To what extent are unstable the maxima of the potential?

Antonio J. Ureña*

Abstract. The classical Lagrange-Dirichlet stability theorem states that, for natural mechanical systems, the strict minima of the potential are dynamically stable. Its converse, i.e., the instability of the *maxima* of the potential, has been proved by several authors including Liapunov (1892), Hagedorn (1971), or Taliaferro (1980), in various degrees of generality. We complement their theorems by presenting an example of a smooth potential on the plane having a maximum and such that the associated dynamical system has a converging sequence of periodic orbits. This implies that the maximum is *not* unstable in a stronger sense considered by Siegel and Moser.

1 Introduction

Consider the Newtonian system of equations

$$\ddot{q} = -\nabla V(q)\,, \qquad q \in \mathbb{R}^d\,, \tag{1}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a given potential having a local maximum at some point $q_* \in \mathbb{R}^d$. In his 1892 PhD dissertation [6], Liapunov showed that if this maximum is nondegenerate, then the corresponding equilibrium is dynamically unstable. Liapunov's instability theorem was extended by Hagedorn [3], who used variational methods to prove the instability of all isolated maxima of the potential, and Taliaferro [11], who removed the isolatedness assumption from Hagedorn's theorem. Results of this type have been named as *converses of the Lagrange-Dirichlet stability theorem*, and the associated literature is very ample, see for instance [4, 7].

In the above-mentioned papers, the word *instability* is understood as the logical negation of Liapunov stability. In particular, for equations of the form (1) and more generally in the Hamiltonian framework, it means both past and future instability; namely, the two concepts are equivalent. See [9, p. 150] and [8, p. 114-115] (in this latter reference the statement is made in dimension 2, but the proof works for all dimensions).

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However, a stronger notion of instability was considered by Siegel and Moser in [10, §25]. According to their definition, the equilibrium $q_* \in \mathbb{R}^d$ is unstable if there is a neighborhood $\mathcal{N}$ of $(q_*, 0)$ in the phase space such that *every* globally-defined solution $q : \mathbb{R} \to \mathbb{R}^d$ of (1), $q(t) \not\equiv q_*$, satisfies $(q(t_q), \dot{q}(t_q)) \notin \mathcal{N}$ for some $t_q \in \mathbb{R}$. In other words, $\{(q_*, 0)\}$ is the maximal subset of $\mathcal{N}$ which invariant by the flow. Observe, for instance, that a hyperbolic fixed point is always unstable in this stronger sense, whereas a nonisolated fixed point never is.

The question which motivates this paper is the following: *assume that the potential V attains its maximum at $q_* \in \mathbb{R}^d$, and that this maximum is isolated as a critical point of V, does it imply that q_* is unstable in the stronger sense of Siegel and Moser?*

The answer to this question is affirmative in the 1-dimensional case $d = 1$, as an easy conservation-of-energy argument shows. It is also affirmative if one assumes that the Hessian matrix of the (sufficiently smooth) potential V is negative semidefinite on *some neighborhood* of q_*; indeed, under this assumption, the function $t \mapsto -V(q(t))$ is convex as long as $q(t)$ belongs to the neighborhood, as one promptly checks. However, this question turns out to be false in general. The goal of this paper is to prove the following:

Theorem 1.1. *There exists a C^∞ function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfying:*

(a) $V(0,0) = \max_{\mathbb{R}^2} V$,

(b) $\nabla V(q) \neq (0,0)$ *for any* $q \neq 0$,

(c) there exists a sequence $T_n > 0$ of positive numbers and a sequence $q_n : \mathbb{R}/T_n\mathbb{Z} \to \mathbb{R}^2$ of nontrivial periodic solutions of $\ddot{q} = -\nabla V(q)$ such that

$$\max_{t\in\mathbb{R}}(\|q_n(t)\| + \|\dot{q}_n(t)\|) \to 0 \text{ as } n \to +\infty .$$

We do not know whether such an example exists in the analytic case. In our construction, the sequence of periods T_n is divergent; one easily checks that this is necessarily the case if the Hessian matrix of V vanishes at the origin. The closed orbits q_n in our example do not encircle the origin.

A comment may shed some light on how our example can(not) be constructed. If $q : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$, $q = q(t)$ is a parameterized closed *convex* curve of class C^2 and $\ddot{q} = -\nabla V(q)$ for some C^1 potential $V : \mathbb{R}^2 \to \mathbb{R}$, then $\nabla V(q)$ points outwards at the points of our closed curve, so that V has a minimum in the interior of $\mathscr{L} := q(\mathbb{R})$. Therefore, the closed curves q_n of Theorem 1.1 cannot be convex.

For this reason, a key advance towards the proof of Theorem 1.1 will consist in building a closed, simple and nonconvex curve $\mathscr{L}$ in the plane, and a potential U with a non-vanishing gradient which points inwards on the concave section of $\mathscr{L}$ and outwards on the convex part. This construction, formulated more precisely in Proposition 2.1, will occupy us through Sections 2, 3 and 4. In Section 5 we shall find a reparametrization $\gamma = \gamma(t)$ of $\mathscr{L}$ which satisfies a generalized version of (1) with a weight. This result will be used in Section 6 to construct a modified potential function W in the plane, with nonvanishing gradient, and such that the corresponding Newtonian system has a closed orbit. Actually, W will be periodic in the first variable, allowing us to see it as defined on a ring circling the origin. The end of the argument will consist in 'stacking' infinitely many copies of W defined on smaller and smaller rings to obtain the potential V of Theorem 1.1.

2 Admissible closed curves and potentials

The closed curve $\mathscr{L}$ will be defined by a 2π-periodic parametrization $\ell = \ell(t)$. For simplicity reasons, we shall further impose some symmetry conditions on our problem; thus, from now on we denote by $\mathcal{S}$ to the symmetry in $\mathbb{R}^2$ with respect to the y axis, i.e., $\mathcal{S}(x, y) := (-x, y)$.

It will be convenient to introduce some definitions here. The parameterized closed curve $\ell : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2,\ \ell = \ell(t)$ will be termed *admissible* provided that it is C^∞, simple, goes in the counterclockwise direction, and satisfies:

(ℓ_i) $\ell(-t) = \mathcal{S}(\ell(t))\ \forall t \in \mathbb{R}/2\pi\mathbb{Z}\,,$

(ℓ_{ii}) $\ell(\mathbb{R}/2\pi\mathbb{Z}) \subset]-1, 1[^2, \quad \ell(]0, \pi[) \subset]0, +\infty[\times\mathbb{R}\,,$

(ℓ_{iii}) $\dot\ell(t) \neq (0, 0) \quad \forall t \in \mathbb{R}/2\pi\mathbb{Z}\,,$

and, for some $t_* \in]0, \pi/2[$,

(ℓ_{iv}) $\det(\dot\ell(t), \ddot\ell(t)) \begin{cases} < 0 \text{ if } 0 \le t < t_* \\ > 0 \text{ if } t_* < t \le \pi \end{cases}, \qquad \left.\frac{d}{dt}\right|_{t=t_*} \det(\dot\ell(t), \ddot\ell(t)) > 0\,.$

On the other hand, the function $U : \mathbb{R}^2 \to \mathbb{R}$, $U = U(x, y)$, will be said to be an *admissible potential* provided that it is C^∞ and satisfies

(U_i): $U(x + 2, y) = U(-x, y) = U(x, y)$;

(U_{ii}): $U(x, y) = y$ if $|y| \ge 1$;

(U_{iii}): $\nabla U(x, y) \neq (0, 0)\ \forall (x, y) \in \mathbb{R}^2\,.$

Finally, the admissible closed curve ℓ and the admissible potential U will be called *coupled* provided that

($U\ell_i$): $\langle \nabla U(\ell(t)), \dot\ell(t)\rangle > 0\ \forall t \in]0, \pi[\,, \qquad \left.\frac{d^2}{dt^2}\right|_{t=0} U(\ell(t)) > 0 > \left.\frac{d^2}{dt^2}\right|_{t=\pi} U(\ell(t))\,,$

($U\ell_{ii}$): $\det(\nabla U(\ell(t)), \dot\ell(t)) \begin{cases} < 0 \text{ if } 0 \le t < t_* \\ > 0 \text{ if } t_* < t \le \pi \end{cases}, \quad \left.\frac{d}{dt}\right|_{t=t_*} \det(\nabla U(\ell(t)), \dot\ell(t)) > 0,$

where t_* is the same number appearing in assumption (ℓ_{iv}). If only ($U\ell_i$) is ensured we shall say that ℓ and U are *semicoupled*.

Assumption (ℓ_{iii}) means that the closed curve $\mathscr{L} := \{\ell(t) : t \in \mathbb{R}/2\pi\mathbb{Z}\}$ is regular in the geometrical sense. Set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; since the parametrization ℓ goes in the counterclockwise direction we deduce that $J\dot{\ell}(t)$ points in the normal outward direction to $\mathscr{L}$ at $\ell(t)$, for every t. Then, by combining (ℓ_i) and the equality $\det(u,v) = \langle u, Jv\rangle$ we can rephrase (ℓ_{iv}) by saying that the closed curve $\mathscr{L} := \{\ell(t) : t \in \mathbb{R}/2\pi\mathbb{Z}\}$ is convex on the arc $t_* < t < 2\pi - t_*$ and concave for $-t_* < t < t_*$. Similarly, $(U\ell_{ii})$ states that $\nabla U(\ell(t))$ points outwards in the convex section of $\mathscr{L}$ and inwards in the concave part.

Our first task in this paper, announced at the beginning of this section, will consist in checking that the definitions above are not empty:

Proposition 2.1. *There exist an admissible closed curve ℓ and an admissible potential U which are coupled.*

The proof of Proposition 2.1 will be divided into three steps:

- Firstly, we shall present a family $\{\ell_\lambda\}_\lambda$ of admissible closed curves.
- Secondly, we shall describe an admissible potential U which is semicoupled to some curves of this family.
- Finally, we shall modify U so that it becomes coupled to one of these admissible closed curves. This last step will be carried out in the next two sections.

Concerning the *first step*, the curves which we have in mind are called *limaçons of Pascal*[1] and have been known in geometry and the arts for centuries [2]. In polar coordinates (ϑ, ϱ), they are defined by the implicit equation

$$\mathscr{L}_\lambda : \qquad \varrho = \frac{1}{4}(\lambda + \sin\vartheta)\,.$$

Here, $1 < \lambda \leq 2$ is a parameter. Notice that all these closed curves are smooth and simple. For $\lambda = 2$ the corresponding curve is convex, but we will be mostly interested in the case $\lambda \in]1,2[$, for which the curvature changes sign. See Fig. 1. The choice of the scaling coefficient $1/4$ has been made so that these curves fit inside the open square $]-1,1[\times]-1,1[$, as required by condition (ℓ_{ii}). Notice finally that our curves are symmetric with respect to the ordinate axis, i.e. $\mathcal{S}(\mathscr{L}_\lambda) = \mathscr{L}_\lambda$.

One can parameterize these curves as follows:

$$\ell_\lambda : \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}_\lambda\,, \qquad \ell_\lambda(t) := \frac{1}{4}(\lambda - \cos t)\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

For $1 < \lambda < 2$, the parametrization ℓ_λ crosses a curvature-changing point of $\mathscr{L}_\lambda$ at $t_\lambda = \arccos\left(\dfrac{\lambda^2+2}{3\lambda}\right) \in]0,\pi/2[$. One easily arrives to the following result:

[1] The French word 'limaçon' can be translated as 'snail'.

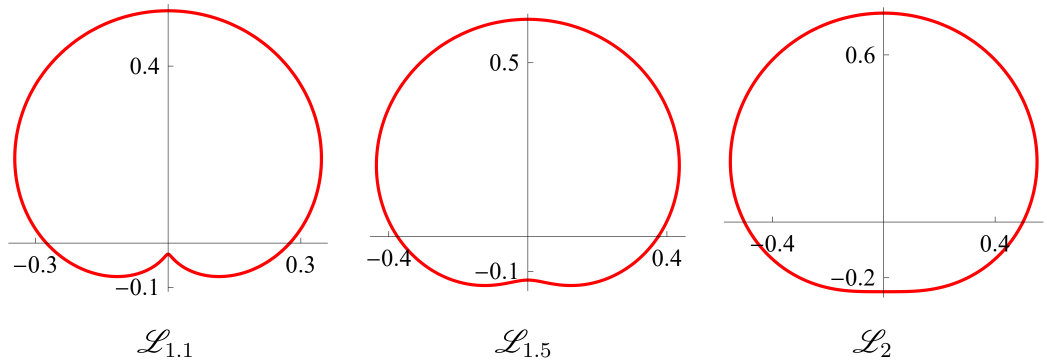

Figure 1: Pascal's limaçons for three possible values of the parameter.

Lemma 2.2. *The closed curves ℓ_λ are admissible for every $\lambda \in]1,2[$.*

This result completes the first step of the proof of Proposition 2.1, so that we turn now our attention to the *second step*. With this purpose we choose some cutoff function $\bar{m} \in C^\infty(\mathbb{R})$ with

$$\bar{m}(r) = 0 \text{ if } r \notin]-1,0[\,, \quad \bar{m}(r) > 0 \text{ if } r \in]-1,0[\,, \quad |\bar{m}'(r)| < 1 \;\forall r \in \mathbb{R}\,,$$

and define $\bar{\psi} : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\bar{\psi}(x, r + \bar{m}(r)) := \big(x, r + \bar{m}(r)\cos(\pi x)\big)\,. \tag{2}$$

This definition makes sense because the map $r \mapsto r + \bar{m}(r)$ is a diffeomorphism of the real line. The diffeomorphism $\bar{\psi} : \mathbb{R}^2 \to \mathbb{R}^2$ moves each point (x, η) downwards inside the vertical line $\{x\} \times \mathbb{R}$. Moreover,

$$\bar{\psi} \circ \mathcal{S} = \mathcal{S} \circ \bar{\psi}, \quad \bar{\psi}(x,\eta) = (x,\eta) \text{ if } x = 0 \text{ or } |\eta| \geq 1, \quad \bar{\psi}(x+2,\eta) = \bar{\psi}(x,\eta) + (2,0)\,. \tag{3}$$

Finally, let the admissible potential $\bar{U} : \mathbb{R}^2 \to \mathbb{R}$, $\bar{U} = \bar{U}(x,y)$ be defined by

$$\bar{U}(\bar{\psi}(x,\eta)) = \eta\,, \qquad (x,\eta) \in \mathbb{R}^2\,. \tag{4}$$

In this way, the level curves of $\bar{U}$ have the form $\bar{\psi}\big(\mathbb{R} \times \{\eta\}\big)$, for $\eta \in \mathbb{R}$. These level curves coincide with the integral trajectories of the vector field $J\nabla\bar{U}$. It follows from (4) that the nonzero vectors $J\nabla\bar{U}(\bar{\psi}(x,\eta))$ and $(\partial\bar{\psi}/\partial x)(x,\eta)$ are always collinear, and since $J\nabla\bar{U}(0,2) = (1,0) = (\partial\bar{\psi}/\partial x)(0,2)$, there exists a continuous function $\bar{c} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\bar{c}(x,\eta) > 0 \;\text{ and }\; \frac{\partial\bar{\psi}}{\partial x}(x,\eta) = \bar{c}(x,\eta)J\nabla\bar{U}(\bar{\psi}(x,\eta))\,, \qquad (x,\eta) \in \mathbb{R}^2\,. \tag{5}$$

This observation will play a role later. We are now ready to show

Lemma 2.3. *$\bar{U} : \mathbb{R}^2 \to \mathbb{R}$ is an admissible potential. In addition, ℓ_λ and $\bar{U}$ are semicoupled provided that $\lambda \in]1,2[$ is close enough to 2.*

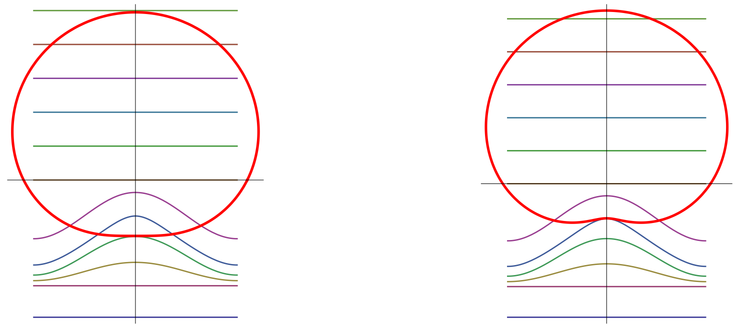

Figure 2: Some of the level curves of the potential $\bar{U}$ of Lemma 2.3 together with $\mathscr{L}_2$ (left picture), and $\mathscr{L}_\lambda$ with $\lambda \in]1,2[$ close to 2 (right picture).

Proof. One easily checks properties *(U_i)-(U_{ii})-(U_{iii})*. In addition, ($U\ell_i$) follows from direct computations for ℓ_2, and from a continuity argument for ℓ_λ with λ close to 2, see Fig. 2. The proof is complete. $\square$

From now on we fix $\lambda \in]1,2[$ sufficiently close to 2 so that Lemma 2.3 holds and set $\ell := \ell_\lambda : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$. Thus, ℓ and $\bar{U}$ are semicoupled, and in order to prove Proposition 2.1 we still need a suitable modification U of $\bar{U}$. We shall construct U by modifying the level curves of $\bar{U}$ on a narrow band $\mathcal{B}$ around $\{\ell(t) : t \in [\epsilon, \pi - \epsilon]\}$ for some small $\epsilon > 0$. The details are given in the next two sections.

3 The level curves of $\bar{U}$ as integral trajectories

Let the functions $l_1, l_2 : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ be the components of $\bar{\psi}^{-1} \circ \ell$, i.e.

$$\bar{\psi}(l_1(t), l_2(t)) = \ell(t)\,, \qquad t \in \mathbb{R}/2\pi\mathbb{Z}\,.$$

In other words, $l_1(t)$ is the first (horizontal) component of $\ell(t)$ and $l_2(t) = \bar{U}(\ell(t))$ (by (4)). The function l_2 is strictly increasing on $[0,\pi]$ (by ($U\ell_i$)). Set $\eta_- := l_2(0)$ and $\eta_+ := l_2(\pi)$, or, what is the same, $\ell(0) = (0, \eta_-)$ and $\ell(\pi) = (0, \eta_+)$, and notice that $\bar{\psi}(\mathbb{R} \times \{\eta\})$ intersects $\ell(]0,\pi[)$ if and only if $\eta_- < \eta < \eta_+$. This leads us to consider the set

$$\Omega := \{(x,y) \in [0,1] \times \mathbb{R} : \eta_- < \bar{U}(x,y) < \eta_+\} = \{\bar{\psi}(x, l_2(t)) : x \in [0,1],\ 0 < t < \pi\}\,,$$

and the function $\nu : \Omega \to \mathbb{R}$ defined by

$$\nu\big(\bar{\psi}(x, l_2(t))\big) := x - l_1(t)\,, \qquad 0 \le x \le 1,\ \ 0 < t < \pi\,. \tag{6}$$

In words ν measures the distance in the abscissa coordinate from each point of Ω to the intersection point of the corresponding level curve of $\bar{U}$ with $\ell(]0,\pi[)$.

We also consider the vector field

$$\bar{Z} := \frac{\partial \bar{\psi}}{\partial x} \circ \bar{\psi}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2\,.$$

The integral trajectories of $\bar{Z}$ are the level curves of $\bar{U}$ as parameterized by $\bar{\psi}$. We can now rephrase (5) as follows: there exists a continuous function $\hat{c}:\mathbb{R}^2\to\mathbb{R}$ such that

$$\hat{c}(x,y)>0 \quad\text{and}\quad \bar{Z}(x,y)=\hat{c}(x,y)J\nabla\bar{U}(x,y)\,,\qquad (x,y)\in\mathbb{R}^2\,. \tag{7}$$

Other properties of $\bar{Z}$ are listed below:

($\bar{Z}_i$): $\bar{Z}(x+2,y)=\bar{Z}(x,y)$ and $\bar{Z}\circ\mathcal{S}=-\mathcal{S}\circ\bar{Z}$ on $\mathbb{R}^2$;

($\bar{Z}_{ii}$): $\bar{Z}(x,y)\neq(0,0)\ \forall(x,y)\in\mathbb{R}^2$, $\bar{Z}(x,y)=(1,0)$ if $|y|\geq 1$;

($\bar{Z}_{iii}$): There exists some constant $\kappa>0$ such that $\langle\bar{Z}(x,y),\nabla\nu(x,y)\rangle\geq\kappa$ on Ω;

($\bar{Z}\ell_i$): $\langle\bar{Z}(\ell(t)),J\dot{\ell}(t)\rangle>0\ \forall t\in]0,\pi[$.

Notice that ($\bar{Z}_i$)-($\bar{Z}_{ii}$) are direct consequences from (3), while ($\bar{Z}_{iii}$) follows (with $\kappa=1$) from differentiation with respect to x in (6). The combination of (7) and ($\bar{Z}_{iii}$) implies that $\langle J\nabla\bar{U}(x,y),\nabla\nu(x,y)\rangle>0$ on Ω, and, in particular, $\langle J\nabla\bar{U}(\ell(t)),\nabla\nu(\ell(t))\rangle>0$ for any $t\in]0,\pi[$. From the equality $\nu(\ell(t))=0$ we deduce that $\langle\nabla\nu(\ell(t)),\dot{\ell}(t)\rangle=0$ and (by ($U\ell_i$)) we conclude the existence of some continuous function $c:]0,\pi[\to\mathbb{R}$ such that

$$c(t)>0 \quad\text{and}\quad \nabla\nu(\ell(t))=c(t)J\dot{\ell}(t)\,,\qquad 0<t<\pi\,. \tag{8}$$

In combination with ($\bar{Z}_{iii}$), this implies ($\bar{Z}\ell_i$). It should be noticed that, by (7), assumption ($\bar{Z}\ell_i$) expresses the fact that $\bar{U}(\ell(t))$ is strictly increasing for $0<t<\pi$.

Vector fields $\bar{Z}:\mathbb{R}^2\to\mathbb{R}^2$ satisfying properties ($\bar{Z}_i$)-($\bar{Z}_{ii}$)-($\bar{Z}_{iii}$)-($\bar{Z}\ell_i$) will be called *normal* in what follows.

4 Bending the level curves of the potential

Our strategy to construct the announced modification of $\bar{U}$ will first need a suitable modification of the vector field $\bar{Z}$. This modification, denoted $Z:\mathbb{R}^2\to\mathbb{R}^2$, will satisfy

$$Z(x,y)=\bar{Z}(x,y)\quad\text{on } ([0,1]\times\mathbb{R})\backslash\Omega\,. \tag{9}$$

Lemma 4.1. *There exists a normal vector field $Z:\mathbb{R}^2\to\mathbb{R}^2$ with (9) and satisfying*

$$\langle Z(\ell(t)),\dot{\ell}(t)\rangle\begin{cases}>0 \text{ if } 0\leq t<t_*\\ <0 \text{ if } t_*<t\leq\pi\end{cases},\qquad \frac{d}{dt}\Big|_{t=t_*}\langle Z(\ell(t)),\dot{\ell}(t)\rangle<0\,, \tag{10}$$

where t_* is the same number appearing in assumption (ℓ_{iv}).

Remark 4.2. *Condition (10) expresses the fact that, as $\ddot{\ell}$, $JZ(\ell(t))$ points outwards (with respect to $\mathscr{L}=\ell(\mathbb{R}/2\pi\mathbb{Z})$) on the concave part of $\mathscr{L}$, and inwards on the convex part of $\mathscr{L}$. At the end of this section we shall construct an admissible potential U such that $-\nabla U$ has the same property.*

Proof of Lemma 4.1. All four vectors $\dot\ell(0), -\dot\ell(\pi), \bar Z(\ell(0)), \bar Z(\ell(\pi))$ are positive multiples of $(1,0)$. Consequently, $\langle \bar Z(\ell(0)), \dot\ell(0)\rangle > 0 > \langle \bar Z(\ell(\pi)), \dot\ell(\pi)\rangle$, and there exists some $0 < \epsilon < t_*$ such that

$$\langle \bar Z(\ell(t)), \dot\ell(t)\rangle \begin{cases} > 0 & \text{if } 0 \le t \le \epsilon\,, \\ < 0 & \text{if } \pi - \epsilon \le t \le \pi\,. \end{cases}$$

On the other hand, in view of (8) there is some small constant $c_1 > 0$ with

$$\langle \nabla\nu(\ell(t)), J\dot\ell(t) - c_1(t-t_*)\dot\ell(t)\rangle > 0 \text{ for all } t \in [\epsilon, \pi-\epsilon]\,. \tag{11}$$

Choose now $\epsilon' \in]0,\epsilon[$ with $0 < \epsilon - \epsilon' < \min\{l_1(t), 1 - l_1(t) : \epsilon' \le t \le \pi - \epsilon'\}$ and consider the open set

$$\begin{aligned} \mathcal{B} &:= \left\{\bar\psi(s, l_2(t)) : \epsilon' < t < \pi - \epsilon',\ |s - l_1(t)| < \epsilon - \epsilon'\right\} = \\ &= \{(x,y) \in \Omega : l_2(\epsilon') < U(x,y) < l_2(\pi - \epsilon'),\ |\nu(x,y)| < \epsilon - \epsilon'\} \subset]0,1[\times]-1,1[\,. \end{aligned}$$

If $\epsilon - \epsilon'$ is small enough then (by (11)) there exists a C^∞ vector field $\widehat Z : \mathcal{B} \to \mathbb{R}^2$ with

$$\widehat Z(\ell(t)) = J\dot\ell(t) - c_1(t-t_*)\dot\ell(t)\ \forall t \in]\epsilon', \pi-\epsilon'[, \quad \langle \nabla\nu(x,y), \widehat Z(x,y)\rangle > 0 \text{ on } \mathcal{B}\,.$$

To conclude, fix some cutoff function $m \in C^\infty(\mathbb{R}^2)$ satisfying

$$0 \le m \le 1 \text{ on } \mathbb{R}^2\,, \qquad \operatorname{supp}(m) \subset \mathcal{B}\,, \qquad m(\ell(t)) = 1\ \forall t \in [\epsilon, \pi-\epsilon]\,,$$

and define $Z : [0,1]\times\mathbb{R} \to \mathbb{R}^2$ by $Z := \begin{cases} (1-m)\bar Z + m\widehat Z & \text{on } \mathcal{B} \\ \bar Z & \text{on } ([0,1]\times\mathbb{R})\backslash\mathcal{B} \end{cases}.$

The extension of Z to $\mathbb{R}^2$ determined by (Z_i) is normal and satisfies (9)-(10). It completes the proof. □

The vector field Z has been built with the aim of using its integral curves as the level sets of the modified potential U; with this fact in mind we consider the function $\psi = \psi(s,\eta)$, defined by

$$\frac{\partial\psi}{\partial s}(s,\eta) = Z(\psi(s,\eta))\,, \qquad \psi(0,\eta) = (0,\eta)\,.$$

The vector field Z being bounded and smooth, ψ is well defined on $\mathbb{R}^2$. The result below shows some properties of ψ which mirror those of $\bar\psi$ listed in (3). Here, we denote by (Z_i), (Z_{ii}), etc, to properties $(\bar Z_i)$, $(\bar Z_{ii})$... but referred to the vector field Z instead of $\bar Z$.

Lemma 4.3. *$\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving diffeomorphism of class C^∞. Furthermore,*

$$\psi\circ\mathcal{S} = \mathcal{S}\circ\psi\,, \quad \psi(s,\eta) = (s,\eta) \text{ if } |\eta| \ge 1\,, \quad \psi(s + P(\eta), \eta) = \psi(s,\eta) + (2,0)\,, \tag{12}$$

for some smooth function $P : \mathbb{R} \to]0,+\infty[$ with $P(\eta) = 2$ if $|\eta| \ge 1$.

Proof. We start by observing that $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is injective. Indeed, the contrary would lead to the existence of an integral curve of Z crossing twice the y axis. However, (Z_i)-(Z_{ii}) imply that $Z(0, y)$ is a positive multiple of $(1, 0)$ for every $y \in \mathbb{R}$, and this leads to a contradiction.

Next we observe that $\bar{\psi}(s, \eta) \in ([0, 1] \times \mathbb{R})\backslash\Omega$ if $(s, \eta) \in [0, 1] \times (\mathbb{R}\backslash]\eta_-, \eta_+[)$. By (9) we deduce that

$$\psi\big([0, 1] \times (\mathbb{R}\backslash]\eta_-, \eta_+[)\big) = \bar{\psi}\big([0, 1] \times (\mathbb{R}\backslash]\eta_-, \eta_+[)\big) = ([0, 1] \times \mathbb{R})\backslash\Omega\,,$$

so that, by the previously-observed injectivity,

$$\psi([0, +\infty[\times]\eta_-, \eta_+[) \cap ([0, 1] \times \mathbb{R}) \subset \Omega\,. \tag{13}$$

We claim that $\psi([0, +\infty[\times\{\eta\}) \not\subset [0, 1] \times \mathbb{R}$ for any $\eta \in \mathbb{R}$, i.e., no integral curve of Z can remain in $[0, 1] \times \mathbb{R}$ for all future time. Indeed, arguing by contradiction, if the contrary happened we observe (by (9)) that $\eta_- < \eta < \eta_+$. Then (13) implies that $\psi([0, +\infty[\times\{\eta\}) \subset \Omega$. Thus, by (Z_{iii})

$$\frac{d}{ds}\nu\big(\psi(s, \eta)\big) \geq \kappa\,, \qquad s \geq 0\,,$$

for some constant $\kappa > 0$. In particular, $\nu\big(\psi(s, \eta)\big) \to +\infty$ as $s \to +\infty$, which is a contradiction since ν is bounded on Ω.

In particular we see that for every $\eta \in \mathbb{R}$ there exists some $P(\eta) > 0$ such that $\psi(P(\eta)/2, \eta) \in \{1\} \times \mathbb{R}$. In fact, since $Z(1, y)$ is a positive multiple of $(1, 0)$ for every $y \in \mathbb{R}$ (this can be argued as before), we see that the number $P(\eta)$ is unique and is indeed a smooth function of η with $P(\eta) = 2$ if $|\eta| > 1$. Together with (Z_i), this implies (12).

We see now that $|\psi(s, \eta)| \to \infty$ as $|s| \to \infty$ uniformly with respect to $\eta \in \mathbb{R}$. On the other hand, $\psi(s, \eta) = (s, \eta)$ if $|\eta| \geq 1$ (by (Z_{ii})), and we deduce that

$$\lim_{|(s,\eta)|\to\infty} |\psi(s, \eta)| = \infty\,.$$

Now, a well-known argument (see, e.g., [1, p. 23]) implies that $\psi(\mathbb{R}^2) = \mathbb{R}^2$. In consequence, $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism.

Fix now some point $(s_0, \eta_0) \in \mathbb{R}^2$ and denote by $P : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto P(x, y)$ to the Poincaré map associated to the vector field Z and the time period s_0. It is well-known that P is a diffeomorphism from $\mathbb{R}^2$ to itself. From the equality $\psi(s, \eta) = P(\psi(s - s_0, \eta))$ we get

$$\frac{\partial\psi}{\partial s}(s_0, \eta_0) = \gamma\frac{\partial P}{\partial x}(0, \eta_0) \text{ for some } \gamma > 0, \qquad \frac{\partial\psi}{\partial\eta}(s_0, \eta_0) = \frac{\partial P}{\partial y}(0, \eta_0),$$

and we deduce that $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is an (orientation-preserving) local diffeomorphism and thus (since it was already shown to be a homeomorphism), a global diffeomorphism. The proof is complete.

□

The end of the proof of Proposition 2.1. Define

$$U : \mathbb{R}^2 \to \mathbb{R}\,, \qquad U(\psi(s,\eta)) := \eta\,, \tag{14}$$

so that

$$\nabla U(\psi(s,\eta)) = -\frac{1}{\det(\psi'(s,\eta))}\, JZ(\psi(s,\eta))\,. \tag{15}$$

Now, $(U_i) - (U_{ii})$ follow from (12), while (U_{iii}) is clear from the definition of U. Since it was observed (in Lemma 4.3) that ψ is orientation-preserving, statement $(U\ell_i)$ follows from the combination of $(Z\ell_i)$ and (15) (for its first part) and the analogous assumption satisfied by U (in its second part). Finally, $(U\ell_{ii})$ arises from (10)-(15). It completes the proof. □

5 A conformally-Newtonian equation

From now on, we fix an admissible closed curve ℓ and a coupled admissible potential U; the existence such a pair is ensured by Proposition 2.1. The combination of conditions (ℓ_i)-(U_i) on one hand, and (ℓ_{iv})-$(U\ell_{ii})$ on the other, implies that the function $\chi : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ defined by

$$\chi(t) := \frac{\det(\dot\ell(t), \ddot\ell(t))}{\det(\nabla U(\ell(t)), \dot\ell(t))}\,, \qquad t \in (\mathbb{R}/2\pi\mathbb{Z})\backslash\{-t_*, t_*\}\,,$$

and extended to $\pm t_*$ by continuity, is actually C^∞-smooth, even and positive. This fact will be used in the proof of our next result.

Lemma 5.1. *Under the above, there exist some $T > 0$, a C^∞ diffeomorphism $\tau : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ and a C^∞ function $w : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ such that*

(i) $\tau(-t) = -\tau(t)$, $w(-t) = w(t) > 0$ $\forall t \in \mathbb{R}/T\mathbb{Z}$,

and, letting $\gamma(t) := \ell(\tau(t))$, one has:

(ii) $\ddot\gamma(t) = -w(t)\nabla U(\gamma(t))$ $\forall t \in \mathbb{R}/T\mathbb{Z}$.

Proof. Differentiating twice in the equality $\gamma(t) = \ell(\tau(t))$ we see that statement *(ii)* above is equivalent to

$$\ddot\tau\,\dot\ell(\tau) + \dot\tau^2\ddot\ell(\tau) = -w\nabla U(\ell(\tau))\,,$$

where, for simplicity, we have dropped the dependence on the time variable t from the notation. This equation in $\mathbb{R}^2$ can be equivalently rewritten as the system

$$\begin{cases} \ddot\tau\,|\dot\ell(\tau)|^2 + \dot\tau^2\langle\ddot\ell(\tau), \dot\ell(\tau)\rangle = -w\langle\nabla U(\ell(\tau)), \dot\ell(\tau)\rangle \\ \dot\tau^2\det(\ddot\ell(\tau), \dot\ell(\tau)) = -w\,\det(\nabla U(\ell(\tau)), \dot\ell(\tau)) \end{cases},$$

or what is the same (isolating w in the second equation and replacing its value into the first one),

$$\begin{cases} \ddot{\tau}/\dot{\tau} = -h(\tau)\dot{\tau} \\ w = \chi(\tau)\,\dot{\tau}^2 \end{cases},$$

where $h(\tau) := \big(\langle \ddot{\ell}(\tau), \dot{\ell}(\tau)\rangle + \chi(\tau)\langle \nabla U(\ell(\tau)), \dot{\ell}(\tau)\rangle\big)/|\dot{\ell}(\tau)|^2$. Integration of the first equation transforms the system above into

$$\begin{cases} \dot{\tau} = ce^{-H(\tau)} \text{ for some constant } c > 0 \\ w = \chi(\tau)\,\dot{\tau}^2 \end{cases}, \tag{16}$$

where $H(\tau) := \displaystyle\int_0^\tau h(r)dr$. Notice that h is 2π-periodic and odd, implying that H is 2π-periodic and even.

We fix now an arbitrary number $c > 0$ (for instance, $c = 1$), and observe that the solution $\tau = \tau(t)$ of the first equation of (16) with $\tau(0) = 0$ is an odd diffeomorphism from $\mathbb{R}$ into itself satisfying $\tau(t + T) = \tau(t) + 2\pi$ for $T = \tau^{-1}(2\pi)$. We define w as in the second equation of (16), and observe that *(i)-(ii)* hold. This proves the lemma. □

6 From conformally-Newtonian to Newtonian

The following lemma could be taken from an elementary course of real analysis, and we state it without proof.

Lemma 6.1. *Let $a > 0$ be some positive number and $u, w : [-a, a] \to \mathbb{R}$ be even functions of class C^∞. Assume that $\ddot{u}(0) > 0$ and $\dot{u}(t) > 0$ $\forall t \in]0, a]$. Then, there exists a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that $w(t) = f(u(t))$ for any $t \in [-a, a]$.*

Remark: Setting $u_+ := u\big|_{]0,a]} :]0, a] \to]u(0), u(a)]$, it is clear from the assumptions that $f\big|_{]u(0),u(a)]} = w \circ u_+^{-1} \in C^\infty(]u(0), u(a)])$. Consequently, the interest of the lemma is to ensure that the derivatives of all orders of f are continuous up to $u(0)$.

From this moment on we shall work on the cylinder $(\mathbb{R}/2\mathbb{Z}) \times \mathbb{R}$ with coordinates (θ, y), instead of the plane $\mathbb{R}^2$. Notice that, by assumption (U_i), one can actually see the potential U of Proposition 2.1 as being defined on this cylinder.

Corollary 6.2. *There exists a C^∞ potential $W : (\mathbb{R}/2\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$, $W = W(\theta, y)$, with*

(W_1) $W(\theta, y) \equiv 1$ *if* $y \le -1$, $\qquad W(\theta, y) \equiv 0$ *if* $y \ge 1$,

(W_2) $\nabla W(\theta, y) \ne (0, 0)$ *if* $|y| < 1$,

and, for some $0 < \varepsilon_1 < 1$,

(W_3) $(\partial W/\partial y)(\theta, y) < 0$ *if* $1 - \varepsilon_1 < |y| < 1$,

and such that the equation $\ddot{\gamma} = -\nabla W(\gamma)$ *has a closed orbit*

$$\gamma : \mathbb{R}/T\mathbb{Z} \to]-1,1[^2 .$$

Proof. Let $U : (\mathbb{R}/2\mathbb{Z})\times\mathbb{R} \to \mathbb{R}$ and $\ell : \mathbb{R}/2\pi\mathbb{Z} \to]-1,1[^2$ be as given by Proposition 2.1; let $w : \mathbb{R} \to]0,+\infty[$ and $\gamma := \ell\circ\tau : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ be as given by Lemma 5.1, and set $u(t) := U(\gamma(t))$. This is an even function, both with respect to $t = 0$ and with respect to $t = T/2$, and it follows from Lemma 6.1 that there exists a C^∞ function $f : \mathbb{R} \to]0,+\infty[$ with $w(t) = f(u(t)) = f(U(\gamma(t)))$ for any t. Set $F(r) := \int_{-1}^{r} f(u)du$ and choose some smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with

(φ_1) $\varphi(r) \equiv 0$ on $]-\infty,0]$, $\quad\varphi(r) \equiv 1$ on $[F(1),+\infty[$,

(φ_2) $\varphi'(r) > 0$ on $]0,F(1)[$,

(φ_3) $\varphi'(r) \equiv \varphi_* > 0$ on $\big[F(\min_{\gamma(\mathbb{R})} U), F(\max_{\gamma(\mathbb{R})} U)\big]$,

and define $\gamma : \mathbb{R} \to]-1,1[^2$ and $W : (\mathbb{R}/2\mathbb{Z})\times\mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) := \mathfrak{S}(\gamma(\sqrt{\varphi_*}\, t)) , \qquad W := \varphi\circ F\circ U\circ\mathfrak{S} ,$$

where we denote by $\mathfrak{S} : (\mathbb{R}/2\mathbb{Z})\times\mathbb{R} \to (\mathbb{R}/2\mathbb{Z})\times\mathbb{R}$ the symmetry $\mathfrak{S}(\theta,y) := (\theta,-y)$. The result follows. □

7 From the cylinder into a ring

It will be convenient to consider, for any $\rho > 1$, the open ring

$$\mathcal{R}_\rho := \{q \in \mathbb{R}^2 : \rho - 1 < |q| < \rho + 1\} .$$

We observe the following geometrical property: for each $0 < \alpha < 1$ there exists some $\rho(\alpha) > 1$ such that, for any $\rho \geq \rho(\alpha)$, one has

$$[-\alpha,\alpha] \times [\rho-\alpha,\rho+\alpha] \subset \mathcal{R}_\rho . \tag{17}$$

In these situations one can use the following

Lemma 7.1. *Let* $0 < \alpha < 1 < \rho$ *be such that (17) holds. Then, there exists a* C^∞ *diffeomorphism* $\Gamma : (\mathbb{R}/2\mathbb{Z})\times[-1,1] \to \overline{\mathcal{R}_\rho}$ *with* $\Gamma((\mathbb{R}/2\mathbb{Z})\times\{\pm1\}) = \{q \in \mathbb{R}^2 : |q| = \rho \pm 1\}$ *and*

$$\Gamma(x,y) = (x, y+\rho) \text{ if } (x,y) \in [-\alpha,\alpha]\times[-\alpha,\alpha] .$$

This result is elementary and its proof will be skipped. It will play a role in the proof of the following

Proposition 7.2. *There exists some* $\rho > 1$ *and a* C^∞ *potential* $\mathcal{V} : \mathbb{R}^2 \to \mathbb{R}$ *satisfying:*

($\mathcal{V}_1$) $\mathcal{V}(q) = 1$ if $|q| \leq \rho - 1$, $\qquad \mathcal{V}(q) = 0$ if $|q| \geq \rho + 1$,

($\mathcal{V}_2$) $\nabla\mathcal{V}(q) \neq (0,0)$ $\forall q \in \mathcal{R}_\rho$,

and, for some $0 < \varepsilon < 1$,

($\mathcal{V}_3$) $\langle \nabla\mathcal{V}(q), q\rangle \leq 0$ if $\operatorname{dist}(q, \partial\mathcal{R}_\rho) < \varepsilon$,

and such that the equation $\ddot{q} = -\nabla\mathcal{V}(q)$ has a closed orbit $q : \mathbb{R}/T\mathbb{Z} \to \mathcal{R}_\rho$.

Proof. Choose $W : (\mathbb{R}/2\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}/T\mathbb{Z} \to]-1,1[^2$ as given by Corollary 6.2. Pick some $0 < \alpha < 1$ such that $\gamma(\mathbb{R}) \subset]-\alpha,\alpha[\times]-\alpha,\alpha[$, and fix $\rho > 1$ big enough so that (17) holds. Choose Γ as in Lemma 7.1 and set

$$\mathcal{V} : \mathbb{R}^2 \to \mathbb{R}, \qquad \mathcal{V}(q) := \begin{cases} W(\Gamma^{-1}(q)) & \text{if } q \in \mathcal{R}_\rho \\ 1 & \text{if } |q| \leq \rho - 1 \\ 0 & \text{if } |q| \geq \rho + 1 \end{cases},$$

and $q := \Gamma \circ \gamma = (0,\rho) + \gamma$. Now, it is clear that $\mathcal{V}$ and q are both C^∞; furthermore, ($\mathcal{V}_1$), ($\mathcal{V}_2$) and ($\mathcal{V}_3$) follow, respectively, from (W_1), (W_2) and (W_3). The proof is complete. □

Remark. There is an alternative way to deduce Proposition 7.2 (for the ring of radii $e^{\pm\pi}$ instead of $\mathcal{R}_\rho$) from Corollary 6.2 without any need of Lemma 7.1. Indeed, the complex exponential $z \mapsto e^{\pi z}$ is a conformal map, and one can use an argument due to Goursat [5] to transform Newtonian equations on the cylinder into Newtonian equations on the ring. We have opted by the proof above because of its conceptual simplicity.

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. Choose $\mathcal{V} : \mathbb{R}^2 \to \mathbb{R}$, $0 < \varepsilon < 1$ and $q : \mathbb{R}/T\mathbb{Z} \to \mathcal{R}_\rho$ as given by Proposition 7.2. For any nonnegative integer n we define

$$M_n := \max\left\{ |\partial^\alpha\mathcal{V}(q)| : \begin{array}{c} q \in \mathbb{R}^2 \\ \alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2,\ \alpha_i \geq 0,\ \alpha_1 + \alpha_2 \leq n \end{array} \right\}.$$

Set $\mu := (\rho+1)/(\rho-1) > 1$, and $\mathscr{G} := \bigcup_{n\geq 0} \left]\frac{\rho+1-\varepsilon}{\mu^{n+1}}, \frac{\rho-1+\varepsilon}{\mu^n}\right[$ (the intervals are nonempty and nonoverlapping). Choose some C^∞ function $h : [0,+\infty[\to \mathbb{R}$ with

$$h'(r) < 0 \text{ if } r \in \mathscr{G}, \qquad h'(r) = 0 \text{ if } r \in [0,+\infty[\setminus\mathscr{G}, \tag{18}$$

and define $V : \mathbb{R}^2 \to \mathbb{R}$ by:

$$V(q) := h(|q|) + \sum_{n=0}^{\infty} \left(\frac{1}{2^n\, M_n\, \mu^{n^2}}\right) \mathcal{V}(\mu^n q), \qquad q \in \mathbb{R}^2.$$

In this way, it is clear that $V \in C^\infty(\mathbb{R}^2)$. It has a (degenerate) maximum at the origin, which (by ($\mathcal{V}_2$)-($\mathcal{V}_3$)-(18)) is the only critical point of V on $\{q \in \mathbb{R}^2 : |q| \leq \rho + 1\}$. Finally, one checks that, setting $T_n := 2^{n/2}\sqrt{M_n}\,\mu^{n^2/2-n}T$, then $q_n(t) := (1/\mu^n)q(Tt/T_n)$ is a solution of $\ddot{q} = -\nabla V(q)$ for each n. The proof is complete. □

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Author's address: Departamento de Matematica Aplicada, Universidad de Granada, 18071, Granada, Spain. e-mail: `ajurena@ugr.es`