

Zero-curvature subconformal structures and dispersionless integrability in dimension five

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Abstract

We extend the recent paradigm “Integrability via Geometry” from dimensions 3 and 4 to higher dimensions, relating dispersionless integrability of partial differential equations to curvature constraints of the background geometry. We observe that in higher dimensions on any solution manifold, the symbol defines a vector distribution equipped with a subconformal structure, and the integrability imposes a certain compatibility between them. In dimension 5, we express dispersionless integrability via the vanishing of a certain curvature of this subconformal structure. We also obtain a “master equation” governing all second-order dispersionless integrable equations in 5D, and count their functional dimension. It turns out that the obtained background geometry is parabolic of the type (A_3, P_{13}) . We provide its Cartan-theoretic description and compute the harmonic curvature components via the Kostant theorem. Then, we relate it to 3D projective and 4D conformal geometries via twistor theory, discuss symmetry reductions and nested Lax sequences, as well as give another interpretation of dispersionless integrability in 5D through Levi-degenerate CR structures in 7D.

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1 | INTRODUCTION AND MAIN RESULTS

Lax pair formulation is one of the most conventional approaches in the integrability analysis of differential equations. For solitonic-type equations, the integrability is related to zero-curvature representation [18, 29], while for dispersionless equations the situation is different. Namely, dispersionless Lax pair (dLp) no longer consists of higher order differential operators, but is reduced to a pair of vector fields with a spectral parameter, often also with terms including differentiation by the spectral parameter [50]. Such dLp allows convenient geometrization of the integrability via twistor theory [23, 25, 37], which eventually for many classes of dispersionless systems in a more general context has been related to curvature properties of underlying background geometries [5, 13, 14, 16].

The integrability of second-order partial differential equations (PDEs), or more generally systems of PDEs with quadratic characteristic variety, was shown to be equivalent to the Einstein–Weyl property in 3D and the self-duality property in 4D for the canonical conformal structure read off the characteristic variety/symbol of the equation [7]. Here, the integrability is understood as the existence of a nontrivial dLp. Moreover, for several classes of examples, this was also shown to be equivalent to the existence of hydrodynamic reductions, as developed in [19, 48], see [20].

Not much is known in dimensions beyond 4, except for sporadic examples [3, 26, 41, 46]. In this paper, we address the smallest of those dimensions, namely, 5D. Let us note that conformal geometry in 4D is one of the key examples of the so-called parabolic geometry, and that self-duality is expressed as vanishing of a part of its curvature, namely, the anti-self-dual Weyl tensor. Similarly, in 3D, Weyl geometry is a structure reduction of conformal geometry, and the Einstein–Weyl condition is expressed as vanishing of a part of the curvature, namely, the trace-free part of the symmetrized Ricci tensor of the Weyl connection.

It turns out that the situation in 5D is similar, and dispersionless integrability can be written as a zero-curvature condition for a contact parabolic geometry.

1.1 | Subconformal structures on solutions

Let us explain the main ingredients of this article in the case of a scalar second-order PDE, which will be fundamental for the more general case of PDE systems in Section 3.1. A scalar second-order PDE can be expressed as

$$\mathcal{E} : F(x, u, \partial u, \partial^2 u) = 0 \tag{1}$$

for a scalar function u of an independent variable x on a connected manifold M with $\dim M = d$, where $\partial u = (u_i)$ and $\partial^2 u = (u_{ij})$ denote partial derivatives of u in local coordinates $x = (x^i)$.

Let M_u denote the manifold M equipped with a scalar function u ; we may view M_u as the graph of u in $M \times \mathbb{R}$ or, as we will see later, in the appropriate jet space. A tensor on M_u is, by definition, a tensor on M , which may also depend, at each $x \in M$, on finitely many derivatives of u at x .

Let $\ell_F(u)$ be the *linearization* on a solution u of \mathcal{E}

$$\ell_F(u)v = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(u + \epsilon v),$$

which clearly depends on the background solution u if F is nonlinear, and let σ_F be the *symbol*, that is, the top degree of ℓ_F involving only second derivatives

$$\sigma_F = \sum_{i \leq j} \frac{\partial F}{\partial u_{ij}} \partial_i \partial_j = \sum_{i,j} \sigma_{ij}(u) \partial_i \otimes \partial_j.$$

Invariantly, σ_F defines a section of $S^2 TM_u$, hence a quadratic form on $T_x^* M_u$ for each $x \in M_u$. Changing the defining function F of \mathcal{E} changes σ_F by a conformal rescaling on \mathcal{E} . Hence, the conformal class of σ_F along $F = 0$ is an invariant of \mathcal{E} , and so is the *characteristic variety* $\text{Char}(\mathcal{E}, u) \rightarrow M_u$, which is defined as a bundle whose fiber at $x \in M_u$ is the projective variety

$$\text{Char}(\mathcal{E}, u)_x = \{[\theta] \in \mathbb{P}(T_x^* M_u) \mid \sigma_F(\theta) = 0\}.$$

In what follows it is important that this variety is a quadric; however, it is degenerate. Actually, the following is a direct corollary of the characteristic property of [7, Theorem 1], hidden in the discussion of the coisortopic property of dLp [7].

Theorem 1.1 [7]. *For integrable determined PDE systems in dimension $d > 4$ with quadratic characteristic variety, the bilinear form σ_F and hence the characteristic variety are degenerate. In fact, the rank of the symmetric matrix $[\sigma_{ij}]$ does not exceed 4.*

In this paper, we assume that $\text{rank} [\sigma_{ij}]$ is maximal, that is, equal to 4 on generic solutions u of \mathcal{E} . This means that the vector distribution (obtained by contraction)

$$\Delta_{\mathcal{E}} = \langle \sigma_F, T^*M_u \rangle \subset TM_u$$

has rank 4. In general, $\Delta_{\mathcal{E}}$ is a nonintegrable and even nonholonomic distribution of rank 4 on M_u for generic solution u of \mathcal{E} . Moreover, in what follows we assume that $d = \dim M_u = 5$ and that $\Delta_{\mathcal{E}}$ is a *contact distribution* on M_u .

The symmetric bivector σ_F is nondegenerate on $\Delta_{\mathcal{E}}$ and can be inverted to give sub-pseudo-Riemannian structure defined up to scale, that is, subconformal structure $c_{\mathcal{E}} = [g_F]$ on any solution:

$$g_F = (\sigma_F|_{\Delta_{\mathcal{E}}})^{-1} \in S^2\Delta_{\mathcal{E}}^*, \quad \text{where} \quad [g_{ij}] = [\sigma_{ij}]^{-1}.$$

In what follows, we assume that the conformal structure $c_{\mathcal{E}}$ has neutral signature (2,2). In other signatures, one still can use the same approach via complexification, yet the real integrability conditions can be more restrictive. For instance, in 4-dimensional Lorentzian conformal structures, the integrability implies that F is the standard wave operator plus lower order terms.

For any point $x \in M_u$, recall that the neutral signature conformal structure $c_{\mathcal{E}}|_x$ on a 4-dimensional space $\Delta_{\mathcal{E}}(x)$ has 2 one-parameter families of isotropic planes that is $\mathbb{C}\mathbb{P}^1$ in the complex case and $\mathbb{R}\mathbb{P}^1 \simeq S^1$ in the real case. These are traditionally called α - and β -planes, or self-dual or anti-self-dual null planes, which are swapped upon the change of orientation. The structure $(\Delta_{\mathcal{E}}, c_{\mathcal{E}})$ on M_u is called *compatible* if the α -family can be chosen to consist of Lagrangian planes with respect to the conformally symplectic structure on the contact distribution. In this case, a generic β -plane would not be Lagrangian. Note that the existence of dLp implies the existence of a 1-parameter family of surfaces whose tangent plane at each point lies in the contact distribution and is null with respect to the subconformal structure. Since an integral surface of any field of contact planes is necessarily Legendrian, then its tangent planes are Lagrangian with respect to the induced conformal symplectic structure. In other words, the existence of dLp requires the compatibility condition that will be assumed in what follows.

1.2 | An example of integrable PDE

Consider the following example of 5D version of the heavenly equation wherein c is a constant:

$$F = u_{15} + c u_{25} + u_{13}u_{24} - u_{14}u_{23} = 0. \quad (2)$$

This equation is a traveling wave reduction of the well-known Takasaki–Plebansky–Przanowski equation in 6D [41, 46], and it, in turn, reduces to the first Plebansky (also known as heavenly), modified heavenly and Hussain’s equations [12, 40].

The symbol of F is $\sigma_F = v_1 \cdot v_3 + v_2 \cdot v_4$ where $v_1 = \partial_1, v_2 = \partial_2, v_3 = \partial_5 - u_{23}\partial_4 + u_{24}\partial_3, v_4 = c\partial_5 + u_{13}\partial_4 - u_{14}\partial_3$;

$$\Delta = \langle v_1, v_2, v_3, v_4 \rangle = \text{Ker}(\omega^0),$$

$$\begin{aligned} \omega^0 &= (u_{13} + c u_{23}) dx^3 + (u_{14} + c u_{24}) dx^4 + (u_{15} + c u_{25}) dx^5 \\ &= d(u_1 + c u_3) - (u_{11} + c u_{13}) dx_1 - (u_{12} + c u_{23}) dx_2. \end{aligned}$$

Additionally, on the contact plane Δ , we have

$$g = \sigma_F^{-1} = \omega^1 \omega^3 + \omega^2 \omega^4,$$

where

$$\omega^1 = dx^1, \omega^2 = dx^2, \omega^3 = \frac{c dx^3 + u_{14} dx^5}{u_{14} + c u_{24}}, \omega^4 = \frac{dx^4 + u_{23} dx^5}{u_{13} + c u_{23}}.$$

The β -planes $\text{Ker}\{\omega^1 - \lambda\omega^2, \omega^4 + \lambda\omega^3\}$ are not Lagrangian for $\Omega = d\omega^0$ except for two particular values of λ , which can be projectively reparametrized to 0 and ∞ .

The α -planes $\text{Ker}\{\omega^1 + \lambda\omega^4, \omega^2 - \lambda\omega^3\} = \langle v_4 - \lambda v_1, v_3 + \lambda v_2 \rangle$ are a family of Lagrangian planes, and they give rise to the following dLp for (2):

$$\hat{\Pi} = \langle \partial_5 - u_{23}\partial_4 + u_{24}\partial_3 + \lambda\partial_2, c\partial_5 + u_{13}\partial_4 - u_{14}\partial_3 - \lambda\partial_1 \rangle.$$

1.3 | Main results

In general, a *dispersionless Lax pair*, referred to as *dLp*, for (1) is a rank 1 covering system $\hat{\mathcal{E}}$ of \mathcal{E} of a special kind [28, 50]. Namely, there is a fiber bundle $\pi : \hat{M}_u \rightarrow M_u$ with connected rank 1 fibers, and a rank 2 distribution $\hat{\Pi} \subset T\hat{M}_u$ such that its Frobenius integrability condition $[\hat{\Pi}, \hat{\Pi}] = \hat{\Pi}$ is a quasi-linear overdetermined system $\hat{\mathcal{E}}$ whose compatibility is \mathcal{E} . Often, this latter condition is relaxed to the claim that \mathcal{E} is a differential corollary of \hat{M}_u , but it is assumed that \mathcal{E} is restored up to a simple integration/nonlocalities of potentiation kind; see a discussion in [7]. In practice, one can choose generators of $\hat{\Pi}$ to be linearly independent vector fields \hat{X} and \hat{Y} on \hat{M}_u , whose coefficients depend on finitely many derivatives of u .

A fiber coordinate $\lambda : \hat{M}_u \rightarrow \mathbb{R}$ is called a *spectral parameter* and it locally identifies \hat{M}_u with $M_u \times \mathbb{R}$. We may then write $\hat{X} = X + m \partial_\lambda, \hat{Y} = Y + n \partial_\lambda$ where X, Y are λ -parametric vector fields on M_u , and a section of $\pi : \hat{M}_u \rightarrow M_u$ may be written $\lambda = q(x)$ for a function $q : M_u \rightarrow \mathbb{R}$. The dLp $\hat{\Pi}$ then has the geometric interpretation that \mathcal{E} is the integrability condition for the existence of many one-parameter families of foliations of M_u by surfaces that at any $x \in M_u$ are tangent to $\Pi = \pi_*(\hat{\Pi})$ at x , with $\lambda = q(x)$.

By the mentioned (co)isotropic property of the dLp, we may thus identify \hat{M}_u locally with the \mathbb{P}^1 -bundle whose fiber over $x \in M_u$ consists of all α -planes. In this case, at each $x \in M$, Π gives

an immersion $\mathbb{P}^1 \rightarrow \text{Gr}(2, \Delta)$. Under this identification, with Π considered as a \mathbb{P}^1 -bundle over M_u , the pullback $\pi^*\Pi \rightarrow \hat{M}_u$ is the tautological bundle.

Any Weyl connection ∇ on M_u , that is, a torsion-free connection that preserves the conformal class $c_{\mathcal{E}}$ and depends on finitely many derivatives of u , induces a connection on the \mathbb{P}^1 -bundle $\hat{M}_u \rightarrow M_u$. Hence, it defines a horizontal lift of Π to a rank 2 distribution $\hat{\Pi}_{\nabla} \subset T\hat{M}_u$. It is well known [7, 39] that in 4D, the lift $\hat{\Pi}_{\nabla}$ is independent of ∇ , that is, *conformally invariant*; the same applies in higher dimensions provided that $\text{rank}[\sigma_{ij}] = 4$.

Now let $(\Delta_{\mathcal{E}}, c_{\mathcal{E}})$ be a compatible subconformal structure on a contact distribution in 5D. In Section 2, we will associate to such structure a coframe ω^i on M_u that is determined up to a projective reparametrization and rescalings, whose group action depends on a discrete invariant $\text{sgn} \delta = \pm 1$ that distinguishes between the CR and para-CR types. The Cartan–Tanaka prolongation of this structure is of finite type and this implies the fundamental curvature components. Important for us is a tensor $W = (W_i)_0^4$ generalizing the self-dual part of the Weyl tensor in 4D conformal geometry, which we will describe in detail in Sections 2.2–2.4.

Similar to the situation in 4D, there exists a canonical lift of the \mathbb{P}^1 -bundle of α -planes, that is, the α -congruence on M , to a rank 2 distribution in 5D. This will be called a *standard dLp*. Two Lax pairs are \mathcal{E} -equivalent if their restrictions to the (infinitely prolonged) equation \mathcal{E} coincide. A dLp is called *nondegenerate* if it is not \mathcal{E} -equivalent to a dispersionless pair, for which the Frobenius integrability condition holds identically, also referred to *offshell*, that is, not as a corollary of \mathcal{E} , cf. [7, Definition 5].

Theorem 1.2. *Let $\mathcal{E} : F = 0$ be a determined PDE system in 5D, whose characteristic variety $\text{Char}(\mathcal{E})$ is a bundle of quadric hypersurfaces of maximal rank 4 in $\mathbb{P} T^*M_u$, and such that the subconformal structure on contact distribution $(\Delta_{\mathcal{E}}, c_{\mathcal{E}})$ on M_u is compatible for (almost) every solution u . Then, any nondegenerate dLp $\hat{\Pi}$ is \mathcal{E} -equivalent to a standard dLp $\hat{\Pi}_{\nabla}$.*

Our main result establishes an equivalence between the dispersionless integrability of \mathcal{E} and the zero-curvature property of $c_{\mathcal{E}}$.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, \mathcal{E} is integrable by a nondegenerate dLp if and only if the zero-curvature condition holds nontrivially on solutions of \mathcal{E} .*

Finally, let us give a presentation of master equation for dispersionless integrable equations in 5D with contact characteristic distribution. Here, by “master,” we mean two following properties.

- Any integrable background subconformal geometry in 5D is locally described by this equation, as a particular solution of the PDE on the functional parameters.
- Any integrable dispersionless equation with the specified conditions on the characteristic variety is a reduction of this equation, meaning that all solutions of the given PDE are locally reparametrizations of the solutions of the master equation.

Such master equation in 3D was identified in [17] with the Manakov–Santini equation, governing Einstein–Weyl structures, and in 4D a determined PDE was derived in [17], governing the self-dual conformal structures.

This can be considered as the quotient problem of the space of all background integrable geometries in 5D by the pseudogroup of local diffeomorphisms, where the residual gauge

transformations can be eliminated via the differential invariants approach, as was done in [31, 32] for the EW (Einstein-Weyl) and SD (self-duality) systems.

Leaving the precise form of the master equation to Section 3, let us preview the result.

Theorem 1.4. *General zero-curvature integrable subconformal structures on contact distributions in 5D are given locally by the contact form $dr - p dx - q dy$ and the conformal metric*

$$g = (dx + (u + v) dy) \cdot (dq - (u - v) dp + w dx + (z - w(u - v)) dy) + (dx + (u - v) dy) \cdot (dq - (u + v) dp + w dx + (z - w(u + v)) dy),$$

where the functions u, v, w, z of variables x, y, p, q, r satisfy PDEs $\{W_i = 0\}_{i=0}^3$ given by (25). This system is itself integrable via a dLp (21–23). The local moduli space of zero-curvature integrable subconformal structures is parametrized by 8 functions of 4 variables.

Structure of the paper. In Section 2, we investigate the background geometry in dimension 5, that is, the subconformal contact structure on M_u with appropriate compatibility conditions, express the structure equations and fundamental invariants, and link the integrability via dLp to the zero-curvature condition. In Section 3, we describe the systems of PDEs to which our results extend, introduce the necessary tools from jet machinery, and prove the main results. In Section 4, we relate our results to other geometries via twistorial techniques and symmetry reductions. We conclude in Section 5 with an overview of the main results and discuss possible generalizations. In the Appendix, we classify integrable parabolic background geometries, in addition to the geometry discussed in this paper and the previously known parabolic geometries in 4D and 3D, namely, Cartan geometries of type $(SO(3, 3), P_1)$ and $(SO(2, 3), P_1)$, the latter of which is equipped with a choice of Weyl structure.

Conventions. In this article, all manifolds are real and smooth. Given a (system of) differential equation, we work locally away from singularities. For a foliation of a manifold, we restrict to open sets in which the leaf space of the foliation is a smooth manifold. Given a set of 1-forms $\{\alpha^1, \dots, \alpha^n\}$ on a manifold M , their span is denoted by $I = \langle \alpha^i \rangle_{i=1}^n$ and their kernel (annihilator) is denoted by $\text{Ker } I$ or I^\perp interchangeably. For a distribution $\Delta \subset TM$, its annihilator will be denoted as $I = \text{Ann}(\Delta)$. For a pseudo-Riemannian metric $g \in S^2T^*M$, the corresponding conformal structure is denoted by either $[g]$ or c_g . Given a bundle $\mathcal{G} \rightarrow M$ with a coframe $\langle \alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^k \rangle$ on \mathcal{G} , the dual frame is denoted by $\langle \partial_{\alpha^1}, \dots, \partial_{\alpha^n}, \partial_{\beta^1}, \dots, \partial_{\beta^k} \rangle$. Furthermore, if $\langle \alpha^i \rangle_{i=1}^n$ are semibasic with respect to a fibration $\mathcal{G} \rightarrow M$, then given a function $f : \mathcal{G} \rightarrow \mathbb{R}$, the iterative coframe derivatives of f are defined as

$$f_{;i} = \partial_{\alpha^i} \lrcorner df, \quad f_{;ij} = \partial_{\alpha^j} \lrcorner df_{;i}, \quad \text{and so on.}$$

2 | GEOMETRY OF SUBCONFORMAL STRUCTURES IN 5D

Here, we discuss compatible subconformal structures on contact 5D manifolds (M, Δ) . Let $\omega^0 \in \text{Ann}(\Delta) \setminus 0$ be a contact form. Then, $\Omega = d\omega^0|_\Delta$ defines a conformal symplectic structure. We will operate with structures on Δ via a coframe $\omega^i, 1 \leq i \leq 4$, modulo ω^0 . The dual frame on Δ will be denoted by $\partial_{\omega^1}, \partial_{\omega^2}, \partial_{\omega^3}, \partial_{\omega^4}$, and it is complemented by the transversal vector ∂_{ω^0} (which may be chosen to be the Reeb vector field but we do not require it).

2.1 | Compatibility

A subconformal structure on (M, Δ) is the conformal class of a nondegenerate bilinear form $g \in \Gamma(S^2\Delta^*)$, which will be assumed of neutral signature $(2,2)$. In null-diagonal coframe, we get

$$g = \omega^1\omega^3 + \omega^2\omega^4. \quad (3)$$

There are two \mathbb{P}^1 -bundles of contact planes, denoted as $\mathcal{A}, \mathcal{B} \subset \text{Gr}_2(\Delta)$ and referred to as α - and β -planes, respectively, which are totally null with respect to $[g]$, that is, at every point $x \in M$

$$\begin{aligned} \mathcal{A}_x &:= \{ \Pi = \text{Ker}\{\alpha_0\omega^1 - \alpha_1\omega^2, \alpha_0\omega^4 + \alpha_1\omega^3, \omega^0\} \mid [\alpha_0 : \alpha_1] \in \mathbb{P}^1 \}, \\ \mathcal{B}_x &:= \{ \Pi = \text{Ker}\{\beta_0\omega^1 - \beta_1\omega^4, \beta_0\omega^2 + \beta_1\omega^3, \omega^0\} \mid [\beta_0 : \beta_1] \in \mathbb{P}^1 \}. \end{aligned} \quad (4)$$

With respect to the Hodge operator $*$ on (Δ, g) , these are self-dual and anti-self-dual planes, respectively, and they are swapped upon the change of orientation on Δ . In what follows, we choose to focus on the family of α -planes.

Definition 2.1. A subconformal contact structure $(M, \Delta, [g])$ is *compatible* if all α -planes of $[g]$

$$\Pi = \langle \partial_{\omega^1} + \lambda \partial_{\omega^2}, \lambda \partial_{\omega^3} - \partial_{\omega^4} \rangle, \quad \lambda = \frac{\alpha_0}{\alpha_1} \in \mathbb{R} \cup \infty, \quad (5)$$

are Lagrangian with respect to the induced conformal symplectic structure $[\Omega]$ on Δ .

We also call Ω compatible under the same conditions with respect to g on Δ .

Lemma 2.2. A compatible symplectic structure has the form

$$\Omega = p \omega^1 \wedge \omega^2 + q (\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4) + r \omega^3 \wedge \omega^4$$

with a relative invariant $\delta = q^2 - pr \neq 0$.

Proof. Indeed, writing $\Omega = \sum_{i < j} c_{ij} \omega^i \wedge \omega^j$ modulo ω^0 , evaluating this on Π and expanding by λ gives $c_{23} = 0, c_{13} = c_{24}, c_{14} = 0$. Then, $\frac{1}{2}\Omega^2 = -\delta \text{vol}_g$, where $\text{vol}_g = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$. \square

Proposition 2.3. If Ω is compatible with (5), then it is conformally equivalent to one of the forms

$$\delta > 0 : \Omega = \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4, \quad (6)$$

$$\delta < 0 : \Omega = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4. \quad (7)$$

Proof. Let us introduce an operator $J = g^{-1}\Omega$ on Δ . Normalizing $|\delta| = 1$ by rescaling Ω is equivalent to the spectrum of J belonging to the unit circle $S^1 \subset \mathbb{C}$. More explicitly, if $\delta = +1$, then eigenvectors of J are ± 1 and if $\delta = -1$, then eigenvectors of J are $\pm i$, in both cases both eigenvalues have multiplicity 2 and the operator J is semisimple and is related to g by

$$g(Jv, w) + g(v, Jw) = 0 \quad \forall v, w \in \Delta.$$

In the case $\delta = +1, J^2 = \mathbf{1}$ and the g -null Ω -Lagrangian planes are generated by J -eigenvectors, and so are either eigenspaces $L_{\pm} = E_J(\pm 1)$ or belong to the family Π generated by an orthogonal pair of vectors from L_- and L_+ . Choosing null-orthogonal basis of eigenvectors of J as in (3), we get the required formula (6).

In the case $\delta = -1, J^2 = -\mathbf{1}$ and the g -null Ω -Lagrangian planes form a 1-parameter family Π , but no such singular planes as in the case $\delta = +1$ exist. Then, a pair of J -invariant null planes yields null-orthogonal basis (3) and the required formula (7) follows. \square

Note that in the family of β -planes given by

$$\Pi' = \langle \partial_{\omega^1} + \lambda \partial_{\omega^4}, \partial_{\omega^2} - \lambda \partial_{\omega^3} \rangle, \quad \lambda = \frac{\beta_0}{\beta_1} \in \mathbb{R} \cup \infty,$$

Ω -Lagrangian planes for $\delta > 0$ correspond to $\lambda = 0$ and ∞ , that is, $L_- = \langle \partial_{\omega^1}, \partial_{\omega^2} \rangle$ and $L_+ = \langle \partial_{\omega^3}, \partial_{\omega^4} \rangle$, while for $\delta < 0$, the equation for such planes is $\lambda^2 + 1 = 0$ and it has no real solutions.

Corollary 2.4. *A compatible subconformal structure $(M, \Delta, [g])$ possesses an adapted coframe $\{\omega^i\}_{i=0}^4$, in which $\Delta = \{\omega^0 = 0\}$ and formulae (3) together with either (6) or (7) hold. The part $\{\omega^i\}_{i=1}^4$ modulo ω^0 is defined up to an action of $\mathbb{R}_\times GL(2, \mathbb{R}) \subset \text{End}(\Delta)$.*

Proof. The first statement is the direct corollary of the proposition. The second follows from the formulae (3)–(6)–(7). In matrix terms, the action is given as follows:

$$A_+ = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b a_{22} & -b a_{21} \\ 0 & 0 & -b a_{12} & b a_{11} \end{pmatrix}, \quad A_- = \begin{pmatrix} a_{11} & a_{12} & -b a_{12} & b a_{11} \\ a_{21} & a_{22} & -b a_{22} & b a_{21} \\ b a_{21} & b a_{22} & a_{22} & -a_{21} \\ -b a_{11} & -b a_{12} & -a_{12} & a_{11} \end{pmatrix}, \quad (8)$$

where the first matrix corresponds to $\delta > 0$ and the second to $\delta < 0$.

Note that those algebras preserve $J_+ = -\partial_{\omega^1} \otimes \omega^1 - \partial_{\omega^2} \otimes \omega^2 + \partial_{\omega^3} \otimes \omega^3 + \partial_{\omega^4} \otimes \omega^4$ and $J_- = \partial_{\omega^1} \otimes \omega^4 - \partial_{\omega^4} \otimes \omega^1 - \partial_{\omega^2} \otimes \omega^3 + \partial_{\omega^3} \otimes \omega^2$, respectively. \square

Formula (8) shows the action of $GL(2, \mathbb{R})$ on $\Delta \simeq \mathbb{R}^4$, which is reducible for $\delta > 0$ and irreducible for $\delta < 0$. It induces the action of $PSL(2, \mathbb{R})$ on the projective line \mathbb{P}^1 of α -planes.

2.2 | Curvature in subconformal geometry: Naïve approach

We consider the case $\delta > 0$ given by (3) and (6) and the case $\delta < 0$ given by (3) and (7) simultaneously.

The preimage of α -planes to \hat{M} , which is the total space of the \mathbb{P}^1 -bundle with fibers \mathcal{A}_x in (4), is given by

$$\pi_*^{-1}\Pi = \langle \partial_{\omega^1} + \lambda \partial_{\omega^2}, \lambda \partial_{\omega^3} - \partial_{\omega^4}, \partial_\lambda \rangle,$$

where λ is a local coordinate along the fibers. Given the compatibility condition, this distribution has growth vector (3, 5, 6), and hence, it possesses the radical (dLp) uniquely given by the

condition $[\hat{\Pi}, \hat{\Pi}] \subset \pi_*^{-1}\Pi$, and so, we get

$$\hat{\Pi} = \sqrt{\pi_*^{-1}\Pi} = \langle \partial_{\omega^1} + \lambda \partial_{\omega^2} + m \partial_{\lambda}, \lambda \partial_{\omega^3} - \partial_{\omega^4} + n \partial_{\lambda} \rangle.$$

This can be considered as a lift of the configuration Π on M , and the coefficients m, n are uniquely determined as follows. Let the structure equation of the frame be

$$d\omega^k = -\frac{1}{2}c_{ij}^k \omega^i \wedge \omega^j \Leftrightarrow [\partial_{\omega^i}, \partial_{\omega^j}] = c_{ij}^k \partial_{\omega^k}.$$

We compute

$$\begin{aligned} & [\partial_{\omega^1} + \lambda \partial_{\omega^2} + m \partial_{\lambda}, \lambda \partial_{\omega^3} - \partial_{\omega^4} + n \partial_{\lambda}] \\ &= \lambda c_{13}^k \partial_{\omega^k} - c_{14}^k \partial_{\omega^k} + \lambda^2 c_{23}^k \partial_{\omega^k} - \lambda c_{24}^k \partial_{\omega^k} + m \partial_{\omega^3} - n \partial_{\omega^2} \text{ mod} \langle \partial_{\lambda} \rangle \\ &\equiv (-\lambda^2 c_{13}^1 + \lambda c_{13}^2 + \lambda c_{14}^1 - c_{14}^2 - \lambda^3 c_{23}^1 + \lambda^2 c_{23}^2 + \lambda^2 c_{24}^1 - \lambda c_{24}^2 - n) \partial_{\omega^2} \\ &\quad + (\lambda c_{13}^3 + \lambda^2 c_{13}^4 - c_{14}^3 - \lambda c_{14}^4 + \lambda^2 c_{23}^3 + \lambda^3 c_{23}^4 - \lambda c_{24}^3 - \lambda^2 c_{24}^4 + m) \partial_{\omega^3} \text{ mod} \langle \partial_{\lambda} \rangle \end{aligned}$$

whence

$$\begin{aligned} m &= -\lambda^3 c_{23}^4 + \lambda^2 (c_{24}^4 - c_{23}^3 - c_{13}^4) + \lambda (c_{24}^3 + c_{14}^4 - c_{13}^3) + c_{14}^3, \\ n &= -\lambda^3 c_{23}^1 + \lambda^2 (c_{24}^1 + c_{23}^2 - c_{13}^1) - \lambda (c_{24}^2 - c_{14}^1 - c_{13}^2) - c_{14}^2. \end{aligned}$$

Now the curvature of the subconformal structure, considered as the obstruction to integrability of dLp, is $d\lambda([\partial_{\omega^1} + \lambda \partial_{\omega^2} + m \partial_{\lambda}, \lambda \partial_{\omega^3} - \partial_{\omega^4} + n \partial_{\lambda}])$, which equals

$$W = \partial_{\omega^1}(n) + \lambda \partial_{\omega^2}(n) - \lambda \partial_{\omega^3}(m) + \partial_{\omega^4}(m) + m n_{\lambda} - n m_{\lambda}, \tag{9}$$

where $m_{\lambda} = \partial_{\lambda} m$. This is clearly a quartic in λ responsible for Frobenius integrability of dLp. In fact, with the notations

$$\begin{aligned} m_0 &= c_{14}^3, & m_1 &= c_{24}^3 + c_{14}^4 - c_{13}^3, & m_2 &= c_{24}^4 - c_{23}^3 - c_{13}^4, & m_3 &= -c_{23}^4, \\ n_0 &= -c_{14}^2, & n_1 &= c_{13}^2 + c_{14}^1 - c_{24}^2, & n_2 &= c_{24}^1 + c_{23}^2 - c_{13}^1, & n_3 &= -c_{23}^1, \end{aligned}$$

we have $W = W_0 + W_1 \lambda + W_2 \lambda^2 + W_3 \lambda^3 + W_4 \lambda^4$, where

$$\begin{aligned} W_0 &= \partial_{\omega^1}(n_0) + \partial_{\omega^4}(m_0) + m_0 n_1 - m_1 n_0, \\ W_1 &= \partial_{\omega^1}(n_1) + \partial_{\omega^2}(n_0) - \partial_{\omega^3}(m_0) + \partial_{\omega^4}(m_1) + 2(m_0 n_2 - m_2 n_0), \\ W_2 &= \partial_{\omega^1}(n_2) + \partial_{\omega^2}(n_1) - \partial_{\omega^3}(m_1) + \partial_{\omega^4}(m_2) + 3(m_0 n_3 - m_3 n_0) + m_1 n_2 - m_2 n_1, \\ W_3 &= \partial_{\omega^1}(n_3) + \partial_{\omega^2}(n_2) - \partial_{\omega^3}(m_2) + \partial_{\omega^4}(m_3) + 2(m_1 n_3 - m_3 n_1), \\ W_4 &= \partial_{\omega^2}(n_3) - \partial_{\omega^3}(m_3) + m_2 n_3 - m_3 n_2. \end{aligned}$$

2.3 | Associated parabolic geometry

A compatible subconformal structure can be equivalently described as a parabolic geometry of type (A_3, P_{13}) . We refer to [9] for the basics of parabolic geometries and to many examples, including those related to ours.

Remark 2.5. Among A_3 -type parabolic geometries, the following are well known and considered in [9]: 3D projective geometry has type (A_3, P_1) , 4D conformal geometry has type (A_3, P_2) , geometry of systems of second-order ordinary differential equations (ODEs) with two dependent variables has type (A_3, P_{12}) and vanishing torsion of the lowest weight, CR structures in 5D or its paraversion integrable Legendrian structures have type (A_3, P_{13}) and vanishing torsion. This latter case is different from ours, which allows torsion but requires vanishing curvature.

Here, we only note that the underlying geometric structure for (A_3, P_{13}) , in the complex case, is a contact distribution Δ on 5-manifold M , and this distribution is split $\Delta = L_- \oplus L_+$ into the sum of two Lagrangian subbundles, which are thus conformally dual to each other. The matrix form of the parabolic subalgebras $\mathfrak{p} = \text{Lie}(P)$ corresponds to the part of nonnegative grading, as follows (we also show other parabolics that will be relevant later):

$$\begin{array}{cccc}
 \underbrace{\begin{bmatrix} 0 & +1 & +1 & +1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}}_{\mathfrak{p}_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1} &
 \underbrace{\begin{bmatrix} 0 & 0 & +1 & +1 \\ 0 & 0 & +1 & +1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}}_{\mathfrak{p}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_1} &
 \underbrace{\begin{bmatrix} 0 & +1 & +2 & +2 \\ -1 & 0 & +1 & +1 \\ -2 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 \end{bmatrix}}_{\mathfrak{p}_{12} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2} &
 \underbrace{\begin{bmatrix} 0 & +1 & +1 & +2 \\ -1 & 0 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ -2 & -1 & -1 & 0 \end{bmatrix}}_{\mathfrak{p}_{13} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}
 \end{array}$$

There are three different real versions, corresponding to the following real groups G of type A_3 : $SL(4, \mathbb{R})$, $SU(1, 3)$, and $SU(2, 2)$, each of which has a parabolic subgroup of type $P_{1,3}$.

However, real compatible subconformal structures correspond only to the first and the last ones. The middle real parabolic geometry has induced conformal structure on Δ of definite signature, hence does not possess null 2-planes, which can be taken as dLp candidates. The other two cases have the following notations as crossed Dynkin diagrams:

$$\mathfrak{p}_{1,3} \subset \mathfrak{sl}(4, \mathbb{R}) : \quad \times \text{---} \circ \text{---} \times \qquad \mathfrak{p}_{1,3} \subset \mathfrak{su}(2, 2) : \quad \times \text{---} \overset{\curvearrowright}{\circ} \text{---} \times$$

In the first case, corresponding to $\delta > 0$, the parabolic geometry on M^5 results in a splitting of Δ and the conformal duality $L_+ \simeq L_-^*$. Conversely, given such geometric structure, we define the subconformal structure via pairing null planes L_- and L_+ . The configuration of α -planes is restored as $\Pi = \ell_\lambda^- \oplus \ell_\lambda^+$, where $L_- \supset \ell_\lambda^- \perp \ell_\lambda^+ \subset L_+$ (so ℓ_λ^- determines ℓ_λ^+). The splitting can be encoded via an almost product structure on the contact distribution Δ given by $J|_{L_\pm} = \pm \text{Id}_\Delta$. With no requirement of integrability of distributions L_\pm , this geometry is almost para-CR.

Similarly, in the last case, corresponding to $\delta < 0$, the parabolic geometry on M^5 reads off the splitting of $\Delta^{\mathbb{C}}$ into a pair of complex conjugated 2-distributions L_{10} and $L_{01} = L_{10}^*$ (holomorphic and antiholomorphic parts of the complexified contact distribution). This can be encoded via an almost complex structure J on Δ , compatible with the conformally symplectic structure. However, we again do not require integrability of the complex distributions (or vanishing of the Nijenhuis tensor), and hence, this geometry is almost CR of split Levi signature. Thus, we proved:

Proposition 2.6. *There is a bijective correspondence in 5D between compatible subconformal contact structures ($\delta > 0$, resp., $\delta < 0$) and (A_3, P_{13}) -type parabolic geometries (Legendrian contact structures, resp, almost CR structures of split Levi signature).*

Now we would like to define the notion of a Cartan geometry. Given Lie groups $P \subset G$, let denote their Lie algebras by \mathfrak{p} and \mathfrak{g} , respectively. A *Cartan geometry* $(\mathcal{G} \rightarrow M, \psi)$ of type (G, P) is given by a (right) principal P -bundle $\tau : \mathcal{G} \rightarrow M$ together with a *Cartan connection* $\psi \in \Omega^1(\mathcal{G}, \mathfrak{g})$, that is, a \mathfrak{g} -valued 1-form on \mathcal{G} with the following properties:

- ψ is P -equivariant, that is, $r_g^* \psi = \text{Ad}_{g^{-1}} \circ \psi$ for any $g \in P$,
- ψ maps fundamental vector fields to their generators, that is, $\psi(\zeta_X) = X$ for any $X \in \mathfrak{p}$,
- ψ defines an isomorphism $\psi : T_u \mathcal{G} \rightarrow \mathfrak{g}$ for any $u \in \mathcal{G}$.

The *curvature* of a Cartan connection ψ is the 2-form $\Psi \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined as

$$\Psi(X, Y) = d\psi(X, Y) + [\psi(X), \psi(Y)]$$

for $X, Y \in \Gamma(T\mathcal{G})$.

Every parabolic geometry possesses a Cartan connection $\psi \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on the principal bundle \mathcal{G} over the base M , where $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of the corresponding Lie group discussed above. The curvature of this connection

$$K = d\psi + \frac{1}{2}[\psi, \psi] \in \Omega^2(\mathcal{G}, \mathfrak{g})$$

can be identified with the curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ via the Killing form identification $\mathfrak{g}_-^* = (\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}_+$, where $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is the grading corresponding to the choice of parabolic $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ and $\mathfrak{p}_+ = \mathfrak{g}_+$. The normality condition $\partial^* \kappa = 0$, where $\partial^* : \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \rightarrow \mathfrak{p}_+ \otimes \mathfrak{g}$ is the Kostant codifferential, uniquely determines the Cartan connection [9].

The harmonic curvature κ_H is the quotient part of κ taking values in the \mathfrak{g}_0 submodule $\text{Ker}(\square) = \frac{\text{Ker}(\partial^*)}{\text{Im}(\partial^*)}$ of $\Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, where $\square = \partial \partial^* + \partial^* \partial$ is the Kostant Laplacian [27]. This κ_H uniquely restores κ via invariant differentiations, and is a simpler object, as it takes values in the Lie algebra cohomology $H_+^2(\mathfrak{g}_-, \mathfrak{g})$, where the subscript “+” indicates positive homogeneity with respect to the grading element $Z \in \mathfrak{g}_0$.

Computation of this cohomology is straightforward from the Kostant’s version of the Bott–Borel–Weyl theorem [27]. For the complex Lie algebra A_3 and its parabolic subalgebra \mathfrak{p}_{13} , we have $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{heis}(5)$, and the cohomology H_+^2 decomposes into \mathfrak{g}_0 -irreps as follows:

$$H_+^2(\mathfrak{g}_-, \mathfrak{g}) = \mathbb{V}_1^{(2,-1)} \oplus \mathbb{V}_1^{(-1,2)} \oplus \mathbb{V}_4^{(1,1)}, \tag{10}$$

where \mathbb{V}_s indicates $\mathfrak{g}_0^{ss} = \mathfrak{sl}(2, \mathbb{C})$ module and superscripts show the weight with respect to $\mathfrak{z}(\mathfrak{g}_0) = \mathbb{C} \oplus \mathbb{C}$. The first two summands correspond to the torsion τ_{\pm} and can be identified with $\Lambda^2 L_-^* \otimes L_+$ and $\Lambda^2 L_+^* \otimes L_-$. The last module corresponds to the curvature wherein W takes values. We thus have the following.

Proposition 2.7. *The harmonic curvature of a compatible subconformal structure splits as above $\kappa_H = \tau_- + \tau_+ + W$ and the zero-curvature condition is $W = 0$.*

The correspondence space, as the total space of the \mathbb{P}^1 -bundle $\pi : \hat{M}^6 \rightarrow M^5$ (with λ coordinate in fibers), is a parabolic geometry of type (A_3, P_{123}) and its underlying rank 3-distribution $\pi_*^{-1}\Pi$ with growth vector $(3,5,6)$ has the radical $\hat{\Pi} = \sqrt{\pi_*^{-1}\Pi}$. The harmonic curvature of the lifted structure on \hat{M}^6 has 5 irreducible components

$$\hat{\kappa}_H = \zeta_- + \zeta_+ + \tau_- + \tau_+ + W, \tag{11}$$

all of which are torsion, that is, the 2-cocycles take values in the corresponding \mathfrak{g}_- . Because the distribution is a lift of a (A_3, P_{13}) structure, one has $\zeta_{\pm} = 0$. The zero-curvature condition $W = 0$ corresponds to integrability of $\hat{\Pi}$, so there is a local quotient, $\hat{M}^6 \rightarrow \mathcal{T}^4$, where \mathcal{T}^4 is referred to as the *twistor space*. However, the induced structure on \mathcal{T} is not a parabolic geometry of type (A_3, P_2) , that is, a conformal structure, unless $\tau_{\pm} = 0$, that is, unless the subconformal contact structure is flat.

In the real split (para CR) case $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$, the above expressions hold literally by changing \mathbb{C} to \mathbb{R} with $\mathfrak{g}_0^{ss} = \mathfrak{sl}(2, \mathbb{R})$, and so on. In the real CR case $\mathfrak{g} = \mathfrak{su}(2, 2)$, the first two summands of (10) form an irreducible module over $\mathfrak{g}_0^{ss} = \mathfrak{su}(1, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$. Thus, $\tau_- + \tau_+$ in Proposition 2.7 is the indecomposable torsion, and similarly, in (11), curvature components $\zeta_- + \zeta_+$ and $\tau_- + \tau_+$ are indecomposable.

2.4 | Structure equations and fundamental invariants

The equivalence method for subconformal compatible contact structures, reformulated as Cartan geometries $(\mathcal{G} \rightarrow M, \psi)$ of type (A_3, P_{13}) , yields the full curvature K of the problem. For our purposes, it suffices to compute only selected entries of the Cartan curvature, which we do using the structure equations.

In this section, for brevity, we consider only the almost para-CR case, corresponding to $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$. The CR case can be treated similarly. The Cartan connection and curvature have the following form:

$$\psi = \begin{pmatrix} \phi_2 - \rho & \xi_2 & \xi_1 & \xi_0 \\ \omega^2 & \phi_1 - \rho & \xi_5 & \xi_4 \\ \omega^1 & \omega^5 & \phi_0 - \rho & \xi_3 \\ \omega^0 & \omega^4 & \omega^3 & \rho \end{pmatrix}, \quad \Psi = d\psi + \psi \wedge \psi = \begin{pmatrix} \Phi_2 - R & \Xi_2 & \Xi_1 & \Xi_0 \\ \Omega^2 & \Phi_1 - R & \Xi_5 & \Xi_4 \\ \Omega^1 & \Omega^5 & \Phi_0 - R & \Xi_3 \\ 0 & \Omega^4 & \Omega^3 & R \end{pmatrix}, \tag{12}$$

where $\rho = \frac{1}{2}(\phi_0 + \phi_1 + \phi_2)$ and $R = \frac{1}{2}(\Phi_0 + \Phi_1 + \Phi_2)$.

Imposing the normality conditions and Bianchi identities, modulo $\{\omega^0\}$, one has

$$\begin{aligned} \Omega^1 &\equiv T_-^1 \omega^3 \wedge \omega^4, & \Omega^2 &\equiv T_-^2 \omega^3 \wedge \omega^4, & \Omega^3 &\equiv T_+^1 \omega^1 \wedge \omega^2, & \Omega^4 &\equiv T_+^2 \omega^1 \wedge \omega^2, \\ \Omega^5 &\equiv -\frac{1}{6}(T_-^1 T_+^1 + T_-^2 T_+^2 - W_2) \omega^1 \wedge \omega^4 + \frac{1}{4}(T_+^2 T_-^1 - W_3) \omega^1 \wedge \omega^3 + \frac{1}{4}(T_+^2 T_-^1 + W_3) \omega^2 \wedge \omega^4 \\ &\quad - W_4 \omega^2 \wedge \omega^3 + x_6 \omega^1 \wedge \omega^2 + x_4 \omega^3 \wedge \omega^4, \\ \Xi^5 &\equiv -\frac{1}{6}(T_-^1 T_+^1 + T_-^2 T_+^2 - W_2) \omega^2 \wedge \omega^3 + \frac{1}{4}(T_+^1 T_-^2 - W_1) \omega^2 \wedge \omega^4 + \frac{1}{4}(T_+^1 T_-^2 + W_1) \omega^1 \wedge \omega^3 \\ &\quad - W_0 \omega^1 \wedge \omega^4 + x_2 \omega^3 \wedge \omega^4 + x_8 \omega^1 \wedge \omega^2, \end{aligned}$$

$$\begin{aligned}
 \Phi_0 &\equiv \frac{1}{24} (T_-^1 T_+^1 - 5T_-^2 T_+^2 - 4W_2) \omega^1 \wedge \omega^3 - \frac{1}{24} (T_-^2 T_+^2 - 5T_-^1 T_+^1 - 4W_2) \omega^2 \wedge \omega^4 \\
 &\quad + \frac{1}{4} W_1 \omega^1 \wedge \omega^4 - \frac{1}{4} W_3 \omega^2 \wedge \omega^3 - x_{13} \omega^1 \wedge \omega^2 + x_{16} \omega^3 \wedge \omega^4, \\
 \Phi_1 &\equiv -\Phi_0 - (x_{13} + x_{15}) \omega^1 \wedge \omega^2 + (x_{14} + x_{16}) \omega^3 \wedge \omega^4, \\
 \Phi_2 &\equiv \frac{1}{12} (T_-^1 T_+^1 - 5T_-^2 T_+^2) \omega^1 \wedge \omega^3 + \frac{1}{12} (T_-^2 T_+^2 - 5T_-^1 T_+^1) \omega^2 \wedge \omega^4 + \frac{1}{2} T_+^1 T_-^2 \omega^1 \wedge \omega^4 \\
 &\quad + \frac{1}{2} T_-^1 T_+^2 \omega^2 \wedge \omega^3 - (x_{13} + x_{15}) \omega^1 \wedge \omega^2 - (x_{14} + x_{16}) \omega^3 \wedge \omega^4,
 \end{aligned} \tag{13}$$

for some functions T_{\pm}^a, W_i, x_j on \mathcal{G} . We omit long expressions for other curvature functions as they are not relevant for this work.

The harmonic invariants from Proposition 2.7 can be represented as two torsion components (here and below we omit pullback s^* via a section $s : M \rightarrow \mathcal{G}$ for forms on the structure bundle)

$$\tau_+ = (T_+^1 \partial_{\omega^3} + T_+^2 \partial_{\omega^4}) \otimes (\omega^1 \wedge \omega^2), \quad \tau_- = (T_-^1 \partial_{\omega^1} + T_-^2 \partial_{\omega^2}) \otimes (\omega^3 \wedge \omega^4). \tag{14}$$

and the curvature component (this and similar invariants will be treated as tensors on M)

$$\begin{aligned}
 W &= (W_0(\omega^1)^4 + W_1(\omega^1)^3 \omega^2 + W_2(\omega^1)^2 (\omega^2)^2 + W_3 \omega^1 (\omega^2)^3 + W_4 (\omega^2)^4) \otimes U \otimes (V_-)^{-2} \\
 &\quad + (W_0(\omega^4)^4 - W_1(\omega^4)^3 \omega^3 + W_2(\omega^4)^2 (\omega^3)^2 - W_3 \omega^4 (\omega^3)^3 + W_4 (\omega^4)^4) \otimes U \otimes (V_+)^{-2}
 \end{aligned} \tag{15}$$

where $U = \omega^0, V_- = \omega^1 \wedge \omega^2, V_+ = \omega^3 \wedge \omega^4$. As a result,

$$W \in \Gamma((S^4 L_-^*) \otimes (\Lambda^2 L_-^*)^{-2} + (S^4 L_+^*) \otimes (\Lambda^2 L_+^*)^{-2}) \otimes \Delta^\perp$$

and $\tau_{\pm} \in \Gamma(\Lambda^2 L_{\pm}^* \otimes L_{\pm})$.

Note that for $v = a(\partial_{\omega^1} + \lambda \partial_{\omega^2}) + b(\lambda \partial_{\omega^3} - \partial_{\omega^4}) \in \Pi$, we get $W(v, v, v, v) = a^4 W_i \lambda^i + b^4 W_i (-\lambda)^i$ and each summand, in turn, can represent the harmonic curvature.

The correspondence space \hat{M} , as a parabolic geometry $(\mu : \mathcal{G} \rightarrow \hat{M}, \psi)$ of type (A_3, P_{123}) , is the leaf space of the Pfaffian system $\{\omega^0, \dots, \omega^5\}$. The structure equations imply that the zero-curvature condition $W = 0$ is equivalent to the Frobenius integrability of the Pfaffian system $\{\omega^0, \omega^1, \omega^4, \omega^5\}$ on \hat{M} . Denote its 4-dimensional leaf space by \mathcal{T} . Thus, the zero-curvature condition gives a fibration $\hat{M} \rightarrow \mathcal{T}$ with 2-dimensional fibers.

Equivalently, as was discussed in Section 2.3, \hat{M} has a (3,5,6) distribution, where the rank 3 distribution has a splitting into a line field and an integrable corank 1 subdistribution. In [35], such (3,5,6) distributions are referred to as *causal structures* on \mathcal{T} wherein the fibers of $\hat{M} \rightarrow \mathcal{T}$ at each point $x \in \mathcal{T}$ can be locally realized as the projectivization of a cone of codimension 1 in $T_x \mathcal{T}$ whose Gauss map has maximal rank.

In terms of the Cartan geometry $(\mu : \mathcal{G} \rightarrow \hat{M}, \psi)$, the rank 3 distribution is given by $\hat{\Pi} \oplus \hat{\ell} = \mu_* \langle \omega^0, \omega^1, \omega^4 \rangle^\perp$, where $\hat{\Pi} := \mu_* \langle \omega^0, \omega^1, \omega^4, \omega^5 \rangle^\perp \subset T\hat{M}$ is integrable and equipped with an indefinite bilinear form. In terms of the structure equations (13), the conformal class of the bilinear form $\omega^2 \circ \omega^3 \in S^2 \hat{\Pi}^*$ is well defined. Moreover, the line field $\hat{\ell} = \mu_* \langle \omega^0, \omega^1, \omega^2, \omega^3, \omega^4 \rangle^\perp$ is the characteristic direction of ω^0 on \hat{M} , that is, $\hat{\ell} = \langle v \rangle \subset T\hat{M}$, where $\omega^0(v) = 0$ and $d\omega^0(v, \cdot) = 0$. The integral curves of $\hat{\ell}$ foliate \hat{M} and can be thought of as a generalization of the null geodesic spray in conformal pseudo-Riemannian structures to causal structures. On \hat{M} the conformal class $[s^* h]$, where $h = \omega^0 \omega^5 - \omega^1 \omega^4$ and $s : \hat{M} \rightarrow \mathcal{G}$ is a section, is well defined. If the fibers of the causal structure $\hat{M} \rightarrow \mathcal{T}$ are the projective quadric, that is, $\tau_{\pm} = 0$, then $[s^* h]$ defines an indefinite conformal structure on \mathcal{T} whose projectivized null cone bundle coincides with \hat{M} .

The almost CR case is similar, except that instead of having two components τ_{\pm} in (14), the torsion has only one irreducible component, so we omit the respective arguments.

Lastly, we point out that subconformal structures on contact 5-manifolds that we consider in this article are also referred to as 5-dimensional Lie contact structures of signature (1,1) for $\delta > 0$ and (2,0) for $\delta < 0$.

3 | PROOF OF THE MAIN RESULTS

Let $\mathcal{E} : F = 0$ be a PDE system in terms of u . Following [7], we introduce dLp's as follows.

Definition 3.1. A dispersionless pair is a bundle $\pi : \hat{M}_u \rightarrow M_u$, called the correspondence space, whose fibers are connected curves, together with a rank 2 distribution $\hat{\Pi} \subset T\hat{M}_u$ such that:

- for all $\hat{x} \in \hat{M}_u$, $\hat{\Pi}_{\hat{x}} \subset T_{\hat{x}}\hat{M}_u$ depends on a finite jet of u at $x = \pi(\hat{x}) \in M_u$;
- $\hat{\Pi}$ is transverse to the fibers of π , that is, $\hat{\Pi} \cap \text{Ker } \pi_* = 0$.

A spectral parameter is a local fiber coordinate $\lambda = \lambda(\hat{x}) : \hat{M}_u \rightarrow \mathbb{R}$ for $\hat{M}_u \rightarrow M_u$.

Definition 3.2. Two dispersionless pairs $\hat{\Pi}, \hat{\Pi}' \subset T\hat{M}_u$ are \mathcal{E} -equivalent if $\hat{\Pi} = \hat{\Pi}'$ whenever $F(u) = 0$. $\hat{\Pi}$ is a dispersionless Lax pair (dLp) for \mathcal{E} if for any $\hat{\Pi}'$ \mathcal{E} -equivalent to $\hat{\Pi}$, the integrability condition $[\hat{\Pi}', \hat{\Pi}'] = \hat{\Pi}'$ is a nontrivial differential corollary of \mathcal{E} .

To be precise with the notion of a differential corollary and to encompass systems of PDEs, we introduce some jet formalism, for which we refer to [28, 30] for further details.

3.1 | Jets, symbols, and characteristics

Consider a (vector) bundle $\nu : \mathcal{V} \rightarrow M$ of rank m with local fiber coordinates $u = (u^j)$, so that sections have coordinate expression $u = u(x)$, $x = (x^i)$ ($1 \leq i \leq d$, $1 \leq j \leq m$). A k -jet of u is an equivalence class of sections by tangency of order $> k$ relation, and in coordinates, it can be written as $j^k u = (x, u, \partial u, \dots, \partial^k u)$, where $\partial^l u = (\partial_{\sigma} u^j)$ with the multi-index $\sigma = (i_1, \dots, i_d)$ of length $|\sigma| = \sum_1^d i_s = l \leq k$.

This yields the space of k -jets $J^k \nu$ and its projective (inverse) limit $J^{\infty} \nu$. There are natural projections $\nu_k : J^k \nu \rightarrow M$, $k = 0, 1, \dots, \infty$, and also $\nu_{k,l} : J^k \nu \rightarrow J^l \nu$ for $k > l$. The fibers of $\nu_{k,k-1}$ for $k \geq 2$ have a natural affine structure associated with fibers of $S^k T^* M \otimes \mathcal{V}$. Any section $u : M \rightarrow \mathcal{V}$ canonically lifts to the jet section $j^k u$ of $J^k \nu$.

By $f \in C^{\infty}(J^{\infty} \nu)$, we mean a function f on $J^k \nu$ for some finite k , which corresponds to a (non-linear) differential operator of order k . A collection of such functions $F = (F_1, \dots, F_m)$ can be seen as a vector-valued differential operator $F : J^k \nu \rightarrow \mathcal{W}$, where the latter is another (vector) bundle over M .

The bundle $J^{\infty} \nu$ has a canonical flat connection, the Cartan distribution, for which the horizontal lift of a vector field X on M is the total derivative D_X characterized by $(D_X f) \circ j^{\infty} u = X(f \circ j^{\infty} u)$ for any smooth function f on $J^{\infty} \nu$. More generally, any section X of $\nu_{\infty}^* TM$ has a lift to a vector field D_X on $J^{\infty} \nu$, given in local coordinates by $D_X = \sum_i a_i D_i$, where $X = \sum_i a_i \partial_i$, in which $\partial_i = \partial_{x^i}$, and $D_i = \partial_i + \sum_{\alpha} u_{i\alpha}^j \partial_{u_{\alpha}^j}$.

Higher order operators \square in total derivatives, also known as \mathcal{E} -differential operators, are generated as compositions of the derivations D_X with coefficients being smooth functions on $J^\infty \nu$. In local coordinates, $\square = \sum a_\alpha D_\alpha$, where $a_\alpha \in C^\infty(J^\infty \nu)$ and $D_\alpha = D_{i_1} \cdots D_{i_j}$ for a multi-index $\alpha = (i_1, \dots, i_j)$ with entries in $\{1, 2, \dots, d\}$.

A PDE of order k is defined as an equation of the form

$$F(j^k u) = 0, \tag{16}$$

where $F \in C^\infty(J^k \nu, \mathcal{W})$ is a vector function.

Let \mathcal{I}_F be the ideal in $C^\infty(J^\infty \nu)$ generated by the pullback of $F \in C^\infty(J^k \nu)$ and its total derivatives of arbitrary order. Then, the zero set $\mathcal{E}_\infty \subset J^\infty \nu$ of \mathcal{I}_F is the space of formal solutions of (16): u is a solution of (16) if and only if $j^\infty u$ is a section of \mathcal{E}_∞ . This embeds M_u to \mathcal{E}_∞ .

In this formalism, a differential corollary of $\mathcal{E} : F = 0$ is a differential ideal $\mathcal{I} \subset \mathcal{I}_F$; it is non-trivial, provided that it is not a subset of $\mathcal{I}_{F'}$ for any F' with the zero locus being a proper (closed) subset of that for F . Thus, in Definition 3.2, the integrability condition for a dLp $\hat{\Pi}$ for $\mathcal{E} : F = 0$ need not generate \mathcal{I}_F : indeed, the freedom to replace a dLp by an \mathcal{E} -equivalent one may change the ideal $\mathcal{I} \subset \mathcal{I}_F$ that its integrability conditions generate.

For a function $F \in C^\infty(J^k \nu)$, the vertical part of the 1-form $dF \in \Omega^1(J^\infty \nu)$ may be viewed in coordinates as a (vector-valued) polynomial on $\nu_\infty^* T^* M$ given by

$$\sum_{j=0}^k F_{(j)} \quad \text{where} \quad F_{(j)} = \sum_{|\alpha|=j} (\partial_{u_\alpha} F) \partial_\alpha \quad \text{is a section of} \quad \nu_\infty^* S^j T M \otimes \mathcal{V}^*.$$

The top degree term $\sigma_F = F_{(k)}$, called the (order k) symbol of F , is independent of coordinates. We assume it is nonvanishing: if it vanishes, F has order $\leq k - 1$ and σ_F has lower degree.

Similarly, for a \mathcal{W} -valued vector-function F on $J^k \nu$, the symbol σ_F is a homogeneous degree k polynomial on $\pi_\infty^* T^* M$ with values in $\text{Hom}(\mathcal{V}, \mathcal{W})$. For a PDE system $\mathcal{E} : F = 0$ of order k it is not identically zero, and the characteristic variety is defined by [43]

$$\text{Char}(\mathcal{E}, u) = \{[\theta] \in \mathbb{P}(\pi_\infty^* T^* M_u) \mid \sigma_F(\theta) \text{ is not injective}\}.$$

If \mathcal{V} and \mathcal{W} have the same rank m , then σ_F is represented by a $m \times m$ matrix $[\sigma_{ij}]$ of polynomials of order k , and the (projective) covector $[\theta]$ is characteristic if and only if $\sigma_F(\theta)$ is not bijective, whence

$$\text{Char}(\mathcal{E}, u) = \{[\theta] \in \mathbb{P}(\pi_\infty^* T^* M_u) \mid \det[\sigma_{ij}(\theta)] = 0\}.$$

Thus, a determined system is defined by the condition $\text{codim Char}(\mathcal{E}) = 1$.

3.2 | Normality condition

In this section, we prove Theorem 1.2.

In order for $\hat{\Pi}$ to be a dLp for an equation $\mathcal{E} : F = 0$, we require that the integrability condition $[\hat{\Pi}, \hat{\Pi}] = \hat{\Pi}$ holds modulo \mathcal{E} , that is, when $F = 0$, or, to use physics terminology, *on shell*.

Definition 3.3. We say that the dispersionless pair $\hat{\Pi} \subset T\hat{M}_u$ is normal if $[\hat{\Pi}, \hat{\Pi}] \subset \pi_*^{-1} \Pi$ off shell, that is, without assuming $F = 0$. In other words, $\pi_*[\hat{\Pi}, \hat{\Pi}] = \Pi$.

Consider local Darboux coordinates (x, y, p, q, r) on M_u so that for $\omega^0 = dr - p dx - q dy$, we get $\Delta = \text{Ker}(\omega^0) = \langle v_1, v_2, w_1, w_2 \rangle$ with $v_1 = \partial_x + p\partial_r$, $v_2 = \partial_y + q\partial_r$, $w_1 = \partial_p$, $w_2 = \partial_q$. Let λ be a local coordinate on the fiber of $\pi : \hat{M}_u \rightarrow M_u$. Then, because a dLp is characteristic [7], we get that $\Pi \subset \Delta$ can be chosen *isotropic* off-shell, and so, in an open region of \hat{M}_u generators of the distribution $\hat{\Pi}$, as well as Darboux coordinates, can be chosen so that

$$\hat{X} = v_1 + a w_1 + b w_2 + m \partial_\lambda, \quad \hat{Y} = v_2 + b w_1 + c w_2 + n \partial_\lambda, \tag{17}$$

for some coefficients a, b, c, m, n depending on coordinates of $J^\infty v$ and λ .

Then, $\hat{\Pi} = \langle \hat{X}, \hat{Y} \rangle$ is normal if and only if $[\hat{X}, \hat{Y}]$ is a multiple of ∂_λ . In this case, the integrability condition reduces to the vanishing of the ∂_λ -component $\hat{X}(n) - \hat{Y}(m)$ of the vector field $[\hat{X}, \hat{Y}]$ with no λ -dependency. The genericity condition we need here is as follows.

Definition 3.4. An isotropic 2-plane congruence $\Pi = \text{Ker}\{\omega^0, \zeta, \theta\} \subset \Delta$ is called *nondegenerate* if

$$\theta \wedge \zeta \wedge \theta_\lambda \wedge \zeta_\lambda \wedge \omega^0 \neq 0. \tag{18}$$

This condition depends only on Π , and not on a choice of generators ω^0, ζ, θ of $\text{Ann}(\Pi)$. Indeed, the nondegeneracy can be expressed, in terms of the generators $X = v_1 + a w_1 + b w_2$, $Y = v_2 + b w_1 + c w_2$ of Π , as

$$(d\omega^0 \wedge d\omega^0)(X, Y, X_\lambda, Y_\lambda) \neq 0,$$

or in terms of their coefficients as follows:

$$a_\lambda c_\lambda - b_\lambda^2 \neq 0, \tag{19}$$

where $a_\lambda = \partial_\lambda a$.

Lemma 3.5. Any nondegenerate isotropic 2-plane congruence Π has a unique normal lift.

Proof. If \hat{X} and \hat{Y} are given by (17), then equalities $dx([\hat{X}, \hat{Y}]) = dy([\hat{X}, \hat{Y}]) = dr([\hat{X}, \hat{Y}]) = 0$ hold identically, while $dp([\hat{X}, \hat{Y}]) = dq([\hat{X}, \hat{Y}]) = 0$ form two linear equations on m, n :

$$\begin{bmatrix} a_\lambda & b_\lambda \\ b_\lambda & c_\lambda \end{bmatrix} \begin{bmatrix} n \\ -m \end{bmatrix} = \begin{bmatrix} (X(b) - Y(a)) \\ (X(c) - Y(b)) \end{bmatrix};$$

these have a unique solution by the nondegeneracy condition (19). □

Proposition 3.6. Let $\hat{\Pi}$ be a dLp such that $\Pi = \pi_*(\hat{\Pi})$ is nondegenerate. Then, $\hat{\Pi}$ is \mathcal{E} -equivalent to a normal dLp. Such a dLp is unique.

Proof. The on shell Lax pair condition implies

$$dp \circ \pi_*[\hat{X}, \hat{Y}] = \square_1 F, \quad dq \circ \pi_*[\hat{X}, \hat{Y}] = \square_2 F$$

for some operators \square_1, \square_2 in total derivatives. Let us modify $\tilde{X} = \hat{X} + A(F)\partial_\lambda, \tilde{Y} = \hat{Y} + B(F)\partial_\lambda$, where A, B are operators in total derivatives. The new commutation equation modulo ∂_λ is

$$\begin{aligned} dp \circ \pi_* [\tilde{X}, \tilde{Y}] &= (\square_1 - a_\lambda B + b_\lambda A)F, \\ dq \circ \pi_* [\tilde{X}, \tilde{Y}] &= (\square_2 - b_\lambda B + c_\lambda A)F. \end{aligned}$$

Vanishing of these, equivalent to normality, can be achieved by a unique choice of the operators in total derivatives A, B due to nondegeneracy condition (19). □

This finishes the proof of Theorem 1.2.

3.3 | Zero-curvature condition

As was noted above, due to the characteristic condition [7], the 2-plane congruence is isotropic both with respect to conformal symplectic and subconformal structures. In other words, for every $x \in M, \Pi \in \text{Gr}(2, T_x M)$ as a function of λ is a section of $\mathcal{A}_x \cup \mathcal{B}_x$. However, as we saw in Section 2.1, isotropic planes in \mathcal{B}_x are discrete (two points for $\delta > 0$ and empty for $\delta < 0$). Thus, if we postulate essential dependence on λ , that is, the spectral parameter is *nonremovable*, the congruence has to take values in the bundle of α -planes.

Definition 3.7. A dLp $\hat{\Pi}$ is called *immersed* if the underlying 2-plane congruence consists of α -planes and the map $\lambda \mapsto \Pi$ is an immersion to $\mathbb{P}^1 = \mathcal{A}_x \subset \text{Gr}(2, T_x M)$ for every $x \in M$.

Lemma 3.8. *A dLp $\hat{\Pi}$ is immersed if and only if it is nondegenerate.*

Proof. Choosing a frame so that the subconformal structure has form (3) and the conformal symplectic is (6) or (7), the 2-plane congruence of α -planes takes form (5). Then, we can choose the symplectic basis $v_1 = \partial_{\omega^1}, v_2 = \partial_{\omega^4}, w_1 = -\partial_{\omega^3}, w_2 = \partial_{\omega^2}$. With respect to these choices, one obtains $a = c = 0$ and $b = \lambda$, so that condition (19) holds. In other words, in the immersed case, we can take the coordinate λ on \mathbb{P}^1 to be a spectral parameter, which implies nondegeneracy, and the converse follows from the same computation if we assume, for instance, $b_\lambda \neq 0$. □

Proof of Theorem 1.3. Let us first note that if F has order k , then the subconformal structure (Δ, c_F) has order $\leq k$ in u , and therefore, is well defined and, moreover, is nondegenerate for almost any u , which may not necessarily be a solution. Note that the order is $< k$, for example, if F is quasi-linear. By Lemma 3.5 and Proposition 3.6, the normal lift to the correspondence space is a first-order operator, and hence, the standard dLp has order $\leq k + 1$ in u .

Suppose next that $\hat{\Pi} \subset T\hat{M}_u$ is a dLp for \mathcal{E} . Then, $\Pi = \pi_*(\hat{\Pi})$ is characteristic, and hence, it is a nondegenerate congruence of α -planes for generic u . Then, by Lemma 3.8, Π immerses into $\text{Gr}(2, TM_u)$, and hence, by Theorem 1.2, $\hat{\Pi}$ is \mathcal{E} -equivalent to a standard dLp over any open subset of M_u . By the results of Section 2.2, the curvature is zero on M_u for every solution u of \mathcal{E} . Hence, the zero-curvature condition is a nontrivial differential corollary of \mathcal{E} , as required.

Conversely, suppose that the condition $W = 0$ is a nontrivial differential corollary of \mathcal{E} , and let $\hat{\pi} : \hat{M}_u \rightarrow M_u$ be the bundle of α -planes. Then, if $\hat{\Pi}$ is \mathcal{E} -equivalent to a standard dLp on an open subset of M_u , the integrability of $\hat{\Pi}$ is a differential corollary of \mathcal{E} on that open subset, since this is

true for the standard dLp. If any such $\hat{\Pi}$ is a differential corollary of a proper subsystem \mathcal{E}' of \mathcal{E} , then the first part of the argument implies that the zero-curvature condition is also a consequence of \mathcal{E}' , contradicting nontriviality. \square

3.4 | Master equation

We restrict to the almost para-CR case $\delta > 0$ and work in the framework of Section 2.1. The spectral parameter λ is defined up to projective transformation, depending on the base point $x \in M_u$. By integrability, there are infinitely many α -surfaces, that is, 2-dimensional submanifolds of M_u whose tangent plane at each point coincides with $\Pi = \Pi_\lambda$ for some λ . They can be thought of as projected integral surfaces of $\hat{\Pi}$, and hence, there are 4-parameter family of them. Thus, using the freedom of projective reparametrization of \mathbb{P}^1 for α -planes, we can arrange a null and Legendrian foliation corresponding to the value $\lambda = \infty$. In other words, we can assume the Legendrian $c_{\mathcal{E}}$ -isotropic distribution Π_∞ to be integrable.

Now we straighten this distribution Π_∞ , that is, choose Darboux coordinates (x, y, p, q, r) such that a contact form $\omega^0 \in \text{Ann}(\Delta)$ can be expressed as $\omega^0 = dr - p dx - q dy$ and that $\Pi_\infty = \langle \partial_p, \partial_q \rangle$. Note that straightening a Legendrian foliation is possible by a canonical transformation. Then,

$$g = a_{11} dx dp + a_{12} dx dq + a_{21} dy dp + a_{22} dy dq + b_{11} dx^2 + 2 b_{12} dx dy + b_{22} dy^2.$$

Computing the operator J , the compatibility conditions between g and Ω , which is equivalent to $J^2 = c \cdot \mathbf{1}$ for a positive constant c on Δ , are the following:

$$a_{12}(a_{11} + a_{22}) = 0, \quad a_{21}(a_{11} + a_{22}) = 0, \quad a_{11}^2 = a_{22}^2, \quad a_{21}b_{11} - a_{12}b_{22} = (a_{11} - a_{22})b_{12}. \quad (20)$$

together with the normalization $a_{12}a_{21} + a_{22}^2 = 4$ coming from the constraint $\det J = 1$ required to compute $L_\pm = \text{Ker}(J \mp \mathbf{1})$. Equations (20) branch as follows:

- $a_{11} = a_{22}, a_{12} = a_{21} = 0;$
- $a_{11} = -a_{22}, a_{12}b_{22} - a_{21}b_{11} = 2 a_{22}b_{12}.$

The first branch has less parameters and can be transformed to a particular case of the second branch. Henceforth, we proceed with the latter. Using the conformal freedom for the metric, we impose a different conformal normalization $a_{12} = 1$ to simplify computations. We get

$$g = b_{11} dx^2 + 2 b_{12} dx dy + (a_{21}b_{11} + 2 a_{22}b_{12}) dy^2 + dx dq + a_{21} dy dp + a_{22} (dy dq - dx dp).$$

Introducing a change of dependent variables $u = a_{22}, v = \sqrt{a_{22}^2 + a_{21}}, w = b_{11}, z = 2b_{12}$, and using the coframe below, in addition to the contact form ω^0 ,

$$\begin{aligned} \omega^1 &= dx + (u + v) dy, & \omega^2 &= dq - (u + v) dp + w dx + (z - w(u + v)) dy, \\ \omega^3 &= dq - (u - v) dp + w dx + (z - w(u - v)) dy, & \omega^4 &= dx + (u - v) dy, \end{aligned}$$

we get the following form for a conformal representatives on Δ :

$$2g = \omega^1 \omega^3 + \omega^2 \omega^4, \quad 2v \Omega = \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4.$$

Then, with the notations $\tilde{\delta}_x = \partial_x + p\partial_r$ and $\tilde{\delta}_y = \partial_y + q\partial_r$, we get

$$\begin{aligned} L_- &= \langle \partial_{\omega^1}, \partial_{\omega^2} \rangle = \langle (u-v)\tilde{\delta}_x - \tilde{\delta}_y + w\partial_p - (w(u-v) - z)\partial_q, \partial_p + (u-v)\partial_q \rangle, \\ L_+ &= \langle \partial_{\omega^4}, \partial_{\omega^3} \rangle = \langle (u+v)\tilde{\delta}_x - \tilde{\delta}_y + w\partial_p - (w(u+v) - z)\partial_q, \partial_p + (u+v)\partial_q \rangle. \end{aligned}$$

These define the para-CR structure on M_u , whose family of α -planes is $\Pi_\lambda = \langle \partial_{\omega^1} + \lambda\partial_{\omega^2}, \lambda\partial_{\omega^3} - \partial_{\omega^4} \rangle$. A linear combination with the change of the parameter $\lambda \mapsto -v\lambda$ yields new generators

$$X = \tilde{\delta}_x + \lambda\partial_p + (\lambda u - w)\partial_q, \quad Y = \tilde{\delta}_y + (\lambda u - w)\partial_p + (\lambda(u^2 - v^2) - z)\partial_q. \quad (21)$$

The corresponding (standard) dLp is

$$\hat{X} = X + m\partial_\lambda, \quad \hat{Y} = Y + n\partial_\lambda, \quad (22)$$

where, by the normality condition,

$$\begin{aligned} m &= \frac{1}{v^2} \left((uX(u) - 2vX(v) - Y(u))\lambda + uX(w) - X(z) + Y(w) \right), \\ n &= \frac{1}{v^2} \left(((u^2 + v^2)X(u) - 2uvX(v) - uY(u))\lambda + (u^2 - v^2)X(w) - uX(z) + uY(w) \right). \end{aligned} \quad (23)$$

Now the curvature, which is the obstruction to integrability, is given by

$$W = X(n) - Y(m) + mn_\lambda - nm_\lambda = W_0 + W_1\lambda + W_2\lambda^2 + W_3\lambda^3 + W_4\lambda^4.$$

In general, $W \in \mathcal{O}^4\mathbb{R}^2$ is a polynomial of degree 4 in λ , but due to straightening of Π_∞ , we get $W_4 = 0$, so integrability is given by 4 second-order PDEs

$$W_0 = W_1 = W_2 = W_3 = 0 \quad (24)$$

on 4 functions u, v, w, z of x, y, p, q, r . Using decompositions,

$$\begin{aligned} X &= X_0 + \lambda X_1, & X_0 &= \partial_x + p\partial_r - w\partial_q, & X_1 &= \partial_p + u\partial_q, \\ Y &= Y_0 + \lambda Y_1, & Y_0 &= \partial_y + q\partial_r - w\partial_p - z\partial_q, & Y_1 &= u\partial_p + (u^2 - v^2)\partial_q, \end{aligned}$$

and $m = m_0 + \lambda m_1 + \lambda^2 m_2$, $n = n_0 + \lambda n_1 + \lambda^2 n_2$, where

$$\begin{aligned} m_0 &= v^{-2}(uX_0(w) - X_0(z) + Y_0(w)), & m_2 &= v^{-2}(uX_1(u) - 2vX_1(v) - Y_1(u)), \\ m_1 &= v^{-2}(uX_0(u) - 2vX_0(v) - Y_0(u) + uX_1(w) - X_1(z) + Y_1(w)), \\ n_0 &= v^{-2}((u^2 - v^2)X_0(w) - uX_0(z) + uY_0(w)), \\ n_1 &= v^{-2}((u^2 + v^2)X_0(u) - 2uvX_0(v) - uY_0(u) + (u^2 - v^2)X_1(w) - uX_1(z) + uY_1(w)), \\ n_2 &= v^{-2}((u^2 + v^2)X_1(u) - 2uvX_1(v) - uY_1(u)), \end{aligned}$$

the components of (24) are given by

$$\begin{aligned}
 W_0 &= X_0(n_0) - Y_0(m_0) + m_0n_1 - n_0m_1, \\
 W_1 &= X_0(n_1) + X_1(n_0) - Y_0(m_1) - Y_1(m_0) + 2(m_0n_2 - n_0m_2), \\
 W_2 &= X_0(n_2) + X_1(n_1) - Y_0(m_2) - Y_1(m_1) + m_1n_2 - n_1m_2, \\
 W_3 &= X_1(n_2) - Y_1(m_2).
 \end{aligned}
 \tag{25}$$

Proof of Theorem 1.4. Master equation (25) is a system of 4 second-order PDEs on 4 functions. Its symbol given by the 4×4 matrix B , whose row number $k + 1$ ($0 \leq k \leq 3$), consists of symbols of W_k by variables u, v, w, z in turn. (The symbol of m th order differential operator F by u is $\sum \frac{\partial F}{\partial u_\alpha} \partial_\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_5)$ is a multi-index of length $|\alpha| = m$ and $\partial_\alpha = \partial_1^{\alpha_1} \dots \partial_5^{\alpha_5}$.)

Up to a nonzero factor, $\det(B) \propto Q^4$, where $Q = \sum b^{ij} \partial_i \partial_j$ is a quadric in the generators $\partial_1 + p\partial_3, \partial_2 + q\partial_3, \partial_4, \partial_5$ and hence has rank 4. The matrix g^{ij} of this bilinear form in the given basis is the inverse to the matrix g_{ij} of the subconformal metric g representing $c_\mathcal{E}$. Thus:

- this system of equation recovers the subconformal structure it describes;
- the symbol is nondegenerate, that is, $\det B \neq 0$ as a function on T^*M ;
- the characteristic variety is a (degenerate quadratic) hypersurface.

Hence, system (25) is determined, and, assuming analyticity, the generality of its local solutions can be read off from the Cauchy data: $4 \cdot 2 = 8$ functions of 4 variables.

To determine functional freedom of zero-curvature subconformal structures, let us compute the equivalence pseudogroup consisting of transformations that leave our normalizations invariant and act as a symmetry on the considered PDE system; a priori the quotient by this symmetry can reduce the naïve count of functional parameters.

Consider the contact vector field X_f on M_u , given by 1 generating function f of 5 arguments, and its action of the most general conformal metric g , parametrized by 4 functions u, v, w, z . The flow of X_f preserves the family of such metrics (shape preserving transformation, cf. [31, 32]) if and only if

$$L_{X_f}g = c g + g' \text{ mod } \omega^0.$$

Here, c is a scalar function and g' is the metric in the family with parameters u, v, w, z changed to parameters u', v', w', z' . Taking components and eliminating c, u', v', w', z' , one obtains five equations, of which only two are linearly independent:

$$f_{y_1, y_1} = (u^2 - v^2)f_{y_2, y_2}, \quad f_{y_1, y_2} = -uf_{y_2, y_2}.$$

Concentrating for a moment on dependence on y_1, y_2 only, the characteristic variety of this system $\{[\theta] = [\theta_1 : \theta_2] \in \mathbb{P}^1 : \theta_1^2 = (u^2 - v^2)\theta_2^2, \theta_1\theta_2 = -u\theta_2^2\}$ is empty for $v \neq 0$, hence it is of finite type. In other words, f depends on functions of 3 arguments only.

In fact, first prolongation of this system of PDEs is complete in third derivatives by y_1, y_2 ; hence, its solution depends on at most 4 functions of 3 arguments; and indeed, it depends exactly on 4 such functions: the compatibility condition for the above second-order system on f (so-called Mayer bracket [30]) is precisely the component W_3 of the curvature, which is one of PDEs in $\{W_i = 0\}_{i=0}^3$. Since local solutions of (25) depend on 8 functions of 4 arguments, the equivalence pseudogroup (shape-preserving transformations) cannot change this count. \square

3.5 | One more example

The following integrable PDE is a direct reduction of the 6D equation by Ferapontov-Khusnutdinova [20], rewritten in the second-order form by Sergyeyev [42]:

$$F = u_5 u_{13} - u_3 u_{15} + u_5 u_{24} - u_4 u_{25} = 0. \tag{26}$$

The symbol of F equals $\sigma_F = v_1 \cdot v_3 + v_2 \cdot v_4$ for $v_1 = \partial_1$, $v_2 = \partial_2$, $v_3 = u_5 \partial_3 - u_3 \partial_5$, $v_4 = u_5 \partial_4 - u_4 \partial_5$, and the nonholonomic distribution is

$$\Delta = \langle v_1, v_2, v_3, v_4 \rangle = \text{Ker}(\omega^0).$$

In coordinates $x = x^1, y = x^2, r = u, p = u_1, q = u_2$, the contact form is canonical

$$\omega^0 = dr - p dx - q dy,$$

while the subconformal structure (mod ω^0) is canonical in other coordinates

$$g = u_5 \cdot \sigma_F^{-1} = dx^1 \cdot dx^3 + dx^2 \cdot dx^4.$$

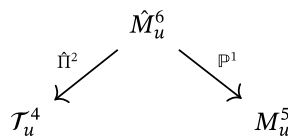
Passing from these to the Darboux coordinates above brings $c_{\mathcal{E}} = [g_F]$ to the form of Theorem 1.4, which realizes (26) as a reduction of the master equation (25); we skip the long explicit formulae.

The α -planes $\langle v_3 - \lambda v_2, v_4 + \lambda v_1 \rangle$ give rise to the dLp:

$$\hat{\Pi} = \langle \partial_3 - \frac{u_3}{u_5} \partial_5 - \lambda \partial_2, \partial_4 - \frac{u_4}{u_5} \partial_5 + \lambda \partial_1 \rangle.$$

4 | SYMMETRY REDUCTIONS AND TWISTOR CORRESPONDENCES

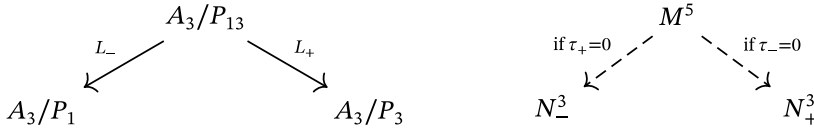
The integrability via dLp can be conveniently described by the double fibration



where the top is the correspondence space and the left-bottom is the twistor space, that is, the leaf space of the distribution $\hat{\Pi}^2$ on \hat{M} . Note that in the parabolic twistorial picture M_u^5 corresponds to a Cartan geometry of type (A_3, P_{13}) , and \hat{M}_u^6 has a Cartan geometry of type (A_3, P_{123}) . As discussed at the end of Section 2.4, the projection to \mathcal{T}_u^4 gives a Cartan geometry of type (A_3, P_2) only if the causal structure arises from a pseudoconformal structure, that is, the 2-dimensional fibers of $\hat{M}_u \rightarrow \mathcal{T}$ are quadratic, which is equivalent to the vanishing of the torsions τ_{\pm} . However, the vanishing of τ_{\pm} implies flatness of the subconformal structure on M , which, in turn, induces flat pseudoconformal structure on \mathcal{T}_u . As a result, in general, the causal structure on \mathcal{T}_u that is encoded by the fibration $\hat{M}_u \rightarrow \mathcal{T}$, defines a field of cones on \mathcal{T}_u that are not quadratic almost everywhere. In this section, we discuss other twistor approaches and symmetry reductions.

4.1 | Subconformal geometry in 5D from 3D projective structures

Parabolic twistor correspondence [8] is represented as double fibration below, where M^5 of type (A_3, P_{13}) with $\delta = +1$ is assumed to be of zero curvature: $W = 0$. The quotient to both sides exists only in the flat case: the quotient by L_{\pm} gives a projective geometry on a 3D manifold N_{\pm} only if $\tau_{\mp} = 0$ and the Weyl curvature of the projective structure on N_{\pm} is generated by the torsion component τ_{\pm} by Proposition 2.7 and the parabolic theory of correspondence spaces [8].



Conversely, given a projective structure in 3D, which can be taken to be either N_+ or N_- , as they are projectively dual to each other, we can lift it to a subconformal structure in 5D as follows. A choice of affine symmetric connection ∇ on N induces a connection on the bundle $\pi : T^*N \rightarrow N$ and hence a splitting $T(T^*N) = H \oplus V$ into a pair of Lagrangian 3-planes, where $V = \text{Ker}(\pi_*)$ and H is the lift of TN .

For $M = \mathbb{P}T^*N$, it also induces a connection $\bar{\nabla}$ on the bundle $\bar{\pi} : M \rightarrow N$, which depends only on the projective class $[\nabla]$ [47], whence a splitting $T_{\bar{x}}M = H'_{\bar{x}} \oplus \bar{V}_{\bar{x}}$ for any $\bar{x} \in M$, where $\bar{V}_{\bar{x}} = \text{Ker}(d_{\bar{x}}\bar{\pi})$ and $H'_{\bar{x}}$ is the lift of T_xN . Since $\bar{x} = (x, [p])$ for $p \in T_x^*N$, the latter space contains the subspace $\bar{H}_{\bar{x}}$ corresponding to $\text{Ann}(p) \subset T_xN$. Thus, we get the splitting of the contact structure: $\Delta_{\bar{x}} = d_{\bar{x}}\bar{\pi}^{-1}(p) = \bar{H}_{\bar{x}} \oplus \bar{V}_{\bar{x}}$ into a pair of Lagrangian 2-planes, giving the L_{\pm} planes of the required subconformal structure on M .

At the level of the PDE system arising from the zero-curvature condition, the additional condition for half of the torsion to vanish results in an overdetermined system of PDEs, whose general solution corresponds to generic 3-dimensional projective structure. Thus, we get:

Theorem 4.1. *Projective lift provides explicit solutions of the master equation (25). Moduli of the corresponding zero-curvature subconformal structures depend on 12 functions of 3 variables.*

Remark 4.2. This twistor reduction can be thought of as a 5-dimensional analog of the so-called Dunajski–West construction and its generalization due to Calderbank [6, 15] wherein self-dual 4-manifolds with a null conformal Killing vector are shown to have a foliation by null surfaces containing the conformal Killing field. Consequently, the 2-dimensional leaf space of such null surfaces is equipped with a projective structure.

In coordinates, the projective connection is given through a representative symmetric affine connection with Christoffel symbols Γ_{ij}^k depending on coordinates (x^0, x^1, x^2) of N . Projective transformations $\Gamma_{ij}^k \mapsto \Gamma_{ij}^k + \frac{1}{2}(Y_i\delta_j^k + Y_j\delta_i^k)$, depending on arbitrary 1-form Y_i , leave the Thomas symbols $\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{4}(\Gamma_{li}^l\delta_j^k + \Gamma_{lj}^l\delta_i^k)$ invariant. Note that $\Pi_{kj}^k = 0$.

Alternatively, using the lift of (A_3, P_1) structure to (A_3, P_{12}) structure, a projective connection can be considered as the second-order ODE system of the type (derivation by $t = x^0$)

$$\begin{aligned} \dot{x}^1 &= \Pi_{11}^0(x^1)^3 + 2\Pi_{12}^0(x^1)^2x^2 + \Pi_{22}^0x^1(x^2)^2 + (2\Pi_{01}^0 - \Pi_{11}^1)(x^1)^2 \\ &+ 2(\Pi_{02}^0 - \Pi_{12}^1)x^1x^2 - \Pi_{22}^1(x^2)^2 + (\Pi_{00}^0 - 2\Pi_{01}^1)x^1 - 2\Pi_{02}^1x^2 - \Pi_{00}^1, \end{aligned}$$

$$\begin{aligned} \ddot{x}^2 &= \Pi_{11}^0 (\dot{x}^1)^2 \dot{x}^2 + 2\Pi_{12}^0 \dot{x}^1 (\dot{x}^2)^2 + \Pi_{22}^0 (\dot{x}^2)^3 - \Pi_{11}^2 (\dot{x}^1)^2 + 2(\Pi_{01}^0 - \Pi_{12}^2) \dot{x}^1 \dot{x}^2 \\ &+ (2\Pi_{02}^0 - \Pi_{22}^2) (\dot{x}^2)^2 - 2\Pi_{01}^2 \dot{x}^1 + (\Pi_{00}^0 - 2\Pi_{02}^2) \dot{x}^2 - \Pi_{00}^2. \end{aligned}$$

Choosing local coordinates $[p] = [-1 : p_1 : p_2]$ in the fiber $\mathbb{P}T_x^*N$, we get local coordinates (t, x^1, x^2, p_1, p_2) on M , in which a contact form is expressed as $\omega^0 = dt - p_1 dx^1 - p_2 dx^2$ and the contact Legendrian structure L_{\pm} is given by 2-distributions

$$H = \langle \eta_1, \eta_2 \rangle, \quad V = \langle \partial_{p_1}, \partial_{p_2} \rangle, \tag{27}$$

where V is integrable and

$$\begin{aligned} \eta_i &= \partial_{x^i} + p_i \partial_t - \Pi_{00}^k p_i p_j p_k \partial_{p_j} - \sum_{k \neq i} \Pi_{0i}^k p_j p_k \partial_{p_j} - \sum_{k \neq j} \Pi_{0j}^k p_i p_k \partial_{p_j} + (\Pi_{00}^0 - \Pi_{0k}^k) p_i p_j \partial_{p_j} \\ &- \sum_{k \neq i, j} \Pi_{ij}^k p_k \partial_{p_j} + \sum_{j \neq i} (\Pi_{0j}^0 - \Pi_{ij}^i) p_i \partial_{p_j} + \sum_{j \neq i} (\Pi_{0i}^0 - \Pi_{ij}^j) p_j \partial_{p_j} + (2\Pi_{0i}^0 - \Pi_{ii}^i) p_i \partial_{p_i} + \Pi_{ij}^0 \partial_{p_j}, \end{aligned}$$

in which the summation is over repeated indices $1 \leq j, k \leq 2$, with no summation over $i = 1, 2$.

Denoting $\omega^1, \dots, \omega^4$ the coframe (mod ω^0) dual to the contact frame $\eta_1, \eta_2, \partial_{p_1}, \partial_{p_2}$, we obtain formulae (3) and (6) for the subconformal structure (M, Δ, g) .

Note that the coordinate freedom can constrain 3 out of 15 coefficients, for example, $\Pi_{00}^0 = \Pi_{01}^0 = \Pi_{02}^0 = 0$; the remaining 12 Thomas’s coefficients give functional parameter on the moduli space.

4.2 | Symmetry reductions and integrable systems in 3D and 4D

Let us assume that a subconformal contact structure $(M, \Delta, [g])$ has an infinitesimal symmetry $\zeta \in \Gamma(TM)$. Since Δ is contact, ζ is transverse to Δ almost everywhere, so localizing in M , we can assume the leaf space of the integral curves of ζ to be a 4-manifold Q with the fibration $q : M \rightarrow Q$.

For $x \in M$, we can identify $\Delta_x \simeq T_{q(x)}Q$, and thus, Q gets equipped with conformal metric $[\tilde{g}] = q_*[g]$ and conformal symplectic structure $[\tilde{\Omega}] = q_*[\Omega]$, as well as the endomorphism $\tilde{J} = \tilde{g}^{-1} \tilde{\Omega} = q_* J$ satisfying $\tilde{J}^2 = \delta \mathbf{1}$, where $\delta = \pm 1$.

A contact form on M can be fixed by the relation

$$\omega^0(\zeta) = c \text{ for } c \in \mathbb{R}_{\neq 0}. \tag{28}$$

This induces a homothety class of closed 2-forms $\tilde{\Omega}$ on Q by $q^* \tilde{\Omega} = d\omega^0$. As a result, the 4-manifold Q is conformally symplectic and has a homothety class of *almost δ -Kähler* of neutral signature, that is, almost pseudo-Kähler for $\delta < 0$ and almost para-Kähler for $\delta > 0$. The converse statement is true together with the integrability constraint:

Theorem 4.3. *There is a bijective local correspondence between zero-curvature subconformal contact structures in 5D with an infinitesimal symmetry and (homothety classes of) self-dual almost (pseudo-/para-)Kähler structures in 4D.*

The inverse construction is given by the so-called *contactification* and the correspondence between zero-curvature and self-duality condition follows from the existence of α -surfaces. Below we briefly describe the construction.

The homothety class of an almost para-Kähler structure on Q defines a Cartan geometry $(\mathcal{K} \rightarrow Q, \tilde{\psi})$ of type $(\mathbb{R}_\times GL(2, \mathbb{R}) \ltimes \mathbb{R}^4, \mathbb{R}_\times GL(2, \mathbb{R}))$, where $\mathbb{R}_\times GL(2, \mathbb{R}) \subset GL(4, \mathbb{R})$ was described in Corollary 2.4. The two cases $\delta < 0$ and $\delta > 0$ involve different representations but are similar, so we proceed with the latter. The Cartan connection takes the form

$$\tilde{\psi} = \begin{pmatrix} \tilde{\rho} - \tilde{\phi}_0 & \tilde{\xi}_5 & 0 & 0 \\ \tilde{\omega}^5 & \tilde{\rho} - \tilde{\phi}_1 & 0 & 0 \\ \tilde{\omega}^4 & \tilde{\omega}^3 & \tilde{\rho} - \tilde{\phi}_2 & 0 \\ \tilde{\omega}^1 & -\tilde{\omega}^2 & 0 & \tilde{\rho} + \tilde{\phi}_2 \end{pmatrix}, \quad \tilde{\rho} = \frac{1}{4}(\tilde{\phi}_0 + \tilde{\phi}_1). \tag{29}$$

Remark 4.4. Fixing $c = 1$ in (28) results in $\tilde{\rho} = 0$ and corresponds to a representative of the homothety class in Theorem 4.3. In the treatment of [10], the authors keep the homothety factor so that the structure group G_0 of the symmetry reduction remains the same as that of the corresponding parabolic geometry in order to exploit the parabolic theory of Weyl connections.

From now on, we reduce the scaling factor \mathbb{R}_\times in the structure group, as in the remark above, which implies $\tilde{\phi}_0 = -\tilde{\phi}_1$ in (29). The Cartan connection and curvature are given by:

$$\tilde{\psi} = \begin{pmatrix} \tilde{\phi}_1 & \tilde{\xi}_5 & 0 & 0 \\ \tilde{\omega}^5 & -\tilde{\phi}_1 & 0 & 0 \\ \tilde{\omega}^4 & \tilde{\omega}^3 & -\tilde{\phi}_2 & 0 \\ \tilde{\omega}^1 & -\tilde{\omega}^2 & 0 & \tilde{\phi}_2 \end{pmatrix}, \quad \tilde{\Psi} = \begin{pmatrix} \tilde{\Phi}_1 & \tilde{\Xi}_5 & 0 & 0 \\ \tilde{\Omega}^5 & -\tilde{\Phi}_1 & 0 & 0 \\ \tilde{\Omega}^4 & \tilde{\Omega}^3 & -\tilde{\Phi}_2 & 0 \\ \tilde{\Omega}^1 & -\tilde{\Omega}^2 & 0 & \tilde{\Phi}_2 \end{pmatrix}, \tag{30}$$

where

$$\begin{aligned} \tilde{\Omega}^1 &= t_-^1 \tilde{\omega}^3 \wedge \tilde{\omega}^4, & \tilde{\Omega}^2 &= t_-^2 \tilde{\omega}^3 \wedge \tilde{\omega}^4, & \tilde{\Omega}^3 &= t_+^1 \tilde{\omega}^1 \wedge \tilde{\omega}^2, & \tilde{\Omega}^4 &= t_+^2 \tilde{\omega}^1 \wedge \tilde{\omega}^2, \\ \tilde{\Omega}^5 &= -s_0(\tilde{\omega}^1 \wedge \tilde{\omega}^3 + \tilde{\omega}^2 \wedge \tilde{\omega}^4) - 2r\tilde{\omega}^1 \wedge \tilde{\omega}^4 - P_{22}\tilde{\omega}^1 \wedge \tilde{\omega}^2 - P_{33}\tilde{\omega}^3 \wedge \tilde{\omega}^4, \\ \tilde{\Phi}_1 &= -2r\tilde{\omega}^2 \wedge \tilde{\omega}^4 + (r + s_1)(\tilde{\omega}^1 \wedge \tilde{\omega}^3 + \tilde{\omega}^2 \wedge \tilde{\omega}^4) + P_{12}\tilde{\omega}^1 \wedge \tilde{\omega}^2 - P_{34}\tilde{\omega}^3 \wedge \tilde{\omega}^4, \\ \tilde{\Phi}_2 &= C_1\tilde{\omega}^1 \wedge \tilde{\omega}^2 - (3r + s_1 - t_-^2 t_+^2)\tilde{\omega}^1 \wedge \tilde{\omega}^3 + (t_-^2 t_+^1 + s_2)\tilde{\omega}^1 \wedge \tilde{\omega}^4 \\ &\quad + (t_-^1 t_+^2 + s_0)\tilde{\omega}^2 \wedge \tilde{\omega}^3 + (3r - s_1 - t_-^1 t_+^1)\tilde{\omega}^2 \wedge \tilde{\omega}^4 + C_3\tilde{\omega}^3 \wedge \tilde{\omega}^4, \\ \tilde{\Xi}_5 &= -s_2(\tilde{\omega}^1 \wedge \tilde{\omega}^3 + \tilde{\omega}^2 \wedge \tilde{\omega}^4) - 2r\tilde{\omega}^2 \wedge \tilde{\omega}^3 + P_{11}\tilde{\omega}^1 \wedge \tilde{\omega}^2 + r\tilde{\omega}^2 \wedge \tilde{\omega}^3 + P_{44}\tilde{\omega}^3 \wedge \tilde{\omega}^4 \end{aligned} \tag{31}$$

for some functions $t_-^1, t_-^2, t_+^1, t_+^2, s_0, s_1, s_2, r$ and C_1, C_3, P_{ij} on \mathcal{K} .

The fundamental invariants of such Cartan geometries are $\tilde{\tau}_\pm, S, R$ given by

$$\begin{aligned} \tilde{\tau}_+ &= (t_+^1 \partial_{\tilde{\omega}^3} + t_+^2 \partial_{\tilde{\omega}^4}) \otimes (\tilde{\omega}^1 \wedge \tilde{\omega}^2), & \tilde{\tau}_- &= (t_-^1 \partial_{\tilde{\omega}^1} + t_-^2 \partial_{\tilde{\omega}^2}) \otimes (\tilde{\omega}^3 \wedge \tilde{\omega}^4), \\ S &= (s_0(\tilde{\omega}^2)^2 + 2s_1\tilde{\omega}^2\tilde{\omega}^1 - s_2(\tilde{\omega}^1)^2) \otimes (\tilde{\omega}^1 \wedge \tilde{\omega}^2)^{1/2} \otimes (\tilde{\omega}^3 \wedge \tilde{\omega}^4)^{-1/2}, & R &= r\tilde{\Omega}. \end{aligned} \tag{32}$$

We also have a closed 2-form

$$\tilde{\Omega} = \tilde{\omega}^1 \wedge \tilde{\omega}^3 + \tilde{\omega}^2 \wedge \tilde{\omega}^4.$$

The conformal structure of the homothetic representative is

$$\tilde{g}_0 = \tilde{\omega}^1 \tilde{\omega}^3 + \tilde{\omega}^2 \tilde{\omega}^4 \tag{33}$$

for which the Schouten tensor in the coframe $(\tilde{\omega}^1, \dots, \tilde{\omega}^4)$ is given by the quantities P_{ij} in (31), where

$$s_0 = P_{23}, \quad s_1 = \frac{1}{2}(P_{13} - P_{24}), \quad s_2 = P_{14}, \quad r = \frac{1}{2}(P_{13} + P_{24}).$$

The remaining entries of the Schouten tensor are derived from the torsions $\tilde{\tau}_\pm$ as follows:

$$\begin{aligned} P_{11} &= -\partial_{\tilde{\omega}^4} t_+^1, & P_{22} &= \partial_{\tilde{\omega}^3} t_+^2, & P_{21} &= \frac{1}{2}(\partial_{\tilde{\omega}^3} t_+^1 - \partial_{\tilde{\omega}^4} t_+^2), & C_1 &= \frac{1}{2}(\partial_{\tilde{\omega}^3} t_+^1 + \partial_{\tilde{\omega}^4} t_+^2), \\ P_{33} &= -\partial_{\tilde{\omega}^2} t_-^1, & P_{44} &= \partial_{\tilde{\omega}^1} t_-^2, & P_{43} &= \frac{1}{2}(\partial_{\tilde{\omega}^1} t_-^1 - \partial_{\tilde{\omega}^2} t_-^2), & C_3 &= -\frac{1}{2}(\partial_{\tilde{\omega}^1} t_-^1 + \partial_{\tilde{\omega}^2} t_-^2). \end{aligned} \tag{34}$$

Note that the fundamental invariant S in (32) is a $GL(2, \mathbb{R})$ -component part of the Ricci curvature of the almost para-Kähler metric \tilde{g}_0 in (33), and the scalar curvature is $24r$. The anti-self-dual Weyl curvature of \tilde{g}_0 is zero; the self-dual part is represented by the quartic

$$(t_{-,3}^2 - t_{-,4}^1)\lambda^4 + 4C_3\lambda^3 + 6(t_-^1 t_+^1 + t_-^2 t_+^2 - 2r)\lambda^2 + 4C_1\lambda + t_{+,1}^2 - t_{+,2}^1 \tag{35}$$

on the bundle of α -planes (5) with parameter $\lambda \in \mathbb{R} \cup \{\infty\}$, where $t_{\varepsilon;l}^k = \partial_{\tilde{\omega}^l} t_\varepsilon^k$.

To contactify, take a local nonvanishing primitive 1-form $\tilde{\omega}^0$ of the closed representative $\tilde{\Omega} \in [\tilde{\Omega}]$, that is, $d\tilde{\omega}^0 = \tilde{\Omega}$. Defining $M = Q \times \mathbb{R}$ with projection $q : M \rightarrow Q$, the 1-form $\beta = dt + q^* \tilde{\omega}^0$ satisfies $d\beta = q^* \tilde{\Omega}$, whence $\Delta := \text{Ker } \beta \subset TM$ is a contact distribution. Moreover, by the construction, the splitting $TQ = \tilde{L}_- \oplus \tilde{L}_+$ induces a splitting on Δ with the appropriate compatibility conditions for the induced (para-)complex structure. This shows that M is equipped with a compatible subconformal contact structure with infinitesimal symmetry ∂_t .

To relate the Cartan connections $\tilde{\psi}$ and ψ , let us denote by $\hat{q} : \hat{\mathcal{K}} \rightarrow M$ the pull-back bundle $q^* \mathcal{K}$ of the principal bundle $\mathcal{K} \rightarrow Q$ using the projection $q : M \rightarrow Q$. Let ω^0 denote the lift of the primitive 1-form β to $\hat{\mathcal{K}}$ by the scaling action. The symmetry reduction gives an inclusion $\iota : \hat{\mathcal{K}} \rightarrow \mathcal{G}$, where $(\mathcal{G} \rightarrow M, \psi)$ is the Cartan geometry for the corresponding subconformal contact 5-manifold. Using the expressions (30) and (12), one has

$$\begin{aligned} \omega^a &= \tilde{\omega}^a \quad (0 \leq a \leq 4), & \omega^5 &= \tilde{\omega}^5 + \frac{1}{2}s_0\omega^0, & \phi_1 &= \tilde{\phi}_1 - \frac{1}{2}s_1\omega^0, \\ \phi_2 &= \tilde{\phi}_2 + (\frac{1}{6}t_-^1 t_+^1 + \frac{1}{6}t_-^2 t_+^2 - r)\omega^0, & \xi_5 &= \tilde{\xi}_5 + \frac{1}{2}s_2\omega^0. \end{aligned} \tag{36}$$

wherein we have suppressed t^* on the left-hand side and \hat{q}^* on the right-hand side.

Using the relations above, one can find the Schouten tensor of the reduced subconformal geometry. More precisely, defining the Schouten tensor as $\xi_i = Q_{ij}\omega^j$, one obtains that the symmetric part $Q_{(ij)}$, $1 \leq i, j \leq 4$, is given by

$$\begin{aligned} Q_{(11)} &= \frac{1}{2}P_{11}, & Q_{(12)} &= \frac{1}{2}P_{21}, & Q_{(22)} &= \frac{1}{2}P_{22}, & Q_{(33)} &= \frac{1}{2}P_{33}, & Q_{(34)} &= \frac{1}{2}P_{43}, & Q_{(44)} &= \frac{1}{2}P_{44} \\ Q_{(13)} &= r + \frac{1}{2}s_1 + \frac{1}{24}(t_-^1 t_+^1 - 5t_-^2 t_+^2), & Q_{(24)} &= r - \frac{1}{2}s_1 + \frac{1}{24}(t_-^2 t_+^2 - 5t_-^1 t_+^1) \\ Q_{(23)} &= \frac{1}{2}s_0 + \frac{1}{4}t_-^1 t_+^2, & Q_{(14)} &= \frac{1}{2}s_2 + \frac{1}{4}t_-^2 t_+^1, \end{aligned}$$

and the only nonzero entries of the skew-symmetric part $Q_{[ij]}$ for $1 \leq i, j \leq 4$ are

$$Q_{[12]} = \frac{1}{4}C_1, \quad Q_{[43]} = \frac{1}{4}C_3.$$

The expressions of the entries Q_{i0} and Q_{0i} involved first and second derivatives of the fundamental invariants of the almost para-Kähler structure. Since their expressions are long and will not be important for us, we will not provide them. Using relations (36), one can immediately relate invariants (14)–(32) by

$$\tau_{\pm} = q^* \tilde{\tau}_{\pm}.$$

An interpretation for the vanishing of S was given in [36], using the natural lift $\hat{\zeta}$ of the infinitesimal symmetry ζ to its correspondence space \hat{M}_u that would be its corresponding causal structure as discussed at the end of Section 2.4. It was shown that the infinitesimal symmetry $\hat{\zeta}$ is *null* with respect to the canonical symmetric bilinear form $h = \omega^0 \omega^5 - \omega^1 \omega^4$ of the causal structure in terms of (12) if and only if the induced structure on Q satisfies $S = 0$.

On the level of differential equations and their canonical subconformal structure on solutions, vanishing of the fundamental invariants is a constraint, not changing the integrability. For instance, if in addition to the symmetry on M_u , we impose vanishing of the Ricci curvature of \tilde{g}_0 , namely, $P_{ij} = 0$ for all $0 \leq i, j \leq 4$, then we get the Plebański equation (first or second depending on coordinization) describing self-dual gravity [17, 40]. On the other hand, fewer constraints yield reductions to other geometries as will be discussed in the next section.

Symmetry reduction of integrable PDEs in 5D clearly gives an integrable PDE in 4D, for instance, by the results of [7, 20]. Imposing more symmetries on M_u , we get lower-dimensional reductions, as in [5]. In particular, reduction by a 2-dimensional Abelian symmetry group whose intersection with the contact distribution is not null with respect to the indefinite metric on Δ yields integrable background geometry in 3D, which is Einstein–Weyl. Split as two successive 1-dimensional symmetry reductions, this yields the Jones–Tod correspondence [25]. Again, vanishing of extra invariants leads to further reductions, for example, almost para-Kähler 4-manifolds with $\tilde{\tau}_- = 0$ give symmetry reduction of 3-dimensional projective structures, which can be viewed as an analog of the Dunajski–West construction [15]. (The local generality of such almost para-Kähler structures is 12 functions of 2 variables.)

4.3 | Further reductions: Nested Lax sequences

In this section, let us restrict to the case $\delta > 0$. We consider a subconformal structure $(M, \Delta, [g])$ with infinitesimal symmetry ζ and its symmetry reduction, namely, a homothety class almost para-Kähler 4-manifold $(Q, [\tilde{g}], \tilde{J})$, with various overdeterminations of the zero-curvature condition.

We interpret such vanishing conditions as the extendability of the Lax pair to a Lax triple, Lax quadruple, and so on. Since a differential subsystem of an integrable system is an integrable system itself, we obtain overdetermined systems that are integrable via Lax distributions of higher rank. The candidates for these distributions are as follows.

Consider at first the reduced space Q and its correspondence space \hat{Q} , that is, the bundle of α -planes and the projection $\pi_Q : \hat{Q} \rightarrow Q$ with \mathbb{P}^1 -fibers. The α -planes $(\tilde{L}_-, \tilde{L}_+)$, where $\tilde{L}_{\pm} \subset \tilde{L}_{\pm}$, $\tilde{L}_- \perp \tilde{L}_+$, lift to the rank 2 distribution $\tilde{\Pi}^2 \subset T\hat{Q}$ defined using the Levi–Civita connection of \tilde{g} ; this lift is independent of the representative metric in the homothety/conformal class. Similarly, the 3-planes $\langle \tilde{L}_- \rangle \oplus \tilde{L}_+$ parametrized by $\lambda \in \mathbb{P}^1$ lift to the rank 3 distribution $\tilde{\Pi}^3 \subset T\hat{Q}$. Finally, the tangent bundle TQ lifts via the Levi–Civita connection to the rank 4 distribution $\tilde{\Pi}^4 \subset T\hat{Q}$.

Proposition 4.5. *Frobenius integrability of $\tilde{\Pi}^2$ is equivalent to the self-duality of \tilde{g} . Assuming self-duality, $\tilde{\Pi}^3$ is integrable if and only if we have $\tau_- = 0$ and $S = 0$. Assuming the integrability of $\tilde{\Pi}^2$ and $\tilde{\Pi}^3$, then $\tilde{\Pi}^4$ is integrable if and only if the Ricci curvature vanishes.*

Proof. In terms of Cartan connection (30) on the principal bundle $\pi_{\hat{Q}} : \mathcal{K} \rightarrow \hat{Q}$, we have $\tilde{\Pi}^2 = \pi_{\hat{Q}*} \langle \tilde{\omega}^1, \tilde{\omega}^4, \tilde{\omega}^5 \rangle^\perp$, $\tilde{\Pi}^3 = \pi_{\hat{Q}*} \langle \tilde{\omega}^1, \tilde{\omega}^5 \rangle^\perp$, and $\tilde{\Pi}^4 = \pi_{\hat{Q}*} \langle \tilde{\omega}^5 \rangle^\perp$. The first claim on the Frobenius condition for $\tilde{\Pi}^2$ is well known [39]. Assuming self-duality, the conditions for this Lax pair to be extendable to the Lax triple are straightforward: using (31) and (34), the Frobenius integrability of $\tilde{\Pi}^3$ is equivalent to $\tau_- = 0$ and $S = 0$. Similarly, assuming $\tau_- = S = 0$, the condition $\text{Ric}_{\tilde{g}} = 0$ is necessary and sufficient for the rank 4 distribution $\tilde{\Pi}^4$ to be a Lax quadruple. \square

Remark 4.6. Note that by relation (34), when $\tau_- = 0$, then the binary quartic (35) has a repeated root of multiplicity at least 2 at $\lambda = \infty$. The condition $\tau_- = 0$ and $\text{Ric}_{\tilde{g}} = 0$ implies that (35) has a repeated root of multiplicity at least 3.

Now consider the subconformal structure with the projection $q : M \rightarrow Q$ and the corresponding projection of the correspondence spaces $\hat{q} : \hat{M} \rightarrow \hat{Q}$. We define $\hat{\Pi}^{i+1} = \hat{q}_*^{-1} \tilde{\Pi}^i$ for $i = 2, 3, 4$ which together with the rank 2 distribution $\hat{\Pi}^2$ give a flag of subspaces of $T\hat{M}$. This can also be defined as a lift of subspaces of TM via Weyl connection. Indeed, the symmetry ζ determines a reduction of the Cartan structure algebra $\mathfrak{sl}(4, \mathbb{R})$ to the opposite parabolic $\mathfrak{p}_{13}^{\text{op}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathfrak{gl}(2, \mathbb{R})$, $\mathfrak{g}_{-1} = \mathbb{R}^4$ and $\mathfrak{g}_{-2} = \mathbb{R}$, the latter generated by ζ . This reduction is given by a Weyl structure.

Proposition 4.7. *Given a zero-curvature subconformal structure (M, Δ, g) , the rank 3 distribution $\hat{\Pi}^3 \subset T\hat{M}$ is integrable if and only if $S = 0$ holds for the induced structure on Q . Assuming that $\hat{\Pi}^3$ is integrable, the integrability of $\hat{\Pi}^4 \subset T\hat{M}$ is equivalent to $\tau_- = 0$ and $S = 0$ on Q . Finally, assuming that $\hat{\Pi}^3$ and $\hat{\Pi}^4$ are integrable, the integrability of $\hat{\Pi}^5 \subset T\hat{M}$ is equivalent to the vanishing of τ_- and $\text{Ric}_{\tilde{g}}$.*

Proof. Infinitesimal symmetry yields a reduction of the structure bundle given by $\iota : \hat{\mathcal{K}} \rightarrow \mathcal{G}$, as discussed before. Using the pull-back of the Cartan connection (12) to \mathcal{K} and the projection $\pi_{\hat{\mathcal{K}}} : \mathcal{K} \rightarrow \hat{M}$, one has $\hat{\Pi}^2 = \pi_{\hat{\mathcal{K}}*} \langle \omega^0, \omega^1, \omega^4, \omega^5 \rangle^\perp$, $\hat{\Pi}^3 = \pi_{\hat{\mathcal{K}}*} \langle \omega^1, \omega^4, \omega^5 \rangle^\perp$, $\hat{\Pi}^4 = \pi_{\hat{\mathcal{K}}*} \langle \omega^1, \omega^5 \rangle^\perp$, and $\hat{\Pi}^5 = \pi_{\hat{\mathcal{K}}*} \langle \omega^5 \rangle^\perp$. Now the proof is a straightforward inspection of Frobenius integrability conditions using the structure equations (31) and the relations (36) on \mathcal{K} . \square

In the proposition above, the Frobenius condition for $\hat{\Pi}^3$ given by $S = 0$ has no analogs on \hat{Q} . This provides another interpretation of the vanishing of S .

Remark 4.8. Our hierarchy of reductions, when a Lax pair is extended to a Lax triple and up to a Lax quintuple, is nested: the smaller equation (larger Lax distribution) is defined when its larger counterpart is integrable. It turns out that given a zero-curvature subconformal structure with an infinitesimal symmetry and integrable $\hat{\Pi}^i$ for $i = 2, 3, 4, 5$, its Cartan Holonomy is reduced to $\mathfrak{p}_{13}^{\text{op}} \subset \mathfrak{sl}(4, \mathbb{R})$ via a reduction $\iota : \hat{\mathcal{K}} \rightarrow \mathcal{G}$ if and only if $C_1 = 0$, that is, $\tilde{\tau}_- = 0$ and $\text{Ric}_{\tilde{g}} = 0$ and the binary quartic (35) has a repeated root of multiplicity 4 or is zero on Q .

4.4 | 2-nondegenerate CR structures on the twistor bundle

For our purposes in this section, involving (para-)CR structures, define $\mathbb{k}_\varepsilon := \mathbb{R}[\sqrt{\varepsilon}]$ for $\varepsilon = \pm 1$ (also written as $\varepsilon = \pm$), that is, $\mathbb{k}_- = \mathbb{C}$ and $\mathbb{k}_+ = \mathbb{R}$.

Definition 4.9. An almost ε -CR structure of hypersurface type on a manifold N consists of a corank 1 distribution $D \subset TN$ equipped with a field of compatible ε -complex structures

$$J_\varepsilon : D \rightarrow D, \quad J_\varepsilon^2 = \varepsilon \mathbf{1}, \quad \text{Tr}(J_\varepsilon) = 0, \quad \mathcal{L}(J_\varepsilon X, J_\varepsilon Y) = -\varepsilon \mathcal{L}(X, Y), \tag{37}$$

where $\mathcal{L} : \Lambda^2 D \rightarrow TN/D$ is the Levi bracket of D defined by

$$\mathcal{L}(X, Y) = [\tilde{X}, \tilde{Y}]_x \text{ mod } D_x \text{ for } X, Y \in D_x, \tilde{X}, \tilde{Y} \in \Gamma(D), \tilde{X}_x = X, \tilde{Y}_x = Y.$$

An ε -CR structure is characterized by the condition that D' and D'' are Frobenius integrable where D', D'' are eigenspaces of J_ε corresponding to $\sqrt{\varepsilon}$ and $-\sqrt{\varepsilon}$.

In the Levi degenerate case, denote the kernel of \mathcal{L} by $K \subset D$, which is equipped with the splitting $K' \oplus K'' = K \otimes \mathbb{k}_\varepsilon$ (with obvious notations) and the higher Levi bracket $\mathcal{L}'_2 : K' \otimes D'' \rightarrow D'/K'$ defined by

$$\mathcal{L}'_2(X, Y) = [\tilde{X}, \tilde{Y}]_x \text{ mod } (K'_x \oplus D''_x) \text{ for } X \in K'_x, Y \in D''_x,$$

where as above $\tilde{X} \in \Gamma(K'), \tilde{Y} \in \Gamma(D''), \tilde{X}_x = X, \tilde{Y}_x = Y$; its conjugate $\mathcal{L}''_2 : K'' \otimes D' \rightarrow D''/K''$ is defined similarly. An ε -CR structure is called 2-nondegenerate if $\mathcal{L}'_2(X, D'') = 0$ implies $X = 0$ and similarly for the conjugate \mathcal{L}''_2 .

Remark 4.10. Note that when the Levi form is nondegenerate the trace condition in (37), which is automatic for $\varepsilon = -1$, follows also for $\varepsilon = +1$. This condition implies, in turn, that the eigenspaces D', D'' of J_ε corresponding to $\sqrt{\varepsilon}$ and $-\sqrt{\varepsilon}$ have equal dimensions, which gives $D' \oplus D'' = D \otimes \mathbb{k}_\varepsilon$, and hence $\dim N$ is odd.

For CR-structures ($\varepsilon = -1$), the conjugate conditions follow automatically (they are required for para-CR case $\varepsilon = +1$). Such structures are a subclass of k -nondegenerate CR-structures (for some finite k), which in the analytic case enjoy the following properties.

- They cannot be CR-straightened, that is, they are not CR-equivalent to a product [21].
- Their algebra of infinitesimal holomorphic symmetries is finite-dimensional [2].

Recall that a compatible subconformal structure on a contact 5-manifold possesses a naturally associated Cartan bundle $(\mathcal{G} \rightarrow M, \psi)$ with parabolic structure group $P_{13} \cong G_0 \ltimes P_+$, where $G_0 \cong (\mathbb{R}_\times) \times SL(2, \mathbb{R}) \times (\mathbb{R}_\times)$ is the reductive part and P_+ is the nilradical; note that G_0 action on P_+ factors through $R_\times GL(2, \mathbb{R})$. Via the $GL(2, \mathbb{R})$ action, the contact structure Δ can be equipped with an almost ε -complex structure using $J_\varepsilon \in GL(2, \mathbb{R})$ given by

$$J_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{38}$$

which, as we will show, gives rise to a 2-nondegenerate ε -CR structure.

The stabilizer of J_ε in P_{13} is equal to $H^\varepsilon = ((\mathbb{R}_\times) \times H_0^\varepsilon \times (\mathbb{R}_\times)) \ltimes P_+$, where H_0^ε is $SO(2), SO(1, 1) \subset SL(2, \mathbb{R})$ for $\varepsilon = -1, 1$, respectively. Define the ε -twistor bundle as

$$\hat{\mathcal{T}}_\varepsilon = \mathcal{G}/H^\varepsilon = \mathcal{G} \times_P (P/H^\varepsilon).$$

This bundle can be identified as the bundle of all g -compatible ε -complex structures, cf. (37), that is, endomorphisms on Δ satisfying:

$$J_\varepsilon : \Delta \rightarrow \Delta, \quad J_\varepsilon^2 = \varepsilon \mathbf{1}, \quad \text{Tr}(J_\varepsilon) = 0, \quad g(J_\varepsilon X, J_\varepsilon Y) = -\varepsilon g(X, Y). \tag{39}$$

Thus, $\nu_\varepsilon : \hat{\mathcal{T}}_\varepsilon \rightarrow M$ is a fiber bundle with the fibers $SL(2, \mathbb{R})/H_0^\varepsilon$.

Lemma 4.11. *Any fiber $SL(2, \mathbb{R})/H_0^\varepsilon$ is diffeomorphic to the disk \mathbb{D}^2 when $\varepsilon = -1$ and to the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$ when $\varepsilon = +1$.*

Proof. This is obvious from the adjoint action of $SL(2, \mathbb{R})$ preserving the Killing form (of Lorentzian signature): in the Minkowski coordinates $\mathbb{R}^{1,2}(t, x, y)$, $ds^2 = dt^2 - dx^2 - dy^2$, the models of Lobachevski and de Sitter planes are given by $t^2 - x^2 - y^2 = \pm 1$ with the stabilizers of points being conjugate to $SO(2)$ and $SO(1, 1)$, respectively.

It is instructive to note that this isomorphism reflects, actually, the twistor picture:

$$\begin{aligned} Z_-(\mathbb{R}^{2,2}) &= SO(2, 2)/U(2) = SL(2, \mathbb{R})/SO(2) \simeq \mathbb{D}^2, \\ Z_+(\mathbb{R}^{2,2}) &= SO(2, 2)/U(1, 1) = SL(2, \mathbb{R})/SO(1, 1) \simeq \mathbb{S}^1 \times \mathbb{R}^1, \end{aligned}$$

which are, respectively, the spaces of orthogonal complex/product structures in four-dimensional space of split signature, reflecting (39). □

Note that the ε -twistor bundle $\hat{\mathcal{T}}_\varepsilon$ has a codimension 1 distribution, namely, the preimage of the contact distribution $D = d\nu_\varepsilon^{-1}(\Delta)$. Its space of Cauchy characteristics, or the kernel of the Levi form, is $K = \text{Ker}(d\nu_\varepsilon)$. Similar to the classical twistor theory, D has the induced almost complex/product structure \mathcal{J}_ε , so $(\hat{\mathcal{T}}_\varepsilon, D, \mathcal{J}_\varepsilon)$ is an almost ε -CR manifold.

The following is a nonholomorphic higher-dimensional analog of the classical Atiyah–Hitchin–Singer version [1] of Penrose’s nonlinear graviton construction.

Theorem 4.12. *A compatible subconformal structure $(M, \Delta, [g])$ in 5D has zero curvature if and only if the corresponding almost ε -CR manifold $(\hat{\mathcal{T}}_\varepsilon, D, \mathcal{J}_\varepsilon)$ in 7D is integrable, which makes it a 2-nondegenerate ε -CR manifold.*

Note that $\varepsilon = \pm 1$ here is independent of the choice of the invariant $\text{sgn } \delta = \pm 1$ of $(M, \Delta, [g])$. Furthermore, if the result holds for either $\varepsilon = +1$ or -1 , then it is true for both $\varepsilon = \pm 1$.

Proof. Note that we have $T\hat{\mathcal{T}}_\varepsilon = \mathcal{G} \times_{H^\varepsilon} (\mathfrak{g}/\mathfrak{h}^\varepsilon)$, where $\mathfrak{h}^\varepsilon = \text{Lie}(H^\varepsilon)$. In the \mathfrak{p}_{13} (contact) grading of \mathfrak{g} , the distribution D corresponds to $\mathfrak{g}_{-1} \oplus (\mathfrak{g}_0/\mathfrak{h}^\varepsilon)$ and K to $\mathfrak{g}_0/\mathfrak{h}^\varepsilon$.

For $\varepsilon = -1$, in terms of matrix expression (12), the 1-forms $(\omega^0, \dots, \omega^4, \omega^5 + \xi_5, \phi_0 - \phi_1)$ give a coframe on $\hat{\mathcal{T}}_-$, and $D = \text{Ker}(\nu_-^* \omega^0) \subset T\hat{\mathcal{T}}_-$. Using the action of $GL(2, \mathbb{R})$ and the almost complex structure J_- as in (38), it follows that the holomorphic (1,0)-type distribution in Definition 4.9 is

given by $D' = (\nu_-)_* \text{Ker}\{\omega^0, \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3\} \subset D^{\mathbb{C}}$ and $D'' = \overline{D'}$, where

$$\zeta^1 = \omega^1 + i\omega^2, \quad \zeta^2 = \omega^3 + i\omega^4, \quad \zeta^3 = \phi_0 - \phi_1 + i(\omega^5 + \xi_5).$$

A straightforward inspection of the structure equation (13) shows that D' is Frobenius integrable if and only if W vanishes. Thus, $W = 0$ implies CR-integrability of \mathcal{J}_- . Moreover, one obtains

$$d\omega^0 \equiv \frac{1}{2}(\zeta^1 \wedge \bar{\zeta}^2 - \zeta^2 \wedge \bar{\zeta}^1) \text{ mod}\{\omega^0\}. \tag{40}$$

As a result, the Levi bracket is degenerate along $K = K' \oplus K''$, where $K' = \langle \partial_{\zeta^3} \rangle$. Lastly, the 2-nondegeneracy of (D, \mathcal{J}_-) follows from the symbol algebra of $(\mathfrak{sl}(4, \mathbb{R}), \mathfrak{p}_{13})$ since

$$d\bar{\zeta}^1 \equiv \frac{1}{2}\zeta^1 \wedge \bar{\zeta}^3, \quad d\bar{\zeta}^2 \equiv -\frac{1}{2}\zeta^2 \wedge \bar{\zeta}^3 \text{ mod}\left\{\omega^0, \bar{\zeta}^1, \bar{\zeta}^2\right\}. \tag{41}$$

For $\varepsilon = +1$, one can proceed similarly. In terms of matrix expression (12), the 1-forms $(\omega^0, \dots, \omega^4, \omega^5 - \xi_5, \phi_0 - \phi_1)$ give a coframe on $\hat{\mathcal{T}}_+$ and $D = \text{Ker}(\nu_+^* \omega^0) \subset T\hat{\mathcal{T}}_+$. Using the $GL(2, \mathbb{R})$ action and the expression of J_+ as in (38), the corresponding para-holomorphic distributions are $D' = (\nu_+)_* \text{Ker}\{\omega^0, \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3\} \subset D$, $D'' = (\nu_+)_* \text{Ker}\{\omega^0, \zeta^1, \zeta^2, \zeta^3\} \subset D$, where

$$\begin{aligned} \zeta^1 &= \omega^1 - \omega^2, & \zeta^2 &= \omega^3 + \omega^4, & \zeta^3 &= \phi_0 - \phi_1 + (\omega^5 - \xi_5), \\ \bar{\zeta}^1 &= \omega^1 + \omega^2, & \bar{\zeta}^2 &= \omega^3 - \omega^4, & \bar{\zeta}^3 &= \phi_0 - \phi_1 - (\omega^5 - \xi_5), \end{aligned}$$

with respect to which relation (40) remains valid. It is again a matter of straightforward inspection of the structure equations (13) to show that distributions D' and D'' are Frobenius integrable if and only if W is zero. Lastly, the 2-nondegeneracy of the para-CR structure follows from the symbol algebra since (41) remains valid in the para-CR case as well. □

The geometric picture presented here has a direct counterpart on the level of equations: the dispersionless integrability of PDE \mathcal{E} , which by Theorem 1.3 is given by the zero-curvature condition of the induced subconformal geometry on M_u for a generic background solution u of \mathcal{E} , is also equivalent to (para)CR-integrability of the corresponding structures on $\hat{\mathcal{T}}_\varepsilon$ with any choice of $\varepsilon = \pm 1$.

Remark 4.13. We point out that the 7-dimensional 2-nondegenerate CR structures, associated to zero-curvature parabolic geometries of type (A_3, P_{13}) in Theorem 4.12, are generalized in [22] to other types of parabolic geometries.

Furthermore, the integrability of the complex structure \mathcal{J}_- on D can be considered as the odd-dimensional counterpart of the same phenomenon on the twistor bundles for para-quaternionic structures, known as β -integrable $(2, n)$ -Segré structures, [38, 49].

4.5 | Models of maximal and submaximal symmetries

Let us begin with the zero-curvature subconformal structure of maximal symmetry, and then, we will discuss the constructions of this section for submaximal symmetric structures.

(1) The unique zero-curvature subconformal structure of maximal symmetry for $\delta > 0$ is $SL(4, \mathbb{R})/P_{13}$ and that for $\delta < 0$ is $SU(2, 2)/P_{13}$; the symmetry is 15-dimensional as indicated.

Let us consider the flat para-CR case ($\delta > 0$) in details. The model, corresponding via the construction from Section 4.1 to the flat projective structure, is given by $M = \mathbb{R}^5(x^1, x^2, u, p_1, p_2)$ and the L_{\pm} splitting of the contact distribution

$$\Delta = \text{Ker}(\omega^0), \quad \omega^0 = du - p_1 dx^1 - p_2 dx^2,$$

as follows (we again redenote the subdistributions by H, V):

$$\Delta = H \oplus V \quad \text{with} \quad H = \langle \partial_{x^1} + p_1 \partial_u, \partial_{x^2} + p_2 \partial_u \rangle \quad \text{and} \quad V = \langle \partial_{p_1}, \partial_{p_2} \rangle.$$

The subconformal structure is represented by the metric $g = dx^1 dp_1 + dx^2 dp_2$ on Δ .

The symmetry algebra $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$ contains an element $\xi = u\partial_u + p_1\partial_{p_1} + p_2\partial_{p_2}$. To obtain the corresponding symmetry reduction, we pass to coordinates $(x^1, x^2, \ln(1/u), p_1/u, p_2/u)$, where the first two and last two are invariants of the flow of ξ . Keeping the same notations (x^1, x^2, u, p_1, p_2) for the new coordinates, the quotient conformal metric on $\bar{M} = \mathbb{R}^4(x^1, p_1, x^2, p_2)$ is given by the formula

$$\bar{g} = dx^1 dp_1 + dx^2 dp_2 + (p_1 dx^1 + p_2 dx^2)^2.$$

This metric is self-dual but not conformally flat. A straightforward computation of conformal Killing vectors shows that this quotient corresponds to $SL(3)/GL(2)$. (This can also be seen from Lie-theoretic arguments, since the normalizer of ξ in \mathfrak{g} is $\mathfrak{gl}(3, \mathbb{R})$ with ξ in the center.)

The corresponding CR-reduction (the case $\delta < 0$) with $\mathfrak{g} = \mathfrak{su}(2, 2)$ gives $SU(1, 2)/U(1, 1)$.

Note that for any $\xi \in \mathfrak{g}$, the symmetry reduction allows to obtain not only 4D conformal structure, but also its canonical metric representative. Indeed, for the symmetry ξ (in the open dense set, where it is transversal to the contact distribution), the contact form can be normalized by the condition $\omega^0(\xi) = 1$, whence we get a symplectic form Ω on Δ and normalization of the metric $\|\Omega^{-1}g\| = 1$. This quotient metric \bar{g} is almost (pseudo/para)-Kähler of neutral signature.

Actually, since in the flat model, the torsion is zero, the metric \bar{g} for $\delta < 0$ is pseudo-Kähler and for $\delta > 0$ is para-Kähler, that is, the almost (para-)complex structure J is integrable. The space of self-dual (pseudo/para)-Kähler structures (also known as Bochner–Kähler or Bochner–bi-Lagrangian structures, cf. [4]) arising as the symmetry reduction of the flat 5D subconformal contact structure, depends on 3 parameters. More precisely, among the fundamental invariants (32), one has $\tilde{\tau}_{\pm} = 0$, and, by (34), the self-dual Weyl curvature, given by the quartic (35), is zero if $r = 0$, or otherwise has Petrov type D if $r \neq 0$, that is, has two repeated roots of multiplicity 2. The unique (up to homothety) such nonconformally flat self-dual (para)-Kähler metric that is Einstein, that is, satisfies $S = 0$ and $R \neq 0$ in (32), is the canonically defined pseudo-Kähler metric on $SU(1, 2)/U(1, 1)$ and para-Kähler metric on $SL(3, \mathbb{R})/GL(2, \mathbb{R})$.

(2) Now consider submaximally symmetric structures. For A_3/P_{13} -type geometry, the submaximal symmetry dimension is 8, achieved both in pure curvature and in pure torsion modules [33]. In the split real case $\delta > 0$, the submaximal zero-curvature subconformal structure is unique and is given by the lift of the Egorov projective connection. We write it as a deformation of the trivial system of ODEs:

$$\ddot{x}^1 = 0, \quad \ddot{x}^2 = \epsilon x^1. \tag{42}$$

By the construction of Section 4.1, this generates the zero-curvature structure in 5D, which, as above, in coordinates (x^1, x^2, u, p_1, p_2) is given by the L_{\pm} splitting of the contact distribution:

$$\Delta = H \oplus V \quad \text{with} \quad H = \langle \partial_{x^1} + p_1 \partial_u + \epsilon x^1 \partial_{p_2}, \partial_{x^2} + p_2 \partial_u + \epsilon x^1 \partial_{p_1} \rangle \quad \text{and} \quad V = \langle \partial_{p_1}, \partial_{p_2} \rangle. \quad (43)$$

The subconformal structure is represented by the metric $g = \omega^1 \omega^3 + \omega^2 \omega^4$ in the coframe $\omega^1 = dx^1, \omega^2 = dx^2, \omega^3 = dp_1 - \epsilon x^1 dx^2, \omega^4 = dp_2 - \epsilon x^1 dx^1$ on Δ . The point symmetry algebra of (42) coincides with the symmetry algebra \mathfrak{g} of the subconformal structure; it is solvable and generated by

$$\begin{aligned} \xi_1 &= x^1 \partial_{x^1} - x^2 \partial_{x^2} + u \partial_u + 2p_2 \partial_{p_2}, \quad \xi_2 = x^1 \partial_{x^2} + \frac{1}{3} \epsilon (x^1)^3 \partial_u + (\epsilon (x^1)^2 - p_2) \partial_{p_1}, \\ \xi_3 &= x^2 \partial_{x^2} + u \partial_u + p_1 \partial_{p_1}, \quad \xi_4 = \partial_{x^1} + \epsilon (x^1 x^2 \partial_u + x^2 \partial_{p_1} + x^1 \partial_{p_2}), \\ \xi_5 &= \partial_{x^2}, \quad \xi_6 = \partial_u, \quad \xi_7 = x^1 \partial_u + \partial_{p_1}, \quad \xi_8 = x^2 \partial_u + \partial_{p_2}. \end{aligned}$$

Consider three Abelian subalgebras $\mathfrak{h}_1 = \langle \xi_4, \xi_7 \rangle, \mathfrak{h}_2 = \langle \xi_4, \xi_5 \rangle, \mathfrak{h}_3 = \langle \xi_4, \xi_3 \rangle$. All of them are nonnull, meaning that the span of the fields does not intersect with the g -null cone on Δ .

Let us first do symmetry reduction along ξ_4 , which is the field common to all subalgebras. Passing to coordinates $(u - \frac{1}{2} \epsilon (x^1)^2 (x^2), x^2, x^1, p_1 - \epsilon x^1 x^2, p_2 - \frac{1}{2} \epsilon (x^1)^2)$ where the first two and last two are invariants of the flow of ξ_4 and keeping the same notations (x^1, x^2, u, p_1, p_2) for the new coordinates, the quotient conformal metric on $\bar{M} = \mathbb{R}^4(x^1, p_1, x^2, p_2)$ is given by the formula

$$\bar{g} = dx^1 dp_1 + dx^2 (p_1 dp_2 - p_2 dp_1) + \epsilon \frac{x^2}{p_1} (dx^1 - p_2 dx^2)^2.$$

This metric is self-dual but not conformally flat.

We can take further reduction from 4D to 3D, and in new coordinates, the remaining generators of $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ have the form: $\bar{\xi}_7 = \partial_{x^1}, \bar{\xi}_5 = x^2 \partial_{x^1} + \partial_{p_2}, \bar{\xi}_3 = x^1 \partial_{x^1} + x^2 \partial_{x^2} + p_1 \partial_{p_1}$. Reduction along both $\bar{\xi}_7$ and $\bar{\xi}_5$ yields conformally flat 3D metrics, but the symmetry reduction along $\bar{\xi}_3$ gives in new coordinates/invariants (x_1, x_2, x_3) the following conformal metric:

$$\check{g} = (x_2 x_3 dx_1 - (x_1 x_3 - 1) dx_2 + x_1 x_2 dx_3)^2 + 4 (\epsilon x_1 (x_1 x_3 - 1) - x_2^2) dx_1 dx_3.$$

This metric on the quotient $\check{M} = \mathbb{R}^3(x_1, x_2, x_3)$ is not conformally flat, but is Einstein–Weyl.

(3) Let us describe the geometry on the twistor space \mathcal{T}^4 obtained from the correspondence space \hat{M}^6 via quotient by the foliation of the dLp $\hat{\Pi}^2$. For the flat (A_3, P_{13}) geometry, the induced geometry on the twistor space is flat conformal, that is, of type (A_3, P_2) . This follows from the parabolic twistor correspondence [8], see the discussion at the beginning of Section 4.

Otherwise the projection of tangents ∂_λ to the fibers of $\hat{M}^6 \rightarrow M^5$ along the foliation $\hat{\Pi}^2$ yields a field of surfaces on \mathcal{T} , that is, a causal structure (see the discussion at the end of Section 2.4); thus, the correspondence space can be identified with a codimension 1 subbundle of the projectivized tangent bundle: $\hat{M} \subset \mathbb{P}T(\mathcal{T})$. In general, such a causal structure (also known as a cone structure) may not be *isotrivial*, in the sense of [24]. Being isotrivial means that all the fibers of $\hat{M} \rightarrow \mathcal{T}$ are projectively equivalent to a fixed projective surface $C \subset \mathbb{P}^3$, and being isotrivially flat means trivialization of the bundle $\hat{M}^6 \simeq C \times \mathcal{T}$ as a subbundle of $\mathbb{P}T(\mathcal{T})$.

In the case of large symmetry, the causal structures are isotrivial. For the maximal symmetric case, the causal structure is isotrivially flat with quadratic fibers. For the submaximal symmetric case, it was demonstrated in [34] that $\hat{M} \rightarrow \mathcal{T}$ is the so-called *Cayley structure*, namely, the fiber

C is the projectivized ruled Cayley cubic; moreover, it was proved that there are precisely two other 3D projective connections inducing a Cayley structure on \mathcal{T} . Let us show it directly for the zero-curvature subconformal structure, derived from the Egorov projective connection.

Denoting the vector generators in (43) by h_1, h_2 for H and v_1, v_2 for V , the dLp is

$$\hat{\Pi} = \langle h_1 + \lambda h_2 = \partial_{x^1} + \lambda \partial_{x^2} + (p_1 + \lambda p_2) \partial_u + \epsilon x^1 (\partial_{p_2} + \lambda \partial_{p_1}), v_2 - \lambda v_1 = \partial_{p_2} - \lambda \partial_{p_1} \rangle.$$

This distribution is Frobenius integrable on \hat{M}^6 . Passing to new coordinates $(p_1 + \lambda(p_2 - \epsilon(x^1)^2), x^2 - \lambda x^1, u + \lambda(\frac{2}{3}\epsilon(x^1)^3 - x^1 p_2) - x^1 p_1, x^1, p_2, \lambda)$, among which the first three and the last one are $\hat{\Pi}$ -invariants, and keeping old notations for the new coordinates, we get

$$\hat{\Pi}_{\text{new}} = \langle \partial_{p_1}, \partial_{p_2} \rangle, \quad \langle \partial_\lambda \rangle_{\text{new}} = \left\langle \partial_\lambda - (\epsilon p_1^2 - p_2) \partial_{x^1} - p_1 \partial_{x^2} + \left(\frac{2}{3} \epsilon p_1^3 - p_1 p_2 \right) \right\rangle.$$

Taking the coefficients of the latter vector field with simple rescaling $[1 : p_1 : p_2 - \epsilon \frac{p_1^2}{2}, p_1 p_2 - \epsilon \frac{p_1^3}{3}]$ and changing coordinates once again, we get (isotrivially flat) ruled Cayley cubic

$$z = \frac{1}{3} x^3 - xy.$$

(4) The submaximal symmetry algebra for subconformal structures in 5D with $\delta < 0$ has dimension 7 [33, 35]. There is no uniqueness in this case, and the models are obtained as follows. Consider a projective surface $C \subset \mathbb{P}^2$ with 2-dimensional symmetry algebra; for the classification, see [11] and references therein. Consider $\mathcal{T}^4 = \mathbb{R}^4(x^1, x^2, x^3, x^4)$ and let $\hat{M}^6 = C \times \mathcal{T}^4 \subset \mathbb{P}T(\mathcal{T})$ be the isotrivially flat cone structure with the fiber C .

By the construction, the correspondence space \hat{M}^6 has a rank 3 distribution split into two integrable subdistributions $\hat{\Pi}^2 \oplus \ell$. The projection along $\ell = \langle \partial_\lambda \rangle$ yields M^5 . Surfaces C with positive definite second fundamental form correspond to zero-curvature subconformal structures with $\delta < 0$, while those with Lorentzian signature correspond to the para-CR case $\delta > 0$.

The 7-dimensional symmetry is composed of four translations ∂_{x^i} and the homothety $x^i \partial_{x^i}$ on \mathcal{T} as well as two projective symmetries of C . By functoriality, these pass to \hat{M} and M . Explicit formulae can be obtained as follows. Choose a coordinate system (z^0, z^1, z^2, z^3) on \mathcal{T} and let (w^0, w^1, w^2) be the corresponding affine coordinate chart for $\mathbb{P}^3 = \mathbb{P}T_z(\mathcal{T})$. An isotrivially flat causal structure $\hat{M} \subset \mathbb{P}T(\mathcal{T})$, adapted to these coordinates, can be written as a graph

$$w^0 = G(w^1, w^2).$$

The induced (3,5,6) distribution on \hat{M} , following Section 2.4, is given as $\hat{\Pi}^2 \oplus \ell$ with

$$\hat{\Pi} = \langle \partial_{w^1}, \partial_{w^2} \rangle, \quad \ell = \left\langle G \frac{\partial}{\partial z^0} + w^1 \frac{\partial}{\partial z^1} + w^2 \frac{\partial}{\partial z^2} + \frac{\partial}{\partial z^3} \right\rangle.$$

The first summand $\hat{\Pi}$ is the vertical tangent bundle to $\hat{M} \rightarrow \mathcal{T}$; the second summand ℓ is the characteristic line field for the odd contact 1-form

$$\alpha = dz^0 - G_{w^1} dz^1 - G_{w^2} dz^2 + (w^1 G_{w^1} + w^2 G_{w^2} - G) dz^3,$$

which at $(z; w) \in \hat{M}$ belongs to the pullback of the annihilator to the affine tangent space of the cone $\hat{C}_z \subset T_z \mathcal{T}$ along $w \in C_z$.

(5) Finally, we consider the twistorial construction of 2-nondegenerate (para-)CR structure in dimension 7. It is a bundle $\hat{\mathcal{T}}_\epsilon$ over M^5 with the two-dimensional fiber \mathbb{D}^2 or $\mathbb{S}^1 \times \mathbb{R}^1$ according to Lemma 4.11. Denoting the coordinates in the fiber by q_1, q_2 and keeping the notations h_1, h_2 for the generators of H and v_1, v_2 for the generators of V in (43), the induced para-CR structure ($\epsilon = +1$) corresponding to the Egorov structure (42) is given by the splitting

$$D = \langle h_1 + q_1 h_2, v_2 - q_1 v_1, \partial_{q_2} \rangle \oplus \langle h_2 + q_2 h_1, v_1 - q_2 v_2, \partial_{q_2} \rangle$$

into a pair of integrable subdistributions. Similarly, one gets a CR structure ($\epsilon = -1$) corresponding to the Egorov structure via a pair of complex conjugated subdistributions in $D \otimes \mathbb{C}$.

This twistorial construction is fully functorial: an equivalence downstairs lifts from M^5 to $\hat{\mathcal{T}}_\epsilon$ and, conversely, any symmetry of $\hat{\mathcal{T}}_\epsilon$ projects along the Levi kernel K to M^5 . The distribution D projects to the contact structure Δ . The (para-)CR structure \mathcal{J}_ϵ is not projectable, but there exists a unique up to sign (para-)complex structure \hat{J} , commuting with it, cf. the proof of Theorem 4.12. (This is analogous to the left and right (split-)quaternionic multiplications.) Such \hat{J} projects along K to J on M^5 making it into a zero-curvature subconformal manifold.

We note that 7-dimensional 2-nondegenerate CR structures arising from zero-curvature subconformal structures are *recoverable* in the sense of [45]. A CR structure is called recoverable if the space

$$\text{ad}(K''_x) := \left\{ \text{ad}_v \mid v \in K''_p \right\} \subset \text{Hom}(D'_x/K'_x, D''_x/K''_x)$$

has vanishing first prolongation. Recall that the first prolongation is defined as the kernel of the Spencer operator $\partial : \text{Hom}(V, W) \rightarrow \text{Hom}(V \wedge V, W)$ where $\partial f(v, u) = f(v)u - f(u)v$. Using the relation (41), it is straightforward to show that the first prolongation of $\text{ad}(K''_x)$ is zero for all $x \in \hat{\mathcal{T}}_\epsilon$. This gives isomorphism of symmetry algebras.

In particular, for the family derived from (42) with $\epsilon \neq 0$, we obtain 7-dimensional 2-nondegenerate CR structure with 8D symmetry, while for the flat case $\epsilon = 0$, we get the structure with 15D symmetry algebra. This latter can be either $SL(4, \mathbb{R})$ or $SU(2, 2)$ realizing the submaximal symmetry dimension for 7-dimensional 2-nondegenerate CR structures.

5 | OUTLOOK

In this paper, we generalized the paradigm ‘‘Integrability via Geometry’’ [7] from 3D and 4D to dispersionless equations and Lax pairs in 5D, provided that the symbol of the equation has conformal symmetric bivector of rank 4 and the corresponding distribution is contact at generic solution u of the equation \mathcal{E} . Surprisingly, this case has an underlying parabolic geometry, similar to the well-studied lower-dimensional cases. This allowed us to identify the required curvature component of the corresponding subconformal structure.

Let us note that parabolic geometries of type (A_3, P_{13}) have been studied in the literature, for instance, subconformal structures of neutral signature in 5D appeared both as integrable Legendrian and Levi-split CR structures. However, in that class, the torsion vanishes while the curvature may be nonzero. In our case, the situation is opposite: the dispersionless integrability is equivalent to the vanishing of the curvature, while we allow torsion.

Some of the concepts we introduced in this paper have direct higher dimensional generalizations. The fact that the underlying geometry is parabolic does not persist in higher dimensions, as we indicate in the Appendix. In our next paper, we will address those and explain how to define the proper curvature and connect it to the integrability.

If the rank of $c_{\mathcal{E}}$ drops below the maximal value 4 or the distribution becomes (partially) integrable for nongeneric u , this yields a class of algebraically special solutions. If, however, this happens for generic solution u of the PDE \mathcal{E} , then the underlying geometry is different.

For instance, investigating rank 3 subconformal structures on a 5D background generically meets the following aspects: (i) the distribution Δ has growth vector (3,5) and the radical $\sqrt{\Delta}$, a rank 2 distribution with growth vector (2,3,5) on a generic solution u , has higher order tensorial invariants; (ii) a partial Weyl connection associated to the subconformal structure $c_{\mathcal{E}}$ has curvature components on its own; and (iii) a compatibility condition relates both (i) and (ii).

We expect that integrability for such a class of 5-dimensional dispersionless PDEs is also expressible via certain “zero-curvature condition,” involving the above invariants. We do not address these questions here, but raise the problem of describing the corresponding integrable background geometry as a parabolic geometry. We also expect that 5D zero-curvature systems discussed in this paper pass the test for hydrodynamic integrability as in [20] as well as adaptation for the deformation scheme as in [44].

APPENDIX: INTEGRABLE PARABOLIC BACKGROUND GEOMETRIES

Let us define a class of parabolic geometries that may serve as *integrable background geometry* by first specifying the compatibility properties discussed in Section 1.1, which only constrain the algebraic type of the geometry and does not restricts the curvature properties.

Definition A.1. Given a semisimple Lie group G and a parabolic subgroup $P \subset G$, a parabolic geometry of type (G, P) is called a *compatible background geometry* if it admits a subconformal structure (\mathfrak{g}_{-1}, c) with the following compatibility conditions:

- $\dim \mathfrak{g}_{-1} = 3$ or 4 ,
- there exists a (unique) G_0 -invariant conformal structure c on \mathfrak{g}_{-1} ,
- there exists a 1-parameter family of c -null 2-planes that are Abelian in $\mathfrak{m} = \mathfrak{g}_{-}$.

These 2-planes are henceforth called α -planes.

Definition A.2. A compatible parabolic background geometry is called an *integrable background geometry* if every α -plane is tangent to an α -surface, that is, a surface whose tangents are α -planes.

Note that we included the case of rank 3 subconformal structure, whose integrability requires a choice of Weyl connection, which is an additional constraint on the parabolic geometry. Geometries of rank 2 and 1 may also be considered as an instance of integrable background geometry, within the paradigm of [5]; however, in this paper, we restrict to ranks 3 and 4.

The uniqueness claim in Definition A.1 is not obvious a priori but is a by-product of the following classification result.

Theorem A.3. *The only compatible parabolic background geometries are those of the types (B_2, P_1) , (D_3, P_1) , and $(D_3, P_{2,3})$, as well as (B_3, P_{13}) and (C_3, P_2) .*

Here, we used the exceptional isomorphism $(A_3, P_{1,3}) = (D_3, P_{2,3})$ over \mathbb{C} . Over \mathbb{R} , the compatible parabolic background geometries of the above type are: $(\mathfrak{so}(2, 3), \mathfrak{p}_1)$, $(\mathfrak{so}(3, 3), \mathfrak{p}_1)$, $(\mathfrak{so}(3, 3), \mathfrak{p}_{2,3})$, $(\mathfrak{su}(2, 2), \mathfrak{p}_{13})$, $(\mathfrak{so}(3, 4), \mathfrak{p}_{13})$, $(\mathfrak{sp}(6, \mathbb{R}), \mathfrak{p}_2)$.

Proof. We will work with Lie algebras. The constraint of \mathfrak{g}_0 -equivariance is enough to exclude false candidates. For the remaining five, the equivariance on the group level is straightforward. Until the end of the proof, we work over \mathbb{C} (the claim over \mathbb{R} follows by direct inspection).

We start with simple Lie algebras. Their structure root theory implies $\dim \mathfrak{g}_{-1} \geq n = \text{rank}(\mathfrak{g})$ (the equality is achieved on the Borel parabolic subalgebra $\mathfrak{p}_{1\dots n}$). Indeed, if α_i are simple roots, then $\sum_{i \in S} \alpha_i$ is also a root for any connected piece S of the Dynkin diagram. Thus, there are at least n roots of this form, where S contains only one crossed node in Σ for $\mathfrak{p} = \mathfrak{p}_\Sigma$.

This already makes the list of candidates finite. Further restrictions are:

- parabolics given by more than 2 crosses (\mathfrak{p}_Σ with $|\Sigma| > 2$) have 3 independent scalings, and so cannot conformally preserve a bilinear form of rank ≥ 3 ;
- the Dynkin diagram of the Levi part \mathfrak{g}_0^{ss} , obtained by excluding crosses, must be $B_1 = A_1$ or $D_2 = A_1A_1$, because in other cases, \mathfrak{g}_0^{ss} cannot preserve a conformal structure.

Consequently, for $\dim \mathfrak{g}_{-1} = 3$, the candidates for compatible parabolic background geometries are (we use outer automorphisms of \mathfrak{g} to exclude repetitions):

$$(A_3, P_{12}), (B_2, P_1), (B_3, P_{23}), (C_3, P_{12}), (C_3, P_{23}).$$

For $\dim \mathfrak{g}_{-1} = 4$, the candidates are (we use exceptional isomorphisms $B_2 = C_2, A_3 = D_3$):

$$(A_3, P_2), (A_3, P_{13}), (A_4, P_{23}), (B_3, P_{12}), (B_3, P_{13}), (C_3, P_2), (C_4, P_{23}), (G_2, P_2), (F_4, P_{23}).$$

Among these only, the following have \mathfrak{g}_0 -invariant conformal structure:

$$(B_2, P_1); (A_3, P_2), (A_3, P_{13}), (B_3, P_{13}), (C_3, P_2). \tag{A.1}$$

The recipe to do so is as follows: determine the \mathfrak{g}_0 -module \mathfrak{g}_{-1} and compute its [2] plethysm; for instance, in the case (B_3, P_{23}) , we have $\mathfrak{g}_0^{ss} = A_1$ and $\mathfrak{g}_{-1} = 1 \times [0] + 1 \times [1]$, whence $S^2 \mathfrak{g}_{-1} = 1 \times [0] + 1 \times [1] + 1 \times [2]$ but the trivial representation corresponds to rank 1 bilinear form, and hence, an invariant conformal metric does not exist.

The first case in (A.1) has 3-dimensional irreducible manifold ($\mathfrak{g}_{-1} = TM$), and the remaining cases have 4-dimensional \mathfrak{g}_{-1} : on the manifolds of dimensions 4, 5, 8, and 7, respectively. A straightforward computation shows that each is a compatible background geometry.

If \mathfrak{g} is semisimple but not simple, it must be a product of

$$(A_1, P_1), (A_2, P_1), (A_2, P_{12}), (B_2, P_2), (B_2, P_{12}), (G_2, P_1), (G_2, P_{12}) \tag{A.2}$$

with $\dim \mathfrak{g}_{-1} = 1 \vee 2$. In this list, only geometries with the Borel subgroup $B \subset G$ possess \mathfrak{g}_0 -invariant conformal structure on \mathfrak{g}_{-1} . However, in the product, that is, for nonsimple G , with factors from (A.1) and (A.2), no such geometry can have \mathfrak{g}_0 -invariant conformal structure on \mathfrak{g}_{-1} (there are, however, \mathfrak{g}_0^{ss} -invariant such). This finishes the proof. \square

Corollary A.4. *The only integrable parabolic background geometries are those listed in Theorem A.3 with their respective corresponding zero-curvature condition.*

These so-called “zero-curvature” conditions were already discussed for the standard conformal geometries (B_2, P_1) in 3D, with a choice of Weyl structure, and (D_3, P_1) in 4D, as well as for the geometry (A_3, P_{13}) studied in this paper. The novel candidates are (B_3, P_{13}) and (C_3, P_2) in 8D and

7D, respectively. While higher dimensions will be considered in a forthcoming paper (where the corresponding curvature W will be introduced) let us briefly comment on the two new cases.

8D case. The grading of B_3 corresponding to parabolic P_{13} is $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ with dimensions of components $(2,2,4,5,4,2,2)$. In terms of root vectors, \mathfrak{g}_0 is generated by the Cartan subalgebra \mathfrak{h} and $e_{\pm\alpha_2}$, while

$$\mathfrak{g}_{-1} = \langle e_{-\alpha_1}, e_{-\alpha_1-\alpha_2}, e_{-\alpha_3}, e_{-\alpha_2-\alpha_3} \rangle \text{ and } \mathfrak{g}_{-2} = \langle e_{-\alpha_1-\alpha_2-\alpha_3}, e_{-\alpha_2-2\alpha_3} \rangle.$$

The only \mathfrak{g}_0 -invariant conformal metric (after proper normalization of root vectors) is

$$g = e_{\alpha_1} \cdot e_{\alpha_2+\alpha_3} + e_{\alpha_1+\alpha_2} \cdot e_{\alpha_3}$$

The only congruence of α -planes (g -null and $[\cdot, \cdot]$ -isotropic) is

$$\langle e_{-\alpha_1} + \lambda e_{-\alpha_1-\alpha_2}, e_{-\alpha_3} - \lambda e_{-\alpha_2-\alpha_3} \rangle.$$

This lifts to a canonical rank 2 distribution in the correspondence space, which is a parabolic geometry of type (B_3, P_{123}) and a bundle over (B_3, P_{13}) with fiber \mathbb{P}^1 . However, the curvature W , corresponding to the Frobenius condition, takes value in the component of the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$, which is a \mathfrak{g}_0^{SS} irreducible module with the lowest weight vector

$$e_{\alpha_1} \wedge e_{\alpha_3} \otimes e_{-\alpha_1-2\alpha_2-2\alpha_3}.$$

This corresponds to torsion of negative homogeneity -1 with respect to the grading element $Z \in \mathfrak{g}_0$. We refer for the technique behind this computation and the general theory of parabolic geometries to [9]. In nontrivial case $W \neq 0$, this means that the parabolic geometry is nonregular, and hence, the zero-curvature condition cannot be obtained as a component of the harmonic curvature in the standard parabolic formalism.

7D case. The grading of C_3 corresponding to parabolic P_2 is $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with dimensions of components $(3,4,7,4,3)$. In terms of root vectors, \mathfrak{g}_0 is generated by the Cartan subalgebra \mathfrak{h} and $e_{\pm\alpha_1}, e_{\pm\alpha_3}$, while

$$\mathfrak{g}_{-1} = \langle e_{-\alpha_2}, e_{-\alpha_1-\alpha_2}, e_{-\alpha_2-\alpha_3}, e_{-\alpha_1-\alpha_2-\alpha_3} \rangle \text{ and } \mathfrak{g}_{-2} = \langle e_{-2\alpha_2-\alpha_3}, e_{-\alpha_1-2\alpha_2-\alpha_3}, e_{-2\alpha_1-2\alpha_2-\alpha_3} \rangle.$$

The only \mathfrak{g}_0 -invariant conformal structure, after proper normalization of root vectors, is

$$g = e_{\alpha_2} \cdot e_{\alpha_1+\alpha_2+\alpha_3} + e_{\alpha_1+\alpha_2} \cdot e_{\alpha_2+\alpha_3}$$

The only congruence of α -planes (g -null and $[\cdot, \cdot]$ -isotropic) is

$$\langle e_{-\alpha_2} + \lambda e_{-\alpha_2-\alpha_3}, e_{-\alpha_1-\alpha_2} - \lambda e_{-\alpha_1-\alpha_2-\alpha_3} \rangle.$$

This lifts to a canonical rank 2 distribution in the correspondence space, which is a parabolic geometry of type (C_3, P_{23}) and a bundle over (C_3, P_2) with fiber \mathbb{P}^1 . In fact, the preimage of the above congruence is the distribution with growth $(3,2,3)$ in the correspondence space. The curvature W , corresponding to the Frobenius condition, takes value in the component of the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$, which is a \mathfrak{g}_0^{SS} irreducible module with the lowest weight vector

$$e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \otimes e_{-\alpha_3}.$$

This has homogeneity 2 with respect to the grading element $Z \in \mathfrak{g}_0$, so the geometry is regular. Thus, the corresponding W component of κ_H^2 can be computed through the technique of parabolic geometry, similar as we did with the 5D subconformal case in this paper.

The zero-curvature condition $W = 0$ of the subconformal geometry is, however, not sufficient to identify the twistor space with the parabolic geometry of type (C_3, P_3) unless the initial parabolic geometry of type (C_3, P_2) is flat. Indeed, there is another harmonic curvature component κ_H^1 of homogeneity 1, which must vanish in order for parabolic geometry to descent. This is yet another analog of the classical conformal geometry in 4D, in which case the twistor space is never projective unless the conformal structure is flat.

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