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Minimum Cost Consensus Modeling under Various Linear Uncertain-constrained Scenarios

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Abstract:	Group decision-making combined with uncertainty theory is verified as a more conclusive theory, by building a bridge between deterministic and indeterministic group decision-making in this paper. Due to the absence of sufficient historical data, reliability of decisions are mainly determined by experts rather than some prior probability distributions, easily leading to the problem of subjectivity. Thus, belief degree and uncertainty distribution are used in this paper to fit individual preferences, and five scenarios of uncertain chance-constrained minimum cost consensus models are further discussed from the perspectives of the moderator, individual decision-makers and non-cooperators. Through deduction, reaching conditions for consensus and analytic formulas of the minimum total cost are both theoretically given. Finally, with the application in carbon quota negotiation, the proposed models are demonstrated as a further extension of the crisp number or interval preference-based minimum cost consensus models. In other words, the basic conclusions of the traditional models are some special cases of the uncertain minimum cost consensus models under different belief degrees.
Response to Reviewers:	Thanks a lot for all the important comments. Suggestion proposed by Reviewer #4 is practical and significant, so we totally agree with him/her. The title of Section 4 has been revised into "Application in carbon quota negotiation" to be more informative and precise about our paper (please see Page 13).

Responses to Editors' and Reviewers' Comments

Authors truly appreciate editor’s and all reviewers’ valuable suggestions regarding our manuscript entitled “Minimum Cost Consensus Modeling under Various Linear Uncertain-constrained Scenarios” (ID: INFFUS_2020_104R2). All comments are of great importance for revising and improving the paper. **Changes are highlighted in blue in the new version.** Detailed responses are as follows.

Editor and Reviewer comments:
EiC: The paper is near to be accepted.

Please, could you check the reviewer 4 suggestion?

The title may also be updated to be more informative about the paper content.
Response: Thanks a lot for these important comments. Suggestion proposed by Reviewer #4 is practical and significant, so we totally agree with him/her. The title of Section 4 has been revised into “Application in carbon quota negotiation” (please see Page 13).

Reviewer #2: I think this paper could be accepted.
Response: Thank you so much for this positive comment.

Reviewer #4: I think that this version of the paper is acceptable. A minor suggestion: the title of Section 4 may be revised to "Application in carbon quota negotiation" since the case study in Section 4 is not a real-life case and may confuse the readers.
Response: We appreciate this important comment, and totally agree with your suggestion. Thus, the title of Section 4 has been revised to “Application in carbon quota negotiation” to be more informative and precise about our paper (please see Page 13).

We earnestly appreciate Editors’/ Reviewers’ constructive work, and wish the new version could meet with approval. Thank you so much for all valuable comments and suggestions!

Highlights:

Uncertain minimum cost consensus models (MCCMs) are built from multiple perspectives.
Belief degree and uncertain distributions are denoted as participants' preferences.
Optimal reaching conditions for consensus and total cost are theoretically given.
Carbon emission quota allocation negotiation is analyzed as case background.
Proposed models are verified as a further extension of the traditional MCCMs.

Minimum Cost Consensus Modeling under Various Linear Uncertain-constrained Scenarios

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Abstract: Group decision-making combined with uncertainty theory is verified as a more conclusive theory, by building a bridge between deterministic and indeterministic group decision-making in this paper. Due to the absence of sufficient historical data, reliability of decisions are mainly determined by experts rather than some prior probability distributions, easily leading to the problem of subjectivity. Thus, belief degree and uncertainty distribution are used in this paper to fit individual preferences, and five scenarios of uncertain chance-constrained minimum cost consensus models are further discussed from the perspectives of the moderator, individual decision-makers and non-cooperators. Through deduction, reaching conditions for consensus and analytic formulas of the minimum total cost are both theoretically given. Finally, with the application in carbon quota negotiation, the proposed models are demonstrated as a further extension of the crisp number or interval preference-based minimum cost consensus models. In other words, the basic conclusions of the traditional models are some special cases of the uncertain minimum cost consensus models under different belief degrees.
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1. Introduction

Group decision-making (GDM) mainly solves unstructured decision-making problems, involving subjective participation of various experts [1, 2]. In GDM, through communication and multiple rounds of effective feedback/adjustment, decision-makers (DMs) eventually form a clear support or objection towards a certain issue. Then, a relatively consistent consensus is reached [3]. Consensus decision-making is a prerequisite for effective GDM and widely exists in our daily lives, such as online P2P lending [2], emergency decision support [4], and trans-boundary water pollution control [5, 6]. In general, factors affecting the consensus reaching process (CRP) include DMs' preference structures or psychological expectation [6, 7, 8, 9], convergence rules [10, 11], decision environment [12, 13, 14, 15, 16, 17], and leaders' [2, 18, 19, 20] or non-cooperators' influence [1, 21, 22, 23, 24]. Urda and Loch [25] indicated that individual behaviours in GDM are driven by both their own economically rational deliberation and decision biases and social preferences (e.g. status achievement, reciprocal

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relations, or group identity). Thus, a moderator [2, 19, 20], on behalf of collective interest, is often introduced to improve the speed and efficiency of CRP. He/she possesses prominent skills in leadership and negotiation, and can persuade/tempt DMs to continually adjust their opinions into consensus through different effective means (collectively referred to as “consensus cost”).

The concept of minimum cost consensus model (MCCM) was first proposed by Ben-Arieh and Easton [20], to explore a single and a multi-criteria decision consensus problem with a linear cost using linear-time algorithms. Afterward, they built models based on quadratic cost functions by taking account of consensus cost, opinion elasticity and the maximum number of experts [26]. Meanwhile, Dong et al. [10] investigated the internal relations of several OWA-based linguistic operators based on position indexes, and originally presented the optimisation-based minimum adjustment consensus models (MACMs). Subsequently, Zhang et al. [27] proposed a new framework for consensus models under aggregation operators, and illustrated that a link existed between MCCMs and MACMs. To further explore the original MCCMs, Gong et al. [19, 28] and Zhang et al. [2] adopted the linear prime-dual theory and presented the economic interpretations of their new consensus models. Wu et al. [29] discussed the scheme recommendation and users' trust measure using the feedback mechanism in MCCMs with social network analysis. Meanwhile, considering that the cost coefficients are asymmetric due to the adjustment direction of DMs' opinions, Cheng et al. [5] analysed the impact of individual limited compromises and tolerance behaviours on MCCMs. Research paradigms about the MCCMs/MACMs with feedback mechanism during the last decade were concluded by Zhang et al. [30], and they further pointed out new directions for the future research. So far, most extant MCCMs/MACMs assume DMs' preferences denoted by crisp numbers or intervals, making the stochastic distribution for DMs' opinions seldom considered. Thus, uncertainty distributions are used to fit DM's preferences in this paper.

Actually, even if there exists a moderator acting as a leader in GDM, the DMs involved still cannot account for all factors; besides, diversity widely exists in individuals' research background, knowledge reserve, and the amount of private information. Thus, GDM is full of uncertainty, making it unable to accurately predict the outcome in advance. GDM essentially includes providing decision support for solving uncertainty. Without loss of generality, theoretical methods for dealing with uncertainty include probability theory, interval analysis, fuzzy sets, rough sets and grey systems. However, it is often difficult to obtain a precise probability for a natural state in real-life GDM, especially when there is little information available for evaluating probabilities, usable information is insufficient, or when several information sources conflict with each other [31]; then, the reliability (or probability) that certain event will occur is primarily determined by experts. To handle situations where the reliable prediction that one event would occur has to be determined by individual subjectivity due to the inability to obtain its actual frequency [32], uncertainty theory was proposed by Liu [33], which gradually extended into a systematic subject, from a theoretical perspective [34, 35, 36, 37, 38] and an application perspective [39, 40]. As an important branch of mathematics [41], uncertainty theory is mainly used to deal with human beings' subjective reliability and has been successfully applied into trust measure in social networks [42]. To the best of the authors' knowledge, compared with multiple prominent theories dealing with indeterminacy, efficiencies and advantages of uncertainty theory in GDM are concluded in [43, 44].

Due to the uncertainty in DMs' opinions, traditional probability and statistics methods are no longer suitable for the preference analysis of individual behaviours involved in GDM, because frequency distribution, probability distribution, and density function for individual opinions are difficult to obtain. However, we can always grasp a certain degree of certainty, such as 95% confidence/belief, to achieve consensus, and when the consensus is reached with a certain degree of belief, CRP is more consistent with actual GDM situations. Therefore, this paper introduces the belief degree and uncertain variables to simulate DMs' judgement behaviours, and by combining the MCCMs and uncertainty theory, this paper extends traditional MCCMs into five scenarios from diverse roles as moderator, individual DMs, and non-cooperators. In GDM, decisions are always made before the realisation of individual preferences (i.e., random variables), so we suppose that the belief degree of the constraints satisfied is no less than a specified value. Such problems can be solved by chance-constrained goal programming [45]. As a stochastic programming method, chance-constrained problems can always be transformed into an equivalent deterministic mathematical model, making it convenient to obtain Pareto optimal solutions toward the original problems. In short, our main contributions are:

- Uncertain MCCMs are discussed from the perspectives of multiple roles, such as the moderator, individual DMs and non-cooperators;
- Since interval preference-based MCCMs take only endpoints into account, belief degree and uncertainty distribution are introduced as a whole to fit individual judgements, making the proposed models more feasible;
- Analytic formulas for both the optimal consensus and the total cost (i.e., the optimal solutions) under each scenario are presented, through linear transformation of the uncertain MCCMs.
- Feasibility of the new uncertain MCCMs is verified by the carbon emission quota negotiation conducted between the heavily polluting enterprises and the local government.

The rest of the paper is organised as follows. Section 2 recalls preliminaries on traditional consensus models (i.e., MCCMs or MACMs) and uncertainty theory. Inspired by the consensus modeling in [10, 19, 28], Section 3 adopts belief degree and uncertain variables to characterise DMs' preferences. In addition, by discussing five GDM scenarios, a series of optimisation-based consensus models are developed. General reaching conditions for the consensus under each scenario are also provided in this section through theoretical deduction. Subsequently, Section 4 verifies the feasibility of the proposed models through the optimal carbon quota allocation negotiation between heavily polluting enterprises and the local government. Finally, concluding remarks and future research directions are presented in Section 5.

2. Preliminaries

2.1. Consensus models with a minimum cost or adjustment

Suppose there exist n DMs participating in GDM, $o_i \in R$ is the original opinion of DM d_i, $i \in N = \{1, 2, \cdots, n\}$ and o' is the collective opinion reached by the whole group (i.e., consensus). Let $f_i(o') = |o' - o_i|$ be the rectilinear distance measure between d_i's original opinion and the consensus [20]. Generally, reaching a consensus depends largely on behaviours of DMs [46], meanwhile high-impact moderators [2] or opinion leaders [18] can effectively promote the speed and efficiency of CRP. Particularly, by exercising significant leadership skills or scheduling limited resources (e.g., human, material, or financial resources), moderators are capable of guiding or coordinating with DMs to change individual inconsistent opinions towards a relatively consistent group opinion. Based on the above distance measure, Ben-Arieh and Easton [20] first put forward the concept of the minimum cost consensus, aiming at minimizing resources consumption during decision-making process; meanwhile, Dong et al. [10] initially proposed consensus models with minimum preference adjustment (i.e., MACMs) by introducing aggregation operators, aiming to preserve DMs' original preference information as much as possible. Subsequently, the two aforementioned modeling ideas become an important foundation of most extant consensus works (e.g., [2, 5, 6, 23, 27, 29, 30, 46]).

This paper mainly pursue the goal of minimizing the total consensus cost instead of keeping the original preference information as much as possible. Without loss of generality, let ω_i denote the cost for moving d_i's original opinion o_i towards the consensus o' one unit. In fact, the main difference between

MCCMs and MACMs lies in whether considering the unit cost or not. Mathematically, if we normalize these unit costs, then they become the weighted arithmetic mean operators, which can also be understood as each individual's influence on CPR [28]. In reality, too many uncertain factors need to be considered in GDM, making the above parameters difficult to quantify, hence, ω_i is subjectively determined in the follow-up discussion of this paper. Zhang et al. [46] presented a bi-level optimization model to describe the interaction behaviors within CRP based on Stackelberg game, and further provided an optimal unit cost from a pre-defined reasonable range rather than assuming ω_i as a known parameter. Anyway, $\omega_i f_i(o')$ and $\sum\limits_{i=1}^{n} \omega_i f_i(o')$ indicate the costs paid by the moderator for persuading individual d_i and all DMs to change their inconsistent opinions during GDM, respectively.

$$\begin{array}{l} Min\ \phi = \sum\limits_{i=1}^{n} \omega_i f_i(o') \\ s.t.\ \left\{ \begin{array}{l} f_i(o') = |o' - o_i| \\ |o' - o_i| \leq \varepsilon_i,\ i \in N \end{array} \right. \end{array} \tag{1}$$

Since the less the total cost the better, a MCCM based on the above principles is built as Model (1) [19, 20, 27], where ϕ represents the total consensus cost for the whole GDM, and ε_i is the upper bound of the deviation (i.e., distance measure) between d_i's opinion and the optimal collective opinion, implying that we want to obtain an acceptable consensus (i.e., soft consensus [27, 47]).

2.2. Uncertainty theory

Uncertainty widely exists in real-life GDM, for instance, when faced with emergency, human beings usually cannot determine the occurrence frequency of certain events due to the absence of historic data, making it difficult to accurately estimate the probability distribution of such events. Aiming at the above limitations in classical probability theory, uncertainty theory proposed by Liu [33, 48] is an important and useful mathematical instrument to handle uncertain phenomenon with non-randomness and non-fuzziness. Next, some basic concepts in uncertainty theory are introduced.

Let Γ be a nonempty set (sometimes referred as universal set), and a collection $\mathcal{L}$ consisting of subsets of Γ is an algebra over Γ, if it meets the following three conditions: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^C \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we have $\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}$, where, if condition (c) is replaced by closure under countable union, that is, if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we obtain $\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}$, then $\mathcal{L}$ is referred as a σ-algebra over Γ. Element Λ in $\mathcal{L}$ is called a measurable set, which also can be interpreted as an event in uncertainty theory. M is defined as an uncertain measure over the σ-algebra $\mathcal{L}$. Without loss of generality, real number $M\{\Lambda_i\}$ corresponds to event Λ_i one by one, representing the belief degree with which we belief event Λ_i will occur. There exist no doubt that such assignment is not arbitrary, and the uncertain measure M satisfies the following four axioms [33, 48].

***Axiom* 1.** *(Normality Axiom): $M\{\Gamma\} = 1$ holds for the universal set Γ.*

***Axiom* 2.** *(Duality Axiom): $M\{\Lambda\} + M\{\Lambda^c\} = 1$ holds for any event Λ.*

***Axiom* 3.** *(Subadditivity Axiom): For every countable sequence of event Λ_1, Λ_2, $\cdots$, we have:*

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$$

***Axiom* 4.** *(Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertain space for $k \in N^+$, then the product of uncertain measure M is still an uncertain measure, and satisfies:*

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}$$

where Λ_k are events arbitrarily chosen from $\mathcal{L}_k, (k \in N^+)$, respectively.

***Definition* 1.** [33] An uncertain variable ξ is a function from an uncertain space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, and $\{\xi \in B\}$ is an event for any Borel set B of real numbers. For any real number x, the uncertainty distribution Φ of an uncertain variable ξ can be defined as: $\Phi(x) = M\{\xi \leq x\}$.

$M\{\xi \leq x\}$ is the belief degree for the event $\xi \leq x$ may occur, and it is denoted as α, where $0 \leq \alpha \leq 1$. In other words, we have $\Phi(x) = M\{\xi \leq x\} = \alpha$. According to Axiom 2, we obtain $M\{\xi > x\} = 1 - \Phi(x) = 1 - \alpha$.

***Definition* 2.** [41] An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and satisfies $\lim\limits_{x \to -\infty} \Phi(x) = 0$ and $\lim\limits_{x \to +\infty} \Phi(x) = 1$.

Note that, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution are all common regular uncertainty distributions.

***Theorem* 1.** *[41] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \cdots, \xi_m$, and strictly decreasing with respect to $\xi_{m+1}, \cdots, \xi_n$, then $f(\xi_1, \cdots, \xi_n)$ has an inverse uncertainty distribution of $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))$.*

***Theorem* 2.** *[33] Let ξ be an uncertain variable with its inverse uncertainty distribution denoted as $\Phi^{-1}(\alpha)$, if and only if $\Phi^{-1}(\alpha) \leq c$, then $M\{\xi \leq c\} \geq \alpha$, where α, c are constants within $[0, 1]$.*

***Theorem* 3.** *Let uncertain variables ξ_1 and ξ_2 be independent with inverse uncertainty distribution Φ_1 and Φ_2, respectively, then the inverse uncertainty distribution for the difference between these two variables (denoted by $\xi_1 - \xi_2$) can be defined as: $\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha)$.*

Actually, uncertain measure can be understood as DMs' personal belief degree (not frequency) of an event may occur, so the real meanings of belief degree and uncertain measure appear to be the same. Generally, regular uncertainty distributions include linear uncertainty distribution, normal uncertainty distribution and so on. Hereafter, we only discuss the linear type since it can be easily transformed when the analytic formulas of the proposed models are to be obtained.

***Definition* 3.** [33] Uncertain variable ξ satisfies a linear uncertainty distribution (see Fig. 1), denoted as $\xi \sim \mathcal{L}(a, b)$, where a, b are both real numbers and $a < b$, then linear uncertainty distribution function is presented as:

$$\Phi(x) = \begin{cases} 0, & if\, x \leq a \\ \dfrac{x-a}{b-a}, & if\, a \leq x \leq b \\ 1, & if\, x \geq b \end{cases}$$

***Definition* 4.** [33] An uncertain variable ξ satisfies $\xi \sim \mathcal{L}(a, b)$, then its inverse uncertainty distribution function (see Fig. 2) is expressed as:

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$$

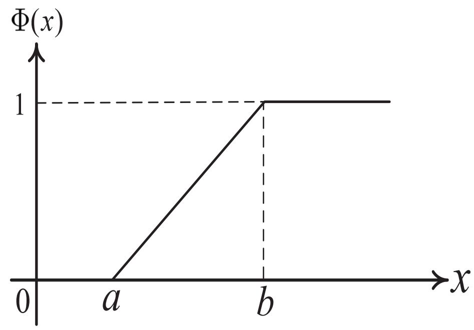


Fig. 1 Linear uncertainty distribution

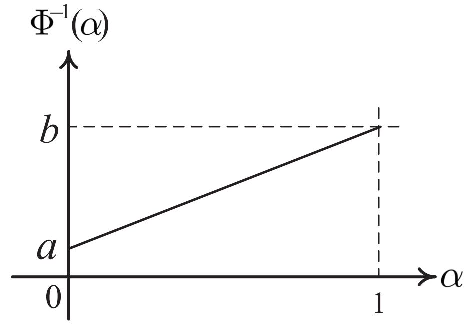


Fig. 2 Inverse linear uncertainty distribution

3. MCCMs with uncertain preferences

Participants' preferences in original MCCMs or MACMs are usually denoted by crisp numbers, without taking into account that their opinions fit by random distributions. In fact, it is often difficult for individuals to provide exact values as their preference, especially in some complex GDM contexts (e.g. social network GDM [6, 22, 29], large-scale GDM [1, 12, 14, 24] or GDM with dynamic opinions [18, 49]). Thus, DMs are more likely to present their decisions by intervals with upper and lower bounds or various uncertainty distributions (e.g. uniform uncertainty distribution or normal uncertainty distribution). Previous research focus on either role combination with moderator and individual DMs [2, 5, 20, 28] or independent decision-making status as moderator [18, 50], individual DM [10] or non-cooperators [1, 22, 23, 24]. Few extant works have build MCCMs by simultaneously taking account on three roles altogether. Given the above points, we utilise uncertainty distributions to denote participants' decision preferences, and by discussing five scenarios from multiple decision-making roles (i.e., moderator, individual DMs and non-cooperators), we aim to investigate a more general form of Model (1).

$$\begin{aligned} & Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\ & s.t.\ \{\ |o' - o_i| \leq \varepsilon_i, \varepsilon_i \geq 0,\ i \in N \end{aligned} \tag{2}$$

To introduce uncertainty theory into soft consensus decision-making, we obtain a further abstracted form from Model (1), which is denoted as Model (2). In specific, decision variable in Model (1) only includes o', while ε_i is a pre-defined threshold set over the distance measure between o' and o_i, $i \in N$. Meanwhile, decision variables in Model (2) include both o' and ε_i, and ε_i is bound by the deterministic threshold given in Model (1) under the premise that these parameters are set as same in both models. Obviously, the feasible domain of the solution set of Model (2) is larger than that of Model (1), making the optimal value of the objective function in Model (2) be no larger than that in Model (1). As a result, although the form of Model (2) is simpler, its scope of application is wider than Model (1). Furthermore, Model (2) becomes the basis of the following consensus models with uncertain variables.

3.1. Moderator with uncertain preference

Assume the original opinion o_i is a known crisp number presented by individual d_i and ω_i is a pre-defined unit cost paid by the moderator for d_i's change amount towards consensus o', $i \in N$. Since the moderator needs to consider many uncertain factors for the final convergent opinion, we assume that the moderator's opinion o' obeys uncertainty distribution. Based on Liu's uncertainty theory, if the deviation between the consensus o' and the individual opinion o_i is no more than ε_i under the belief degree α, then it can be denoted as $M\{o' - o_i \leq \varepsilon_i\} \geq \alpha$ and $M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha$, where M represents the uncertain measure in uncertainty theory, and the variable $\alpha \in [0, 1]$ indicates the belief degree of the constraint of $|o' - o_i| \leq \varepsilon_i$ holding, $i \in N$. Accordingly, an MCCM with uncertain chance constraints can be constructed as follows:

$$Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i$$
$$s.t. \begin{cases} M\{o' \leq o_i + \varepsilon_i\} \geq \alpha \\ M\{o' \geq o_i - \varepsilon_i\} \geq \alpha \\ \varepsilon_i \geq 0,\ i \in N \end{cases} \tag{3}$$

***Theorem* 4.** *Model (3) is equal to the non-linear goal programming Model (4).*

$$Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i$$
$$s.t. \begin{cases} \varepsilon_i \geq \Phi^{-1}(\alpha) - o_i & (4-1) \\ \varepsilon_i \geq -\Phi^{-1}(1-\alpha) + o_i & (4-2) \\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N & (4-3) \end{cases} \tag{4}$$

where ϕ is the total budget for the consensus reached; ω_i is the unit-persuading cost paid by the moderator to DM d_i; the consensus o' obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, where a and b are decision variables obeying an uncertainty distribution; and constraints (4-1) and (4-2) mean that the deviation between individual original opinion o_i and consensus o' is no more than ε_i under the premise of no less than an uncertain belief degree α. Clearly, the belief variable α ($\alpha \in [0, 1]$) can be a pre-determined fixed value or a decision variable to be solved.

Thus, if we reconsider the uncertain belief degree α, Model (3) or Model (4) essentially includes two issues: α is a pre-determined value or α is a parameter to be determined. As for the latter situation, the variable α solved by Model (3) or Model (4) will be an optimal belief degree in GDM. Besides, when $o' \sim \mathcal{L}(a, b)$, Model (4) can be further transformed into a linear programming model as in Corollary 1.

***Corollary* 1.** *Assuming that the consensus opinion obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, Model (4) is equivalent to the following optimisation model:*

$$Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i$$
$$s.t. \begin{cases} \varepsilon_i \geq (1-\alpha)a + \alpha b - o_i \\ \varepsilon_i \geq -\alpha a - (1-\alpha)b + o_i \\ a \leq b,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N \end{cases} \tag{5}$$

***Corollary* 2.** *If and only if $\alpha = 1$ and o' is a crisp number (i.e. $a = b$), Model (5) degenerates into Model (2), and under this situation, the two models have identical values of the optimal consensus and minimum consensus cost.*

With the constraint of $o' \sim \mathcal{L}(a, b)$, we are to discuss when the total budget and consensus opinion are exactly the same as that solved by Model (2), and to gain the threshold of the belief degree α once the consensus is reached. In fact, by conducting sensitivity analysis on variable α, we obtain the analytical formulas of the optimal solutions for Model (5), namely, we aim to explore the conditions under which Model (5) and Model (2) have identical optimal consensus and total budget.

***Theorem* 5.** *Assume DM's original opinions in Model (5) are arranged in order (i.e. $o_1 \leq o_2 \leq \cdots \leq o_n$), weights attached to each DM (i.e. $\omega_{i,\ i \in N}$) are different, and moderator's opinion obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, where a and b are decision variables (see Section 2.2). Once the belief degree satisfies $\alpha \geq 0.5$, the optimal objective value and consensus reached conditions for Model (5)* *are:*

$$\phi^* = min\sum_{i=1}^{n}\omega_i\varepsilon_i = \begin{cases} \sum\limits_{i=m+1}^{n}\omega_i o_i - \sum\limits_{i=1}^{m}\omega_i o_i,\ a=b\in[o_m, o_{m+1}] \\ \qquad iff\ \sum\limits_{i=1}^{m}\omega_i = \sum\limits_{i=m+1}^{n}\omega_i \\ \sum\limits_{i=m+1}^{n}\omega_i(o_i-o_m)+\sum\limits_{i=1}^{m}\omega_i(o_m-o_i),\ a=b=o_m \\ \qquad iff\ \sum\limits_{i=1}^{m-1}\omega_i < \sum\limits_{i=m}^{n}\omega_i, \sum\limits_{i=1}^{m}\omega_i > \sum\limits_{i=m+1}^{n}\omega_i \end{cases}$$

Proof. See Appendix A. □

Remark 1. Theorem 5 shows that once $\alpha \geq 0.5$, the optimal consensus and total cost will be constants and irrelevant with the belief degree any more. By then, Model (5) with linear uncertain preferences is equivalent to Model (2) with preferences denoted by crisp numbers. That is, the two models have identical minimum budget and optimal collective opinions. Above findings verify that the uncertain MCCMs proposed do have practical meanings.

3.2. Non-cooperators considered and moderator with uncertain preference

So far, non-cooperators' impact on MCCMs has gradually become an intriguing topic [23], particularly under some complex GDM contexts [1, 21, 22, 24], and most of those research are analysed by theoretical modeling and simulation experiments. Thus, without loss of generality, suppose multiple individuals have similar preferences or interest in GDM, while some non-cooperators insist on their own opinions for certain reasons, who may have authority power within industries or districts, making the moderator unable to ignore their demands. Under this scenario, moderator's budget is mainly used to persuade these non-cooperators for compromising. MCCMs discussed here correspond to the decision rule of minority being subordinate to majority. For example, a certain district is stepping into the final stage of China's urban demolition process, a large amount of local citizens have agreed to move while few nail-house holders insist to stay put, probably for more compensation from the government or for some stuff hard to let go. Then, the government has to schedule some extra budget to pursue better development for the whole district. Such phenomenon can be modeled as:

$$\begin{aligned} &Min\ Z = \sum_{i=1}^{t}\omega_k\varepsilon_k \\ &s.t. \begin{cases} f_k(o') \leq \varepsilon_k, k\in\{1,2,\ldots,t\} & (6-1) \\ f_i(o') \leq \varepsilon_i, i\in N\backslash k & (6-2) \\ \varepsilon_i \geq 0,\ i\in N & (6-3) \end{cases} \end{aligned} \tag{6}$$

Model (6) assumes there exist a total of t non-cooperators (denoted as d_k). Once a consensus is reached, the change amount of d_k's opinion is $f_k(o') = |o' - o_k|$ and his/ her unit cost paid by the moderator is ω_k, then the total consensus cost for this GDM scenario is Z. Note, $i \in N\backslash k$ means that excluding those non-cooperators, individuals belong to a small alliance where they may have similar interest or have already reached a temporary consensus. Model (6) is a general form of GDM with non-cooperators, however, situations with only one non-cooperator is discussed hereafter (i.e., Model (7)), for simplicity and for easy to obtain the analytic formulas of the uncertain MCCMs. In fact, when there exist no less than two non-cooperators, the modeling mechanism is similar and the optimal solutions can be easy to get by using softwares such as MATLAB.

$$\begin{aligned} &Min\ Z = \omega_k\varepsilon_k \\ &s.t. \begin{cases} f_k(o') \leq \varepsilon_k & (7-1) \\ f_i(o') \leq \varepsilon_i, i\in N, i\neq k & (7-2) \\ \varepsilon_i \geq 0,\ i\in N & (7-3) \end{cases} \end{aligned} \tag{7}$$

By introducing uncertain chance constraints based on the removal of the absolute value symbols, (7-1) can be transformed as $M\{o' - o_k \leq \varepsilon_k\} \geq \beta$ and $M\{o' - o_k \geq -\varepsilon_k\} \geq \beta$, and (7-2) becomes the uncertain constraints as $M\{o' - o_i \leq \varepsilon_i\} \geq \alpha$ and $M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha$, where β and α are the belief degrees imposed on d_k's and other DMs' opinion deviations with the consensus, respectively. Obviously, constraints (7-1) and (7-2) simultaneously define the threshold of the variable o'. Next, we obtain an equivalent non-linear consensus model.

$$\begin{aligned}
&Min\ Z = \omega_k \varepsilon_k \\
&s.t. \begin{cases}
\Phi^{-1}(\beta) \leq \varepsilon_k + o_k \\
\Phi^{-1}(1-\beta) \geq -\varepsilon_k + o_k \\
\Phi^{-1}(\alpha) \leq \varepsilon_i + o_i, i \in N, i \neq k \\
\Phi^{-1}(1-\alpha) \geq -\varepsilon_i + o_i, i \in N, i \neq k \\
0 \leq \alpha,\ \beta \leq 1,\ \varepsilon_i \geq 0,\ i \in N
\end{cases}
\end{aligned} \tag{8}$$

If the consensus obeys a linear uncertainty distribution with unknown parameters of a and b, namely $o' \sim \mathcal{L}(a, b)$, then Model (8) can be further extended as Model (9). For the convenience of comparative analysis with [19], this paper sets Model (9) as an MCCM with a soft-consensus constraint. That is, except for the non-cooperator, all other threshold constraints $\varepsilon_{i, i \in N, i \neq k}$ imposed on DMs' opinions and the final consensus are pre-determined. To make up for the deficiency of hard consensus [51], soft consensus, which allows for a certain range between individual opinions and the collective opinion, is proposed [2, 47, 20]. Generally, soft consensus can be measured by consensus level [22, 51]. Therefore, Z, a, b and ε_k are all decision variables in Model (9).

$$\begin{aligned}
&Min\ Z = \omega_k \varepsilon_k \\
&s.t. \begin{cases}
(1-\beta)a + \beta b - o_k \leq \varepsilon_k & (9-1) \\
-\beta a + (\beta - 1)b + o_k \leq \varepsilon_k & (9-2) \\
(1-\alpha)a + \alpha b - o_i \leq \varepsilon_i, i \in N, i \neq k & (9-3) \\
-\alpha a + (\alpha - 1)b + o_i \leq \varepsilon_i, i \in N, i \neq k & (9-4) \\
a \leq b,\ 0 \leq \alpha,\ \beta \leq 1, \varepsilon_i \geq 0,\ i \in N & (9-5)
\end{cases}
\end{aligned} \tag{9}$$

Constraints (9-1)-(9-4) create bounds on the parameters of a and b, which may lead to an empty solution space, that is, a feasible solution maybe no longer exist in Model (9). However, this situation makes sense in real-life GDM. For example, if a non-cooperator is no longer rational enough, then the urban demolition negotiation may bring to an end. Furthermore, we should note that once no feasible solution exists, then the roles of different DMs will change. Specifically, DMs other than d_k now have a veto power, and in fact are then more powerful than d_k, which may result in a new iteration for reaching a consensus. Currently, such scenarios haven't been analysed in this paper, but it will be an interesting topic in our future research. However, using conclusions in Theorem 6, we can always set certain pre-defined parameters in Model (9) to guarantee that a feasible solution exist.

Theorem 6. *When belief degrees α and β in Model (9) satisfy the constraint (10). Then, if and only if $a = b$, Model (9) degenerates into the $P_k(\varepsilon)$ problem in [19] (i.e. Model (11)), meaning that Model (9) and Model (11) have identical optimal solutions, then the final collective opinion (i.e. the consensus) for Model (9) is also obtained.*

$$\begin{aligned}
&P_k(\varepsilon): \ Min\ Z = \omega_k |o' - o_k| \\
&s.t. \begin{cases}
|o' - o_i| \leq \varepsilon_i, i \in N, i \neq k \\
o' \geq 0
\end{cases}
\end{aligned} \tag{11}$$

Proof. See Appendix B. □

Remark 2. Theorem 6 provides consensus reaching conditions for MCCMs in light of the non-cooperator d_k and the consensus o' obeying a linear uncertainty distribution. And when $\alpha = \beta = 1$, Model (9) is equivalent to Model (7).

$$
\frac{1}{2} \le \beta \le 1 \begin{cases}
0 \le \alpha \le \frac{1}{2} \begin{cases}
o_k > min\ (o_i+\varepsilon_i) \begin{cases}
\varepsilon_k^* = min\{(\beta-\frac{1}{2})(b-a), o_k - min(o_i+\varepsilon_i)\} \\
\ if\ \varepsilon_k^* = (\beta-\frac{1}{2})(b-a): \\
\quad a = \frac{min(o_i+\varepsilon_i)-2\alpha o_k}{1-2\alpha},\ b = \frac{(2-2\alpha)o_k - min(o_i+\varepsilon_i)}{1-2\alpha} \\
\ if\ \varepsilon_k^* = o_k - min(o_i+\varepsilon_i): \\
\quad a = b = min(o_i+\varepsilon_i)
\end{cases} \\
max\ (o_i-\varepsilon_i) \le o_k \le min\ (o_i+\varepsilon_i) \begin{cases}
a = b = o_k \\
\varepsilon_k^* = 0
\end{cases} \\
o_k < max\ (o_i-\varepsilon_i) \begin{cases}
\varepsilon_k^* = min\{(\beta-\frac{1}{2})(b-a), max(o_i-\varepsilon_i) - o_k\} \\
\ if\ \varepsilon_k^* = (\beta-\frac{1}{2})(b-a): \\
\quad a = \frac{max(o_i-\varepsilon_i)+(2-2\alpha)o_k}{1-2\alpha},\ b = \frac{max(o_i-\varepsilon_i)-2\alpha o_k}{1-2\alpha} \\
\ if\ \varepsilon_k^* = max(o_i-\varepsilon_i) - o_k: \\
\quad a = b = max(o_i-\varepsilon_i)
\end{cases}
\end{cases} \\
\frac{1}{2} \le \alpha \le 1 \begin{cases}
o_k > min\ (o_i+\varepsilon_i) \begin{cases}
a = b = min\ (o_i+\varepsilon_i) \\
\varepsilon_k^* = o_k - min\ (o_i+\varepsilon_i)
\end{cases} \\
max\ (o_i-\varepsilon_i) \le o_k \le min\ (o_i+\varepsilon_i) \begin{cases}
a = b = o_k \\
\varepsilon_k^* = 0
\end{cases} \\
o_k < max\ (o_i-\varepsilon_i) \begin{cases}
a = b = max\ (o_i-\varepsilon_i) \\
\varepsilon_k^* = max\ (o_i-\varepsilon_i) - o_k
\end{cases}
\end{cases}
\end{cases} \tag{10}
$$

3.3. DMs with uncertain preferences

Suppose individual opinion $o_i = [a_i, b_i]$ obeys an uncertainty distribution, while the random distribution characteristics of the consensus is not considered (i.e., o' denoted as a crisp number). Similar to the aforementioned research idea, deviation between o_i and o' can be expressed using uncertain measure based on the removal of the absolute value symbols as $M\{o' - o_i \le \varepsilon_i\} \ge \alpha$ and $M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha$, $(i \in N)$. Therefore, an optimisation-based MCCM with uncertain preferences is built as follows.

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
\Phi_i^{-1}(\alpha) \le o' + \varepsilon_i,\ i \in N \\
\Phi_i^{-1}(1-\alpha) \ge o' - \varepsilon_i,\ i \in N \\
o' \ge 0,\ 0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N
\end{cases}
\end{aligned} \tag{12}
$$

If an individual opinion specifically obeys a linear uncertainty distribution, denoted as $o_i \sim \mathcal{L}(a_i, b_i)$, where a_i and b_i are predetermined parameters of d_i's original uncertain preference, exhibiting certain extent of indetermination, $i \in N$. Other variables are similarly defined as in Section 3.1. Model (12) then equals to Model (13).

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
(1-\alpha)a_i + \alpha b_i - o' \le \varepsilon_i,\ i \in N & (13-1) \\
o' - \alpha a_i - (1-\alpha) b_i \le \varepsilon_i,\ i \in N & (13-2) \\
o' \ge 0,\ 0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N & (13-3)
\end{cases}
\end{aligned} \tag{13}
$$

Theorem 7. *Suppose original individual opinions in GDM satisfy linear uncertainty distributions as $o_i \sim \mathcal{L}(a_i, b_i)$, $i \in N$. If and only if $\frac{1}{2} \le \alpha \le 1$ and all opinions are organised in order as $\frac{a_1+b_1}{2} \le$*

$\frac{a_2+b_2}{2} \leq \cdots \leq \frac{a_n+b_n}{2}$, *analytic formulas of the objective function and the consensus are obtained as:*

$$\phi^* = \begin{cases} \sum\limits_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum\limits_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \; where \; o' \in [\frac{a_m+b_m}{2}, \frac{a_{m+1}+b_{m+1}}{2}], \\ \quad iff \; \sum\limits_{i=1}^{m} \omega_i = \sum\limits_{i=m+1}^{n} \omega_i \\ (\sum\limits_{i=1}^{m} \omega_i - \sum\limits_{i=m+1}^{n} \omega_i)\frac{a_m+b_m}{2} + \sum\limits_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum\limits_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \; where \; o' = \frac{a_m+b_m}{2}, \\ \quad iff \; \sum\limits_{i=1}^{m-1} \omega_i < \sum\limits_{i=m}^{n} \omega_i, \sum\limits_{i=1}^{m} \omega_i > \sum\limits_{i=m+1}^{n} \omega_i \end{cases}$$

Theorem 7 can be verified by a similar mechanism as for Theorem 5, thereby its relevant proof is omitted here due to space limitation. Note that when $0 \leq \alpha \leq \frac{1}{2}$, there is no general conclusion for Model (13). In addition, from practical perspective, if the belief degree belongs to the threshold of $[0, \frac{1}{2}]$, the CRP discussed makes no sense.

Remark 3. Theorem 7 indicates that once the value of the belief degree α is large enough, the consensus in Section 3.3 is only related to the mean values of individual opinions expressed by linear uncertainty distributions. Essentially, Theorem 7 and Theorem 5 are equivalent in forms.

3.4. Non-cooperators considered and DMs with uncertain preferences

Assume there exist a total of t non-cooperators in GDM process (referred to as d_k), and all the individual opinions $o_{i,\ i\in N}$ obey uncertainty distributions while the consensus is presented as a crisp number. Since DMs other than $d_k, k \in \{1, 2, \ldots, t\}$ are like-minded and form a small alliance, then the whole group will mostly emphasize on d_k's interest, thus, an optimisation-based uncertain MCCM is constructed as

$$\begin{array}{l} Min \; Z = \sum\limits_{i=1}^{t} \omega_k \varepsilon_k \\ s.t. \begin{cases} M\{o' - o_k \leq \varepsilon_k\} \geq \beta, M\{o' - o_k \geq -\varepsilon_k\} \geq \beta, k \in \{1, 2, \ldots, t\} & (14-1) \\ M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha, i \in N \backslash k & (14-2) \\ o' \in O, o' \geq o, \varepsilon_k \geq 0, k \in \{1, 2, \ldots, t\} & (14-3) \end{cases} \end{array} \tag{14}$$

In Model (14), d_k, $(k \in \{1, 2, \ldots, t\})$ is non-cooperated with the small alliance in GDM. That is, other $(n-t)$ DMs have basically reached a temporary consensus, or the $(n-t)$ DMs may have similar interest or like-minded, so this scenario aims to minimise the total consensus cost (i.e. Z) on d_k for adjusting their opinions. Constraint (14-1) denotes the uncertain measure for those non-cooperators with belief degree β, while constraint (14-2) represents other individual opinions obeying an uncertainty distribution under the belief degree of α. Consensus o' belongs to the feasible set of O, and all opinions are greater than zero by default. For the logical consistency of this paper and easy to obtain the analytic formulas of the uncertain MCCMs, hereafter, we still discuss the GDM scenario with only one non-cooperator considered, then, the above uncertain MCCM is further transformed as Model (15).

Namely, if individual opinions satisfy linear uncertainty distributions as $o_i \sim \mathcal{L}(a_i, b_i)$, then Model (14) with only one non-cooperator considered is equivalent to Model (15), where $\forall i \in N$, a_i and b_i are pre-determined. Similar in Section 3.2, $\varepsilon_{i,\ i\neq k}$ are some known soft-consensus thresholds, α, β are belief degrees for different DMs, and o', ε_k are decision variables. Similarly, when GDM situation involves more than two non-cooperators, corresponding optimization models with linear uncertain preferences can be easily solved by software as MATLAB, however, the analytic formulas of their optimal solutions will be difficult to obtain then, thus, this paper mainly focuses on the simplest GDM context.

$$
\begin{aligned}
& Min\ Z = \omega_k \varepsilon_k \\
& s.t. \begin{cases}
(1-\beta)a_k + \beta b_k - o' \le \varepsilon_k & (15-1) \\
o' - [\beta a_k + (1-\beta) b_k] \le \varepsilon_k & (15-2) \\
(1-\alpha)a_i + \alpha b_i - o' \le \varepsilon_i, i \in N, i \ne k & (15-3) \\
o' - [\alpha a_i + (1-\alpha) b_i] \le \varepsilon_i, i \in N, i \ne k & (15-4) \\
o' \in O, o' \ge o, \varepsilon_k \ge 0 & (15-5)
\end{cases}
\end{aligned} \tag{15}
$$

Theorem 8. *If d_i's opinion ($i \in N$) obeys a linear uncertainty distribution as $o_i \sim \mathcal{L}(a_i, b_i)$, the analytic formulas of the objective function and the consensus in Model (15) satisfy constraint (16).*

$$
\begin{cases}
0 \le \beta \le \frac{1}{2} \begin{cases}
\beta a_k + (1-\beta) b_k] < G: & \varepsilon_k^* = o' - \beta a_k - (1-\beta) b_k,\ o' = G \\
(1-\beta)a_k + \beta b_k > H: & \varepsilon_k^* = (1-\beta)a_k + \beta b_k - o',\ o' = H \\
Otherwise: & \varepsilon_k^* = 0,\ o' \in [(1-\beta)a_k + \beta b_k, \beta a_k + (1-\beta) b_k] \bigcap [G, H]
\end{cases} \\
\frac{1}{2} \le \beta \le 1 \begin{cases}
\frac{a_k + b_k}{2} < G: & \varepsilon_k^* = o' - \beta a_k - (1-\beta) b_k,\ o' = G \\
\frac{a_k + b_k}{2} \in [G, H]: & \varepsilon_k^* = (\beta - \frac{1}{2})(b_k - a_k),\ o' = \frac{a_k + b_k}{2} \\
\frac{a_k + b_k}{2} > H: & \varepsilon_k^* = (1-\beta)a_k + \beta b_k - o',\ o' = H
\end{cases}
\end{cases} \tag{16}
$$

where $G = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\}$ and $H = min\{\alpha a_i + (1-\alpha) b_i + \varepsilon_i\}$, $i \in N, i \ne k$.

Proof. See Appendix C. □

3.5. Moderator and DMs with uncertain preferences

Suppose all participants' opinions (including moderator and individual DMs) obey uncertainty distributions. Once individuals in the group obey diverse uncertainty distributions, the MCCM constructed aims to solve heterogeneous GDM problems [13, 14]. However, this is not the focus we intend to explore, in other words, this paper assumes that all participants obey the same type of uncertainty distribution. Therefore, a corresponding CRP can be mathematically constructed as:

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
M\{o' - o_i \le \varepsilon_i\} \ge \alpha \\
M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha \\
0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N
\end{cases}
\end{aligned} \tag{17}
$$

As both individual opinion o_i and consensus o' obey uncertainty distributions, then based on Theorem 3, Model (17) can be further extended as

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
\Phi_{o'}^{-1}(\alpha) - \Phi_{o_i}^{-1}(1-\alpha) \le \varepsilon_i & (18-1) \\
\Phi_{o_i}^{-1}(\alpha) - \Phi_{o'}^{-1}(1-\alpha) \le \varepsilon_i & (18-2) \\
0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N & (18-3)
\end{cases}
\end{aligned} \tag{18}
$$

Specifically, suppose DM's opinion $o_i = [a_i, b_i]$ obeys a linear uncertainty distribution (denoted as $o_i \sim \mathcal{L}(a_i, b_i)$), and moderator's opinion, on behalf of the interest of the whole group, also obeys a linear uncertainty distribution by default as $o' \sim \mathcal{L}(a, b)$. Where a_i and b_i are known parameters of d_i's uncertain preference, while a and b are unknowns to be solved. Model (18) is equivalent to

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
a + (b-a)\alpha + (b_i - a_i)\alpha - b_i \le \varepsilon_i \\
(b_i - a_i)\alpha + (b-a)\alpha - b + a_i \le \varepsilon_i \\
a \le b,\quad 0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N
\end{cases}
\end{aligned} \tag{19}
$$

***Theorem* 9.** *Assume all individual DMs' opinions satisfy $o_i \sim \mathcal{L}(a_i, b_i)$ and moderator's opinion satisfies $o' \sim \mathcal{L}(a, b)$, and adjust individual original opinions in sequence as $\frac{a_1+b_1}{2} \leq \frac{a_2+b_2}{2} \leq \cdots \leq \frac{a_n+b_n}{2}$, then if and only if $\frac{1}{2} \leq \alpha \leq 1$, the optimal solution for Model (19) exists, which satisfies the following conditions:*

$$\phi^* = \begin{cases} \sum_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \\ \quad where\ a = b \in [\frac{a_m+b_m}{2}, \frac{a_{m+1}+b_{m+1}}{2}],\ iff\ \sum_{i=1}^{m} \omega_i = \sum_{i=m+1}^{n} \omega_i \\ (\sum_{i=1}^{m} \omega_i - \sum_{i=m+1}^{n} \omega_i)\frac{a_m+b_m}{2} + \sum_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \\ \quad where\ a = b = \frac{a_m+b_m}{2},\ iff\ \sum_{i=1}^{m-1} \omega_i < \sum_{i=m}^{n} \omega_i, \sum_{i=1}^{m} \omega_i > \sum_{i=m+1}^{n} \omega_i \end{cases}$$

Theorem 9 can be proved by a same mechanism as for Theorem 5, therefore, its relevant proof is omitted.

Remark 4. Theorem 9 verifies that once the participants' opinions obey linear uncertainty distributions in GDM, the final consensus is only related to the weight allocation and the mean values of initial opinions for all DMs. Besides, Theorem 9, Theorem 7 and Theorem 5 are formally equivalent.

3.6. Flowchart of MCCMs with uncertain preferences

For clarity, a flowchart of this paper is given as Fig. 3, and the relations between the aforementioned five GDM scenarios are also summarized in detail.

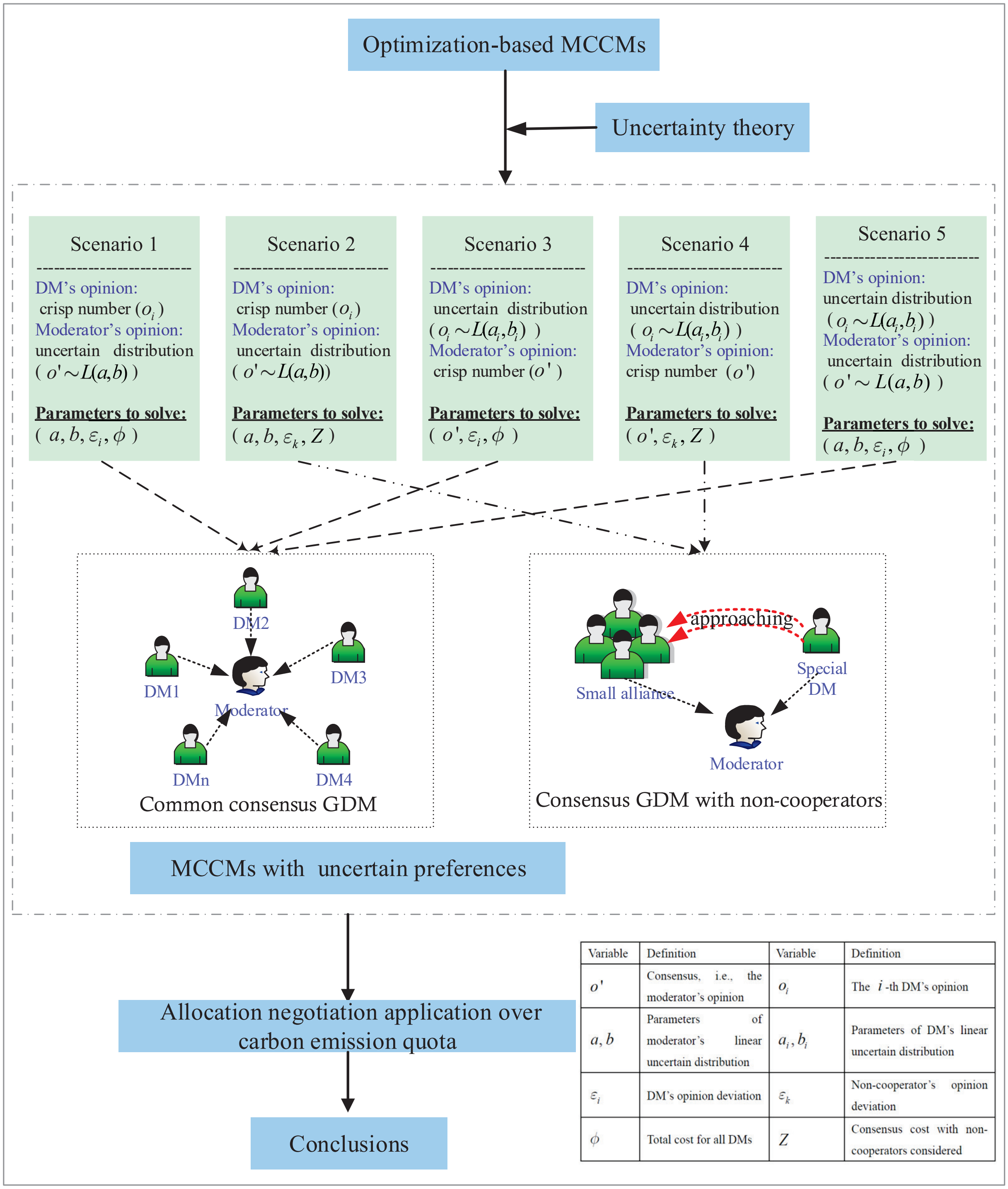
In specific, we differentiate all the GDM participants into three roles as moderator, individual DMs, and non-cooperators. Considering that participants usually have disparate standpoints or interests when facing real-life GDM, uncertain preferences will be accordingly expressed by different roles under various decision contexts. Thus, Section 3.1 and 3.2 assume that moderator's opinion is expressed as uncertain preference (denoted by belief degree and uncertainty distribution) while individuals present crisp number preferences, and then preference structures of those two roles are reversed in Section 3.3 and 3.4. Finally, in Section 3.5, participants involved in GDM all present their judgements by uncertain preferences. For more in line with real-life GDM problems, we also deeply explored the influence of non-cooperators in uncertain MCCMs in Section 3.2 and 3.4, simultaneously aiming to conduct an association research with previous MCCMs in [19].

4. Application in carbon quota negotiation

A negotiation abstracted from real-life GDM is conducted in this section, over the carbon emission quota issue between the government and four local heavily polluting enterprises, so as to further illustrate the validity of the above five uncertain chance-constrained MCCMs and the proposed theorems. In addition, this section also deeply investigates the relations between the newly constructed models with the traditional MCCMs through data analysis.

4.1. Research background

The fifth assessment report of the intergovernmental panel on climate change (IPCC) clearly states that global warming is intensified according to the observable data over the global surface temperatures and the rising sea levels, which is largely due to human activities [52]. Therefore, how to reduce the impact of human activities on the environment through greenhouse gas emission reduction has become a priority for entire mankind. Jiang et al. [53] proved that the most cost-efficient way to deal with



Variable	Definition	Variable	Definition
o'	Consensus, i.e., the moderator's opinion	o_i	The i-th DM's opinion
a, b	Parameters of moderator's linear uncertain distribution	a_i, b_i	Parameters of DM's linear uncertain distribution
ε_i	DM's opinion deviation	ε_k	Non-cooperator's opinion deviation
ϕ	Total cost for all DMs	Z	Consensus cost with non-cooperators considered

Fig. 3 Flowchart of MCCMs with uncertain preferences

global warming is to build a carbon market under which the key problem becomes to allocate carbon emission quota (hereafter carbon quota). "Carbon emission right" refers to the right of enterprises to legally discharge greenhouse gases such as carbon dioxide into the atmosphere according to relevant laws, and "carbon quota" refers to the legal amount for each enterprise within a certain period through some bargains with the government. Based on the concept of the carbon market, if the actual emission amounts of enterprises are more than their quotas, then they need to pay for extra quotas to make up their illegal amounts; conversely, if the actual emission amounts are smaller, then the balance can be sold out in the carbon market (http://www.tanpaifang.com). Given the above background, a negotiation is usually conducted over specific carbon quotas between the government and different enterprises, and thus, the allocation of carbon quotas for various enterprises is essentially a CRP.

How to allocate carbon quotas? Take a certain region as an example, local government can always provide a rough carbon quota allocation scheme for each enterprise, through comprehensively considering their historical emission data, advanced emission reduction measures, and future development strategies. Then, through bargain or negotiation, the government and all enterprises can reach a carbon quota consensus. In fact, enterprises are mostly profit-oriented and usually believe that environmental actions lead to financial cost increasing, because their proactive huge investment in green technology may not pay off for decades [54]. Therefore, it is relatively difficult for enterprises to provide an exact emission index. Although an exact emission number sometimes needs to be given by enterprises, it is highly likely to have some deception in that index from the view of enterprises' interest [49]. Acting as a macro-moderator, government needs to take into account both economic and social benefits and always stick in line with the principle of fairness and effectiveness, so as to help enterprises more accurately determine carbon quotas through multiple means (e.g. game, negotiation, and implementation of relevant administrative orders or incentives). Obviously, such carbon quota negotiations involving the government (i.e. moderator) and enterprises (i.e. DMs) can constitute a cost consensus GDM problem. As mentioned earlier, we will not discuss the economic benefits of enterprises resulting from their subsequent carbon quota transferring (i.e. trading behaviour in carbon market), which can be viewed as post-consensus decision-making problems [55], such as how to use tiered pricing after overrun for different heavily polluting enterprises.

4.2. Numerical discussion and sensitivity analysis

Assume four heavily polluting enterprises located in different regions within a same province, with similar qualifications and scales in the same industry, denoted as d_i, $i \in N = \{1, 2, 3, 4\}$. o_i is their original carbon quota (unit: 10,000 tons/year). We assume that to facilitate unified management, the provincial government needs to set a unified standard (i.e. an optimal carbon quota) for these enterprises. The optimal carbon quota negotiated above is not only the consensus reached but also the final value expected by the government, which can be marked as o'. For simplicity, we conduct comparative analysis with the data in [19, 28] (see Table 1). If there exists a non-cooperator, we might as well assume that $d_k = d_3$, and it holds a special position different from the others (e.g. d_3 is a pillar industry within its region, receiving special support from the government, while others aren't). Note that, by considering the moderator's preference on some specific factors, we can always easily identify such non-cooperators in real-life GDM.

Case 1. Assume that the original carbon quotas required by four enterprises are $o_1 = 0$, $o_2 = 3$, $o_3 = 6$, and $o_4 = 10$ (unit: 10,000 tons/year). To promote the allocation of optimal carbon quotas, unit costs that the provincial government is willing to pay are: $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, and $\omega_4 = 1$ (unit:10,000 yuan/ton). Here, we suppose that the optimal quota expected by the government (i.e. o')

Table 1 Summary of original decision information

Cases	(1) $o' \sim \mathcal{L}(a,b)$		(2) $o' \sim \mathcal{L}(a,b)$			(3) $o_i \sim \mathcal{L}(a_i,b_i)$		(4) $o_i \sim \mathcal{L}(a_i,b_i)$			(5) $o' \sim \mathcal{L}(a,b)$;$o_i \sim \mathcal{L}(a_i,b_i)$	
Variables	o_i	ω_i	o_i	ω_i	ε_i	o_i	ω_i	o_i	ω_i	ε_i	o_i	ω_i
d_1	0	1	0	-	5	[14,37]	1	[14,37]	-	12	[14,37]	1
d_2	3	2	3	-	4	[22,30]	2	[22,30]	-	5	[22,30]	2
d_3	6	3	6	3	-	[64,153]	3	[64,153]	3	-	[64,153]	3
d_4	10	1	10	-	6	[8,61]	1	[8,61]	-	36	[8,61]	1
Unknown	a,b,ε_i,ϕ		a,b,ε_3,Z			o',ε_i,ϕ		o',ε_3,Z			a,b,ε_i,ϕ	

Note: ϕ indicates the total consensus cost for all plants d_i, $(i \in N)$; Z indicates the consensus cost for the non-cooperator d_k. Unit for o_i, o': 10,000 tons/year; unit for w_i: 10,000 yuan/ton.

obeys a linear uncertainty distribution, represented as $o' \sim \mathcal{L}(a,b)$, where a and b are unknown.

$$Model(4-1): \ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$$
$$s.t. \begin{cases} M\{o' - 0 \le \varepsilon_1\} \ge \alpha, M\{o' - 0 \ge -\varepsilon_1\} \ge \alpha \\ M\{o' - 3 \le \varepsilon_2\} \ge \alpha, M\{o' - 3 \ge -\varepsilon_2\} \ge \alpha, \\ M\{o' - 6 \le \varepsilon_3\} \ge \alpha, M\{o' - 6 \ge -\varepsilon_3\} \ge \alpha, \\ M\{o' - 10 \le \varepsilon_4\} \ge \alpha, M\{o' - 10 \ge -\varepsilon_4\} \ge \alpha \\ o' \sim \mathcal{L}(a,b),\ \varepsilon_i \ge 0,\ i = 1,2,3,4 \end{cases}$$

Taking the negotiation cost initiated by the government to minimise as our main goal, an MCCM based on the carbon quota is constructed as Model (4-1), which is finally transformed as Model (4-11) (see Appendix D). Without regard to the random distribution for the consensus opinion, optimal solution obtained by [19] is $o^* = 6$ and $\phi^* = 16$. Table 2 provides the sensitivity results for Model (4-1) when the step length for the belief degree α is 0.1. In addition, Table 2 indicates that if and

Table 2 Sensitivity results for Case 1

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	1.11	2.5	3	4	6	6	6	6	6	6
ε_2	0	0	0	0	1	3	3	3	3	3	3
ε_3	0	0	0	0	0	0	0	0	0	0	0
ε_4	0	0	0	3	4	4	4	4	4	4	4
$o^* = [a,b]$	[0,10]	[0,11.11]	[0,12.5]	[0,10]	[0,10]	6	6	6	6	6	6
ϕ^*	0	1.11	2.5	6	10	16	16	16	16	16	16

only if $\alpha \ge 0.5$, the optimal carbon quota o^* reached by polluters and the government evolves from an uncertainty distribution to a single real value 6, and then, the consensus budget ϕ^* also reaches a stable level of 16. Moreover, when we take the belief degree α as an unknown variable, then its optimal solution solved by Model (4-1) is $\alpha^* = [0.5, 1]$. Compared to the results in [19], Corollary 2 holds. As polluters' original carbon quotas have already been ranked in an ascending order, and due to $\omega_1 + \omega_2 < \omega_3 + \omega_4$, $\omega_1 + \omega_2 + \omega_3 > \omega_4$, the optimal quota for minimising the objective function is $o^* = o_3 = 6$, so Theorem 5 is verified.

Combined with the research background, when the belief degree is rather low (i.e. $\alpha < 0.5$), the provincial government can only obtain a threshold for the optimal carbon quota. However, when the belief degree is no less than 0.5, the optimal carbon quota is constant at 60,000 tons/year, meaning that once the government's belief degree reaches a critical value (i.e. 0.5), the optimal quota will stabilize to a fixed value. Next, we economically explain the changes in carbon quotas for each polluter. Enterprises d_1 and d_2 require a relatively low quota at the beginning, possibly due to the overconfidence or lack of comprehensive verification of their emission capacity, but the government believes that they should get 60,000 tons of quota per year from the perspective of their previous emission situation or future development demands. As for d_4, a rather high quota is given on the account of excessive conservatism or the desire to obtain more economic subsidies from the government. However, d_4 should finally lower its emission standard to balance the environmental and economic benefits.

As we stated earlier, if the actual emission amounts of enterprises are less than the allocated quotas, the balance can produce certain economic benefits in the carbon market, which can be regarded as an incentive to promote emission reduction. Conversely, for polluters whose real emission amounts exceed the allocated quota, only by purchasing extra quotas, can they complete their business targets, and then, the transaction can be considered as a negative incentive. In short, irrespective of how much the polluters actually emit, the government can always achieve the emission reduction target by setting an optimal carbon quota.

Case 2. As shown in Table 1, enterprise d_3 acts as a non-cooperator, and the government provides it a unit negotiation cost as $\omega_3 = 3$ (unit: 10,000 yuan/ton). Meanwhile, to make the CRP more flexible, final optimal carbon quotas for other three polluters have soft- consensus thresholds as $\varepsilon_1 = 5$, $\varepsilon_2 = 4$ and $\varepsilon_4 = 6$ (unit: 10,000 tons/year). Assume the optimal quota o' obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$ by default. Aiming to minimise the total budget, an uncertain carbon quota MCCM is built as Model (4-2) and its linear equivalent form is Model (4-21) (see Appendix D). Ref [19] provided an optimal solution as $o^* = 5, Z^* = 3$ without considering DMs' opinions characterizing random distributions.

$$Model(4-2): \ Min \ Z = 3 * \varepsilon_3$$
$$s.t. \begin{cases} M\{o' - 6 \leq \varepsilon_3\} \geq \beta, M\{o' - 6 \geq -\varepsilon_3\} \geq \beta \\ M\{o' - 0 \leq \varepsilon_1\} \geq \alpha, M\{o' - 0 \geq -\varepsilon_1\} \geq \alpha \\ M\{o' - 3 \leq \varepsilon_2\} \geq \alpha, M\{o' - 3 \geq -\varepsilon_2\} \geq \alpha \\ M\{o' - 10 \leq \varepsilon_4\} \geq \alpha, M\{o' - 10 \geq -\varepsilon_4\} \geq \alpha \\ 0 \leq \varepsilon_1 \leq 5, \ 0 \leq \varepsilon_2 \leq 4, \ 0 \leq \varepsilon_4 \leq 6, \ \varepsilon_3 \geq 0 \\ o' \sim \mathcal{L}(a, b), a \leq b, \ 0 \leq \alpha, \ \beta \leq 1 \end{cases}$$

In fact, Case 2 introduces the soft-consensus constraints based on Case 1, and analyses the consensus GDM with only considering some non-cooperators instead of all DMs. Table 3 provides sensitivity results for the variable α in Model (4-2) when another belief degree set for the non-cooperator d_3 is fixed as $\beta = 0.6$. For detailed analysis, we identify the consensus reaching conditions with all DMs' carbon quotas being crisp numbers, while the emission index for the local government obeys a linear uncertainty distribution. Namely, we draw conclusions by adapting the belief degree α within the interval of [0,1] for the other three polluters during the carbon quota negotiation. Through calculation, we find that once the soft-consensus thresholds are given in advance, the optimal values for variables $\varepsilon_{i, \ i \neq k, \ i \in N}$ always take the upper limits as $\varepsilon_1^* = 5, \varepsilon_2^* = 4$ and $\varepsilon_4^* = 6$, so these values are omitted in Table 3.

Table 3 Sensitivity results for Case 2.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
β	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
ε_3	0.2	0.25	0.33	0.5	1	1	1	1	1	1	1
$o^* = [a, b]$	[5,7]	[4.75,7.25]	[4.33,7.67]	[3.5,8.5]	[1,11]	5	5	5	5	5	5
Z^*	0.6	0.75	1	1.5	3	3	3	3	3	3	3

Table 3 indicates that when $\alpha \geq 0.5$, the optimal carbon quota becomes a fixed constant from an uncertainty distribution. $\forall i \in N, i \neq k$, $max(o_i - \varepsilon_i) = 4$, $min(o_i + \varepsilon_i) = 5$, we have $o_{k=3} = 6 > min(o_i + \varepsilon_i)$. Therefore, when $\alpha \leq 0.5$, we obtain the optimal value for ε_k^* and further obtain accurate values for a and b, by comparing the sizes of $(\beta - 0.5)(b - a)$ and $o_k - min(o_i + \varepsilon_i)$. Taking the situation of $\alpha = 0.3, \beta = 0.6$ as an example, we have $a = [min(o_i + \varepsilon_i) - 2\alpha * o_k]/(1 - 2\alpha) = (5 - 2 * 0.3 * 6)/(1 - 2 * 0.3) =$

3.5 and $b = [(2-2\alpha)o_k - min(o_i+\varepsilon_i)]/(1-2\alpha) = [(2-2*0.3)*6-5]/(1-2*0.3) = 8.5$. Thus, $\varepsilon_k^* = min\{(\beta-0.5)(b-a), o_k - min(o_i+\varepsilon_i)\} = 0.5$. Meanwhile if $\alpha \geq 0.5$, for $o_k = 6 > min(o_i+\varepsilon_i)$ always holds, thereby $a^* = b^* = 5$, $Z^* = \omega_3 * (o_3 - min(o_i+\varepsilon_i)) = 3$. Obviously, above calculations are in accordance with the data in Table 3, so Theorem 6 holds.

Owing to the randomness of data selection, Case 2 only validates the conclusion of $o_k > min(o_i+\varepsilon_i)$. By adjusting d_3's original carbon quota, the rest of Theorem 6 can always be validated by a similar mechanism.

Case 3. Initial emission quotas of the four polluters are listed in Table 1. The local government, for obtaining an optimal allocation with unified standards, provides each enterprise a unit cost as $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, and $\omega_4 = 1$ (unit:10,000 yuan/ton). Here, the final collective carbon quota is defaulted as a crisp number, while the original emission indexes for the four polluters obey linear uncertainty distributions. Thus, an uncertain MCCM is constructed as Model (4-3), whose optimal solution is solved by Model (4-31) (see Appendix D).

$$
\begin{aligned}
&Model(4-3): \ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4 \\
&s.t. \begin{cases} M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\ M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\ o_1 \sim \mathcal{L}(14,37), o_2 \sim \mathcal{L}(22,30), \\ o_3 \sim \mathcal{L}(64,153), o_4 \sim \mathcal{L}(8,61) \\ o' \geq 0, \ 0 \leq \alpha \leq 1, \ \varepsilon_i \geq 0, \ i = 1,2,3,4 \end{cases}
\end{aligned}
$$

With no uncertain chance-constraints, an optimal solution presented in [28] is $o^* = [37,61], \phi^* = 95$. Previous work only regards DMs' opinions as intervals, without considering the characteristics of opinions obeying random distributions.

Table 4 Sensitivity results for Case 3.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	0	0	0	1.4	9	11.3	13.6	15.9	18.2	20.5
ε_2	7	5.5	4	2.5	2.4	8.5	9.3	10.1	10.9	11.7	12.5
ε_3	27	38.2	49.4	60.6	70.4	74	82.9	91.8	100.7	109.6	118.5
ε_4	0	0	0	0	0	0	5.3	10.6	15.9	21.2	26.5
o^*	[37,61]	[34.7,55.7]	[32.4,50.4]	[30.1,45.1]	[29.2,39.8]	34.5	34.5	34.5	34.5	34.5	34.5
ϕ^*	95	125.6	156.2	186.8	217.4	248	283.9	319.8	355.7	391.6	427.5

When belief degree α is an unknown parameter, we obtain $\alpha^* = 0$ and $o^* = [37,61], \phi^* = 95$ (see Column 2 in Table 4), corresponding to the optimal solution in [28]. Note that, keeping $\alpha^* = 0, \phi^* = 95$ constant, the variable o^* can be any optimal value within the interval [37,61] and variables $\varepsilon_{i,\ i\in N}$ will change with the specific value of o^*. Due to $\varepsilon_{i,\ i\in N}$ have no effect on the final results, relevant analysis is omitted in the paper.

Table 4 intuitively shows that once the uncertain belief degree of the carbon quota negotiation satisfies $\alpha \geq 0.5$, the optimal consensus expected by the moderator becomes a real value from an uncertainty distribution. As original emission indexes of the four DMs haven't been ranked in an ascending order, Table 5 gives the updated values to conveniently verify the effectiveness of relevant theorems. Since $\omega_{(1)} + \omega_{(2)} < \omega_{(3)} + \omega_{(4)}$, $\omega_{(1)} + \omega_{(2)} + \omega_{(3)} > \omega_{(4)}$, the optimal carbon quota is obtained as $o^* = \frac{a_{(3)}+b_{(3)}}{2} = \frac{69}{2} = 34.5$. Then, using the analytic formula of the objective function in Theorem 7, values of ϕ^* calculated are identical with the data in Table 4. Thus, Theorem 7 holds.

Case 4. Similar as in Case 2, d_3 is assumed as the non-cooperating enterprise. Data of polluters' original carbon quotas, unit cost for d_3 as well as the soft-consensus thresholds for the other three enterprises are all listed in Table 1. Here, the consensus o' obtained for the government is defaulted as a crisp number, while DMs' preferences obey linear uncertainty distributions. Then, Model (4-4) is built

Table 5 Updated opinions in Case (3-5) with an ascending order

Updated	Origin	a_i	b_i	$a_i + b_i$	weight
$o_{(1)}$	o_1	14	37	51	1
$o_{(2)}$	o_2	22	30	52	2
$o_{(3)}$	o_4	8	61	69	1
$o_{(4)}$	o_3	64	153	217	3

Note: o_i represents the opinion for the original i-th DM;
$o_{(i)}$ represents the opinion for the updated i-th DM in an ascending order.

with great emphasis on d_3's interest. Table 6 and Table 7 are obtained by solving Model (4-41) (see Appendix D).

$$Model(4-4):\ Min\ Z = 3 * \varepsilon_3$$
$$s.t.\begin{cases} M\{o' - o_3 \leq \varepsilon_3\} \geq \beta, M\{o' - o_3 \geq -\varepsilon_3\} \geq \beta \\ M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha,\ i = 1, 2, 4 \\ o_1 \sim \mathcal{L}(14, 37), o_2 \sim \mathcal{L}(22, 30), \\ o_3 \sim \mathcal{L}(64, 153), o_4 \sim \mathcal{L}(8, 61) \\ 0 \leq \varepsilon_1 \leq 12,\ 0 \leq \varepsilon_2 \leq 5,\ 0 \leq \varepsilon_4 \leq 36 \\ o' \geq o,\ \varepsilon_3 \geq 0 \end{cases}$$

Table 6 provides the sensitivity results over the belief degree α, which is set for the small alliance (including polluters d_1, d_2 and d_4), meantime, the belief degree imposed on o_3 and o' is fixed as $\beta = 0.75$. Table 7 gives the sensitivity results over the belief degree β for d_3. Similar in Case 2, once $\varepsilon_{i, i\in N, i\neq k}$ are pre-defined, their optimal values are exactly the pre-determined upper limits, so values of $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_4^*$ are omitted in Table 6 and Table 7.

Table 6 Sensitivity results for Case 4 on α when $\beta = 0.75$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
β	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
ε_3	95.75	96.55	97.35	98.15	98.95	99.75	100.55	101.35	102.15	102.95	104.75
o^*	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	26
z^*	287.25	289.65	292.05	294.45	296.85	299.25	301.65	304.05	306.45	308.85	314.25
$i = 1$	2	4.3	6.6	8.9	11.2	13.5	15.8	18.1	20.4	22.7	25
$i = 2$	17	17.8	18.6	19.4	20.2	21	21.8	22.6	23.4	24.2	25
$i = 4$	-28	-22.7	-17.4	-12.1	-6.8	-1.5	3.8	9.1	14.4	19.7	25
$G = Max$	17	17.8	18.6	19.4	20.2	21	21.8	22.6	23.4	24.2	25
$i = 1$	49	46.7	44.4	42.1	39.8	37.5	35.2	32.9	30.6	28.3	26
$i = 2$	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	27
$i = 4$	97	91.7	86.4	81.1	75.8	70.5	65.2	59.9	54.6	49.3	44
$H = Min$	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	26

Note: $G = Max\{(1-\alpha) * a_i + \alpha * b_i - \varepsilon_i\}$, $H = Min\{\alpha * a_i + (1-\alpha) * b_i + \varepsilon_i\}$, $i \in N, i \neq k$

Table 7 Sensitivity results for Case 4 on β when $\alpha = 0.86$.

α	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86
β	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_3	35.88	44.78	53.68	62.58	71.48	80.38	89.28	98.18	107.08	115.98	124.88
o^*	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12
Z^*	107.64	134.34	161.04	187.74	214.44	241.14	267.84	294.54	321.24	347.94	374.64

Values of the total cost and the consensus calculated by the analytic formula in Theorem 8 are exactly the same as the data in Table 6 and Table 7. However, detailed analysis for Case 4 is omitted here due to space limitation. Note, Case 4 only validates part of the conclusions in Theorem 8, but by adjusting the value of o_3, the remaining parts can also be verified.

Case 5. Assume all participants involved in this negotiation obey linear uncertainty distributions. Relevant data is provided as Table 1, so an uncertain chance-constrained MCCM is constructed as Model (4-5) and its equivalent linear transformation is Model (4-51) (see Appendix D).

$$Model(4-5): \ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$$
$$s.t. \begin{cases} M\{o' - o_i \le \varepsilon_i\} \ge \alpha, \ i = 1,2,3,4 \\ M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha, \ i = 1,2,3,4 \\ o' \sim \mathcal{L}(a,b), o_1 \sim \mathcal{L}(14,37), o_2 \sim \mathcal{L}(22,30), \\ o_3 \sim \mathcal{L}(64,153), o_4 \sim \mathcal{L}(8,61) \\ a \le b, \ \ 0 \le \alpha \le 1, \ \varepsilon_i \ge 0, \ i = 1,2,3,4 \end{cases}$$

By solving Model (4-51), optimal solutions under different belief degrees are obtained as Table 8. Simultaneously, Table 8 provides the changes for both the optimal carbon quota o^* and the optimal total cost ϕ^* with the variable α.

Table 8 Sensitivity results for Case 5.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	0	0	8.77	38.6	9	11.3	13.6	15.9	18.2	20.5
ε_2	0	0	0	11.27	39.6	8.5	9.3	10.1	10.9	11.7	12.5
ε_3	0	0	0	0	0	74	82.9	91.8	100.7	109.6	118.5
ε_4	0	0	0	0	26.6	0	5.3	10.6	15.9	21.2	26.5
$o^* = [a,b]$	[0,64]	[0,81]	[0,102.25]	[0,129.57]	[0,166]	34.5	34.5	34.5	34.5	34.5	34.5
ϕ^*	0	0	0	31.31	144.4	248	283.9	319.8	355.7	391.6	427.5

Once $\alpha \ge 0.5$, the optimal allocation values evolves from an uncertainty distribution to the real value of 34.5. Similar as the analysis in Case 3, $o^* = \frac{a_{(3)}+b_{(3)}}{2} = \frac{69}{2} = 34.5$ is obtained by referring to Table 5. Negotiation costs calculated by the analytic expression of ϕ^* are in accordance with the data in Table 8. Taking $\alpha = 0.7$ as an example, $\phi^* = (\omega_{(1)} + \omega_{(2)} + \omega_{(3)} - \omega_{(4)}) * 34.5 + \omega_{(4)} * (0.3*64 + 0.7*153) - \omega_{(1)} * (0.7*14 + 0.3*37) - \omega_{(2)} * (0.7*22 + 0.3*30) - \omega_{(3)} * (0.7*8 + 0.3*61) = 319.8$. So Theorem 9 holds.

4.3. Comparison and discussion

Due to serious deterioration of the global environment, the reduction of carbon emission has become a key measure to improve the ecological system, so we choose the application in carbon quota negotiation to verify the feasibility of the proposed models. Results show that the calculated values correspond to the analytic formulas of the optimal solutions under each scenario, verifying the correctness of the theorems obtained by theoretical deduction. Moreover, findings in the application indicate that traditional crisp number- or interval preference-based MCCMs are some special cases of the new uncertain MCCMs, suggesting that uncertainty theory can build a bridge between deterministic and indeterministic GDM. Finally, we find that once the belief degree, set for the deviation of polluters' and government's quota indexes, is larger than the critical value of 0.5, then the optimal carbon quota consensus will be crisp numbers and no longer obey uncertainty distributions. The above conclusion implies that only belief degree is large enough, GDM can achieve a deterministic consensus and the carbon quota negotiation can then be effectively conducted to some extent.

To illustrate the novelty of our research, we conduct a comparative analysis (see Table 9). Distinguished from previous research, we build the consensus models from three decision roles, by introducing non-cooperators into traditional MCCMs. Meanwhile, we first introduce Liu's uncertainty theory into consensus modeling, by adopting belief degree and uncertainty distribution as a whole to fit individual preferences, and find out the relations between the deterministic and indeterministic GDM through theoretical deduction. Finally, we apply the proposed models into the carbon emission quota allocation negotiation problem to verify their feasibility. However, it is undeniable that some important contributions in relevant MCCMs/ MACMs may be neglected in this paper, such as setting an aggregation function over the adjusted individual opinions to obtain a consensus [2, 10, 27], using consensus level to measure the efficiency of CRP [2, 10], or considering the asymmetric characteristic of unit costs [5].

Table 9 Comparative analysis on relevant MCCMs/ MACMs.

Consensus models	**Decision roles**	**DM's preference**	**Application**
This paper	Moderator; individual DMs; non-cooperators	Uncertainty distributions and belief degree	Carbon emission quota allocation
Ben-Arieh and Easton [20]	Moderator; individual DMs	Crisp numbers	None-numerical examples
Dong et al. [10]	Individual DMs	Linguistic preferences	None-numerical examples
Zhang et al. [27]	Individual DMs	Crisp numbers	Apartment selection
Gong et al. [28]	Moderator; individual DMs	Interval preferences	None-numerical examples
Gong et al. [19]	Moderator; individual DMs	Crisp numbers	None-numerical examples
Zhang et al. [2]	Moderator; individual DMs	Crisp numbers	Loan consensus problems in Online P2P lending
Cheng et al. [5]	Moderator; individual DMs	Crisp numbers	Trans-boundary pollution control

Inspired by the fact that flexible management has been a premiere goal pursued by Chinese government, in order to encourage high-quality development of enterprises, the negotiation over the carbon emission quota allocation problem is chosen as our case background. In fact, when setting carbon emission reduction quotas for different enterprises with similar scales, it can better reflect the government's humanized management by setting uncertain indicators rather than some deterministic and fixed ones, which may also be understood as the practical significance of the uncertainty constraints in this paper. Without doubt, our newly proposed uncertain MCCMs can provide significant managerial implications for moderators to deal with real-life GDM problems with flexible requirements, such as targeted recommendation system purchasing based on advertisers' market share, and second-hand housing selection bargain from different agencies.

5. Conclusion

Compared to traditional deterministic preferences, fitting DMs' preferences with uncertainty distributions is more suitable for real-life decision-making contexts, especially for complex GDM. In this paper, linear uncertainty distributions are adopted to fit individual judgements, and a series of uncertain MCCMs are proposed. Through transformation into equivalent linear programming models, the analytic formulas of the optimal consensus and minimum total cost under each scenario are given in the paper. We find out that the uncertain preference-based MCCMs are more inclusive than those traditional ones, in other words, the basic conclusions of the crisp number- or interval preference-based models are some special cases of uncertain MCCMs under different belief degrees, thus our research is more flexible in actual GDM. In addition, optimal solution of each uncertain chance-constrained

MCCM is theoretically provided, and the feasibility of the proposed models are further verified with the application in carbon quota negotiation between enterprises and the local government.

Main contributions of this paper are as follows. Firstly, this paper builds the optimisation-based MCCMs from multiple decision roles (i.e., the moderator, individual DMs, and non-cooperators). Secondly, belief degree and uncertainty distributions are used as a whole to simulate DMs' preference structure, making the new models more feasible than those traditional ones (i.e., crisp number- or interval preference-based MCCMs), better avoiding the paradox in interval operations (e.g. $[1,3]-[1,3]=[-2,2]\neq[0,0]$), and maintaining the integrity of decision information by analyzing individual uncertain opinions as a whole instead of only endpoints being considered. Thirdly, consensus reaching conditions under different GDM scenarios are presented through mathematical deduction. By taking the application in carbon quota negotiation, the proposed models are verified as a more general paradigm of the traditional MCCMs.

This paper explores consensus reaching conditions in homogenous GDM, but real-life decision is rather complex and changeable, making it highly possible for involved participants to simultaneously present completely different preference structures. So, in the future, we may deal with heterogeneous GDM problems [14, 56] by modeling non-linear uncertain chance-constrained MCCMs. At present, unit costs attached to DMs are subjectively given in the paper, afterwards, we may adopt some robust methods, such as game theory [46], to assure those parameters to be more reasonable. In specific, we may need to set variable unit costs for DMs to deal with the situation under which tiered pricing is set for heavily polluting enterprises after overrun. Finally, in this paper, we aim to figure out how an optimal consensus can be reached within each certain stage of the whole GDM process under the uncertain chance constraints, neglecting the dynamic characteristics for the whole process, which are definitely of great significance for GDM, so our subsequent research may also focus on dynamic uncertain MCCMs with feedback mechanism [6, 30, 57] or social interactions [12, 29, 49].

Acknowledgements

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Appendix A. Proof of Theorem 5.

Proof. Without loss of generality, we suppose n DMs participate in GDM, and their decisions satisfy $o_1\leq o_2\leq\cdots\leq o_m\leq o_{m+1}\leq\cdots\leq o_n$. Furthermore, let $o_m\leq\frac{a+b}{2}\leq o_{m+1}$, then when $\alpha\geq 0.5$, the constraints of Model (5) are simplified as

$$\Phi^{-1}(\alpha)-o_i\geq\tfrac{a+b}{2}-o_i\geq 0\quad(i=1,2,\cdots m);$$
$$-\Phi^{-1}(1-\alpha)+o_i\geq-\tfrac{a+b}{2}+o_i\geq 0\quad(i=m+1,\cdots n).$$

At this point, the objective function satisfies

$$\begin{aligned}min\sum_{i=1}^{n}\omega_i\varepsilon_i \quad &= \quad \omega_1(\Phi^{-1}(\alpha)-o_1)+\cdots+\omega_m(\Phi^{-1}(\alpha)-o_m)\\ &\quad\quad +\omega_{m+1}(-\Phi^{-1}(1-\alpha)+o_{m+1})+\cdots\\ &\quad\quad +\omega_n(-\Phi^{-1}(1-\alpha)+o_n)\\ &\geq \quad \sum_{i=m+1}^{n}\omega_io_i-\sum_{i=1}^{m}\omega_io_i+\tfrac{a+b}{2}(\sum_{i=1}^{m}\omega_i-\sum_{i=m+1}^{n}\omega_i)\end{aligned}$$

If and only if $\Phi^{-1}(\alpha) = \Phi^{-1}(1-\alpha) = \frac{a+b}{2}$, the above inequality takes the mark of equality, then we obtain $a = b$ and $min \sum_{i=1}^{n} \omega_i \varepsilon_i = \sum_{i=m+1}^{n} \omega_i o_i - \sum_{i=1}^{m} \omega_i o_i + a(\sum_{i=1}^{m} \omega_i - \sum_{i=m+1}^{n} \omega_i)$.

Next, the optimal analytic expression of the objective function is derived by comparing $\sum_{i=1}^{m} \omega_i$ and $\sum_{i=m+1}^{n} \omega_i$.

- If $\sum_{i=1}^{m} \omega_i = \sum_{i=m+1}^{n} \omega_i$, then $a = b$ can be any value in the interval of $[o_m, o_{m+1}]$, and $\phi^* = \sum_{i=m+1}^{n} \omega_i o_i - \sum_{i=1}^{m} \omega_i o_i$.
- If $\sum_{i=1}^{m-1} \omega_i < \sum_{i=m}^{n} \omega_i$ and $\sum_{i=1}^{m} \omega_i > \sum_{i=m+1}^{n} \omega_i$, because $\omega_{i,i \in N}$ are positive constants, the objective function ϕ first decreases and then increases with the variable a. Thus, when $a = b = o_m$, the optimal value for the objective function will be $\phi^* = \sum_{i=m+1}^{n} \omega_i (o_i - o_m) + \sum_{i=1}^{m} \omega_i (o_m - o_i)$.

This completes the proof for Theorem 5. □

Appendix B. Proof of Theorem 6.

Proof. Theorem 6 is derived in two steps: (1) Determination of the analytic formula of the objective function $min\ Z = \omega_k \varepsilon_k(a, b)$. As the parameter ω_k is pre-defined, only the formula of $\varepsilon_k^* = min\ \varepsilon_k(a, b)$ actually needs solving. Then (2), determination of the optimal solutions for variables a, b and ε_k^*.

Part 1. Determination of $\varepsilon_k^* = min\ \varepsilon_k(a, b)$.

As the value of ε_k only depends on the constraints of (9-1) and (9-2), let $A = (1-\beta)a + \beta b - o_k$ and $B = -\beta a + (\beta - 1)b + o_k$. Compared to the sizes of A and B, the following three situations are discussed:

- Case 1: If $A = B$, then $a + b = 2o_k$, making $\varepsilon_k \geq A = (\beta - \frac{1}{2})(b - a)$ hold.
  - If $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k^* = (\beta - \frac{1}{2})(b - a)$;
  - If $0 \leq \beta \leq \frac{1}{2}$, then $(\beta - \frac{1}{2})(b - a) \leq 0$; also due to $\varepsilon_k \geq 0$, then $\varepsilon_k^* = 0$ is obtained.
- Case 2: if $A > B$, we obtain $a + b > 2o_k$.
  - If $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k = A \geq \frac{a+b}{2} - o_k$, so if and only if $a = b$, the above inequality takes the mark of equality, making $\varepsilon_k^* = \frac{a+b}{2} - o_k$;
  - If $0 \leq \beta \leq \frac{1}{2}$, we obtain $\varepsilon_k = A \leq \frac{a+b}{2} - o_k$. However, due to $\varepsilon_k \geq 0$ and $\frac{a+b}{2} - o_k > 0$, we get $0 \leq \varepsilon_k \leq \frac{a+b}{2} - o_k$. Thus, $\varepsilon_k^* = 0$.
- Case 3: If $A < B$, we obtain $a + b < 2o_k$.
  - If $\frac{1}{2} \leq \beta \leq 1$, we obtain $\varepsilon_k = B \geq o_k - \frac{a+b}{2}$, so if and only if $a = b$, the above inequality takes the mark of equality, then $\varepsilon_k^* = o_k - \frac{a+b}{2}$ holds;
  - If $0 \leq \beta \leq \frac{1}{2}$, we get $\varepsilon_k = B \leq o_k - \frac{a+b}{2}$; by taking both $\varepsilon_k \geq 0$ and $o_k - \frac{a+b}{2} > 0$ into consideration, we obtain $0 \leq \varepsilon_k \leq o_k - \frac{a+b}{2}$. Thus, $\varepsilon_k^* = 0$.

Above all,

$$\varepsilon_k^* = \begin{cases} (\beta - \frac{1}{2})(b-a), iff\ \beta \in [\frac{1}{2}, 1], a+b = 2o_k & (B1-1) \\ \frac{a+b}{2} - o_k, iff\ \beta \in [\frac{1}{2}, 1], a = b, a+b > 2o_k & (B1-2) \\ o_k - \frac{a+b}{2}, iff\ \beta \in [\frac{1}{2}, 1], a = b, a+b < 2o_k & (B1-3) \\ 0, iff\ \beta \in [0, \frac{1}{2}] & (B1-4) \end{cases} \tag{B1}$$

Part 2. Determination of optimal solutions for a, b and ε_k^*.

As the analytic formula for $min\ \varepsilon_k$ is determined in Part 1 and the original opinion o_k for the non-cooperator d_k with greater influence is known in advance, the value of ε_k^* mainly depends on those of $a+b$ and $b-a$. Without loss of generality, let $a+b=m$ and $b-a=n$, then we have $a=\frac{m-n}{2}$ and $b=\frac{m+n}{2}$. By substituting constraints (9-3) and (9-4), we get

$$\begin{cases} (1-\alpha)\cdot\frac{m-n}{2} + \alpha\cdot\frac{m+n}{2} - o_i \le \varepsilon_i \\ -\alpha\cdot\frac{m-n}{2} + (\alpha-1)\cdot\frac{m+n}{2} + o_i \le \varepsilon_i \end{cases} \tag{B2}$$

After simplifying the inequality (B2), $\forall i \in N, i \neq k$, the range of m is solved as

$$(2\alpha-1)n + 2(o_i - \varepsilon_i) \le m \le 2(o_i + \varepsilon_i) - (2\alpha-1)n \tag{B3}$$

By fully considering the values of o_i, ε_i, and n, the above formula is equivalent to

$$min\ [max\ 2(o_i - \varepsilon_i) + (2\alpha-1)n] \le m \le max\ [min\ 2(o_i + \varepsilon_i) - (2\alpha-1)n] \tag{B4}$$

Taking both the actual GDM and the construction mechanism of uncertainty theory into account, opinions of the non-cooperator d_k with great influence cannot be ignored, so the belief degree β for d_k's original opinion o_k and the finally reached consensus o' should satisfy the condition of $\beta \ge \frac{1}{2}$. Then, the CRP makes sense. Based on the conclusion derived from Part 1, the optimal value of the objective function is always equal to zero when $0 \le \beta \le \frac{1}{2}$. Thus, only the situation of $\frac{1}{2} \le \beta \le 1$ will be discussed below. For simplicity, let $E = max\ 2(o_i - \varepsilon_i) + (2\alpha-1)n$ and $F = min\ 2(o_i + \varepsilon_i) - (2\alpha-1)n$.

Situation 1: When $0 \le \alpha \le \frac{1}{2}$ and $\frac{1}{2} \le \beta \le 1$, we get $2\alpha - 1 \le 0$, considering $n = b - a \ge 0$, so E monotonically decreases with respect to n, while F monotonically increases with respect to n. On account of inequality constraints (B4), if $\exists a, b,\ 0 \le a \le b$ such that $a + b = 2o_k$, then $min\ \varepsilon_k$ exists, satisfying $\varepsilon_k^* = (\beta - \frac{1}{2})(b-a)$. In view of α, the optimal values of a and b are gained from three scenarios.

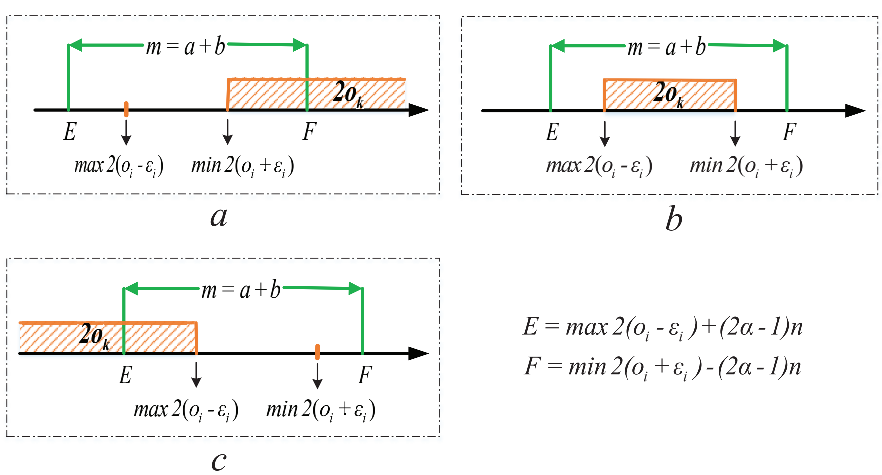

Fig. B1 Discussion on $0 \le \alpha \le \frac{1}{2}$ and $\frac{1}{2} \le \beta \le 1$

- As shown in Fig. B1(a), when $o_k > min\ (o_i+\varepsilon_i)$, then $2o_k > min\ 2(o_i+\varepsilon_i)$ is obtained.

  1. If $2o_k \le F$, then $min2(o_i+\varepsilon_i) < 2o_k \le min2(o_i+\varepsilon_i) - (2\alpha-1)n$, that is, when $n \ge \frac{2o_k - min2(o_i+\varepsilon_i)}{1-2\alpha}$, then $a+b = 2o_k$ holds; thus, $min\varepsilon_k = (\beta - \frac{1}{2})(b-a) = (\beta-\frac{1}{2})n$ is derived. Obviously, ε_k is a monotonically increasing function of n, so once n takes the minimum value, namely $n = b-a = \frac{2o_k - min2(o_i+\varepsilon_i)}{1-2\alpha}$, ε_k^* exists. On the basis of the formulas of $a+b$ and $b-a$, we get

  $$\begin{cases} a = \frac{min\ (o_i+\varepsilon_i) - 2\alpha o_k}{1-2\alpha} \\ b = \frac{(2-2\alpha)o_k - min\ (o_i+\varepsilon_i)}{1-2\alpha} \\ \varepsilon_k^* = (\beta-\frac{1}{2})(b-a) \end{cases}$$

  2. If $2o_k > F$, then $a+b < 2o_k$, such that $min\varepsilon_k = o_k - \frac{a+b}{2}$. Obviously, ε_k monotonically decreases with the variable of $(a+b)$. Therefore, once $a+b$ takes the maximum value, namely $a+b = m = F = min\ 2(o_i+\varepsilon_i) - (2\alpha-1)n$, then ε_k^* exists and $a=b$ holds. In other words, $n=0$ is obtained. Thus, if $a=b=min\ (o_i+\varepsilon_i)$, the optimal objective function will be $\varepsilon_k^* = o_k - min\ (o_i+\varepsilon_i)$.

  As a result, when $o_k > min\ (o_i+\varepsilon_i)$, we obtain $\varepsilon_k^* = min\ \{(\beta-\frac{1}{2})(b-a), o_k - min\ (o_i+\varepsilon_i)\}$, where if $\varepsilon_k^* = (\beta-\frac{1}{2})(b-a)$, then $a = \frac{min\ (o_i+\varepsilon_i)-2\alpha o_k}{1-2\alpha}$ and $b = \frac{(2-2\alpha)o_k - min\ (o_i+\varepsilon_i)}{1-2\alpha}$; if $\varepsilon_k^* = o_k - min\ (o_i+\varepsilon_i)$, then $a=b=min\ (o_i+\varepsilon_i)$ holds.

- As shown in Fig. B1(b), $max(o_i-\varepsilon_i) \le o_k \le min(o_i+\varepsilon_i)$, for $n = b-a \ge 0$, so $2o_k \in [max2(o_i-\varepsilon_i), min2(o_i+\varepsilon_i)] \subseteq [E,F]$. Clearly, $\exists m$ such that $m = a+b = 2o_k$. That is, when $min\varepsilon_k = (\beta-\frac{1}{2})(b-a) = (\beta-\frac{1}{2})n$, then $n = b-a = 0$ holds. Thus, if $a=b=o_k$, we obtain $\varepsilon_k^* = 0$.

- As shown in Fig. B1(c), $o_k < max(o_i-\varepsilon_i)$, so $2o_k < max2(o_i-\varepsilon_i)$.

  1. If $2o_k \ge E$, then $max2(o_i-\varepsilon_i) + (2\alpha-1)n \le 2o_k < max(o_i-\varepsilon_i)$, and we have $n \ge \frac{max2(o_i-\varepsilon_i)-2o_k}{1-2\alpha}$. Because $min\varepsilon_k = (\beta-\frac{1}{2})(b-a) = (\beta-\frac{1}{2})n$ can be obtained when $a+b=2o_k$, obviously, once n takes the minimum value, namely when $n = \frac{max2(o_i-\varepsilon_i)-2o_k}{1-2\alpha} = b-a$, ε_k^* exists. Due to the formulas of $a+b$ and $b-a$, we obtain

  $$\begin{cases} a = \frac{max\ (o_i-\varepsilon_i)+(2-2\alpha)o_k}{1-2\alpha} \\ b = \frac{max\ (o_i-\varepsilon_i)-2\alpha o_k}{1-2\alpha} \\ \varepsilon_k^* = (\beta-\frac{1}{2})(b-a) \end{cases}$$

  2. If $2o_k < E$, then $a+b > 2o_k$, so we get $min\ \varepsilon_k = \frac{a+b}{2} - o_k$. Obviously, ε_k is increasing with $(a+b)$. Thus, if $a+b$ takes the minimum value, that is, $a+b = m = E = max\ 2(o_i-\varepsilon_i) + (2\alpha-1)n$, ε_k^* exists. Based on the constraint of (B1-2), $a=b$ holds (i.e. $n=0$). Therefore, once $a=b=max\ (o_i-\varepsilon_i)$, we always have $\varepsilon_k^* = max\ (o_i-\varepsilon_i) - o_k$.

  As a result, if $o_k < max\ (o_i-\varepsilon_i)$, $\varepsilon_k^* = min\ \{(\beta-\frac{1}{2})(b-a), max\ (o_i-\varepsilon_i)-o_k\}$, where if $\varepsilon_k^* = (\beta-\frac{1}{2})(b-a)$, we have $a = \frac{max\ (o_i-\varepsilon_i)+(2-2\alpha)o_k}{1-2\alpha}$ and $b = \frac{max\ (o_i-\varepsilon_i)-2\alpha o_k}{1-2\alpha}$; if $\varepsilon_k^* = max\ (o_i-\varepsilon_i)-o_k$, we get $a=b=max\ (o_i-\varepsilon_i)$.

Situation 2: When $\frac{1}{2} \le \alpha \le 1$ and $\frac{1}{2} \le \beta \le 1$, we obtain $2\alpha - 1 \ge 0$, as $n = b-a \ge 0$, making E monotonically increase with respect to n and F monotonically decrease with respect to n. Then, we divide a similar discussion into three scenarios.

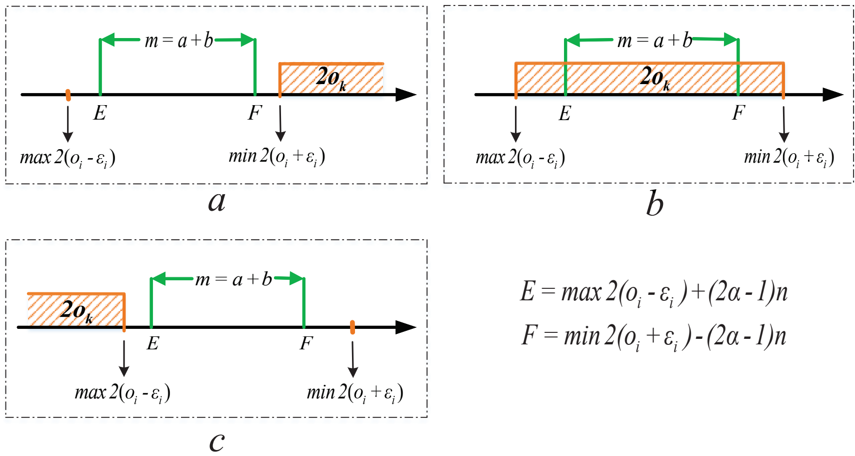

Fig. B2 Discussion on $\frac{1}{2} < \alpha \leq 1$ and $\frac{1}{2} \leq \beta \leq 1$.

- As shown in Fig. B2(a), if $o_k > min\ (o_i + \varepsilon_i)$, then $2o_k > min\ 2(o_i + \varepsilon_i)$; thus, $m = a + b \leq min\ 2(o_i + \varepsilon_i) - (2\alpha - 1)n \leq min\ 2(o_i + \varepsilon_i) < 2o_k$ and then $min\ \varepsilon_k = o_k - \frac{a+b}{2}$. Obviously, ε_k is a monotonically decreasing function of $(a + b)$. In view of formula (B1-3) in Part 1, as $a = b$ and $a + b = min\ 2(o_i + \varepsilon_i)$, we have $a = b = min\ (o_i + \varepsilon_i)$ and $\varepsilon_k^* = o_k - min\ (o_i + \varepsilon_i)$.

- As shown in Fig. B2(b), if $max\ (o_i - \varepsilon_i) \leq o_k \leq min\ (o_i + \varepsilon_i)$, then $max\ 2(o_i - \varepsilon_i) \leq 2o_k \leq min\ 2(o_i + \varepsilon_i)$; if and only if $n = 0$, then $m = a + b = 2o_k$ holds. Thus, $a = b = o_k$ and $\varepsilon_k^* = (\beta - \frac{1}{2})(b - a) = 0$.

- As shown in Fig. B2(c), if $o_k < max\ (o_i - \varepsilon_i)$, then $2o_k < max\ 2(o_i - \varepsilon_i) \leq max\ 2(o_i - \varepsilon_i) + (2\alpha - 1)n \leq m = a + b$. Considering the formula (B1-2) in Part 1, we obtain $a = b$ and $min\ \varepsilon_k = \frac{a+b}{2} - o_k$. Obviously, ε_k increases with $(a + b)$, so if $a = b = max\ (o_i - \varepsilon_i)$, $\varepsilon_k^* = max\ (o_i - \varepsilon_i) - o_k$ holds.

Above all, conditions of the existence for ε_k^* and o^* in Model (9) are obtained as Theorem 6. □

Appendix C. Proof of Theorem 8.

Proof. From constraints (15-3) and (15-4), we have $(1 - \alpha)a_i + \alpha b_i - \varepsilon_i \leq o' \leq \alpha a_i + (1 - \alpha)b_i + \varepsilon_i$, where $i \in N, i \neq k$. The above inequalities are equivalent to $max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\} \leq o' \leq min\{\alpha a_i + (1 - \alpha)b_i + \varepsilon_i\}$. For simplicity, let $G = max\{(1 - \alpha)a_i + \alpha b_i - \varepsilon_i\}$ and $H = min\{\alpha a_i + (1 - \alpha)b_i + \varepsilon_i\}$, $(i \in N, i \neq k)$, such that $G \leq o' \leq H$.

From constraints (15-1) and (15-2), we have

- If $(1 - \beta)a_k + \beta b_k - o' = o' - [\beta a_k + (1 - \beta)b_k]$, we have $a_k + b_k = 2o'$, then $min\ \varepsilon_k = (1 - \beta)a_k + \beta b_k - \frac{a_k+b_k}{2} = (\beta - \frac{1}{2})(b_k - a_k)$. When $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k^* = (\beta - \frac{1}{2})(b_k - a_k) \geq 0$ holds, and when $0 \leq \beta \leq \frac{1}{2}$, $(\beta - \frac{1}{2})(b_k - a_k) \leq 0$, for $\varepsilon_k \geq 0$, we have $\varepsilon_k^* = 0$.

- If $(1 - \beta)a_k + \beta b_k - o' < o' - [\beta a_k + (1 - \beta)b_k]$, we have $a_k + b_k < 2o'$, and if and only if $\frac{1}{2} \leq \beta \leq 1$, $\varepsilon_k^* = o' - [\beta a_k + (1 - \beta)b_k] > 0$ holds; when $0 \leq \beta \leq \frac{1}{2}$, for $\varepsilon_k \in [0, o' - \frac{a_k+b_k}{2}]$; thus, $\varepsilon_k^* = 0$.

- If $(1 - \beta)a_k + \beta b_k - o' > o' - [\beta a_k + (1 - \beta)b_k]$, we have $a_k + b_k > 2o'$, and if and only if $\frac{1}{2} \leq \beta \leq 1$, $\varepsilon_k^* = (1 - \beta)a_k + \beta b_k - o' > 0$ holds; when $0 \leq \beta \leq \frac{1}{2}$, then $\varepsilon_k \in [0, \frac{a_k+b_k}{2} - o']$; thus, $\varepsilon_k^* = 0$.

If $\frac{1}{2} \le \beta \le 1$, we should discuss the sizes of $(a_k + b_k)$ and $2o'$, so as to obtain the existing conditions for o'.

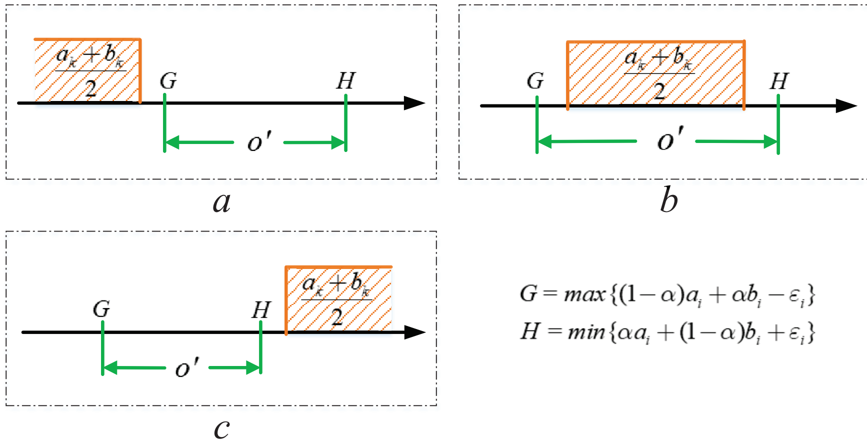

Fig. C1 Comparative analysis between o' and $\frac{a_k+b_k}{2}$.

- As shown in Fig. C1(a), if $\frac{a_k+b_k}{2} < G$, namely $a_k + b_k < 2o'$, then $min\varepsilon_k = o' - [\beta a_k + (1-\beta)b_k]$ is a monotonically increasing function with respect to o'. Therefore, when $o' = G = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\}$, ε_k^* exists, and $\varepsilon_k^* = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\} - \beta a_k - (1-\beta)b_k$.

- As shown in Fig. C1(b), if $G \le \frac{a_k+b_k}{2} \le H$, then $a_k + b_k = 2o'$ holds; thus, when $o' = \frac{a_k+b_k}{2}$, $\varepsilon_k^* = (\beta - \frac{1}{2})(b_k - a_k)$.

- As shown in Fig. C1(c), if $\frac{a_k+b_k}{2} > H$, namely $a_k + b_k > 2o'$, then $min\varepsilon_k = (1-\beta)a_k + \beta b_k - o'$ decreases with o'. Therefore, when $o' = H = min\{\alpha a_i + (1-\alpha)b_i + \varepsilon_i\}$, ε_k^* exists and $\varepsilon_k^* = (1-\beta)a_k + \beta b_k - min\{\alpha a_i + (1-\alpha)b_i + \varepsilon_i\}$.

When $0 \le \beta \le \frac{1}{2}$, due to the constraints of (15-1) and (15-2), we have $o' \in [\frac{a_k+b_k}{2}, \beta a_k + (1-\beta)b_k]$ or $o' \in [(1-\beta)a_k + \beta b_k, \frac{a_k+b_k}{2}]$. Then, $o' \in [(1-\beta)a_k + \beta b_k, \beta a_k + (1-\beta)b_k]$ and $\varepsilon_k^* = 0$. However, because of $G \le o' \le H$, we need to comprehensively discuss the final threshold of o'.

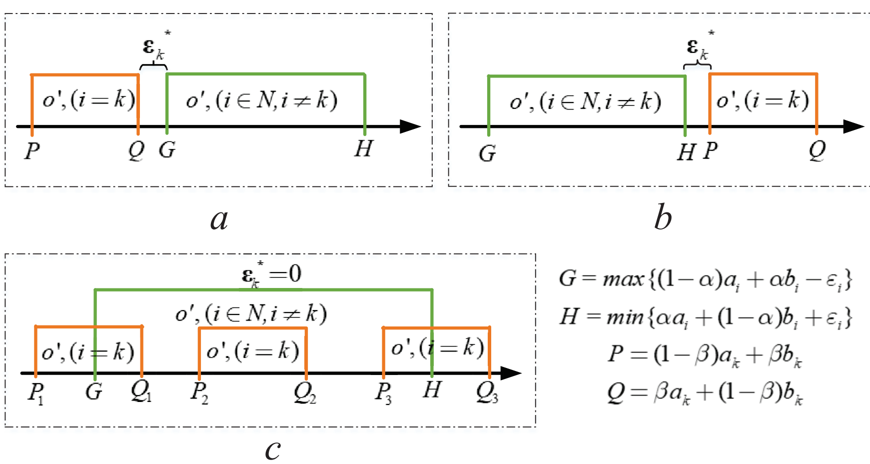

Fig. C2 Discussion on the analytic formulas of the objective function depends on o'.

- As shown in Fig. C2(a), if $\beta a_k + (1-\beta)b_k < G$, then $\varepsilon_k^* = o' - [\beta a_k + (1-\beta)b_k]$, $o' = G$.

- As shown in Fig. C2(b), if $(1-\beta)a_k + \beta b_k > H$, then $\varepsilon_k^* = (1-\beta)a_k + \beta b_k - o'$, $o' = H$.
- As shown in Fig. C2(c), if other circumstances are met, then $\varepsilon_k^* = 0$ and $o' \in [(1-\beta)a_k + \beta b_k, \beta a_k + (1-\beta)b_k] \bigcap [G, H]$.

Thus, this completes the proof for Theorem 8. □

Appendix D. Equivalent forms of carbon quota MCCMs in Case (1-5)

Table D1 Equivalent forms of carbon quota MCCMs in Case (1-5)

Cases	Models
Case 1	$Model(4-11):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} \varepsilon_1 \geq (1-\alpha)a + \alpha b - 0,\ \varepsilon_1 \geq -(1-\alpha)b - \alpha a + 0 \\ \varepsilon_2 \geq (1-\alpha)a + \alpha b - 3,\ \varepsilon_2 \geq -(1-\alpha)b - \alpha a + 3 \\ \varepsilon_3 \geq (1-\alpha)a + \alpha b - 6,\ \varepsilon_3 \geq -(1-\alpha)b - \alpha a + 6 \\ \varepsilon_4 \geq (1-\alpha)a + \alpha b - 10,\ \varepsilon_4 \geq -(1-\alpha)b - \alpha a + 10 \\ a \leq b,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$
Case 2	$Model(4-21):\ Min\ Z = 3*\varepsilon_3$ $s.t. \begin{cases} (1-\beta)a + \beta b - 6 \leq \varepsilon_3,\ -\beta a + (\beta-1)b + 6 \leq \varepsilon_3 \\ (1-\alpha)a + \alpha b - 0 \leq \varepsilon_1,\ -\alpha a + (\alpha-1)b + 0 \leq \varepsilon_1 \\ (1-\alpha)a + \alpha b - 3 \leq \varepsilon_2,\ -\alpha a + (\alpha-1)b + 3 \leq \varepsilon_2 \\ (1-\alpha)a + \alpha b - 10 \leq \varepsilon_4,\ -\alpha a + (\alpha-1)b + 10 \leq \varepsilon_4 \\ 0 \leq \varepsilon_1 \leq 5,\ 0 \leq \varepsilon_2 \leq 4,\ 0 \leq \varepsilon_4 \leq 6,\ \varepsilon_3 \geq 0 \\ a \leq b,\ 0 \leq \alpha,\ \beta \leq 1 \end{cases}$
Case 3	$Model(4-31):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} 14(1-\alpha) + 37\alpha - o' \leq \varepsilon_1,\ o' - 14\alpha - 37(1-\alpha) \leq \varepsilon_1 \\ 22(1-\alpha) + 30\alpha - o' \leq \varepsilon_2,\ o' - 22\alpha - 30(1-\alpha) \leq \varepsilon_2 \\ 64(1-\alpha) + 153\alpha - o' \leq \varepsilon_3,\ o' - 64\alpha - 153(1-\alpha) \leq \varepsilon_3 \\ 8(1-\alpha) + 61\alpha - o' \leq \varepsilon_4,\ o' - 8\alpha - 61(1-\alpha) \leq \varepsilon_4 \\ o' \geq 0,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$
Case 4	$Model(4-41):\ Min\ Z = 3*\varepsilon_3$ $s.t. \begin{cases} 64(1-\beta) + 153\beta - o' \leq \varepsilon_3, o' - [64\beta + 153(1-\beta)] \leq \varepsilon_3 \\ 14(1-\alpha) + 37\alpha - o' \leq \varepsilon_1,\ o' - [14\alpha + 37(1-\alpha)] \leq \varepsilon_1 \\ 22(1-\alpha) + 30\alpha - o' \leq \varepsilon_2,\ o' - [22\alpha + 30(1-\alpha)] \leq \varepsilon_2 \\ 8(1-\alpha) + 61\alpha - o' \leq \varepsilon_4,\ o' - [8\alpha + 61(1-\alpha)] \leq \varepsilon_4 \\ 0 \leq \varepsilon_1 \leq 12,\ 0 \leq \varepsilon_2 \leq 5,\ 0 \leq \varepsilon_4 \leq 36 \\ o' \geq o,\ \varepsilon_3 \geq 0 \end{cases}$
Case 5	$Model(4-51):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} a + (b-a)\alpha + (37-14)\alpha - 37 \leq \varepsilon_1,\ (37-14)\alpha + (b-a)\alpha - b + 14 \leq \varepsilon_1 \\ a + (b-a)\alpha + (30-22)\alpha - 30 \leq \varepsilon_2,\ (30-22)\alpha + (b-a)\alpha - b + 22 \leq \varepsilon_2 \\ a + (b-a)\alpha + (153-64)\alpha - 153 \leq \varepsilon_3,\ (153-64)\alpha + (b-a)\alpha - b + 64 \leq \varepsilon_3 \\ a + (b-a)\alpha + (61-8)\alpha - 61 \leq \varepsilon_4,\ (61-8)\alpha + (b-a)\alpha - b + 8 \leq \varepsilon_4 \\ a \leq b,\ \ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$

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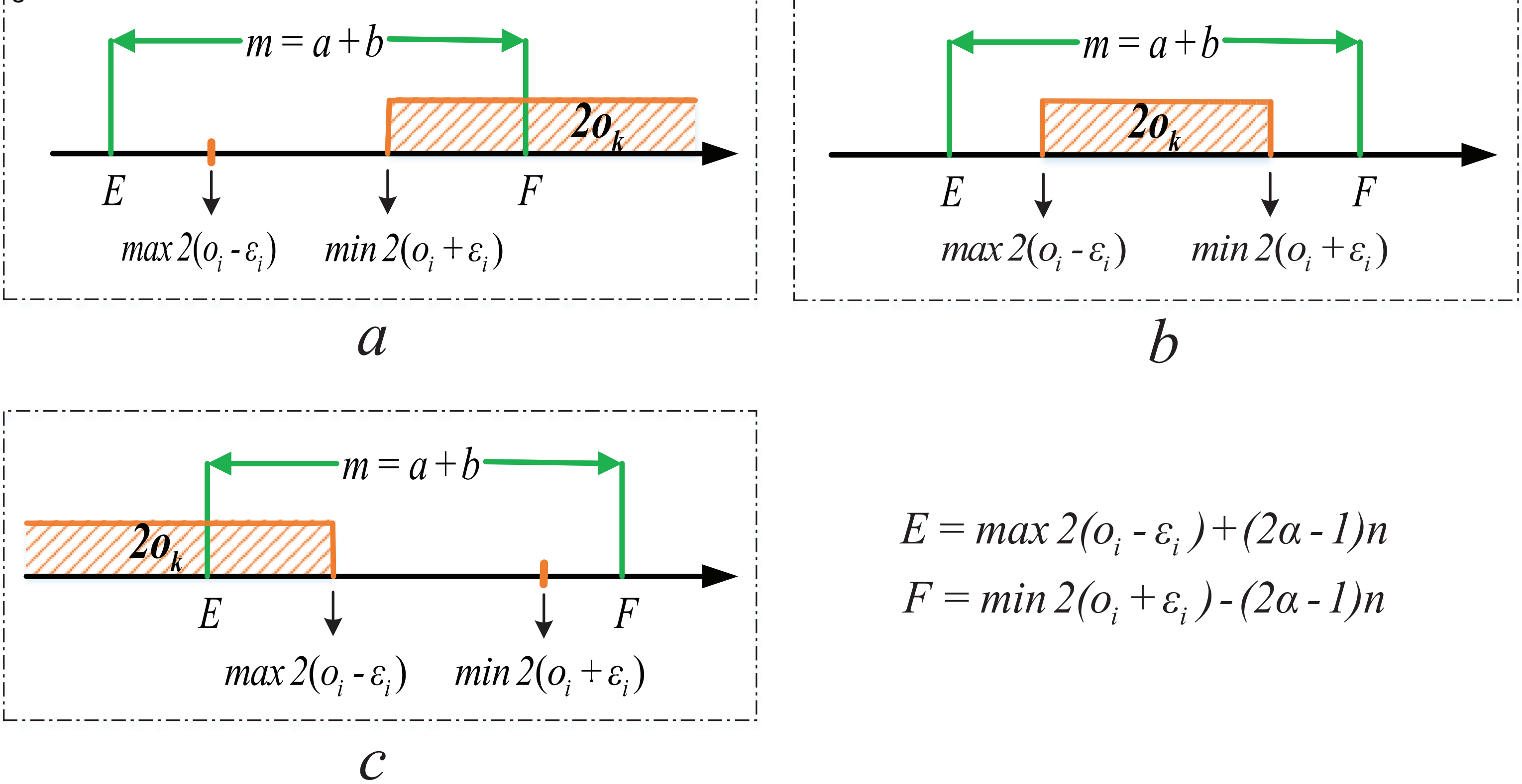
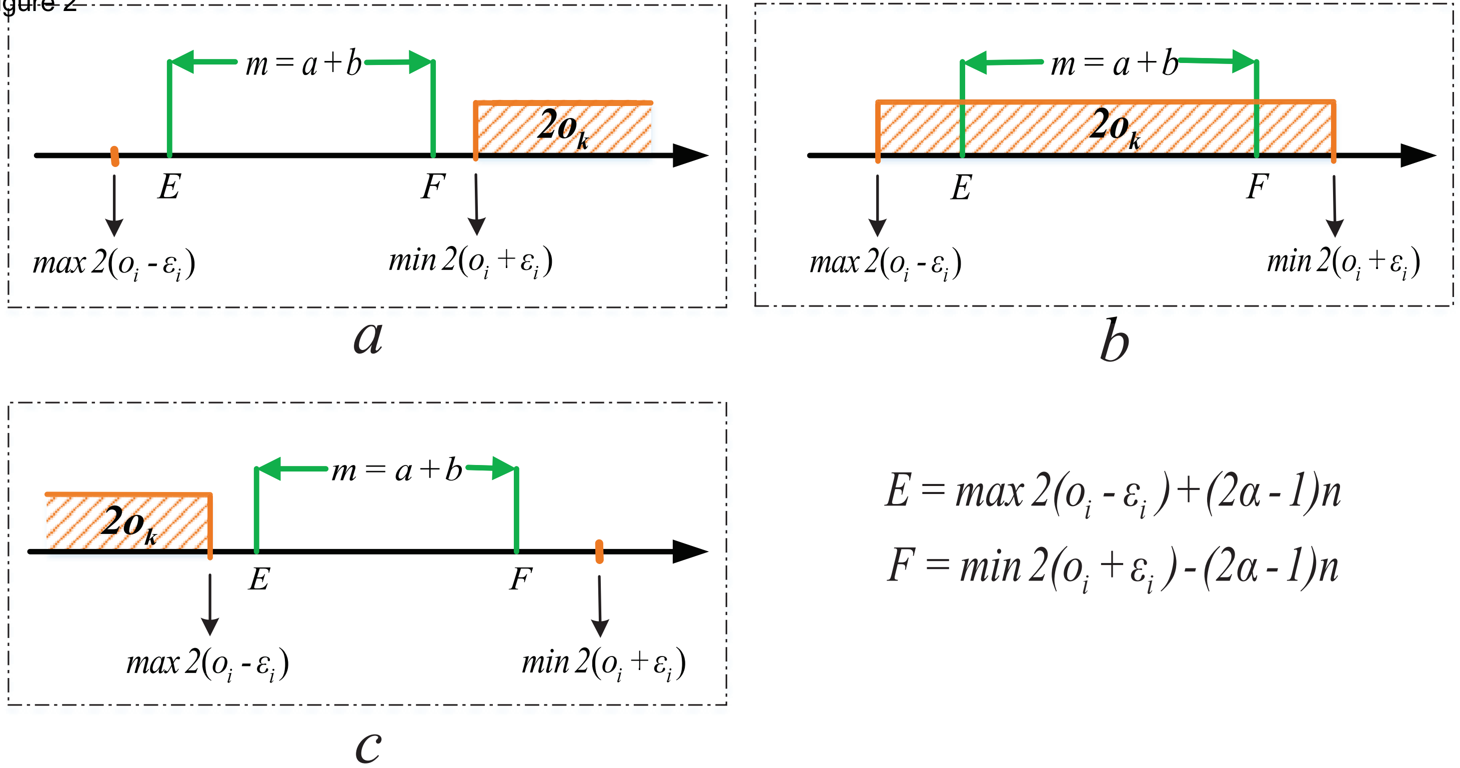
Figure 1



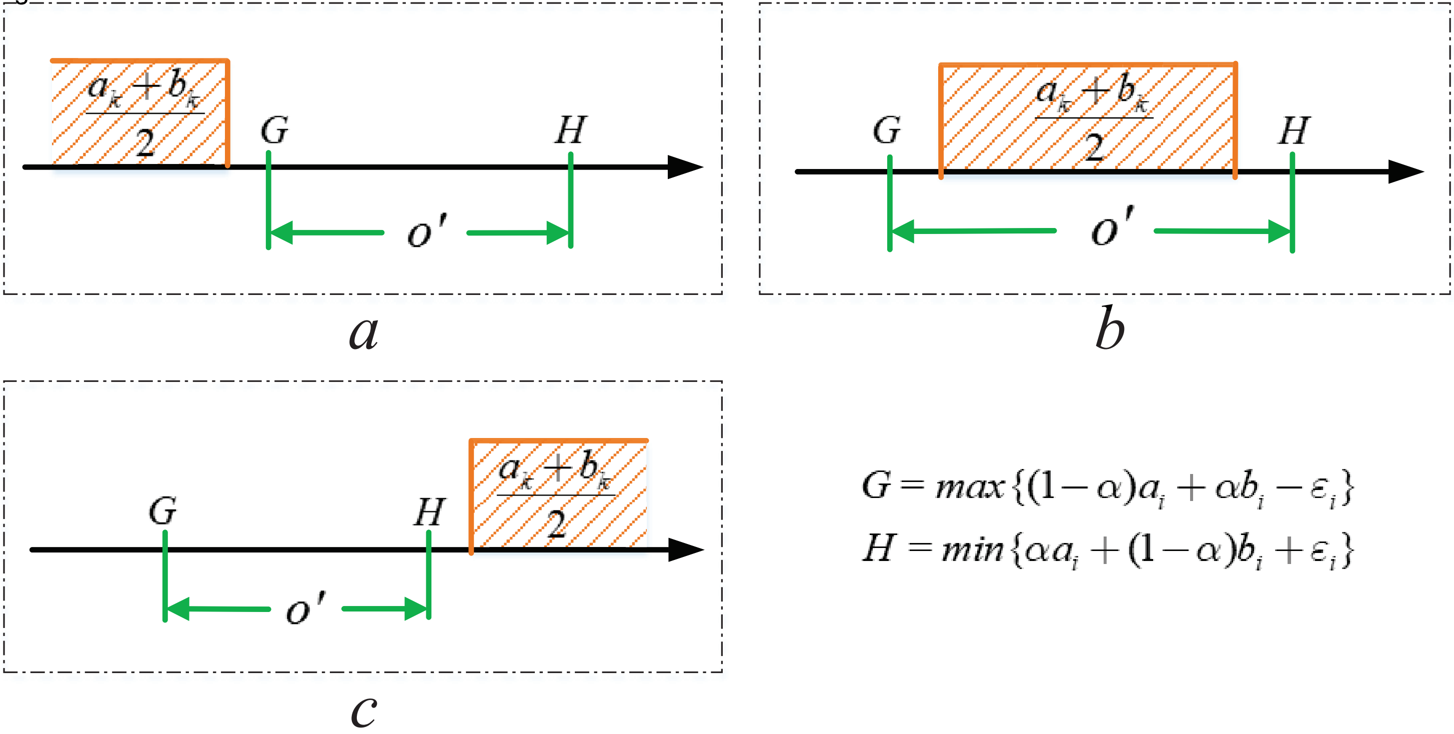
m = a + b
2o_k
E
F
max 2(o_i - ε_i)
min 2(o_i + ε_i)
a
b
c
E = max 2(o_i - ε_i) + (2α - 1)n
F = min 2(o_i + ε_i) - (2α - 1)n

Figure 2



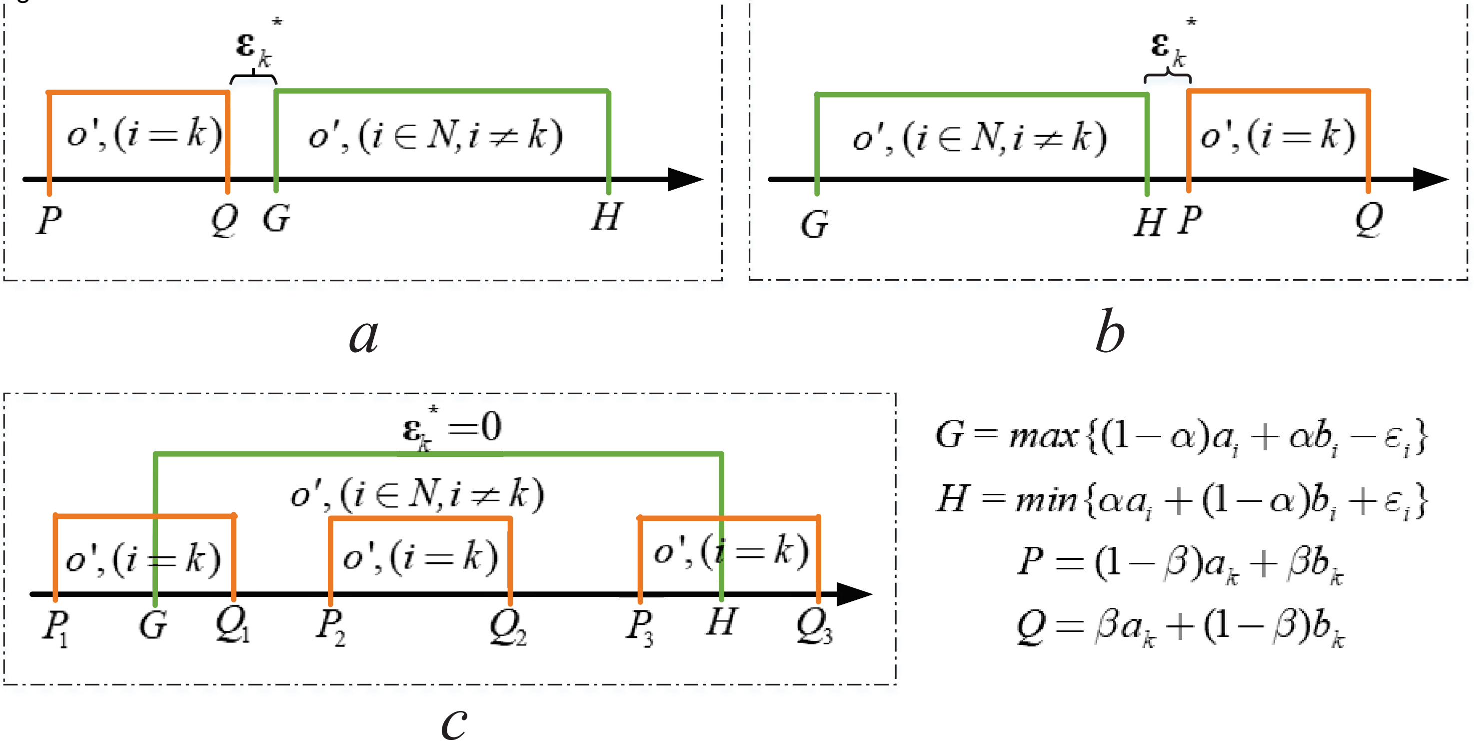
m = a + b
2o_k
E
F
max 2(o_i - ε_i)
min 2(o_i + ε_i)
a
m = a + b
2o_k
E
F
max 2(o_i - ε_i)
min 2(o_i + ε_i)
b
m = a + b
2o_k
E
F
max 2(o_i - ε_i)
min 2(o_i + ε_i)
c
E = max 2(o_i - ε_i) + (2α - 1)n
F = min 2(o_i + ε_i) - (2α - 1)n

Figure 3



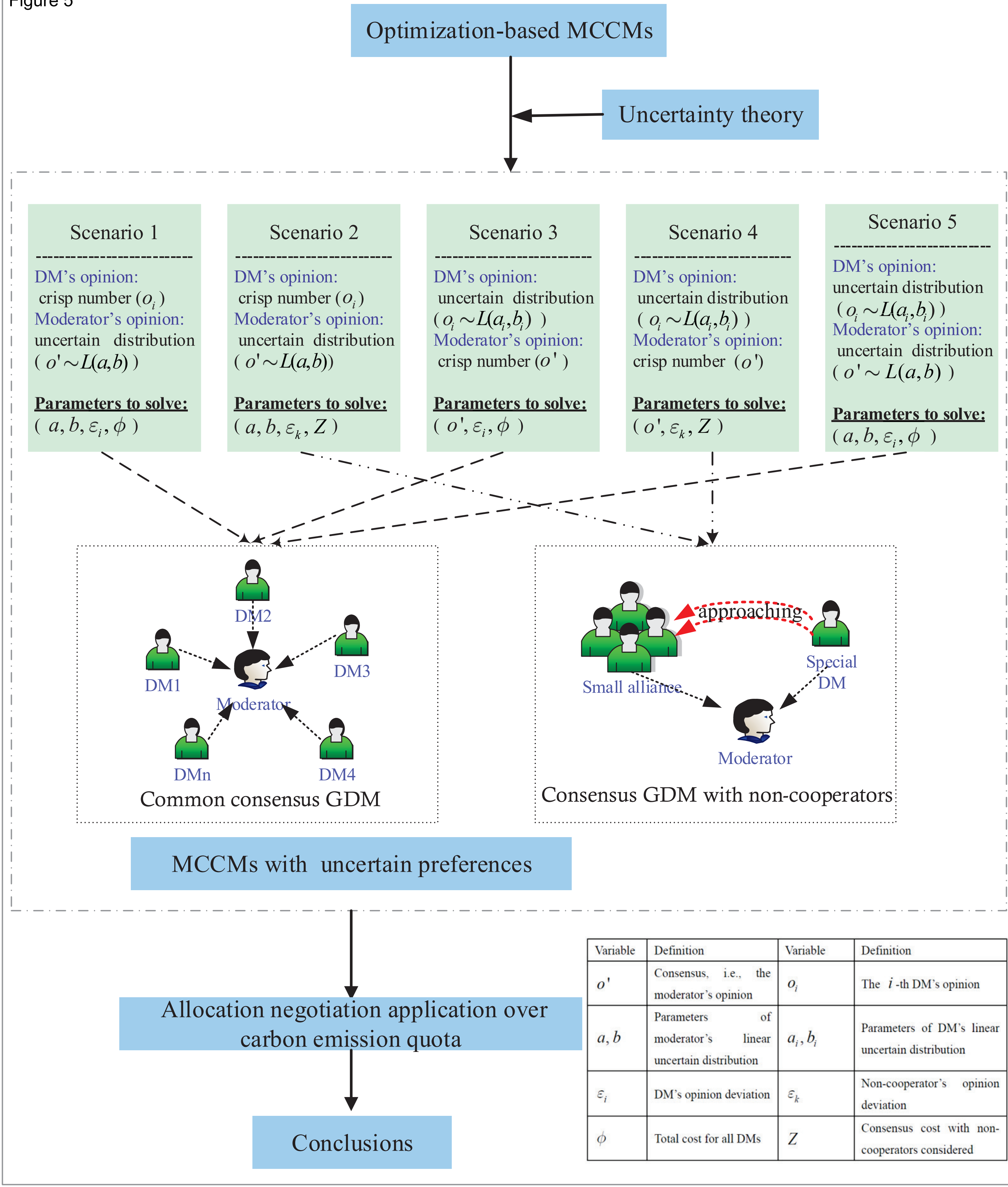
$\frac{a_k+b_k}{2}$
G
H
o'
a
b
c
$G = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\}$
$H = min\{\alpha a_i + (1-\alpha)b_i + \varepsilon_i\}$

Figure 4



ε_k^*
$o', (i = k)$
$o', (i \in N, i \neq k)$
P
Q
G
H
a
ε_k^*
$o', (i \in N, i \neq k)$
$o', (i = k)$
G
H
P
Q
b
$\varepsilon_k^* = 0$
$o', (i \in N, i \neq k)$
$o', (i = k)$
$o', (i = k)$
$o', (i = k)$
P_1
G
Q_1
P_2
Q_2
P_3
H
Q_3
c
$G = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\}$
$H = min\{\alpha a_i + (1-\alpha)b_i + \varepsilon_i\}$
$P = (1-\beta)a_k + \beta b_k$
$Q = \beta a_k + (1-\beta)b_k$

Figure 5



Variable	Definition	Variable	Definition
o'	Consensus, i.e., the moderator's opinion	o_i	The i-th DM's opinion
a, b	Parameters of moderator's linear uncertain distribution	a_i, b_i	Parameters of DM's linear uncertain distribution
ε_i	DM's opinion deviation	ε_k	Non-cooperator's opinion deviation
ϕ	Total cost for all DMs	Z	Consensus cost with non-cooperators considered

Figure 6

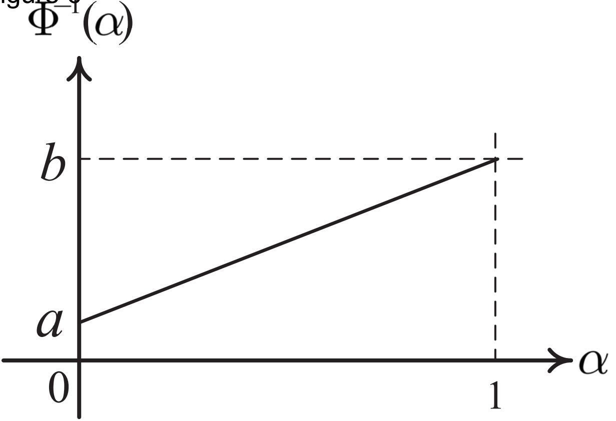
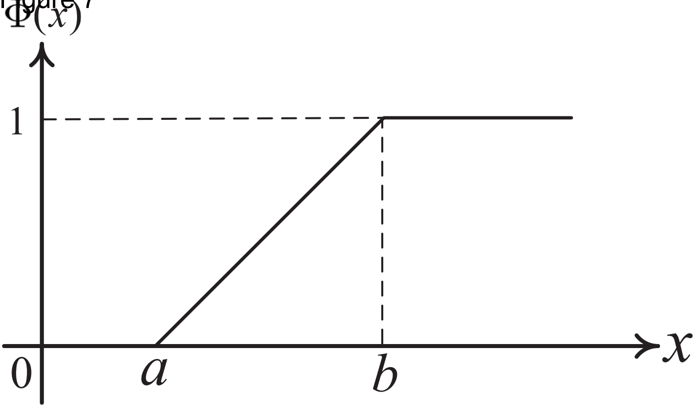


Figure 7



Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Zaiwu Gong: Conceptualization, Methodology, Supervision, Writing-Review & Editing, Funding acquisition

Xiaoxia Xu: Writing-Original Draft, Software, Data Curation, Formal analysis, Validation, Writing-Review & Editing, Funding acquisition

Weiwei Guo: Methodology, Formal analysis, Validation

Enrique Herrera-Viedma: Supervision, Writing-Review & Editing, Funding acquisition

Francisco Javier Cabrerizo: Formal analysis, Writing-Review & Editing