L-ORTHOGONALITY, OCTAHEDRALITY AND DAUGAVET PROPERTY IN BANACH SPACES

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ABSTRACT. We prove that the abundance of almost L-orthogonal vectors in a Banach space X (almost Daugavet property) implies the abundance of nonzero vectors in X^{**} being L-orthogonal to X. In fact, we get that a Banach space X verifies the Daugavet property if, and only if, the set of vectors in X^{**} being L-orthogonal to X is weak-star dense in X^{**}. In contrast with the separable case, we prove that the existence of almost L-orthogonal vectors in a nonseparable Banach space X (octahedrality) does not imply the existence of nonzero vectors in X^{**} being L-orthogonal to X, which shows that the answer to an environment question is negative (see [7, Section 9] and [13, Section 4]). Also, in contrast with the separable case, we obtain that the existence of almost L-orthogonal vectors in a nonseparable Banach space X (octahedrality) does not imply the abundance of almost L-orthogonal vectors in Banach space X (almost Daugavet property), which solves an open question in [20]. Some consequences on Daugavet property in the setting of L-embedded spaces are also obtained.

1. INTRODUCTION

The concept of orthogonality in the setting of Banach spaces has been a central topic in the theory of Banach spaces. There are important and different concepts of orthogonality in Banach spaces in the literature as the given ones in [11] and [23]. For example, B. Maurey proved in [21] that a separable Banach space contains an isomorphic copy of ℓ_1 if and only if, there is a nonzero element $x^{**} \in X^{**}$ being symmetric orthogonal to X, in the terminology of [23], that is, $\|x^{**}+x\| = \|x^{**}-x\|$ for every $x \in X$. One of the strongest concepts of orthogonality is the L-orthogonality: two vectors x and y in a Banach space X are called L-orthogonal if $\|x+y\| = \|x\|+\|y\|$. An element x in X will be called L-orthogonal to a subspace Y of X if x

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is L-orthogonal to every element in Y. In the setting of Hilbert spaces, it is well known that for every closed and proper subspace there is a non-zero orthogonal vector to that subspace. In this sense, G. Godefroy proved in [6, Theorem II.4] that a separable Banach space X containing isomorphic copies of ℓ_1 can be equivalently renormed so that there is a vector x^{**} in the unit sphere of X^{**} being L-orthogonal to X. The aim of this note is to study the existence and abundance of vectors in the bidual space X^{**} of a Banach space X being L-orthogonal to X, in terms of the existence and abundance of vectors in X which are almost L-orthogonal to finite-dimensional subspaces of X. It is natural to say that a Banach space X contains almost L-orthogonal vectors if, for every $x_1, \ldots, x_n$ vectors in the unit sphere of X and for every $\varepsilon > 0$, there is some vector x in the unit ball of X such that $\|x+x_i\| > 2-\varepsilon$ for every $1 \leqslant i \leqslant n$. This is exactly equivalent to say that the norm of X is octahedral, a concept considered by N. Kalton and G. Godefroy in [7]. In fact, it was proved in [16] that a Banach space X containing isomorphic copies of ℓ_1 can be equivalently renormed so that the new bidual norm is octahedral, and so a bidual renorming of X^{**} contains almost L-orthogonal vectors. Similarly, we will say that a Banach space X has abundance of L- orthogonal vectors with respect to a norming subspace Y of X^* if, for every $x_1, \ldots, x_n$ vectors in the unit sphere of X, for every nonempty $\sigma(X, Y)$-open subset U of the unit ball of X and for every $\varepsilon > 0$, there is some vector x in the unit ball of X such that $\|x + x_i\| > 2 - \varepsilon$ for every $1 \leqslant i \leqslant n$. This is exactly equivalent to say that X satisfies the almost Daugavet property with respect Y (see [12, 13] and Lemma 2.4).

Recall that X has the *Daugavet property with respect to* Y if every rank one operator $T : X \longrightarrow X$ of the form $T = y^* \otimes x$, for $x \in X$ and $y \in Y$, satisfies the equation

$$\|T + I\| = 1 + \|T\|,$$

where I denotes the identity operator. If Y is a norming of X, we say that X has the *almost Daugavet property.* We will say that X has the Daugavet property if $Y = X^*$.

It is then natural to ask if for Banach spaces X containing or having abundance of almost L-orthogonal vectors one can find some or many elements in X^{**} being L-orthogonal to X. For example, in the case that X is separable, G. Godefroy and N. Kalton proved in [7, Lemma 9.1] that if X contains almost L-orthogonal vectors, that is, the norm of X is octahedral, then there are elements in X^{**} being L-orthogonal to X, opening the question in the nonseparable setting.

After some preliminary results in Section 2, we prove in Section 3 that the above question has a negative answer (Theorem 3.2), exhibiting examples of Banach spaces X containing almost L-orthogonal vectors, that is, Banach spaces with an octahedral norm, whose bidual space lacks of nonzero vectors being L-orthogonal to X. In contrast with the above, we also prove in Section 3 that the abundance of almost L-orthogonal vectors in a Banach space X implies the abundance of vectors in X^{**} being L-orthogonal to X (Theorem 3.4). In other more precise words, if X is a Banach space with the almost Daugavet property with respect to some norming subspace Y of X^*, then the set of elements in X^{**} being L-orthogonal to X is $\sigma(X^{**}, Y)$-dense in X^{**}. Then, as an immediate consequence, we get that a Banach space satisfies de Daugavet property if, and only if, the set of elements in X^{**} being L-orthogonal to X is w^*-dense in X^{**} (Theorem 3.5).

We finish the Section 3 showing that the known equivalence of almost Daugavet property and octahedrality is no longer true in the nonseparable setting (Theorem 3.6), solving a question posed in [20]. That is, the existence of almost L-orthogonal elements does not imply the abundance of such elements in the nonseparable setting.

In Section 4 we get some consequences on Daugavet property for Banach spaces being L-embedded. In particular we get that $X\widehat{\otimes}_\pi Y$ has the Daugavet property, whenever X is an L-embedded Banach space and Y is a nonzero Banach space such that either X^{**} or Y has the metric approximation property (Theorem 4.2). Also we get that a Banach space with the Daugavet property can not be an u-ideal in its bidual (Theorem 4.4).

2. Preliminaries

We will consider only real Banach spaces. Given a Banach space X, we will denote the unit ball and the unit sphere of X by B_X and S_X respectively. Moreover, given $x \in X$ and $r > 0$, we will denote $B(x, r) = x + rB_X = \{y \in X : \|x - y\| \leqslant r\}$. We will also denote by X^* the topological dual of X. If Y is a subspace of X^*, $\sigma(X, Y)$ will denote the coarsest topology on X so that elements of Y are continuous. Also, Y is *norming* if $\|x\| = \sup_{y\in Y, \|y\|\leqslant 1} |y(x)|$. Given a bounded subset C of X, we will mean by a *slice of* C a set of the following form

$$S(C, x^*, \alpha) := \{x \in C : x^*(x) > \sup x^*(C) - \alpha\}$$

where $x^* \in X^*$ and $\alpha > 0$. If X is a dual Banach space, the previous set will be called a w^*-*slice* if x^* belongs to the predual of X. Note that finite intersections of slices of C (respectively of w^*-slices of C) form a basis for the inherited weak (respectively weak-star) topology of C.

According to [10], a Banach space X is said to be an L-*embedded space* if there exists a subspace Z of X^{**} such that $X^{**} = X \oplus_1 Z$. Examples of L-embedded Banach spaces are $L_1(\mu)$ spaces, preduals of von Neumann algebras, duals of M-embedded spaces or the dual of the disk algebra (see [10, Example IV.1.1] for formal definitions and details).

Given two Banach spaces X and Y we will denote by $L(X, Y)$ (respectively $K(X, Y)$) the space of all linear and bounded (respectively linear and compact) operators from X to Y, and we will denote by $X\widehat{\otimes}_\pi Y$ and $X\widehat{\otimes}_\varepsilon Y$ the projective and injective tensor product of X and Y, respectively. Moreover, we will say that X has the *metric approximation property* if there exists a net of finite rank and norm-one operators $S_\alpha : X \longrightarrow X$ such that

$S_\alpha(x) \to x$ for all $x \in X$. See [25] for a detailed treatment of the tensor product theory and approximation properties.

Let Z be a subspace of a Banach space X. We say that Z is an *almost isometric ideal* (ai-ideal) in X if X is locally complemented in Z by almost isometries. This means that for each $\varepsilon > 0$ and for each finite-dimensional subspace $E \subseteq X$ there exists a linear operator $T : E \to Z$ satisfying

(1) $T(e) = e$ for each $e \in E \cap Z$, and
(2) $(1-\varepsilon)\|e\| \leqslant \|T(e)\| \leqslant (1+\varepsilon)\|e\|$ for each $e \in E$,

i.e. T is a $(1+\varepsilon)$ isometry fixing the elements of E. If the T satisfies only (1) and the right-hand side of (2) we get the well-known concept of Z being an *ideal* in X [8].

Note that the Principle of Local Reflexivity means that X is an ai-ideal in X^{**} for every Banach space X. Moreover, there are well known Banach spaces properties, as the Daugavet property, octahedrality and all of the diameter two properties, being inherited by ai-ideals (see [1] and [2]). Furthermore, given two Banach spaces X and Y and given an ideal Z in X, then $Z\widehat{\otimes}_\pi Y$ is a closed subspace of $X\widehat{\otimes}_\pi Y$ (see e.g. [22, Theorem 1]). It is also known that whenever X^{**} or Y has the metric approximation property then $X^{**}\widehat{\otimes}_\pi Y$ is an isometric subspace of $(X\widehat{\otimes}_\pi Y)^{**}$ (see [15, Proposition 2.3] and [22, Theorem 1]).

Throughout the text we will make use of the following two results, which we include here for the sake of completeness and for easy reference.

Theorem 2.1. [2, Theorem 1.4] *Let X be a Banach space and let Z be an almost isometric ideal in X. Then there is a linear isometry $\varphi : Z^* \longrightarrow X^*$ such that*

$$\varphi(z^*)(z) = z^*(z)$$

holds for every $z \in Z$ and $z^* \in Z^*$ and satisfying that, for every $\varepsilon > 0$, every finite-dimensional subspace E of X and every finite-dimensional subspace F of Z^*, we can find an operator $T : E \longrightarrow Z$ satisfying

(1) *$T(e) = e$ for every $e \in E \cap Z$,*
(2) *$(1-\varepsilon)\|e\| \leqslant \|T(e)\| \leqslant (1+\varepsilon)\|e\|$ holds for every $e \in E$, and;*
(3) *$f(T(e)) = \varphi(f)(e)$ holds for every $e \in E$ and every $f \in F$.*

Following the notation of [1], to such an operator φ we will refer as *an almost-isometric Hahn-Banach extension operator*. Notice that if $\varphi : Z^* \longrightarrow X^*$ is an almost isometric Hahn-Banach extension operator, then $\varphi^* : X^{**} \longrightarrow Z^{**}$ is a norm-one projection.

Another central result in our main theorems will be the following, coming from [1, Remark 2.3]

Theorem 2.2. *Let X be a Banach space, let Y be a subspace of X such that* $\mathrm{dens}(Y) = \alpha$ *and let $W \subseteq X^*$ be such that* $\mathrm{dens}(W) \leqslant \alpha$*. Then there exists an almost isometric ideal Z in X containing Y and an almost isometric Hahn-Banach extension operator $\varphi : Z^* \longrightarrow X^*$ such that $\varphi(Z^*) \supset W$.*

According to [7], given a Banach space X, the *ball topology*, denoted by b_X, is defined as the coarsest topology on X so that every closed ball is closed in b_X. As a consequence, a basis for the topology b_X is formed by the sets of the following form

$$X \setminus \bigcup_{i=1}^{n} B(x_i, r_i),$$

where $x_1, \ldots, x_n$ are elements of X and $r_1, \ldots, r_n$ are positive numbers.

Let us end by giving a pair of technical results which will be used in the proof of Theorem 3.3. The first one can be seen as a kind of generalisation of the classical Bourgain Lemma [5, Lemma II.1], which asserts that, given a Banach space X, then every non-empty weakly open subset of B_X contains a convex combination of slices of B_X. The following result already appeared in [13] without a complete proof. However, let us provide a proof here for the sake of completeness.

Lemma 2.3. *Let X be a Banach space and $Y \subseteq X^*$ be a norming subspace for X. Let U be a non-empty $\sigma(X,Y)$ open subset of B_X. Then U contains a convex combination of $\sigma(X,Y)$-slices of B_X.*

Proof. Let $\hat{U}$ be the $\sigma(X^{**},Y)$-open subset of $B_{X^{**}}$ defined by U. Notice that

$$B_{X^{**}} = \overline{\text{co}}^{w^*}(\text{Ext}(B_{X^{**}})) \subseteq \overline{\text{co}}^{\sigma(X^{**},Y)}(\text{Ext}(B_{X^{**}}))$$

by Krein-Milman theorem, so we can find a convex combination of extreme points $\sum_{i=1}^{n} \lambda_i e_i \in \hat{U}$. Since the sum in X^{**} is $\sigma(X^{**},Y)$ continuous we can find, for every $i \in \{1, \ldots, n\}$, a $\sigma(X^{**},Y)$ open subset of $B_{X^{**}}$ such that $e_i \in V_i$ holds for every i and such that $\sum_{i=1}^{n} \lambda_i V_i \subseteq \hat{U}$. Since the following chain of inclusions hold

$$\sum_{i=1}^{n} \lambda_i (V_i \cap B_X) \subseteq \left(\sum_{i=1}^{n} \lambda_i v_i\right) \cap B_X \subseteq \hat{U} \cap B_X = U,$$

the following claim finishes the proof.

Claim: Given $i \in \{1, \ldots, n\}$ we can find a slice S_i such that $S_i \subseteq V_i \cap B_X$.

Proof of the Claim. By the definition of the $\sigma(X^{**},Y)$ we can assume that $V_i = \bigcap\limits_{j=1}^{k_i} T_j$ where every T_j is a $\sigma(X^{**},Y)$-slice of $B_{X^{**}}$. Since $e_i \in V_i$ it follows that $e_i \notin \bigcup\limits_{j=1}^{k_i} B_{X^{**}} \setminus T_j$. Now e_i is an extreme point of $B_{X^{**}}$ and then $e_i \notin \text{co}\left(\bigcup\limits_{j=1}^{k_i} B_{X^{**}} \setminus T_j\right)$. Notice that $B_{X^{**}} \setminus T_j$ is $\sigma(X^{**},Y)$-closed in the $\sigma(X^{**},Y)$-compact space $B_{X^{**}}$ for every j and, since it is additionally convex, it follows that $\text{co}\left(\bigcup\limits_{j=1}^{k_i} B_{X^{**}} \setminus T_j\right)$ is $\sigma(X^{**},Y)$ compact too. Since $e_i \notin \operatorname{co}\left(\bigcup_{j=1}^{k_i} B_{X^{**}} \setminus T_j\right)$ then we can find $x \in B_X$ such that $x \notin \overline{\operatorname{co}}^{\sigma(X,Y)}\left(\bigcup_{j=1}^{k_i} B_{X^{**}} \setminus T_j\right)$. By a separation argument we can find $y^* \in S_Y$ and $\alpha > 0$ such that

$$y^*(x) > \alpha > \sup_{z \in Z} y^*(z)$$

for $Z = \bigcup_{j=1}^{k_i} (B_{X^{**}} \setminus T_j) \cap B_X$. If we define

$$S_i := \{z \in B_X : y^*(z) > \alpha\}$$

it follows that S_i a $\sigma(X,Y)$-slice. Furthermore, given $z \in S_i$ it follows that $y^*(z) > \alpha$, so $z \in \bigcap_{j=1}^{k_i} T_j \cap B_X = V_i \cap B_X$, which completes the proof of the claim. ∎

∎

Let us end by giving a brief sketch of proof of the following lemma, which is an easy extension of [12, Corollary 3.4].

Lemma 2.4. *Let X be a Banach space and assume that X has the almost Daugavet property with respect to a norming subspace $Y \subseteq X^*$. Then, for every $x_1, \ldots, x_n \in S_X$, every $\varepsilon > 0$ and every non-empty $\sigma(X,Y)$-open subset U of B_X there exists $z \in U$ such that*

$$\|x_i + z\| > 2 - \varepsilon$$

for every $i \in \{1, \ldots, n\}$.

Proof. We will prove the lemma by induction on n. The case $n = 1$ is just [12, Corollary 3.4].

Hence, assume by induction that the lemma holds for n, and let us prove it for $n+1$. To this end, pick $x_1, \ldots, x_{n+1} \in S_X$, $\varepsilon > 0$ and U to be a non-empty $\sigma(X,Y)$-open subset of B_X. By induction hypothesis we can find $z \in U$ such that

$$\|x_i + z\| > 2 - \frac{\varepsilon}{2}$$

holds for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n\}$ choose $f_i \in S_Y$ such that $f_i(x_i + z) > 2 - \frac{\varepsilon}{2}$. Since $z \in U$ and $f_i \in Y$, it follows that

$$z \in W := U \cap \bigcap_{i=1}^{n} S(B_X, f_i, \frac{\varepsilon}{2}).$$

Since W is a non-empty $\sigma(X,Y)$ open subset of B_X we can find $u \in W$ such that

$$\|x_{n+1} + u\| > 2 - \varepsilon.$$

Also, given $i \in \{1, \ldots, n\}$, since $f_i(x_i) > 1 - \frac{\varepsilon}{2}$ and $f_i(u) > 1 - \frac{\varepsilon}{2}$ we get that

$$\|x_i + u\| \geqslant f_i(x_i + u) > 2 - \varepsilon,$$

which concludes the proof. ∎

3. Main results

Our first goal will be to show that, in contrast with the result in [7, Lemma 9.1], where it is proved that octahedrality of a separable Banach space X is equivalent to the existence of elements in X^{**} being L-orthogonal to X, this is no longer true in the nonseparable setting. That is, the existence of almost L-orthogonal vectors in a Banach space X, as defined in the introduction, does not imply the existence of nonzero vectors in X^{**} being L-orthogonal to X. For this, we need the following result.

Proposition 3.1. *Let X be a uniformly smooth Banach space. Assume that there exists an element $T \in (X\widehat{\otimes}_\varepsilon \ell_1)^{**} = (X^*\widehat{\otimes}_\pi \ell_1^*)^* = L(X^*, \ell_1^{**})$ such that $\|T\| = 1$ and such that*

$$\|T + S\| = 2$$

for every norm-one element $S \in X\widehat{\otimes}_\varepsilon \ell_1 = K(X^*, \ell_1)$. Then T is an isometry.

Proof. Pick an arbitrary $x^* \in S_{X^*}$ and let us prove that $\|T(x^*)\| = 1$. This is enough in view of the homogeneity of T. To this end, pick $x \in S_X$ such that $x^*(x) = 1$. Define $S := x \otimes e_1$, which is a norm-one element of $X\widehat{\otimes}_\varepsilon \ell_1$. By assumptions we have that $\|T + S\| = 2$ so we can find, for every $n \in \mathbb{N}$, an element $x^* \in S_{X^*}$ such that

$$2 - \frac{1}{n} < \|(T + S)(x_n^*)\| \leqslant \|T(x_n^*)\| + |x_n^*(x)|.$$

From the previous inequality it is clear that $\|T(x_n^*)\| \to 1$ and $|x_n^*(x)| \to 1$. Now, up taking a suitable subsequence, we can assume that the sign of $x_n^*(x)$ is constant, so $x_n^*(x)$ converges to 1 or to -1. Since X is uniformly smooth we deduce that either $x_n^* \to x^*$ or $x_n^* \to -x^*$. With no loss of generaly, assume that $x_n^* \to x^*$. Now $T(x_n^*) \to T(x^*)$ which in turn implies that $\|T(x_n^*)\| \to \|T(x^*)\|$. Since $\|T(x_n^*)\| \to 1$ then $\|T(x^*)\| = 1$, so the lemma follows. ∎

The previous lemma together with [17, Theorem 3.2] yield the desired counterexample.

Theorem 3.2. *Let I be an infinite set with $\operatorname{card}(I) > \operatorname{dens}(\ell_1^{**})$ and let $2 < p < \infty$. Then the norm of $\ell_p(I)\widehat{\otimes}_\varepsilon \ell_1$ is octahedral but there is no $T \in (\ell_p(I)\widehat{\otimes}_\varepsilon \ell_1)^{**}$ such that $\|T\| = 1$ and such that*

$$\|T + S\| = 1 + \|S\|$$

for every $S \in \ell_p(I)\widehat{\otimes}_\varepsilon \ell_1$.

Proof. Since $2 < p < \infty$ it follows that $\ell_q(I)$ is finitely representable in ℓ_1, where $\frac{1}{p}+\frac{1}{q}=1$, and has the MAP. By [17, Theorem 3.2] it follows that the norm of $\ell_p(I)\widehat{\otimes}_\varepsilon\ell_1$ is octahedral. However, notice that there is no isometry $T:\ell_q(I)\longrightarrow\ell_1^{**}$ since $\operatorname{dens}(\ell_q(I))\geqslant\operatorname{card}(I)>\operatorname{dens}(\ell_1^{**})$: According to Proposition 3.1, there is no $T\in S_{(\ell_p(I)\widehat{\otimes}_\varepsilon\ell_1)^{**}}$ such that $\|T+S\|=2$ holds for every $S\in S_{\ell_p(I)\widehat{\otimes}_\varepsilon\ell_1}$, so we are done. ∎

Now, our goal will be to get nonzero vectors in the bidual of a Banach space X being L-orthogonal to X from the existence of almost L-orthogonal vectors in X. Let us show the main result of the paper.

Theorem 3.3. *Let X be a Banach space with the almost Daugavet property with respect to the norming subspace $Y\subseteq X^*$. Let $u\in B_{X^{**}}$. Then, for every almost isometric ideal Z in X and for every $\{g_\beta:\beta\leqslant\alpha\}\subseteq S_Y$ such that $g_\beta\in\varphi(Z^*)$ for every $\beta\leqslant\alpha$, where $\alpha=\operatorname{dens}(Z)$, we can find $v\in S_{X^{**}}$ satisfying the following two assertions:*

1. *$\|x+v\|=1+\|x\|$ for every $x\in Z$.*
2. *$v(g_\beta)=u(g_\beta)$ for every $\beta\leqslant\alpha$.*

Proof. The proof will be done by induction in $\alpha=\operatorname{dens}(Z)$.

Case $\alpha=\omega_0$.

Let $\{g_n:n\in\mathbb{N}\}\subseteq S_Y$ and let Z be a separable almost isometric ideal in X and $\varphi:Z^*\longrightarrow X^*$ such that $\{g_n:n\in\mathbb{N}\}\subseteq\varphi(Z^*)$. Let us construct v_α. To this end, since Z is separable, there exists a basis $\{O_n:n\in\mathbb{N}\}$ of the b_Z-topology restricted to B_Z. For every $n\in\mathbb{N}$ consider $\tilde{O_n}$ to be the b_X-open subset of B_X which defines O_n (i.e. if $O_n:=\bigcap\limits_{i=1}^{k_n}B_Z\setminus B(z_i^n,r_i)$ then $\tilde{O}_n:=\bigcap\limits_{i=1}^{k_n}B_X\setminus B(z_i^n,r_i)$). Since X has the Daugavet property with respect to Y it follows that, for every $n\in\mathbb{N}$, there exists by Lemma 2.4 an element

$$x_n\in\bigcap_{k=1}^n\tilde{O}_k\cap\bigcap_{k=1}^n\left\{x\in B_X:|g_k(x)-u(g_k)|<\frac{1}{n}\right\}.$$

Now, for every $\delta>0$, there exists a δ-isometry

$$T:E:=\operatorname{span}\{z_1,\ldots,z_{k_n},x_n\}\longrightarrow Z$$

such that $T(z_i)=z_i$ and that $g_k(T(v))=\varphi(g_k)(v)$ holds for every $v\in E$ and every $k\in\{1,\ldots,n\}$. Taking into account the property defining x_n and the fact that δ can be taken as small as we wish we can ensure the existence of

$$z_n\in\bigcap_{k=1}^n O_k\cap\bigcap_{k=1}^n\left\{z\in B_Z:|\varphi^{-1}(g_k)(z)-u(g_k)|<\frac{1}{n}\right\}.$$

Now [7, Lemma 9.1] ensures the existence of a suitable w^*-cluster point $u_\alpha \in S_{X^{**}}$ of $\{z_n\}$ such that

$$\|z+u_\alpha\| = 1+\|z\|$$

holds for every $z \in Z$. If we take $v_\alpha \in (\varphi^*)^{-1}(u_\alpha)$ then we have that

$$\|x+v_\alpha\| \geqslant \|\varphi^*(x+v_\alpha)\| = \|x+u_\alpha\| = 1+\|x\|$$

holds for every $x \in X$. Also, it is clear, by definition of the sequence $\{x_n\}$ and the fact that u is a w^*-cluster point, that $v_\alpha(g_k) = u(g_k)$ holds for every $k \in \mathbb{N}$. This completes the case $\alpha = \omega_0$.

Assume now that $\omega_0 < \alpha \leqslant \operatorname{dens}(X)$ *and that the thesis of the theorem holds for every almost-isometric ideal in* X *whose density character is smaller than* α*.*

Let Z be an almost isometric ideal in X of density character equal to α and let $\varphi : Z^* \longrightarrow X^*$ be a almost isometric Hahn-Banach extension operator such that $\{g_\beta : \beta \leqslant \alpha\} \subseteq \varphi(Z^*) \cap S_Y$. In order to construct v, pick $\{x_\beta : \beta \leqslant \alpha\} \subseteq S_X$ to be a dense subset of S_Z. Let us construct by transfinite induction on $\omega_0 \leqslant \beta < \alpha$ a family $\{(Z_\beta, \varphi_\beta, \{f_{\beta,\gamma} : \gamma < \beta\}, v_\beta : \beta < \alpha\}$ satisfying the following assertions:

(1) Z_β is an almost isometric ideal in X containing $\bigcup\limits_{\gamma<\beta} Z_\gamma \cup \{x_\beta\}$ and such that $\operatorname{dens}(Z_\beta) = \operatorname{card}(\beta)$.

(2) $\varphi_\beta : Z_\beta^* \longrightarrow X^*$ is an almost isometric Hahn-Banach operator such that $\{f_{\gamma,\delta} : \delta < \gamma < \beta\} \cup \{g_\gamma : \gamma < \beta\} \subseteq \varphi_\beta(Z_\beta^*)$.

(3) $v_\beta \in S_{X^{**}}$ satisfies that

$$\|z+v_\beta\| = 1+\|z\|$$

for every $z \in Z_\beta$ and $\{f_{\beta,\gamma} : \gamma \leqslant \beta\} \subseteq S_Y$ is norming for $Z_\beta \oplus \mathbb{R}v_\beta$.

(4) For every $\delta < \gamma < \beta < \alpha$ it follows

$$v_\beta(f_{\gamma,\delta}) = v_\gamma(f_{\gamma,\delta}),$$

and

$$v_\beta(g_\gamma) = u(g_\gamma).$$

The construction of the family will be completed by transfinite induction on β. To this end, notice that the case $\beta = \omega_0$ runs similarly to the case that Z is separable. So, assume that $(Z_\gamma, \varphi_\gamma, \{f_{\gamma,\delta} : \delta \in \gamma\}, v_\gamma)$ has already been constructed for every $\gamma < \beta$, and let us construct $(Z_\beta, \varphi_\beta, \{f_{\beta,\gamma} : \gamma \in \beta\}, v_\beta)$. Pick v to be a w^*-cluster point of the net $\{v_\gamma : \gamma < \beta\}$ (where the order in $[0, \beta[$ is the classical order). Notice that, by induction hypothesis, for every $\delta_0 < \gamma_0 < \gamma < \beta$ we have that

$$v_\gamma(f_{\gamma_0,\delta_0}) = v_{\gamma_0}(f_{\gamma_0,\delta_0}).$$

Then, since v is a w^*-cluster point of $\{v_\gamma\}_{\gamma<\beta}$, we get that

$$v(f_{\gamma_0,\delta_0}) = v_{\gamma_0}(f_{\gamma_0,\delta_0}). \tag{3.1}$$

Because of the same reason we obtain that

$$v(g_\beta) = v_{\gamma+1}(g_\gamma) = u(g_\gamma) \tag{3.2}$$

for every $\gamma < \beta$. Now

$\text{card}(\{f_{\gamma,\delta} : \delta < \gamma < \beta\} \cup \{g_\gamma : \gamma \in \beta\}) \leqslant \max\{\text{card}(\beta\times\beta), \text{card}(\beta)\} = \text{card}(\beta)$.

Also $\text{dens}(\bigcup_{\gamma<\beta} Z_\gamma) = \text{card}(\beta)$. Then, by [1, Remark 2.3] there exists an almost isometric ideal Z_β in X containing $\bigcup_{\gamma<\beta} Z_\gamma \cup \{x_\beta\}$ and an almost isometric Hahn-Banach extension operator $\varphi_\beta : Z_\beta \longrightarrow X^*$ such that

$$\varphi_\beta(Z_\beta^*) \supset \{f_{\gamma,\delta} : \delta < \gamma < \beta\} \cup \{g_\gamma : \gamma \leqslant \beta\}.$$

Let us construct v_β. To this end, consider $\varphi_\beta^*(v) \in Z_\beta^{**}$. Since $\text{dens}(Z_\beta) = \text{card}(\beta) < \alpha$, then the induction hypothesis applies. Consequently, we can find $v_\beta \in S_{X^{**}}$ such that

(1) $\|z + v_\beta\| = 1 + \|x\|$ for every $x \in Z_\beta$, and
(2) $v_\beta(f_{\gamma,\delta}) = v(f_{\gamma,\delta})$ for $\delta < \gamma < \beta$, and $v_\beta(g_\gamma) = v(g_\gamma)$ for every $\gamma < \beta$.

Take $\{f_{\beta,\gamma} : \gamma < \beta\} \subseteq S_Y$ being norming for $Z_\beta \oplus \mathbb{R}v_\beta$. It follows as before that $\{(Z_\gamma, \varphi_\gamma, \{\varphi_\gamma(f_{\gamma,\delta}) : \delta < \gamma\}, v_\gamma) : \gamma \leqslant \beta\}$ satisfies our purposes.

Now consider v_α to be a w^*-cluster point of $\{v_\beta\}_{\beta\in\alpha}$. Let us prove that v_α satisfies the desired properties.

(1) Let us prove that $v_\alpha(g_\beta) = u(g_\beta)$ for every $\beta < \alpha$. To this end pick $\varepsilon > 0$ and find $\gamma > \beta + 1$ so that $|(v_\alpha - v_\gamma)(g_\beta)| < \varepsilon$. Since $v_\delta(g_\beta) = u(g_\beta)$ holds for every $\delta \geqslant \beta + 1$ it follows that

$$|(v_\alpha - u)(g_\beta)| = |(v - v_\gamma)(g_\beta)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we are done.

(2) Given $x \in S_Z$ it follows that

$$\|x + u\| = 2.$$

To this end, pick $\varepsilon > 0$. Since $\{x_\beta : \beta < \alpha\}$ is dense in S_Z find $\beta < \alpha$ such that $\|x - x_\beta\| < \frac{\varepsilon}{3}$. Since $\|x_\beta + v_\beta\| = 2$ find $\gamma < \beta$ such that

$$(z_\beta + v_\beta)(f_{\beta,\gamma}) > 2 - \frac{\varepsilon}{3}.$$

Now, given any $\beta' > \beta$ we have that

$$(z_\beta + v_{\beta'})(f_{\beta,\gamma}) = (z_\beta + v_\beta)(f_{\beta,\gamma}) > 2 - \frac{\varepsilon}{3}.$$

Since v is a w^*-cluster point of $\{v_\beta : \beta < \alpha\}$ we obtain that

$$2 - \frac{\varepsilon}{3} \leqslant (z_\beta + v)(f_{\beta,\gamma}) \leqslant \|x_\beta + v\| \leqslant \|x + v\| + \frac{\varepsilon}{3},$$

so $\|x+v\| > 2-\varepsilon$. Since $\varepsilon > 0$ was arbitrary we also conclude that $\|x+v\| = 2$. Finally, since $x \in S_Z$ was arbitrary, a convexity argument yields that

$$\|x+v\| = 1 + \|x\|$$

holds for every $x \in Z$.

This completes the proof of the theorem by transfinite induction on $\alpha = \operatorname{dens}(X)$. ∎

Since every Banach space is trivially an almost isometric ideal in itself, the following result follows.

Theorem 3.4. *Let X be a Banach space with the almost Daugavet property with respect to $Y \subseteq X^*$. Let $u \in B_{X^{**}}$ and $\{g_\beta : \beta \leqslant \alpha\} \subseteq S_Y$, where $\alpha = \operatorname{dens}(Z)$. Then we can find $v \in S_{X^{**}}$ satisfying the following two assertions:*

1. *$\|x+v\| = 1 + \|x\|$ for every $x \in X$.*
2. *$v(g_\beta) = u(g_\beta)$ for every $\beta \in \alpha$.*

As a consequence we obtain the following strenghtening of the Daugavet property, which extends [24, Theorem 3.2] to the non-separable case.

Theorem 3.5. *Let X be a Banach space. The following assertions are equivalent:*

1. *X has the Daugavet property, that is, for every $x \in S_X$, every non-empty relatively weakly open subset of B_X and every $\varepsilon > 0$ there exists $y \in W$ such that $\|x+y\| > 2-\varepsilon$.*
2. *For every non-empty relatively weakly-star open subset W of $B_{X^{**}}$ there exists $v \in S_{X^{**}} \cap W$ such that*

$$\|x+v\| = 1 + \|x\|$$

holds for every $x \in X$.

Proof. (2)⇒(1) is obvious. For the converse, take a non-empty weakly-star open set W of $B_{X^{**}}$ and $u \in W \cap S_{X^{**}}$. With no loss of generality we can assume that $W = \bigcap_{i=1}^{n} S(B_{X^{**}}, f_i, \alpha_i)$, for suitable $f_i \in X^*$ and $\alpha_i > 0$. By Theorem 3.4 we can find an element $v \in S_{X^{**}}$ such that

1. $\|x+v\| = 1 + \|x\|$ for every $x \in X$ and,
2. $v(f_i) = u(f_i)$ for every $i \in \{1, \ldots, n\}$.

Now condition (2) above implies that $v \in W$ since $u \in W$, so we are done. ∎

Notice that, from the results of [13] together with [7, Lemma 9.1], it is known that, given a separable Banach space X, then the following assertions are equivalent:

1. X has the almost Daugavet property.

(2) There exists an element $u \in S_{X^{**}}$ such that
$$\|x+u\| = 1 + \|x\|$$
holds for every $x \in X$.

(3) The norm of X is octahedral.

Notice that a consequence of Theorem 3.3 is that (1) implies (2), which in turn implies (3). Note also that from the results [13] it is unclear whether the implication (3) implies (1) holds in the non-separable context (it is indeed explicitly posed as an open question in [20, P. 89]). Note that Theorems 3.2 and 3.4 imply that (3) does not imply (1). However, as application of Theorem 3.3, we even obtain that (2) does not imply (1), as the following theorem shows.

Theorem 3.6. *Let α be a cardinal number so that $\alpha > \text{dens}(\ell_1^{**})$. Then $X = \ell_1 \oplus_1 \ell_2(\alpha)$ fails the almost Daugavet property.*

Proof. Assume by contradiction that X has the almost Daugavet property with respect to a norming subspace Y of X^*. Notice that $\text{dens}(X) = \alpha = w^* - \text{dens}(X^*)$. Since any dense subset of S_Y is dense for X we obtain that $\text{dens}(Y) \geqslant \alpha$. Pick a cardinal number β so that $\text{dens}(\ell_1^{**}) < \beta < \alpha$. By transfinite induction together with Riesz lemma [4, Lemma 1.23] we can find a set $\{f_\gamma : \gamma \leqslant \beta\} \subseteq S_Y$ so that $\text{dist}(f_\gamma, \overline{\text{span}}\{f_\delta : \delta < \gamma\}) \geqslant 1/2$. Consequently, by Hahn-Banach theorem we can get, for every $\gamma \leqslant \beta$, an element $u \in S_{X^{**}}$ such that $u_\gamma(f_\delta) = 0$ for every $\delta < \gamma$ and $u_\gamma(f_\gamma) \geqslant \frac{1}{2}$. By Theorem 3.3 we can find, for every $\gamma \leqslant \beta$, an element $v_\gamma \in S_{X^{**}}$ such that
$$\|x + v_\gamma\| = 1 + \|x\|$$
for every $x \in X$ and such that $v_\gamma = u_\gamma$ on $\{f_\delta : \delta \leqslant \gamma\}$. Notice that the first condition implies, from the equality $X^{**} = \ell_1^{**} \oplus_1 \ell_2(\alpha)$, that $\{v_\gamma : \gamma \leqslant \beta\} \subseteq \ell_1^{**}$. On the other hand, given $\delta < \gamma$ arbitrary we get that
$$\|v_\gamma - v_\delta\| \geqslant |(v_\gamma - v_\delta)(f_\delta)| = |v_\delta(f_\delta)| \geqslant \frac{1}{2}.$$
This implies that $\text{card}(\{f_\gamma : \gamma \leqslant \beta\}) = \beta \leqslant \text{dens}(\ell_1^{**})$, which entails a contradiction with the choice of β. Consequently, X fails the almost Daugavet proeprty, as desired. ∎

Now some comments are pertinent.

Remark 3.7. Notice that the space X exposed in Theorem 3.6, which fails to enjoy the almost Daugavet property, is a Banach space whose norm is octahedral (see e.g. [9, Corollary 2.3]). Consequently, octahedrality of the norm does not imply almost Daugavet property, which gives a negative answer to [20, Section VI,8]. Furthermore, since $X^{**} = \ell_1^{**} \oplus_1 \ell_2(\alpha)$, we obtain even that (2)⇒(1) is false.

Remark 3.8. Let X and Y be two Banach spaces with the almost Daugavet property. S. Lucking proved in [18, Proposition 2.2], by making use of the characterisation of the almost Daugavet property given in [13], that if X and Y are separable and X has the almost Daugavet property then $X \oplus_1 Y$ has the almost Daugavet property. However, Theorem 3.6 shows that this result is not longer true if we remove separability assumption on the space Y.

Remark 3.9. In [19, Corollary 3.3] it is proved that if Y is a non-reflexive separable subspace of a non-reflexive L-embedded Banach space X then Y has the almost Daugavet property. Note that this result is not true in the non-separable context since the space X considered in Theorem 3.6 is L-embedded by [10, Example IV.1.1 and Proposition IV.1.5].

4. Daugavet property and L-embedded spaces

In order to obtain more consequences from Theorem 3.5 we consider the following characterisation of the Daugavet property in L-embedded spaces, which is an extension to the non-separable case of [24, Theorem 3.3].

Theorem 4.1. *Let X be an L-embedded Banach space. Assume that $X^{**} = X \oplus_1 Z$. Then, the following are equivalent:*

1. *X^* has the Daugavet property.*
2. *X has the Daugavet property.*
3. *B_Z is weak-star dense in $B_{X^{**}}$.*

Proof. (1)$\Rightarrow$ (2) is obvious.

(2)$\Rightarrow$ (3). Let W be a non-empty weakly-star open subset of $B_{X^{**}}$ and let us prove that $B_Z \cap W \neq \emptyset$. By Theorem 3.5 we can find $u \in W \cap S_{X^{**}}$ such that

$$\|x + u\| = 1 + \|x\|$$

for every $x \in X$. Since $u \in X^{**}$ we can find $x \in X$ and $z \in Z$ such that $u = x + z$. Now

$$1 \geqslant \|z\| = \| - x + (x + z)\| = 1 + \|x\|.$$

This implies that $x = 0$ and, consequently, $u \in B_Z$. So $W \cap B_Z \neq \emptyset$, as desired.

(2)$\Rightarrow$ (3) follows from [3, Theorem 2.2]. ∎

This result generalises [3, Theorem 3.2], where the authors proved that a real or complex JBW^*-triple X has the Daugavet property if, and only if, its predual X_* (which is an L-embedded Banach space) has the Daugavet property.

Now, following word-by-word the proof of [24, Theorem 3.7], we get the next result, which gives an affirmative answer to [24, Problem 5.2]

Theorem 4.2. *Let X be an L-embedded Banach space with the Daugavet property and let Y be a non-zero Banach space. If either X^{**} or Y has the metric approximation property then $X \widehat{\otimes}_\pi Y$ has the Daugavet property.*

Proof. Assume with no loss of generality that $X^{**} = X \oplus_1 Z$. We will follow the ideas of [24, Theorem 3.7]. To this end, pick $G \in S_{L(X,Y^*)}$ and $\alpha > 0$ and, to prove the theorem, it suffices to find an element $u \in S_{X^{**}}$ and $y \in S_Y$ such that $u(y \circ G) > 1 - \alpha$ and such that

$$\|z + u \otimes y\|_{(X\widehat{\otimes}_\pi Y)^{**}} = 1 + \|z\|$$

for every $z \in X\widehat{\otimes}_\pi Y$. To do so, by the assumption that either X^{**} or Y has the MAP, it follows that $X^{**}\widehat{\otimes}_\pi Y$ is an isometric subspace of $(X\widehat{\otimes}_\pi Y)^{**}$ by [15, Proposition 2.3], so it suffices to prove that

$$\|z + u \otimes y\|_{X^{**}\widehat{\otimes}_\pi Y} = 1 + \|z\|$$

for every $z \in X\widehat{\otimes}_\pi Y$. To this end, find $x \in S_X$ and $y \in S_Y$ such that $G(x)(y) > 1 - \alpha$. This means that

$$x \in S(B_X, y \circ G, \alpha).$$

Since $S(B_{X^{**}}, y \circ G, \alpha)$ is a non-empty weakly-star open subset of $B_{X^{**}}$ and X is an L-embedded Banach space with the Daugavet property then by Theorem 4.1 we can find $u \in S_Z$ such that $u(y \circ G) > 1 - \alpha$. Let us prove that

$$\|z + u \otimes y\|_{X^{**}\widehat{\otimes}_\pi Y} = 1 + \|z\|$$

for every $z \in X\widehat{\otimes}_\pi Y$. To this end pick $z \in X\widehat{\otimes}_\pi Y$, $\varepsilon > 0$, and take $T \in S_{L(X,Y^*)}$ such that $T(z) = \|z\|$. Since $\|u\| = 1$ choose $x^* \in S_{X^*}$ such that $u(x^*) > 1 - \varepsilon$. Pick $y^* \in S_{Y^*}$ such that $y^*(y) = 1$ and define $\hat{T} : X^{**} = X \oplus_1 Z \longrightarrow Y^*$ by the equation

$$\hat{T}(x + z) = T(x) + z(x^*)y^*.$$

It is not difficult to prove that $\|\hat{T}\| = 1$. Hence

$$\|z + u \otimes y\|_{X^{**}\widehat{\otimes}_\pi Y} \geqslant \hat{T}(z + u \otimes y) = T(z) + u(x^*)y^*(y) = 1 + u(x^*) > 2 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude the theorem. ∎

Let us end with some consequences about *u-structure* in Banach spaces with the Daugavet property. To this end, according to [8], given a Banach space X and a subspace Y, we say that Y *is a u-summand in* X if there exists a subspace Z of X such that $X = Y \oplus Z$ and such that the projection $P : X \longrightarrow X$ such that $P(X) \subseteq Y$ satisfies that $\|I - 2P\| \leqslant 1$ (in such a case we say that P is a u-projection). We say that Y is an *u-ideal in* X if there exists a u-projection $P : X^* \longrightarrow Y^*$ such that $\mathrm{Ker}(P) = Y^\perp$. Finally, we say that X is an u-ideal if X is an u-ideal in X^{**} (under the canonical inclusion).

Let us end the section with the following two consequences of Theorem 3.5 about u structure in Banach spaces.

Proposition 4.3. *Let X be a Banach space with the Daugavet property. Assume that X is an u-summand in its bidual, say $X^{**} = X \oplus Z$. Then Z is w^*-dense in X^*.*

Proof. By Theorem 3.5 it is enough to prove that every $u \in S_{X^{**}}$ such that

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$ satisfies that $u \in Z$. To this end, pick such an element $u \in S_{X^{**}}$. By the decomposition $X^{**} = X \oplus Z$ we get that there exist (unique) $x \in X$ and $z \in Z$ such that $u = x + z$. Let us prove that $x = 0$. Notice that

$$1 + 2\|x\| = \|u - 2x\| = \|u - 2P(u)\| \leqslant \|I - 2P\| \leqslant 1.$$

By the above inequality we obtain $\|x\| = 0$ or, equivalently, that $u = z \in Z$, as we wanted. ■

Theorem 4.4. *Let X be a Banach space with the Daugavet property. Then X can not be an u-ideal in its bidual.*

Proof. Assume, by contradiction, that there exists $P : X^{***} \longrightarrow X^{***}$ a u-projection such that $\operatorname{Ker}(P) = X^{\perp}$. Pick $x \in S_X$. Since X has the Daugavet property then X^* has the almost Daugavet property with respect to the norming subspace X of X^{**} [14, Lemma 2.1], so an application of Theorem 3.4 implies the existence of an element $u \in S_{X^{***}}$ such that $u(x) = 1$ and such that

$$\|x^* + u\| = 1 + \|x^*\|$$

holds for every $x^* \in X$. Let us prove that $P(u) = 0$. Indeed,

$$1 + 2\|P(u)\| = \|u - 2P(u)\| \leqslant \|I - 2P\| \leqslant 1,$$

so $P(u) = 0$. This implies that $u \in \operatorname{Ker}(P) = X^{\perp}$, which entails a contradiction with the fact that $u(x) = 1$. Consequently, X can not be a u-ideal in its bidual. ■

Remark 4.5. In [8, Theorem 5.1] it is proved that if a separable Banach space X contains ℓ_1 then X can not be a strict u-ideal. The previous theorem makes use of the (quite stronger) assumption that X has the Daugavet property. However, the separability assumption is removed. Also, it is proved that X can not be even a u-ideal in its bidual.

As $L_1[0,1]$ is an L-embedded space with the Daugavet property, the above theorem fails in the setting of L-embedded spaces. However, we don't know if there is an L-embedded dual space satisfying the Daugavet property. From Theorem 4.1, this is equivalent to ask about the existence of an L-embedded dual space X so that its L-complement in X^{**} has a w^*-dense unit ball in the unit ball of X^{**}. Also, the above is again equivalent, from Theorem 4.1, to the existence of a Banach space X whose dual space, X^*, is L-embedded and so that X^{**} satisfies de Daugavet property. Recall that the existence of a bidual space with the Daugavet property is unknown (see [26, Section 6, Question (2)]).

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