Some rigorous results on the 1:1 resonance of the spin-orbit problem*

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Abstract

We study the classical planar spin-orbit model from an analytical point of view, with no requirements of smallness of the orbital eccentricity and taking into account dissipative forces. The problem depends on e, the eccentricity of the orbit, and on Λ, the oblateness of the spinning body. Our main concern is the capture into the 1:1 resonance for points of the (e, Λ)-plane. First, we find a region of uniqueness of the 1:1 resonance, which is the continuation from the solution for $e = 0$. Then, a subregion of linear stability is estimated. We also study a separatrix close to the line $e = e_* \approx 0.682$, beyond which the resonance is unstable. Finally, we study the dissipative case by estimating regions of asymptotic stability of the solution (capture into resonance) depending on the strength of the dissipation applied.

Keywords: Spin-orbit problem, Forced pendulum, Dissipative systems, Synchronous resonance, Capture into resonance, Asymptotic stability

1 Introduction

Consider a satellite whose center of mass is moving around a planet in a Keplerian elliptical orbit of eccentricity e. We are interested in the spin of the satellite around its center of mass, so, we will identify the satellite with a tridimensional object and the planet with a point mass. Let the satellite be triaxial, with principal moments of inertia $\mathcal{A} < \mathcal{B} < \mathcal{C}$. Assume that the spin axis of the satellite is perpendicular to the orbital plane and coincides with the smallest of its physical axes, which is associated to $\mathcal{C}$. Then, the parameter $\epsilon = \frac{3}{2}\frac{\mathcal{B}-\mathcal{A}}{\mathcal{C}}$ measures the oblateness of the satellite in the orbital plane.

Let us identify the orbital plane with the complex plane $\mathbb{C}$, consider the planet fixed in the 0 and let the position of the center of mass of the satellite, in the exponential notation, be $\mathcal{P} = r \exp[if] \in \mathbb{C}$, where $r > 0$ and f are real functions of the time. Note that this point describes an ellipse with focus at the origin, so, the polar coordinates r and f vary periodically with time. Let us take convenient units so that the period is 2π and the semi-major axis of

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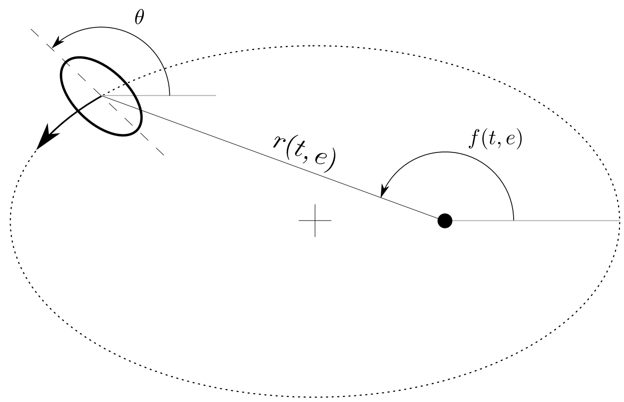

Figure 1: The spin-orbit problem

the ellipse is 1. In the usual terminology, f is called *true anomaly* and the time t is the *mean anomaly*. There is a third useful angle u, the *eccentric anomaly*, which is defined by the famous Kepler's equation

$$t = u - e\sin u, \tag{1}$$

and which let us determine the Keplerian ellipse simply by

$$r = 1 - e\cos u. \tag{2}$$

We can write the position of the planet also in terms of the eccentric anomaly as

$$r\exp[if] = \cos u - e + i\sqrt{1-e^2}\sin u. \tag{3}$$

Note that for $t = 0$ we assumed that $f = u = 0$, and consequently, $f = u = \pi$ when $t = \pi$. The expressions eqs. (2) and (3) relate the true and the eccentric anomalies. Moreover, Equations (1) to (3) define $u = u(t,e)$, $r = r(t,e)$ and $f = f(t,e)$ as analytic functions in both entries.

Let θ be the angle that determines the direction of the body's axis of the major elongation with respect to the major axis of the ellipse. See Figure 1. The motion of the satellite is modeled by the following biparametric equation

$$\ddot{\theta} + \frac{\epsilon}{r(t,e)^3}\sin[2(\theta - f(t,e))] = \mathcal{T}_d(t,\dot{\theta}), \quad e \in [0,1),\ \epsilon > 0, \tag{4}$$

where $\mathcal{T}_d$ is a dissipative torque, which has different forms depending on the model. One of the most popular is the MacDonald torque

$$\mathcal{T}_d(t,\dot{\theta}) = -\frac{C_M}{r(t,e)^6}\sin[2\Delta t(\dot{\theta} - \dot{f}(t,e))] \approx -\frac{\delta}{r(t,e)^6}(\dot{\theta} - \dot{f}(t,e)), \tag{5}$$

where $1 \gg \delta = 2C_M\Delta t \geq 0$, and C_M is a constant depending on the parameters of the bodies. This formula has been extensively used, taking as reference [15] or [21], for example. According to [13], to obtain (5), the dissipation is modeled by assuming that there is a time delay between the deforming disturbance and the actual deformation of the body. That delay is a small fixed amount Δt (time lag), which leads to an angular lag of $(\dot{f}(t,e) - \dot{\theta})\Delta t$ (geometric lag).

This is the so-called spin-orbit problem, it has been widely studied, see for example [7], [3] and [22] for the conservative case, or [15] and [9] for the dissipative case.

We are going to deal with the capture into the 1:1 resonance of equation (4), i.e., solutions that satisfy $\theta(t + 2\pi) = \theta(t) + 2\pi$. It is also known as synchronous resonance. This is the resonance of most important practical interest because all the tidally evolved satellites in the Solar System are trapped on it. For us it is indeed a very familiar phenomenon: from the Earth we always look at the same side of the Moon. Mercury is the only body in the Solar System that is captured in a different spin-orbit resonance, the 3:2, for which $\theta(t+4\pi) = \theta(t)+6\pi$. This is also a very interesting resonance, see for example [4] or [8]. It is accepted that the capture into the resonance is an effect driven by the dissipation. We will approach this phenomenon from an analytical point of view, by looking for conditions resulting in the existence of an asymptotically stable synchronous resonance.

Let us take the change of variable $\Theta = 2(\theta - f)$ and a more general dissipation proportional to $(\dot\theta - \dot f)$, such that equation (4) turns into

$$\ddot\Theta + \delta D(t,e)\dot\Theta + \frac{\Lambda}{r(t,e)^3}\sin\Theta = -2\ddot f(t,e), \qquad e \in [0,1),\ \Lambda = 2\epsilon \gg \delta \geq 0, \tag{6}$$

where $D(t,e)$ is a positive analytic function, which is 2π-periodic in t. Note that, for the MacDonald torque, $D(t,e) = r(t,e)^{-6}$. Equation (6) models a damped and forced pendulum of variable length. Since $f(t+2\pi,e) = f(t,e)+2\pi$, then, $1:1$ resonances correspond to solutions of (6) satisfying $\Theta(t + 2\pi) = \Theta(t)$.

We will start discussing, in Section 2, the linear stability of a particular 2π-periodic solution for the non-dissipative problem

$$\ddot\Theta + \frac{\Lambda}{r(t,e)^3}\sin\Theta = -2\ddot f(t,e), \qquad e \in [0,1),\ \Lambda > 0. \tag{7}$$

This solution is the analytic continuation of the trivial solution for $e = 0$. This will lead us to a region of linear stability in the (e,Λ)-plane. Section 3 will be devoted to prove the linear instability of the solution of the previous section for high eccentricities, no matter how small Λ is considered. Our main concern about linear stability is because it will allow us to find an asymptotically stable solution for the dissipative case (6) for $\delta > 0$, which is a continuation of the solution considered in previous sections. This will be proved in Section 4, provided that δ is smaller than a certain quantifiable value $\bar\delta$. We will compute such value for a few systems, including the Earth-Moon system, using the MacDonald torque. Finally, in Section 5, we will present a discussion, putting our results in context with other previous works.

2 Linear stability of the synchronous resonance

The main result of this section is at the end of it, in Proposition 2. It determines a region of linear stability written in terms of the functions $\Lambda_1(e)$, defined in the first subsection, and $\Lambda_2(e)$, defined in the second one.

2.1 Uniqueness of the odd 2π-periodic solution

Note that equation (7) is invariant under the change $(t,\Theta) \to (-t,-\Theta)$, since $f(-t,e) = -f(t,e)$ and $r(-t,e) = r(t,e)$. Then, if $\Theta(t)$ is a solution of (7), so it is $-\Theta(-t)$. On the other hand, for $e = 0$, the equation (7) becomes the free pendulum equation $\ddot\Theta + \Lambda\sin\Theta = 0$. In this case we know that for $\Lambda \leq 1$, the only 2π-periodic solution are the equilibria $\Theta \equiv 0$ and $\Theta \equiv \pi$. Since the trivial solution is the stable one, it is natural to look for the 2π-periodic continuation of such solution for $e \neq 0$ in the family of the odd solutions of (7), which is equivalent to solve the Dirichlet problem

$$\begin{cases} \ddot{\Theta} + \frac{\Lambda}{r(t,e)^3} \sin \Theta = -2\ddot{f}(t,e), \\ \Theta(0) = \Theta(\pi) = 0. \end{cases} \tag{8}$$

It is well known from nonlinear analysis that this problem has at least one solution because equation (7) can be written as

$$\ddot{\Theta} = F(t, \Theta), \tag{9}$$

with $F(t, \Theta)$ bounded. Making explicit[1] the dependence with the parameters we can write

$$F(t, \Theta; e, \Lambda) = -\frac{\Lambda}{r(t,e)^3} \sin \Theta - 2\ddot{f}(t,e). \tag{10}$$

We are going to present a simple proof of the existence of solution, based on the shooting method. This will be a convenient way to introduce some notation.

Let $\Theta(t) = \vartheta(t, v)$ be the solution of (9) satisfying initial conditions $\Theta(0) = 0$, $\dot{\Theta}(0) = v \in \mathbb{R}$. Solutions of the problem (8) are in correspondence with the solutions of the equation $\vartheta(\pi, v) = 0$. As solution of (9), we know that ϑ satisfies the following integral equation

$$\vartheta(t, v) = vt + \int_0^t (t-s) F(s, \vartheta(s, v)) ds.$$

Moreover, since there exists a positive number $M \geq |F(t, \Theta)|$, then

$$|\vartheta(t, v) - vt| \leq M \frac{t^2}{2},$$

for each $t \in \mathbb{R}$. Using this estimate for $t = \pi$, we conclude that

$$\lim_{v \to \pm\infty} \vartheta(\pi, v) = \pm\infty.$$

In consequence, the equation $\vartheta(\pi, v) = 0$ must have at least one solution.

We know now that the Dirichlet problem (8) has a solution, however, it is not necessarily unique. For instance, in the circular case ($e = 0$), the number of solutions of (8) becomes arbitrarily large as Λ tends to infinity. We would like to determine a region of parameters (e, Λ) where there is uniqueness for the problem (8).

The shooting method is also useful to prove uniqueness by proving that $\vartheta(\pi, v)$ is monotone, which is equivalent to say that the partial derivative $\partial_v \vartheta(\pi, v)$ never vanishes. From the theorem of differentiability with respect to initial conditions, we know that $y(t) = \partial_v \vartheta(t, v)$ is the solution of the variational equation

$$\ddot{y} + \left(\frac{\Lambda}{r(t,e)^3} \cos[\vartheta(t, v; e, \Lambda)] \right) y = 0, \tag{11}$$

with initial conditions $y(0) = 0, \dot{y}(0) = 1$. Note that in (11) we have made explicit the dependence with the parameters of ϑ. We conclude that the problem (8) has a unique solution as soon as the equation (11) has the trivial solution $y(t) \equiv 0$ as the unique solution satisfying the Dirichlet conditions $y(0) = y(\pi) = 0$. This condition must be checked for every $v \in \mathbb{R}$.

[1] Sometimes we will make explicit the dependence on the parameters of the problem in this way, for example, a solution $\Theta(t)$ of (9) would be referred as $\Theta(t; e, \Lambda)$

To do this we will employ the Sobolev inequality

$$K_l(p)||\xi||_p^2 \leq ||\dot{\xi}||_2^2, \tag{12}$$

where ξ is any function in the space $H_0^1[0,l]$, $||\cdot||_p$ is the L^p-norm, which is defined by

$$||\xi||_p = \begin{cases} \left(\int_0^l |\xi(t)|^p dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,l]} |\xi(t)|, & \text{if} \quad p = \infty, \end{cases}$$

and the constant $K_l(p)$ is optimal for (12) by definition,

$$K_l(p) = \inf_{\xi \in H_0^1[0,l]\setminus\{0\}} \frac{||\dot{\xi}||_2^2}{||\xi||_p^2},$$

see [24], [6]. This constant has an explicit expression given in the Appendix A, with which we can check that $K_l(p)$ is continuous for all $p \in [1,\infty]$.

In order to prove the following lemma let us also recall the Hölder's inequality for $\xi \in L^p[0,l]$ and $\chi \in L^q[0,l]$, with $1/p + 1/q = 1$,

$$||\xi \cdot \chi||_1 \leq ||\xi||_p ||\chi||_q. \tag{13}$$

Lemma 1 *Let $a \in C[0,l]$ be a function such that its positive part*

$$a^+(t) = \max\{0, a(t)\},$$

satisfies

$$||a^+||_\alpha < K_l\left(\frac{2\alpha}{\alpha-1}\right), \tag{14}$$

with $\alpha \in [1,\infty]$, then, the unique solution of the Dirichlet problem

$$\begin{cases} \ddot{y} + a(t)y = 0, \\ y(0) = y(l) = 0, \end{cases} \tag{15}$$

is the trivial solution.

This Lemma is a particular case of Corollary 2.2 in [6], in its Dirichlet version (Section 2.3). As we see in the Remark 2.1 and in Section 2.3 of [6], we do not need to impose the condition $\int_0^l a(t)dt > 0$, since our problem is of a type that they call *nonresonant*.

Proof. We proceed by contradiction. Multiply the equation in (15) by $y(t) \not\equiv 0$ and integrate by parts,

$$\int_0^l \dot{y}(t)^2 dt = \int_0^l a(t)y(t)^2 dt \leq \int_0^l a^+(t)y(t)^2 dt, \tag{16}$$

i.e.

$$||\dot{y}||_2^2 \leq ||a^+ y^2||_1,$$

from (13) we get

$$||\dot{y}||_2^2 \leq ||a^+||_p ||y^2||_q,$$

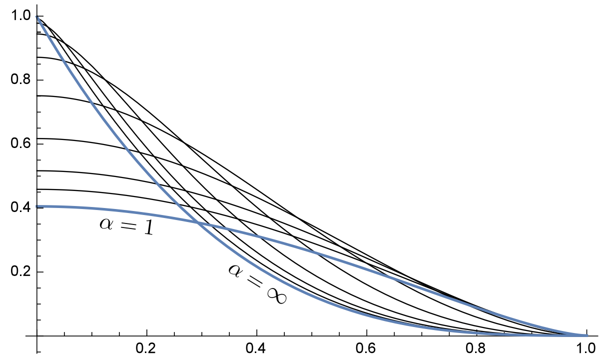

Figure 2: The functions $e \mapsto \Lambda_0(e,\alpha)$ for different values of α.

for any numbers such that $1 \leq p, q \leq \infty$ satisfying $1/p + 1/q = 1$, additionally, from (12) we get

$$K_l(\beta)||y||_\beta^2 \leq ||a^+||_p ||y^2||_q,$$

for any number such that $1 \leq \beta \leq \infty$. Let us take $q = \beta/2$ in the last inequality and assume that $y(t) \not\equiv 0$, consequently

$$K_l(\beta) \leq ||a^+||_{\frac{\beta}{\beta-2}},$$

where we have used that $||y||_\beta^2 = ||y^2||_{\beta/2}$. The last inequality contradicts the hypothesis (14). Then, it must be satisfied that $y(t) \equiv 0$. ■

The previous Lemma can be applied to equation (11) in the interval $[0,\pi]$ for each $\alpha \in [1,\infty]$. To do this we define the following function

$$\Lambda_0(e,\alpha) = \frac{K_\pi(\frac{2\alpha}{\alpha-1})}{||r(\cdot,e)^{-3}||_\alpha}, \tag{17}$$

which has an explicit expression in terms of the hypergeometric function and the Γ function, and, it is continuous in both of its variables (see the Appendix A). The graphs of $\Lambda_0(\cdot,\alpha)$, for some values of α, are plotted in Figure 2.

Now we are able to define the function

$$\Lambda_1(e) = \max_{\alpha\in[1,\infty]} \Lambda_0(e,\alpha), \tag{18}$$

which leads us to the following Proposition. See some properties of $\Lambda_1(e)$ in Lemma 7, Appendix A.

Proposition 1 *Assume that $e \in (0,1]$ and $0 \leq \Lambda < \Lambda_1(e)$. Then, there exists a unique solution of the Dirichlet problem (8), denoted by $\Theta^*(t;e,\Lambda)$. The function*

$$(t,e,\Lambda) \in [0,\pi]\times[0,1)\times[0,\Lambda_1(e)) \quad \mapsto \quad \Theta^*(t;e,\Lambda),$$

is analytic in the real sense.

Proof. The previous discussions lead directly to the existence and uniqueness of the solution. To prove the analytic character of it, we observe that, in terms of the previous notation,

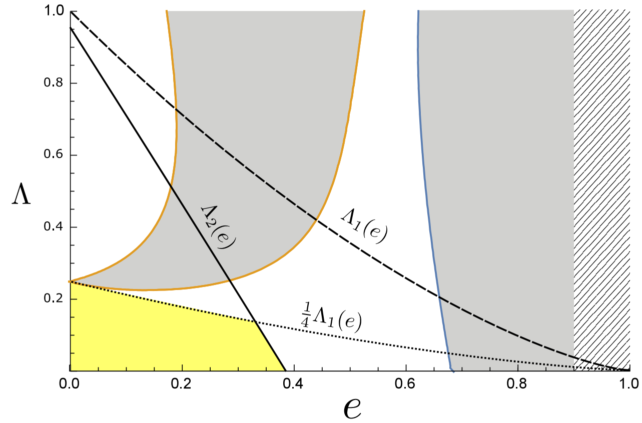

Figure 3: Stability diagram of $\Theta^*(t; e, \Lambda)$ computed numerically: the gray regions are linearly unstable. The lines pattern indicates that for high eccentricity $e \geq 0.9$ we did not compute the linear stability due to the closeness to the singularity. The yellow region is the stable region by a Lyapunov-type criterion.

the solution corresponds to $\Theta^*(t; e, \Lambda) = \vartheta(t, v(e, \Lambda); e, \Lambda)$, where $v = v(e, \Lambda)$ is the unique solution of

$$\vartheta(\pi, v; e, \Lambda) = 0.$$

Now we can apply the implicit function theorem, in its real analytic version (Theorem 2.3.5 in [16]), because the solution ϑ is analytic in all the entries due to the analytic character of the equation (7). Also, the derivative $\partial_v \vartheta(\pi, v; e, \Lambda)$ does not vanish as long as $v \in \mathbb{R}$, $e \in [0, 1)$ and $0 \leq \Lambda < \Lambda_1(e)$, due to Lemma 1. These considerations lead easily to the proof of the claim. ■

Remark 1 *Being more precise, in terms of complex analysis, the function* $\Theta^*(t; e, \Lambda)$ *has a holomorphic extension to some open subset of* $\mathbb{C}^3$ *containing* $[0, \pi] \times [0, 1) \times [0, \Lambda_1(e))$.

Remark 2 *Note that there are two special cases for which* Θ^* *can be computed*

$$\Theta^*(t; e, 0) = 2(t - f(t, e)), \quad \Theta^*(t; 0, \Lambda) = 0. \tag{19}$$

Now we are interested in the stability properties of the solution $\Theta^*(t; e, \Lambda)$, which should be seen as 2π-periodic and odd from now on. In the following we will find a region of parameters where the linearized equation of (7) at Θ^*, say,

$$\ddot{y} + \left(\frac{\Lambda}{r(t, e)^3} \cos[\Theta^*(t; e, \Lambda)] \right) y = 0, \tag{20}$$

is stable (linear stability). See Figure 3.

For this purpose we are going to apply Theorem 1 from [26], which is a generalization of the classical Lyapunov criterion for the stability of a Hill's equation using L^α norms. According to it, given a Hill's equation

$$\ddot{y} + a(t)y = 0, \quad a(t + T) = a(t),$$

with $a \in L^\alpha[0, T]$, the equation is stable if

$$\int_0^T a(t)dt > 0 \quad \text{and} \quad ||a^+||_\alpha < K_T \left(\frac{2\alpha}{\alpha - 1} \right). \tag{21}$$

See also a similar result by Borg in [18], Section 5.2.

It is important to observe that in our current situation we want to consider $T = 2\pi$, then, we have to do the computations in the interval $[0, 2\pi]$, instead of $[0, \pi]$, as we did so far. We know that the following relations hold

$$K_{2l}\left(\frac{2\alpha}{\alpha-1}\right) = \frac{2^{\frac{1}{\alpha}}}{4} K_l\left(\frac{2\alpha}{\alpha-1}\right), \quad \int_0^{2\pi} \frac{dt}{r(t,e)^{3\alpha}} = 2\int_0^{\pi} \frac{dt}{r(t,e)^{3\alpha}}. \tag{22}$$

The first identity comes from the definition of $K_l(p)$ in (48). In consequence, the second inequality in (21) is satisfied for equation (20) if $0 < \Lambda < \frac{1}{4}\Lambda_1(e)$. We rule out the case $\Lambda = 0$ because it does not satisfy the second inequality.

2.2 Upper and lower solutions

We see from Figure 3 that the condition $0 < \Lambda < \frac{1}{4}\Lambda_1(e)$ is not sufficient to obtain stability. We are going to define another function $\Lambda_2(e)$, which let us guarantee that the first inequality in (21) is satisfied for equation (20). Let us impose that $\cos[\Theta^*(t; e, \Lambda)] > 0$, or, equivalently,

$$|\Theta^*(t; e, \Lambda)| < \frac{\pi}{2}, \quad t \in [0, 2\pi]. \tag{23}$$

Due to the symmetry of this solution, it is sufficient to find the estimate on the half-interval $[0, \pi]$. This can be done with the method of upper and lower solutions. See for example [11].

Let $\psi(t)$ be a solution of the Dirichlet problem

$$\ddot{\psi} = -\frac{\Lambda}{r(t,e)^3}, \quad \psi(0) = \psi(\pi) = 0.$$

By the maximum principle, the function ψ is positive on $(0, \pi)$ and can be expressed as

$$\psi(t) = -\Lambda \int_0^{\pi} \frac{G(t,s)}{r(s,e)^3} ds,$$

where $G(t, s)$ is the Green's function associated to the operator $L[\psi] = \ddot{\psi}$ with Dirichlet conditions $\psi(0) = \psi(\pi) = 0$, and whose expression is

$$G(t,s) = \begin{cases} -s(\pi - t)/\pi & \text{if} \quad s \in [0, t], \\ -t(\pi - s)/\pi & \text{if} \quad s \in [t, \pi]. \end{cases}$$

Note that $G(t, s) \le 0$, and that $|G(t, s)| \le |G(s, s)|$, then,

$$\psi(t) \le \frac{\Lambda}{\pi} \int_0^{\pi} \frac{s(\pi - s)}{r(s,e)^3} ds. \tag{24}$$

We can use ψ to produce the following functions

$$\psi_\pm(t) = 2(t - f(t,e)) \pm \psi(t).$$

These functions are upper and lower solutions of our problem, since they satisfy

$$\psi_+(t) > \psi_-(t), \quad \ddot{\psi}_-(t) \ge F(t, \psi_-(t)), \quad \ddot{\psi}_+(t) \le F(t, \psi_+(t)),$$

where $F(t,\Theta)$ was defined by the expressions eqs. (9) and (10), and contains the nonlinear terms of our equation. Consequently,

$$\psi_+(t) \geq \Theta^*(t;e,\Lambda) \geq \psi_-(t), \quad t \in [0,\pi].$$

Let us look for bounds that do not depend on t. Consider the function $\chi(t) = f(t,e) - t$. Since $\chi(0) = \chi(\pi) = 0$ and computing that

$$\ddot{f}(t,e) = -\frac{2e\sqrt{1-e^2}\sin[u(t,e)]}{(1-\cos[u(t,e)])^4} < 0, \quad t \in (0,\pi),$$

we can say that $\chi(t), t \in [0,\pi]$, is positive and has a unique maximum $m(e)$ in the interval $(0,\pi)$, say,

$$m(e) = 2\arctan\sqrt{\frac{(1+e)(e-1+(1-e^2)^{\frac{1}{4}})}{(1-e)(e+1-(1-e^2)^{\frac{1}{4}})}} - \arccos\left(\frac{1-\sqrt[4]{1-e^2}}{e}\right) + \sqrt{e^2-(1-\sqrt[4]{1-e^2})^2}.$$

According to this discussion, define the function

$$\Lambda_2(e) = \frac{\frac{\pi^2}{2} - 2\pi m(e)}{\int_0^\pi \frac{t(\pi-t)}{r(t,e)^3}dt}, \tag{25}$$

then, the condition (23) is satisfied if $0 < \Lambda < \Lambda_2(e)$. The stability result is summarized in the following Proposition.

Proposition 2 *For each value $e \in (0,1)$ such that $m(e) < \pi/4$, if $0 < \Lambda < \frac{1}{4}\Lambda_1(e)$ and $0 < \Lambda < \Lambda_2(e)$, then, the solution $\Theta^*(t;e,\Lambda)$ is linearly stable.*

3 Instability for high eccentricity

The numerical calculation of the instability regions in Figure 3 shows that there exists instability for high eccentricities, whose boundary bifurcates from $\Lambda = 0$ at some value of the eccentricity that we will call $e = e_*$. As it is described in [3], Chapter 2, Section 7.3, Zlatoustov and collaborators already showed this behavior with computer simulations in [27]. In this section we will prove the existence of such bifurcation branch for small Λ. Assume that e and Λ are in the conditions of Proposition 1 and let $\Theta^*(t;e,\Lambda)$ be the odd 2π-periodic solution obtained for the Dirichlet problem (8). The next result also concerns the stability of the variational equation (20).

Theorem 1 *For some $\varepsilon > 0$, there exists a function $E : [0,\varepsilon) \to (0,1)$, $\Lambda \mapsto E(\Lambda)$, such that the equation (20) is unstable and has a non-trivial 4π-periodic solution if $e = E(\Lambda)$. Moreover, $E(0) = e_* \in (0,1)$ and for each $\bar{e} \in (e_*,1)$ there exists a $\bar{\Lambda} = \bar{\Lambda}(\bar{e}) \in (0,\varepsilon)$, such that the equation (20) is unstable for the points (e,Λ) satisfying $E(\Lambda) < e < \bar{e}$, $0 < \Lambda < \bar{\Lambda}$. In addition, the function E can be expressed as $E(\Lambda) = \xi(\Lambda^{1/p})$, where $\xi(\zeta)$ is real analytic at $\zeta = 0$ and $p \geq 1$ is an integer.*

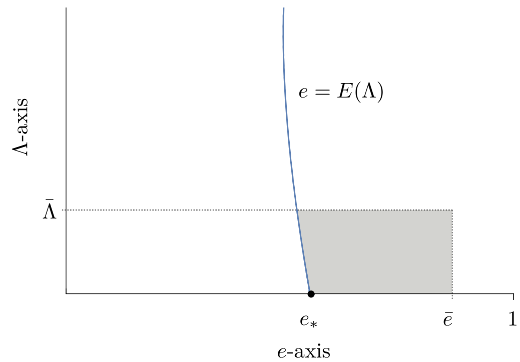

Figure 4: Bifurcation of stability at $e = e_*$. The shaded region is unstable.

The proof of this result will provide some additional information. The number e_* solves the equation $I(e) = 0$, where,

$$I(e) := \int_{-\pi}^{\pi} \frac{\cos[2(t - f(t,e))]}{r(t,e)^3} dt. \tag{26}$$

Numerical computations suggest that e_* is the only root of $I(e)$ and $e_* \approx 0.682...$

The general theory of Hill's equation deals with the study of

$$\ddot{y} + a(t)y = 0, \tag{27}$$

where a is a continuous and 2π-periodic function, see [18]. As it is well known, the discriminant $\Delta = \Delta[a]$ is a real number such that (27) is stable if $|\Delta| < 2$ (elliptic case) and unstable if $|\Delta| > 2$ (hyperbolic case). When $|\Delta| = 2$ (parabolic case) the equation might be stable or unstable. In this last case, there is at least one 4π-periodic solution and, only if all the solutions of (27) are 4π-periodic (coexistence), the equation is stable.

According to Theorem 1, the specific equation (20) is parabolic-unstable on the curve $e = E(\Lambda)$ and hyperbolic-unstable on the shaded region of the Figure 4. In particular, this hyperbolicity implies that $\Theta^*(t; e, \Lambda)$ is unstable, in the Lyapunov sense, as solution of the nonlinear equation (7).

Incidentally, we notice that Zhang's conditions (21) really imply that (27) is elliptic-stable. In our particular case, it means that equation (20) is elliptic on the conditions of Proposition 2.

To describe the strategy for the proof of Theorem 1 we first recall some facts on the linear equation (27) when the coefficient $a(t)$ is even. Note that this is the case for (20). Let y_1 and y_2 be the normalized solutions, i.e., solutions obtained with initial conditions $y_1(0) = 1$, $\dot{y}_1(0) = 0$, $y_2(0) = 0$, $\dot{y}_2(0) = 1$. The discriminant is expressed in terms of these solutions by the formula

$$\Delta^2 - 4 = 4\dot{y}_1(2\pi)y_2(2\pi), \tag{28}$$

which let us discriminate between the cases of stability in a convenient way.

For the equation (20), we have that $y_1(t) = y_1(t; e, \Lambda)$ and $y_2(t) = y_2(t; e, \Lambda)$. Particularly, for $\Lambda = 0$ we observe that for all $t \in \mathbb{R}$, $e \in [0, 1)$,

$$y_1(t; e, 0) \equiv 1, \quad y_2(t; e, 0) \equiv t, \tag{29}$$

Assuming that $\Lambda < \frac{1}{4}\Lambda_1(e)$, let us prove that $y_2(2\pi; e, \Lambda) > 0$. Since y_2 is a non-trivial solution, we can apply Lemma 1 for $l = 2\pi$, and conclude that $y_2(2\pi; e, \Lambda) \neq 0 = y_2(0; e, \Lambda)$. From (29), $y_2(2\pi; e, 0) > 0$ for each $e \in [0, 1)$, then, by continuity, $y_2(2\pi; e, \Lambda) > 0$.

In consequence, by (28), if $\dot{y}_1(2\pi; e, \Lambda)$ is negative/positive, the equation will be stable/unstable. Additionally, the fact that $y_2(2\pi; e, \Lambda) \neq 0$ implies, by Theorems 1.1 and 1.2 in [18], that y_2 is not 4π-periodic. Therefore, if $\dot{y}_1(2\pi; e, \Lambda) = 0$, then, the equation (27) will be parabolic-unstable.
Since $\dot{y}_1(2\pi; e, 0) = 0$ for each e, the division formula can be applied to write

$$\dot{y}_1(2\pi; e, \Lambda) = \Lambda \Psi(e, \Lambda), \tag{30}$$

with

$$\Psi(e, \Lambda) = \int_0^1 \partial_\Lambda \dot{y}_1(2\pi; e, s\Lambda) ds,$$

which is a real analytic function in both variables. This comes from the fact that any solution $y = y(t; e, \Lambda)$ of (20) is analytic in all its entries due to the real analytic version of the theorem of differentiability of solutions with respect to the parameters. Fix a value $e \in [0, 1)$ and note that, according to the expansion of $\dot{y}_1(2\pi, e, \Lambda)$ around $\Lambda = 0$, we have

$$\Psi(e, 0) = \partial_\Lambda \dot{y}_1(2\pi; e, 0).$$

Differentiating the equation (20) with respect to Λ and evaluating at $\Lambda = 0$, we obtain

$$\partial_\Lambda \ddot{y}(t; e, 0) + \frac{\cos[\Theta^*(t; e, 0)]}{r(t, e)^3}\, y(t; e, 0) = 0,$$

which can be integrated with initial condition $\partial_\Lambda \dot{y}_1(0; e, 0) = 0$ and give as a result that

$$\partial_\Lambda \dot{y}_1(2\pi; e, 0) = -\int_0^{2\pi} \frac{\cos[2(t - f(t, e))]}{r(t, e)^3} dt.$$

By 2π-periodicity of the integrand and considering the definition (26), we can change the interval of integration from $[0, 2\pi]$ to $[-\pi, \pi]$ and identify $\Psi(e, 0) = -I(e)$.

The standard theory of integrals depending on parameters implies that $I(e)$ is a real analytic function defined on $e \in [0, 1)$. Moreover, since $f(t, 0) = t$ and $r(t, 0) = 1$, we see that $I(0) = 2\pi > 0$. The following result implies that $I(e)$ has a change of sign. Hence, $I(e_*) = 0$ for some $e_* \in (0, 1)$.

Lemma 2 *The function $I(e)$ has a negative finite limit as $e \to 1^-$.*

The proof of this result is delicate because it is not possible to interchange the limit with the integral sign. Let us explain this point. Since the integrand is even, we just consider the integral (26) on the interval $(0, \pi)$. At first glance we see that the limit of the integrand is

$$\lim_{e \to 1^-} \frac{\cos[2(t - f(t, e))]}{r(t, e)^3} = \frac{\cos[2t]}{r(t, 1)^3}$$

because, as $e \to 1^-$, $f(t, e) \to \pi$ for all $t \in (0, \pi)$. From its definition, it can be computed the expansion $r(t, 1)^3 = 9t^2/2 + O(t^4)$, i.e., the integrand has a pole of order 2 at $t = 0$. Therefore,

$$\int_0^\pi \frac{\cos[2t]}{r(t, 1)^3} dt = +\infty.$$

However, the delicate point comes from the fact that, for e close to 1, the value of $f(t, e)$ increases from 0 to π very fast. This results in a fast-changing argument of the cosine and, ultimately, a change of sign of the integrand in (26) for smaller and smaller $t > 0$. In this situation we cannot apply any classical technique, such as the dominated convergence theorem or the Fatou's lemma.

In order to prove Lemma 2, first, we will apply the Residue Theorem to compute $I(e)$ for $e \in (0,1)$, then, let e tend to 1. All this hard work is postponed to the end of the section.

Once we have found e_* such that $\Psi(e_*, 0) = 0$, it seems natural to find the function $e = E(\Lambda)$ as a solution of the implicit function problem

$$\Psi(E(\Lambda), \Lambda) = 0, \quad E(0) = e_*.$$

A direct application of the implicit function theorem does not seem easy. The number e_* is not known explicitly and the transversality condition $\partial_\Lambda \Psi(e_*, 0) \neq 0$ leads to a complicated integral with no clear sign.

Taking advantage of the analytic character of the function Ψ, we will apply the following parametric version of Bolzano's Theorem. The proof of Theorem 1 will follow as a direct consequence.

Lemma 3 *Let $\Upsilon : [0, l_1) \times [0, l_2) \to \mathbb{R}$, $l_1, l_2 > 0$, be a real analytic function of two variables $\Upsilon = \Upsilon(x, y)$ such that,*

$$\Upsilon(0,0) < 0 < \liminf_{x \to l_1^-} \Upsilon(x, 0). \tag{31}$$

Then, there exists a value $x_* \in (0, l_1)$ and a function $\varphi : [0, \varepsilon) \to (0, l_1)$, such that

$$\varphi(0) = x_*, \quad \Upsilon(\varphi(y), y) = 0, \quad \text{for each } y \in [0, \varepsilon), \tag{32}$$

and, for each $\bar{x} \in (x_*, l_1)$, there exists a $\bar{y} > 0$ such that

$$\Upsilon(x, y) > 0, \quad \text{if} \quad \varphi(y) < x,\ x < \bar{x},\ 0 < y < \bar{y}. \tag{33}$$

Moreover, there exists some positive integer $p \geq 1$ such that $\varphi(y) = \tilde{\varphi}(y^{1/p})$, where $\tilde{\varphi}(\zeta)$ is analytic at $\zeta = 0$.

Proof. The function $\Upsilon(\cdot, 0)$ is analytic in $[0, l_1)$ and changes sign, then, it has a finite number of zeros. We will say that a zero $x_0 \in (0, l_1)$ of this function is *transversal* if

$$\Upsilon(x_0 + \sigma, 0)\Upsilon(x_0 - \sigma, 0) < 0,$$

for every small enough $\sigma > 0$. The function $\Upsilon(\cdot, 0)$ has at least one transversal zero due to the condition (31).

Define the set of zeros

$$Z = \{(x, y) \in [0, l_1) \times [0, l_2) \ : \ \Upsilon(x, y) = 0\}.$$

We say that a zero $x_0 \in (0, l_1)$ of $\Upsilon(\cdot, 0)$ has a *continuation* if the point $(x_0, 0)$ is non-isolated in Z. Transversal zeros have always a continuation. This is a consequence of Bolzano's Theorem. Given a transversal zero x_0, for small $\varepsilon > 0$,

$$\Upsilon(x_0 + \sigma, y)\Upsilon(x_0 - \sigma, y) < 0 \quad \text{if} \quad 0 < y < \varepsilon.$$

Therefore, Z has a point lying in the segment $[x_0 - \sigma, x_0 + \sigma] \times \{y\}$. The converse is not true, sometimes non-transversal zeros have a continuation. We illustrate the previous definitions with the example

$$\Upsilon(x, y) = (y - x + 1)^2 (y + x - 2)(x - 3)^2.$$

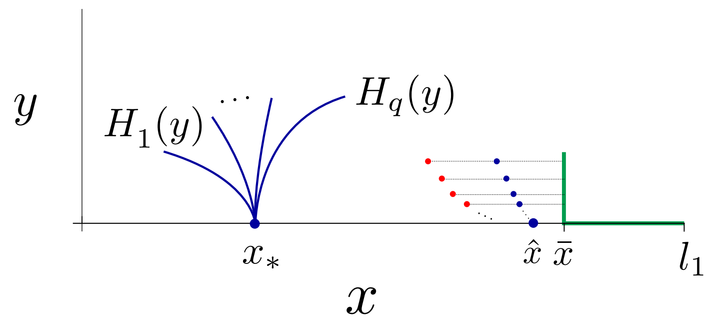

Figure 5: Illustration of the contradiction argument. Green, red and blue correspond to values of Υ that are positive, negative and zero, respectively.

This function satisfies the conditions of the Lemma if $l_1 > 3, l_2 > 0$. Additionally, $\Upsilon(\cdot, 0)$ has three zeros, say, $x_0 = 2$ (transversal), $x_1 = 1$ (non-transversal with continuation) and $x_2 = 3$ (non-transversal without continuation).

Let $x_* \in (0, l_1)$ be the largest zero of $\Upsilon(\cdot, 0)$ having a continuation. Now, we are going to use several theorems on real analytic functions of two variables in order to characterize the continuation set of the point $(x_*, 0)$.

First, we apply the Weierstrass Preparation Theorem (Theorem 6.3.1 in [16]) to the function Υ at the point $(x_*, 0)$. To do this, note that some coefficient of the power series expansion of $\Upsilon(x, 0)$ at $x = x_*$ does not vanish (otherwise $\Upsilon(x, 0) \equiv 0$). Then, the function $\Upsilon(x, y)$ can be decomposed as

$$\Upsilon(x, y) = W(x - x_*, y)Y(x, y), \qquad (x, y) \in U,$$

where U is a small neighborhood of $(x_*, 0)$, $Y(x, y)$ is a non-vanishing real analytic function defined on U and $W(x, y)$ is a Weierstrass polynomial. This means that there exists an integer $N \geq 1$ such that

$$W(x, y) = x^N + A_{N-1}(y)x^{N-1} + \cdots + A_1(y)x + A_0(y),$$

where the functions $A_n(y)$, $n = 0, ..., N-1$, are real analytic at $y = 0$ and $A_n(0) = 0$. As a result, the equation $\Upsilon = 0$ is equivalent to $W = 0$ in U.

Second, we can apply the Decomposition Theorem (Theorem 4.2.7 in [16]) to W. We deduce that there exists a finite number q of functions $H_1, H_2, ..., H_q$, defined on $[0, \varepsilon)$ where $H_j(0) = x_*$, $j = 1, ..., q$; $H_1(y) < H_2(y) < ... < H_q(y)$ if $y \in (0, \varepsilon)$, and such that, for some neighborhood $V \subset U$ of $(x_*, 0)$, we can characterize the continuation set as

$$Z \cap V = \{(H_j(y), y) \;:\; y \in [0, \varepsilon),\ j = 1, ..., q\}.$$

Moreover, there exists an integer $p \geq 1$ such that $H_j(y) = \tilde{H}_j(y^{1/p})$, where each $\tilde{H}_j(\zeta)$ is analytic at $\zeta = 0$. We define $\varphi = H_q$ and $\tilde{\varphi} = \tilde{H}_q$. Then, the identities (32) are automatically satisfied.

It remains to check that the inequality (33) holds. We start with a preliminary observation: given a point $(x, y) \in V$ with $x > \varphi(y)$, then, $\Upsilon(x, y) > 0$. This is a consequence of the way we have chosen $\varphi = H_q$. Consider $\chi \in (x_*, x_* + \sigma)$, with $\sigma > 0$ small enough such that the point $(\chi, 0) \in V$. The point $(\chi, 0)$ is connected to every point within the region $\{(x, y) \in V : \ x > \varphi(y)\}$. Since $\Upsilon(\chi, 0) > 0$, the same is true for all the points in the region.

Let us now prove (33) by a contradiction argument concerning the definition of x_*. Assume the existence of a number $\bar{x} \in (x_*, l_1)$ and a sequence of points $\{(x_n, y_n)\}$ satisfying

$$\Upsilon(x_n, y_n) \le 0, \quad \varphi(y_n) < x_n,\ x_n < \bar{x},\ y_n > 0,\ y_n \underset{n\to\infty}{\longrightarrow} 0.$$

Figure 5 illustrates the argument for strict inequalities $\Upsilon(x_n, y_n) < 0$. It is not restrictive to assume that $\bar{x}$ is sufficiently close to l_1 in order to assure that $\bar{x} \notin V$ and $\Upsilon(\bar{x}, 0) > 0$. Let us fix $\varepsilon_1 > 0$ such that $\Upsilon(\bar{x}, y) > 0$ if $y \in [0, \varepsilon_1]$. Letting $n \to \infty$ in the inequality $x_n > \varphi(y_n)$, we deduce that $\liminf_{n\to\infty} x_n \ge x_*$.

From the previous discussions we know that $(x_n, y_n) \notin V$. Then, there exists some positive number $\nu > 0$ such that $x_n > x_* + \nu$ for large n. Also, we can assume $y_n \le \varepsilon_1$. For each n, the function $\Upsilon(\cdot, y_n)$ must have a zero in the interval $[x_n, \bar{x})$, say $\hat{x}_n \in [x_n, \bar{x})$. After extracting a subsequence we can assume that $\hat{x}_n$ converges to some $\hat{x} \in [x_* + \nu, \bar{x}]$. Then, $(\hat{x}, 0)$ is a non-isolated point in Z. Which contradicts the definition of x_* as the largest of such points. ∎

Proof of Lemma 2. We will employ some techniques from complex analysis. They are motivated by the following observation: after the change of variable $t = u - e\sin u$, we can express the integral in the form

$$I(e) = \int_{-\pi}^{\pi} \frac{\cos[2(u - e\sin u - f(u,e))]}{(1 - e\cos u)^2} du,$$

where the true anomaly f is written in terms of the eccentric anomaly u, as defined by eqs. (2) and (3). Then, it is indeed the composition of $f(t, e)$ with $t = u - e\sin u$.

Using some trigonometric identities $I(e)$ can be written as an integral in the family

$$\int_{-\pi}^{\pi} \left(R_1(\cos u, \sin u)\sin(2e\sin u) + R_2(\cos u, \sin u)\cos(2e\sin u)\right) du,$$

where $R_1(x, y)$ and $R_2(x, y)$ are rational functions. The simpler family of trigonometric integrals

$$\int_{-\pi}^{\pi} R(\cos u, \sin u) du,$$

is often analyzed using the change of variables $z = \exp[iu]$ and the Residue Theorem. We will show that this trick also works in our situation.

First it is convenient to observe that

$$\int_{-\pi}^{\pi} \frac{\sin[2(u - e\sin u - f(u,e))]}{(1 - e\cos u)^2} du = 0,$$

because the integrand is odd. Therefore, our integral can be expressed as

$$I(e) = \int_{-\pi}^{\pi} \frac{\exp[2i(u - e\sin u - f(u,e))]}{(1 - e\cos u)^2} du. \tag{34}$$

After the change of variable $z = \exp[iu]$, we can interpret $I(e)$ as an integral over the curve γ, where γ is the unit circle run counter-clockwise. To find the integrand we employ the formulas

$$\cos u = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin u = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

leading to

$$1 - e\cos u = -\frac{e}{2z}(z-\zeta_-)(z-\zeta_+),$$

where $\zeta_\pm = \frac{1\pm\sqrt{1-e^2}}{e}$. Note that ζ_- and ζ_+ are positive real numbers with $\zeta_-\zeta_+ = 1$ and $\zeta_- < 1$.

From (3) we can compute that

$$\exp[if(u,e)] = -\zeta_+ \frac{z-\zeta_-}{z-\zeta_+}.$$

Straightforward computations show that

$$I(e) = \frac{4}{ie^2\zeta_+^2}\int_\gamma h(z,e)dz,$$

where

$$h(z,e) = \frac{z^3\exp[-e\left(z-\frac{1}{z}\right)]}{(z-\zeta_-)^4}.$$

For each $e \in (0,1)$, the meromorphic function $h(\cdot,e)$ has two singularities at $z = \zeta_-$ and $z = 0$, both inside the unit circle. Therefore, making explicit the dependence on e and using the Residue Theorem,

$$I(e) = \frac{8\pi}{e^2\zeta_+(e)^2}[\mathrm{Res}(h(\cdot,e),\zeta_-(e)) + \mathrm{Res}(h(\cdot,e),0)].$$

The singularity at $z = \zeta_-(e)$ is a pole of order 4. Then,

$$\mathrm{Res}(h(\cdot,e),\zeta_-(e)) = \frac{1}{6}\frac{d^3g}{dz^3}(\zeta_-(e),e),$$

with $g(z,e) = z^3\exp[-e\left(z-\frac{1}{z}\right)]$. The function $g(\cdot,1)$ is holomorphic in $|z| > 0$. Moreover,

$$g(z,e) \to g(z,1) \quad \text{as} \quad e \to 1^-, \ \ z \neq 0,$$

and the convergence is uniform for z lying in any compact subset of $\mathbb{C}\backslash\{0\}$. In particular,

$$\frac{d^3g}{dz^3}(z,e) \to \frac{d^3g}{dz^3}(z,1) \quad \text{as} \quad e \to 1^-.$$

This shows that the residue $\mathrm{Res}(h(\cdot,e),\zeta_-(e))$ has a limit when $e \to 1^-$. We are going to prove that this is also the case for the residue at the origin.

The function $h(\cdot,1)$ is holomorphic in $0 < |z| < 1$ and

$$h(z,e) \to h(z,1) \quad \text{as} \quad e \to 1^-, \ \ 0 < |z| < 1.$$

The convergence is uniform on compact subsets. In particular on the circle $\tilde{\gamma} = \{z \in \mathbb{C} \ : \ |z| = 1/2\}$. Then, as $e \to 1^-$,

$$\mathrm{Res}(h(\cdot,e),0) = \frac{1}{2\pi i}\int_{\tilde{\gamma}} h(z,e)dz \quad \to \quad \frac{1}{2\pi i}\int_{\tilde{\gamma}} h(z,1)dz = \mathrm{Res}(h(\cdot,1),0).$$

We conclude that $I(e)$ has a limit, namely,

$$\lim_{e\to 1^-} I(e) = 8\pi\left[\frac{1}{6}\frac{d^3g}{dz^3}(1,1) + \mathrm{Res}(h(\cdot,1),0)\right]. \tag{35}$$

To complete the proof we must show that this number is negative. This will involve some computations, first,

$$\frac{1}{6}\frac{d^3 g}{dz^3}(1,1) = -\frac{1}{3}. \tag{36}$$

The function $h(\cdot,1)$ has an essential singularity at $z = 0$. To compute the residue we factorize $h(\cdot,1)$ in the form

$$h(z,1) = \mu(z)\nu(z),$$

with

$$\mu(z) = \frac{z^3}{(z-1)^4}\exp[-z], \quad \nu(z) = \exp\frac{1}{z}.$$

Then, μ is holomorphic in the disk $|z| < 1$ and has an expansion

$$\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad |z| < 1.$$

The function ν has an essential singularity at $z = 0$ with Laurent expansion

$$\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}, \quad |z| > 0.$$

The residue of $h(\cdot,1)$ can be computed from the Laurent expansion. More precisely,

$$\operatorname{Res}(h(\cdot,1),0) = \sum_{n=0}^{\infty} \frac{\mu_n}{(n+1)!}. \tag{37}$$

From the binomial series $(1-z)^{-d} = \sum_{n=0}^{\infty} \binom{n+d-1}{n} z^n$, we know that

$$\frac{1}{(z-1)^4} = \sum_{n=0}^{\infty} \binom{n+3}{n} z^n, \quad |z| < 1,$$

and

$$z^3 \exp[-z] = \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{(n-3)!} z^n, \quad z \in \mathbb{C},$$

we deduce that

$$\mu_0 = \mu_1 = \mu_3 = 0, \quad \mu_n = \sum_{k=0}^{n-3} \binom{k+3}{k} \frac{(-1)^{n-k+1}}{(n-k-3)!}, \;\; n \geq 3.$$

Combining this formula with (37),

$$\operatorname{Res}(h(\cdot,1),0) = \sum_{n=0}^{\infty}\sum_{k=0}^{n-3} \binom{k+3}{k} \frac{(-1)^{n-k+1}}{(n+1)!(n-k-3)!}$$

Letting $n-k-3=j$,

$$\begin{aligned}
\operatorname{Res}(h(\cdot,1),0) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{k+3}{k}\frac{(-1)^j}{j!(j+k+4)!} \\
&< \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{k+3}{k}\frac{1}{j!(j+k+4)!} \\
&< \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{k=0}^{\infty}\binom{k+3}{k}\frac{1}{(k+4)!} = \exp[1]\sum_{k=0}^{\infty}\binom{k+3}{k}\frac{1}{(k+4)!}
\end{aligned}$$

Finally, we observe that

$$\sum_{k=0}^{\infty}\binom{k+3}{k}\frac{1}{(k+4)!} = \frac{1}{3!}\sum_{k=0}^{\infty}\frac{1}{(k+4)k!} < \frac{1}{4!}\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{\exp[1]}{24}.$$

Thus,

$$\operatorname{Res}(h(\cdot,1),0) < \frac{\exp[2]}{24} < \frac{1}{3},$$

and the proof follows from (35), (36) and this inequality. ■

4 Asymptotic stability of the synchronous resonance in the dissipative case

Recall that the dissipative spin-orbit problem is modeled by the equation

$$\ddot{\Theta} + \delta D(t,e)\dot{\Theta} + \frac{\Lambda}{r(t,e)^3}\sin\Theta = -2\ddot{f}(t,e), \qquad e\in[0,1),\ \Lambda \gg \delta \geq 0, \tag{38}$$

with $D(t,e)$ positive, analytic and 2π-periodic in t. We know from Proposition 2 that, for $\delta=0$, there exists an odd 2π-periodic solution $\Theta^*(t;e,\Lambda)$, which is linearly stable in the set

$$\Omega = \{(e,\Lambda):\ 0<e<1,\ 0<\Lambda<\frac{1}{4}\Lambda_1(e),\ 0<\Lambda<\Lambda_2(e)\}.$$

Furthermore, this solution is elliptic in the following sense, the discriminant associated to the linearized equation at $\Theta^*(t;e,\Lambda)$, say $\Delta_0=\Delta_0(e,\Lambda)$, satisfies $|\Delta_0|<2$.

We will prove that this periodic solution can be continued in the presence of friction, although the odd symmetry is lost.

Theorem 2 *Assume that $(e,\Lambda)\in\Omega$. Then, there exists a number $\bar{\delta}>0$ and a real analytic function*

$$(t,\delta)\in\mathbb{R}\times[0,\bar{\delta}] \quad\mapsto\quad \Theta^*_\delta(t)\in\mathbb{R},$$

satisfying

i) $\Theta^*_\delta(t)$ is an asymptotically stable 2π-periodic solution of (38),

ii) $\Theta^*_0(t)=\Theta^*(t;e,\Lambda)$ for each $t\in\mathbb{R}$.

In principle, this theorem is consequence of well known classical results (see for instance Theorem 1.1 and 1.2 in Chapter 14, [10]). However, we will prove it independently because our proof will provide an explicit formula for $\bar{\delta}$. This formula will involve the quantities e, Λ and Δ_0 only.

Let $\Theta_\delta(t)$ be a solution of (38) satisfying $\Theta_\delta(0) = \Theta_0 \in \mathbb{R}$, $\dot{\Theta}_\delta(0) = \omega_0 \in \mathbb{R}$. To make explicit the dependence on initial conditions, consider $x = (\Theta_0, \omega_0)^\mathsf{T} \in \mathbb{R}^2$ and

$$\phi_t(\delta, x) = \begin{pmatrix} \Theta_\delta(t) \\ \omega_\delta(t) \end{pmatrix}, \quad \text{with} \quad \omega_\delta(t) = \dot{\Theta}_\delta(t).$$

Define the function

$$\mathcal{F}(\delta, x) = \phi_{2\pi}(\delta, x) - x.$$

The zeros of this function are in correspondence with the 2π-periodic solutions of (38). Since we know that $\Theta^*(t; e, \Lambda)$ is the odd 2π-periodic solution for $\delta = 0$, we are going to study the implicit function problem

$$\mathcal{F}(\delta, \chi(\delta)) = 0, \quad \chi(0) = \begin{pmatrix} 0 \\ \dot{\Theta}^*(0; e, \Lambda) \end{pmatrix}.$$

The solution $x = \chi(\delta)$ of this problem will produce a branch of periodic solutions in the conditions of Theorem 2.

The proof of Theorem 2 will consists of two steps. First, we will apply a quantitative version of the Implicit Function Theorem in order to find $\chi(\delta)$, defined on $\delta \in [0, \bar{\delta}]$. Once the branch is constructed, we will prove the asymptotic stability of the solution.

In the next lemma we will employ the following notation.

Given a function $(\delta, x) \in [0,1] \times \mathbb{R}^n \mapsto \mathcal{G}(\delta, x) \in \mathbb{R}^n$. The partial derivative $\partial_\delta \mathcal{G}(\delta, x)$ will be interpreted as a vector in $\mathbb{R}^n$, whereas $\partial_x \mathcal{G}(\delta, x)$ and $\partial_{\delta x} \mathcal{G}(\delta, x) = \partial_\delta(\partial_x \mathcal{G}(\delta, x))$ are linear maps represented by matrices in $\mathbb{R}^{n \times n}$. Let x_i be the i-th component of $x \in \mathbb{R}^n$, then, $\partial_{xx}\mathcal{G}(\delta, x)$ is a bilinear map given by

$$\partial_{xx}\mathcal{G}(\delta, x)[u, v] = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\partial^2 \mathcal{G}(\delta, x)}{\partial x_i \partial x_j} u_i v_j, \qquad u, v \in \mathbb{R}^n.$$

The norm in $\mathbb{R}^n$ is denoted by $||\cdot||$. The same notation will be employed for the induced norm in spaces of multilinear forms. See [12], Chapter 5, Section 7. Given a point $\zeta \in \mathbb{R}^n$ and a positive number $r > 0$, the closed ball centered at ζ of radius r is denoted by $\bar{B}_r(\zeta)$.

Lemma 4 *Let $\mathcal{G} : [0, \delta_*] \times \mathbb{R}^n \to \mathbb{R}^n$, with $\delta_* > 0$, and $\mathcal{G} = \mathcal{G}(\delta, x)$, be a function of class C^2 such that there exists $(\partial_x \mathcal{G}(\delta_0, x_0))^{-1}$ and $\mathcal{G}(\delta_0, x_0) = 0$ for a certain point $(\delta_0, x_0) \in [0, \delta_*] \times \mathbb{R}^n$.*

Assume that there exist uniform bounds $C_1 \geq ||\partial_\delta \mathcal{G}||$, $C_{12} \geq ||\partial_{\delta x}\mathcal{G}||$, $C_{22} \geq ||\partial_{xx}\mathcal{G}||$ for all $(\delta, x) \in [0, \delta_*] \times \mathbb{R}^n$ and $C_0 > ||(\partial_x \mathcal{G}(\delta_0, x_0))^{-1}||$.

Define the positive constants

$$\rho = \begin{cases} \frac{C_{12} - C_0 C_1 C_{22}}{C_0 C_{12}^2} & \text{if} \quad 2C_0 C_1 C_{22} < C_{12}, \\ \frac{1}{4C_0^2 C_1 C_{22}} & \text{if} \quad 2C_0 C_1 C_{22} \geq C_{12}, \end{cases} \tag{39}$$

and

$$R = R(\rho) = \frac{1 - \sqrt{1 - 4C_0^2 C_1 C_{22}\rho}}{2C_0 C_{22}}, \tag{40}$$

then, there exists a C^2 *function* $\chi : \bar{B}_\rho(\delta_0) \cap [0, \delta_*] \to \bar{B}_R(x_0)$ *satisfying* $x_0 = \chi(\delta_0)$ *and*

$$\mathcal{G}(\delta, \chi(\delta)) = 0, \quad \delta \in \bar{B}_\rho(\delta_0) \cap [0, \delta_*].$$

Proof. The proof follows along the standard methods using the Contraction mapping theorem. See for instance Section 3.4 in [17]. We give some hints to reproduce the values of ρ and R in eqs. (39) and (40). Define

$$L(\delta, x) = x - M^{-1}\mathcal{G}(\delta, x), \quad M = \partial_x \mathcal{G}(\delta_0, x_0),$$

so that our problem is equivalent to the fixed point equation $x = L(\delta, x)$. We see that

$$\partial_x \mathcal{G}(\delta, x) - M = \int_0^1 \partial_{\delta x}\mathcal{G}(\delta_\lambda, x_\lambda)(\delta - \delta_0)d\lambda + \int_0^1 \partial_{xx}\mathcal{G}(\delta_\lambda, x_\lambda)(x - x_0)d\lambda,$$

$$\partial_x \mathcal{G}(\delta_0, x) - M = \int_0^1 \partial_{xx}\mathcal{G}(\delta_0, x_\lambda)(x - x_0)d\lambda,$$

where $\delta_\lambda = \lambda\delta + (1-\lambda)\delta_0$, $x_\lambda = \lambda x + (1-\lambda)x_0$, $\lambda \in [0,1]$. Consequently, since $\partial_x L(\delta, x) = \mathbb{1} - M^{-1}\partial_x \mathcal{G}(\delta, x)$, we can use the bounds of the derivatives of $\mathcal{G}$ to get that

$$||\partial_x L(\delta, x)|| \le C_0(C_{12}\rho + C_{22}R), \qquad ||\partial_x L(\delta_0, x)|| \le C_0 C_{22} R, \tag{41}$$

for each $x \in \bar{B}_R(x_0)$ and $\delta \in \bar{B}_\rho(\delta_0)$. From the second expression in (41) and the generalized version of the mean-value theorem for vector-valued functions, see [2], we get

$$||L(\delta_0, x) - L(\delta_0, x_0)|| \le C_0 C_{22} R^2 \qquad x \in \bar{B}_R(x_0).$$

Proceeding analogously with $\partial_\delta L$, we obtain

$$||L(\delta, x) - L(\delta_0, x)|| \le C_0 C_1 \rho \qquad x \in \bar{B}_R(x_0),\ \delta \in \bar{B}_\rho(\delta_0).$$

Note that $L(\delta_0, x_0) = x_0$ by definition, then,

$$\begin{aligned} ||L(\delta, x) - x_0|| &\le ||L(\delta, x) - L(\delta_0, x)|| + ||L(\delta_0, x) - L(\delta_0, x_0)|| \\ &\le C_0(C_1\rho + C_{22}R^2). \end{aligned}$$

Let X be the complete metric space composed by continuous functions

$$\chi : \bar{B}_\rho(\delta_0) \cap [0, \delta_*] \to \bar{B}_R(x_0), \quad \chi(\delta_0) = x_0.$$

The distance in X is induced by the uniform norm. Consider the operator $L(\delta, \cdot) : X \to X$. From the previous computations, this operator is well defined, i.e., $L(\delta, X) \subseteq X$, as long as

$$C_0(C_1\rho + C_{22}R^2) \le R.$$

Moreover, from the first inequality in (41), $L(\delta, \cdot)$ is a contraction if

$$C_0(C_{12}\rho + C_{22}R) < 1.$$

The parameters ρ and R in (39) and (40) are the values such that the last two inequalities are satisfied and the value of ρ is the largest possible. The Banach principle leads to a continuous solution of the functional equation. The implicit function theorem can be applied at each $(\delta, \chi(\delta))$ to deduce that this solution is indeed C^2. ∎

Remark 3 *Note that χ is a real analytic function if $\mathcal{G}$ is real analytic.*

Remark 4 *By direct substitution of (39), we get that $0 \le 1 - 4C_0^2 C_1 C_{22}\rho < 1$, then,*

$$\rho \in \left(0, \frac{1}{4C_0^2 C_1 C_{22}}\right], \quad R \in \left(0, \frac{1}{2C_0 C_{22}}\right].$$

We will work with $n = 2$ and the maximum norm

$$||x|| = \max\{|x_1|, |x_2|\}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The corresponding norms in the spaces of multilinear maps (see [12]) are given by

$$||\partial_\delta \mathcal{G}(\delta, x)|| = \max_{i\in\{1,2\}} |\partial_\delta \mathcal{G}_i(\delta, x)|,$$

$$||\partial_{\delta x} \mathcal{G}(\delta, x)|| = \max_{i\in\{1,2\}} \left\{ \sum_{j\in\{1,2\}} \left| \frac{\partial^2 \mathcal{G}_i(\delta, x)}{\partial\delta\partial x_j} \right| \right\},$$

$$||\partial_{xx} \mathcal{G}(\delta, x)|| = \max_{i\in\{1,2\}} \left\{ \sum_{j,k\in\{1,2\}} \left| \frac{\partial^2 \mathcal{G}_i(\delta, x)}{\partial x_k\partial x_j} \right| \right\}.$$

It will be clear from the computations in the Appendix C that Lemma 4 cannot be directly applied to the function $\mathcal{F}$. Actually, the norm of $\partial_{\delta x}\mathcal{F}(\delta, x)$ has not a uniform bound in $(\delta, x) \in \mathbb{R}^2$. To overcome this difficulty, we will observe that there is a partial a priori bound for the periodic solutions of (38).

Lemma 5 *Let $\Theta(t)$ be a 2π-periodic solution of (38). Then,*

$$|\dot{\Theta}(t)| \le C \quad \text{for each } t \in \mathbb{R},$$

with $C := \Lambda \int_0^{2\pi} \frac{dt}{r(t,e)^3} + 2\int_0^{2\pi} |\ddot{f}(t, e)| dt$.

The proof is postponed to the Appendix B. Note that an analogous bound cannot be obtained for $\Theta(t)$. Due to the periodicity of the equation, $\Theta(t) + 2n\pi$ is also a solution for each $n \in \mathbb{Z}$.

Let us define the function $\mathcal{M} : \mathbb{R} \to \mathbb{R}$, and the map $\mathcal{R} : \mathbb{R}^2 \to \mathcal{R}^2$ such that

$$\mathcal{M}(\zeta) = \begin{cases} \arctan(\zeta + C) - C & \text{if} \quad \zeta < -C, \\ \zeta & \text{if} \quad |\zeta| \le C, \\ \arctan(\zeta - C) + C & \text{if} \quad \zeta > C, \end{cases} \qquad \mathcal{R}(x) = \begin{pmatrix} x_1 \\ \mathcal{M}(x_2) \end{pmatrix}. \tag{42}$$

We observe that $\mathcal{R}$ is C^2 and satisfies

- $\mathcal{R}$ is the identity on the strip $S_C = \{(x_1, x_2)^\mathsf{T} : \ |x_2| \le C\}$.
- $\mathcal{R}(\mathbb{R}^2 \backslash S_C) \cap S_C = \emptyset$.

From these properties and Lemma 5 it is easy to deduce that, if $x \in S_C$, the equation $\mathcal{F}(\delta, x) = 0$ is equivalent to $\mathcal{G}(\delta, x) = 0$, where,

$$\mathcal{G}(\delta, x) = \mathcal{F}(\delta, \mathcal{R}(x)).$$

Here by equivalence we mean that both equations have the same solutions.

We can now find estimates for the norms of the derivatives of $\mathcal{G}$. In the Appendix C we will obtain

$$C_0 = \frac{1+\kappa_0}{|2-\Delta_0|}, \qquad C_1 = \kappa ||D(\cdot, e)||_1 (2C + \pi/2),$$

$$C_{12} = \kappa^2 \left(C_1 \Lambda ||r(\cdot, e)^{-3}||_1 + ||D(\cdot, e)||_1 \right), \qquad C_{22} = 4\kappa^3 \Lambda ||r(\cdot, e)^{-3}||_1 + \frac{3\sqrt{3}(1+\kappa)}{8},$$

where $|| \cdot ||_1$ is the $L^1[0, 2\pi]$-norm and the constants κ_0 and κ are defined by

$$\kappa_0 = \exp \left(\max \left\{ 2\pi, \Lambda ||r(\cdot, e)^{-3}||_1 \right\} \right),$$

$$\kappa = \exp \left(\max \left\{ 2\pi, \Lambda ||r(\cdot, e)^{-3}||_1 + \delta_* ||D(\cdot, e)||_1 \right\} \right),$$

where the constant $\delta_* \in (0, 1/4)$ can be chosen arbitrarily making sure that the computed value $\bar{\delta}$ must be smaller than the chosen δ_*. We will take $\delta_* = 0.01$ for the MacDonald torque in Section 4.2.

The parameters ρ and R are now determined by (39) and (40). We find the function $\chi(\delta)$ defined on $[0, \rho]$, such that

$$||\chi(\delta) - \chi(0)|| \leq R. \tag{43}$$

Once we have constructed the branch of periodic solutions $\Theta = \Theta^*_\delta(t)$, we analyze the stability properties. The next Lemma on linear equations is tailored for our purposes.

Lemma 6 *Let $a, b, c : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions, also, $c \in C^1$ and $\int_0^T c(t)dt > 0$. Let $K > 0$ be a constant such that*

$$\left| b(t) - \frac{1}{4}c(t)^2 - \frac{1}{2}\dot{c}(t) \right| < K, \quad t \in [0, T].$$

Let Δ_0 be the discriminant of $\ddot{y} + a(t)y = 0$ and assume that $|\Delta_0| < 2$. Assume that

$$||\Phi_0(t)\Phi_0(s)^{-1}|| \leq \kappa_0, \quad 0 \leq s < t \leq T,$$

where $\Phi_0(t)$ is the matrix solution of $\ddot{y} + a(t)y = 0$, such that $\Phi_0(0) = \mathbb{1}$, and $|| \cdot ||$ denotes the matrix norm induced by the maximum norm in $\mathbb{R}^2$. Then, the equation

$$\ddot{y} + c(t)\dot{y} + (a(t) + b(t))y = 0 \tag{44}$$

is asymptotically stable if

$$K < \frac{1}{\kappa_0 T} \ln \left(1 + \frac{2 - |\Delta_0|}{2\kappa_0} \right). \tag{45}$$

The proof of this result is postponed to the Appendix D.

The variational equation of (38) at $\Theta^*_\delta(t)$ is

$$\ddot{y} + \delta D(t, e)\dot{y} + \frac{\Lambda}{r(t, e)^3} \cos[\Theta^*_\delta(t)]y = 0.$$

We can interpret this equation as a perturbation of the equation for $\delta = 0$. In the framework of (44),

$$a(t) = \frac{\Lambda}{r(t,e)^3}\cos[\Theta_0^*(t)],$$

$$b(t) = \frac{\Lambda}{r(t,e)^3}\left(\cos[\Theta_\delta^*(t)] - \cos[\Theta_0^*(t)]\right), \quad c(t) = \delta D(t,e).$$

To estimate $|b(t)|$ we observe that $\eta(t) = \Theta_\delta^*(t) - \Theta_0^*(t)$ satisfies the linear equation

$$\ddot{\eta} + P(t)\eta = Q(t), \tag{46}$$

where

$$P(t) = \frac{\Lambda}{r(t,e)^3}\frac{\sin[\Theta_\delta^*(t)] - \sin[\Theta_0^*(t)]}{\Theta_\delta^*(t) - \Theta_0^*(t)}, \qquad Q(t) = -\delta D(t,e)\dot{\Theta}_\delta^*(t)$$

In view of Lemma 5 we deduce that $|Q(t)| \leq \rho C D(t,e)$. Also, $|P(t)| \leq \frac{\Lambda}{r(t,e)^3}$. These estimates, together with (43) and (40) lead (see Appendix E) to

$$|\Theta_\delta^*(t) - \Theta_0^*(t)| \leq \kappa_0 R(\rho) + \kappa_0 \rho C ||D(\cdot,e)||_1, \tag{47}$$

where $||\cdot||_1$ is the $L^1[0,2\pi]$-norm. Consequently,

$$|b(t)| \leq \frac{\kappa_0 \Lambda}{r(t,e)^3}(R(\rho) + \rho C||D(\cdot,e)||_1),$$

and we can take a suitable $K = K(\rho)$, say,

$$K(\rho) = \kappa_0 \Lambda ||r(\cdot,e)^{-3}||_\infty \left(R(\rho) + \rho C||D(\cdot,e)||_1\right) + \frac{\rho^2}{4}||D(\cdot,e)||_\infty^2 + \frac{\rho}{2}||\dot{D}(\cdot,e)||_\infty.$$

Note that, by (40), $K(\rho)$ is an increasing continuous function for $\rho \in (0, \frac{1}{4C_0^2 C_1 C_{22}}]$ and such that $K(\rho) \to 0^+$ as $\rho \to 0^+$. In consequence, the function $\Theta_\delta^*(t)$ obtained by Lemma 4 is asymptotically stable as long as

$$\rho < K^{-1}\left(\frac{1}{\kappa_0 T}\ln\left(1 + \frac{2 - |\Delta_0|}{2\kappa_0}\right)\right).$$

In principle, we do not know if the value of ρ defined by (39) satisfies this inequality. However, since the solution $\Theta_\delta^*(t)$ is defined for all $\delta \in [0,\rho]$, we can always take a smaller value of ρ, say $\bar{\delta}$, satisfying the previous inequality so that $\Theta_\delta^*(t)$ is asymptotically stable for all $\delta \in [0,\bar{\delta}]$. With this, we have proved Theorem 2.

4.1 The dissipative function $D(t,e)$

In addition of being a planar model, equation (38) models the dissipative spin-orbit with the strong assumption that the dissipative torque is proportional to $\dot{\Theta}$. This is obvious for the so-called MacDonald torque in (5), for which $D(t,e) = r(t,e)^{-6}$, in this case we will be able to find analytically the constants of our estimates, except for Δ_0, which can be found numerically.

Let us sketch out a procedure through which our results are applicable to other dissipative torques. We take [13] as reference. In general, to compute the dissipative torque $\mathcal{T}_d$ we start from a potential U depending on the position of the perturbing body, so that $\mathcal{T}_d = rF_z$, where F_z is the z-component of the force $F = -\operatorname{grad} U$. It is common to expand U in power

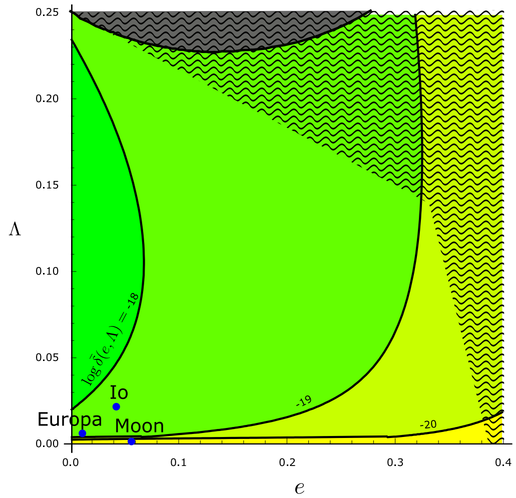

Figure 6: Dissipative diagram for the MacDonald torque. The more green is the region, the higher δ admissible. We do not guarantee the existence of $\Theta^*_\delta(t; e, \Lambda)$ for the region filled with the wavy pattern.

series of $1/r$, via Legendre polynomials, and assume that the dissipation is introduced by including a constant small time delay Δt in the position of the perturber. This gives rise to the torque of equation (28) in [13]. If we take only the leading term of the expansion we get the MacDonald torque, equation (30), [13]. Note that, in the expanded torque (28), [13], we work with $i = 0$, $M_1 = M_1^*$, $r = r^*$, and $\lambda = \lambda^*$. Consequently, we can see that each term of the expansion is proportional to $\sin(-m\Delta t\dot{\Theta}/2) \approx -m\Delta t\dot{\Theta}/2$ and we can write the torque in the form $-\delta D(t, e)\dot{\Theta}$. Actually, different orders of approximation would give rise to different functions $D(t, e)$. However, we must mention that this procedure does not guarantee that $D(t, e)$ is positive, which is important to find an upper bound $C \geq |\dot{\Theta}^*_\delta(t)|$, see Lemma 5, involved in the computation of other constants of our estimates, as we can check in the Appendix C.

4.2 Quantitative estimates for the MacDonald torque

In general we see that

$$||r(\cdot, e)^{-3}||_\infty = \frac{1}{(1-e)^3}, \quad ||r(\cdot, e)^{-3}||_1 = \frac{2\pi}{(1-e^2)^{3/2}}, \quad ||\ddot{f}(\cdot, e)||_1 = \frac{8e}{\sqrt{1-e^2}},$$

while for the MacDonald torque

$$||D(\cdot, e)||_\infty = \frac{1}{(1-e)^6}, \quad ||D(\cdot, e)||_1 = \frac{8 + 24e^2 + 3e^4}{4(1-e^2)^{9/2}}\pi, \quad ||\dot{D}(\cdot, e)||_\infty \leq \frac{6e}{(1-e)^8}.$$

Taking $\delta_* = 0.01$, we can compute the maximum admissible $\bar{\delta} = \bar{\delta}(e, \Lambda)$ and divide the (e, Λ)-diagram in regions corresponding to different orders of magnitude of $\bar{\delta}$. See Figure 6.

In the case of the Moon-Earth system, for which $e = 0.0549$, $\Lambda = 0.00069$, we obtain that the $1:1$ resonance is asymptotically stable for all δ smaller than $\bar{\delta} = 2.06 \cdot 10^{-20}$. This is actually a very small value, however, if we evaluate the corresponding maximum admissible delay we get $\Delta t(\bar{\delta}) = 11$ min. Which means that, if the Moon's response is delayed 11 minutes or less, its asymptotic stability is guaranteed by our computations. Consider a seismic event

Satellite (Planet)	e	Λ	$\bar{\delta}$	$\Delta t(\bar{\delta})$
Moon (Earth)	0.0549	0.00069	$2.06 \cdot 10^{-20}$	11 min
Io (Jupiter)	0.0041	0.021	$9.69 \cdot 10^{-18}$	0.00057 min
Europa (Jupiter)	0.0094	0.0055	$2.85 \cdot 10^{-19}$	0.0064 min

Table 1: Estimates for some satellite-planet systems with strong spin-orbit interaction. The parameters e and Λ have been taken from [5] and other constants from [1]. The corresponding $\bar{\delta}$ have been obtained numerically. The dependence of Δt with respect to δ only depends on the parameters of the system.

occurring at the center of the Moon and reaching the surface in 11 minutes. If the Moon were homogeneous, the necessary velocity of propagation would be of 2.74 km/s. This is reasonably consistent with the available data from the interior of the Moon, for which the speed of p-waves ranges from 1.0 km/s to 8.5 km/s, according to Table 24.2 in [25]. On the contrary, as we see in Table 1, we are less optimistic with respect to the direct applicability of these estimates for Io and Europa, since we get too small values of Δt.

5 Discussion

In this paper we have obtained some rigorous results concerning the existence and stability of the 1:1 resonant solution for the spin orbit problem, which is closely related to the capture into the resonance. The dissipative as well as the conservative version of the problem have attracted much attention. We have considered equation (4) as the reference model for the problem. Despite that the MacDonald torque introduces a rough simplification in the model, it looks for us reasonable to take the dissipative torque $\mathcal{T}_d$ proportional to $\dot{\theta} - \dot{f}$, which inevitably leads us to the intricate pendulum-like equation (6). This starting point, though suggested in many articles, was never fully exploited, as far as we know. That is why we wanted to study this equation with an analytical point of view and without further modifications. This contrasts to the literature, where the numerical approach prevails, or, where it is usual to average equation over a period or expand it in powers of the eccentricity e in order to focus the study on small e. The pioneer works of Goldreich and Pale [15], on one hand, and Beletskii [3], on the other hand, set the main research directions. The articles following [3] work in the conservative regime with f as independent variable, obtaining the so-called Beletskii equation.

In the literature we find different approaches to the stability of the solution. For instance, [15] poses the rough stability condition that the averaged dissipative torque does not exceed the maximum conservative torque. There are also more sophisticated approaches by A. Celletti and her collaborators, as the KAM stability in [7] for the conservative case, or the existence of quasiperiodic attractors in [9] for the dissipative problem, which bifurcate from the KAM tori of the conservative case. See also other related articles like [8] and [14]. The articles that employ the Beletskii equation, such as [3], [20] and [19], do not average the equation, they study the 2π-periodic solutions (there is not uniqueness) and consider the linear stability, particularly for the even solution $\Theta^*(t)$. They produce similar numerical stability diagrams as that of Figure 3, and notice, see for example [3], the complexity of the region of linear stability of $\Theta^*(t)$ for high eccentricities.

The most similar approach to ours is that of [22], which studies the Beletskii equation with analytical tools. Its main result (Theorem 5) estimates a region of existence and Lyapunov stability of $\Theta^*(t)$. The authors use the method of upper and lower solutions to prove the existence of solution, but they do not guarantee the uniqueness as odd 2π-periodic solution as we did in Proposition 1. Since we consider the dissipative model, we are interested in the asymptotic stability of the solution instead of the Lyapunov stability. For this purpose we need a region of elliptic linear stability, which is larger than the region obtained in Theorem 5, [22]. Using their computations, which correspond to L^∞-norm estimates (recall that we used all the L^α-norms to estimate our region), the resulting region of linear stability is given by

$$0 < \Lambda < \frac{(1-e)^3}{4}, \quad 0 < \Lambda < \frac{1}{\pi}\left(\frac{(1-e^2)^{3/2}}{2} - 8e\right).$$

We can check that this region is in fact included in our Ω.

A Properties of $\Lambda_0(e,\alpha)$ and $\Lambda_1(e)$

The function $\Lambda_0(e,\alpha)$ defined in (17) has an explicit expression in terms of special functions since, according to [26],

$$K_l(p) = \begin{cases} \frac{2\pi}{p\, l^{1+2/p}}\left(\frac{2}{2+p}\right)^{1-2/p}\left(\frac{\Gamma(1/p)}{\Gamma(1/2+1/p)}\right)^2, & \text{if} \quad 1 \le p < \infty, \\ \frac{4}{l}, & \text{if} \quad p = \infty, \end{cases} \tag{48}$$

where Γ is the usual Gamma function. We can compute that

$$||r(\cdot,e)^{-3}||_\alpha = \begin{cases} \dfrac{\pi}{(1-e^2)^{3/2}}, & \text{if} \quad \alpha = 1, \\ \left(\dfrac{\pi\ {}_2F_1(1/2, 3\alpha-1; 1; 2e/(1+e))}{(1+e)^{3\alpha-1}}\right)^{1/\alpha}, & \text{if} \quad 1 < \alpha < \infty, \\ (1-e)^{-3}, & \text{if} \quad \alpha = \infty, \end{cases} \tag{49}$$

where $||\cdot||_\alpha$ denotes the $L^\alpha[0,\pi]$-norm and ${}_2F_1$ is the hypergeometric function. The continuous dependence of the function $||r(\cdot,e)^{-3}||_\alpha$ with respect to e and $\alpha < \infty$ is guaranteed by the classical theorem in integration theory. Moreover, it is also guaranteed for $\alpha = \infty$, because it is well known that the L^α-norm of a function converges to its L^∞-norm as $\alpha \to \infty$. The function $K_l(p)$ is continuous for each $p \in [1,\infty)$ since $\Gamma(x)$ is continuous for all $x > 0$. It is also continuous for $p = \infty$ because $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1/p) = p + \mathcal{O}(1)$ as $p \to \infty$. Consequently, $\Lambda_0(e,\alpha)$ is continuous in both entries.

Lemma 7 *The function Λ_1 defined in (18) is continuous, strictly decreasing and*

$$\Lambda_1(0) = 1, \quad \lim_{e\to 1^-} \Lambda_1(e) = 0.$$

Proof. The continuity of $\Lambda_0(e,\alpha)$ implies that the function $\Lambda_1(e)$ is bounded in $[0,\bar{e}]$, for all $\bar{e} \in (0,1)$, since,

$$|\Lambda_1(e)| \le \max_{[0,\bar{e}]\times[1,\infty]} |\Lambda_0(e,\alpha)|.$$

Then, the continuity of $\Lambda_1(e)$ is equivalent to say that the set $\{(e, \Lambda_1(e)) : e \in [0, \bar{e}]\}$ is closed in $\mathbb{R}^2$.

Since Λ_1 is bounded, we can take a sequence $e_n \in [0, \bar{e}]$ converging to e such that $\Lambda_1(e_n)$ converges to a some ζ. Then we have to prove that $\Lambda_1(e) = \zeta$.

By definition of Λ_1, for each n there exists $\alpha_n \in [1, \infty]$ such that $\Lambda_1(e_n) = \Lambda_0(e_n, \alpha_n)$. We can take a subsequence $\alpha_{\sigma(n)}$ converging to a some $\hat{\alpha}$. Due to the continuity of Λ_0 and because $\Lambda_1(e_{\sigma(n)}) = \Lambda_0(e_{\sigma(n)}, \alpha_{\sigma(n)})$, we get that $\Lambda_1(e) \geq \Lambda_0(e, \hat{\alpha}) = \zeta$. On the other hand, we can take a $\alpha_* \in [1, \infty]$ such that $\Lambda_1(e) = \Lambda_0(e, \alpha_*)$. Again, since $\Lambda_1(e_n) \geq \Lambda_0(e_n, \alpha_*)$, then, $\zeta \geq \Lambda_0(e, \alpha_*) = \Lambda_1(e)$. Consequently, $\Lambda_1(e) = \zeta$.

To prove that Λ_1 is monotone, recall the definitions made in (2). We can evaluate the integral $||r(\cdot, e)^{-3}||_\alpha$, $\alpha \in [1, \infty)$ using the change of variable of the eccentric anomaly $t = u - e \sin u$ and get

$$||r(\cdot, e)^{-3}||_\alpha^\alpha = \int_0^\pi \frac{du}{(1 - e \cos u)^{3\alpha - 1}}, \tag{50}$$

differentiating with respect to e and applying properties of the cosine we obtain

$$\begin{aligned} \frac{d}{de}||r(\cdot, e)^{-3}||_\alpha^\alpha &= (3\alpha - 1) \int_0^\pi \frac{\cos u}{(1 - e \cos u)^{3\alpha}} du \\ &= (3\alpha - 1) \int_0^{\pi/2} \left(\frac{1}{(1 - e \cos u)^{3\alpha}} - \frac{1}{(1 + e \cos u)^{3\alpha}} \right) \cos u \; du. \end{aligned}$$

The last integral is clearly positive and, since

$$\frac{d}{de}||r(\cdot, e)^{-3}||_\alpha^\alpha = \alpha ||r(\cdot, e)^{-3}||_\alpha^{\alpha - 1} \frac{d}{de}||r(\cdot, e)^{-3}||_\alpha,$$

the function $||r(\cdot, e)^{-3}||_\alpha$ is increasing in e. In consequence, according to the definition (17), each $\Lambda_0(\cdot, \alpha)$ is strictly decreasing for $\alpha \in [1, \infty)$. The same can be said about $\Lambda_0(\cdot, \infty)$ since, according to (49), $||r(\cdot, e)^{-3}||_\infty = (1 - e)^{-3}$. This implies also that Λ_1 is monotone.

Now let us prove that $\Lambda_1(0) = 1$. First note that it is easy to compute that $\Lambda_0(0, \infty) = K_\pi(2) = 1$, then $\Lambda_1(0) \geq 1$. On the other hand, we can apply Theorem 5 in [26] to our equation (20), which for $e = 0$ is simply $\ddot{y} + \Lambda y = 0$. In this case, the first anti-periodic eigenvalue is $1/4$, then, using the identities (22) again and the definition of $\Lambda_0(0, \alpha)$, we get the following inequalities

$$1 > \Lambda_0(0, 1), \quad 1 \geq \Lambda_0(0, \alpha), \quad \alpha \in (1, \infty],$$

which lead to $\Lambda_1(0) \leq 1$. This proves the claim.

Now consider the asymptotic behavior. Note that the integrand in (50) is positive and has a singularity at $u = 0$ for $e = 1$, actually, it behaves as $u^{-2(3\alpha - 1)}$ as $u \to 0^+$, making the integral diverges. This is the reason why $||r(\cdot, e)^{-3}||_\alpha \to \infty$ as $e \to 1^-$ for each $\alpha \in [1, \infty)$, and, as a result, $\Lambda_0(e, \alpha) \to 0$. The same happens for the case $\alpha = \infty$ as we did before. To be able to take the maximum, the limit $\Lambda_0(e, \alpha) \to 0$ as $e \to 1^-$ should be uniform, but so far we have shown only the pointwise convergence. To obtain this property we can apply Dini's Theorem (see Theorem 7.13 in [23]) thanks to the fact that $\Lambda_0(e, \alpha)$ is continuous in the compact set of values $\alpha \in [1, \infty]$ for each e, and it is monotone in e for each α. We can take any set of values $\{e_n\} \subset [0, 1)$, such that $e_n < e_{n+1}$, and $e_n \to 1^-$ as $n \to \infty$, and apply the mentioned theorem to the functions $f_n(\alpha) = \Lambda_0(e_n, \alpha)$. As a result, $\Lambda_0(e, \alpha) \to 0$ uniformly in α as $e \to 1^-$, which guarantees that $\Lambda_1(e) \to 0$. ■

B Proof of Lemma 5

Note that if $\Theta(t)$ satisfies (38), then $\omega(t) = \dot{\Theta}(t)$ satisfies the equation

$$\dot{\omega} + \delta D(t,e)\omega = b_0(t), \quad b_0(t) = -2\ddot{f}(t,e) - \frac{\Lambda}{r(t,e)^3}\sin[\Theta(t)],$$

by variation of constants we see that if $\omega_0 = \omega(t_0)$,

$$\omega(t) = \omega_0 \exp\left(-\delta \int_{t_0}^{t} D(s,e)ds\right) + \int_{t_0}^{t} b_0(s)\exp\left(-\delta\int_s^t D(\tau,e)d\tau\right)ds,$$

and, since D is positive,

$$|\omega(t)| \leq |\omega_0| + \int_{t_0}^{t_0+2\pi} |b_0(s)|ds \leq |\omega_0| + C, \tag{51}$$

where we defined

$$C = 2||\ddot{f}(\cdot,e)||_1 + \Lambda||r(\cdot,e)^{-3}||_1,$$

where $||\cdot||_1$ is the $L^1[0,2\pi]$-norm.

Since the solution $\Theta(t)$ is 2π-periodic, then, we could choose t_0 such that $\dot{\Theta}(t_0) = \omega_0 = 0$, then, C is a bound for $|\dot{\Theta}(t)|$.

C Computation of constants $\kappa_0, \kappa, C_0, C_1, C_{12}, C_{22}$

Consider a vector $x = (\Theta_0, \omega_0)^{\mathsf{T}} \in \mathbb{R}^2$ and let $\Theta = \Theta(t;\delta,x)$ be the solution of (38) with initial conditions $\Theta(0) = \Theta_0$, $\dot{\Theta}(0) = \omega_0$ for fixed (e,Λ). Let us call $\omega(t;\delta,x) = \dot{\Theta}(t;\delta,x)$ and

$$\phi_t(\delta,x) = \begin{pmatrix} \Theta(t;\delta,x) \\ \omega(t;\delta,x) \end{pmatrix}.$$

Note that $\Phi(t) = \partial_x\phi_t(\delta,x) \in \mathbb{R}^{2\times 2}$ is the matrix solution of

$$\dot{y} = A(t)y, \quad y(0) = \mathbb{1},$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{\Lambda}{r(t,e)^3}\cos[\Theta(t;\delta,x)] & -\delta D(t,e) \end{pmatrix},$$

then, for $0 \leq s \leq t \leq T = 2\pi$,

$$\Phi(t)\Phi(s)^{-1} = \mathbb{1} + \int_s^t A(\tau)\Phi(\tau)\Phi(s)^{-1}d\tau,$$

taking matrix norms and using Gronwall's inequalities, we have that

$$||\Phi(t)\Phi(s)^{-1}|| \leq \exp\left(\int_s^t ||A(\tau)||d\tau\right) \leq \exp\left(\int_0^T ||A(\tau)||d\tau\right).$$

Using the maximum norm we get

$$||A(\tau)|| \leq \begin{cases} \max\{T, \Lambda|r(\tau,e)^{-3}|\} & \text{if} \quad \delta = 0, \\ \max\left\{T, \Lambda|r(\tau,e)^{-3}| + \delta|D(\tau,e)|\right\} & \text{if} \quad \delta > 0, \end{cases}$$

in consequence, using the subscript 0 for the case $\delta = 0$, we have that

$$||\Phi_0(t)\Phi_0(s)^{-1}|| \leq \kappa_0 = \exp\left(\max\left\{T, \Lambda||r(\cdot,e)^{-3}||_1\right\}\right),$$

where $||\cdot||_1$ is the $L^1[0,T]$-norm. Note that, for $\delta = 0$, it is also true that $||A(\tau)^{\mathsf{T}}|| \leq \max\{T, \Lambda|r(\tau,e)^{-3}|\}$, then, $||\Phi_0(t)^{\mathsf{T}}|| \leq \kappa_0$. Fixing a value $\delta_* > 0$, for $\delta \in (0, \delta_*]$ we have

$$||\Phi(t)\Phi(s)^{-1}|| \leq \kappa = \exp\left(\max\left\{T, \Lambda||r(\cdot,e)^{-3}||_1 + \delta_*||D(\cdot,e)||_1\right\}\right).$$

The previous bounds are valid for all the initial conditions $x \in \mathbb{R}^2$.

For

$$\mathcal{G}(\delta, x) = \mathcal{F}(\delta, \mathcal{R}(x)) = \phi_T(\delta, \mathcal{R}(x)) - \mathcal{R}(x),$$

where $\mathcal{R}$ is defined in (42), we want to find some positive constants such that

$$C_1 \geq ||\partial_\delta \mathcal{G}(\delta, x)||, \quad C_{12} \geq ||\partial_{\delta x} \mathcal{G}(\delta, x)||, \quad C_{22} \geq ||\partial_{xx} \mathcal{G}(\delta, x)||.$$

Computing the derivatives,

$$\partial_\delta \mathcal{G}(\delta, x) = \partial_\delta \phi_T(\delta, \mathcal{R}(x)),$$

$$\partial_{\delta x} \mathcal{G}(\delta, x) = \partial_{\delta x} \phi_T(\delta, \mathcal{R}(x)) \partial_x \mathcal{R}(x),$$

$$\frac{\partial^2 \mathcal{G}}{\partial x_i \partial x_j}(\delta, x) = \partial_{xx} \phi_T(\delta, \mathcal{R}(x)) \left[\frac{\partial \mathcal{R}}{\partial x_i}, \frac{\partial \mathcal{R}}{\partial x_j}\right] + (\partial_x \phi_T(\delta, \mathcal{R}(x)) - \mathbb{1}) \left[\frac{\partial^2 \mathcal{R}}{\partial x_i \partial x_j}\right],$$

where, in the last expression, $\partial_{xx}\phi_T$ is considered as a bilinear map and $(\partial_x \phi_T - \mathbb{1})$ as a linear map. We can see that the non-vanishing derivatives of $\mathcal{R}$ appearing above are

$$\frac{\partial \mathcal{R}}{\partial x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{\partial \mathcal{R}}{\partial x_2} = \begin{pmatrix} 0 \\ \mathcal{M}'(x_2) \end{pmatrix}, \quad \frac{\partial^2 \mathcal{R}}{\partial x_2^2} = \begin{pmatrix} 0 \\ \mathcal{M}''(x_2) \end{pmatrix},$$

then, particularly,

$$\frac{\partial^2 \mathcal{G}}{\partial x_1^2}(\delta, x) = \frac{\partial^2 \phi_T}{\partial x_1^2}(\delta, \mathcal{R}(x)), \quad \frac{\partial^2 \mathcal{G}}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi_T}{\partial x_1 \partial x_2}(\delta, \mathcal{R}(x))\mathcal{M}'(x_2),$$

$$\frac{\partial^2 \mathcal{G}}{\partial x_2^2}(\delta, x) = (\mathcal{M}'(x_2))^2 \frac{\partial^2 \phi_T}{\partial x_2^2}(\delta, \mathcal{R}(x)) + (\partial_x \phi_T(\delta, \mathcal{R}(x)) - \mathbb{1}) \begin{pmatrix} 0 \\ \mathcal{M}''(x_2) \end{pmatrix}.$$

Taking into account that $|\mathcal{M}'(x_2)| \leq 1$ and $|\mathcal{M}''(x_2)| \leq 3\sqrt{3}/8$, the constants C_1, C_{12} and C_{22} are going to be defined in terms of bounds for the derivatives of $\phi_T(\delta, \mathcal{R}(x))$.

The function $\partial_\delta \phi_t(\delta, x)$ is a solution of

$$\dot{y} = A(t)y + b_1(t), \quad y(0) = 0,$$

where

$$b_1(t) = \begin{pmatrix} 0 \\ -D(t,e)\omega(t;\delta,x) \end{pmatrix}.$$

By variation of constants

$$\partial_\delta \phi_t(\delta, x) = \int_0^t \Phi(t)\Phi(s)^{-1} b_1(s) ds,$$

then, in order to find a bound for $||\partial_\delta \phi_T(\delta, \mathcal{R}(x))||$, first we need to find a bound for $|\omega(t;\delta,\mathcal{R}(x))|$. Note that the expression (51) is valid for all the initial conditions $\omega_0 = \omega(0;\delta,x)$, but in our case $|\mathcal{M}(\omega_0)| \leq C + \pi/2$, then,

$$|\omega(t;\delta,\mathcal{R}(x))| \leq 2C + \pi/2.$$

Consequently,

$$||\partial_\delta \phi_T(\delta, \mathcal{R}(x))|| \leq \int_0^T ||\Phi(T)\Phi(s)^{-1}||\, ||b_1(s)|| ds \leq C_1 = \kappa ||D(\cdot, e)||_1 (2C + \pi/2).$$

We see that $\partial_{\delta x}\phi_t(\delta, x)$ is a matrix solution of

$$\dot{y} = A(t)y + b_{12}(t), \quad y(0) = 0,$$

where

$$b_{12}(t) = \begin{pmatrix} 0 & 0 \\ \frac{\Lambda}{r(t,e)^3}\partial_\delta \Theta(t;\delta,x)\sin[\Theta(t;\delta,x)] & -D(t,e) \end{pmatrix} \Phi(t),$$

then,

$$||\partial_{\delta x}\phi_T(\delta, \mathcal{R}(x))|| \leq \int_0^T ||\Phi(T)\Phi(s)^{-1}||\, ||b_{12}(s)|| ds \leq C_{12},$$

with $C_{12} = \kappa^2 (C_1 \Lambda ||r(\cdot, e)^{-3}||_1 + ||D(\cdot, e)||_1)$.

Finally, we observe that $\frac{\partial^2 \phi_t}{\partial x_i \partial x_j}$ is solution of

$$\dot{y} = A(t)y + b_{22}^{ij}(t), \quad y(0) = 0,$$

where

$$b_{22}^{ij}(t) = \begin{pmatrix} 0 \\ \frac{\Lambda}{r(t,e)^3}\frac{\partial \Theta}{\partial x_i}\frac{\partial \Theta}{\partial x_j}\sin\Theta \end{pmatrix},$$

then,

$$\left|\left|\frac{\partial^2 \phi_T}{\partial x_i \partial x_j}(\delta, \mathcal{R}(x))\right|\right| \leq \int_0^T ||\Phi(T)\Phi(s)^{-1}||\, ||b_{22}^{ij}(s)|| ds \leq \kappa^3 \Lambda ||r(\cdot, e)^{-3}||_1.$$

In consequence,

$$||\partial_{xx}\mathcal{G}(\delta, x)|| \leq \sum_{i,j\in\{1,2\}} \left|\left|\frac{\partial^2 \mathcal{G}}{\partial x_i \partial x_j}\right|\right| \leq C_{22} = 4\kappa^3 \Lambda ||r(\cdot, e)^{-3}||_1 + \frac{3\sqrt{3}(1+\kappa)}{8}.$$

Finally, in our case, $\delta_0 = 0$ and $x_0 = (0, \dot{\Theta}^*(0))^\top$. We know that $|\dot{\Theta}^*(0)| \leq C$, since $\Theta^*(t)$ is a T-periodic solution, then, $\mathcal{R}(x_0) = x_0$ and $\partial_x \mathcal{R}(x_0) = \mathbb{1}$, so, we have to find a bound for the norm of

$$(\partial_x \mathcal{G}(0, x_0))^{-1} = (\partial_x \phi_T(0, x_0) - \mathbb{1})^{-1}.$$

For any matrix $M \in \mathbb{R}^{2\times 2}$,

$$M^{-1} = -\frac{JM^\top J}{\det M}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

since $||J|| = 1$, then,

$$||M^{-1}|| \leq \frac{||M^\top||}{|\det M|},$$

on the other hand, define $\Delta_0 = \mathrm{Tr}(\partial_x \phi_T(0, x_0))$, since $\partial_x \phi_T(0, x_0)$ is a symplectic matrix, we have that $\det(\partial_x \phi_T(0, x_0)^\top - \mathbb{1}) = 2 - \Delta_0$. In consequence,

$$||(\partial_x \mathcal{G}(0, x_0))^{-1}|| \leq C_0 = \frac{1 + \kappa_0}{|2 - \Delta_0|}.$$

D Proof of Lemma 6

The change of variables $\eta(t) = \exp\left(\frac{1}{2}\int_0^t c(s)ds\right) y(t)$, transforms (44) into

$$\ddot{\eta} + \left(a(t) + \tilde{b}(t)\right)\eta = 0,$$

where

$$\tilde{b}(t) = b(t) - \frac{1}{4}c(t)^2 - \frac{1}{2}\dot{c}(t).$$

We observe that (44) is asymptotically stable if the equation for η is stable. If we call $Y = \begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix}$, we obtain the equation

$$\dot{Y} = \left(A_0(t) + \tilde{b}(t)N\right)Y,$$

where

$$A_0(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Let $\Phi(t)$ be the matrix solution with $\Phi(0) = \mathbb{1}$. Then, by variation of constants, the equation is equivalent to

$$\Phi(t) = \Phi_0(t) + \int_0^t \Phi_0(t)\Phi_0(s)^{-1}\tilde{b}(t)N\Phi(s)ds.$$

Since the maximum norm of N is 1, then,

$$||\Phi(t)|| \leq \kappa_0 + \kappa_0\mu\int_0^t ||\Phi(s)||ds,$$

using Gronwall's Lemma,

$$||\Phi(t)|| \le \kappa_0 \exp\left(\kappa_0 \mu t\right),$$

in consequence,

$$||\Phi(t) - \Phi_0(t)|| \le \kappa_0^2 \mu \int_0^t \exp\left(\kappa_0 \mu s\right) ds = \kappa_0(\exp\left(\kappa_0 \mu t\right) - 1).$$

For any real square matrix M, $|\operatorname{Tr} M| \le 2 r_s(M) \le 2||M||$, where r_s is the spectral radius. If $\Delta = \operatorname{Tr}\Phi(T)$ and, using hypothesis (45),

$$|\Delta - \Delta_0| \le 2\kappa_0(\exp\left(\kappa_0 \mu T\right) - 1) < 2 - |\Delta_0|.$$

We conclude that $|\Delta| < 2$.

E Computation of equation (47)

By variation of constants in (46) we get that

$$\begin{pmatrix} \eta(t) \\ \dot\eta(t) \end{pmatrix} = \hat\Phi(t) \begin{pmatrix} \eta(0) \\ \dot\eta(0) \end{pmatrix} + \int_0^t \hat\Phi(t)\hat\Phi^{-1}(s) \begin{pmatrix} 0 \\ Q(s) \end{pmatrix} ds,$$

where $\hat\Phi(t)$ is the matrix solution of the homogeneous equation $\ddot\eta + P(t)\eta = 0$. Note that, since $|P(t)| \le \frac{\Lambda}{r(t,e)^3}$, we have that

$$||\hat\Phi(t)|| \le \kappa_0 = \exp\left(\max\left\{2\pi, \Lambda||r(\cdot,e)^{-3}||_1\right\}\right),$$

exactly as in the computation of κ_0 in Appendix C. Moreover, note that

$$\left|\left|\begin{pmatrix} \eta(0) \\ \dot\eta(0) \end{pmatrix}\right|\right| = ||\chi(\delta) - \chi(0)|| \le R,$$

since $|Q(t)| \le \rho C D(t,e)$, we get that

$$|\Theta_\delta^*(t) - \Theta_0^*(t)| \le \kappa_0 R + \kappa_0 \rho C ||D(\cdot,e)||_1.$$

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