathbb{R}$ such that $x_\sigma := x \circ \sigma$ has constant speed $\|\dot{x}_\sigma(t)\| \equiv v_x$. Moreover, if x is close to x_* then x_σ is also close to x_*, and one has

$$\frac{1}{2}\int_0^1 \|\dot{x}_\sigma(t)\|^2 dt = \frac{1}{2}v_x^2 \geq \frac{1}{2}v^2\,,$$

implying that $v_x \geq v$. Consequently,

$$\int_0^1 \|\dot{x}(t)\|\, dt = \int_0^1 \|\dot{x}_\sigma(t)\|\, dt = v_x \geq v = \int_0^1 \|\dot{x}_*(t)\|\, dt\,,$$

proving this implication.

It remains to check the implication *(iii)* $\Longrightarrow$ *(i)*. With this aim, choose $(T, x) \in \Lambda$ near $(1, x_*)$ and set $x_1(t) := x(tT)$, which is a function in $C^1(\mathbb{R}/\mathbb{Z}, M)$

near x_*. Thus,

$$\int_0^T \|\dot{x}(t)\| \, dt = \int_0^1 \|\dot{x}_1(t)\| \, dt \geq \int_0^1 \|\dot{x}_*(t)\| \, dt = v \, .$$

On the other hand, the Cauchy-Schwarz inequality implies

$$\int_0^T \|\dot{x}(t)\|^2 dt \geq \frac{1}{T} \left(\int_0^T \|\dot{x}(t)\| dt \right)^2 \geq \frac{v^2}{T},$$

and therefore, the inequality $a^2 + b^2 \geq 2ab$ gives

$$\int_0^T (v^2 + \|\dot{x}(t)\|^2) dt \geq T v^2 + \frac{v^2}{T} \geq 2v^2 = \int_0^{T_*} (v^2 + \|\dot{x}_*(t)\|^2) dt \, ,$$

concluding the proof. □

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