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SOME EXACT SEQUENCES ASSOCIATED WITH ADJUNCTIONS IN BICATEGORIES. APPLICATIONS

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Abstract. We prove that the classical result asserting that the relative Picard group of a faithfully flat extension of commutative rings is isomorphic to the first Amitsur cohomology group still is valid in the realm of symmetric monoidal categories. To this end, we build some group exact sequences from an adjunction in a bicategory, which are of independent interest. As a particular byproduct of the evolving theory, we prove a version of Hilbert's theorem 90 for cocommutative coalgebra coextensions (=surjective homomorphisms).

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Introduction

Given a commutative ring R, the classical Chase–Rosenberg seven term exact sequence [6] gives a cohomological interpretation of the Brauer group of Azumaya R-algebras split by a finitely generated faithful projective commutative R-algebra S. This sequence leads to the Chase–Harrison–Rosenberg seven term exact sequence [5] for a Galois extension of commutative rings, that generalizes Hilbert's theorem 90 and the classical cohomological description of the relative Brauer group. Then Chase–Rosenberg exact sequence relates the Picard and (relative) Brauer groups, as well as some low degree Amitsur cohomology groups with coefficients in the unit and Picard functors. Villamayor and Zelinsky showed [30] that, by replacing the Brauer group by a suitable group of analogs of descent data, a generalization of the Chase–Rosenberg sequence can be built for any commutative ring extension.

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Indeed, this construction allows one to connect the aforementioned Amitsur cohomology groups of the extension by means of the Villamayor–Zelinsky infinite exact sequence [30, Theorem 4.14]. These results received a great deal of attention and various generalizations of the sequence and its parts have been considered in different contexts (see, for example, [29], [12], [13], [14], [31]). We are interested in such a generalization given by Masuoka in [23]. For any extension of (non-necessarily commutative) rings, Masuoka built an exact sequence of groups involving the automorphism and Picard groups of the rings and the group of invertible subbimodules of the extension. In this paper we give categorical generalizations of the main results of [23].

By means of abstractions to general categorical contexts we gain a deeper understanding of these results and find precise expressions for the analogies existing between various exact sequences. On the other hand, once the general theory is well developed, we can come down to the concrete situations and make use of that general theory. As a relevant example, we prove that the classical result asserting that the relative Picard group of a faithfully flat extension of commutative rings is isomorphic to the first Amitsur cohomology group is still valid in the realm of symmetric monoidal categories. To this end, we prove that for a commutative algebra $\mathbf{A} = (A, m, e)$ in a symmetric monoidal category $\mathcal{V}$ satisfying some technical conditions, there is an exact sequence of groups

(1)

$$0 \to \mathbf{Aut}_{\mathcal{V}}(I) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\kappa_A} \mathbf{Aut}_{\mathbf{A}-\mathsf{cor}}(A \otimes A) \xrightarrow{o_A} \mathbf{Pic}^c(\mathbf{I}) \xrightarrow{\mathbf{Pic}^c(e)} \mathbf{Pic}^c(\mathbf{A}),$$

whenever the functor $A \otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is comonadic. Details on the group homomorphisms involved are to be found in Subsection 4.2. The sequence (1) is deduced from an exact sequence

$$(2) \qquad 0 \to \mathbf{Aut}_{\mathcal{V}}(I) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\overline{\mathcal{D}}_A} \mathfrak{I}_A^{\mathbf{I}} \xrightarrow{\Omega_A} \mathbf{Pic}^c(\mathbf{I}) \xrightarrow{\mathbf{Pic}^c(e)} \mathbf{Pic}^c(\mathbf{A}),$$

defined for any commutative algebra $\mathbf{A}$ over $\mathcal{V}$. The sequence (2) turns out to be a generalization of the five term crank-shaped sequence at degree zero from [30, Section 4]. A sequence similar to (1) can be derived from an exact sequence of categorical groups stemming from the monoidal functor $A \otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ (see [31, Corollary 6.1]). For our purposes we prefer to build (2), and, hence, (1) with the aid of an exact sequence of groups associated, under mild conditions, to any adjunction in a bicategory (see Theorem 3.9). This helps to identify the comonodacity as the essential hypothesis generalizing the traditional faithfully flat condition on extensions of commutative rings. For instance, to get the isomorphism between the first Amitsur cohomology group and the relative Picard group, the middle points of the exact sequences (1) and (2) need to be connected by a precise isomorphism, and this requires the comonadicity of the functor $A \otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$. This kind of isomorphism between the group of invertible sub-bimodules and the automorphism group of a suitable coring was proved in [23, Theorem 2.10] for noncommutative ring extensions, under faithfully flatness or separability-like conditions. In Theorem 3.12 we get, with the help of the previous analysis developed in [15], a very general version of [23, Theorem 2.10] that works for any adjunction in a bicategory whenever the functor encoded by the left adjoint 1-cell is comonadic.

The exact sequence displayed in Theorem 3.9 generalizes some useful exact sequences associated to a noncommutative ring extension, used in [19, 25] in the unital case and in [11] in the realm of ring with local units, to take the first steps to derive adequate noncommutative versions of Chase–Harrison–Rosenberg's seven term exact sequence [5, Corollary 5.5]. Thus, our results could be of independent interest for extending to wider contexts the aforementioned seven term sequence. More specifically, we are thinking of a possible extension of [25, Theorem 2.12] and [11, Theorem 4.12] to the realm of firm rings. In this direction, we illustrate how to obtain some preliminary exact sequences (see Examples 3.10 and 3.18) where it is well understood that a major part of the needed work, for instance, the construction of a group of equivalence classes of generalized crossed products similar to that of [25] and [11], is left for future inquiries.

We specialize our general theory to the case of the bicategory of bimodules over a (non-necessarily symmetric) monoidal category $\mathcal{V}$ (see Theorem 4.1). We get in particular an exact sequence of groups associated to a monomorphic homomorphism of $\mathcal{V}$-algebras (see Example 4.2).

Every comonadic homomorphism of commutative $\mathcal{V}$-algebras $\iota : \mathbf{A} \to \mathbf{B}$, where $\mathcal{V}$ is a symmetric monoidal category fulfilling some minimal technical requirements, leads to a homomorphism of abelian groups $\mathbf{Pic}^c(\iota) : \mathbf{Pic}^c(\mathbf{A}) \to \mathbf{Pic}^c(\mathbf{B})$. Our theory applies to obtain (Theorem 4.13) that if the change-of-base functor associated to ι is comonadic, then $\mathbf{Ker}(\mathbf{Pic}^c(\iota))$ is isomorphic to the first Amitsur cohomology group $\mathcal{H}^1(\iota\, , \underline{\mathbf{Aut}}_A^{\mathfrak{Alg}})$ of ι with coefficients in the functor $\underline{\mathbf{Aut}}_A^{\mathfrak{Alg}}$ (which is a generalization of the usual units functor; see Lemma 4.11). The problem is easily reduced to prove that, in (1), the cokernel of κ_A is isomorphic to $\mathcal{H}^1(e\, , \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}})$, whenever $A \otimes -$ is a comonadic functor. Our proof involves some classical results, namely a version of the Bénabou–Roubaud–Beck theorem identifying the category of descent data with an Eilenberg–Moore category (Theorem A.3), and Grothendiek's isomorphism between the Amitsur first cohomology pointed set and the set of descent data of an effective descent morphism (Proposition A.4). A brief account of the required classical theory is given in the Appendix.

In the final section, we apply our general theory to the bicategory of bicomodules. In particular, a version of Hilbert's theorem 90 for cocommutative coalgebra coextensions (=surjective homomorphisms) (Theorem 4.18) is obtained.

1. Preliminaries

In this section, we list some categorical notions and basic constructions that will be needed. Our basic references on categories are [1, 4, 21].

1.1. Subobjects and quotient objects. Let a be an object of a category $\mathcal{A}$. Preorder monomorphisms with range a by setting $j \leq i$ if j is of the form $j = ik$; the equivalence classes for the relation

$$\text{“}j \leq i \ \text{ and } \ i \leq j\text{”}$$

are called *subobjects* of a. We write $\mathbf{Sub}_{\mathcal{A}}(a)$ for the the class of all subobjects of a. We often identify a subobject with a representative monomorphism, and we call the subobject regular, etc., if the monomorphism i is regular, etc.

Dually, one has the collection $\mathbf{Quot}_{\mathcal{A}}(a) = \mathbf{Sub}_{\mathcal{A}^{op}}(a)$ of isomorphism classes of epimorphisms with domain a ($\mathcal{A}^{op}$ denotes the opposite category of $\mathcal{A}$). We shall call an element of $\mathbf{Quot}_{\mathcal{A}}(a)$ a *quotient object* of a. Note that for epimorphisms with domain a we write $j \leq i$ if j is of the form $j = ki$.

1.2. Images and coimages. Recall that a category admits *images* if any morphism f can be written as $f = ip$ with i monomorphic and p regular epimorphic.

The subobject $[i]$ of the codomain of f is called the *image* of f. Dually, a category is said to admit *coimages* if any morphism f can be written as $f = ip$ with p epimorphic and i regular monomorphic. The quotient object $[p]$ of the domain of f is called the *coimage* of f. We say that a monoidal category admits (co)images if its underlying ordinary category does so.

1.3. Subobjects and quotient objects of (co)algebras. Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a fixed monoidal category with underlying ordinary category $\mathcal{V}$, tensor product $\otimes$, and monoidal unit I. Recall that an *algebra* in $\mathcal{V}$ (or $\mathcal{V}$-algebra) consists of an object A of $\mathcal{V}$ endowed with a multiplication $m_A : A \otimes A \to A$ and a unit morphism $e_A : I \to A$, subject to the usual associative and identity conditions. These algebras are the objects of a category $\mathtt{Alg}(\mathcal{V})$ with the obvious morphisms. Dually, one has the notion of $\mathcal{V}$-*coalgebra*; the corresponding category of $\mathcal{V}$-coalgebras is denoted by $\mathtt{Coalg}(\mathcal{V})$.

Given a $\mathcal{V}$-algebra $\mathbf{A} = (A, m_A, e_A)$, we write $\mathfrak{I}^l_{\mathcal{V}}(\mathbf{A})$ for the subclass of $\mathbf{Sub}_{\mathcal{V}}(A)$ consisting of those elements $[(J, i_J : J \to A)]$ for which the composite

$$\xi^l_{i_J} : A \otimes J \xrightarrow{A \otimes i_J} A \otimes A \xrightarrow{m_A} A$$

is an isomorphism. Symmetrically, we let $\mathfrak{I}^r_{\mathcal{V}}(\mathbf{A})$ denote the subclass of $\mathbf{Sub}_{\mathcal{V}}(A)$ consisting of those elements $[(J, i_J : J \to A)]$, for which the composite

$$\xi^r_{i_J} : J \otimes A \xrightarrow{i_J \otimes A} A \otimes A \xrightarrow{m_A} A$$

is an isomorphism.

Dually, for a $\mathcal{V}$-coalgebra $\mathbf{C} = (C, \delta, \varepsilon)$, we write $\mathcal{Q}^l_{\mathcal{V}}(\mathbf{C})$ (resp., $\mathcal{Q}^r_{\mathcal{V}}(\mathbf{C})$) for the subclass of $\mathbf{Quot}_{\mathcal{V}}(C)$ consisting of those elements $[(P, \pi_P : C \to P)]$ for which the composite

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{C \otimes \pi_P} C \otimes P$$

(resp.,

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{\pi_P \otimes C} P \otimes C)$$

is an isomorphism.

1.4. Adjunctions in bicategories. We begin by recalling from [2] that a bicategory $\mathbb{B}$ consists of :

- a class $\mathrm{Ob}(\mathbb{B})$ of objects, or 0-cells;
- a family $\mathbb{B}(A, B)$, for all $A, B \in \mathrm{Ob}(\mathbb{B})$, of hom-categories, whose objects and morphisms are respectively called 1-cells and 2-cells;
- a (horizontal) composition operation, given by a family of functors

  $$\mathbb{B}(B, C) \times \mathbb{B}(A, B) \to \mathbb{B}(A, C)$$

  whose action on a pair $(g, f) \in \mathbb{B}(B, C) \times \mathbb{B}(A, B)$ is written $g \circ f$;
- identities, given by 1-cells $1_A \in \mathbb{B}(A, A)$, for $A \in \mathrm{Ob}(\mathbb{B})$;
- natural isomorphisms

$$\alpha_{h,g,f} : (h \circ g) \circ f \simeq h \circ (g \circ f), l_f : 1_A \circ f \simeq f \text{ and } r_f : f \circ 1_A \simeq f,$$

subject to two coherence axioms (see [2]).

Whenever the context is clear, we write $[A, B]$ instead of $\mathbb{B}(A, B)$.

We review the concept of adjunction in an arbitrary bicategory along with some of the general theory needed later on.

Fix a bicategory $\mathbb{B}$. An adjunction $(\eta, \varepsilon : f \dashv f^* : B \to A)$ in $\mathbb{B}$ consists of objects A and B, 1-cells $f : A \to B$ and $f^* : B \to A$, and 2-cells $\eta : 1_A \to f^* \circ f$, called the *unit*, and $\varepsilon : f \circ f^* \to 1_B$, called the *counit*, such that the following diagrams commute in $[A, B]$ and $[B, A]$, respectively:

(3)
$$\begin{tikzcd}
f \circ 1_A \arrow[r, "f\circ\eta"] \arrow[d, "r_f"'] & f \circ (f^* \circ f) \arrow[r, "\alpha^{-1}_{f,f^*,f}"] & (f \circ f^*) \circ f \arrow[d, "\varepsilon\circ f"] \\
f \arrow[rr, "l_f^{-1}"'] & & 1_B \circ f
\end{tikzcd}$$

and

(4)
$$\begin{tikzcd}
1_A \circ f^* \arrow[r, "\eta\circ f^*"] \arrow[d, "l_{f^*}"'] & (f^* \circ f) \circ f^* \arrow[r, "\alpha_{f^*,f,f^*}"] & f^* \circ (f \circ f^*) \arrow[d, "f^*\circ\varepsilon"] \\
f^* \arrow[rr, "r_{f^*}^{-1}"'] & & f^* \circ 1_B
\end{tikzcd}$$

Let $\eta, \varepsilon : f \dashv f^* : B \to A$ be an adjunction in $\mathbb{B}$, and let X be an arbitrary 0-cell of $\mathbb{B}$. Then the functor

$$[X, f] = f \circ - : [X, A] \to [X, B]$$

admits as a right adjoint the functor

$$[X, f^*] = f^* \circ - : [X, B] \to [X, A] .$$

The unit η^X and counit ε^X of this adjunction are given by the formulas

$$\eta^X_g : g \xrightarrow{l_g^{-1}} 1_A \circ g \xrightarrow{\eta\circ g} (f^* \circ f) \circ g \xrightarrow{\alpha_{f^*,f,g}} f^* \circ (f \circ g), \text{ for all } g \in [X, A]$$

and

$$\varepsilon^X_h : f \circ (f^* \circ h) \xrightarrow{\alpha^{-1}_{f,f^*,h}} (f \circ f^*) \circ h \xrightarrow{\varepsilon\circ h} 1_B \circ h \xrightarrow{l_h} h, \text{ for all } h \in [X, B].$$

The situation may be pictured as

$$\begin{tikzcd}
{[X,A]} \arrow[r, bend left, "{[X,f]=f\circ -}"] \arrow[r, phantom, "\perp"] & {[X,B]} \arrow[l, bend left, "{[X,f^*]=f^*\circ -}"]
\end{tikzcd}$$

Definition 1.5. A 1-cell $f : A \to B$ in $\mathbb{B}$ is called *invertible* if there exist a 1-cell $g : B \to A$ and isomorphisms $g \circ f \simeq 1_A$ and $f \circ g \simeq 1_B$. The 1-cell g is called *a pseudoinverse* of f.

Recall that an *adjoint equivalence* in $\mathbb{B}$ is an adjunction in which both the unit and counit are isomorphisms, and that any equivalence is part of an adjoint equivalence.

Remark 1.6. If a 1-cell $h : A \to A$ is invertible, then, for any object $X \in \mathbb{B}$, both functors $[X, h] = h \circ - : [X, A] \to [X, A]$ and $[h, X] = - \circ h : [A, X] \to [A, X]$ are equivalences of categories, and thus they preserve existing limits and colimits. In particular, they preserve monomorphisms and epimorphisms.

The following is an example of a bicategory to which some of our general results will be applied.

Example 1.7. Firm bimodules. Let S be a ring, which is not assumed to be unital. A right S-module M is said to be *firm* [27] if the map $M \otimes_S S \to M$ sending $m \otimes_S s$ to ms is an isomorphism. Thus, the ring S is said to be firm if the multiplication map $S \otimes_S S \to S$ is an isomorphism. Firm left modules and firm bimodules are defined analogously. We denote by **Firm** the bicategory whose 0-cells are firm rings, the 1-cells are firm bimodules, and the 2-cells are homomorphisms of firm bimodules. The horizontal composition in **Firm** is given by the tensor product of bimodules. Given a homomorphism $\varphi : R \to S$, where R and S are firm rings, we may consider the bimodules ${}_RS_S$ and ${}_SS_R$ in the usual way. We say that φ is a *homomorphism of firm rings* if ${}_RS_S$ and ${}_SS_R$ are firm bimodules. In this case, we have 1-cells ${}_SS_R : R \to S$ and ${}_RS_S : S \to R$ which form an adjunction ${}_SS_R \dashv {}_RS_S$ in **Firm**. Its counit is the multiplication map $\mu : S \otimes_R S \to S$, while the unit is given by the composite $R \xrightarrow{\varphi} S \xrightarrow{\nu} S \otimes_S S$, where ν denotes the inverse of the multiplication map $S \otimes_S S \xrightarrow{\cong} S$.

1.8. Mates. Recall from [20] that for adjunctions $(\eta, \varepsilon : f \dashv g : B \to A)$ and $(\eta', \varepsilon' : f' \dashv g' : B \to A)$ in $\mathbb{B}$, there is a bijection between 2-cells

$$\sigma : f \to f' \text{ and } \overline{\sigma} : g' \to g,$$

where $\overline{\sigma}$ is obtained as the composite

$$g' \overset{l_{g'}^{-1}}{\simeq} 1_A \circ g' \xrightarrow{\eta \circ g'} (g \circ f) \circ g' \xrightarrow{(g \circ \sigma) \circ g'} (g \circ f') \circ g' \overset{\alpha}{\simeq} g \circ (f' \circ g') \xrightarrow{g \circ \varepsilon'} g \circ 1_B \overset{r_g}{\simeq} g$$

and σ is given as the composite

$$f \overset{r_f^{-1}}{\simeq} f \circ 1_A \xrightarrow{f \circ \eta'} f \circ (g' \circ f') \xrightarrow{f \circ (\overline{\sigma} \circ f')} f \circ (g \circ f') \overset{\alpha^{-1}}{\simeq} (f \circ g) \circ f' \xrightarrow{\varepsilon \circ f'} 1_B \circ f' \overset{l_{f'}}{\simeq} f'.$$

In this situation, σ and $\overline{\sigma}$ are called *mates* under the given adjunctions, and this is denoted by $\sigma \dashv \overline{\sigma}$.

Lemma 1.9. *If $\sigma \dashv \overline{\sigma}$ under adjunctions $(f \dashv g : B \to A)$ and $(f' \dashv g' : B \to A)$, then σ is an isomorphism if and only if $\overline{\sigma}$ is an isomorphism.*

2. Invertible cells associated to an adjunction

Let A be an object of a bicategory $\mathbb{B}$. We call a (co)algebra in the monoidal category $[A, A]$ an A-*(co)ring* and write A-$\mathtt{ring} = \mathtt{Alg}([A, A])$ (resp., A-$\mathtt{cor} = \mathtt{Coalg}([A, A])$) for the category of A-(co)rings.

Any 1-cell with a right adjoint generates a ring as well as a coring as follows. If $\eta_f, \varepsilon_f : f \dashv f^* : B \to A$ is an adjunction in $\mathbb{B}$, then the triple

$$\mathcal{S}_f = (f^* \circ f, m_f, \eta_f), \tag{5}$$

where

(6)

$$m_f = (r_{f^*} \circ f) \cdot ((f^* \circ \varepsilon_f) \circ f) \cdot (\alpha_{f^*, f, f^*} \circ f) \cdot (\alpha_{f^* \circ f, f^*, f})^{-1} : (f^* \circ f) \circ (f^* \circ f) \to f^* \circ f,$$

is an A-ring, while the triple

$$\mathfrak{C}_f = (f \circ f^*, \delta_f, \varepsilon_f), \tag{7}$$

where δ_f is the composite

$$(\alpha_{f^*\circ f,f^*,f})\cdot(\alpha^{-1}_{f,f^*,f}\circ f^*)\cdot((f\circ\eta_f)\circ f^*)\cdot(r_f^{-1}\circ f^*):f\circ f^*\to(f\circ f^*)\circ(f\circ f^*),$$

is a B-coring.

Since $\mathcal{S}_f$ is an algebra in the monoidal category $[A,A]$, one has the sets $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ and $\mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$ described in Subsection 1.3. Recall from [15, Remark 4.2] that for any monomorphic 2-cell $i_h: h\to f^*\circ f$,

$$\xi^l_{i_h}=(f^*\circ\xi_{i_h})\cdot\alpha_{f^*,\,f,\,h}$$

and

$$\xi^r_{i_h}=(\xi^*_{i_h}\circ f)\cdot\alpha^{-1}_{h,\,f^*,\,f}$$

where ξ_{i_h} and $\xi^*_{i_h}$ are the composites

$$\xi_{i_h}:f\circ h\xrightarrow{f\circ i_h}f\circ(f^*\circ f)\xrightarrow{\alpha^{-1}_{f,f^*,f}}(f\circ f^*)\circ f\xrightarrow{\varepsilon_f\circ f}1_B\circ f\xrightarrow{l_f}f$$

and

$$\xi^*_{i_h}:h\circ f^*\xrightarrow{i_h\circ f^*}(f^*\circ f)\circ f^*\xrightarrow{\alpha_{f^*,f,f^*}}f^*\circ(f\circ f^*)\xrightarrow{f^*\circ\varepsilon_f}f^*\circ 1_B\xrightarrow{r_{f^*}}f^*,$$

respectively.

We write $\mathfrak{I}^{A,\,l}_f$ (resp., $\mathfrak{I}^{A,r}_f$) for the subset of $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ (resp., $\mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$) determined by those subobjects $[(h,i_h)]$ with h invertible.

Proposition 2.1. *Let $\eta_f,\varepsilon_f: f\dashv f^*: B\to A$ be an adjunction in $\mathbb{B}$ such that η_f is monomorphic in $[A,A]$, and let $h:A\to A$ be an invertible 1-cell. If there is an isomorphism $\sigma: f\circ h\to f$ in $[A,B]$, then $[(h,i_h)]\in\mathfrak{I}^{A,\,l}_f$, where i_h is the composite $h\xrightarrow{\eta_f\circ h}f^*\circ f\circ h\xrightarrow{f^*\circ\sigma}f^*\circ f$.*[1]

Proof. Suppose that h is invertible and that there is an isomorphism $\sigma: f\circ h\to f$ in $[A,B]$. Since η_f is assumed to be monomorphic in $[A,A]$, it follows from Remark 1.6 that $h\xrightarrow{\eta_f\circ h}f^*\circ f\circ h$ is monomorphic in $[A,A]$. Then, since σ is an isomorphism, i_h must be a monomorphism too. Now, since the functor $f^*\circ-:[A,B]\to[A,A]$ is right adjoint to the functor $f\circ-:[A,A]\to[A,B]$ with $\eta_f\circ-$ as unit and $\varepsilon_f\circ-$ as counit, it follows that $\sigma=(\varepsilon_f\circ f)\cdot(f\circ i_h)=\xi_{i_h}$. Therefore, $\xi^l_{i_h}=f^*\circ\xi_{i_h}=f^*\circ\sigma$ is an isomorphism too, and hence $[(h,i_h)]\in\mathfrak{I}^l_{[A,\,A]}(\mathcal{S}_f)$. □

Proposition 2.2. *In the situation of Proposition 2.1, suppose that h^* is a pseudo-inverse of h. Then there is a monomorphic 2-cell $i_{h^*}: h^*\to f^*\circ f$ such that $[(h^*,i_{h^*})]\in\mathfrak{I}^{A,r}_f$.*

Proof. Composing the adjunction $f\dashv f^*$ with $h\dashv h^*$ yields an adjunction $f\circ h\dashv h^*\circ f^*$. Since $f\circ h\simeq f$ and since adjoints are unique up to a unique isomorphism, one has an isomorphism $\tau: h^*\circ f^*\simeq f^*$. Now, if we take i_{h^*} to be the composite $i_{h^*}: h^*\xrightarrow{h^*\circ\eta_f}h^*\circ f^*\circ f\xrightarrow{\tau\circ f}f^*\circ f$, then the result is proved in exactly the same way as Proposition 2.1, but this time using the adjunction

$$-\circ f^*\dashv-\circ f:[B,A]\to[A,A].$$ □

[1]For simplicity of exposition we sometimes treat $\mathbb{B}$ as a 2-category which is justified by the coherence theorem (see [22]) asserting that every bicategory is biequivalent to a 2-category. Consequently, we sometimes omit brackets in the horizontal compositions and suppress the associativity constraints α and the unitality constraints l and r.

Proposition 2.3. *Let $\eta, \varepsilon : f \dashv f^* : B \to A$ be an adjunction, and let $(\eta_h, \varepsilon_h : h \dashv h^* : A \to A)$ be an adjoint equivalence in $\mathbb{B}$. Then for any 2-cell $i_h : h \to f^*\circ f$, the following are equivalent:*

(i) $\xi_{i_h} : f\circ h \to f$ *is an isomorphism;*
(ii) $\xi^l_{i_h} : (f^*\circ f)\circ h \to f^*\circ f$ *is an isomorphism;*
(iii) $\xi^*_{i_h} : h\circ f^* \to f^*$ *is an isomorphism;*
(iv) $\xi^r_{i_h} : h\circ(f^*\circ f) \to f^*\circ f$ *is an isomorphism.*

Moreover, i_h is monomorphism in $[A, A]$ provided any (and hence all) of the above conditions hold.

Proof. Since (i) is equivalent to (ii) and (iii) is equivalent to (iv) by [15, Remark 4.2] and its dual, we have only to show that (i) and (iii) are equivalent.

Note first that composing the adjunction $(\eta_f, \varepsilon_f : f \dashv f^*)$ with $(\eta_h, \varepsilon_h : h \dashv h^*)$ yields an adjunction $(\overline{\eta}, \overline{\varepsilon} : f\circ h \dashv h^*\circ f^*)$, where $\overline{\eta}$ and $\overline{\varepsilon}$ are the composites

$$1_A \xrightarrow{\eta_h} h^*\circ h \xrightarrow{h^*\circ\eta_f\circ h} h^*\circ f^*\circ f\circ h$$

and

$$f\circ h\circ h^*\circ f^* \xrightarrow{f\circ\varepsilon_h\circ f^*} f\circ f^* \xrightarrow{\varepsilon_f} 1_B,$$

respectively.

Consider now the composite

$$\overline{\xi}_{i_h} : f^* \xrightarrow{\overline{\eta}\circ f^*} h^*\circ f^*\circ f\circ h\circ f^* \xrightarrow{h^*\circ f^*\circ\xi_{i_h}\circ f^*} h^*\circ f^*\circ f\circ f^* \xrightarrow{h^*\circ f^*\circ\varepsilon_f} h^*\circ f^*,$$

which is the mate of ξ_{i_h} under the adjunctions $(f\circ h \dashv h^*\circ f^*)$ and $(f \dashv f^*)$. A straightforward calculation, using the expression for $\overline{\eta}$ and ξ_{i_h}, shows that $\overline{\xi}_{i_h}$ is the composite

$$f^* \xrightarrow{\eta_h\circ f^*} h^*\circ h\circ f^* \xrightarrow{h^*\circ i_h\circ f^*} h^*\circ f^*\circ f\circ f^* \xrightarrow{h^*\circ f^*\circ\varepsilon_f} h^*\circ f^*,$$

and therefore

$$\overline{\xi}_{i_h} = (h^*\circ\xi^*_{i_h})\cdot(\eta_h\circ f^*),$$

implying—since both h^* and η_h are invertible 1-cells—that $\overline{\xi}_{i_h}$ is an isomorphism iff $\xi^*_{i_h}$ is an isomorphism. In light of Lemma 1.9 one now concludes that (i) and (iii) are equivalent.

Finally, each of the conditions (i)–(iv) implies that $\xi^l_{i_h}$ is an isomorphism, and then i_h is a monomorphism by Proposition 2.1. This completes the proof. □

Proposition 2.4. $\mathfrak{I}^{A,\,l}_f = \mathfrak{I}^{A,\,r}_f$.

Proof. By symmetry, it suffices to prove the inclusion $\mathfrak{I}^{A,\,l}_f \subseteq \mathfrak{I}^{A,\,r}_f$. To this end consider an arbitrary element $[(h, i_h)] \in \mathfrak{I}^{A,\,l}_f$. Since h is an invertible 1-cell, we need only show that $[(h, i_h)] \in \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$. Since $[(h, i_h)] \in \mathfrak{I}^{A,\,l}_f \subseteq \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$, the 2-cell $\xi^l_{i_h} : (f^*\circ f)\circ h \to f^*\circ f$ is an isomorphism, and then $\xi_{i_h} : f\circ h \to h$ is also an isomorphism by [15, Remark 4.2]. Now applying Proposition 2.3 gives that both $\xi^*_{i_h} : h\circ f^* \to f^*$ and $\xi^r_{i_h} = \xi^*_{i_h}\circ f : h\circ(f^*\circ f) \to f^*\circ f$ are isomorphisms. Thus, $[(h, i_h)] \in \mathfrak{I}^r_{[\mathcal{A},\mathcal{A}]}(\mathcal{S}_f)$, and hence $[(h, i_h)] \in \mathfrak{I}^{A,\,r}_f$. □

Definition 2.5. We write $\mathfrak{I}^A_f$ to denote either $\mathfrak{I}^{A,\,l}_f$ or $\mathfrak{I}^{A,\,r}_f$.

3. Exact sequences of groups related to adjunctions in bicategories

Fix an adjunction $\eta_f, \varepsilon_f : f \dashv f^* : B \to A$ in $\mathbb{B}$. In this section we suppose, with the exception of Subsection 3.4, that $\mathbb{B}$ is a bicategory such that each hom-category admits finite limits and images[2] and that the 2-cell $\eta_f : 1_A \to f^* \circ f$ is a monomorphism in $[A, A]$. Then, according to [15, Proposition 3.2], $\mathbf{Sub}_{[A,A]}(f^* \circ f)$ can be endowed with the structure of a monoid by defining the product $[(J_1, i_{J_1})] \cdot [(J_2, i_{J_2})]$ of any two elements of $\mathbf{Sub}_{[A,A]}(f^* \circ f)$ to be the image of the composite $J_1 \circ J_2 \xrightarrow{i_{J_1} \circ i_{J_2}} S_f \circ S_f \xrightarrow{m_f} S_f$. The unit of this monoid is $[(1_A, \eta_f : 1_A \to f^* \circ f)]$. Throughout this paper, when considering $\mathbf{Sub}_{[A,A]}(f^* \circ f)$ as a monoid, we always mean this monoid structure. Note that since the image of any monomorphic 1-cell is (isomorphic to) itself, it follows that if $[(J_1, i_{J_1})], [(J_2, i_{J_2})] \in \mathbf{Sub}_{[A,A]}(f^* \circ f)$ are such that the composite $m_f \cdot (i_{J_1} \circ i_{J_2})$ is monomorphic, then

$$[(J_1, i_{J_1})] \cdot [(J_2, i_{J_2})] = [(J_1 \circ J_2, m_f \cdot (i_{J_1} \circ i_{J_2}))].$$

Our motivation for defining such a monoid structure on $\mathbf{Sub}_{[A,A]}(f^* \circ f)$ came initially from the fact that for any (noncommutative) ring extension $R \subseteq S$, the set of all R-sub-bimodules of S has a monoid structure given as follows. For arbitrary R-sub-bimodules $I, J \subseteq R$, one defines the product IJ to be the image of the multiplication map

$$I \otimes_R J \to S, \quad s \otimes_R t \to st.$$

With respect to this product, the R-sub-bimodules of S form a monoid with unit R.

3.1. Automorphisms and invertible subobjects.

One can easily verify that the assignment of taking a 2-cell $s : 1_A \to 1_A$ to the composite

$$f^* \xrightarrow{l_{f^*}^{-1}} 1_A \circ f^* \xrightarrow{s \circ f^*} 1_A \circ f^* \xrightarrow{l_{f^*}} f^*$$

yields a monoid morphism

$$\varpi : [A, A](1_A, 1_A) \to [B, A](f^*, f^*),$$

which gives, by restriction, a homomorphism of groups

$$\varpi_0 : \mathbf{Aut}_{[A,\,A]}(1_A) \to \mathbf{Aut}_{[B,\,A]}(f^*)$$

between the groups of automorphisms of the objects 1_A and f^*, respectively.

Proposition 3.1. *The map ϖ_0 is a monomorphism of groups.*

Proof. If $s \in \mathbf{Aut}_{[A,\,A]}(1_A)$ is such that $\varpi_0(s) = 1_{f^*}$, then $1_{f^*} = l_{f^*} \cdot (s \circ f^*) \cdot l_{f^*}^{-1}$, and hence $l_{f^*} = l_{f^*} \cdot (s \circ f^*)$. But since $l_{f^*} = l_{f^*} \cdot (1_{1_A} \circ f^*)$ and since l_{f^*} is invertible, it follows that $1_{1_A} \circ f^* = s \circ f^*$, and hence $1_{1_A} \circ (f^* \circ f) = s \circ (f^* \circ f)$. Direct calculation then shows that $\eta_f \cdot 1_{1_A} = \eta_f \cdot s$. Now, since η_f is assumed to be monomorphic in $[A, A]$, it follows that $1_{1_A} = s$. Thus, ϖ_0 is a monomorphism of groups. $\square$

[2] Indeed, we need this assumption only for the hom-category $[A, A]$.

For any $\lambda \in \mathbf{Aut}_{[B,\,A]}(f^*)$, form the pullback in $[A, A]$

$$\begin{tikzcd} f_\lambda \arrow[r, "i_\lambda"] \arrow[d, "p_\lambda"'] & f^*\circ f \arrow[d, "\lambda\circ f"] \\ 1_A \arrow[r, "\eta_f"'] & f^*\circ f \end{tikzcd} \tag{8}$$

Since, by hypothesis, η_f is a monomorphism in $[A,\,A]$, so too is i_λ, and thus (f_λ, i_λ) represents an element of $\mathrm{Sub}_{[A,\,A]}(f\circ f^*)$, implying—since pullbacks are unique up to isomorphism—that the assignment $\lambda \longmapsto [(f_\lambda, i_\lambda)]$ yields a map $\mathcal{D}_f : \mathbf{Aut}_{[B,\,A]}(f^*) \to \mathbf{Sub}_{[A,\,A]}(f^*\circ f)$.

Now consider the diagram

$$\begin{tikzcd}[column sep=small] (f^*\circ f)\circ(f^*\circ f) \arrow[r, "\alpha^{-1}"] \arrow[d, "(\lambda\circ f)\circ(f^*\circ f)"] \arrow[dr, phantom, "(1)"] & ((f^*\circ f)\circ f^*)\circ f \arrow[r, "\alpha\circ f"] \arrow[d, "((\lambda\circ f)\circ f^*)\circ f"] \arrow[dr, phantom, "(2)"] & (f^*\circ(f\circ f^*))\circ f \arrow[r, "(f^*\circ\varepsilon_f)\circ f"] \arrow[d, "(\lambda\circ(f\circ f^*))\circ f"] \arrow[dr, phantom, "(3)"] & (f^*\circ 1_A)\circ f \arrow[r, "r_{f^*}\circ f"] \arrow[d, "(\lambda\circ 1_A)\circ f"] \arrow[dr, phantom, "(4)"] & f^*\circ f \arrow[d, "\lambda\circ f"] \\ (f^*\circ f)\circ(f^*\circ f) \arrow[r, "\alpha^{-1}"'] & ((f^*\circ f)\circ f^*)\circ f \arrow[r, "\alpha\circ f"'] & (f^*\circ(f\circ f^*))\circ f \arrow[r, "(f^*\circ\varepsilon_f)\circ f"'] & (f^*\circ 1_A)\circ f \arrow[r, "r_{f^*}\circ f"'] & f^*\circ f \end{tikzcd}$$

in which rectangles (1) and (2) commute by naturality of α, rectangle (3) commutes by naturality of the composition, while rectangle (4) commutes by naturality of r. Thus the outer rectangle of the diagram is also commutative, and now using that, by (6),

$$m_f = (r_{f^*}\circ f)\cdot((f^*\circ\varepsilon_f)\circ f^*)\cdot(\alpha_{f^*,f,f^*}\circ f)\cdot\alpha^{-1}_{f^*\circ f,f^*,f}$$

we get

$$(\lambda\circ f)\cdot m_f = m_f\cdot((\lambda\circ f)\circ(f^*\circ f)). \tag{9}$$

It then follows from (8) that

$$\begin{aligned} (\lambda\circ f)\cdot m_f\cdot(i_\lambda\circ(f^*\circ f)) &= m_f\cdot((\lambda\circ f)\circ(f^*\circ f))\cdot(i_\lambda\circ(f^*\circ f)) \\ &= m_f\cdot(\eta_f\circ(f^*\circ f))\cdot(p_\lambda\circ(f^*\circ f)) \\ &= l_{f^*\circ f}\cdot(p_\lambda\circ(f^*\circ f)). \end{aligned} \tag{10}$$

Since the morphisms $\lambda, l_{f^*\circ f}$, and p_λ are all isomorphisms, one concludes that the composite $m_f\cdot(i_\lambda\circ(f^*\circ f))$ is also an isomorphism and hence we have the following.

Proposition 3.2. *Under the hypotheses above, $\mathcal{D}_f(\lambda) \in \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$ for all $\lambda \in \mathbf{Aut}_{[B,\,A]}(f^*)$.*

We shall need the following easy lemma.

Lemma 3.3. *In an arbitrary category, a commutative diagram $gf = yx$ with g isomorphism is a pullback iff x is an isomorphism.*

Proposition 3.4. *The map $\mathcal{D}_f : \mathbf{Aut}_{[B,\,A]}(f^*) \to \mathbf{Sub}_{[A,\,A]}(f^*\circ f)$ is a homomorphism of monoids.*

Proof. Quite obviously, the diagram

$$\begin{tikzcd}
1_A \arrow[r, "\eta_f"] \arrow[d, "1_{1_A}"'] & f^*\circ f \arrow[d, "1_{f^*}\circ f = 1_{f^*\circ f}"] \\
1_A \arrow[r, "\eta_f"'] & f^*\circ f
\end{tikzcd}$$

is a pullback, showing that $\mathcal{D}_f(1_{f^*}) = [(1_A, \eta_f)] = 1$.

Next, for any two elements $\lambda, \lambda' \in \mathbf{Aut}_{[B,\,A]}(f^*)$, consider the diagram

$$\begin{tikzcd}
f_\lambda\circ f_{\lambda'} \arrow[rr, "f_\lambda\circ p_{\lambda'}"] \arrow[dd, "f_\lambda\circ i_{\lambda'}"'] \arrow[drr, phantom, "(I)"] & & f_\lambda\circ 1_A \arrow[rr, "p_\lambda\circ 1_A"] \arrow[d, "f_\lambda\circ\eta_f"'] \arrow[drr, phantom, "(IV)"] & & 1_A\circ 1_A \arrow[dd, "1_A\circ\eta_f"] \\
& & f_\lambda\circ(f^*\circ f) \arrow[drr, "p_\lambda\circ(f^*\circ f)"] \arrow[d, phantom, "(V)"] & & \\
f_\lambda\circ(f^*\circ f) \arrow[urr, "f_\lambda\circ(\lambda'\circ f)"] \arrow[rr, "p_\lambda\circ(f^*\circ f)"'] \arrow[d, "i_\lambda\circ(f^*\circ f)"'] \arrow[drr, phantom, "(II)"] & & 1_A\circ(f^*\circ f) \arrow[rr, "1_A\circ(\lambda'\circ f)"'] \arrow[d, "\eta_f\circ(f^*\circ f)"] \arrow[dd, bend left=60, "l_{f^*\circ f}"] \arrow[drr, phantom, "(VII)"] & & 1_A\circ(f^*\circ f) \arrow[dd, "l_{f^*\circ f}"] \\
(f^*\circ f)\circ(f^*\circ f) \arrow[rr, "(\lambda\circ f)\circ(f^*\circ f)"'] \arrow[d, "m_f"'] \arrow[drr, phantom, "(III)"] & & (f^*\circ f)\circ(f^*\circ f) \arrow[d, "m_f"] \arrow[r, phantom, "(VI)"] & & \\
f^*\circ f \arrow[rr, "\lambda\circ f"'] & & f^*\circ f \arrow[rr, "\lambda'\circ f"'] & & f^*\circ f
\end{tikzcd}$$

in which diagrams (I) and (II) commute by (8), diagram (III) commutes by (9), diagrams (IV), (V) commute by naturality of composition, diagram (VII) commutes by naturality of l, and diagram (VI) commutes since $\eta_f : 1_A \to f^*\circ f$ is the unit for the multiplication m_f. Thus the outer diagram, which by naturality of l can be rewritten as

$$\begin{tikzcd}
f_\lambda\circ f_{\lambda'} \arrow[r, "i_\lambda\circ i_{\lambda'}"] \arrow[d, "p_\lambda\circ p_{\lambda'}"'] & (f^*\circ f)\circ(f^*\circ f) \arrow[r, "m_f"] & f^*\circ f \arrow[dd, "(\lambda\lambda')\circ f"] \\
1_A\circ 1_A \arrow[d, "l_{1_A}=r_{1_A}"'] & & \\
1_A \arrow[rr, "\eta_f"'] & & f^*\circ f
\end{tikzcd}$$

commutes, and since all the 2-cells l_{1_A}, λ, λ', p_λ, and $p_{\lambda'}$ (and hence also $l_{1_A}\cdot(p_\lambda\circ p_{\lambda'})$ and $(\lambda\lambda')\circ f$) are isomorphisms, it follows from Lemma 3.3 that the diagram is a pullback. Then, in particular, the composite $m_f\cdot(i_\lambda\circ i_{\lambda'})$ is a monomorphism, and thus

$$\mathcal{D}_f(\lambda\lambda') = [(f_\lambda \circ f_{\lambda'}, m_f \cdot (i_\lambda \circ i_{\lambda'}))].$$

Moreover,

$$[(f_\lambda, i_\lambda)] \cdot [(f_{\lambda'}, i_{\lambda'})] = [(f_\lambda \circ f_{\lambda'}, m_f \cdot (i_\lambda \circ i_{\lambda'}))]$$

in $\mathbf{Sub}_{[\mathcal{A},\,\mathcal{A}]}(f^* \circ f)$ by our remark at the beginning of this section. Thus

$$\mathcal{D}_f(\lambda\lambda') = [(f_\lambda, i_\lambda)] \cdot [(f_{\lambda'}, i_{\lambda'})] = \mathcal{D}_f(\lambda) \cdot \mathcal{D}_f(\lambda'),$$

and hence $\mathcal{D}_f$ is a homomorphism of monoids. □

Remark 3.5. Putting $\lambda' = \lambda^{-1}$ in the proof of Proposition 3.4 gives that for any $\lambda \in \mathbf{Aut}_{\,[B,\,A]}(f^*)$, the 1-cell f_λ defined in (8) is invertible.

Proposition 3.6. *The monoid structure on $\mathbf{Sub}_{[A,\,A]}(f^* \circ f)$ restricts to a group structure on $\mathfrak{I}^A_f$. Moreover, $\mathcal{D}_f$ induces a group homomorphism*

$$\overline{\mathcal{D}}_f : \mathbf{Aut}_{[B,\,A]}(f^*) \to \mathfrak{I}^A_f.$$

Proof. The 2-cell η_f is monomorphic by assumption. Since, quite obviously, 1_A is invertible, and $[(1_A, \eta_f)] \in \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$, it follows that $[(1_A, \eta_f)] \in \mathfrak{I}^A_f$.

Next, if $[(h, i_h)], [(g, i_g)] \in \mathfrak{I}^A_f$, then clearly $h \circ g$ is invertible. Observe that $\xi^l_{i_g} : \mathcal{S}_f \circ g \to \mathcal{S}_f$ is an isomorphism as $[(g, i_g)] \in \mathfrak{I}^A_f \subseteq \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$. Since i_h is a monomorphism, we get from Remark 1.6 that the 2-cell $i_h \circ g : h \circ g \to \mathcal{S}_f \circ g$ is a monomorphism. On the other hand, $m_f \cdot (i_h \circ i_g) = \xi^l_{i_g} \cdot (i_h \circ g)$, and it follows that the 2-cell $i_{h\circ g} := m_f \cdot (i_h \circ i_g)$ is monomorphic. Thus, as we have remarked at the beginning of this section, $[(h, i_h)] \cdot [(g, i_g)] = [(h \circ g, i_{h\circ g})]$ in $\mathbf{Sub}_{[A,\,A]}(f^* \circ f)$. Moreover, $[(h \circ g, i_{h\circ g})]$ lies in $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ by exactly the same argument as in the proof of [15, Proposition 3.5]. Thus $[(h \circ g, i_{h\circ g})] \in \mathfrak{I}^A_f$, and hence $\mathfrak{I}^A_f$ inherits the structure of a monoid from $\mathbf{Sub}_{[A,\,A]}(f^* \circ f)$. In view of Proposition 2.1, it is easy to see that if $[(h, i_h)] \in \mathfrak{I}^A_f$, then its two-sided inverse is $[(h^*, i_{h^*})]$, where h^* is the pseudo-inverse of h. Therefore, $\mathfrak{I}^A_f$ is in fact a group.

In light of Proposition 3.2 and Remark 3.5, it follows from Proposition 2.4 that $\mathcal{D}_f(\lambda) \in \mathfrak{I}^A_f$, for any $\lambda \in \mathbf{Aut}_{\,[B,\,A]}(f^*)$. Proposition 3.4 then guarantees that $\mathcal{D}_f$ induces a homomorphism of groups $\overline{\mathcal{D}}_f : \mathbf{Aut}_{[B,\,A]}(f^*) \to \mathfrak{I}^A_f$. □

Theorem 3.7. *The sequence of groups*

$$1 \to \mathbf{Aut}_{[A,\,A]}(1_A) \xrightarrow{\overline{\omega}_0} \mathbf{Aut}_{[B,\,A]}(f^*) \xrightarrow{\overline{\mathcal{D}}_f} \mathfrak{I}^A_f$$

is exact.

Proof. To say that the sequence is exact at $\mathbf{Aut}_{[A,\,A]}(1_A)$ is to say that $\overline{\omega}_0$ is injective, which is indeed the case by Proposition 3.1.

To prove the exactness at $\mathbf{Aut}_{[B,\,A]}(f^*)$, we have to show that $\mathrm{Ker}(\overline{\mathcal{D}}_f) = \mathrm{Im}(\overline{\varpi}_0)$. For any $s \in \mathbf{Aut}_{[A,\,A]}(1_A)$, the diagram

(11)
$$\begin{tikzcd}
1_A \arrow[r, "\eta_f"] \arrow[d, "(l_{1_A})^{-1}"'] & f^*\circ f \arrow[d, "(l_{f^*})^{-1}\circ f"] \\
1_A\circ 1_A \arrow[d, "s\circ 1_A"'] & (1_A\circ f^*)\circ f \arrow[d, "(s\circ f^*)\circ f"] \\
1_A\circ 1_A \arrow[d, "l_{1_A}"'] & (1_A\circ f^*)\circ f \arrow[d, "l_{f^*}\circ f"] \\
1_A \arrow[r, "\eta_f"'] & f^*\circ f
\end{tikzcd}$$

is commutative, as can be easily seen by using the naturality of l and of α and the fact that

$$\begin{tikzcd}
(1_A\circ u)\circ v \arrow[r, "\alpha_{1_A,u,v}"] \arrow[dr, "l_u\circ v"'] & 1_A\circ(u\circ v) \arrow[d, "l_{u\circ v}"] \\
 & u\circ v
\end{tikzcd}$$

is a commutative diagram for all 1-cells $u, v : A \to A$ (e.g., [18, Proposition 1.1]). Since $s = r_{1_A} \cdot (s\circ 1_A) \cdot (r_{1_A})^{-1}$ by the naturality of r, $l_{1_A} = r_{1_A}$, and $\overline{\varpi}_0(s) = l_{f^*}\circ(s\circ f^*)\circ(l_{f^*})^{-1}$, it follows that (11) may be rewritten in the form

$$\begin{tikzcd}
1_A \arrow[r, "\eta_f"] \arrow[d, "s"'] & f^*\circ f \arrow[d, "\overline{\varpi}_0(s)\circ f"] \\
1_A \arrow[r, "\eta_f"'] & f^*\circ f
\end{tikzcd}$$

Since both s and $\overline{\varpi}_0(s)$ are invertible 2-cells, it follows from Lemma 3.3 that the diagram above is a pullback, implying that $\overline{\mathcal{D}}_f(\overline{\varpi}_0(s)) = [(1_A, \eta_f)] = 1$ in $\mathfrak{I}_f^A$. Since $s \in \mathbf{Aut}_{[A,\,A]}(1_{\mathcal{A}})$ was arbitrary, $\mathrm{Im}(\overline{\varpi}_0) \subseteq \mathrm{Ker}(\overline{\mathcal{D}}_f)$.

Next, if $\lambda \in \mathbf{Aut}_{[B,\,A]}(f^*)$ is such that $\overline{\mathcal{D}}_f(\lambda) = 1$, then there is an automorphism $s : 1_A \to 1_A$ such that the diagram

$$\begin{tikzcd}
1_A \arrow[r, "\eta_f"] \arrow[d, "s"'] & f^*\circ f \arrow[d, "\lambda\circ f"] \\
1_A \arrow[r, "\eta_f"'] & f^*\circ f
\end{tikzcd}$$

is a pullback, implying that in the diagram

$$\begin{tikzcd}
f \arrow[r, "r_f^{-1}"] & f\circ 1_A \arrow[r, "f\circ\eta_f"] \arrow[d, "f\circ s"'] & f\circ(f^*\circ f) \arrow[r, "f\circ(\lambda\circ f)"] & f\circ(f^*\circ f) \arrow[r, "\alpha^{-1}_{f,f^*,f}"] & (f\circ f^*)\circ f \arrow[d, "\varepsilon_f\circ f"] \\
& f\circ 1_A \arrow[r, "r_f"'] \arrow[urr, "f\circ\eta_f"'] & f \arrow[rr, "l_f^{-1}"'] & & 1_A\circ f
\end{tikzcd}$$

the triangle commutes, while the trapezoid commutes by (3). It then follows that the mate of λ under the adjunction $f \dashv f^*$, which is the composite

$$l_f \cdot (\varepsilon_f\circ f)\cdot \alpha^{-1}_{f,f^*,f}\cdot (f\circ(\lambda\circ f))\cdot (f\circ\eta_f)\cdot r_f^{-1},$$

is in fact equal to the composite $r_f\cdot(f\circ s)\cdot r_f^{-1}$. Direct inspection using the fact that the diagram

$$\begin{tikzcd}
(f\circ 1_A)\circ f^* \arrow[rr, "\alpha_{f,1_A,f^*}"] \arrow[dr, "r_f\circ f^*"'] & & f\circ(1_A\circ f^*) \arrow[dl, "f\circ l_{f^*}"] \\
& f\circ f^* &
\end{tikzcd}$$

commutes, shows that the mate of the last composite under the adjunction $f \dashv f^*$ is just $\overline{\varpi}_0(s) = l_{f^*}\cdot(s\circ f^*)\cdot(l_{f^*})^{-1}$. This proves that $\overline{\varpi}_0(s)=\lambda$. Thus $\mathrm{Ker}(\overline{\mathcal{D}}_f)\subseteq \mathrm{Im}(\overline{\varpi}_0)$, and hence $\mathrm{Ker}(\overline{\mathcal{D}}_f)=\mathrm{Im}(\overline{\varpi}_0)$. □

3.2. **An exact sequence involving the Picard group.** For any object A of $\mathbb{B}$, define the *Picard Group* of A, denoted $\mathbf{Pic}(A)$, to be the collection of isomorphism-classes $[h]$ of invertible 1-cells $h: A\to A$ with product and inverses defined by

$$[h]\cdot[g]=[h\circ g] \text{ and } [h]^{-1}=[h^*],$$

where h^* is a pseudoinverse of h. As is easily seen, $\mathbf{Pic}(A)$ is a well-defined group with identity element $[1_A]$.

Proposition 3.8. *The assignment that takes $[(h,i_h)]\in\mathfrak{I}_f^A$ to $[h]$ defines a group homomorphism*

$$\Omega_f : \mathfrak{I}_f^A \to \mathbf{Pic}(A).$$

Proof. For any $[(h,i_h)]\in\mathfrak{I}_f^A$, $[h]\in\mathbf{Pic}(A)$ by the very definition of $\mathfrak{I}_f^A$. The product $[(h,i_h)]\cdot[(h',i_{h'})]$ of $[(h,i_h)]$, $[(h',i_{h'})]\in\mathfrak{I}_f^A$ is the pair $([h\circ h'], i_{h\circ h'})$, where $i_{h\circ h'}$ is the composite

$$i_{h\circ h'} : h\circ h' \xrightarrow{i_h\circ i_{h'}} (f^*\circ f)\circ(f^*\circ f) \xrightarrow{m_f} f^*\circ f$$

(see the proof of Proposition 3.6). Therefore, Ω_f preserves the product and hence is a group homomorphism. □

Theorem 3.9. *The sequence of groups*

$$1\to \mathbf{Aut}_{[A,A]}(1_A) \xrightarrow{\overline{\varpi}_0} \mathbf{Aut}_{[B,A]}(f^*) \xrightarrow{\overline{\mathcal{D}}_f} \mathfrak{I}_f^A \xrightarrow{\Omega_f} \mathbf{Pic}(A)$$

is exact.

Proof. By Theorem 3.7, it suffices to show that the sequence is exact at $\mathfrak{I}_f^A$. So, suppose $[(h, i_h)] \in \mathfrak{I}_f^A$ is such that $\Omega_f([(h, i_h)]) = [h] = [1_A]$. Then there exists an isomorphism $\tau : h \to 1_A$ in $[A, A]$. Define λ to be the composite

$$f^* \xrightarrow{(\xi^*_{i_h})^{-1}} h\circ f^* \xrightarrow{\tau\circ f^*} 1_A\circ f^* \xrightarrow{l_{f^*}} f^*.$$

It is clear that $\lambda \in \mathbf{Aut}_{[B,\,A]}(f^*)$. We claim that $\overline{\mathcal{D}}_f(\lambda) = [(h, i_h)]$. Indeed, we know that the diagram

$$\begin{tikzcd}
(h\circ f^*)\circ f \arrow[r, "\alpha_{h,f^*,f}"] \arrow[d, "(\tau\circ f^*)\circ f"'] & h\circ(f^*\circ f) \arrow[d, "\tau\circ(f^*\circ f)"] \\
(1_A\circ f^*)\circ f \arrow[r, "\alpha_{1_A,f^*,f}"'] & 1_A\circ(f^*\circ f)
\end{tikzcd}$$

commutes by naturality of α, and $l_{f^*\circ f}\cdot\alpha_{1_A,f^*,f} = l_{f^*}\circ f$ by one of the two coherence axioms (see [18, Proposition 1.1]). Since $(\xi^r_{i_h})^{-1} = \alpha_{h,f^*,f}\cdot((\xi^*_{i_h})^{-1}\circ f)$ by the dual of [15, Remark 4.2]), the 2-cell $\lambda\circ f$ can be rewritten as follows:

$$f^*\circ f \xrightarrow{(\xi^r_{i_h})^{-1}} h\circ(f^*\circ f) \xrightarrow{\tau\circ(f^*\circ f)} 1_A\circ(f^*\circ f) \xrightarrow{l_{f^*\circ f}} f^*\circ f.$$

In the following diagram

$$\begin{tikzcd}
h \arrow[r, "(r_h)^{-1}"] \arrow[d, "i_h"'] \arrow[dr, phantom, "(1)"] & h\circ 1_A \arrow[r, "\tau\circ 1_A"] \arrow[d, "h\circ\eta_f"] \arrow[dr, phantom, "(2)"] & 1_A\circ 1_A \arrow[r, "l_{1_A}=r_{1_A}"] \arrow[d, "1_A\circ\eta_f"] \arrow[dr, phantom, "(3)"] & 1_A \arrow[d, "\eta_f"] \\
f^*\circ f \arrow[r, "(\xi^r_{i_h})^{-1}"'] & h\circ(f^*\circ f) \arrow[r, "\tau\circ(f^*\circ f)"'] & 1_A\circ(f^*\circ f) \arrow[r, "l_{f^*\circ f}"'] & f^*\circ f
\end{tikzcd}$$

square (2) commutes by naturality of composition, while rectangle (3) commutes by naturality of l. We claim that rectangle (1) is also commutative. Indeed, using that

$$m_f\cdot((f^*\circ f)\circ\eta_f) = r_{f^*\circ f} \tag{12}$$

since m_f is the multiplication for the A-ring $\mathcal{S}_f$, we have

$$\begin{aligned}
&\xi^r_{i_h}\cdot(h\circ\eta_f)\cdot(r_h)^{-1} && \text{since } \xi^r_{i_h} = m_f\cdot(i_h\circ(f^*\circ f))\\
&= m_f\cdot(i_h\circ(f^*\circ f))\cdot(h\circ\eta_f)\cdot(r_h)^{-1} && \text{by naturality of composition}\\
&= m_f\cdot((f^*\circ f)\circ\eta_f)\cdot(i_h\circ 1_A)\cdot(r_h)^{-1} && \text{by (12)}\\
&= r_{f^*\circ f}\cdot(i_h\circ 1_A)\cdot(r_h)^{-1} && \text{by naturality of } r\\
&= i_h\cdot r_h\cdot(r_h)^{-1}\\
&= i_h.
\end{aligned}$$

Thus the diagram

$$\begin{tikzcd}
h \arrow[r, "i_h"] \arrow[d, "r_{1_A}\cdot(\tau\circ 1_A)\cdot(r_h)^{-1}"'] & f^*\circ f \arrow[d, "\lambda\circ f"] \\
1_A \arrow[r, "\eta_f"'] & f^*\circ f
\end{tikzcd}$$

commutes, and since the composite $r_{1_A} \cdot (\tau \circ 1_A) \cdot (r_h)^{-1}$ is an isomorphism, it follows from Lemma 3.3 that the diagram is a pullback. Hence $\overline{\mathcal{D}}_f(\lambda) = [(h, i_h)]$, and thus $\mathrm{Ker}(\Omega_f) \subseteq \mathrm{Im}(\overline{\mathcal{D}}_f)$.

Now, if $i_h : h \to f^* \circ f$ is such that there are an automorphism $\lambda \in \mathbf{Aut}_{[B,\,A]}(f^*)$ and a pullback

$$\begin{tikzcd} h \arrow[r, "i_h"] \arrow[d, "p_h"'] & f^* \circ f \arrow[d, "\lambda \circ f"] \\ 1_A \arrow[r, "\eta_f"'] & f^* \circ f \end{tikzcd}$$

then clearly the 2-cell $p_h : h \to 1_A$ is an isomorphism and thus $\Omega_f([(h, i_h)]) = [h] = [1_A]$. Thus $\mathrm{Im}(\overline{\mathcal{D}}_f) \subseteq \mathrm{Ker}(\Omega_f)$, and hence $\mathrm{Ker}(\Omega_f) = \mathrm{Im}(\overline{\mathcal{D}}_f)$. This completes the proof. □

Example 3.10. Let $\varphi : R \to S$ be a homomorphism of firm rings as in Example 1.7. If φ is injective, then we can apply Theorem 3.9 to the adjunction ${}_SS_R \dashv {}_RS_S$ in **Firm**, and we get the exact sequence of groups

$$1 \longrightarrow \mathbf{Aut}({}_RR_R) \longrightarrow \mathbf{Aut}({}_SS_R) \longrightarrow \mathbf{Inv}_R(S) \longrightarrow \mathbf{Pic}(R), \tag{13}$$

where $\mathbf{Aut}({}_RR_R)$, (resp., $\mathbf{Aut}({}_SS_R)$) denotes the group of (R, R)-bimodule (resp., (S, R)-bimodule) automorphisms of R (resp., S), $\mathbf{Inv}_R(S)$ is the group of invertible R-sub-bimodules of S, and $\mathbf{Pic}(R)$ is the Picard group of the ring R. The exact sequence (13) was obtained, as a generalization of the unital case [25], in [10, Proposition 1.4] for any extension of rings with the same set of local units. Every such an extension is clearly an injective homomorphism of firm rings.

Given an arbitrary category $\mathbf{C}$, we write $\pi_0(\mathbf{C})$ for the collection of the isomorphism classes of objects of $\mathbf{C}$. For any $C \in \mathbf{C}$, $[C]$ denotes the class of C. Clearly, for any functor $S : \mathbf{C} \to \mathbf{D}$, the assignment $[C] \to [S(C)]$ yields a map $\pi_0(S) : \pi_0(\mathbf{C}) \to \pi_0(\mathbf{D})$.

Quite obviously, the assignment $[h] \to [f \circ h]$ yields a map $\mathbf{Pic}(A) \xrightarrow{[f\circ -]} \pi_0([A, B])$, where $\pi_0([A, B])$ denotes the pointed set of the isomorphism classes $[g]$ of 2-cells $g : A \to B$ with a distinguished class $[f]$. Since $f \circ 1_A \simeq f$, $[f \circ -]$ is a morphism of pointed sets.

Theorem 3.11. *The following sequence of pointed sets,*

$$\mathfrak{I}_f^A \xrightarrow{\Omega_f} \mathbf{Pic}(A) \xrightarrow{[f\circ -]} \pi_0([A, B]),$$

is exact.

Proof. Since $\mathfrak{I}_f^A \subseteq \mathfrak{I}^l_{[A,A]}(S_f)$, it is clear that $([f \circ -] \cdot \Omega_f)([(h, i_h)]) = [f]$ for all $[(h, i_h)] \in \mathfrak{I}_f^A$. So it remains to show that if $[g] \in \mathbf{Pic}(A)$ is such that

$$[f \circ -]([g]) = [f \circ g] = [f],$$

then there exists $[(h, i_h)] \in \mathfrak{I}_f^A$ with $[g] = \Omega_f([(h, i_h)]) = [h]$. Since $[f \circ g] = [f]$, there is an isomorphism $\sigma : f \circ g \to f$ in $[A, B]$. It then follows from Proposition 2.1 that $[(g, i_g)] \in \mathfrak{I}_f^A$, where i_g is the composite $g \xrightarrow{\eta_f \circ g} f^* \circ f \circ g \xrightarrow{f^* \circ \sigma} f^* \circ f$. Then clearly $\Omega_f([(g, i_g)]) = [g]$. □

3.3. **Comonadicity.** Recall from Section 2 that from the adjunction $\eta_f, \varepsilon_f : f \dashv f^* : B \to A$ in $\mathbb{B}$ we get, besides the A-coring $\mathcal{S}_f$, the B-coring $\mathfrak{C}_f$ (see (7)). We know from [15, p. 172] that there is a map

$$\Gamma_f : \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f) \to \mathbf{End}_{B\text{-}\mathsf{cor}}(\mathfrak{C}_f)$$

that takes $[(h, i_h)] \in \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ to the composite

$$f \circ f^* \xrightarrow{\xi^{-1}_{i_h} \circ f^*} (f \circ h) \circ f^* \xrightarrow{\alpha_{f,h,f^*}} f \circ (h \circ f^*) \xrightarrow{f \circ \xi^*_{i_h}} f \circ f^*.$$

Theorem 3.12. *Suppose that the functor $[A, f] = f \circ - : [A, A] \to [A, B]$ is comonadic. Then Γ_f restricts to an isomorphism of groups*

$$\overline{\Gamma}_f : \mathfrak{I}^A_f \to \mathbf{Aut}_{B\text{-}cor}(\mathfrak{C}_f).$$

Proof. The functor $[A, f]$ is precomonadic if and only if the unit of the adjunction $[A, f] \dashv [A, f^*]$ is a componentwise monomorphism. So $\eta_f : 1 \to f^* \circ f$ is right pure in the monoidal category $[A, A]$ (meaning that $\eta_f \circ h : 1 \circ h \to (f^* \circ f) \circ h$ is monomorphic for all 1-cells $h : A \to A$), provided the functor $[A, f]$ is (pre)comonadic. Consequently, according to [15, Proposition 4.4], $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ inherits the structure of a monoid from $\mathbf{Sub}_{[A,\,A]}(f^* \circ f)$. Moreover, the map $\Gamma_f : \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f) \to \mathbf{End}_{B\text{-}\mathsf{cor}}(\mathfrak{C}_f)$ is an isomorphism of monoids by [15, Theorem 4.9]. If $[(h, i_h)] \in \mathfrak{I}^A_f$, then $\xi^*_{i_h}$ is an isomorphism and hence is so $\Gamma_f([(h, i_h)])$. Thus, Γ_f restricts to a monomorphism $\overline{\Gamma}_f : \mathfrak{I}^A_f \to \mathbf{Aut}_{B\text{-}\mathsf{cor}}(\mathfrak{C}_f)$ of groups. To show that $\overline{\Gamma}_f$ is surjective, note first that if $[(h, i_h)] \in \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ is such that $\Gamma_f([(h, i_h)]) \in \mathbf{Aut}_{B\text{-}\mathsf{cor}}(\mathfrak{C}_f)$, then $[(h, i_h)] \in \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$. Indeed, if the composite

$$\Gamma_f([(h, i_h)]) : f \circ f^* \xrightarrow{\xi^{-1}_{i_h} \circ f^*} (f \circ h) \circ f^* \xrightarrow{\alpha_{f,h,f^*}} f \circ (h \circ f^*) \xrightarrow{f \circ \xi^*_{i_h}} f \circ f^*$$

is an isomorphism, then $f \circ \xi^*_{i_h}$ is also an isomorphism. But by hypothesis the functor $[A, f] = f \circ -$ is comonadic, and in particular conservative. Hence $\xi^*_{i_h}$ is an isomorphism too. Thus $[(h, i_h)] \in \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$. Consider now any $\alpha \in \mathbf{Aut}_{B\text{-}\mathsf{cor}}(\mathfrak{C}_f)$. Then, since Γ^{-1}_f is a morphism of monoids, one has the following equalities in $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$:

$$\Gamma^{-1}_f(\alpha) \cdot \Gamma^{-1}_f(\alpha^{-1}) = \Gamma^{-1}_f(\alpha \cdot \alpha^{-1}) = \Gamma^{-1}_f(1_{\mathcal{S}_f}) = [(1_A, \eta_f)].$$

Similarly, $\Gamma^{-1}_f(\alpha^{-1}) \cdot \Gamma^{-1}_f(\alpha) = [(1_A, \eta_f)]$. If now $\Gamma^{-1}_f(\alpha) = [(h, i_h)]$ and $\Gamma^{-1}_f(\alpha^{-1}) = [(h', i_{h'})]$, then since by [15, Proposition 4.4] the product $[(h, i_h)] \cdot [(h', i_{h'})]$ in $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ is the pair $([h \circ h'], i_{h \circ h'})$, where $i_{h \circ h'}$ is the composite

$$i_{h \circ h'} : h \circ h' \xrightarrow{i_h \circ i_{h'}} (f^* \circ f) \circ (f^* \circ f) \xrightarrow{m_f} f^* \circ f,$$

it follows that $h \circ h' \simeq 1_A$ and $h' \circ h \simeq 1_A$. Hence $[h] \in \mathbf{Pic}(A)$. Since $\Gamma_f(\Gamma^{-1}_f(\alpha)) = \alpha$ is an isomorphism, $\Gamma^{-1}_f(\alpha) \in \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f)$, as we have shown above. Thus, $\Gamma^{-1}_f(\alpha) \in \mathfrak{I}^A_f$, and hence $\overline{\Gamma}_f$ is surjective. This completes the proof. $\square$

Remark 3.13. We have proved in passing that, when the functor

$$[A, f] = f \circ - : [A, A] \to [A, B]$$

is comonadic, then

$$\mathfrak{I}^A_f = \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f) \cap \mathfrak{I}^r_{[A,A]}(\mathcal{S}_f).$$

As a corollary, we get the following.

Proposition 3.14. *Whenever the functor*

$$[A, f] = f\circ - : [A, A] \to [A, B]$$

is comonadic, we have an equality of groups

$$\mathfrak{I}_f^A = (\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f))^\times,$$

where $(-)^\times$ *is the functor taking a monoid to its group of invertible elements.*

Remark 3.15. According to [15, Theorem 4.9], if the functor $f\circ - : [A, A] \to [A, B]$ is comonadic, then $\mathfrak{I}^l_{[A,A]}(\mathcal{S}_f)$ is endowed with the structure of a monoid in such a way that the map $\Gamma_f : \mathfrak{I}^l_{[A,A]}(\mathcal{S}_f) \to \mathbf{End}_{B\text{-}\mathtt{cor}}(\mathfrak{C}_f)$ is an isomorphism of monoids. Theorem 3.12 might be considered as a refinement of this result, although we know that $\mathfrak{I}_f^A$ is a group without the comonadicity condition. In [15, Section 5] some sufficient conditions for the comonadicity of the functor $[A, f] = f\circ - : [A, A] \to [A, B]$ are investigated. Concretely, if $[A, f]$ preserves equalizers and η_f is right regular A-pure (that is, $\eta_f \circ h : 1_A \circ h \to (f^* \circ f) \circ h$ is a regular monomorphism in $[A, A]$ for every object h of $[A, A]$), then $[A, f]$ is comonadic, according to [15, Proposition 5.2]. A classical example is to consider an extension $R \subseteq S$ of unital rings such that S_R is faithfully flat. Then it is easily proved that S_R is pure and, hence, the functor $S \otimes_R -$ becomes comonadic. Therefore, we get from Theorem 3.12, applied to the adjunction ${}_SS_R \dashv {}_RS_S$ in the bicategory of bimodules, the isomorphism of groups $\mathfrak{I}_S^R \cong \mathbf{Aut}_{S-\mathrm{cor}}(S \otimes_R S)$ proved in [23, Theorem 2.10]. The functor $[A, f]$ also becomes comonadic if f is a separable 1-cell (that is, if η_f is a split monomorphism in the category $[A, A]$) (see [15, Proposition 5.5]). In the case of a ring extension $R \subseteq S$, we get that if R is a direct summand of S as an R-bimodule, then the isomorphism from [23, Theorem 2.10] is also obtained.

3.4. **Duality.** Let $\mathbb{B}$ be a bicategory whose hom-categories admit finite colimits and coimages, and let $\eta_f, \varepsilon_f : f \dashv f^* : B \to A$ be an adjunction in $\mathbb{B}$ such that $\varepsilon_f : f\circ f^* \to 1_B$ is epimorphic in $[B, B]$. Let $\mathfrak{C}_f$ be the corresponding B-coring. Write $\mathcal{Q}_f^{B,l}$ (resp., $\mathcal{Q}_f^{B,r}$) for the subset of $\mathcal{Q}^l_{[B,B]}(\mathfrak{C}_f)$ (resp., $\mathcal{Q}^l_{[B,B]}(\mathfrak{C}_f)$) determined by the elements $[(h, i_h)]$ with $h \in \mathbf{Pic}(B)$. Then, by the dual form of Proposition 2.4, $\mathcal{Q}_f^{B,l} = \mathcal{Q}_f^{B,r}$, and we write $\mathcal{Q}_f^B$ to denote either $\mathcal{Q}_f^{B,l}$ or $\mathcal{Q}_f^{B,r}$.

Recall that for any bicategory $\mathbb{B}$, $\mathbb{B}^{\mathrm{co}}$ is a bicategory obtained from $\mathbb{B}$ by reversing 2-cells, i.e., $\mathbb{B}^{\mathrm{co}}(A, B) = \mathbb{B}(A, B)^{\mathrm{op}}$. Now applying Theorems 3.9 and 3.11 to the bicategory $\mathbb{B}^{\mathrm{co}}$ gives the following.

Theorem 3.16. *We have an exact sequence of groups*

$$1 \to \mathbf{Aut}_{[B,\,B]}(1_B) \xrightarrow{\widehat{\omega}_0} \mathbf{Aut}_{[A,\,B]}(f) \xrightarrow{\overline{\mathcal{D}}_{f^*}} \mathcal{Q}_f^B \xrightarrow{\Omega_{f^*}} \mathbf{Pic}(B),$$

and an exact sequence of pointed sets

$$\mathcal{Q}_f^B \xrightarrow{\Omega_{f^*}} \mathbf{Pic}(B) \xrightarrow{[f^*\circ -]} \pi_0([B, A]).$$

Here,

- $\widehat{\omega}_0(1_B \xrightarrow{s} 1_B) = f \xrightarrow{(l_f)^{-1}} 1_B\circ f \xrightarrow{s\circ f} 1_B\circ f \xrightarrow{l_f} f,$

- $\overline{\mathcal{D}}_{f^*}(f \xrightarrow{\sigma} f) = [(P, \pi_P)]$, *where*

$$\begin{tikzcd} f \circ f^* \arrow[r, "\varepsilon_f"] \arrow[d, "\sigma \circ f^*"'] & 1_B \arrow[d] \\ f \circ f^* \arrow[r, "\pi_P"'] & P \end{tikzcd}$$

  is a pushout, and
- $\Omega_{f^*}([P, \pi_P]) = [P]$.

When the functor $[B, f^*]$ is monadic, we have, by [15, Theorem 4.11], that the map

$$\overline{\Gamma}_{f^*} : \mathcal{Q}^l_{[B,B]}(\mathfrak{C}_f) \to \mathrm{End}_{A-ring}(\mathcal{S}_f),$$

given by

$$[(p, \pi_p)] \longrightarrow f^* \circ f \xrightarrow{f^* \circ \xi_{\pi_p}} f^* \circ (p \circ f) \xrightarrow{\alpha^{-1}_{f^*, p, f}} (f^* \circ p) \circ f \xrightarrow{(\xi^*_{\pi_p})^{-1} \circ f} f^* \circ f,$$

is an isomorphism of monoids.

Now, the dual version of Theorem 3.12 yields the following.

Proposition 3.17. *Suppose that the functor* $[B, f^*] = f^* \circ - : [B, B] \to [B, A]$ *is monadic. Then* Γ_{f^*} *restricts to an isomorphism of groups*

$$\overline{\Gamma}_{f^*} : \mathcal{Q}^B_f \to \mathbf{Aut}_{A\text{-}ring}(\mathcal{S}_f).$$

Example 3.18. Let $\varphi : R \to S$ be a homomorphism of firm rings as in Example 1.7. Now, the adjunction ${}_S S_R \dashv {}_R S_S$ in **Firm** leads to the functor $S \otimes_S - :$ $\mathbf{Firm}(S, S) \to \mathbf{Firm}(S, R)$, which is monadic according to Beck's Theorem. Moreover, the isomorphism $S \otimes_S S \cong S$ becomes an isomorphism of R–rings, so that, by Proposition 3.17, we get an isomorphism of groups $\mathcal{Q}^S_S \cong \mathbf{Aut}_{R-\mathtt{ring}}(S)$. We thus get from Theorem 3.16 an exact sequence of groups

$$1 \longrightarrow \mathbf{Aut}({}_S S_S) \longrightarrow \mathbf{Aut}({}_S S_R) \longrightarrow \mathbf{Aut}_{R-ring}(S) \longrightarrow \mathbf{Pic}(S),$$

which generalizes [10, Proposition 2.3].

4. Applications

In this section, we apply the results from Section 3 to an adjoint pair in a bicategory of bimodules. This bicategory is built over an abstract monoidal category subject to some requirements which, of course, are fulfilled by the category of abelian groups, recovering the usual bicategory of bimodules. With this tool at hand, we treat the case of a homomorphism of commutative algebras. In particular, the group isomorphism involving first Amitsur cohomology and the Picard group is proved.

4.1. **The bicategory of bimodules.** Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a monoidal category such that the category $\mathcal{V}$ admits reflexive coequalizers and the latter are preserved, as in the biclosed case, for instance, by the functors $M \otimes -, - \otimes M :$ $\mathcal{V} \to \mathcal{V}$, for all $M \in \mathcal{V}$. We will briefly recall basic notions and results about (commutative) monoids and modules over them in monoidal categories; all can be found in [21].

For simplicity of exposition we treat $\otimes$ as strictly associative and I as a strict unit, which is justified by Mac Lane's coherence theorem [21].

Recall that, for a $\mathcal{V}$-algebra $\mathbf{A} = (A, m_A, e_A)$, a *left* $\mathbf{A}$*-module* is a pair (M, ρ_M), where M is an object of $\mathcal{V}$ and $\rho_M : A \otimes M \to M$ is a morphism in $\mathcal{V}$, called the *action* (or the $\mathbf{A}$*-action*) on M, such that $\rho_M(m_A \otimes M) = \rho_M(A \otimes \rho_M)$ and $\rho_M(e_A \otimes M) = 1$.

The left $\mathbf{A}$-modules are the objects of a category ${}_{\mathbf{A}}\mathcal{V}$. A morphism of left $\mathbf{A}$-modules is a morphism in $\mathcal{V}$ of the underlying $\mathcal{V}$-objects that commutes with the actions of $\mathbf{A}$. In a similar manner, one defines the category $\mathcal{V}_{\mathbf{A}}$ of right $\mathbf{A}$-modules.

If $\mathbf{A}$ and $\mathbf{B}$ are algebras in $\mathcal{V}$, then an $(\mathbf{A}, \mathbf{B})$-bimodule M in $\mathcal{V}$ is an object of $\mathcal{V}$ with commuting left $\mathbf{A}$-module and right $\mathbf{B}$-module structures. The category of $(\mathbf{A}, \mathbf{B})$-bimodules is denoted ${}_{\mathbf{A}}\mathcal{V}_{\mathbf{B}}$.

If $(M, \varrho_M) \in \mathcal{V}_{\mathbf{A}}$ and $(N, \rho_N) \in {}_{\mathbf{A}}\mathcal{V}$, then the *tensor product of* (M, ϱ_M) *and* (N, ρ_N) *over* $\mathbf{A}$ is the object part of the following (reflexive) coequalizer:

$$M \otimes A \otimes N \overset{\varrho_M \otimes N}{\underset{M \otimes \rho_N}{\rightrightarrows}} M \otimes N \xrightarrow{q_{M,N}} M \otimes_{\mathbf{A}} N.$$

Moreover, if $M \in {}_{\mathbf{B}}\mathcal{V}_{\mathbf{A}}$ and $N \in {}_{\mathbf{A}}\mathcal{V}_{\mathbf{C}}$, then $M \otimes_{\mathbf{A}} N \in {}_{\mathbf{B}}\mathcal{V}_{\mathbf{C}}$. It then follows, in particular, that for a fixed $\mathcal{V}$-algebra $\mathbf{A}$, the category ${}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}$ of $(\mathbf{A}, \mathbf{A})$-bimodules in $\mathcal{V}$ is a (nonsymmetric) monoidal category with tensor product of two $(\mathbf{A}, \mathbf{A})$-bimodules being their tensor product over $\mathbf{A}$ and the unit for this tensor product being the $(\mathbf{A}, \mathbf{A})$-bimodule A.

This allows us (see, for example, [3] or [31]) to construct the bicategory $\mathbf{Bim}(\mathcal{V})$ in which:

- objects are $\mathcal{V}$-algebras,
- $\mathbf{Bim}(\mathcal{V})(\mathbf{A}, \mathbf{B}) = {}_{\mathbf{B}}\mathcal{V}_{\mathbf{A}}$,
- 2-cells are bimodule morphisms.

Although the 1-cells in a bicategory are usually denoted using the arrow symbols, we sometimes, as here, find it convenient to write $\mathbf{A} \rightsquigarrow \mathbf{B}$ instead of $\mathbf{A} \to \mathbf{B}$. Thus, $M : \mathbf{A} \rightsquigarrow \mathbf{B}$ means that M is a $(\mathbf{B}, \mathbf{A})$-bimodule.

The horizontal composite $N \circ M$ of $M : \mathbf{A} \rightsquigarrow \mathbf{B}$ and $N : \mathbf{B} \rightsquigarrow \mathbf{C}$ is the $(\mathbf{C}, \mathbf{A})$-bimodule $N \otimes_{\mathbf{B}} M$, while the vertical composition of two 2-cells is the ordinary composition of bimodule morphisms.

We write $\mathbf{I}$ for the trivial $\mathcal{V}$-algebra $(I, r_I = l_I : I \otimes I \to I, 1_I : I \to I)$. Then, for any $\mathcal{V}$-algebra $\mathbf{A}$, the category $\mathbf{Bim}(\mathcal{V})(\mathbf{I},\, \mathbf{A})$ is (isomorphic to) the category of left $\mathbf{A}$-modules ${}_{\mathbf{A}}\mathcal{V}$, while the category $\mathbf{Bim}(\mathcal{V})(\mathbf{A},\, \mathbf{I})$ is (isomorphic to) the category of right $\mathbf{A}$-modules $\mathcal{V}_{\mathbf{A}}$. Moreover, if $M : \mathbf{A} \rightsquigarrow \mathbf{B}$ is a $(\mathbf{B}, \mathbf{A})$-bimodule, then the diagrams

$$\begin{tikzcd}
\mathbf{Bim}(\mathcal{V})(\mathbf{I},\, \mathbf{A}) \arrow[r, "M\circ -"] \arrow[d, equal] & \mathbf{Bim}(\mathcal{V})(\mathbf{I},\, \mathbf{B}) \arrow[d, equal] \\
{}_{\mathbf{A}}\mathcal{V} \arrow[r, "M \otimes_{\mathbf{A}} -"'] & {}_{\mathbf{B}}\mathcal{V}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Bim}(\mathcal{V})(\mathbf{B},\, \mathbf{I}) \arrow[r, "-\circ M"] \arrow[d, equal] & \mathbf{Bim}(\mathcal{V})(\mathbf{A},\, \mathbf{I}) \arrow[d, equal] \\
\mathcal{V}_{\mathbf{B}} \arrow[r, "- \otimes_{\mathbf{B}} M"'] & \mathcal{V}_{\mathbf{A}}
\end{tikzcd}$$

where the vertical morphisms are the isomorphisms, are both commutative.

We henceforth suppose in addition that the category $\mathcal{V}$ admits, besides reflexive coequalizers, all finite limits and image factorizations. Then, for any two $\mathcal{V}$-algebras $\mathbf{A}$ and $\mathbf{B}$, the category ${}_{\mathbf{A}}\mathcal{V}_{\mathbf{B}}$, being the Eilenberg–Moore category for the monad

$$A\otimes - \otimes B : \mathcal{V} \to \mathcal{V},$$

also admits finite limits (see, for example, [4]) and image factorizations (see [1]). So we are in a position to apply Theorems 3.9 and 3.11 to obtain the following result.

Theorem 4.1. *Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category with $\mathcal{V}$ admitting finite limits, image factorizations, and reflexive coequalizers. Assume that the latter are preserved by the tensor product, and let $\mathbf{A}, \mathbf{B}$ be two $\mathcal{V}$-algebras. Then for any adjunction $\eta_M, \varepsilon_M : M \dashv M^* : \mathbf{B} \rightsquigarrow \mathbf{A}$ in $\mathbf{Bim}(\mathcal{V})$ with monomorphic $\eta_M : A \to M^*\otimes_{\mathbf{B}} M = M^*\circ M$, the sequence of groups*

$$1 \to \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(A) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{B}}}(M^*) \xrightarrow{\overline{\mathcal{D}}_M} \mathfrak{I}_M^{\mathbf{A}} \xrightarrow{\Omega_M} \mathbf{Pic}(\mathbf{A})$$

is exact. Moreover, the sequence of pointed sets

$$\mathfrak{I}_M^{\mathbf{A}} \xrightarrow{\Omega_M} \mathbf{Pic}(\mathbf{A}) \xrightarrow{[M\otimes_{\mathbf{A}}-]} \pi_0({}_{\mathbf{B}}\mathcal{V}_{\mathbf{A}})$$

is exact.

Example 4.2. Each morphism of $\mathcal{V}$-algebras $\iota : \mathbf{A} \to \mathbf{B}$ leads to two bimodules $B_\iota : \mathbf{A} \rightsquigarrow \mathbf{B}$ and $B^\iota : \mathbf{B} \rightsquigarrow \mathbf{A}$ which are both equal to B as objects of $\mathcal{V}$ but with the bimodule structures defined by

$$(B \otimes B \xrightarrow{m_B} B, B\otimes A \xrightarrow{B\otimes\iota} B \otimes B \xrightarrow{m_B} B)$$

and

$$(A\otimes B \xrightarrow{\iota\otimes B} B \otimes B \xrightarrow{m_B} B, B \otimes B \xrightarrow{m_B} B).$$

In fact B^ι is right adjoint for B_ι in $\mathbf{Bim}(\mathcal{V})$ with

$$A \xrightarrow{\iota} B \simeq B\otimes_{\mathbf{B}} B = B^\iota\circ B_\iota$$

as unit and

$$B_\iota\circ B^\iota = B\otimes_{\mathbf{A}} B \xrightarrow{\overline{m}_B} B$$

as counit. Here $B\otimes_{\mathbf{A}} B \xrightarrow{\overline{m}_B} B$ is the unique morphism making the triangle

$$\begin{tikzcd} B\otimes B \arrow[r, "q_{B,B}"] \arrow[dr, "m_B"'] & B\otimes_{\mathbf{A}} B \arrow[d, "\overline{m}_B"] \\ & B \end{tikzcd}$$

commute.

It therefore follows that every morphism $\iota : \mathbf{A} \to \mathbf{B}$ of $\mathcal{V}$-algebras gives rise to two functors:

- the *forgetful functor* $\iota_* = B^\iota\circ- : {}_{\mathbf{B}}\mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$, where for any left $\mathbf{B}$-module (M, ϱ_M), $\iota_*(M, \varrho_M) = M$ is a left $\mathbf{A}$-module via the action
$$A \otimes M \xrightarrow{\iota\otimes M} B \otimes M \xrightarrow{\varrho_M} M;$$
- the *change-of-base functor* $\iota^* = B_\iota\circ- : {}_{\mathbf{A}}\mathcal{V} \to {}_{\mathbf{B}}\mathcal{V}$, where for any (left) $\mathbf{A}$-module (N, ρ_N), $\iota^*(N, \rho_N) = B \otimes_A N$ and $B \otimes_A N$ is a (left) $\mathbf{B}$-module via the action
$$B \otimes B \otimes_A N \xrightarrow{m_B\otimes_A N} B \otimes_A N.$$

Since B^ι is right adjoint to B_ι in $\mathbf{Bim}(\mathcal{V})$, it follows that the forgetful functor ι_* is right adjoint to the change-of-base functor functor ι^*.

If we specialize Theorem 4.1 to the adjunction $B_\iota \dashv B^\iota$ in $\mathbf{Bim}(\mathcal{V})$, we obtain the following exact sequence of groups:

$$1 \to \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(A) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{B}}}(B^\iota) \xrightarrow{\overline{\mathcal{D}}_{B_\iota}} \mathfrak{I}^{\mathbf{A}}_{B_\iota} \xrightarrow{\Omega_{B_\iota}} \mathbf{Pic}(\mathbf{A}) \tag{14}$$

and the following exact sequence of pointed sets:

$$\mathfrak{I}^{\mathbf{A}}_{B_\iota} \xrightarrow{\Omega_M} \mathbf{Pic}(\mathbf{A}) \xrightarrow{[B_\iota \otimes_{\mathbf{A}} -]} \pi_0({}_{\mathbf{B}}\mathcal{V}_{\mathbf{A}}). \tag{15}$$

Let us now assume that $\mathcal{V}$ is a symmetric monoidal category with symmetry τ. Recall that a $\mathcal{V}$-algebra is called *commutative* if the multiplication morphism is unchanged when composed with the symmetry.

Given a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathcal{V}$-algebras, consider the associated adjunction $B_\iota \dashv B^\iota$ in $\mathbf{Bim}(\mathcal{V})$. Write $\mathcal{S}_\iota$ for $\mathcal{S}_{\mathbf{B}_\iota}$. Then $\mathcal{S}_\iota = B^\iota \otimes_{\mathbf{B}} B_\iota \simeq \mathbf{B}$, where the left and right actions of $\mathbf{A}$ on $\mathbf{B}$ are given by the compositions

$$\rho_l : A \otimes B \xrightarrow{\iota \otimes B} B \otimes B \xrightarrow{m_B} B \text{ and } \rho_r : B \otimes A \xrightarrow{B \otimes \iota} B \otimes B \xrightarrow{m_B} B,$$

respectively. Since ι is a morphism of commutative $\mathcal{V}$-algebras, these actions coincide (in the sense that $\rho_r = \rho_l \cdot \tau_{B,A}$), and one concludes that $\mathbf{Sub}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(\mathcal{S}_\iota) = \mathbf{Sub}_{{}_{\mathbf{A}}\mathcal{V}}(\mathcal{S}_\iota) = \mathbf{Sub}_{\mathcal{V}_{\mathbf{A}}}(\mathcal{S}_\iota)$. Therefore $\mathfrak{I}^l_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(\mathcal{S}_\iota) = \mathfrak{I}^l_{{}_{\mathbf{A}}\mathcal{V}}(\mathcal{S}_\iota)$.

Similarly, write $\mathfrak{C}_\iota$ for $\mathfrak{C}_{\mathbf{B}_\iota}$. Then $\mathfrak{C}_\iota$ is the $(\mathbf{B}, \mathbf{B})$- bimodule $(B \otimes_{\mathbf{A}} B, m_B \otimes_{\mathbf{A}} B, B \otimes_{\mathbf{A}} m_B)$ equipped with the coproduct

$$B \otimes_{\mathbf{A}} e_{\mathbf{B}} \otimes_{\mathbf{A}} B : B \otimes_{\mathbf{A}} B \to (B \otimes_{\mathbf{A}} B) \otimes_{\mathbf{B}} (B \otimes_{\mathbf{A}} B) \simeq B \otimes_{\mathbf{A}} B \otimes_{\mathbf{A}} B$$

and counit $\overline{m}_B : B \otimes_{\mathbf{A}} B \to B$.

The unit e of $\mathbf{A}$ can be seen as a morphism of commutative $\mathcal{V}$-algebras $\mathbf{I} \to \mathbf{A}$. If e is a monomorphism, using that ${}_{\mathbf{I}}\mathcal{V}_{\mathbf{A}} = \mathcal{V}_{\mathbf{A}}$ and that ${}_{\mathbf{A}}\mathcal{V}_{\mathbf{I}} = {}_{\mathbf{A}}\mathcal{V}$, we get from (14) and (15) the following exact sequences of groups:

$$1 \to \mathbf{Aut}_{\mathcal{V}}(I) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\overline{\mathcal{D}}_A} \mathfrak{I}^{\mathbf{I}}_A \xrightarrow{\Omega_I} \mathbf{Pic}(\mathbf{I}) \tag{16}$$

and of pointed sets:

$$\mathfrak{I}^{\mathbf{I}}_A \xrightarrow{\Omega_A} \mathbf{Pic}(\mathbf{I}) \xrightarrow{[A \otimes -]} \pi_0({}_{\mathbf{A}}\mathcal{V}). \tag{17}$$

It is easy to see that $\mathfrak{I}^l_{\mathcal{V}}(\mathcal{S}_e) = \mathfrak{I}^l_{\mathcal{V}}(A)$ and that $\mathfrak{I}^r_{\mathcal{V}}(\mathcal{S}_e) = \mathfrak{I}^r_{\mathcal{V}}(A)$.

Proposition 4.3. *Let $\mathbf{A}$ be a commutative $\mathcal{V}$-algebra with monomorphic unit $e : I \to A$. Then $\mathfrak{I}^l_{\mathcal{V}}(A)$ is a commutative monoid, while $\mathfrak{I}^{\mathbf{I}}_A$ is an abelian group.*

Proof. Since $\mathcal{V}$ is symmetric, the monoid structure on $\mathfrak{I}^l_{\mathcal{V}}(A)$ is easily seen to be commutative. This implies—since by Proposition 3.6 the monoid structure on $\mathfrak{I}^l_{\mathcal{V}}(A)$ restricts to the group structure on $\mathfrak{I}^{\mathbf{I}}_A$—that the group $\mathfrak{I}^{\mathbf{I}}_A$ is abelian. □

Lemma 4.4. *For any commutative $\mathcal{V}$-algebra $\mathbf{A}$, $\mathfrak{I}^l_{\mathcal{V}}(A) = \mathfrak{I}^r_{\mathcal{V}}(A)$.*

Proof. For any subobject $i_J : J \to A$ of A, consider the diagram

$$\begin{tikzcd} A\otimes J \arrow[r, "A\otimes i_J"] \arrow[d, "\tau_{A,J}"'] & A\otimes A \arrow[d, "\tau_{A,A}"] \\ J\otimes A \arrow[r, "i_J\otimes A"] \arrow[dr, dashed, "\xi^r_{i_J}"'] & A\otimes A \arrow[d, "m_A"] \\ & A \end{tikzcd}$$

in which the rectangle commutes by naturality of τ. Since $\mathbf{A}$ is commutative, $m_A \cdot \tau_{A,A} = m_A$, and hence $\xi^r_{i_J} \cdot \tau_{A,J} = m_A \cdot \tau_{A,A} \cdot (A\otimes i_J) = m_A \cdot (A\otimes i_J) = \xi^l_{i_J}$. Thus $\xi^r_{i_J} \cdot \tau_{A,J} = \xi^l_{i_J}$, and hence $\xi^r_{i_J}$ is an isomorphism (i.e., $[(i_J, J)] \in \mathfrak{I}^r_{\mathcal{V}}(A)$) iff $\xi^l_{i_J}$ is so (i.e., $[(i_J, J)] \in \mathfrak{I}^l_{\mathcal{V}}(A)$). Therefore, $\mathfrak{I}^l_{\mathcal{V}}(A) = \mathfrak{I}^r_{\mathcal{V}}(A)$. $\square$

Proposition 4.5. *Let $\mathbf{A}$ be a commutative $\mathcal{V}$-algebra such that the functor $A\otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is comonadic. Then*

$$\mathbf{End}_{\mathbf{A}\text{-}cor}(\mathfrak{C}_e) = \mathbf{Aut}_{\mathbf{A}\text{-}cor}(\mathfrak{C}_e).$$

Proof. Since the functor $A\otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is assumed to be comonadic, the map

$$\Gamma_A : \mathfrak{I}^l_{\mathcal{V}}(A) = \mathfrak{I}^l_{\mathcal{V}}(\mathcal{S}_e) \to \mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$$

is an isomorphism of monoids by [15, Theorem 4.9]. But since

- the monoid isomorphism $\Gamma_A : \mathfrak{I}^l_{\mathcal{V}}(A) \to \mathbf{End}_{\mathbf{A}-\mathtt{cor}}(\mathfrak{C}_e)$ restricts to an isomorphism $\overline{\Gamma}_A : \mathfrak{I}^{\mathbf{I}}_A \to \mathbf{Aut}_{\mathbf{A}-\mathtt{cor}}(\mathfrak{C}_e)$ of groups by Theorem 3.12, and
- $\mathfrak{I}^{\mathbf{I}}_A = \mathfrak{I}^l_{\mathcal{V}}(A) \cap \mathfrak{I}^r_{\mathcal{V}}(A) = \mathfrak{I}^l_{\mathcal{V}}(A)$ by Remark 3.13 and by Lemma 4.4,

it follows that $\Gamma_A = \overline{\Gamma}_A$, and hence $\mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathcal{S}_e) = \mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$. $\square$

Remark 4.6. Since for any commutative $\mathcal{V}$-algebra $\mathbf{A}$, ${}_{\mathbf{A}}\mathcal{V}$ is a symmetric monoidal category with tensor product $-\otimes_{\mathbf{A}}-$ and monoidal unit (A, m_A), and since the monoid of endomorphisms of the monoidal unit of any monoidal category is commutative (e.g., ([28, 1.3.3.1])), it follows that $\mathbf{End}_{{}_{\mathbf{A}}\mathcal{V}}(A, m_A)$ is a commutative monoid and $\mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A)$ is an abelian group. Since $\mathbf{A} = \mathbf{A}^{\mathrm{op}}$ for any commutative $\mathcal{V}$-algebra $\mathbf{A}$, it follows that ${}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}} = {}_{\mathbf{A}\otimes\mathbf{A}}\mathcal{V}$; and since $\mathbf{A}\otimes\mathbf{A}$ is again a commutative $\mathcal{V}$-algebra, we get that $\mathbf{End}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(A\otimes A)$ is a commutative monoid. Then the inclusions

$$\mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e) \subseteq \mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e) \subseteq \mathbf{End}_{{}_{\mathbf{A}}\mathcal{V}_{\mathbf{A}}}(A\otimes A)$$

imply that $\mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$ is a commutative monoid, and that $\mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$ is an abelian group.

4.2. **Amitsur cohomology and Picard group.** We still assume that $\mathcal{V}$ is symmetric with symmetry τ, and also that $\mathcal{V}$ admits reflexive coequalizers that are preserved by the tensor product, and all finite limits and image factorizations.

For a commutative algebra $\mathbf{A} = (A, m, e)$ in $\mathcal{V}$, write $\mathbf{Pic}^c(\mathbf{A})$ for the subgroup of $\mathbf{Pic}(\mathbf{A})$ consisting of all classes of invertible $(\mathbf{A}, \mathbf{A})$-bimodules $(M, \rho_l : A\otimes M \to M, \rho_r : M\otimes A \to M)$ such that $\rho_r \cdot \tau_{A,M} = \rho_l$. Then $\mathbf{Pic}^c(\mathbf{A})$ is easily seen to be an abelian group. Moreover, given a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathcal{V}$-algebras,

$$\mathbf{Pic}^c(\iota) : \mathbf{Pic}^c(\mathbf{A}) \to \mathbf{Pic}^c(\mathbf{B}),$$

defined by $\mathbf{Pic}^c(\iota)([P]) = [B\otimes_{\mathbf{A}} P]$, is a homomorphism of abelian groups.

It is clear that $\mathbf{Pic}^c(\mathbf{I}) = \mathbf{Pic}(\mathbf{I})$. It is also clear that $[A\otimes -]$ factors through $\mathbf{Pic}^c(e) : \mathbf{Pic}^c(\mathbf{I}) \to \mathbf{Pic}^c(\mathbf{A})$, i.e., the diagram

$$\begin{tikzcd} \mathbf{Pic}^c(\mathbf{I}) \arrow[r, "{[A\otimes -]}"] \arrow[d, "\mathbf{Pic}^c(e)"'] & \pi_0({}_{\mathbf{A}}\mathcal{V}) \\ \mathbf{Pic}^c(\mathbf{A}) \arrow[ur, hook] & \end{tikzcd}$$

where the unlabeled morphism is the canonical embedding, is commutative. It then follows—since (17) is an exact sequence of pointed sets—that

$$\mathfrak{I}^{\mathbf{I}}_A \xrightarrow{\Omega_A} \mathbf{Pic}^c(\mathbf{I}) \xrightarrow{\mathbf{Pic}^c(e)} \mathbf{Pic}^c(\mathbf{A}) \tag{18}$$

is an exact sequence of abelian groups, provided $e : I \to A$ is monomorphic.

Theorem 4.7.

(1) *For any algebra $\mathbf{A} = (A, m, e)$ in $\mathcal{V}$, we have the exact sequence of abelian groups*

$$0 \to \mathbf{Aut}_{\mathcal{V}}(I) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\overline{\mathcal{D}}_A} \mathfrak{I}^{\mathbf{I}}_A \xrightarrow{\Omega_A} \mathbf{Pic}^c(\mathbf{I}) \xrightarrow{\mathbf{Pic}^c(e)} \mathbf{Pic}^c(\mathbf{A}). \tag{19}$$

(2) *If $\mathbf{A}$ is such that the functor $A\otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is comonadic, then there exists an exact sequence of abelian groups*

(20)

$$0 \to \mathbf{Aut}_{\mathcal{V}}(\mathbf{I}) \xrightarrow{\varpi_0} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\kappa_A} \mathbf{Aut}_{\mathbf{A}-cor}(\mathfrak{C}_e) \xrightarrow{o_A} \mathbf{Pic}^c(\mathbf{I}) \xrightarrow{\mathbf{Pic}^c(e)} \mathbf{Pic}^c(\mathbf{A}).$$

Proof.

(1) The exact sequence (19) is obtained by combining (16) with (18).

(2) By Theorem 3.12, the isomorphism $\Gamma_A : \mathfrak{I}^l_{\mathcal{V}}(A) \to \mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$ of monoids restricts to an isomorphism

$$\overline{\Gamma}_A : \mathfrak{I}^{\mathbf{I}}_A \to \mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e).$$

Write κ_A for $\overline{\Gamma}_A\overline{\mathcal{D}}_A$ and o_A for $\Omega_A(\overline{\Gamma}_A)^{-1}$. Then the exact sequence (20) is deduced from (19). $\square$

As an immediate consequence we deduce the following.

Proposition 4.8. *Suppose that $\mathbf{A} = (A, m, e)$ is a commutative $\mathcal{V}$-algebra such that the functor $A\otimes - : \mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is comonadic. Then one has an isomorphism of groups*

$$\mathbf{Coker}(\kappa_A) \simeq \mathbf{Ker}(\mathbf{Pic}^c(e)),$$

where $\mathbf{Coker}(\kappa_A)$ denotes the cokernel of the homomorphism κ_A.

We will need the following description of κ_A.

Proposition 4.9. *In the circumstances above, $\kappa_A(\lambda) = (A\otimes\lambda^{-1})\cdot(\lambda\otimes A)$ for every $\lambda \in \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A)$.*

Proof. Recall first that for any $[(J, i_J)] \in \mathfrak{I}^{\mathbf{I}}_A$, $\overline{\Gamma}_A([(J, i_J)])$ is the composite

$$A\otimes A \xrightarrow{(\xi^l_{i_J})^{-1}\otimes A} A\otimes J\otimes A \xrightarrow{A\otimes\xi^r_{i_J}} A\otimes A.$$

Now, take any $\lambda \in \mathbf{Aut}_{_\mathbf{A}\mathcal{V}}(A)$ and form the pullback

$$\begin{tikzcd} A_\lambda \arrow[r, "i_\lambda"] \arrow[d, "p_\lambda"'] & A \arrow[d, "\lambda"] \\ I \arrow[r, "e"'] & A \end{tikzcd}$$

Then $\overline{\mathcal{D}}_A(\lambda) = [(A_\lambda, i_\lambda)]$. Moreover, $m \cdot (i_\lambda \otimes A) = \lambda^{-1} \cdot (p_\lambda \otimes A)$ by (10). Thus $\xi^r_{i_\lambda} = \lambda^{-1} \cdot (p_\lambda \otimes A)$. Then, since $\xi^l_{i_\lambda} = \xi^r_{i_\lambda} \cdot \tau_{A,\, A_\lambda}$, it follows that $\xi^l_{i_\lambda} = \lambda^{-1} \cdot (A \otimes p_\lambda)$, and hence $(\xi^l_{i_\lambda})^{-1} = (A \otimes p_\lambda^{-1}) \cdot \lambda$. We now calculate:

$$(A\otimes\xi^r_{i_\lambda})\cdot((\xi^l_{i_\lambda})^{-1}\otimes A) = (A\otimes\lambda^{-1})\cdot(A\otimes p_\lambda\otimes A)\cdot(A\otimes p_\lambda^{-1}\otimes A)\cdot(\lambda\otimes A) = (A\otimes\lambda^{-1})\cdot(\lambda\otimes A).$$

Thus, $\kappa_A(\lambda) = (A \otimes \lambda^{-1}) \cdot (\lambda \otimes A)$. □

Remark 4.10. An exact sequence equivalent to (19) can be derived from [31, Corollary 6.1]. Next, we sketch how to prove this claim. To this end, let $\mathcal{V}$ be a symmetric monoidal category with equalizers and coequalizers stable under the tensor product, and write F for the monoidal functor $A \otimes - : \mathcal{V} \to {_\mathbf{A}\mathcal{V}}$. Then F satisfies the conditions of [31, Section 6]. In the following diagram of abelian groups the top row is the exact sequence (19), while the bottom row is a complex obtained from the sequence of symmetric cat-groups from [31, Section 6] by applying the functor π_1 from cat-groups to abelian groups sending each cat-group to the endomorphism group of its unit object:

$$\begin{tikzcd} \mathbf{Aut}_\mathcal{V}(\mathbf{I}) \arrow[r, "\varpi_0"] \arrow[d, equal] & \mathbf{Aut}_{_\mathbf{A}\mathcal{V}}(A) \arrow[r, "\overline{\mathcal{D}}_A"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & \mathfrak{I}^\mathbf{I}_A \arrow[r, "\Omega_A"] \arrow[d, dashed, "t"'] \arrow[dr, phantom, "(2)"] & \mathbf{Pic}^c(\mathbf{I}) \arrow[r, "\mathbf{Pic}^c(e)"] \arrow[d, equal] & \mathbf{Pic}^c(\mathbf{A}) \arrow[d, equal] \\ \pi_1(\mathscr{P}(\mathcal{V})) \arrow[r, "\pi_1(\mathscr{P}(F))"'] & \pi_1(\mathscr{P}(_\mathbf{A}\mathcal{V})) \arrow[r, "\pi_1(\mathscr{P}F_1)"'] & \pi_1(\mathscr{F}) \arrow[r, "\pi_1(F_2)"'] & \pi_1(\mathscr{B}(\mathcal{V})) \arrow[r, "\pi_1(\mathscr{B}(F))"'] & \pi_1(\mathscr{B}(_\mathbf{A}\mathcal{V})) \end{tikzcd}$$

Following [31, Section 5], one constructs a bicategory $\mathbf{F}$ from the functor F with classifying category $cl\mathbf{F}$, endowed with a suitable monoidal structure. Since the unit object of $cl\mathbf{F}$ (and hence also of the symmetric cat-group $\mathscr{F} = \mathscr{P}(cl\mathbf{F})$) is (I, A, I), the elements of the group $\pi_1(\mathscr{F})$ are automorphisms of (I, A, I) in $cl\mathbf{F}$. But, according to [31], any endomorphism of (I, A, I) in $\mathbf{F}$ is a triple (V, f, W), where $V, W \in \mathcal{V}$ and

$$f : A \otimes W \simeq A \otimes_A (A \otimes W) \to (A \otimes V) \otimes_A A \simeq A \otimes V$$

is a morphism in ${_\mathbf{A}\mathcal{V}}$, and its 2-isomorphism class $[(V, f, W)]$ is an isomorphism in $cl\mathbf{F}$ iff ${_I V_I}$ and ${_I W_I}$ are equivalences in $\mathbf{Bim}(\mathcal{V})$ (or, equivalently, $V, W \in \mathscr{P}(\mathcal{V})$) and f is an isomorphism in ${_\mathbf{A}\mathcal{V}}$.

It follows that the assignment that sends $[(J, i_J)] \in \mathfrak{I}^\mathbf{I}_A$ to $[(J, (\xi^l_{i_J})^{-1}, I)]$, where $\xi^l_{i_J}$ is the composite $J \otimes A \xrightarrow{i_J \otimes A} A \otimes A \xrightarrow{m} A$, yields a map $t : \mathfrak{I}^\mathbf{I}_A \to \pi_1(\mathscr{F})$. It follows from the definitions of the map Ω_A and the functor F_2 (see [31, Section 6]) that t makes the square (2) commute. We claim that the square (1) also commutes. Indeed, let $\lambda : A \to A$ be an isomorphism in ${_\mathbf{A}\mathcal{V}}$. Then $\overline{\mathcal{D}}_A(\lambda) = [(A_\lambda, i_\lambda)]$, where

A_λ and i_λ are from the pullback

$$\begin{tikzcd} A_\lambda \arrow[r, "i_\lambda"] \arrow[d, "p_\lambda"'] & A \arrow[d, "\lambda"] \\ I \arrow[r, "e"'] & A \end{tikzcd}$$

Then $(t \cdot \overline{\mathcal{D}}_A)(\lambda) = [(A_\lambda, (\xi^l_{i_\lambda})^{-1}, I)]$, where $\xi^l_{i_\lambda}$ is the composite $A_\lambda \otimes A \xrightarrow{i_\lambda \otimes A} A \otimes A \xrightarrow{m} A$. On the other hand, $(\mathscr{P}F_1)(\lambda)$ is the 2-isomorphism class of the 1-arrow $(I, \overline{\lambda}, I)$ in $\mathbf{F}$, where $\overline{\lambda}$ is the composite $A \otimes_A A \simeq A \xrightarrow{\lambda} A \simeq A \otimes_A A$. We claim that $[(A_\lambda, (\xi^l_{i_\lambda})^{-1}, I)] = [(I, \overline{\lambda}, I)]$, or equivalently, that $(A_\lambda, (\xi^l_{i_\lambda})^{-1}, I)$ and $(I, \overline{\lambda}, I)$ are 2-isomorphic 1-cells in $\mathbf{F}$. Direct inspection, using commutativity of the above square, shows that the pair $(p_\lambda, 1_I)$ is a 2-isomorphism in $\mathbf{F}$. Thus, (1) is also commutative. Then the following diagram is also commutative, which implies that $\pi_1(P_{\mathscr{I}}) \cdot t : \mathfrak{I}^{\mathbf{I}}_A \to \pi_1(\overline{\mathscr{F}})$ is an isomorphism:

$$\begin{tikzcd}[column sep=huge]
\mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \arrow[r, "\overline{{}_{\mathbf{A}}\mathcal{V}_A}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & \mathfrak{I}^{\mathbf{I}}_A \arrow[r, "\Omega_A"] \arrow[d, dashed, "\pi_1(P_{\mathscr{I}})\cdot t"] \arrow[dr, phantom, "(2)"] & \mathbf{Pic}^c(\mathbf{I}) \arrow[d, equal] \\
\pi_1(\mathscr{P}({}_{\mathbf{A}}\mathcal{V})) \arrow[r, "\pi_1(P_{\mathscr{I}}\cdot(\mathscr{P}F_1))"'] & \pi_1(\overline{\mathscr{F}}) \arrow[r, "\pi_1(F_2')"'] & \pi_1(\mathscr{B}(\mathcal{V}))
\end{tikzcd}$$

Our next objective is to prove that the group $\mathbf{Ker}(\mathbf{Pic}^c(e))$ is isomorphic to a suitable Amitsur cohomology group $\mathcal{H}^1(e, \underline{\mathbf{Aut}}^{\mathfrak{Alg}}_I)$. In order to describe this cohomology group, and to prove the existence of the aforementioned isomorphism, we need to use some classical results which, for the convenience of the reader, are recalled in the Appendix. Let us first describe the functor $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_I$, which is a particular case of the given at the beginning of the Appendix.

Let $\mathcal{E}$ be the opposite of the category of commutative $\mathcal{V}$-algebras, $\mathtt{CAlg}(\mathcal{V})$. It is well known (e.g., [17, Corollary C.1.1.9]) that, under our assumptions on $\mathcal{V}$, $\mathcal{E}$ has pullbacks and they are constructed as tensor products. It is routine to check that the assignments

$$\mathbf{A} \longmapsto {}_{\mathbf{A}}\mathcal{V} \quad \text{and} \quad \mathbf{A} \xrightarrow{\iota} \mathbf{B} \longmapsto {}_{\mathbf{A}}\mathcal{V} \xrightarrow{\iota^*} {}_{\mathbf{B}}\mathcal{V},$$

where $\iota^* : {}_{\mathbf{A}}\mathcal{V} \to {}_{\mathbf{B}}\mathcal{V}$ is the *change–of–base* functor induced by ι (see Example 4.2), give rise to an $\mathcal{E}$-indexed category (see the Appendix)

$$\mathfrak{Alg} : \mathcal{E}^{\mathrm{op}} \to \mathbf{CAT}.$$

Since for any morphism $\iota : \mathbf{B} \to \mathbf{A}$ in $\mathcal{E}$ (i.e., a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathcal{V}$-algebras) the functor $\iota^* : {}_{\mathbf{A}}\mathcal{V} \to {}_{\mathbf{B}}\mathcal{V}$ admits as a right adjoint the forgetful functor $\iota_* : {}_{\mathbf{B}}\mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ (see Example 4.2), the $\mathcal{E}$-indexed category $\mathfrak{Alg}$ has products (in the sense of Definition A.2) if and only if, for any morphism $\kappa : \mathbf{C} \to \mathbf{A}$ in $\mathcal{E}$, there is an isomorphism $q_* p^* \to \iota^* \kappa_*$ of functors, where $p = \iota \otimes_{\mathbf{A}} C : C \to B \otimes_{\mathbf{A}} C$ and $q = B \otimes_{\mathbf{A}} \kappa : B \to B \otimes_{\mathbf{A}} C$. It is easy to see that this condition is equivalent to saying that for any $V \in {}_{\mathbf{C}}\mathcal{V}$, one has an isomorphism

$$(B \otimes_{\mathbf{A}} C) \otimes_{\mathbf{C}} V \simeq B \otimes_{\mathbf{A}} V,$$

and this is certainly the case, since the tensor product in $\mathcal{V}$ preserves reflexive coequalizers by our assumption on $\mathcal{V}$. Thus, $\mathfrak{Alg}$ admits products.

Lemma 4.11. *The functor* $\underline{\mathbf{Aut}}_A^{\mathfrak{Alg}} : (\mathcal{E} \downarrow \mathbf{A})^{op} = \mathbf{A} \downarrow \mathtt{CAlg}(\mathcal{V}) \to \mathbf{Group}$ *is described on objects as*

$$\underline{\mathbf{Aut}}_A^{\mathfrak{Alg}}(\mathbf{A} \xrightarrow{\iota} \mathbf{B}) = \mathbf{Aut}_{\mathbf{B}\mathcal{V}}(B) \tag{21}$$

and on morphisms as

$$\underline{\mathbf{Aut}}_A^{\mathfrak{Alg}}(f : (\mathbf{B}', \iota') \to (\mathbf{B}, \iota))(\sigma) = m_{B'} \cdot (B' \otimes f) \cdot (B' \otimes \sigma) \cdot (B' \otimes e_B) \tag{22}$$

for all $\sigma \in \mathbf{Aut}_{\mathbf{B}\mathcal{V}}(B)$.

Proof. Fix a commutative $\mathcal{V}$-algebra $\mathbf{A}$ and a left $\mathbf{A}$-module M_0. We have the functor

$$\underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}} : (\mathcal{E} \downarrow \mathbf{A})^{\mathrm{op}} = \mathbf{A} \downarrow \mathtt{CAlg}(\mathcal{V}) \to \mathbf{Group}$$

sending each object $\iota : \mathbf{B} \to \mathbf{A}$ of $\mathcal{E} \downarrow \mathbf{A}$ (i.e., a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathcal{V}$-algebras) to the group

$$\underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(\iota) = \mathbf{Aut}_{\mathbf{B}\mathcal{V}}(\iota^*(M_0)).$$

Note that since $\iota^*(M_0) = (B \otimes_{\mathbf{A}} M_0, m_B \otimes_{\mathbf{A}} M_0)$, $\underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(\iota) = \mathbf{Aut}_{\mathbf{B}\mathcal{V}}(B \otimes_{\mathbf{A}} M_0)$, where $B \otimes_{\mathbf{A}} M_0$ is a left $\mathbf{B}$-module via $m_B \otimes_{\mathbf{A}} M_0 : B \otimes B \otimes_{\mathbf{A}} M_0 \to B \otimes_{\mathbf{A}} M_0$.

Let us describe explicitly the action of $\underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}$ on morphisms. If $f : (\mathbf{B}', \iota') \to (\mathbf{B}, \iota)$ is a morphism in $(\mathcal{E} \downarrow \mathbf{A})^{\mathrm{op}}$ making the triangle

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ \mathbf{B} \arrow[rr, "f"'] & & \mathbf{B}' \end{tikzcd}$$

commute, then

$$\underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(f) : \underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(\iota) \to \underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(\iota')$$

takes $\sigma \in \underline{\mathbf{Aut}}_{M_0}^{\mathfrak{Alg}}(\iota)$ to the composite

$$B' \otimes_{\mathbf{A}} M_0 \simeq B' \otimes_{\mathbf{B}} (B \otimes_{\mathbf{A}} M_0) \xrightarrow{B' \otimes_{\mathbf{B}} \sigma} B' \otimes_{\mathbf{B}} (B \otimes_{\mathbf{A}} M_0) \simeq B' \otimes_{\mathbf{A}} M_0.$$

Since the split coequalizer diagram

$$\begin{tikzcd} B' \otimes B \otimes B \otimes_{\mathbf{A}} M_0 \arrow[r, shift left, "B' \otimes m_B \otimes_{\mathbf{A}} M_0"] \arrow[r, shift right, "a"'] & B' \otimes B \otimes_{\mathbf{A}} M_0 \arrow[l, bend right, "B' \otimes B \otimes e_B \otimes_{\mathbf{A}} M_0"'] \arrow[r, "B' \otimes f \otimes_{\mathbf{A}} M_0"'] & B' \otimes B' \otimes_{\mathbf{A}} M_0 \arrow[r, "m_{B'} \otimes_{\mathbf{A}} M_0"'] & B' \otimes_{\mathbf{A}} M_0 \arrow[ll, bend right, "B' \otimes e_B \otimes_{\mathbf{A}} M_0"'] \end{tikzcd}$$

(where a is the composite $(m_{B'}\otimes B\otimes_{\mathbf{A}}M_0)\cdot(B'\otimes f\otimes B\otimes_{\mathbf{A}}M_0))$ is the defining coequalizer for $B'\otimes_{\mathbf{B}}(B\otimes_{\mathbf{A}}M_0)$, it follows that $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_{M_0}(f)(\sigma)$ is the unique (iso)-morphism $B'\otimes_{\mathbf{A}}M_0\to B'\otimes_{\mathbf{A}}M_0$ making the diagram

$$\begin{tikzcd}
B'\otimes B\otimes_{\mathbf{A}}M_0 \arrow[r, "B'\otimes\sigma"] \arrow[d, "B'\otimes f\otimes_{\mathbf{A}}M_0"'] & B'\otimes B\otimes_{\mathbf{A}}M_0 \arrow[d, "B'\otimes f\otimes_{\mathbf{A}}M_0"] \\
B'\otimes B'\otimes_{\mathbf{A}}M_0 \arrow[d, "m_{B'}\otimes_{\mathbf{A}}M_0"'] & B'\otimes B'\otimes_{\mathbf{A}}M_0 \arrow[d, "m_{B'}\otimes_{\mathbf{A}}M_0"] \\
B'\otimes_{\mathbf{A}}M_0 \arrow[r, "\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_{M_0}(f)(\sigma)"'] & B'\otimes_{\mathbf{A}}M_0
\end{tikzcd}$$

commute. But since $(m_{B'}\otimes_{\mathbf{A}}M_0)\cdot(B'\otimes f\otimes_{\mathbf{A}}M_0)\cdot(B'\otimes e_B\otimes_{\mathbf{A}}M_0)=1$, it follows that

$$\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_{M_0}(f)(\sigma)=(m_{B'}\otimes_{\mathbf{A}}M_0)\cdot(B'\otimes f\otimes_{\mathbf{A}}M_0)\cdot(B'\otimes\sigma)\cdot(B'\otimes e_B\otimes_{\mathbf{A}}M_0).$$

Our interest is in the case where $M_0=A$ with the left regular action of $\mathbf{A}$ on A. Since $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_A(\mathbf{A}\xrightarrow{\iota}\mathbf{B})=\mathbf{Aut}_{\mathbf{B}\mathcal{V}}(B\otimes_{\mathbf{A}}A)$ and since the defining coequalizer diagram for $B\otimes_{\mathbf{A}}A$ is the split diagram

$$\begin{tikzcd}
B\otimes A\otimes A \arrow[r, shift left, "B\otimes m_A"] \arrow[r, shift right, "(m_B\otimes A)\cdot(B\otimes\iota\otimes A)"'] & B\otimes A \arrow[l, bend right=60, "B\otimes B\otimes e_A"'] \arrow[r, "B\otimes\iota"'] & B\otimes B \arrow[r, "m_B"'] & B \arrow[ll, bend right=60, "B\otimes e_A"']
\end{tikzcd}$$

it follows that the group $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_A(\iota)$ is canonically isomorphic to $\mathbf{Aut}_{\mathbf{B}\mathcal{V}}(B)$. $\square$

Note that since for each morphism $\iota:\mathbf{A}\to\mathbf{B}$ of commutative $\mathcal{V}$-algebras, $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_A(\iota)$ is an abelian group by Remark 4.6, it follows that the functor $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_A$ takes values in the category of abelian groups.

Now consider the augmented simplicial object in $\mathcal{E}=\mathtt{CAlg}(\mathcal{V})^{op}$

$$\begin{tikzcd}
(A/I)_*:\ I \arrow[r, "e"] & A \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & A\otimes A \arrow[l, bend right=60, "s_0"'] \arrow[r, shift left=2, "i_0"] \arrow[r, "i_1" description] \arrow[r, shift right=2, "i_2"'] & A\otimes A\otimes A \arrow[l, bend right=60, "{s_0, s_1}"'] \quad \cdots
\end{tikzcd} \tag{23}$$

associated to the morphism $e:I\to A$, which is a particular case of (27) in the Appendix. By applying the functor $\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_I$ to $(A/I)_*$, and computing cohomology, we get the first Amitsur cohomology group $\mathcal{H}^1(e\,,\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_I)$. The reader is referred to the Appendix for details.

Theorem 4.12. *Suppose that $\mathbf{A}=(A,m,e)$ is a $\mathcal{V}$-algebra such that the functor*

$$A\otimes-:\mathcal{V}\to{}_{\mathbf{A}}\mathcal{V}$$

is comonadic. Then there is a natural isomorphism

$$\mathcal{H}^1(e\,,\underline{\mathbf{Aut}}^{\mathfrak{Alg}}_I)\simeq\mathbf{Ker}(\mathbf{Pic}^c(e)).$$

Proof. According to Proposition 4.8, it is enough to show that there is a natural isomorphism $\mathbf{Coker}(\kappa_A) \simeq \mathcal{H}^1(e\,, \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}})$.

Write $\mathcal{G}_e$ for the comonad on ${}_{\mathbf{A}}\mathcal{V}$ generated by the adjunction

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "e^*=A\otimes -"] \arrow[r, phantom, "\top"] & {}_{\mathbf{A}}\mathcal{V} \arrow[l, bend left, "e_*=U"] \end{tikzcd}$$

and write $\mathcal{G}_e\mathbf{-Coalg}(A, m_A)$ for the set of all $\mathcal{G}_e$-coalgebra structures on $(A, m_A) \in {}_{\mathbf{A}}\mathcal{V}$. We know from [15, Proposition 4.5] that $\mathcal{G}_e\mathbf{-Coalg}(A, m_A) = \mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$ and that $\mathbf{End}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e) = \mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$ by Proposition 4.5. On the other hand, $\mathrm{Des}_{\mathfrak{Alg}}(A, m_A) = \mathcal{Z}^1(\iota\,, \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}})$ by Proposition A.4, and $\mathrm{Des}_{\mathfrak{Alg}}(A, m_A) = \mathcal{G}_e\mathbf{-Coalg}(A, m_A)$ by Theorem A.3. It follows that $\mathcal{Z}^1(e\,, \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}) = \mathbf{Aut}_{\mathbf{A}\text{-}\mathtt{cor}}(\mathfrak{C}_e)$.

Applying the functor $\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}$ to (23), and using the fact that for any commutative $\mathcal{V}$-algebra $\mathbf{S}$, $\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(S) = \mathbf{Aut}_{{}_{\mathbf{S}}\mathcal{V}}(S)$ is an abelian group by Remark 4.6, we get the simplicial abelian group

$$\begin{tikzcd}[column sep=large] (A/I, \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}})_* : \mathbf{Aut}_{\mathcal{V}}(I) \arrow[r, "\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(e)"] & \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \arrow[r, shift left, "\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_0)"] \arrow[r, shift right, "\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_1)"'] & \mathbf{Aut}_{{}_{\mathbf{A}\otimes\mathbf{A}}\mathcal{V}}(A\otimes A) \ldots \arrow[l, bend right=60, "\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(s_0)"'] \end{tikzcd}$$

and the corresponding complex $C(A/I, \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}})$ of abelian groups

$$0 \to \mathbf{Aut}_{\mathcal{V}}(I) \xrightarrow{\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(e)} \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \xrightarrow{\Delta_1} \mathbf{Aut}_{{}_{\mathbf{A}\otimes\mathbf{A}}\mathcal{V}}(A\otimes A) \xrightarrow{\Delta_2} \cdots$$

where

$$\Delta_n = \prod_{i=0}^{n} \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_n)^{(-1)^n},\ n \geq 1.$$

Since $i_0 = A\otimes e$, $i_1 = e\otimes A$ and since the multiplication in the tensor product $\mathcal{V}$-algebra $\mathbf{A}\otimes\mathbf{A}$ is given by the composite $(m_A\otimes m_A)\cdot(A\otimes\tau_{A,A}\otimes A)$, it follows from (22) that

$$\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_0)(\lambda) = (m_A\otimes m_A)\cdot(A\otimes\tau_{A,\,A}\otimes A)\cdot(A\otimes A\otimes A\otimes e)\cdot(A\otimes A\otimes\lambda)\cdot(A\otimes A\otimes e)$$

and

$$\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_1)(\lambda) = (m_A\otimes m_A)\cdot(A\otimes\tau_{A,\,A}\otimes A)\cdot(A\otimes A\otimes e\otimes A)\cdot(A\otimes A\otimes\lambda)\cdot(A\otimes A\otimes e)$$

for all $\lambda \in \mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A, m_A)$.

But since in the diagram

$$\begin{tikzcd}
A\otimes A \arrow[r, "A\otimes A\otimes e"] \arrow[dr, "A\otimes e\otimes A"'] \arrow[ddr, bend right, "1_{A\otimes A}"'] & A\otimes A\otimes A \arrow[r, "A\otimes A\otimes \lambda"] \arrow[d, "A\otimes \tau_{A,\,A}"] & A\otimes A\otimes A \arrow[r, "A\otimes A\otimes A\otimes e"] \arrow[d, "A\otimes \tau_{A,\,A}"] & A\otimes A\otimes A\otimes A \arrow[d, "A\otimes \tau_{A,\,A}\otimes A"] \\
 & A\otimes A\otimes A \arrow[r, "A\otimes \lambda\otimes A"'] \arrow[d, "m_A\otimes A"'] & A\otimes A\otimes A \arrow[r, "A\otimes A\otimes A\otimes e"'] \arrow[dr, equal] & A\otimes A\otimes A\otimes A \arrow[d, "A\otimes A\otimes m_A"] \\
 & A\otimes A \arrow[r, "\lambda\otimes A"'] & A\otimes A & A\otimes A\otimes A \arrow[l, "m_A\otimes A"]
\end{tikzcd}$$

- the top left triangle commutes since τ is symmetry,
- the middle rectangle commutes by naturality of τ,
- the right rectangle commutes by naturality of composition,
- the curved triangle and the bottom right triangle commute since e is the unit of $\mathbf{A}$, and
- the trapezoid commutes since λ is an automorphism of the left $\mathbf{A}$-module (A, m),

while in the diagram

$$\begin{tikzcd}
A\otimes A \arrow[r, "A\otimes A\otimes e"] \arrow[ddr, equal] & A\otimes A\otimes A \arrow[r, "A\otimes A\otimes \lambda"] \arrow[dd, "A\otimes m_A"] & A\otimes A\otimes A \arrow[r, "A\otimes A\otimes e\otimes A"] \arrow[d, equal] \arrow[dr, "A\otimes e\otimes A\otimes A"'] & A\otimes A\otimes A\otimes A \arrow[d, "A\otimes \tau_{A,\,A}\otimes A"] \\
 & & A\otimes A\otimes A \arrow[d, "A\otimes m_A"'] & A\otimes A\otimes A\otimes A \arrow[l, "m_A\otimes A\otimes A"] \\
 & A\otimes A \arrow[r, "A\otimes \lambda"'] & A\otimes A &
\end{tikzcd}$$

- the top right triangle commutes since τ is symmetry,
- the left and the bottom right triangles commute since e is the unit of $\mathbf{A}$, and
- the rectangle commutes since λ is an automorphism of the left $\mathbf{A}$-module (A, m),

it follows that

$$\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_0) = \lambda\otimes A \text{ and } \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_1) = A\otimes\lambda,$$

and since $\Delta_1 = \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_0) \cdot (\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_1))^{-1} = (\underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_1))^{-1} \cdot \underline{\mathbf{Aut}}_I^{\mathfrak{Alg}}(i_0)$, it follows that $\Delta_1 = (A\otimes\lambda^{-1})\cdot(\lambda\otimes A)$. From Proposition 4.9 we get the commutativity in

$$\begin{tikzcd}
\mathbf{Aut}_{\mathcal{V}_{\mathbf{A}}}(A) \arrow[r, "\kappa_A"] \arrow[d, equal] & \mathbf{Aut}_{\mathbf{A}-\mathtt{cor}}(\mathfrak{C}_e) \arrow[d, equal] \\
\mathbf{Aut}_{{}_{\mathbf{A}}\mathcal{V}}(A) \arrow[r, "\Delta_1"'] & \mathcal{Z}^1(\iota\,, \underline{\mathbf{Aut}}^{\mathfrak{Alg}}(I)) \subseteq \mathbf{Aut}_{{}_{\mathbf{A}\otimes\mathbf{A}}\mathcal{V}}(A\otimes A)
\end{tikzcd}$$

implying that $\mathbf{Coker}(\kappa_A) \simeq \mathcal{H}^1(e\,, \underline{\mathbf{Aut}}\,_I^{\mathfrak{Alg}})$. This completes the proof of the theorem. □

It is well known (e.g., [17]) that for any commutative $\mathcal{V}$-algebra $\mathbf{A}$, one has

$$\mathcal{E}{\downarrow}\mathbf{A} = (\mathbf{A}{\downarrow}\mathtt{CAlg}(\mathcal{V}))^{\mathrm{op}}.$$

Moreover, the coslice category $\mathbf{A}{\downarrow}\mathtt{CAlg}(\mathcal{V})$ is isomorphic to the category $\mathtt{CAlg}({}_{\mathbf{A}}\mathcal{V})$. In other words, to give a commutative algebra $\mathbf{B}$ in the symmetric monoidal category ${}_{\mathbf{A}}\mathcal{V}$ is to give a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative algebras in $\mathcal{V}$. The latter morphism serves as the unit morphism of the ${}_{\mathbf{A}}\mathcal{V}$-monoid $\mathbf{B}$. Write $(\iota\,)$ for the corresponding commutative algebra in the symmetric monoidal category ${}_{\mathbf{A}}\mathcal{V}$. Then a (left) $(\iota\,)$-module in ${}_{\mathbf{A}}\mathcal{V}$ consists of a (left) $\mathbf{A}$-module structure $A{\otimes}M \to M$ together with a morphism $\rho : B{\otimes}_{\mathbf{A}}M \to M$ in ${}_{\mathbf{A}}\mathcal{V}$. A straightforward calculation shows that the composite

$$B{\otimes}M \xrightarrow{q_{B,M}} B{\otimes}_{\mathbf{A}}M \xrightarrow{\rho} M$$

makes M into a $\mathbf{B}$-module. In other direction, if $\varrho : B{\otimes}M \to M$ is a $\mathbf{B}$-module structure on M, then the pair $(M, A{\otimes}M \xrightarrow{\iota\otimes M} B{\otimes}M \xrightarrow{\varrho} M)$ is a left $\mathbf{A}$-module and $\varrho = \varrho' \cdot q_{B,M}$ for a unique $\varrho' : B{\otimes}_{\mathbf{A}}M \to M$. Then (M, ϱ') is a left $(\iota\,)$-module in ${}_{\mathbf{B}}\mathcal{V}$. It is easily checked that the above constructions are inverse to each other, and hence give an isomorphism ${}_{(\iota)}({}_{\mathbf{A}}\mathcal{V}) \simeq {}_{\mathbf{B}}\mathcal{V}$ of categories. This allows us to identify the change-of-base functor $\iota^* = B{\otimes}_{\mathbf{A}}- : {}_{\mathbf{A}}\mathcal{V} \to {}_{\mathbf{B}}\mathcal{V}$ with the functor $(\iota\,){\otimes}_{\mathbf{A}}- : {}_{\mathbf{A}}\mathcal{V} \to {}_{(\iota\,)}({}_{\mathbf{A}}\mathcal{V})$.

One then constructs an $\mathcal{E}{\downarrow}\mathbf{A}$-indexed category

$$\mathfrak{Alg} : (\mathcal{E}{\downarrow}\mathbf{A})^{\mathrm{op}} \to \mathbf{CAT}$$

as follows: if $(\iota : \mathbf{A} \to \mathbf{B})$ is an object of $(\mathcal{E}{\downarrow}\mathbf{A})^{\mathrm{op}}$, then $\mathfrak{Alg}(\iota) = {}_{\mathbf{B}}\mathcal{V}$, and if

$$\begin{tikzcd} & \mathbf{B} \arrow[d, "f"] \\ \mathbf{A} \arrow[ur, "\iota"] \arrow[r, "\iota'"'] & \mathbf{B}' \end{tikzcd}$$

is a morphism in $(\mathcal{E}{\downarrow}\mathbf{A})^{\mathrm{op}}$, then f^* is the functor $B'{\otimes}_{\mathbf{B}}- : {}_{\mathbf{B}}\mathcal{V} \to {}_{\mathbf{B}'}\mathcal{V}$. This $\mathcal{E}{\downarrow}\mathbf{A}$-indexed category satisfies the Beck–Chevalley condition (see the Appendix) and applying Theorem 4.12 gives us the following.

Theorem 4.13. *Suppose that $\iota : \mathbf{A} \to \mathbf{B}$ is a morphism of commutative $\mathcal{V}$-algebras such that the change-of-base functor $B{\otimes}_{\mathbf{A}}- : {}_{\mathbf{A}}\mathcal{V} \to {}_{\mathbf{B}}\mathcal{V}$ is comonadic. Then there is a natural isomorphism*

$$\mathcal{H}^1(\iota\,, \underline{\mathbf{Aut}}_A^{\mathfrak{Alg}}) \simeq \mathbf{Ker}(\mathbf{Pic}^c(\iota)).$$

Let $R \subseteq A$ be an extension of commutative rings. If $\iota : R \to A$ denotes the inclusion map, then $\mathcal{H}^1(\iota\,, \underline{\mathbf{Aut}}_R^{\mathfrak{Alg}})$ is just the first Amitsur cohomology group $H^1(A/R, U)$, where U denotes the "units" functor. When ι is a faithfully flat extension of commutative rings, then the change-of-base functor $A{\otimes}_R - : {}_R\mathbf{Mod} \to {}_A\mathbf{Mod}$ is comonadic (see, for example, [24]). Moreover, specializing Theorem 4.13 to this case gives the following well-known result (see, for example, [6, Corollary 4.6]).

Corollary 4.14. *Let A be a faithfully flat commutative R-algebra, and let $\iota : R \to A$ be the inclusion map. Then there is a natural isomorphism*

$$H^1(A/R, U) \to \mathbf{Ker}(\mathbf{Pic}^c(\iota)).$$

Theorem 4.13 also implies the following, in view of Remark 3.15.

Corollary 4.15. *Let A be a commutative R-algebra such that R is a direct summand of A as an R-module, and let $\iota : R \to A$ be the inclusion map. Then there is a natural isomorphism*

$$H^1(A/R, U) \to \mathbf{Ker}(\mathbf{Pic}^c(\iota)).$$

4.3. **The category of bicomodules.** As observed in Subsection 3.4, there are dual versions of the exact sequences built in Subsections 3.1, 3.2, and 3.3 from an adjunction in a bicategory. One reason for explicitly recording them is to have statements tailored to concrete situations, as Examples 3.10 and 3.18 illustrate. Next, with the same motivation, we will consider the bicategory of bicomodules, and we will record some exact sequences derived from an adjunction in this setting. We close with some applications.

In this subsection, suppose that $\mathcal{V} = (V, \otimes, I)$ is a monoidal category with equalizers such that all the functors $X\otimes - : V \to V$, as well as $-\otimes X : V \to V$ for $X \in \mathcal{V}$, preserve equalizers. Coalgebras and (*left, right, bi-*) *comodules* in $\mathcal{V}$ can be defined as algebras and left (right, bi-) modules in the *opposite* monoidal category $(\mathcal{V}^{\mathrm{op}}, \otimes, I)$. The resulting categories are denoted by $\mathtt{Coalg}(\mathcal{V})$, ${}^{\mathbf{C}}\mathcal{V}$, $\mathcal{V}^{\mathbf{C}}$, and ${}^{\mathbf{C}}\mathcal{V}^{\mathbf{D}}$, with $\mathbf{C}$ and $\mathbf{D}$ being coalgebras in $\mathcal{V}$.

Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be $\mathcal{V}$-coalgebras. Dualizing the tensor product of bimodules, the *cotensor product* $X\square_{\mathbf{C}}Y$ of a $(\mathbf{D}, \mathbf{C})$-bicomodule $(X, \vartheta^l, \vartheta^r)$ and a $(\mathbf{C}, \mathbf{E})$-bicomodule (Y, θ^l, θ^r) over $\mathbf{C}$ is defined to be the equalizer of the pair of morphisms

$$X\square_{\mathbf{C}}Y \xrightarrow{\kappa_{X,Y}} X\otimes Y \underset{1\otimes\theta^l}{\overset{\vartheta^r\otimes 1}{\rightrightarrows}} X\otimes C\otimes Y.$$

Note that $X\square_{\mathbf{C}}Y$ is a $(\mathbf{D}, \mathbf{E})$-bicomodule.

Recall (for example, from [9]) that there is a bicategory $\mathbf{Bicom}(\mathcal{V})$, called the *bicategory of $\mathcal{V}$-bicomodules*, in which

- 0-cells are $\mathcal{V}$-coalgebras,
- for $\mathbf{C}, \mathbf{D} \in \mathtt{Coalg}(\mathcal{V})$, the hom-category $\mathbf{Bicom}(\mathcal{V})(\mathbf{C}, \mathbf{D})$ is the category ${}^{\mathbf{C}}\mathcal{V}^{\mathbf{D}}$ of $(\mathbf{C}, \mathbf{D})$-bicomodules,
- 2-cells are morphisms of bicomodules, with obvious vertical composition and identities, and
- horizontal composition is the opposite of the cotensor product of bicomodules; the identity 1-cell $\iota_{\mathbf{C}}$ for $\mathbf{C} \in \mathtt{Coalg}(\mathcal{V})$ is the *regular* $(\mathbf{C}, \mathbf{C})$-bicomodule $\mathbf{C}$, i.e., the $(\mathbf{C}, \mathbf{C})$-bicomodule $(\mathbf{C}, \delta_{\mathbf{C}}, \delta_{\mathbf{C}})$.

Suppose in addition that $\mathcal{V}$ admits, besides equalizers, finite colimits and coimage factorizations. In this case, for any two $\mathcal{V}$-coalgebras $\mathbf{C}$ and $\mathbf{C}'$, the category ${}^{\mathbf{C}}\mathcal{V}^{\mathbf{C}'} = \mathbf{Bicom}(\mathcal{V})(\mathbf{C}, \mathbf{C}')$, being the category of Eilenberg–Moore algebras for the comonad $\mathbf{C}\otimes - \otimes\mathbf{C}'$, also admits coequalizers (see, for example, [4]) and coimage factorizations (see [1]).

Applying Theorem 3.16 gives the following.

Theorem 4.16. *In the circumstances above, let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* $\mathcal{V}$*-coalgebras and let* $\Lambda : \mathbf{D} \rightsquigarrow \mathbf{C}$ *be a* 1*-cell admitting a right adjoint* $\Lambda^* : \mathbf{C} \rightsquigarrow \mathbf{D}$ *with unit* $\eta_\Lambda : \iota_{\mathbf{D}} \to \Lambda^* \circ \Lambda = \Lambda \Box_{\mathbf{C}} \Lambda^*$ *and counit* $\varepsilon_\Lambda : \Lambda \circ \Lambda^* = \Lambda^* \Box_{\mathbf{D}} \Lambda \to \iota_{\mathbf{C}}$*. If* ε_Λ *is epimorphic, then*

$$1 \to \mathbf{Aut}_{\mathbf{C}\mathcal{V}\mathbf{C}}(C) \xrightarrow{\widehat{\omega}_0} \mathbf{Aut}_{\mathbf{D}\mathcal{V}\mathbf{C}}(\Lambda) \xrightarrow{\overline{\mathcal{D}}_{\Lambda^*}} \mathcal{Q}_\Lambda^{\mathbf{C}} \xrightarrow{\Omega_{\Lambda^*}} \mathbf{Pic}(\mathbf{C})$$

is an exact sequence of groups, while

$$\mathcal{Q}_\Lambda^{\mathbf{C}} \xrightarrow{\Omega_{\Lambda^*}} \mathbf{Pic}(\mathbf{C}) \xrightarrow{[-\Box_{\mathbf{C}}\Lambda^*]} \pi_0({}^{\mathbf{C}}\mathcal{V}^{\mathbf{D}})$$

is an exact sequence of pointed sets.

Each morphism $\phi : \mathbf{D} \to \mathbf{C}$ of $\mathcal{V}$-coalgebras determines a bicomodule $\mathbf{D}_\phi : \mathbf{D} \rightsquigarrow \mathbf{C}$ defined to be D together with the coactions

$$(D \xrightarrow{\delta_D} D \otimes D, D \xrightarrow{\delta_D} D \otimes D \xrightarrow{D \otimes \phi} D \otimes C)$$

and a bicomodule $\mathbf{D}^\phi : \mathbf{C} \rightsquigarrow \mathbf{D}$ defined to be C together with the coactions

$$(D \xrightarrow{\delta_D} D \otimes D \xrightarrow{\phi \otimes D} C \otimes D, D \xrightarrow{\delta_D} D \otimes D).$$

$\mathbf{D}^\phi$ is right adjoint to $\mathbf{D}_\phi$ in $\mathbf{Bicom}(\mathcal{V})$ with

$$\mathbf{D}_\phi \circ \mathbf{D}^\phi = D \Box_{\mathbf{D}} D \simeq D \xrightarrow{\phi} C$$

as counit and

$$D \xrightarrow{\overline{\delta}_D} D \Box_{\mathbf{C}} D = \mathbf{D}^\phi \circ \mathbf{D}_\phi$$

as unit. Here $D \xrightarrow{\overline{\delta}_D} D \Box_{\mathbf{C}} D$ is the unique morphism making the triangle

$$\begin{tikzcd} D \arrow[r, "\delta_D"] \arrow[dr, "\overline{\delta}_D"'] & D \otimes D \\ & D \Box_{\mathbf{C}} D \arrow[u, "\kappa_{D,D}"'] \end{tikzcd}$$

commute.

It follows that a morphism $\phi : \mathbf{D} \to \mathbf{C}$ of $\mathcal{V}$-coalgebras gives rise to two functors

$$\phi^* = -\Box_{\mathbf{C}} D^\phi : \mathcal{V}^{\mathbf{C}} \to \mathcal{V}^{\mathbf{D}},$$

$$(Y, \vartheta_Y) \in \mathcal{V}^{\mathbf{C}} \longmapsto (Y \Box_{\mathbf{C}} D, Y \Box_{\mathbf{C}} \delta_D),$$

known as the *change-of-cobase functor*, and

$$\phi_* = -\Box_{\mathbf{D}} D_\phi : \mathcal{V}^{\mathbf{D}} \to \mathcal{V}^{\mathbf{C}},$$

where for any right $\mathbf{D}$-comodule (X, θ_X), $\phi_*(X, \theta_X) = X$ is a right $\mathbf{C}$-comodule via the coaction

$$X \xrightarrow{\theta_X} X \otimes D \xrightarrow{X \otimes \phi} X \otimes C.$$

Since $\mathbf{D}^\phi$ is right adjoint to $\mathbf{D}_\phi$ in $\mathbf{Bicom}(\mathcal{V})$, it follows that ϕ^* is left adjoint to ϕ_*.

Assume further that $\mathcal{V}$ is symmetric with symmetry τ. It is well known that the category of cocommutative $\mathcal{V}$-coalgebras, `CCoalg`$(\mathcal{V})$, has pullbacks and they are constructed as cotensor products: For any two morphism $\phi : \mathbf{D} \to \mathbf{C}$ and $\phi' : \mathbf{D}' \to \mathbf{C}$ in $\mathtt{CCoalg}(\mathcal{V})$, the diagram

$$\begin{tikzcd}
\mathbf{D}'\Box_{\mathbf{C}}\mathbf{D} \arrow[r, "p_{\mathbf{D}}"] \arrow[d, "p_{\mathbf{D}'}"'] & \mathbf{D} \arrow[d, "\phi"] \\
\mathbf{D}' \arrow[r, "\phi'"'] & \mathbf{C}
\end{tikzcd}$$

where $p_{\mathbf{D}} = \varepsilon_{D'}\Box_{\mathbf{C}}\mathbf{D}$ and $p_{\mathbf{D}'} = \mathbf{D}'\Box_{\mathbf{C}}\varepsilon_D$, is a pullback in $\mathtt{CCoalg}(\mathcal{V})$.

Define a $\mathtt{CCoalg}(\mathcal{V})$-indexed category

$$\mathfrak{Coalg} : (\mathtt{CCoalg}(\mathcal{V}))^{\mathrm{op}} \to \mathbf{CAT}$$

by setting

$$\mathfrak{Coalg}(\mathbf{C}) = \mathcal{V}^{\mathbf{C}} \quad \text{and} \quad \mathfrak{Coalg}(\mathbf{D} \xrightarrow{\phi} \mathbf{C}) = \mathcal{V}^{\mathbf{C}} \xrightarrow{\phi^*} \mathcal{V}^{\mathbf{D}},$$

where $\phi^* = -\Box_{\mathbf{C}}D : \mathcal{V}^{\mathbf{C}} \to \mathcal{V}^{\mathbf{D}}$ is the change-of-cobase functor. Since for any morphism $\phi : \mathbf{D} \to \mathbf{C}$ in $\mathtt{CCoalg}(\mathcal{V})$, the functor $\phi^* = -\Box_{\mathbf{C}}D$ admits as a left adjoint the forgetful functor $\mathcal{V}^{\mathbf{D}} \xrightarrow{\phi_* = -\Box_{\mathbf{D}}D} \mathcal{V}^{\mathbf{C}}$, the $\mathtt{CCoalg}(\mathcal{V})$-indexed category $\mathfrak{Coalg}$ admits coproducts iff it satisfies the Beck–Chevalley condition, i.e., for any morphism $\phi' : \mathbf{D}' \to \mathbf{C}$ in $\mathtt{CCoalg}(\mathcal{V})$, the diagram

$$\begin{tikzcd}
\mathcal{V}^{\mathbf{D}'} \arrow[r, "(p_{\mathbf{D}'})^*"] \arrow[d, "(\phi')_*"'] & \mathcal{V}^{\mathbf{D}'\Box_{\mathbf{C}}\mathbf{D}} \arrow[d, "(p_{\mathbf{D}})_*"] \\
\mathcal{V}^{\mathbf{C}} \arrow[r, "\phi^*"'] & \mathcal{V}^{\mathbf{D}}
\end{tikzcd}$$

commutes up to canonical isomorphism. It is easily to check that this condition is equivalent to saying that for any $X \in \mathcal{V}^{\mathbf{D}'}$, one has an isomorphism

$$X\Box_{\mathbf{D}'}(D'\Box_{\mathbf{C}}D) \simeq X\Box_{\mathbf{C}}D,$$

and this is certainly the case, since the tensor product in $\mathcal{V}$ preserves reflexive equalizers by our assumptions on $\mathcal{V}$. Thus, $\mathfrak{Coalg}$ admits coproducts.

As in the case of algebras, given a cocommutative $\mathcal{V}$-coalgebra $\mathbf{C}$, we get the functor

$$\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}} : (\mathtt{CCoalg}{\downarrow}\mathbf{C})^{\mathrm{op}} \to \mathbf{Group}$$

sending each object $\phi : \mathbf{D} \to \mathbf{C}$ of $(\mathtt{CCoalg}{\downarrow}\mathbf{C})^{\mathrm{op}}$ (i.e., a morphism $\phi : \mathbf{D} \to \mathbf{C}$ of cocommutative $\mathcal{V}$-coalgebras) to the group

$$\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\mathbf{D} \xrightarrow{\phi} \mathbf{C}) = \mathbf{Aut}_{\mathcal{V}^{\mathbf{D}}}(D). \tag{24}$$

On morphisms it is defined in the following way: given a morphism $f : (\mathbf{D}, \phi) \to (\mathbf{D}', \phi')$ in $(\mathtt{CCoalg}{\downarrow}\mathbf{C})^{\mathrm{op}}$ (i.e., a commutative diagram

$$\begin{tikzcd}
 & \mathbf{C} & \\
\mathbf{D}' \arrow[ur, "\phi'"] \arrow[rr, "f"'] & & \mathbf{D} \arrow[ul, "\phi"']
\end{tikzcd}$$

in $\mathtt{CCoalg}{\downarrow}\mathbf{C}$), then

$$\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(f:(\mathbf{D},\phi)\to(\mathbf{D}',\phi'))(\sigma)=(\varepsilon_D\otimes D')\cdot(\sigma\otimes D')\cdot(f\otimes D')\cdot\delta_{D'} \tag{25}$$

for all $\sigma\in\mathbf{Aut}_{\mathcal{V}^{\mathbf{D}}}(D)$.

Note that, since for each morphism $\phi:\mathbf{D}\to\mathbf{C}$ of cocommutative $\mathcal{V}$-coalgebras, $\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\phi)$ is an abelian group, it follows that the functor $\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}$ also takes values in the category of abelian groups.

Given a morphism $\phi:\mathbf{D}\to\mathbf{C}$ of cocommutative $\mathcal{V}$-coalgebras, consider the associated augmented simplicial object

$$\begin{tikzcd}
(\mathbf{D}/\mathbf{C})_*: \quad \cdots \; D^2 \arrow[r, shift left=2, "\partial_0"] \arrow[r, "\partial_1" description] \arrow[r, shift right=2, "\partial_2"'] & D^1 \arrow[l, bend right=60, "s_0,s_1"'] \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & D^0 \arrow[l, bend right=60, "s_0"'] \arrow[r, "\phi"] & C
\end{tikzcd}$$

where

- $D^0=D$,
- $D^n=\underbrace{D\Box_{\mathbf{C}}D\cdots\Box_{\mathbf{C}}D}_{(n+1)\text{-times}}$ for all $n\geq 1$,
- $\partial_i=D^i\Box_{\mathbf{C}}\phi\Box_{\mathbf{C}}D^{n-i}:D^{n+1}\to D^n$ for all $0\leq i\leq n$,
- $s_j=D^j\Box_{\mathbf{C}}\overline{\delta}_D\Box_{\mathbf{C}}D^{n-j-1}:D^{n-1}\to D^n$ for all $0\leq j\leq n$.

Applying the functor $\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}:(\mathtt{CCoalg}{\downarrow}\mathbf{C})^{\mathrm{op}}\to\mathbf{Group}$ to $(\mathbf{D}/\mathbf{C})_*$, we obtain the following augmented cosimplicial group

$$\begin{tikzcd}
(\mathbf{D}/\mathbf{C},\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}})_*:\mathbf{Aut}_{\mathcal{V}^{\mathbf{C}}}(C) \arrow[r, "\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\phi)"] & \mathbf{Aut}_{\mathcal{V}^{\mathbf{D}}}(D) \arrow[r, shift left, "\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\partial_0)"] \arrow[r, shift right, "\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\partial_1)"'] & \mathbf{Aut}_{\mathcal{V}^{\mathbf{D}\Box_{\mathbf{C}}\mathbf{D}}}(D\Box_{\mathbf{C}}D)\ldots \arrow[l, bend right=60, "\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(s_0)"']
\end{tikzcd}$$

Since for any $n\geq 0$, $\mathbf{D}^n$ is a cocommutative $\mathcal{V}$-coalgebra, and all the categories $\mathcal{V}^{\mathbf{D}^n}$ are symmetric monoidal with monoidal unit D^n, it follows that $(\mathbf{D}/\mathbf{C},\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}})_*$ is in fact an augmented abelian cosimplicial group.

Write $\mathbf{Pic}^c(\mathbf{C})$ for the subgroup of $\mathbf{Pic}(\mathbf{C})$ consisting of all classes of invertible $(\mathbf{C},\mathbf{C})$-bicomodules $(X,\vartheta_l:X\to C\otimes X,\vartheta_r:\to X\otimes C)$ such that $\vartheta_r=\tau_{C,X}\cdot\vartheta_l$. Then $\mathbf{Pic}^c(\mathbf{C})$ is an abelian group. Moreover, given a morphism $\phi:\mathbf{D}\to\mathbf{C}$ of cocommutative $\mathcal{V}$-coalgebras, the map

$$\mathbf{Pic}^c(\phi):\mathbf{Pic}^c(\mathbf{C})\to\mathbf{Pic}^c(\mathbf{D})$$

defined by $\mathbf{Pic}^c(\phi)([P])=[P\Box_{\mathbf{C}}D]$ is a homomorphism of abelian groups.

Now with the complex of abelian groups

$$0\to\mathbf{Aut}_{\mathcal{V}^{\mathbf{C}}}(C)\xrightarrow{\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\phi)}\mathbf{Aut}_{\mathcal{V}^{\mathbf{D}}}(D)\xrightarrow{\Delta_1}\mathbf{Aut}_{\mathcal{V}^{\mathbf{D}\Box_{\mathbf{C}}\mathbf{D}}}(D\Box_{\mathbf{C}}D)\xrightarrow{\Delta_2}\cdots,$$

$$\Delta_n=\prod_{i=0}^{n}\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}(\partial_n)^{(-1)^n},\ n\geq 1,$$

corresponding to the augmented abelian cosimplicial group $(\mathbf{D}/\mathbf{C},\underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}})_*$, by arguments similar to those used in Theorem 4.13, we derive the following result.

Theorem 4.17. *Suppose that $\phi : \mathbf{D} \to \mathbf{C}$ is a morphism of cocommutative $\mathcal{V}$-coalgebras such that the change-of-cobase functor $\phi^* = -\Box_{\mathbf{C}} D : \mathcal{V}^{\mathbf{C}} \to \mathcal{V}^{\mathbf{D}}$ is monadic. Then there is a natural isomorphism*

$$\mathcal{H}^1(\phi, \underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}) \simeq \mathbf{Ker}(\mathbf{Pic}^c(\phi)).$$

Specializing Theorem 4.17 to the case where $\mathcal{V} = \mathtt{Vect}_k$ is the category of vector spaces over a field k and using that for any cocommutative k-coalgebra $\mathbf{C}$, $\mathbf{Pic}^c(\mathbf{C}) = 0$ (see, for example, [8, Proposition 3.2.14] or [7, Proposition 4.1]), we get the following version of Hilbert's theorem 90 for cocommutative coalgebras.

Theorem 4.18. *Suppose that $\phi : \mathbf{D} \to \mathbf{C}$ is a morphism of cocommutative k-coalgebras such that the change-of-cobase functor $-\Box_{\mathbf{C}} D : \mathtt{Vect}_k^{\mathbf{C}} \to \mathtt{Vect}_k^{\mathbf{D}}$ is monadic. Then*

$$\mathcal{H}^1(\phi, \underline{\mathbf{Aut}}_{\mathbf{C}}^{\mathfrak{Coalg}}) = 0.$$

Appendix A. Some classical results and constructions

Let $\mathcal{A}$ be a category with pullbacks. An $\mathcal{A}$-indexed category $\mathcal{X}$ is a pseudofunctor $\mathcal{A}^{\mathrm{op}} \to \mathbf{CAT}$, where $\mathbf{CAT}$ denotes the 2-category of locally small (but possibly large) categories, explicitly given by the data of a family of categories $\mathcal{X}(a)$, indexed by the objects of $\mathcal{A}$, with *change of base functors*

$$\iota^* : \mathcal{X}(a) \to \mathcal{X}(b)$$

for each morphism $\iota : b \to a$ of $\mathcal{A}$ and with additional structure expressing the idea of a pseudofunctor (see [22], [26]).

A simple but important example of an $\mathcal{A}$-indexed category is the so-called *basic $\mathcal{A}$-indexed category* $\mathcal{A}{\downarrow}- : \mathcal{A}^{\mathrm{op}} \to \mathbf{CAT}$ that to any object $a \in \mathcal{A}$ associates the slice category $\mathcal{A}{\downarrow}a$, and to a morphism $\iota : b \to a$ the functor $\iota^* : \mathcal{A}{\downarrow}a \to \mathcal{A}{\downarrow}b$ given by pulling back along ι.

Fix an object a of $\mathcal{A}$, and let $\mathcal{X} : \mathcal{A}^{\mathrm{op}} \to \mathbf{CAT}$ be an $\mathcal{A}$-indexed category. For each object $x \in \mathcal{X}(a)$, let us define a functor

$$\underline{\mathbf{Aut}}_x^{\mathcal{X}} : (\mathcal{A}{\downarrow}a)^{\mathrm{op}} \to \mathbf{Group}$$

sending each object $b \xrightarrow{\kappa} a$ of $\mathcal{A}{\downarrow}a$ to the group

$$\underline{\mathbf{Aut}}_x^{\mathcal{X}}(\kappa) \overset{\mathrm{def}}{=} \mathbf{Aut}_{\mathcal{X}(b)}(\kappa^*(x)). \tag{26}$$

A.1. **Descent.** Consider the augmented simplicial object in $\mathcal{A}$

$$(b/a)_* : \quad \begin{tikzcd} \cdots (b/a)_2 \arrow[r, shift left=2, "\partial_0"] \arrow[r, "\partial_1" description] \arrow[r, shift right=2, "\partial_2"'] & (b/a)_1 \arrow[l, bend right=50, "s_0{,}s_1"'] \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & (b/a)_0 \arrow[l, bend right=50, "s_0"'] \arrow[r, "\iota"] & a \end{tikzcd} \tag{27}$$

associated to any morphism $\iota : b \to a$ in $\mathcal{A}$, where

- $(b/a)_0 = b$,
- $(b/a)_n = \underbrace{b \times_a b \times_a \cdots \times_a b}_{(n+1)\text{-times}}$ for all $n \geq 1$,
- $\partial_i =< p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1} >: a_n \to a_{n-1}$ for all $0 \leq i \leq n$,
- $s_j = \underbrace{b \times_a b \times_a \cdots \times_a b}_{j\text{-times}} \times_a \Delta_{b/a} \times_a \underbrace{b \times_a b \times_a \cdots \times_a b}_{(n-j-1)\text{-times}} : (b/a)_{n-1} \to (b/a)_n$,

  for all $0 \leq j \leq n$.

Here $p_i : \underbrace{b \times_a b \times_a \cdots \times_a b}_{(n+1)-\text{times}} \to b$ is the projection to the ith factor, while $\Delta_{b/a}$ is the diagonal morphism $b \to b \times_a b$.

Let us recall from [16] the definition of the category $\mathrm{Des}_{\mathcal{X}}(\iota)$ of *$\mathcal{X}$-descent data relative to ι*. Its objects are pairs (x, ϑ), with x an object of $\mathcal{X}(b)$ and $\vartheta : \partial_1^*(x) \simeq \partial_0^*(x)$ an isomorphism in $\mathcal{X}(b \times_a b)$ such that $s_0^*(\vartheta) = 1$ and the diagram

$$\begin{tikzcd}
\partial_2^*\partial_1^*(x) \arrow[r, "\partial_2^*(\vartheta)"] \arrow[d, "\simeq"'] & \partial_2^*\partial_0^*(x) \arrow[r, "\simeq"] & \partial_0^*\partial_1^*(x) \arrow[d, "\partial_0^*(\vartheta)"] \\
\partial_1^*\partial_1^*(x) \arrow[r, "\partial_1^*(\vartheta)"'] & \partial_1^*\partial_0^*(x) \arrow[r, "\simeq"'] & \partial_0^*\partial_0^*(x)
\end{tikzcd}$$

commutes in $\mathcal{X}(b \times_a b \times_a b)$. Here the labeled isomorphisms are the canonical ones of the $\mathcal{A}$-indexed category $\mathcal{X}$ coming from the simplicial identities

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad (i < j).$$

A morphism $f : (x, \vartheta) \to (y, \theta)$ in $\mathrm{Des}_{\mathcal{X}}(\iota)$ is a morphism $f : x \to y$ in $\mathcal{X}(b)$ which commutes with the descent data ϑ and θ in the sense that the diagram

$$\begin{tikzcd}
\partial_1^*(x) \arrow[r, "\vartheta"] \arrow[d, "\partial_1^*(f)"'] & \partial_0^*(x) \arrow[d, "\partial_0^*(f)"] \\
\partial_1^*(y) \arrow[r, "\theta"'] & \partial_0^*(y)
\end{tikzcd}$$

commutes in $\mathcal{X}(b \times_a b)$.

If z is an object of $\mathcal{X}(a)$, then $\iota^*(z)$ comes equipped with *canonical descent datum* given by the composite

$$\partial_1^*(\iota^*(z)) \simeq (\iota\partial_1)^*(z) = (\iota\partial_0)^*(z) \simeq \partial_0^*(\iota^*(z))$$

of canonical isomorphisms. In other words, the functor ι^* factors as

$$\begin{tikzcd}
\mathcal{X}(a) \arrow[r, "K_\iota"] \arrow[dr, "\iota^*"'] & \mathrm{Des}_{\mathcal{X}}(\iota) \arrow[d, "U"] \\
& \mathcal{X}(b)
\end{tikzcd}$$

where U is the evident forgetful functor and K_ι sends $z \in \mathcal{X}(a)$ to $\iota^*(z)$ equipped with the canonical descent datum.

Definition A.1. ι is called an *$\mathcal{X}$-descent morphism* if K_ι is full and faithful, and an *effective $\mathcal{X}$-descent morphism* if K_ι is an equivalence of categories.

Let $x \in \mathcal{X}(b)$. Write $\mathrm{Des}_{\mathcal{X}}(x)$ for the set of all descent data on x. Two descent data (x, ϑ) and (x, ϑ') on x are called *equivalent* if they are isomorphic objects in the category $\mathrm{Des}_{\mathcal{X}}(\iota)$. The set of equivalence classes of descent data on x is denoted by $\mathbf{Des}_{\mathcal{X}}(x)$. If $x = \iota^*(y)$ for some $y \in \mathcal{X}(a)$, then $\mathbf{Des}_{\mathcal{X}}(\iota^*(y))$ is a pointed set with the class of canonical descent datum as a distinguished element.

Definition A.2. An $\mathcal{A}$-indexed category $\mathcal{X}$ *has products* (resp., *coproducts*) if for each morphism $\iota : b \to a$ in $\mathcal{A}$, the change of base functor $\iota^* : \mathcal{X}(a) \to \mathcal{X}(b)$ admits a right (resp., left) adjoint $\Pi_\iota : \mathcal{X}(b) \to \mathcal{X}(a)$ (resp., $\Sigma_\iota : \mathcal{X}(b) \to \mathcal{X}(a)$) and the *Beck–Chevalley condition* is satisfied, i.e., for every pullback diagram

$$\begin{tikzcd} c \arrow[r, "q"] \arrow[d, "p"'] & b \arrow[d, "\iota"] \\ b' \arrow[r, "\iota'"'] & a, \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathcal{X}(b) \arrow[r, "q^*"] \arrow[d, "\Pi_\iota"'] & \mathcal{X}(c) \arrow[d, "\Pi_p"] \\ \mathcal{X}(a) \arrow[r, "(\iota')^*"'] & \mathcal{X}(b') \end{tikzcd} \qquad \left(\text{resp.}\quad \begin{tikzcd} \mathcal{X}(b') \arrow[r, "p^*"] \arrow[d, "\Sigma_{\iota'}"'] & \mathcal{X}(c) \arrow[d, "\Sigma_q"] \\ \mathcal{X}(a) \arrow[r, "\iota^*"'] & \mathcal{X}(b) \end{tikzcd}\right)$$

commutes up to canonical isomorphism.

We shall need the following version of the Bénabou–Roubaud–Beck theorem (cf. [17, Proposition B1.5.5]).

Theorem A.3. *For an $\mathcal{A}$-indexed category $\mathcal{X} : \mathcal{A}^{op} \to$* **CAT** *having products (resp., coproducts) and for an arbitrary morphism $\iota : b \to a$ in $\mathcal{A}$, the category* $\mathrm{Des}_{\mathcal{A}}(\iota)$ *of descent data with respect to ι is isomorphic to the Eilenberg–Moore category of coalgebras (resp., algebras) for the comonad (resp., monad)* $\mathbf{G}_\iota$ *(resp.,* $\mathbf{T}_\iota$*) on $\mathcal{X}(b)$ generated by the adjoint pair $\iota^* \dashv \Pi_\iota : \mathcal{X}(b) \to \mathcal{X}(a)$ (resp., $\Sigma_\iota \dashv \iota^* : \mathcal{X}(a) \to \mathcal{X}(b)$). Moreover, modulo this equivalence, the functor $K_\iota : \mathcal{X}(a) \to$* $\mathrm{Des}_{\mathcal{X}}(\iota)$ *corresponds to the comparison functor $\mathcal{X}(a) \to (\mathcal{X}(b))^{\mathbf{G}_\iota}$ (resp., $\mathcal{X}(a) \to (\mathcal{X}(b))_{\mathbf{T}_\iota}$). Thus, ι is an effective $\mathcal{X}$-descent morphism if and only if the functor $\iota^* : \mathcal{X}(a) \to \mathcal{X}(b)$ is comonadic (resp., monadic).*

A.2. **Amitsur cohomology.** Let F be a functor on the category $(\mathcal{A}\downarrow a)^{\mathrm{op}}$ with values in the category of groups. Applying F to the augmented simplicial object (27), one gets a coaugmented cosimplicial group

$$\begin{tikzcd}[column sep=large] (b/a,F)_* :\ F(a) \arrow[r, "F(\iota)"] & F(b) \arrow[r, shift left, "F(\partial_0)"] \arrow[r, shift right, "F(\partial_1)"'] & F(b\times_a b) \arrow[l, bend right=60, "F(s_0)"'] \arrow[r, shift left=2, "F(\partial_0)"] \arrow[r, "F(\partial_1)" description] \arrow[r, shift right=2, "F(\partial_2)"'] & F(b\times_a b\times_a b) \arrow[l, bend right=60, "{F(s_0),F(s_1)}"'] \cdots \end{tikzcd}$$

with cofaces $F(\partial_i)$ and codegeneracies $F(s_i)$, and hence one has the nonabelian 0-cohomology group $\mathcal{H}^0((b/a,F)_*$ and the nonabelian 1-cohomology pointed set $\mathcal{H}^1((b/a,F)_*)$. More precisely, $\mathcal{H}^0((b/a,F)_*)$ is the equalizer of the pair $(F(\partial_0)$, $F(\partial_1))$.

On the other hand, a 1-*cocycle* is an element $x \in F(b\times_a b)$ such that

$$F(\partial_1)(x) = F(\partial_2)(x)\cdot F(\partial_0)(x)$$

in $F(b\times_a b\times_a b)$. Write $\mathcal{Z}^1((b/a,F)_*)$ for the set of 1-cocycles. This set is pointed with point the unit element of $F(b\times_a b)$. Two 1-cocycles x and x' are equivalent if

$$x' = F(\partial_1)(y)\cdot x\cdot F(\partial_0)(y)^{-1}$$

for some element $y \in F(b)$. This is an equivalence relation on $\mathcal{Z}^1((b/a, F)_*)$ and $\mathcal{H}^1((b/a, F)_*)$ is defined as the factor-set of equivalence classes of 1-cocycles, and it is a pointed set.

We call $\mathcal{H}^0((b/a, F)_*)$ (resp., $\mathcal{H}^1((b/a, F)_*)$ the *zeroth Amitsur cohomology group of* $\iota : a \to b$ *with values in* F (resp., the *first Amitsur cohomology pointed set of* $\iota : a \to b$ *with values in* F) and denote it by $\mathcal{H}^0(\iota, F)$ (resp., $\mathcal{H}^1(\iota, F)$).

If for each $n \geq 1$, $F(\underbrace{b \times_a b \times_a \cdots \times_a b}_{n \text{ times}})$ is an abelian group (which is indeed the case if the functor F factors through the category of abelian groups), it is possible to define higher cohomology groups as follows. Let

$$C(\iota, F) : 0 \to F(a) \xrightarrow{F(\iota)} F(b) \xrightarrow{\Delta_1} F(b \times_a b) \xrightarrow{\Delta_2} \cdots,$$

$$\Delta_n = \prod_{i=0}^{n} (F(\partial_n))^{(-1)^n},\ n \geq 1,$$

be the complex of abelian groups associated to the abelian cosimplicial group $(F/\iota)^*$. The cohomology groups of this complex are called the *Amitsur cohomology groups of* $\iota : a \to b$ *with values in* F and are denoted by $\mathcal{H}^i(\iota\,, F)$.

The following result of Grothendieck is to be found in [16].

Proposition A.4. *Let* $\mathcal{X} : \mathcal{A}^{op} \to \mathbf{CAT}$ *be an* $\mathcal{A}$*-indexed category, let* $\iota : b \to a$ *be a morphism in* $\mathcal{A}$*, and let* $x \in \mathcal{X}(a)$*. Then the assignment that takes* $\alpha \in \mathbf{Aut}_{\mathcal{X}(b \times_a b)}(\partial^*(x))$*, where* ∂ *denotes the common value of* $\iota \cdot \partial_0$ *and* $\iota \cdot \partial_1$*, to the composite*

$$\partial_1^*(\iota^*(x)) \simeq (\iota \cdot \partial_1)^*(x) = \partial^*(x) \xrightarrow{\alpha} \partial^*(x) \simeq (\iota \cdot \partial_0)^*(x) \simeq \partial_0^*(\iota^*(x))$$

yields an isomorphism

$$\Upsilon^{\iota,x} : \mathcal{Z}^1(\iota\,, \underline{\mathbf{Aut}}_x^{\mathcal{X}}) \simeq \mathrm{Des}_{\mathcal{X}}(\iota^*(x))$$

of pointed sets. When ι *is an effective* $\mathcal{X}$*-descent morphism,* $\Upsilon^{\iota,x}$ *induces an isomorphism*

$$\widehat{\Upsilon}^{\iota,x} : \mathcal{H}^1(\iota\,, \underline{\mathbf{Aut}}_x^{\mathcal{X}}) \simeq \mathbf{Des}_{\mathcal{X}}(\iota^*(x))$$

of pointed sets.

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