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Numerical Semigroups with a Fixed Fundamental Gap

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Abstract: A gap a of a numerical semigroup S is fundamental if $\{2a, 3a\} \subseteq S$. In this work, we will study the set $\mathscr{B}(a) = \{S \mid S$ is a numerical semigroup and a is a fundamental gap of $S\}$. In particular, we will give an algorithm to compute all the elements of $\mathscr{B}(a)$ with a given genus. The intersection of two elements of $\mathscr{B}(a)$ is again one element of $\mathscr{B}(a)$. A $\mathscr{B}(a)$-irreducible numerical semigroup is an element of $\mathscr{B}(a)$ that cannot be expressed as an intersection of two elements of $\mathscr{B}(a)$ containing it properly. In this paper, we will study the $\mathscr{B}(a)$-irreducible numerical semigroups. In this sense we will give an algorithm to calculate all of them. Finally, we will study the submonoids of $(\mathbb{N}, +)$ that can be expressed as an intersection (finite or infinite) of elements belonging to $\mathscr{B}(a)$.

Keywords: numerical semigroup; monoids; fundamental gap; genus; Frobenius number; algorithm; $\mathscr{B}(a)$-rank

MSC: 20M14; 11D07

1. Introduction

Let $\mathbb{Z}$ be the set of integers, and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A submonoid of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$ that is closed under addition and contains the element 0. A numerical semigroup is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S = \{x \in \mathbb{N} \mid x \notin S\}$ is a finite set.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S \backslash \{0\})$, $\mathrm{F}(S) = \max\{z \in \mathbb{Z} \mid z \notin S\}$ and $\mathrm{g}(S) = \#(\mathbb{N} \backslash S)$ (where $\#X$ denotes the cardinality of a set X) are three important invariants of S, called multiplicity, Frobenius number, and genus of S, respectively.

If A is a non-empty subset of $\mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is, $\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\}, \{a_1, \ldots, a_n\} \subseteq A$ and $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{N}\}$. In ([1] Lema 2.1) it is shown that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle A \rangle$, then we say that A is a system of generators of M. Moreover, if $M \neq \langle B \rangle$ for all $B \subsetneq A$, then we will say that A is a minimal system of generators of M. In ([1] Corollary 2.8) it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which in addition is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the embedding dimension of M and will be denoted by $\mathrm{e}(M)$.

The Frobenius problem for numerical semigroups (see [2]) focuses on finding formulas to calculate the Frobenius number and the genus of a numerical semigroup from its minimal system of generators. This problem was solved in [3] for numerical semigroups with embedding dimension two. Nowadays, the problem is still open in the case of numerical semigroups with embedding dimension greater than or equal to three. Furthermore, in this case the problem of computing the Frobenius number of a general numerical semigroup becomes NP-hard.

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We say that an integer a is a fundamental gap of a numerical semigroup S if $a \notin S$ and $\{2a, 3a\} \subseteq S$. We denote by $\mathrm{FG}(S)$ the set formed by the fundamental gaps of S.

If $a \in \mathbb{N}\backslash\{0\}$, we denote by

$$\mathscr{B}(a) = \{S \mid S \text{ is a numerical semigroup and } a \in \mathrm{FG}(S)\}.$$

The main aim of this work will be to study the set $\mathscr{B}(a)$.

If $a \in \mathbb{N}\backslash\{0\}$, we denote by

$$\mathscr{F}(a) = \{S \mid S \text{ is a numerical semigroup and } \mathrm{F}(S) = a\}.$$

It is clear that $\mathscr{F}(a) \subseteq \mathscr{B}(a)$.

Define the following equivalence binary relation on $\mathscr{B}(a)$: $S \mathcal{R}_a T$ if and only if $S \cup \{a+1, \rightarrow\} = T \cup \{a+1, \rightarrow\}$, where the symbol $\rightarrow$ means that every integer greater than $a+1$ belongs to the set denoted by $[S]_a = \{T \in \mathscr{B}(a) \mid S \mathcal{R}_a T\}$. In Section 2, we will see that $\{[S]_a \mid S \in \mathscr{F}(a)\}$ is a partition of $\mathscr{B}(a)$.

If $q \in \mathbb{Q}$, we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$.

We will start Section 3, by showing that $\{\mathrm{g}(S) \mid S \in \mathscr{B}(a)\} = \{\lceil \frac{a+1}{2} \rceil, \rightarrow\}$. If $\Delta \in \mathscr{F}(a)$, then we will arrange the elements of $[\Delta]_a$, making a tree with root Δ. This fact allows us to give an algorithm that computes all the elements of $[\Delta]_a$ with a fixed genus. As a consequence, in Section 3, we will show an algorithm to compute all the elements of $\mathscr{B}(a)$ with a given genus.

The intersection of two elements of $\mathscr{B}(a)$ is again one element of $\mathscr{B}(a)$. We say that an element of $\mathscr{B}(a)$ is $\mathscr{B}(a)$-irreducible if it cannot be expressed as an intersection of two elements of $\mathscr{B}(a)$ containing it properly. In Section 4, we describe an algorithmic procedure to compute all the numerical semigroups that are $\mathscr{B}(a)$-irreducible.

A $\mathscr{B}(a)$-monoid is a submonoid of $(\mathbb{N}, +)$ that can be expressed as the interesection of some elements belonging to $\mathscr{B}(a)$. A $\mathscr{B}(a)$-set is a set that is contained in some element of $\mathscr{B}(a)$.

In Section 5 we will see that if X is a $\mathscr{B}(a)$-set, then there exists the least $\mathscr{B}(a)$-monoid (with respect to set inclusion) containing X. This element will be denoted by $\mathscr{B}(a)[X]$ and will be called the $\mathscr{B}(a)$-monoid generated by X. We will prove that $\mathscr{B}(a)[X] = \langle X \cup \{2a, 3a\} \rangle$.

If M is a $\mathscr{B}(a)$-monoid and $M = \mathscr{B}(a)[X]$, then we will say that X is a $\mathscr{B}(a)$-system of generators of M. Moreover, if $M \neq \mathscr{B}(a)[Y]$ for all $Y \subsetneq X$, then we will say that X is a $\mathscr{B}(a)$-minimal system of generators of M. In Section 5, we will see that every $\mathscr{B}(a)$-monoid M has a unique $\mathscr{B}(a)$-minimal system of generators, and its cardinality will be called the $\mathscr{B}(a)$-rank of M. Finally, we will solve the Frobenius problem for the elements belonging to $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank equal to one.

The fundamental gaps of a numerical semigroup totally determine the semigroup, because if S is a numerical semigroup, then $\mathbb{N}\backslash S = \{x \in \mathbb{N} \mid x \text{ divides a fundamental gap of } S\}$. Hence the study carried out in this work helps to understand the structure of the numerical semigroups, and so it is useful to try to solve some very important problems of this theory, as they are the Frobenius Problem (see [2,4–8]), Wilf's conjecture (see [9–15]), and the conjecture of Bras-Amorós (see [16–21]).

2. A Partition of $\mathscr{B}(a)$

Throughout this whole section, a will denote a positive integer,

$$\mathscr{B}(a) = \{S \mid S \text{ is a numerical semigroup and } a \in \mathrm{FG}(S)\}$$

and
$$\mathscr{F}(a) = \{S \mid S \text{ is a numerical semigroup and } \mathrm{F}(S) = a\}.$$
Note that $\mathscr{F}(a) \subseteq \mathscr{B}(a)$.

If $S \in \mathscr{B}(a)$, then we denote by $\Delta_a(S) = S \cup \{a+1, \rightarrow\}$.

The following result has an easy proof.

Lemma 1. *If $S \in \mathscr{B}(a)$, then $\Delta_a(S) \in \mathscr{F}(a)$.*

Define on $\mathscr{B}(a)$ the following equivalence binary relation: $S\mathcal{R}_aT$ if and only if $\Delta_a(S) = \Delta_a(T)$.

If $S \in \mathscr{B}(a)$, then we denote by $[S]_a = \{T \in \mathscr{B}(a) \mid S\mathcal{R}_aT\}$. So, the quotient set of $\mathscr{B}(a)$ by $\mathcal{R}_a$ is $\frac{\mathscr{B}(a)}{\mathcal{R}_a} = \{[S]_a \mid S \in \mathscr{B}(a)\}$.

The next result is straightforward to prove.

Theorem 1. *If $a \in \mathbb{N}\backslash\{0\}$, then $\frac{\mathscr{B}(a)}{\mathcal{R}_a} = \{[S]_a \mid S \in \mathscr{F}(a)\}$. Moreover, if $\{S,T\} \subseteq \mathscr{F}(a)$ and $S \neq T$, then $[S]_a \cap [T]_a = \emptyset$.*

As a consequence of Theorem 1, we have the following result.

Corollary 1. *If $a \in \mathbb{N}\backslash\{0\}$, then $\{[S]_a \mid S \in \mathscr{F}(a)\}$ is a partition of $\mathscr{B}(a)$.*

As a consequence of Corollary 1, we deduce that in order to build all the elements of $\mathscr{B}(a)$, it is enough to perform the following two steps:

(1) Compute all the elements of $\mathscr{F}(a)$.
(2) For every $S \in \mathscr{F}(a)$, compute all the elements of $[S]_a$.

In [22] appears an algorithm that solves the step (1). Now, we will focus on showing an algorithm that solves the step (2).

The following result has an easy proof.

Lemma 2. *With the above notation, we have the following results:*

(1) If S is a numerical semigroup and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is again a numerical semigroup.
(2) If $a \in \mathbb{N}\backslash\{0\}$, $S \in \mathscr{B}(a)$ and $\mathrm{F}(S) \neq a$, then $S \cup \{\mathrm{F}(S)\} \in \mathscr{B}(a)$.

Let $\Delta \in \mathscr{F}(a)$ and $S \in [\Delta]_a$. Define recursively the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:

- $S_0 = S$.
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \Delta, \\ \Delta & \text{otherwise.} \end{cases}$

It is straightforward to see the next result.

Lemma 3. *Let $\Delta \in \mathscr{F}(a)$, $S \in [\Delta]_a$ and $\{S_n\}_{n\in\mathbb{N}}$ be the sequence defined above. Then, $S_n \in [\Delta]_a$ for every $n \in \mathbb{N}$. Moreover, there exists $k \in \mathbb{N}$ such that $S_{k+i} = \Delta$ for every $i \in \mathbb{N}$.*

A graph G is a pair (V,E) where V is a nonempty set and E is a subset of $\{(u,v) \in V \times V \mid u \neq v\}$. The elements of V and E are called vertices and edges, respectively. A path (of length n) that connects the vertices u and v of G is a sequence of different edges of the form $(v_0,v_1),(v_1,v_2),\ldots,(v_{n-1},v_n)$ such that $v_0 = u$ and $v_n = v$.

A graph G is a tree if there exists a vertex r (known as the root of G) such that for any other vertex x of G, there exists a unique path connecting x and r. If (u,v) is an edge of the tree, we say that u is a child of v.

If $\Delta \in \mathscr{F}(a)$, then we define the graph $G([\Delta]_a)$ as follows: $[\Delta]_a$ is its set of vertices, and $(S,T) \in [\Delta]_a \times [\Delta]_a$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$.

By applying Lemma 3, we obtain the following.

Theorem 2. *If $\Delta \in \mathscr{F}(a)$, then $G([\Delta]_a)$ is a tree and Δ is its root.*

A tree can be recurrently built beginning from the root and joining each vertex already built with its children by an edge. Hence, it is very interesting to characterize the children of an arbitrary vertex in the tree $G([\Delta]_a)$.

The next result is well known and easy to prove.

Lemma 4. *Let M be a submonoid of $(\mathbb{N},+)$ and $x \in M$. Then $M\backslash\{x\}$ is a submonoid of $(\mathbb{N},+)$ if and only if $x \in \mathrm{msg}(M)$.*

The following result will tell us who the children of a vertex are, and it is easy to prove.

Proposition 1. *Let $\Delta \in \mathscr{F}(a)$ and $S \in [\Delta]_a$. Then the set formed by the children of S in the tree $G([\Delta]_a)$ is $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S), x > \mathrm{F}(S)$ and $x \notin \{2a,3a\}\}$.*

It is straightforward to see the next result.

Lemma 5. *Let $n \in \{3,\rightarrow\}$. Then $S(n,a) = \langle 2a,3a\rangle \cup \{na+1,\rightarrow\} \in \mathscr{B}(a)$, and $S(n+1,a) \subsetneq S(n,a)$.*

As an immediate consequence of the above lemma, we obtain the next result.

Proposition 2. *Let $a \in \mathbb{N}\backslash\{0\}$. Then $\mathscr{B}(a)$ is a set with infinite cardinality.*

As every element of $\mathscr{F}(a)$ must contain $\{a+1,\rightarrow\}$, then we deduce that $\mathscr{F}(a)$ is a finite set. By applying Corollary 1 and Proposition 2, there exists $\Delta \in \mathscr{F}(a)$ such that $[\Delta]_a$ has infinite cardinality. In the next section, we will characterize the elements $\Delta \in \mathscr{F}(a)$ such that $[\Delta]_a$ has infinite cardinality.

If $\Delta \in \mathscr{F}(a)$ and $[\Delta]_a$ is finite, then the previous results allow us to give an algorithm that computes all the elements of $[\Delta]_a$ (Algorithm 1).

Algorithm 1 Computation of $[\Delta]_a$

INPUT: $\Delta \in \mathscr{F}(a)$ such that $[\Delta]_a$ is a finite set.
OUTPUT: $[\Delta]_a$.

(1) $A = B = \{\Delta\}$.
(2) For every $S \in B$, by using Proposition 1, compute $\mathrm{CHILD}(S) = \{T \mid T$ is a child of S in the tree $G([\Delta]_a)\}$.
(3) $B := \bigcup_{S\in B} \mathrm{CHILD}(S)$.
(4) If $B = \emptyset$, then return A.
(5) $A := A \cup B$ and go to Step (2).

3. Elements of $\mathscr{B}(a)$ with a Fixed Genus

Throughout this section, a denotes a positive integer. Our first aim in this section will be to determine the set$\{\mathrm{g}(S) \mid S \in \mathscr{B}(a)\}$.

By applying ([1] Lemma 2.14), we can deduce the next result.

Lemma 6. *Under the standing notation, we have the following results:*

(1) *If S is a numerical semigroup, then* $\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$.
(2) *If $S \in \mathscr{B}(a)$, then* $\mathrm{g}(S) \geq \frac{a+1}{2}$.

An integer a is a special gap of a numerical semigroup S if $a \notin S$ and $S \cup \{a\}$ is a numerical semigroup. Denote by $\mathrm{SG}(S)$ the set of special gaps of S.

The following result has an immediate proof.

Lemma 7. *Let S be a numerical semigroup. Then,* $\mathrm{SG}(S) \subseteq \mathrm{FG}(S)$.

If a is a positive integer, denote by

$$\mathscr{A}(a) = \{S \mid S \text{ is a numerical semigroup and } a \in \mathrm{SG}(S)\}.$$

As an immediate consequence of Lemma 7, we can announce the following result.

Lemma 8. *Let $a \in \mathbb{N}\backslash\{0\}$. Then $\mathscr{A}(a) \subseteq \mathscr{B}(a)$.*

The next result is Theorem 17 from [23].

Lemma 9. *Let $a \in \mathbb{N}\backslash\{0\}$. Then* $\{\mathrm{g}(S) \mid S \in \mathscr{A}(a)\} = \{\lceil \frac{a+1}{2} \rceil, \rightarrow\}$.

By applying Lemmas 6, 8 and 9, we obtain the following result:

Theorem 3. *If $a \in \mathbb{N}\backslash\{0\}$, then* $\{\mathrm{g}(S) \mid S \in \mathscr{B}(a)\} = \{\lceil \frac{a+1}{2} \rceil, \rightarrow\}$.

Our next purpose in this section will be to show a characterization of the elements $\Delta \in \mathscr{F}(a)$ such that $[\Delta]_a$ is a finite set.

If S is a numerical semigroup, then $\mathbb{N}\backslash S$ is a finite set, and we deduce the following result.

Lemma 10. *If S is a numerical semigroup, then the set*

$$\{T \mid T \textit{ is a numerical semigroup and } S \subseteq T\}$$

is finite.

Proposition 3. *Let $\Delta \in \mathscr{F}(a)$. Then $[\Delta]_a$ is a finite set if and only if* $\gcd(\{x \in \mathrm{msg}(\Delta) \mid x < a\} \cup \{a\}) = 1$.

Proof. Denote by $A = \{x \in \mathrm{msg}(\Delta) \mid x < a\}$.
Necessity: If $\gcd(A \cup \{a\}) = d \neq 1$, then it is clear that $M = \langle A \cup \{a\} \rangle \backslash \{a\}$ is a submonoid of $(\mathbb{N}, +)$ and $dn + 1 \notin M$ for all $n \in \mathbb{N}$. For every $k \in \{\lceil \frac{a-1}{d} \rceil + 1, \rightarrow\}$, denote by $S_k = M \cup \{dk+1, \rightarrow\}$. Clearly, $S_k \in \mathscr{B}(a)$, $\Delta_a(S_k) = \Delta$ and $S_{k+1} \subsetneq S_k$ for all $k \in \{\lceil \frac{a-1}{d} \rceil + 1 \rightarrow\}$. Hence, $[\Delta]_a$ is a set with infinite cardinality.

Sufficiency: If $T \in [\Delta]_a$, then $T \in \mathscr{B}(a)$ and $\Delta_a(T) = \Delta$. So, $A \cup \{2a, 3a\} \subseteq T$. As $\gcd(A \cup \{2a, 3a\}) = \gcd(A \cup \{a\}) = 1$, then $\langle A \cup \{2a, 3a\} \rangle$ is a numerical semigroup. By applying Lemma 10 and that $[\Delta]_a \subseteq \{S \mid S \text{ is a numerical semigroup and } \langle A \cup \{2a, 3a\} \rangle \subseteq S\}$, we deduce that $[\Delta]_a$ is a set with finite cardinality. □

As a consequence of Proposition 3, we obtain the following properties which characterize when $[S]_a$ is an infinite or finite set.

Corollary 2. *With the above notation, we have the following:*

(1) *If* $X \subseteq \{1, \cdots, a-1\}$, $a \notin \langle X \rangle$ *and* $\gcd(X \cup \{a\}) \neq 1$, *then* $S = \langle X \rangle \cup \{a+1, \rightarrow\} \in \mathscr{F}(a)$ *and* $[S]_a$ *is an infinite set. Moreover, if* $T \in \mathscr{F}(a)$ *and* $[T]_a$ *is an infinite set, then* T *has this form.*

(2) *If* $X \subseteq \{1, \cdots, a-1\}$, $a \notin \langle X \rangle$ *and* $\gcd(X \cup \{a\}) = 1$, *then* $S = \langle X \rangle \cup \{a+1, \rightarrow\} \in \mathscr{F}(a)$ *and* $[S]_a$ *is a finite set. Moreover, if* $T \in \mathscr{F}(a)$ *and* $[T]_a$ *is a finite set, then* T *has this form.*

If $G = (V, E)$ is a tree and $v \in V$, then the depth of v is the length of the unique path connecting v with the root. If $k \in \mathbb{N}$, we denote by $\mathrm{N}(G, k) = \{v \in V \mid v \text{ has depth equal to } k\}$. The height of G is

$$\mathrm{h}(G) = \max\{k \in \mathbb{N} \mid \mathrm{N}(G, k) \neq \emptyset\}.$$

The following result has an immediate proof.

Lemma 11. *If* $\Delta \in \mathscr{F}(a)$ *and* $k \in \mathbb{N}$, *then the following is true:*

(1) $\mathrm{N}(G([\Delta]_a), k) = \{S \in [\Delta]_a \mid \mathrm{g}(S) = \mathrm{g}(\Delta) + k\}$.

(2) $\mathrm{N}(G([\Delta]_a), k+1) = \{S \mid S$ *is a child of an element of* $\mathrm{N}(G([\Delta]_a), k)$ *in the tree* $G([\Delta]_a)$ $\}$.

(3) $\{\mathrm{g}(S) \mid S \in [\Delta]_a\} = \{\mathrm{g}(\Delta), \mathrm{g}(\Delta) + 1, \ldots, \mathrm{g}(\Delta) + \mathrm{h}(G([\Delta]_a))\}$.

Note that if $\Delta \in \mathscr{F}(a)$ and $[\Delta]_a$ is a set with infinite cardinality, then the tree $G([\Delta]_a)$ has an infinite depth. Therefore, the point 3) of Lemma 11, asserts that in this case, $\{\mathrm{g}(S) \mid S \in [\Delta]_a\} = \{\mathrm{g}(\Delta), \rightarrow\}$.

Now, we have all the ingredients to show the following algorithm.

By using the Algorithm 2, we can present the announced algorithm.

Algorithm 2 Computation of all elements of $[\Delta]_a$ with a fixed genus.

INPUT: $\Delta \in \mathscr{F}(a)$ and $k \in \mathbb{N}$.

OUTPUT: The set $\{S \in [\Delta]_a \mid \mathrm{g}(S) = \mathrm{g}(\Delta) + k\}$.

(1) $A = \{\Delta\}$, $i = 0$.

(2) If $A = \emptyset$, then return $\emptyset$.

(3) If $i = k$, then return A.

(4) For every $S \in A$ compute, by using Proposition 1, $\mathrm{CHILD}(S) = \{T \mid T$ is a child of S in the tree $G([\Delta]_a)\}$.

(5) $A := \bigcup_{S \in A} \mathrm{CHILD}(S)$, $i := i + 1$ and go to Step (2).

By using the Algorithm 2, we can present the announced Algorithm 3.

Algorithm 3 Computation of all elements of $\mathscr{B}(a)$ with a fixed genus.

INPUT: A positive integer g.

OUTPUT: The set $\{S \in \mathscr{B}(a) \mid \mathrm{g}(S) = g\}$.

(1) If $g < \lceil \frac{a+1}{2} \rceil$, then return $\emptyset$.

(2) Compute $\mathscr{F}(a)$, by using ([23] Algorithm 1).

(3) Compute $\mathscr{C}(g) = \{\Delta \in \mathscr{F}(a) \mid \mathrm{g}(\Delta) \leq g\}$.

(4) For every $\Delta \in \mathscr{C}(g)$ compute, by using Algorithm 2, $\mathrm{L}(\Delta) = \{S \in [\Delta]_a \mid \mathrm{g}(S) = \mathrm{g}(\Delta) + (g - \mathrm{g}(\Delta))\}$.

(5) Return $\bigcup_{\Delta \in \mathscr{C}(g)} \mathrm{L}(\Delta)$.

4. $\mathscr{B}(a)$-Irreducible Numerical Semigroups

Along this section a will denote a positive integer. It is clear that the intersection of two elements of $\mathscr{B}(a)$ is again one element of $\mathscr{B}(a)$. We say that a numerical semigroup is $\mathscr{B}(a)$-irreducible if belongs to $\mathscr{B}(a)$ and can not be expressed as an intersection of two elements of $\mathscr{B}(a)$ properly containing it.

The following result is ([1] Lemma 4.35).

Lemma 12. *Let S and T be numerical semigroups such that $S \subsetneq T$ and $x = \max(T \backslash S)$. Then $S \cup \{x\}$ is again a numerical semigroup.*

Proposition 4. *Let $S \in \mathscr{B}(a)$. Then S is a $\mathscr{B}(a)$-irreducible numerical semigroup if and only if $\#(\mathrm{SG}(S) \backslash \{a\}) \leq 1$.*

Proof. Necessity. If $\#(\mathrm{SG}(S) \backslash \{a\}) \geq 2$, then there exists $\{u, v\} \in \mathrm{SG}(S)$ such that $u \neq v$ and $a \notin \{u, v\}$. It is clear that $\{S \cup \{u\}, S \cup \{v\}\} \subseteq \mathscr{B}(a)$, $S \subsetneq S \cup \{u\}$, $S \subsetneq S \cup \{v\}$ and $(S \cup \{u\}) \cap (S \cup \{v\}) = S$. Therefore, S is not a $\mathscr{B}(a)$-irreducible numerical semigroup.

Sufficiency. If S is not $\mathscr{B}(a)$-irreducible, then there exists $\{P, Q\} \subseteq \mathscr{B}(a)$ such that $S \subsetneq P$, $S \subsetneq Q$ and $S = P \cap Q$. Let $u = \max(P \backslash S)$ and $v = \max(Q \backslash S)$. By Lemma 12, we know that $\{u, v\} \subseteq \mathrm{SG}(S)$. Moreover, it is clear that $u \neq v$ and $a \notin \{u, v\}$. Hence $\#(\mathrm{SG}(S) \backslash \{a\}) \geq 2$. □

As a consequence of Proposition 4, we have the following result.

Corollary 3. *Let $S \in \mathscr{B}(a)$. Then S is a $\mathscr{B}(a)$-irreducible numerical semigroup if and only if just one of the following conditions is verified:*

(1) $\mathrm{SG}(S) = \{a\}$.
(2) *There exists $b \in \mathbb{N} \backslash \{a\}$ such that $\mathrm{SG}(S) = \{b\}$.*
(3) *There exists $b \in \mathbb{N}$ such that $b < a$ and $\mathrm{SG}(S) = \{a, b\}$.*
(4) *There exists $b \in \mathbb{N}$ such that $a < b$ and $\mathrm{SG}(S) = \{a, b\}$.*

Our next purpose in this section will be to show how we can compute all the elements of $\mathscr{B}(a)$ that are $\mathscr{B}(a)$-irreducibles.

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Denote by

$$\mathscr{I}(F) = \{S \mid S \text{ is an irreducible numerical semigroup and } \mathrm{F}(S) = F\}.$$

From ([24] Proposition 2.7), we deduce the next result.

Lemma 13. *If F is a positive integer, then $\mathscr{I}(F) \neq \emptyset$.*

The next result characterizes the irreducible numerical semigroup, and it can be deduced from ([25] Corollary 13).

Lemma 14. *A numerical semigroup S is irreducible if and only if $S = \mathbb{N}$ or $\#(\mathrm{SG}(S)) = 1$.*

As an immediate consequence of Lemma 14, we have the following result, which characterizes the numerical semigroup with an only special gap.

Proposition 5. *Let S be a numerical semigroup and $x \in \mathbb{N} \backslash \{0\}$. Then $\mathrm{SG}(S) = \{x\}$ if and only if S is an irreducible numerical semigroup and $\mathrm{F}(S) = x$.*

As an immediate consequence of the previous proposition, we obtain the following.

Corollary 4. *If* $a \in \mathbb{N}\backslash\{0\}$, *then* $\{S \in \mathscr{B}(a) \mid \mathrm{SG}(S) = \{a\}\} = \mathscr{J}(a)$.

In [24] the authors give an algorithm to compute $\mathscr{J}(a)$. Therefore, by applying Corollary 4, we have an algorithm to obtain all the $\mathscr{B}(a)$-irreducible numerical semigroups verifying the condition (1) of Corollary 3.

If we apply Proposition 5, we obtain the following.

Corollary 5. *Let* S *be a numerical semigroup. Then, the following conditions are equivalent:*

(1) $S \in \mathscr{B}(a)$ *and* $\mathrm{SG}(S) = \{b\}$ *for some* $b \in \mathbb{N}\backslash\{a\}$.

(2) S *is an irreducible numerical semigroup,* $a < \mathrm{F}(S)$, $a \notin S$, *and* $\{2a, 3a\} \subseteq S$.

By applying ([1] Theorem 4.2 and Proposition 4.4), we obtain the next result.

Lemma 15. *Let* S *be a numerical semigroup. Then, the following occurs:*

(1) S *is irreducible if and only if* S *is maximal (with respect to set inclusion) in the set formed by all numerical semigroups with Frobenius number equal to* $\mathrm{F}(S)$.

(2) *If* S *is irreducible,* $x \in \mathbb{N}$ *and* $x \neq \frac{\mathrm{F}(S)}{2}$, *then* $\mathrm{F}(S) - x \in S$.

Lemma 16. *Let* a *and* b *be positive integers such that* $a \neq b$. *Then* $\{S \in \mathscr{B}(a) \mid \mathrm{SG}(S) = \{b\}\} \neq \emptyset$ *if and only if* $b > a$ *and* $b \notin \langle 2a, 3a, b - a\rangle$.

Proof. Necessity: If $\mathrm{SG}(S) = \{b\}$, then by applying Lemma 14, we deduce that S is an irreducible numerical semigroup and $\mathrm{F}(S) = b$. As $a \notin S$ and $a \neq b$, then $a < b$. Moreover, by Lemma 15, $b - a \in S$. Hence $\langle 2a, 3a, b - a\rangle \subseteq S$ and consequently $b \notin \langle 2a, 3a, b - a\rangle$.

Sufficiency: It is clear that $T = \langle 2a, 3a, b - a\rangle \cup \{b + 1, \rightarrow\}$ is a numerical semigroup with Frobenius number b. Let S be a numerical semigroup that is maximal in the set of numerical semigroups having Frobenius number b and that contains T. By applying Lemma 15, we deduce that S is an irreducible numerical semigroup with Frobenius number b. We have that $a \notin S$, because if $a \in S$, then a, $b - a \in S$, and in particular $b \in S$, a contradiction. Finally, by applying Corollary 5, we can assert that $\{S \in \mathscr{B}(a) \mid \mathrm{SG}(S) = \{b\}\} \neq \emptyset$. □

Let a and b be integers. We say that a divides b if there exists an integer c such that $b = ca$, and we denote this by $a \mid b$. Otherwise, a does not divide b, and we denote this by $a \nmid b$.

Lemma 17. *Let* a *and* b *be positive integers such that* $b > a$. *Then* $b \notin \langle 2a, 3a, b - a\rangle$ *if and only if* $b \notin \langle 2a, 3a\rangle \cup \left\{(d+1)\frac{a}{d} \mid d \mid a \text{ and } d \geq 2\right\}$.

Proof. Necessity. If $b \notin \langle 2a, 3a, b - a\rangle$, then $b \notin \langle 2a, 3a\rangle$. If $b = (d+1)\frac{a}{d}$ for some integer $d \geq 2$ such that $d \mid a$, then $b = a + \frac{a}{d}$ and so, $b - a = \frac{a}{d}$. Then $b = (d+1)(b-a)$ and consequently, $b \in \langle 2a, 3a, b - a\rangle$ which is absurd.

Sufficiency. If $b \in \langle 2a, 3a, b - a\rangle$, then by applying that $b \notin \langle 2a, 3a\rangle$, we deduce that $b = k(b - a)$ for some $k \in \mathbb{N}\backslash\{0, 1\}$. Therefore, $ka = (k-1)b$. As $\gcd\{k, k-1\} = 1$, then $(k-1) \mid a$. Finally, note that if $k - 1 = 1$, then $b = 2a$, contradicting that $b \notin \langle 2a, 3a\rangle$. □

By applying Lemmas 16 and 17, we obtain the following result.

Proposition 6. *Let* a *and* b *be positive integers such that* $a \neq b$. *Then* $\{S \in \mathscr{B}(a) \mid \mathrm{SG}(S) = \{b\}\} \neq \emptyset$ *if and only if* $b > a$ *and* $b \notin \langle 2a, 3a\rangle \cup \left\{(d+1)\frac{a}{d} \mid d \mid a \text{ and } d \geq 2\right\}$.

Proposition 6 provides us an algorithmic procedure that computes the set

$$C(a) = \{b \in \mathbb{N} \mid \{S \in \mathscr{B}(a) \mid SG(S) = \{b\}\} \neq \emptyset\}.$$

For every $b \in C(a)$, in [24] appears an algorithm to calculate $\mathscr{J}(b)$. As $\{S \in \mathscr{B}(a) \mid SG(S) = \{b\}\} = \{S \in \mathscr{J}(b) \mid \{2a, 3a, b-a\} \subseteq S\}$, then we can assert that for every $b \in C(a)$, we have an algorithm to obtain the set $\{S \in \mathscr{B}(a) \mid SG(S) = \{b\}\}$.

As $\{S \mid S \text{ is an irreducible numerical semigroup}, a < F(S), a \notin S, \text{and } \{2a, 3a\} \subseteq S\} = \bigcup_{b \in C(a)} \{S \in \mathscr{B}(a) \mid SG(S) = \{b\}\}$, then by applying Corollary 5, we can assert that we have an algorithmic procedure to build all the $\mathscr{B}(a)$-irreducible numerical semigroups verifying condition (2) of Corollary 3.

Observe that if $a \geq 2$, from Proposition 6, it easily follows that $C(a)$ is a set with infinite cardinality, and so the set of $\mathscr{B}(a)$-irreducibles that verifies the condition (2) of Corollary 3 also has an infinite cardinality. Hence, we do not have a proper algorithm to calculate them, although we can say that we have an algorithmic process that builds them recurrently.

A numerical semigroup S is atomic if it cannot be written as an intersection of two elements of $\mathscr{F}(F(S))$ containing it properly.

The following result has an immediate proof.

Lemma 18. *Every irreducible numerical semigroup is atomic.*

By using the notation introduced in [26], an ANI-semigroup is an atomic numerical semigroup that is not irreducible.

The following characterization can be deduced from ([27] Corollary 2.3).

Lemma 19. *A numerical semigroup S is an* ANI*-semigroup if and only if* $\#(SG(S)) = 2$.

The next proposition is obtained by applying the previous lemma.

Proposition 7. *If $a \in \mathbb{N} \backslash \{0\}$, then $\{S \in \mathscr{B}(a) \mid SG(S) = \{a, b\}$ for some $b < a\} = \{S \mid S$ is an* ANI*-semigroup and $F(S) = a\}$.*

Algorithm 3.7 from [27] allows us to compute all the atomic numerical semigroups with a fixed Frobenius number. If we remove step (1) of this algorithm and we change step (4) by

(4) Return $N(F) = \bigcup_{l \in L(F)} N(F, l)$,

then we obtain an algorithm that calculates all the ANI-semigroups with Frobenius number equal to F. Therefore, by applying Proposition 7, we have an algorithm to compute all the $\mathscr{B}(a)$-irreducible numerical semigroups verifying the condition (3) of Corollary 3.

Next, if we apply Lemma 19, we deduce the following result.

Proposition 8. *If $a \in \mathbb{N} \backslash \{0\}$, then $\{S \in \mathscr{B}(a) \mid SG(S) = \{a, b\}$ for some $a < b\} = \{S \mid S$ is an* ANI*-semigroup, $a \in SG(S)$, and $a < F(S)\}$.*

The following result is deduced from ([26] Theorem 19).

Lemma 20. *Let a and b be positive integers such that $a < b$. Then there exits a numerical semigroup S such that $SG(S) = \{a, b\}$ if and only if $\frac{b}{2} < a$ and $(b-a) \mid b$ or $(2a-b) \nmid b$.*

Note that if a and b are positive integers, $a < b$, and $\mathrm{SG}(S) = \{a, b\}$, then by Lemma 20 we know that $a < b < 2a$. Therefore, we have an algorithm to compute the next set,

$$\mathrm{J}(a) = \{b \in \mathbb{N} \mid a < b \text{ and } \{S \in \mathscr{B}(a) \mid \mathrm{SG}(S) = \{a, b\}\} \neq \emptyset\}.$$

Now, for every $b \in \mathrm{J}(a)$, Algorithm 3.5 from [27] allows us to compute the set $\{S \mid S$ is a numerical semigroup and $\mathrm{SG}(S) = \{a, b\}\}$. Hence, by applying Proposition 8, we can assert that we have an algorithm to compute all the $\mathscr{B}(a)$-irreducible numerical semigroups verifying the condition (4) from Corollary 3.

5. $\mathscr{B}(a)$-Monoids

Throughout this section, a denotes a positive integer. We know that the intersection of two elements of $\mathscr{B}(a)$ is again an element of $\mathscr{B}(a)$. Thus, $\mathscr{B}(a)$ is closed by finite intersection. However, it is not closed by infinite intersections, as the following example shows.

Example 1. *Suppose that* $a \in \mathbb{N}\backslash\{0, 1\}$. *For every* $k \in \mathbb{N}\backslash\{0\}$, *denote by* $S(k) = \langle 2a, 3a \rangle \cup \{ka + 1, \rightarrow\}$. *It is clear that* $S(k) \in \mathscr{B}(a)$ *for every* $k \in \mathbb{N}\backslash\{0\}$ *and* $\bigcap_{k \in \mathbb{N}\backslash\{0\}} S(k) = \langle 2a, 3a \rangle \notin \mathscr{B}(a)$ *because* $\gcd\{2a, 3a\} = a \neq 1$, *and so* $\langle 2a, 3a \rangle$ *is not a numerical semigroup.*

The intersections (finite or infinite) of elements belonging to $\mathscr{B}(a)$ are always a submonoid of $(\mathbb{N}, +)$. This fact induces us to give the following definition. A $\mathscr{B}(a)$-monoid is a submonoid of $(\mathbb{N}, +)$ that can be expressed as the intersection of elements belonging to $\mathscr{B}(a)$.

Proposition 9. *Let M be a submonoid of* $(\mathbb{N}, +)$. *Then M is a* $\mathscr{B}(a)$*-monoid if and only if* $a \notin M$ *and* $\{2a, 3a\} \subseteq M$.

Proof. Necessity. Trivial.

Sufficiency. For every $k \in \mathbb{N}\backslash\{0\}$, let $S(k) = M \cup \{ka + 1, \rightarrow\}$. It is clear that $S(k) \in \mathscr{B}(a)$ for all $k \in \mathbb{N}\backslash\{0\}$ and $\bigcap_{k \in \mathbb{N}\backslash\{0\}} S(k) = M$. Consequently, M is a $\mathscr{B}(a)$-monoid. □

A $\mathscr{B}(a)$-set is a subset X of $\mathbb{N}$ verifying that there exists $S \in \mathscr{B}(a)$ such that $X \subseteq S$.

Lemma 21. *Let* $X \subseteq \mathbb{N}$. *Then X is a* $\mathscr{B}(a)$*-set if and only if* $a \notin \langle X \rangle$.

Proof. Necessity. If X is a $\mathscr{B}(a)$-set, then there exists $S \in \mathscr{B}(a)$ such that $X \subseteq S$. Therefore, $\langle X \rangle \subseteq S$. As $a \notin S$, then $a \notin \langle X \rangle$.

Sufficiency. Let $S = \langle X \rangle \cup \{2a, \rightarrow\}$. Then it is clear that $S \in \mathscr{B}(a)$ and $X \subseteq S$. Hence X is a $\mathscr{B}(a)$-set. □

The following result has an immediate proof.

Lemma 22. *The intersection of* $\mathscr{B}(a)$*-monoids is again a* $\mathscr{B}(a)$*-monoid.*

If X is a $\mathscr{B}(a)$-set, then denote by $\mathscr{B}(a)[X]$ the intersection of all $\mathscr{B}(a)$-monoids that contain X.

By applying the previous lemma, we obtain the following.

Lemma 23. *If X is a* $\mathscr{B}(a)$*-set, then* $\mathscr{B}(a)[X]$ *is the smallest (with respect to set inclusion)* $\mathscr{B}(a)$*-monoid containing X. Furthermore,* $\mathscr{B}(a)[X]$ *is the intersection of all the elements of* $\mathscr{B}(a)$ *containing X.*

The following result tells us the condition that a $\mathscr{B}(a)$-set, X, must verify in order for $\mathscr{B}(a)[X]$ to be a numerical semigroup.

Proposition 10. *Let $a \in \mathbb{N}\backslash\{0,1\}$ and X be a $\mathscr{B}(a)$-set. Then, $\mathscr{B}(a)[X] \in \mathscr{B}(a)$ if and only if* $\gcd(X \cup \{a\}) = 1$.

Proof. Necessity. If $\gcd(X \cup \{a\}) \neq 1$, then $\gcd(X \cup \{2a, 3a\}) \neq 1$, and so $\langle X \cup \{2a, 3a\}\rangle$ is not a numerical semigroup. From Proposition 9, we deduce that $\langle X \cup \{2a, 3a\}\rangle$ is a $\mathscr{B}(a)$-monoid containing X. Therefore, $\mathscr{B}(a)[X] \subseteq \langle X \cup \{2a, 3a\}\rangle$ and consequently, $\mathscr{B}(a)[X] \notin \mathscr{B}(a)$.

Sufficiency. If $S \in \mathscr{B}(a)$ and $X \subseteq S$, then $\langle X \cup \{2a, 3a\}\rangle \subseteq S$. As $\gcd(X \cup \{2a, 3a\}) = \gcd(X \cup \{a\}) = 1$, then $\langle X \cup \{2a, 3a\}\rangle$ is a numerical semigroup. By Lemma 10, we know that $\{S \in \mathscr{B}(a) \mid X \subseteq S\}$ is a finite set. Hence $\mathscr{B}(a)[X]$ is an intersection of finite elements belonging to $\mathscr{B}(a)$, and so $\mathscr{B}(a)[X] \in \mathscr{B}(a)$. □

If X is a $\mathscr{B}(a)$-set and $M = \mathscr{B}(a)[X]$, then we will say that X is a $\mathscr{B}(a)$-system of generators of M. In addition, if $M \neq \mathscr{B}(a)[Y]$ for all $Y \subsetneq X$, then X will be called a $\mathscr{B}(a)[X]$-minimal system of generators of M.

The following result has an easy proof.

Lemma 24. *If X is a $\mathscr{B}(a)$-set, then $\mathscr{B}(a)[X] = \langle X \cup \{2a, 3a\}\rangle$.*

Theorem 4. *If M is a $\mathscr{B}(a)$-monoid, then $\mathrm{msg}(M)\backslash\{2a, 3a\}$ is the unique $\mathscr{B}(a)$-minimal system of generators of M.*

Proof. Let $A = \mathrm{msg}(M)\backslash\{2a, 3a\}$. It is clear that every element of $\mathscr{B}(a)$ that contains A must contain $A \cup \{2a, 3a\}$. Therefore, every element of $\mathscr{B}(a)$ containing A must contain $\mathrm{msg}(M)$. Consequently, every element of $\mathscr{B}(a)$ that contains A has to contain M. Hence $\mathscr{B}(a)[A] = M$, and so A is a $\mathscr{B}(a)$-system of generators of M.

To conclude the proof, we will see that if B is a $\mathscr{B}(a)$-system of generators of M, then $A \subseteq B$. Indeed, as $\mathscr{B}(a)[B] = M$, then by Lemma 24, we know that $\langle B \cup \{2a, 3a\}\rangle = M$, and so $A \subseteq B$. □

If M is a $\mathscr{B}(a)$-monoid, denote by $\mathscr{B}(a)\mathrm{msg}(M)$ the unique $\mathscr{B}(a)$-minimal system of generators of M. The cardinality of $\mathscr{B}(a)\mathrm{msg}(M)$ is called the $\mathscr{B}(a)$-rank of M, and it will be denoted by $\mathscr{B}(a)\mathrm{rank}\,(M)$.

The proof of the next result is very easy.

Proposition 11. *If M is a $\mathscr{B}(a)$-monoid, then the following conditions are verified:*

(1) $0 \leq \mathscr{B}(a)\mathrm{rank}\,(M) \leq \mathrm{e}(M)$.
(2) $\mathscr{B}(a)\mathrm{rank}\,(M) = 0$ *if and only if* $M = \langle 2a, 3a\rangle$.
(3) $\mathscr{B}(a)\mathrm{rank}\,(M) = 1$ *if and only if* $M = \langle x, 2a, 3a\rangle$ *for some* $x \in \mathbb{N}\backslash\{0\}$ *such that* $x \nmid a$.

We finish this section by studying the elements of $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one.

By applying Lemma 24 and Proposition 11, we deduce the following result.

Proposition 12. *If $x \in \mathbb{N}\backslash\{0,1\}$ and $\gcd\{x, a\} = 1$, then $\langle x, 2a, 3a\rangle$ is an element of $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one. Moreover, every element of $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one has this form.*

As a consequence of Proposition 12, we obtain a characterization of the element belonging to $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one.

Corollary 6. *S is an element of $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one if and only if just one of the following conditions is verified:*

(1) $S = \langle 2, 3a\rangle$ *and a is odd.*

(2) $S = \langle 3, 2a\rangle$ *and* $3 \nmid a$.

(3) $S = \langle x, 2a, 3a\rangle$, $x \in \{4, \rightarrow\}$ *and* $\gcd\{x, a\} = 1$.

The following result appears in [3].

Lemma 25. *Let S be a numerical semigroup with embedding dimension two and* $\mathrm{msg}(S) = \{n_1, n_2\}$. *Then* $\mathrm{F}(S) = n_1 n_2 - n_1 - n_2$ *and* $\mathrm{g}(S) = \frac{(n_1-1)(n_2-1)}{2}$.

In ([2] Lemma 3.1.7 and Theorem 5.3.2) the next result can be consulted.

Lemma 26. *If S is a numerical semigroup,* $\mathrm{msg}(S) = \{n_1, n_2, \cdots, n_e\}$ *and* $d = \gcd\{n_1, \cdots, n_{e-1}\}$, *then* $\mathrm{F}(S) = d\mathrm{F}(\langle \frac{n_1}{d}, \cdots, \frac{n_{e-1}}{d}, n_e\rangle) + (d-1)n_e$ *and* $\mathrm{g}(S) = d\mathrm{g}(\langle \frac{n_1}{d}, \cdots, \frac{n_{e-1}}{d}, n_e\rangle) + \frac{(n_e-1)(d-1)}{2}$.

By applying Lemmas 25 and 26, we obtain the following result, which solves the Frobenius problem for elements of $\mathscr{B}(a)$ with $\mathscr{B}(a)$-rank one.

Proposition 13. *It verifies the following:*

(1) *If a is odd, then* $\mathrm{F}(\langle 2, 3a\rangle) = 3a - 2$ *and* $\mathrm{g}(\langle 2, 3a\rangle) = \frac{3a-1}{2}$.

(2) *If* $3 \nmid a$, *then* $\mathrm{F}(\langle 3, 2a\rangle) = 4a - 3$ *and* $\mathrm{g}(\langle 3, 2a\rangle) = 2a - 1$.

(3) *If* $x \in \{4, \rightarrow\}$ *and* $\gcd\{x, a\} = 1$, *then* $\mathrm{F}(\langle x, 2a, 3a\rangle) = a + (a-1)x$ *and* $\mathrm{g}(\langle x, 2a, 3a\rangle) = a + \frac{(a-1)(x-1)}{2}$.

6. Conclusions

In this paper, we have studied the set $\mathscr{B}(a)$ of the numerical semigroups for which a is a fundamental gap. In particular, we have given an initialized partition of $\mathscr{B}(a)$ into the set $\mathscr{F}(a)$ of the numerical semigroups with Frobenius number a. As a consequence, an algorithm that computes all the elements of $\mathscr{B}(a)$ with a fixed genus is obtained.

The intersection of two elements of $\mathscr{B}(a)$ is again one element of $\mathscr{B}(a)$. A numerical semigroup is $\mathscr{B}(a)$-irreducible if it belongs to $\mathscr{B}(a)$ and it cannot be expressed as an intersection of two elements of $\mathscr{B}(a)$ containing it properly. In this work we have also studied this kind of semigroups, and in particular we have given an algorithm that calculates all the $\mathscr{B}(a)$-irreducible numerical semigroups.

A $\mathscr{B}(a)$-monoid is a submonoid of $(\mathbb{N}, +)$ that can be expressed as the intersection of elements belonging to $\mathscr{B}(a)$. In this work we have proved that there exists the smallest $\mathscr{B}(a)$-monoid containing a subset X of $\mathbb{N}$. It will be denoted by $\mathscr{B}(a)[X]$. If $M = \mathscr{B}(a)[X]$, then we will say that X is a $\mathscr{B}(a)$-system of generators of M. We have shown that every $\mathscr{B}(a)$-monoid has a unique $\mathscr{B}(a)$-minimal system of generators, and we have studied the $\mathscr{B}(a)$-monoids whose $\mathscr{B}(a)$-minimal system of generators has cardinality one.

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