THE EQUIVALENCE BETWEEN CPCP AND STRONG REGULARITY UNDER KREIN-MILMAN PROPERTY

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Abstract. We obtain a result in the spirit of the well-known W. Schachermeyer and H. P. Rosenthal research about the equivalence between Radon-Nikodym and Krein-Milman properties, by showing that, for closed, bounded and convex subsets C of a separable Banach space, under Krein-Milman property for C, one has the equivalence between convex point of continuity property and strong regularity both defined for every locally convex topology on C, containing the weak topology on C. Then, under Krein-Milman property, not only convex point of continuity property and strong regularity are equivalent as defined for weak topology, but even when they are defined for a locally convex topology containing the weak topology. We also show that while the unit ball B of c_0 fails convex point of continuity property and strong regularity (both defined for the weak topology), threre is a locally convex topology τ on B, containing the weak topology on B, such that B still fails convex point of continuity property for τ, but B surprisingly satisfies strong regularity for $\tau-$open subsets. As a consequence, using the usual norm of c_0, we obtain that B satisfies the diameter two property for the topology τ, that is, every nonempty $\tau-$open subset of B has diameter two, but every $\tau-$open subset of B contains convex combinations of relative $\tau-$open subsets with diameter arbitrarily small, that is, B fails strong diameter two property for topology τ, which stresses the known extreme differences up to now between those diameter two properties from a topological point of view.

1. Introduction

There are three families of subsets in each bounded, convex, and non-empty subset C of a Banach space X that are highly relevant to understanding the weak topology w of the space X and its structural properties: the family of slices of C (a subbasis of the weak topology), the family of relatively weakly open subsets of C (or a basis of the weak topology) and the family of convex combinations of slices or relatively

Date: May 25, 2023.

1991 *Mathematics Subject Classification.* 46B20 (primary), and 46B22 (secondary).

Key words and phrases. Convex point of continuity property, Krein-Milman property, Radon-Nikodym property, topologies on Banach spaces.

This first author research has been supported by MICINN (Spain) Grant CEX2020-001105-M (MCIU, AEI), by MICINN (Spain) Grant PID2021-122126NB-C31 (MCIU, AEI, FEDER, UE), by Junta de Andaluca Grant P20-00255 (FEDER, UE) and by Junta de Andaluca Grant FQM-0185. The second author research has been supported by MIU (Spain) FPU19/04085 Grant, by MICINN (Spain) Grant PID2021-122126NB-C31 (MCIU, AEI, FEDER, UE), by Czech Technical University in Prague Grant SGS21/056/OHK3/1T/13 and by Czech Technical University in Prague Grant CAAS CZ.02.1.01/0.0/0.0/16-019/0000778.

weakly open subsets of C (a π-basis of the weak topology). From the isomorphic point of view, these three families of subsets give rise to three widely studied isomorphic properties in Banach spaces, the Radon-Nikodym property (RNP) for C (existence of slices of arbitrarily small diameter in each bounded and convex subset of C), the convex point of continuity property (CPCP) for C (existence of relatively weakly open subsets of arbitrarily small diameter in each bounded and convex subset of C) and strong regularity (SR) for C (existence of convex combinations of slices or relatively weakly open subsets with arbitrarily small diameter in each bounded and convex subset of C). When C is the unit ball B_X of the Banach space X, it is said that X verifies RNP, CPCP, or is strongly regular, respectively (see [7], [11] for background). Another widely studied property related to the previous ones is the Krein-Milman property (KMP). It is said that the subset C verifies the KMP if each nonempty, closed, bounded and convex subset of C has some extreme point, and when C is the unit ball of the space it is said that the space itself verifies the KMP. It is known that RNP implies CPCP and CPCP implies SR, and that these three properties are different [2]. It is also known that RNP implies KMP, and it is a famous problem still open today whether KMP implies RNP. Research on this problem has obtained some important partial answers (see for example [7], [8], [9], [12], [13], [14], [16]), although in general the most relevant result in this regard has been to be able to relate the properties defined above, obtaining that for a closed, bounded, convex and strongly regular subset C of a Banach space X, C verifies the RNP as long as C verifies the KMP ([17], [15]). Consequently, the problem of the equivalence between the RNP and the KMP is solved for subsets where the RNP and the SR are not equivalent properties, or if desired, this problem will be solved for subsets in which the CPCP and the SR are not equivalent, since RNP implies CPCP. In short, the previous result can be stated by saying that under the KMP, one has that CPCP and SR are equivalent properties. Since the CPCP and the SR are defined in terms of families of subsets relevant to the weak topology, it is then natural to ask if translating the definitions of the CPCP and SR for a topology other than the weak one can obtain a result similar to the previous one. In this note we study this possibility, showing that for a locally convex topology that contains the weak topology, it is also obtained that the new CPCP and SR properties for the above topology are equivalent under the KMP, which generalizes the results known up to now. As a consequence, a new strategy is obtained to obtain the equivalence between the RNP and KMP: starting from a Banach space X failing CPCP for the weak topology (otherwise we already know that RNP and KMP are equivalent for X), it would enough to build a locally convex topology τ on the unit ball B_X of X, in such a way that for the new topology B_X fails CPCP and verifies SR, which implies, applying the previously announced result, that X does not verify KMP. In fact, if one starts with a Banach space whose unit ball has nonempty relatively weakly open subsets with diameter uniformly positive, we prove that it is enough assume SR for open subsets to get the failure of KMP. The above strategy is based on the possibility of building a suitable topology for which the CPCP and SR properties are not equivalent. The more ambitious question would be whether this can be done for any space without the CPCP for the weak topology, but we don't have an answer to this question for now. However, another natural question is whether this can be done in some space where CPCP and SR properties are equivalent for the weak topology, or at least for some space where the unit ball does not satisfy either of the above two properties. The answer in this case is affirmative, since it is shown in this note that there is a locally convex topology on the unit ball B of space c_0, containing the weak topology, such that B is SR for the family of open subsets of this topology, but it does not verify CPCP for the same topology. Of course, such a topology is not the weak topology because B verifies neither CPCP nor is it SR for the weak topology.

From the geometric point of view, there are also widely studied properties as a non-isomorphic counterpart to the properties we have talked about so far, they are the so-called diameter 2 properties, which are also defined through the same families of subsets of the ball unit of a Banach space relevant to the weak topology discussed above. A Banach space verifies the diameter two property for slices (slice-D2P), diameter two property (D2P) or strong diameter two property (SD2P) if each slice, nonempty weak open, or convex combination of slices or nonempty weak open subsets, respectively, of the unit ball of the space has diameter 2. These properties are related to other well-known properties, such as the Daugavet property, octahedrality, and dualize as non-differentiability properties of the dual norm. Furthermore, it is known that the three previous properties are different and it is clear that SD2P implies D2P and D2P implies slice-D2P (see [3], [4], [6]). Precisely to obtain the difference between the D2P and SD2P properties, an equivalent norm was constructed in the space c_0 in such a way that each nonempty weak open subset of the unit ball of c_0 has a diameter 2 for the new norm, while for a such norm it is possible to find convex combinations of slices or nonempty weak open subsets with arbitrarily small diameter [4]. It is natural then, as previously stated for the CPCP and SR properties, to ask if it is possible to obtain a similar result to the above for the unit ball of c_0 using the usual norm of c_0, but changing the definition of D2P and SD2P to another proper topology. The answer is yes, in fact, for the topology built on the unit ball of c_0 we talked about above, it is obtained that the unit ball of c_0 verifies the D2P for this topology and each nonempty open subset for this topology contains convex combinations of relatively open subsets for the same topology with arbitrarily small diameter, and in particular, it does not satisfy SD2P for the same topology. Of course, the diameters are measured here with the usual norm of c_0. Then the differences between the D2P and SD2P properties are even greater than what has been known so far if they are defined for more general topologies, even maintaining the natural norm of the space.

2. Main results

Let X be a Banach space, D is a closed convex and bounded subset of X and τ is a locally convex topology (τ has a countable basis of open subsets) in D containing the relative weak topology on D. For every subset C of D we will denote $\tau|_C$ the induced topology of τ in D. With this notation, we recall the definitions for CPCP and SR. We say that D has the

i) τ-convex point of continuity property (τ-CPCP) if, for every bounded and convex subset C of D and every $\varepsilon > 0$ there is a nonempty $\tau|_C$-open set with diameter less than ε.
ii) τ-strongly regular ($\tau-$SR) if, for every bounded and convex subset C of D and every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_C$-open sets with diameter less than ε.
iii) τ-strongly regular ($\tau-$SR) for open subsets if, for every nonempty and convex $\tau-$open subset O of D and for every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_O$-open subsets with diameter less than ε.
iv) Krein-Milman property (KMP) if every nonempty closed and convex subset of D has some extreme point.

When τ is the weak topology restricted to D, we use CPCP, SR and SR for open subsets, and omit the reference to the topology. Also, when D is the unit ball of the space X and τ is defined on X, we say that X satisfies $\tau-$CPCP, $\tau-$SR or $\tau-$SR for open subsets, respectively.

As we said in the introduction, it is well known that D satisfies CPCP whenever D is SR and satisfies KMP, that is, under KMP, CPCP and SR are equivalent. Our mail goal is to get the same result for another topology other than the weak one. We start with a couple of easy lemmas, omitting the elementary proof of the first one.

Lemma 2.1. Let E be a closed subset of a Banach space X and let (S_n) be a decreasing sequence of subsets of E. If $\operatorname{diam}(S_n) \to 0$ then there is $x \in E$ such that every sequence $(x_n) \subset E$ with $x_n \in \overline{S_n}$ for every $n \in \mathbb{N}$ converges to x.

Lemma 2.2. If D is τ-strongly regular and C is a nonempty convex subset of D, then, for every $\varepsilon > 0$ and every nonempty set $O \in \tau|_C$ there is a a finite family of nonempty sets $O_1, \dots O_n \in \tau|_C$, with $O_i \subset O$, such that

$$\operatorname{diam}\Big(\sum_{i=1}^{n} \frac{1}{n} O_i\Big) < \varepsilon.$$

Proof. Since τ is locally convex and C is convex we get that $\tau|_C$ is locally convex. Hence, there is a nonempty convex set $\widetilde{O} \in \tau|_C$ with $\widetilde{O} \subset O$. Now, as D is τ-strongly regular, we deduce that there are nonempty sets $O_1, \dots, O_n \in \tau|_{\widetilde{O}}$ such that

$$\operatorname{diam}\Big(\sum_{i} \frac{1}{n} O_i\Big) < \varepsilon.$$

We are now done because $O_i \in \tau|_{\widetilde{O}} \subset \tau|_C$ for every $i = 1, \dots, n$. □

Inspired by Schachermayer's work in [17], we introduce in the following definition the subdiameter (SD) in order to estimate from bellow the diameter of convex combinations of weakly open subsets, taking into account the coefficients of the convex combinations.

Definition 2.3. Let C be a convex subset of D and $n \in \mathbb{N}$. Given a n-tuple of nonempty sets $(O_i)_{i=1}^n \subset \tau|_C$ and $\lambda \in S^+_{\ell_1^n}$, we define the sub-diameter of $\sum_i \lambda(i) O_i$ as

$$SD(\lambda, (O_i)) = \inf \left\{ \operatorname{diam}\Big(\sum_i \lambda(i) U_i \Big) \ : \ U_i \in \tau|_{O_i} \setminus \{\emptyset\} \text{ for } 1 \leq i \leq n \right\} \quad \forall \lambda \in S^+_{\ell_1^n}.$$

The next two lemmas use the definition of subdiameter to get a relation between the diameter of a convex combination of weak open subsets and the distance from a finite set to these convex combination.

Lemma 2.4. Let C be a convex subset of D, $\varepsilon > 0$, $n, k \in \mathbb{N}$, $\lambda \in S^+_{\ell_1^n}$, $\{x_1, \ldots, x_k\} \subset X$ and a n-tuple of nonempty sets $(\widetilde{O}_i)_{i=1}^n \subset \tau|_C$. Then, for $1 \leq i \leq n$ there is a nonempty $\tau|_{\widetilde{O}_i}$-open set O_i such that:

$$\operatorname{diam}\Big(\sum_i \lambda(i) O_i \Big) \leq SD(\lambda, (O_i)) + \varepsilon, \tag{2.1}$$

$$d\Big(\{x_1, \ldots, x_k\}, \sum_i \lambda(i) O_i \Big) \geq \frac{SD(\lambda, (O_i))}{2} - \varepsilon. \tag{2.2}$$

Proof. By definition, for each $i = 1, \ldots, n$ there must be $O_i^0 \in \tau|_{\widetilde{O}_i} \setminus \{\emptyset\}$ such that

$$\operatorname{diam}\Big(\sum_i \lambda(i) O_i^0 \Big) \leq SD(\lambda, (\widetilde{O}_i)) + \varepsilon/2 \leq SD(\lambda, (O_i^0)) + \varepsilon/2. \tag{2.3}$$

We will be done if we find $O_i \in \tau|_{O_i^0} \setminus \{\emptyset\}$ for $1 \leq i \leq n$ satisfying (2.2). Indeed, we would have from (2.3) that

$$\operatorname{diam}\Big(\sum_i \lambda(i) O_i \Big) \leq \operatorname{diam}\Big(\sum_i \lambda(i) O_i^0 \Big) \leq SD(\lambda, (O_i^0)) + \varepsilon \leq SD(\lambda, (O_i)) + \varepsilon,$$

and thus (2.1) would also hold.

Let us then prove by induction on $k \in \mathbb{N}$ that there is for every $i \leq n$ a nonempty set $O_i \in \tau|_{O_i^0}$ satisfying (2.2).

First step of the induction ($k = 1$): Clearly, there are $a, b \in \sum_i \lambda(i) O_i^0$ and $x^* \in S_{X^*}$ such that

$$x^*(a - b) \geq SD(\lambda, (O_i^0)) - \varepsilon/2.$$

We may assume without loss of generality that the last inequality holds together with $x^*(x_1) \leq x^*\left(\frac{a+b}{2}\right)$ (changing sign of x^* and swapping the roles of a and b if necessary). We define now

$$O_i^1 = O_i^0 \cap \{x \in D \ : \ x^*(x) > \sup x^*(O_i^0) - \varepsilon/2\}.$$

Clearly, $O_i^1 \in \tau|_{O_i^0} \setminus \{\emptyset\}$ for every i. Let us prove that $O_1^1, \ldots O_n^1$ satisfy (2.2) for $k = 1$:

If $y \in \sum_i \lambda(i)O_i^1$ then $y = \sum_i \lambda(i)y_i$ where $y_i \in O_i^1$. Therefore,

$$x^*(y) = \sum_i \lambda(i)x^*(y_i) > \sum_i \lambda(i) \sup x^*(O_i^0) - \varepsilon/2 \geq \sup x^*\Big(\sum_i \lambda(i)O_i^0\Big) - \varepsilon/2$$
$$\geq x^*(a) - \varepsilon/2.$$

Hence, for every $y \in \sum_i \lambda(i)O_i^1$,

$$\begin{aligned}\|y - x_1\| \geq x^*(y - x_1) \geq x^*(a) - x^*\Big(\frac{a+b}{2}\Big) - \varepsilon/2 =& x^*\Big(\frac{a-b}{2}\Big) - \varepsilon/2 \\ \geq& \frac{SD(\lambda,(O_i^0))}{2} - \frac{3\varepsilon}{4}.\end{aligned} \tag{2.4}$$

Clearly, from (2.3) we have that

$$SD(\lambda,(O_i^1)) \leq \operatorname{diam}\Big(\sum_i \lambda(i)O_i^1\Big) \leq \operatorname{diam}\Big(\sum_i \lambda(i)O_i^0\Big) \leq SD(\lambda,(O_i^0)) + \varepsilon/2.$$

Therefore, putting together this last inequality and (2.4) we finish this step.

Inductive step: Let us assume that for some $m \in \mathbb{N}$ there are nonempty sets $O_1^m, \ldots, O_n^m$ with $O_i^m \in \tau|_{O_i^0}$ for every $i \leq n$ satisfying equation (2.2) for $k = m$. We may then construct $O_1^{m+1}, \ldots O_n^{m+1} \in \tau|_C \setminus \{\emptyset\}$ with $O_i^{m+1} \subset O_i^0$ and satisfying equation (2.2) for $k = m+1$. The construction is analogous to the one given for the case $k = 1$. We just take $x^* \in S_{X^*}$ and $a, b \in \sum_i \lambda(i)O_i^m$ such that $x^*(a-b) \geq SD(\lambda,(O_i)) - \varepsilon$ and $x^*(x_{m+1}) \leq x^*\big(\frac{a+b}{2}\big)$. Then, the sets given by

$$O_i^{m+1} = O_i^m \cap \{x \in D \ : \ x^*(x) > \sup x^*(O_i^m) - \varepsilon/2\}$$

make the trick. Hence, (2.2) holds for $O_i = O_i^k$ for every $i \leq n$. □

Lemma 2.5. Let C be a convex subset of D, $\varepsilon > 0$, $n, k \in \mathbb{N}$ and $\{x_1, \ldots, x_k\} \subset X$. If $(\widetilde{O}_i)_{i=1}^n \subset \tau|_C$, then there is another n-tuple of nonempty $\tau|_C$-open sets $(O_i)_{i=1}^n$ with $O_i \subset \widetilde{O}_i$ such that

$$\operatorname{diam}\Big(\sum_{i=1}^n \lambda(i)O_i\Big) \leq 2d\Big(\{x_1, \ldots, x_k\}, \sum_{i=1}^n \lambda(i)O_i\Big) + \varepsilon \quad \forall \lambda \in S^+_{\ell_1^n}.$$

Proof. Let us consider $(\lambda_i)_{i=1}^N$ an $\frac{\varepsilon}{4n}$-net of $S^+_{\ell_1^n}$. It is now straightforward to construct, by recursively using Lemma 2.4, n-tuples $(O_i^j)_{i=1}^n$ for $j \leq N$ satisfying

(1) $\emptyset \neq O_i^1 \subset \widetilde{O}_i$ and $\emptyset \neq O_i^{j+1} \subset O_i^j$ for every $j < N$ and $i \leq n$.
(2) $\operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i^j\Big) \leq 2d\Big(\{x_1, \ldots, x_k\}, \sum_{i=1}^n \lambda_j(i)O_i^j\Big) + \varepsilon/4$ for $j \leq N$.

We take $O_i = O_i^N$ for $i = 1, \dots, n$ so that for every $j \leq N$,

$$\begin{aligned}\operatorname{diam}\left(\sum_{i=1}^n \lambda_j(i)O_i\right) \leq& \operatorname{diam}\left(\sum_{i=1}^n \lambda_j(i)O_i^j\right)\\ \leq& 2d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i^j\right) + \varepsilon/4 \\ \leq& 2d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\right) + \varepsilon/4.\end{aligned} \tag{2.5}$$

If we take now $\lambda \in S_{\ell_1}^+$ arbitrary, then there must be $j \leq N$ such that $\|\lambda_j - \lambda\|_{\ell_1^n} \leq \frac{\varepsilon}{4n}$. It is straightforward to check that

$$\operatorname{diam}\left(\sum_{i=1}^n \lambda(i)O_i\right) \leq \operatorname{diam}\left(\sum_{i=1}^n \lambda_j(i)O_i\right) + \varepsilon/2,$$

$$d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda(i)O_i\right) \geq d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\right) - \varepsilon/4.$$

We are therefore done since

$$\begin{aligned}\operatorname{diam}\left(\sum_{i=1}^n \lambda(i)O_i\right) \leq& \operatorname{diam}\left(\sum_{i=1}^n \lambda_j(i)O_i\right) + \varepsilon/2 \\ \overset{(2.5)}{\leq}& 2d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\right) + 3\varepsilon/4 \\ \leq& 2d\left(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda(i)O_i\right) + \varepsilon.\end{aligned}$$

□

Now, we are ready to get our main result

Theorem 2.6. Let X be a separable Banach space and D a τ-strongly regular closed, bounded and convex subset of X without the τ-CPCP, where τ is a locally convex topology on D containing the weak topology on D. Then, there is a closed convex subset of D without extreme points, that is D fails KMP.

Proof. Let C be a bounded and convex subset of D and $\delta > 0$ such that every $\tau|_C$-open set has diameter greater than $\delta > 0$. We assume without loss of generality that $C \subset B_X$. Our aim is to find out a closed bounded and convex subset of D without extreme points. We will split the proof into three different steps. The first two steps will deal with the construction of a particular set $E \subset D$ whereas in the final step we will prove that E is the sought set without extreme points.

Since X is separable we may consider a sequence (x_n) dense in X. We are going to inductively define a sequence of indexes sets (Ω_n) satisfying that $\Omega_1 = \{1, \dots, m_1\}$ and $\Omega_{n+1} = \Omega_n \times \{1, \dots, m_{n+1}\}$ for every $n \in \mathbb{N}$ where $(m_n) \subset \mathbb{N}$.

Step 1. We are going to define inductively the sequence (Ω_n) altogether with a family of nonempty $\tau|_C$-open sets indexed by $\Omega = \bigcup_n \Omega_n$, $(O_\omega)_{\omega\in\Omega}$ satisfying the following properties.

(1) $O_{(\omega,i)} \subset O_\omega$ for every $\omega \in \Omega_n$ and $i \in \{1,\dots,m_{n+1}\}$ with $n \in \mathbb{N}$.
(2) For every $n \in \mathbb{N}$ and $\lambda \in S^+_{\ell_1(\Omega_n)}$,

$$\operatorname{diam}\Big(\sum_{\omega\in\Omega_n} \lambda(\omega) O_\omega\Big) \le 2d\Big(\{x_1,\dots,x_n\}, \sum_{\omega\in\Omega_n} \lambda(\omega) O_\omega\Big) + 2^{-n}.$$

(3) For every $n \in \mathbb{N}$ and $\omega \in \Omega_n$,

$$\operatorname{diam}\Big(\sum_{i=1}^{m_{n+1}} \frac{1}{m_{n+1}} O_{(\omega,i)}\Big) \le 2^{-n}.$$

Let us proceed with the inductive construction. For n=1 we consider $m_1 = 1$. We use Lemma 2.5 with $n = k = 1$, $\widetilde{O}_1 = C$, $\varepsilon = 1/2$ and $\lambda \in S^+_{\ell_1(\{1\})} = \{1\}$. Thus, there is a nonempty $\tau|_C$-open set O_1 satisfying (2). Now, provided that for some $n \in \mathbb{N}$ we have $\Omega_n = \{1,\dots,m_1\} \times \cdots \times \{1,\dots,m_n\}$ and O_ω for every $\omega \in \Omega_n$, let us define m_{n+1} and $O_{(\omega,i)}$ for every $(\omega,i) \in \Omega_n \times \{1,\dots,m_{n+1}\} = \Omega_{n+1}$ satisfying properties (1), (2) and (3). First, by Lemma 2.2 there is $m_{n+1} \in \mathbb{N}$ such that for every $\omega \in \Omega_n$ there are m_{n+1} nonempty $\tau|_C$-open sets $(\widetilde{O}_{(\omega,i)})_{i=1}^{m_{n+1}}$ such that

$$\operatorname{diam}\Big(\sum_{i=1}^{m_{n+1}} \frac{1}{m_{n+1}} \widetilde{O}_{(\omega,i)}\Big) \le 2^{-n} \quad \text{and} \quad \widetilde{O}_{(\omega,i)} \subset O_\omega \quad \forall i = 1,\dots,m_{n+1}. \tag{2.6}$$

Now, by Lemma 2.5, for every $\omega \in \Omega_{n+1}$ there is a nonempty $\tau|_C$-open set $O_\omega \subset \widetilde{O}_\omega$ satisfying property (2). Clearly, by (2.6), $(O_\omega)_{\omega\in\Omega_{n+1}}$ also satisfies property (3). We may assume that O_ω is convex for every $\omega \in \Omega$ without loss of generality because τ is locally convex.

Step 2. We are finally ready to construct the closed convex subset of D without extreme points. Let us first pick for every $\omega \in \Omega$ an element $y_\omega \in O_\omega$. Now, if $\omega \in \Omega_n$ for some $n \in \mathbb{N}$ we define for every $k \ge n+3$ the element

$$z^k_\omega = \sum_{\widetilde{\omega}\in\Omega^\omega_k} \frac{1}{m_{n+1}\dots m_k} y_{\widetilde{\omega}},$$

Where $\Omega^\omega_k = \{\widetilde{\omega} \in \Omega_k \ : \ \widetilde{\omega}(i) = \omega(i) \ \ \forall i \le n\}$. Notice that $\#\Omega^\omega_k = m_{n+1}\cdots m_k$. We claim that $(z^k_\omega)_k$ is a Cauchy sequence for every $\omega \in \Omega$. From property (1) we deduce that $O_{\widetilde{\omega}} \subset O_\omega$ for every $\widetilde{\omega} \in \Omega^\omega_k$. Now, for every $q \in \{n+2,\dots,k-1\}$ we have $\Omega_k^\omega = \bigcup_{\widetilde{\widetilde{\omega}}\in\Omega_q^\omega} \Omega_k^{\widetilde{\widetilde{\omega}}}$ and therefore

$$\begin{aligned}\sum_{\widetilde{\omega}\in\Omega_k^\omega} \frac{1}{m_{n+1}\dots m_k} O_{\widetilde{\omega}} &= \sum_{\widetilde{\widetilde{\omega}}\in\Omega_q^\omega} \sum_{\widetilde{\omega}\in\Omega_k^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{n+1}\dots m_q m_{q+1}\dots m_k} O_{\widetilde{\omega}} \\ &= \sum_{\widetilde{\widetilde{\omega}}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} \sum_{\widetilde{\omega}\in\Omega_k^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{q+1}\dots m_k} O_{\widetilde{\omega}}.\end{aligned} \tag{2.7}$$

Hence, it follows that

$$\begin{aligned}\sum_{\widetilde{\omega}\in\Omega_k^\omega} \frac{1}{m_{n+1}\dots m_k} O_{\widetilde{\omega}} &\subset \sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \\ &= \sum_{\widetilde{\omega}\in\Omega_{q-1}^\omega} \frac{1}{m_{n+1}\dots m_{q-1}} \sum_{i=1}^{m_q} \frac{1}{m_q} O_{(\widetilde{\omega},i)}.\end{aligned} \tag{2.8}$$

Equation (2.7) yields that if $n\in\mathbb{N}$ and $\omega\in\Omega_n$,

$$z_\omega^k \in \sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \qquad \forall q,k\in\mathbb{N}\ ,\ n+2\le q\le k-1. \tag{2.9}$$

On the other hand, if $q\ge n+2$,

$$\begin{aligned}&\operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}}\Big) \\ &\overset{(2.8)}{\le} \sum_{\widetilde{\omega}\in\Omega_{q-1}^\omega} \frac{1}{m_{n+1}\dots m_{q-1}} \operatorname{diam}\Big(\sum_{i=1}^{m_q} \frac{1}{m_q} O_{(\widetilde{\omega},i)}\Big) \overset{(3)}{\le} 2^{-q+1}.\end{aligned} \tag{2.10}$$

Therefore, for every $k,k'\ge q+1$,

$$\|z_\omega^k - z_\omega^{k'}\| \le \operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}}\Big) \overset{(2.10)}{\le} 2^{-q+1}.$$

This proves that $(z_\omega^k)_k$ is a Cauchy sequence for every $\omega\in\Omega$. We are then allowed to consider its limit $z_\omega\in D$. Our sough set finally is

$$E = \overline{\operatorname{co}}\big((z_\omega)_{\omega\in\Omega}\big).$$

Step 3. We just need to show that E does not have any extreme point.

Claim 1. We claim that for every $a\in E$ there is a sequence $(a_k)_k$ where $a_k\in \operatorname{co}\big((z_\omega)_{\omega\in\Omega_k}\big)$ for every $k\in\mathbb{N}$ such that $a_k \xrightarrow{\|\cdot\|} a$.

It is enough to show that if $\omega \in \Omega_n$ for some $n \in \mathbb{N}$ then for every $q > n$ it holds that

$$z_\omega = \sum_{\widetilde{\widetilde{\omega}} \in \Omega_k^\omega} \frac{1}{m_{n+1} \cdots m_q} z_{\widetilde{\widetilde{\omega}}} \in \operatorname{co}\big((z_{\widetilde{\omega}})_{\widetilde{\omega} \in \Omega_q}\big). \tag{2.11}$$

Indeed, if (2.11) holds, then for every $x \in \operatorname{co}\big((z_\omega)_{\omega\in\Omega}\big)$ there is $N \in \mathbb{N}$ such that for every $q \geq N$, $x \in \operatorname{co}\big((z_{\widetilde{\omega}})_{\widetilde{\omega}\in\Omega_q}\big)$ and the claim is immediately deduced. Let us show the proof of (2.11). Let $\omega \in \Omega_n$ and $q > n$ be fixed. From (2.9) we know that

$$z^k_{\widetilde{\widetilde{\omega}}} \in \sum_{\widetilde{\omega} \in \Omega_{k-1}^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{q+1} \dots m_{k-1}} O_{\widetilde{\omega}} \qquad \forall \widetilde{\widetilde{\omega}} \in \Omega_q, \ \ \forall k \geq q+3.$$

Therefore, for every $k \geq q+3$ it follows that

$$\begin{aligned}\sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z^k_{\widetilde{\widetilde{\omega}}} \in \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1}} \dots m_q \sum_{\widetilde{\omega} \in \Omega_{k-1}^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{q+1} \dots m_{k-1}} O_{\widetilde{\omega}} \\ \overset{(2.7)}{=} \sum_{\widetilde{\omega} \in \Omega_{k-1}^\omega} \frac{1}{m_{n+1} \dots m_{k-1}} O_{\widetilde{\omega}} =: S_k.\end{aligned} \tag{2.12}$$

Clearly, from (2.9) we know that $z^k_\omega \in S_k$ for every $k \geq q+3$. From (2.8) we get that $S_{k+1} \subset S_k$ for every $k \geq q+3$ and from (2.9) we know that $\operatorname{diam} S_k \to 0$. Hence, by Lemma 2.1

$$z_\omega = \lim_k z^k_\omega = \lim_k \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z^k_{\widetilde{\widetilde{\omega}}} = \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z_{\widetilde{\widetilde{\omega}}},$$

which finally proves the claim.

Let us take $a \in E$ and a sequence (a_k) converging to a as in the claim. Then, for every $k \in \mathbb{N}$ there is $\lambda_k \in S^+_{\ell_1(\Omega_k)}$ such that $a_k = \sum_{\omega \in \Omega_k} \lambda_k(\omega) z_\omega$ for every $k \in \mathbb{N}$. Let us prove that a is not an extreme point in E. First, we extend λ_k to $\bigcup_{n \leq k} \Omega_n$ as follows. If $\omega \in \Omega_n$ with $n \leq k$ we define

$$\lambda_k(\omega) = \sum_{\widetilde{\omega} \in \Omega_k^\omega} \lambda_k(\widetilde{\omega}).$$

We may assume (taking subsequence if necessary) that for every $n \in \mathbb{N}$, the sequence $(\lambda_k|_{\Omega_n})_k$ is pointwise convergent. Therefore, we define $\lambda : \Omega \to [0,1]$ as

$$\lambda(\omega) = \lim_k \lambda_k(\omega) \qquad (\omega \in \Omega).$$

Clearly, $\lambda|_{\Omega_n} \in S^+_{\ell_1(\Omega_n)}$ for every $n \in \mathbb{N}$. Moreover, if $\omega \in \Omega_n$ then

$$\sum_{\widetilde{\omega} \in \Omega_m^\omega} \lambda(\widetilde{\omega}) = \lim_k \sum_{\widetilde{\omega} \in \Omega_m^\omega} \lambda_k(\widetilde{\omega}) = \lim_k \lambda_k(\omega) = \lambda(\omega) \quad \forall m \geq n. \tag{2.13}$$

Let us now define for every $m,n\in\mathbb{N}$ with $m\geq n$ and every $\omega\in\Omega_n$ the element

$$a^m(\omega)=\begin{cases}\sum\limits_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}z_{\widetilde{\omega}} & \text{if } \lambda(\omega)>0,\\ z_\omega & \text{otherwise.}\end{cases}$$

Claim 2. For every $\omega\in\Omega$ the sequence $(a^m(\omega))_m$ is convergent, moreover, if we take $a(\omega)=\lim\limits_m a^m(\omega)$ then $a=\sum\limits_{\omega\in\Omega_n}\lambda(\omega)a(\omega)$ for every $n\in\mathbb{N}$.

If we fix $n\in\mathbb{N}$, we have that $a_k\in\sum\limits_{\omega\in\Omega_n}\lambda_k(\omega)O_\omega$ for every $k\in\mathbb{N}$ so that $a\in\overline{\sum\limits_{\omega\in\Omega_n}\lambda(\omega)O_\omega}$. By property (2) we deduce that

$$\begin{aligned}\operatorname{diam}\Big(\sum_{\omega\in\Omega_n}\lambda(\omega)O_\omega\Big)&\leq 2d\Big(\{x_1,\dots,x_n\},\sum_{\omega\in\Omega_n}\lambda(\omega)O_\omega\Big)+2^{-n}\\&\leq 2d(\{x_1,\dots,x_n\},a)+2^{-n}\xrightarrow{n\to\infty}0.\end{aligned}\tag{2.14}$$

Also, if $m>n$ and $\omega\in\Omega_n$ with $\lambda(\omega)>0$ then

$$\sum_{\widetilde{\omega}\in\Omega_m}\lambda(\widetilde{\omega})O_{\widetilde{\omega}}=\lambda(\omega)\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}+\sum_{\widetilde{\widetilde{\omega}}\in\Omega_n\setminus\{\omega\}}\lambda(\widetilde{\widetilde{\omega}})\sum_{\widetilde{\omega}\in\Omega_m^{\widetilde{\widetilde{\omega}}}}\frac{\lambda(\widetilde{\omega})}{\lambda(\widetilde{\widetilde{\omega}})}O_{\widetilde{\omega}},$$

so that, by (2.14) we have

$$\operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_m}\lambda(\widetilde{\omega})O_{\widetilde{\omega}}\Big)\geq\lambda(\omega)\operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}\Big).$$

Therefore,

$$\lim_m\operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}\Big)=0\quad\forall\omega\in\Omega_n\ \text{ with }\lambda(\omega)>0.\tag{2.15}$$

Let us prove that the sequence of subsets $\Big(\sum\limits_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}\Big)_m$ is decreasing for every $\omega\in\Omega$ in order to use Lemma 2.1. Clearly, by (2.13), property (1) and the fact that $O_{\widetilde{\omega}}$ is convex for every $\widetilde{\omega}\in\Omega$, we have that

$$\sum_{\widetilde{\widetilde{\omega}}\in\Omega_{m+1}^{\widetilde{\omega}}}\frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\widetilde{\omega})}O_{\widetilde{\widetilde{\omega}}}\subset O_{\widetilde{\omega}}\qquad\forall\widetilde{\omega}\in\Omega_m,\quad\forall m\in\mathbb{N}.$$

Then, if $\omega\in\Omega_n$ for some $n\in\mathbb{N}$ and $m\geq n$,

$$\sum_{\widetilde{\omega}\in\Omega_{m+1}^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}=\sum_{\widetilde{\omega}\in\Omega_m^\omega}\sum_{\widetilde{\widetilde{\omega}}\in\Omega_{m+1}^{\widetilde{\omega}}}\frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\omega)}O_{\widetilde{\widetilde{\omega}}}=\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}\sum_{\widetilde{\widetilde{\omega}}\in\Omega_{m+1}^{\widetilde{\omega}}}\frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\widetilde{\omega})}O_{\widetilde{\widetilde{\omega}}}\subset\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}},$$

which means that $\left(\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}\right)_m$ is decreasing for every $\omega\in\Omega$, as we wanted. In particular, taking $\omega=1\in\Omega_1$, this shows that the sequence of sets $\left(\sum_{\omega\in\Omega_m}\lambda(\omega)O_\omega\right)_m$ is a decreasing sequence.

By Lemma 2.1 the sequence $(a^m(\omega))_m$ is convergent in E since $a^m(\omega)\in\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}O_{\widetilde{\omega}}$. Let us denote $a(\omega)$ its limit. We just need to prove that $a=\sum_{\omega\in\Omega_n}\lambda(\omega)a(\omega)$ for every $n\in\mathbb{N}$. If we fix $n\in\mathbb{N}$ and denote $a^m=\sum_{\omega\in\Omega_n}\lambda(\omega)a^m(\omega)$ for every $m\geq n$, it is enough to show that $a^m\to a$. By (2.15), we know that $\lim_m\operatorname{diam}\left(\sum_{\omega\in\Omega_m}\lambda(\omega)O_\omega\right)=0$. For every $m\geq n$, $a\in\overline{\sum_{\omega\in\Omega_m}\lambda(\omega)O_\omega}$ and

$$a^m=\sum_{\substack{\omega\in\Omega_n\\ \lambda(\omega)>0}}\lambda(\omega)a^m(\omega)=\sum_{\substack{\omega\in\Omega_n\\ \lambda(\omega)>0}}\lambda(\omega)\sum_{\widetilde{\omega}\in\Omega_m^\omega}\frac{\lambda(\widetilde{\omega})}{\lambda(\omega)}z_{\widetilde{\omega}}=\sum_{\omega\in\Omega_m}\lambda(\omega)z_\omega\in\sum_{\omega\in\Omega_m}\lambda(\omega)O_\omega.$$

Therefore, by Lemma 2.1, $a^m\to a$ and Claim 2 is proven.

Finally, we wanted to show that E has no extreme point, that is, we want to show that our previously picked element $a\in E$ is not an extreme point in E. Notice that by Claim 2, it is enough to show that there is $\omega\in\Omega$ with $\lambda(\omega)>0$ such that $a(\omega)\neq a$.

Let us argue by contradiction. If we deny the above statement then for every $\omega\in\Omega$ with $\lambda(\omega)>0$ we have $a(\omega)=a$. Therefore, $a\in O_\omega$ for every $\omega\in\Omega$ with $\lambda(\omega)>0$. From (2.13), we know that there must be a sequence $(\omega_n)\subset\Omega$ such that for every $n\in\mathbb{N}$, $\omega_{n+1}\in\Omega_{n+1}^{\omega_n}$ and $\lambda(\omega_n)>0$. Hence, $a\in O_{\omega_n}$ for every $n\in\mathbb{N}$. By property (2) we obtain that $d(a,X)\geq\delta/2$ which is clearly false.

□

We want to remark that while part of the techniques used in the proof of the above theorem appear in [17], we don't use the essential approach in [17] about the positive face of the unit sphere of L_1 via the construction of an operator from L_1 to the space X, in order to get the subset without extreme points.

Observe that the hypothesis on D (non-$\tau-$CPCP) in the above theorem is used first to get a subset C so that every nonempty weakly open subset of C has diameter at least δ, for some positive δ. From this moment, the τ-SR is used to find convex combinations of relatively $\tau-$open subsets with diameter arbitrarily small inside every relative $\tau-$open subset of C. Then, starting with D satisfying that every nonempty $\tau-$open subset of D has diameter at least δ (say D fails $\tau-$CPCP $\delta-$uniformly), we have with the same proof that D fails KMP whenever D is $\tau-$SR for open subsets.

Corollary 2.7. Let X be a separable Banach space and D a closed, bounded and convex subsets of X failing $\tau-$CPCP $\delta-$uniformly for some positive δ, where τ is a locally convex topology on D containing the weak topology on D. If D is $\tau-$SR for open subsets then D fails KMP.

Observe that SR for open subsets is strictly weaker than SR, since every Banach space with a LUR norm, for example, has a unit ball SR for open subsets while not a such ball is necessarily SR, c_0 is an example, that is, SR for open subsets depends on the used equivalent norm and this is not the case for SR. On the other hand, it is known that if a closed, bounded and convex subset C of a Banach space X fails CPCP, then there is an equivalent norm on X so that the new unit ball fails CPCP $\delta-$uniformly [5]. As every separable Banach space can be equivalently renormed with a LUR norm [10], then every separable Banach space can be equivalently renormed so that its new unit ball is SR for open subsets.

In order to get a nonseparable analogous result of the above theorem, recall that a topological space X is said countable tightness if for every $A \subset X$ and for every $x \in \overline{A}$ there is a countable subset $B \subset A$ such that $x \in \overline{B}$. The key to get the nonseparable result is to have the separable determination of $\tau-$CPCP, which is proved in the next

Lemma 2.8. Let X be a Banach space and D a closed, bounded and convex subset of X without $\tau-$CPCP, where τ is a locally convex and countable tightness topology on D containing the weak topology on D. Then there is a separable, closed, bounded and convex subset of D without $\tau-$CPCP.

Proof. If D fails $\tau-$CPCP there is a convex subset A of D and a $\delta > 0$ such that every relatively $\tau-$open subset of A has diameter at least δ, so $a \in \overline{A \setminus B(a, \frac{\delta}{2})}^{\tau}$ for every $a \in A$. As τ is countable tightness, for every $a \in A$ there is D_a a countable subset of $A \setminus B(a, \frac{\delta}{2})$ such that $a \in \overline{D_a}^{\tau}$. For $a_0 \in A$ define $B_1 = \{a_0\} \cup co(D_{a_0})$ and $B_{n+1} = B_n \cup co(\cup_{a \in B_n} D_a)$ for every n. Now put $B = \cup_n B_n$. Then $\overline{B}^{\tau}$ is a separable, closed, bounded and convex subset of D since A it is and $A \subset D$. Observe that B is convex since $B_n \subset co(\cup_{a \in B_n} D_a)$ and $B_{n+1} = B_n \cup co(\cup_{a \in B_n} D_a)$ for every n. Now every relatively $\tau-$open subset of B has diameter, at least, $\frac{\delta}{2}$. Indeed, if U is a relatively $\tau-$open subset of B then there is $x \in U \cap B_n$ for some n and $x \in \overline{D_x}^{\tau}$. Then there is $y \in D_x \cap U$ and $y \in B \cap U$. As $\|xy\| > \frac{\delta}{2}$, one has $diam(U) \geqslant \frac{\delta}{2}$ and we are done, since B is dense in $\overline{B}^{\tau}$. □

Finally, using the above lemma joint to theorem 2.6 we get the following

Corollary 2.9. Let X be a Banach space and D a $\tau-$strongly regular closed, bounded and convex subset of X without $\tau-$CPCP, where τ is a locally convex and countable tightness topology on D containing the weak topology on D. Then, there is a closed and convex subset of D without extreme points, that is D fails KMP.

3. Non-CPCP and SR locally convex topology on the unit ball of c_0

As the problem about the equivalence between RNP and KMP, as we said in the introduction, is solved when CPCP and SR are not equivalent, the above theorem suggests exploring a new way to proceed: starting with a Banach space X failing CPCP, is it possible to construct a locally convex topology τ on the unit ball of X, B_X, containing the weak topology so that B_X still fails $\tau-$CPCP and satisfying $\tau-$SR? We don't know the answer to this question, but a first step could be answer the above question for the unit ball B of c_0, since B fails CPCP, in fact every nonempty weakly open subset of B has diameter 2, that is B fails CPCP 2$-$uniformly and B also fails SR. In this case, we have a positive answer and for that purpose we state and prove the following lemma.

Lemma 3.1. Let $k \in \mathbb{N}$, $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ countable family and $A_0 \in \mathcal{A}$. There is a partition $\{A_0^1, \dots A_0^k\}$ of A_0 satisfying that for every $F \in \mathcal{A}^{<\omega}$ and $1 \leq i \leq k$, if $A_0 \cap \bigcap_{A\in F} A$ is infinite, so is $A_0^i \cap \bigcap_{A \in F} A$.

Proof. Since $\mathcal{A}$ is countable, so is $\mathcal{F} = \{F \in \mathcal{A}^{<\omega} \ : \ A_0 \cap \bigcap_{A\in F} A \text{ is infinite}\}$. Hence, we may label its elements as $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ where $F_1 = \{A_0\}$ and we also label $A_0 = \{a_j\}_{j\in\mathbb{N}}$ (the case when A_0 is not infinite is trivial). We now construct inductively a family $\{A_n^i\}_{\substack{i\leq k\\ n\in\mathbb{N}}}$ of finite subsets of A_0 with the following properties for every $p, q \in \mathbb{N}$,

(1) For $i, j \leq k$, $A_p^i \cap A_q^j = \emptyset$ if either $i \neq j$ or $p \neq q$.
(2) $\#\big(A_p^i \cap \bigcap_{A\in F_m} A\big) \geq 1$ for every $m \leq p$ and $i \leq k$.
(3)
$$A_0 \setminus \Big(\bigcup_{\substack{m\leq p\\ i\leq k}} A_m^i\Big) \subset \{a_j\}_{j>kp}.$$

We take $A_1^i = \{a_i\}$ for $i = 1, \dots, k$ which satisfy the above properties for $p = q = 1$. Assume that A_m^i has been defined for $i \leq k$ and $m \leq n-1$ satisfying the above properties for $p, q \leq n-1$. We aim to define A_n^i for $i \leq k$ such that $\{A_m^i\}_{\substack{i\leq k\\ m\leq n}}$ satisfy properties (1) to (3) for $p, q \leq n$. For that purpose, if $A_0 \setminus \Big(\bigcup_{\substack{i\leq k\\ m\leq n-1}} A_m^i\Big) = \{a_{\sigma(j)}\}_{j\in\mathbb{N}}$ we take $A_{n,1}^i = \{a_{\sigma(i)}\}$ and inductively $A_{n,m}^i = A_{n,m-1}^i \cup \{a_{\tau(i)}\}$ where

$$\Big(A_0 \cap \bigcap_{A\in F_m} A\Big) \setminus \Big(\bigcup_{\substack{i\leq k\\ r\leq n-1}} A_r^i \cup \bigcup_{i\leq k} A_{n,m-1}^i\Big) = \{a_{\tau(j)}\}_{j\in\mathbb{N}}. \tag{3.1}$$

We rename $A_{n,n}^i = A_n^i$ for $i = 1, \dots, k$. Let us prove that we reached our goal, that is, $\{A_m^i\}_{\substack{i\leq k\\ m\leq n}}$ satisfy properties (1) to (3) for $p, q \leq n$.

Property (1): If $p, q \leq n-1$ then by the induction hypothesis it is clear that (1) holds true. Otherwise, we may assume that $p = n$. If $q < p = n$ and $i, j \leq k$ then by construction, $A_{n,1}^i \cap A_q^j = \emptyset$ since $a_{\sigma(i)} \notin A_q^j$. If we assume that $A_{n,m-1}^i \cap A_q^j = \emptyset$, we need to show that $A_{n,m}^i \cap A_q^j = \emptyset$. Indeed, $A_{n,m}^i \cap A_q^j = \big(A_{n,m-1}^i \cup \{a_{\tau(i)}\}\big) \cap A_q^j = \{a_{\tau(i)}\} \cap A_q^j = \emptyset$ since again by construction $a_{\tau(i)} \notin A_q^j$. This shows that $A_n^i \cap A_q^j = A_{n,n}^i \cap A_q^j = \emptyset$. Finally, if $q = n$ and $i \neq j \leq k$ then $a_{\sigma(i)} \neq a_{\sigma(j)}$ and hence $A_{n,1}^i \cap A_{n,1}^j = \emptyset$. We assume that $A_{n,m-1}^i \cap A_{n,m-1}^j = \emptyset$ and show that

$A^i_{n,m} \cap A^j_{n,m} = \emptyset$. Since $a_{\tau(i)} \neq a_{\tau(j)}$ and $a_{\tau(i)}, a_{\tau(j)} \notin A^i_{n,m-1} \cup A^j_{n,m-1}$ it is clear that $A^i_{n,m} \cap A^j_{n,m} = \big(A^i_{n,m-1} \cup \{a_{\tau(i)}\}\big) \cap \big(A^j_{n,m-1} \cup \{a_{\tau(j)}\}\big) = \emptyset$. Hence, $A^i_n \cap A^j_n = A^i_{n,n} \cap A^j_{n,n} = \emptyset$.

Property (2): If $p < n$ then by the induction hypothesis $\#\big(A^i_p \cap \bigcap_{A\in F_m} A\big) \geq 1$ for every $m \leq p$ and $i \leq k$. Let then $p = n$ and $i \leq k$. In this case $a_{\tau(i)} \in A^i_{n,m} \subset A^i_n$. By (3.1), it is clear that $a_{\tau(i)} \in \bigcap_{A\in F_m} A$ and therefore $a_{\tau(i)} \in A^i_n \cap \bigcap_{A\in F_m} A$.

Property (3): By the induction hypothesis we have that

$$\{a_{\sigma(j)}\}_{j\in\mathbb{N}} = A_0 \setminus \Big(\bigcup_{\substack{m\leq n-1\\ i\leq k}} A^i_m\Big) \subset \{a_j\}_{j>k(n-1)}.$$

This means that $\sigma(1) \geq k(n-1)+1$ and hence $\sigma(j) \geq \sigma(1)+j-1 \geq k(n-1)+j$. Taking into account that last inequality, we have that

$$\begin{aligned} A_0 \setminus \Big(\bigcup_{\substack{m\leq n\\ i\leq k}} A^i_m\Big) =& \Big(A_0 \setminus \Big(\bigcup_{\substack{m\leq n-1\\ i\leq k}} A^i_m\Big)\Big) \setminus \Big(\bigcup_{i\leq k} A^i_n\Big) \\ =& \{a_{\sigma(j)}\}_{j\geq 1} \setminus \Big(\bigcup_{i\leq k} A^i_{n,n}\Big) \subset \{a_{\sigma(j)}\}_{j\geq 1} \setminus \Big(\bigcup_{i\leq k} A^i_{n,1}\Big) \\ =& \{a_{\sigma(j)}\}_{j>k} \subset \{a_{k(n-1)+j}\}_{j>k} = \{a_j\}_{j>kn}. \end{aligned}$$

This finishes the induction.

Finally, we define $A^i_0 = \bigcup_{n\in\mathbb{N}} A^i_n$ for every $i \leq k$. It only remains to prove that $A^1_0, \ldots, A^k_0$ satisfy the statement of the lemma. From (1) it is clear that $A^i_0 \cap A^j_0 = \emptyset$ for $i \neq j$. It is also immediate from (3) that $\bigcup_{i=1}^k A^i_0 = A_0$ so that $\{A^1_0, \ldots, A^n_0\}$ forms a partition of A_0. Finally, if $F \in \mathcal{F}$ then combining properties (1) and (2) we have that

$$\#\big(A^i_0 \cap \bigcap_{A\in F} A\big) \overset{(1)}{=} \sum_{n=1}^{\infty} \#\big(A^i_n \cap \bigcap_{A\in F} A\big) \overset{(2)}{=} +\infty.$$

□

Now, we pass to show the existence of a appropriate topology on B_{c_0}. From now on we say that an interval $I \subset [-1,1]$ is a proper open subinterval whenever it is open as a subset of $[-1,1]$ and different from $[-1,1]$.

Theorem 3.2. There exists a locally convex topology τ in B_{c_0} containing the weak topology such that B_{c_0} is τ-strongly regular for open subsets, but B_{c_0} does not enjoy the τ-CPCP, in fact every nonempty relatively $\tau-$open subset of B_{c_0} has diameter 2.

Proof. We will proceed by induction, constructing an increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ of countable convex subbases of topologies in B_{c_0} where β_1 is a subbase of the weak topology and such that for every $n \in \mathbb{N}$ the following properties are satisfied:

(1) For every nonempty $U \in \beta_n$ there is $A_U \subset \mathbb{N}$ such that

$$U = \{(x_k) \in B_{c_0} \ : \ x_k \in I_k, \ \forall k \in \mathbb{N} \setminus A_U\},$$

where I_k is a nonempty proper open subinterval of $[-1, 1]$.

(2) For every $F \in \mathcal{F}(\beta_n) := \{G \in \beta_n^{<\omega} \ : \ \bigcap_{U \in G} U \neq \emptyset\}$, the set $\bigcap_{U \in F} A_U$ is infinite.

(3) If $n > 1$ then for every $F \in \mathcal{F}(\beta_{n-1})$ there exist $U_1, \dots, U_n \in \beta_n \setminus \{\emptyset\}$ such that $U_i \subset \bigcap_{U \in F} U$ for $i = 1, \dots, n$ and

$$\operatorname{diam}\left(\sum_{i=1}^{n} \frac{1}{n} U_i\right) \leq \frac{4}{n}.$$

Clearly, the weak topology has a convex countable subbase β_1 satisfying properties (1) and (2) (notice that property (3) does not apply when $n = 1$). Now, for the inductive step, assume that β_{n-1} is a convex countable subbase containing β_1 and satisfying properties (1) and (2). Since $\mathcal{F}(\beta_{n-1})$ is countable, we label its elements as $\mathcal{F}(\beta_{n-1}) = \{F_m\}_m$. Now we rename $\bigcap_{U \in F_m} A_U = A_m$ so that by (1),

$$\bigcap_{U \in F_m} U = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,m}, \ \forall k \in \mathbb{N} \setminus A_m\},$$

for some proper open subintervals $I_{k,m}$ of $[-1, 1]$. We define β_n by means of another inductive argument. Specifically, we construct now an increasing sequence of subbases $\{\beta_n^p\}_{p \in \mathbb{N}}$ with the following properties,

(4) $\beta_n^p = \{U_p^1, \dots, U_p^n\} \cup \beta_n^{p-1}$ with $U_p^1, \dots, U_p^n$ pairwise disjoint subsets of $\bigcap_{U \in F_p} U$ for every $p \in \mathbb{N}$ (taking $\beta_n^0 = \beta_{n-1}$).

(5) β_n^p satisfies properties (1) and (2) for every $p \in \mathbb{N}$.

(6) For every $p \in \mathbb{N}$,

$$\operatorname{diam}\left(\sum_{i=1}^{n} \frac{1}{n} U_p^i\right) \leq \frac{4}{n}.$$

Let us construct β_n^1.

We use here Lemma 3.1 with $k = n$, $\mathcal{A} = A_1 \cup \{A_U\}_{U \in \beta_{n-1}}$ and $A_0 = A_1$. Therefore, there exists $\{A_1^1, \dots, A_1^n\}$ a partition of A_1 such that

$$A_1^i \cap \bigcap_{U \in F_1 \cup F} A_U \ \text{ is infinite} \quad \forall F_1 \cup F \in \mathcal{F}(\beta_{n-1}) \ , \quad \forall i \in \{1, \dots, n\}. \tag{3.2}$$

We define n pairwise disjoint subsets of $\bigcap_{U \in F_1} U$ as

$$U_1^i = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,1}^i \text{ if } k \in \mathbb{N} \setminus A_1, \ x_k \in (-1/n, 1/n) \text{ if } k \in A_1 \setminus A_1^i\},$$

where $I_{k,1}^1, \dots, I_{k,1}^n \subset I_{k,1}$ are pairwise disjoint nonempty open intervals of the same length.

Claim. We claim that $\beta_n^1 := \{U_1^1, \dots, U_1^n\} \cup \beta_{n-1}$ is a subbase satisfying (4), (5) and (6).

Clearly, β_n^1 satisfies by definition property (4) so that we just have to show (5) and (6). Let us first tackle property (5). It is immediate from the definitions and inductive assumptions that β_n^1 satisfies (1) with $A_1^i = A_{U_1^i}$ for $i = 1, \ldots, n$. Let us focus on property (2). Consider then $F \in \mathcal{F}(\beta_n^1)$. If $F \in \beta_{n-1}^{<\omega}$ then from the induction hypothesis we know that (2) is satisfied. Otherwise, there is $1 \le i \le n$ such that $U_1^i \in F$. Since $\bigcap_{U\in F} U \neq \emptyset$ necessarily $F \setminus \{U_1^i\} \in \beta_{n-1}^{<\omega}$ and $\bigcap_{U \in F_1 \cup F\setminus\{U_1^i\}} U \neq \emptyset$. Hence, $F_1 \cup F \setminus \{U_1^i\} \in \mathcal{F}(\beta_{n-1})$. Therefore, $\bigcap_{U\in F} A_U = A_1^i \cap \bigcap_{U\in F_1\cup F\setminus\{U_1^i\}} A_U$ which is infinite by (3.2). This finally proves that β_n^1 enjoys property (2) and hence (5). Let us finish the proof of the claim by showing that β_n^1 enjoys property (6). Indeed, if we pick $(x_k^{i,1})_k, (y_k^{i,1})_k \in U_1^i$ for $i = 1, \ldots, n$, then $|x_k^{i,1} - y_k^{i,1}| \le \frac{2}{n}$ whenever $k \notin A_1^i$. Therefore,

$$\left\| \sum_{i=1}^n \frac{1}{n}(x_k^{i,1})_k - \sum_{i=1}^n \frac{1}{n}(y_k^{i,1})_k \right\| = \frac{1}{n}\sup_k \left| \sum_{i=1}^n (x_k^{i,1} - y_k^{i,1}) \right| \le \frac{1}{n}\sup_k \left(2 + \sum_{\substack{i=1 \\ k\notin A_1^i}}^n |x_k^{i,1} - y_k^{i,1}| \right) \le \frac{4}{n}.$$

This proves that β_n^1 satisfies the induction hypothesis and hence the claim is proven.

Let us now get into the inductive step. Assume β_n^{p-1} has been defined for some $p > 1$ enjoying properties (4), (5) and (6).We use Lemma 3.1 with $k = n$, the countable set $\mathcal{A} = \{A_p\} \cup \{A_U\}_{U\in\beta_n^{p-1}}$ and $A_0 = A_p$ and obtain a partition $\{A_p^1, \ldots, A_p^n\}$ of A_p such that for every $F_p \cup F \in \mathcal{F}(\beta_n^{p-1})$ we have that $A_p^i \cap \bigcap_{U\in F_p\cup F} A_U$ is infinite (this is due to the fact that β_n^{p-1} satisfy (2)). Now, we define for $i = 1, \ldots, n$ the set,

$$U_p^i = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,p}^i \text{ if } k \in \mathbb{N}\setminus A_p,\ x_k \in (-1/n, 1/n) \text{ if } k \in A_p \setminus A_p^i\},$$

where $I_{k,p}^1, \ldots, I_{k,p}^n \subset I_{k,p}$ are pairwise disjoint nonempty open intervals of the same length. Following the same reasoning as in the case $p = 1$, we know that $\beta_n^p := \{U_p^1, \ldots, U_p^n\} \cup \beta_n^{p-1}$ satisfies properties (4), (5) and (6). Finally, β_n is defined as the union $\beta_n := \bigcup_p \beta_n^p$. It is straightforward to see that β_n satisfies properties (1), (2) and (3) and hence the inductive construction is complete.

We are then ready to define our sought topology τ as the topology generated by the subbase $\beta = \bigcup_n \beta_n$. τ is clearly locally convex and contains the weak topology. Let us check that B_{c_0} is τ-strongly regular but does not enjoy the τ-CPCP.

In order to show that B_{c_0} fails τ-CPCP we will prove that every nonempty $\tau-$open subset of B_{c_0} has diameter 2. For that, observe that the family of finite intersections of elements in β is a basis for τ, so we will be done if we get that every set of the form $\bigcap_{U\in F} U$ is either empty or has diameter 2 for every $F \in \beta^{<\omega}$. There has to be some $n \in \mathbb{N}$ such that $F \in \beta_n^{<\omega}$. If we assume that $\bigcap_{U\in F} U$ is nonempty then $F \in \mathcal{F}(\beta_n)$ and the set $\bigcap_{U\in F} A_U$ is infinite from (2). Now, from (1), if $U \in F$ the elements in U are sequences of B_{c_0} without restrictions on terms of A_U, so there is k (in fact, infinite many terms) such that the elements of U have no restrictions on its values for the term k, other than belongs to B_{c_0}, for every $U \in F$, and then $diam(\bigcap_{U\in F} U) = 2$.

In order to show that B_{c_0} is $\tau-$SR for open subsets, let O be a nonempty τ-open set and $\varepsilon > 0$. Then, there is $F \in \beta^{<\omega}$ such that $\emptyset \neq \bigcap_{U\in F} U \subset O$. Therefore, there is $n \in \mathbb{N}$ such that $F \in \mathcal{F}(\beta_n)$. Consider $m > n$ such that $4/m < \varepsilon$. Since $(\beta_n)_n$ is increasing, $F \in \mathcal{F}(\beta_{m-1})$ and thus by property (3) there are $U_1, \ldots, U_m \in \beta \setminus \{\emptyset\}$ such that $U_i \subset \bigcap_{U \in F} U$ for $i = 1 \ldots, n$ and

$$\operatorname{diam}\left(\sum_{i=1}^{m} \frac{1}{m} U_i\right) \leq \frac{4}{m} < \varepsilon.$$

□

We think that it would be interesting to know if a topology as the constructed one in the above theorem exists for L_1.

Observe that the topology defined in the above theorem can be defined on the whole space c_0 and even it is possible defining a topology on ℓ_∞, following the same above ideas, containing the weak-star topology on ℓ_∞ so that every relatively open subset of the unit ball has diameter 2, while the unit ball contains convex combinations of relatively open subsets with diameter arbitrarily small.

In [4] (see also [1]) is proved that every Banach space containing c_0 can be equivalently renormed satisfying D2P so that the new unit ball contains convex combinations of relatively weakly open susets with diameter arbitrarily small, showing the extreme diference between D2P and SD2P. The abore result stresses this extreme difference from a topological point of view, enlarging the family of open subsets with diameter 2 and keeping the existence of convex combinations of relatively open subsets with diameter arbitrarily small, without renorming.

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