Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 370894, 12 pages
doi:10.1155/2012/370894

Research Article

Fixed-Point Iterative Algorithm for the Linear Fredholm-Volterra Integro-Differential Equation

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Received 23 March 2012; Revised 30 April 2012; Accepted 1 May 2012

Academic Editor: Rudong Chen

With the aid of fixed-point theorem (an equivalent version for the linear case) and biorthogonal systems in adequate Banach spaces, the problem of approximating the solution of a linear Fredholm-Volterra integro-differential equation is turned into a numerical algorithm, so that it can be solved numerically.

1. Introduction

Denoting by $\mathcal{C}([0,1])$ and $\mathcal{C}([0,1]^2)$ the Banach space of all continuous and real-valued functions defined on $[0,1]$ and $[0,1]^2$, respectively, equipped with their usual sup-sup norm, let us consider the following problem associated to the Fredholm-Volterra integro-differential equation: given $\rho \in \mathbb{R}$, $k_1, k_2 \in \mathcal{C}([0,1]^2)$, and $f, x_0 \in \mathcal{C}([0,1])$, find $x \in \mathcal{C}^1([0,1])$ such that

$$x'(t) = x_0(t) + f(t)x(t) + \int_0^1 k_1(t,s)x(s)ds + \int_0^t k_2(t,s)x(s)ds \quad \left((t,s) \in [0,1]^2\right), \\ x(0) = \rho. \tag{1.1}$$

Observe that if $k_1(t,s) = 0$ and $f(t) = 0$ in (1.1), the equation is transformed into a linear Volterra integro-differential equation; and if $k_2(t,s) = 0$ and $f(t) = 0$, it becomes a linear Fredholm integro-differential. Additionally, if $k_1(t,s) = k_2(t,s) = 0$, (1.1) is transformed into a linear differential equation of the first order.

Frequently the mathematical modelling of problems arising from the real world (see [1] and the references therein) deals with problem (1.1). These are usually difficult to solve analytically, and in many cases, the solution must be approximated. Therefore, in recent years, several numerical approaches have been proposed (see, e.g., [2–4]). The numerical methods usually transform the integro-differential equation into a linear system that can be solved by direct or iterative methods. On the other hand, the use of fixed-point techniques in the numerical study of linear differential, integral, and integro-differential equations has also proven successful in some works, as [5–11]. The purpose of this paper is to develop an effective method for approximating the solution of (1.1) using Schauder basis and another classical tool in analysis: an equivalent version of the fixed-point theorem for the linear case (the geometric series theorem). This algorithm generalizes the developed ones in [7, 8, 10] for Volterra integro-differential, Fredholm integro-differential, and differential equation, respectively.

To establish our numerical method, we first need to review some results of a theoretical nature in Section 2. We arrive at a numerical method for approximating the solution of (1.1) in Section 3, and in order to state the results about convergence and to study the error of the proposed algorithm, we will assume that $\gamma = \|f\| + \|k_1\| + \|k_2\|/2 < 1$ and $k_1, k_2 \in C^1([0,1]^2)$ and $f, g \in C^1([0,1])$. Finally, in Section 4, we illustrate the theoretical results with two examples.

2. Two Tools of a Theoretical Nature

Two fundamental tools will be used to establish the algorithm needed to solve the problem (1.1). The first is the following equivalent version (for the linear case) of the Banach fixed-point theorem (see [12]).

Geometric Series Theorem

Let X be a Banach space, and let $T : X \to X$ be a continuous and linear operator such that $\|T\| < 1$. Then, $Id - T$ is a continuous, linear, and bijective operator and $(Id - T)^{-1} = \sum_{n\geq 0} T^n$.

In particular, given $y \in X$, the equation $(Id - T)x = y$ has a unique solution $x = (Id - T)^{-1}y$.

The second tool applied consists of biorthogonal systems in Banach spaces $C([0,1])$ and $C([0,1]^2)$. We will make use of the usual Schauder basis for simplicity in the exposition, although the numerical method given works equally well by replacing it with any complete biorthogonal system in $C([0,1]^2)$. For this reason, we will now briefly recall the main issues and notations regarding Schauder bases in $C([0,1])$ and $C([0,1]^2)$.

Let us consider the usual Schauder basis $\{b_n\}_{n\geq 1}$ in the space $C([0,1])$, that is, for a dense sequence $\{t_i\}_{i\geq 1}$ of distinct real numbers in $[0,1]$ such that $t_1 = 0$ and $t_2 = 1$, we define $b_1(t) := 1$ for $t \in [0,1]$, and for $n \geq 2$, b_n is the piecewise linear continuous function on $[0,1]$ with nodes at $\{t_i : 1 \leq i \leq n\}$, uniquely determined by the relations $b_n(t_n) = 1$ and $b_n(t_k) = 0$ for $k < n$. For each function $z \in C([0,1])$, there exists a unique sequence of scalars $\{\beta_n\}_{n\geq 1}$ such that $z = \sum_{n\geq 1} \beta_n b_n$. We denote by $\{b_n^*\}_{n\geq 1}$ the sequence of (continuous and linear) biorthogonal functionals in the dual space of $C([0,1])$, defined by

$$b_n^*\left(\sum_{k\geq 1} \beta_k b_k\right) = \beta_n, \tag{2.1}$$

whereas the sequence of (continuous and linear) *projections* $\{P_n\}_{n\geq 1}$ is defined by the partial sums

$$P_n\left(\sum_{k\geq 1}\beta_k b_k\right) = \sum_{k=1}^{n}\beta_k b_k. \tag{2.2}$$

For the respective sequence of biorthogonal systems $\{b_n^*\}_{n\geq 1}$ and each $z \in C([0,1])$, the equalities

$$b_1^*(z) = z(t_1), \qquad b_n^*(z) = z(t_n) - \sum_{k=1}^{n-1} b_k^*(z) b_k(t_n), \quad \text{for } n \geq 2 \tag{2.3}$$

are valid (see [13]). As a consequence, the sequence of associated projections $\{P_n\}_{n\geq 1}$ satisfies the following interpolation property:

$$z \in C([0,1]), \quad n \geq 1,\ i \leq n \Longrightarrow P_n(z)(t_i) = z(t_i). \tag{2.4}$$

We now evoke the construction of the usual Schauder basis $\{B_n\}_{n\geq 1}$ for the Banach space $C([0,1]^2)$, endowed with its usual sup-norm (see [13, 14]). To this end, we consider the bijective mapping $\tau : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ (where $[p]$ denotes the integer part of p) given by

$$\tau(n) := \begin{cases} \left(\sqrt{n}, \sqrt{n}\right) & \text{if } \left[\sqrt{n}\right] = \sqrt{n}, \\ \left(n - \left[\sqrt{n}\right]^2, \left[\sqrt{n}\right] + 1\right) & \text{if } 0 < n - \left[\sqrt{n}\right]^2 \leq \left[\sqrt{n}\right], \\ \left(\left[\sqrt{n}\right] + 1, n - \left[\sqrt{n}\right]^2 - \left[\sqrt{n}\right]\right) & \text{if } \left[\sqrt{n}\right] < n - \left[\sqrt{n}\right]^2. \end{cases} \tag{2.5}$$

Then it suffices to define

$$B_n(t,s) := b_i(t) b_j(s), \quad t, s \in [0,1], \tag{2.6}$$

whenever $\tau(n) = (i,j)$. Let $\{B_n^*\}_{n\geq 1}$ and $\{Q_n\}_{n\geq 1}$ stand for the corresponding sequences of biorthogonal functionals and projections, respectively. The Schauder basis $\{B_n\}_{n\geq 1}$ has similar properties to those of the one-dimensional basis $\{b_n\}_{n\geq 1}$:

(a) for all $t, s \in [0,1]$, $B_1(t,s) = 1$ and for $n \geq 2$, $B_n(t_i, t_j) = \begin{cases} 1 & \text{if } \tau(n) = (i,j), \\ 0 & \text{if } \tau^{-1}(i,j) < n, \end{cases}$

(b) if $w \in C([0,1]^2)$, then $B_1^*(w) = w(t_1, t_1)$, and for all $n \geq 2$, if $\tau(n) = (i,j)$, $B_n^*(w) = w(t_i, t_j) - \sum_{k=1}^{n-1} B_k^*(w) B_k(t_i, t_j)$,

(c) the sequence of associated projections $\{Q_n\}_{n\geq 1}$ satisfies $Q_n(w)(t_i, t_j) = w(t_i, t_j)$, whenever $n, i, j \in \mathbb{N}$ and $\tau^{-1}(i,j) \leq n$,

(d) this Schauder basis is monotone, that is, $\sup\{\|Q_n\|\}_{n\in\mathbb{N}} = 1$.

Observe that for each $n \in \mathbb{N}$, the definition of the projections P_n and Q_{n^2} requires only the first $n+1$ points of the sequence $\{t_i\}_{i\geq 1}$. These $n+1$ points, sorted in increasing order, constitute a partition of the interval $[0,1]$, which will be denoted as T_n; let ΔT_n denote the maximum distance between two consecutive points of T_n. Under some weak conditions, from (2.4) and (c), respectively, and the mean-value theorem, we can estimate the rate of convergence of the sequence of projections:

$$\begin{gathered} z \in C^1([0,1]), \quad n \geq 2 \Longrightarrow \|z - P_n(z)\| \leq 2\|z'\| \Delta T_n. \\ w \in C^1\left([0,1]^2\right) n \geq 2 \Longrightarrow \|w - Q_{n^2}(w)\| \leq 4 \max\left\{ \left\| \frac{\partial w}{\partial t} \right\|, \left\| \frac{\partial w}{\partial s} \right\| \right\} \Delta T_n. \end{gathered} \tag{2.7}$$

3. The Algorithm: Convergence and Error

Our starting point is the formulation of (1.1) in terms of a certain operator L as follows: let $L : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ be the linear and continuous operator defined by

$$\begin{aligned} Lx(t) :=& \int_0^t f(u)x(u)du + \int_0^t \int_0^1 k_1(u,s)x(s)ds\,du \\ &+ \int_0^t \int_0^u k_2(u,s)x(s)ds\,du \quad (0 \leq t \leq 1,\ x \in \mathcal{C}([0,1])). \end{aligned} \tag{3.1}$$

It is a simple matter to check that a function $x \in C^1([0,1])$ is the solution of (1.1) if and only if $(Id - L)x = g$, where $g(t) := \rho + \int_0^t x_0(u)du$.

It can be shown by an induction argument and Fubini's theorem that for $j \in \mathbb{N}$, $\|L^j x\| \leq \|x\|\gamma^j$; hence,

$$\sum_{j\geq 0} \left\| L^j x \right\| \leq \|x\| \sum_{j\geq 0} \gamma^j = \frac{\|x\|}{1-\gamma}. \tag{3.2}$$

The condition assumed on γ and the geometric series theorem allow us to establish the existence of one and only one solution x of (1.1), which is

$$x = \sum_{j\geq 0} L^j g. \tag{3.3}$$

Besides,

$$\left\| x - \sum_{j=0}^{m} L^j g \right\| = \left\| \sum_{j\geq m+1} L^j g \right\| \leq \sum_{j\geq m+1} \left\| L^j g \right\| \leq \|g\| \sum_{j\geq m+1} \gamma^j = \frac{\gamma^{m+1}}{1-\gamma} \|g\|. \tag{3.4}$$

In view of (3.3), we consider the sequence $\{y_m\}_{m\geq 0}$ defined as follows: $y_0(t) = L^0 g(t) = g(t)$, and for $m \geq 1$,

$$y_m(t) = \sum_{j=0}^{m} L^j g(t) = g(t) + Ly_{m-1}(t)$$
$$= g(t) + \int_0^t f(u) y_{m-1}(u) du + \int_0^t \int_0^1 k_1(u,s) y_{m-1}(s) ds\, du + \int_0^t \int_0^u k_2(u,s) y_{m-1}(s) ds\, du. \tag{3.5}$$

The sequence $\{y_m\}_{m\geq 0}$ converges to the solution x of (1.1) and

$$\|x - y_m\| \leq \|g\| \frac{\gamma^{m+1}}{1-\gamma}. \tag{3.6}$$

We can then calculate iteratively using (3.5), at least in a theoretical way, the solution of (1.1). From a practical viewpoint, in general these calculations are not possible explicitly. The idea of our numerical method is to use an appropriate Schauder basis in the spaces $\mathcal{C}([0,1])$, $\mathcal{C}([0,1]^2)$ truncating the functions of such spaces by means of the projections of the Schauder bases $\{P_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$, and replacing each Ly_{m-1} $(m \geq 1)$ in (3.5) by a new function $v_m \in \mathcal{C}([0,1])$, easier to calculate, and in such a way that the error $\|x - (g + v_m)\|$ be small enough. Given these functions, each $g + v_m$ will be approximation of the solution of (1.1).

Specifically, we begin with $v_0(t) = 0$ and consider

$$\widetilde{y}_0(t) = y_0(t) = g(t) + v_0(t). \tag{3.7}$$

Letting $m \in \mathbb{N}$, we inductively define the functions

$$v_m(t) = \int_0^t P_{n_m}\big(f(u)\widetilde{y}_{m-1}(u)\big) du + \int_0^t \int_0^1 Q_{n_m^2}\big(k_1(u,s)\widetilde{y}_{m-1}(s)\big) ds du$$
$$+ \int_0^t \int_0^u Q_{n_m^2}\big(k_2(u,s)\widetilde{y}_{m-1}(s)\big) ds du, \tag{3.8}$$
$$\widetilde{y}_m(t) = g(t) + v_m(t),$$

where n_m are natural numbers.

We will show that the sequence $\{\widetilde{y}_m\}_{m\geq 0}$ approximates the solution of (1.1) while in order to study the error $\|x - \widetilde{y}_m\|$, let us assume that $k_1, k_2 \in C^1([0,1]^2)$ and $f, g \in C^1([0,1])$.

In the first place, we show the following.

Lemma 3.1. *The sequences* $\{\|v_m\|\}_{m\in\mathbb{N}}$ *and* $\{\|v'_m\|\}_{m\in\mathbb{N}}$ *are bounded.*

Proof. First we show, using an inductive argument, that for all $t \in [0,1]$, $|v_m(t)| \le \|g\| \sum_{j=1}^{m} \gamma^j$. Since the Schauder basis $\{B_n\}_{n\ge1}$ is monotone, we have

$$|v_1(t)| \le \|g\|\|f\| \int_0^t du + \|g\|\|k_1\| \int_0^t \int_0^1 ds\, du + \|g\|\|k_2\| \int_0^t \int_0^u ds\, du \le \|g\|\gamma. \tag{3.9}$$

Suppose that the result holds for $m-1$, in other words $|v_{m-1}(t)| \le \|g\| \sum_{j=1}^{m-1} \gamma^j$, and using the monotony of $\{B_n\}_{n\ge1}$, we prove for m the following:

$$\begin{aligned} |v_m(t)| &\le \|f\|\|\tilde{y}_{m-1}\| \int_0^t du + \|k_1\|\|\tilde{y}_{m-1}\| \int_0^t \int_0^1 ds\, du \\ &\quad + \|k_2\|\|\tilde{y}_{m-1}\| \int_0^t \int_0^u ds\, du \le \|\tilde{y}_{m-1}\|\gamma \le (\|g\| + \|v_{m-1}\|)\gamma \\ &\le \|g\|\gamma + \left(\|g\| \sum_{j=1}^{m-1} \gamma^j \right)\gamma = \|g\| \sum_{j=1}^{m} \gamma^j. \end{aligned} \tag{3.10}$$

Therefore, $\|v_m\| \le \|g\| \sum_{j\ge1} \gamma^j$.

On the other hand, with similar arguments,

$$\begin{aligned} \|v'_m(t)\| &\le \|f\|\|\tilde{y}_{m-1}\| + \|k_1\|\|\tilde{y}_{m-1}\| \int_0^1 ds \\ &\quad + \|k_2\|\|\tilde{y}_{m-1}\| \int_0^t ds \le \|\tilde{y}_{m-1}\|(\|f\| + \|k_1\| + \|k_2\|) \\ &\le \|\tilde{y}_{m-1}\| 2\gamma \le 2\gamma(\|g\| + \|v_{m-1}\|), \end{aligned} \tag{3.11}$$

and thus, $\|v'_m\| \le 2\gamma(\|g\| + \|v_{m-1}\|)$. □

Remark 3.2. Given that for $i \in \{1,2\}$ and $m \in \mathbb{N}$,

$$\begin{aligned} \frac{\partial(k_i(t,s)\tilde{y}_m(s))}{\partial t} &= \frac{\partial k_i(t,s)}{\partial t}\tilde{y}_m(s), \\ \frac{\partial(k_i(t,s)\tilde{y}_m(s))}{\partial s} &= \frac{\partial k_i(t,s)}{\partial s}\tilde{y}_m(s) + k_i(t,s)\tilde{y}'_m(s), \end{aligned} \tag{3.12}$$

denoting by

$$M_m = \max\left\{ \|f'\tilde{y}_m + f\tilde{y}'_m\|, \left\|\frac{\partial k_i \tilde{y}_m}{\partial t}\right\|, \left\|\frac{\partial k_i \tilde{y}_m}{\partial s}\right\| \right\}_{i\in\{1,2\}}, \tag{3.13}$$

we have that the sequence $\{M_m\}_{m\in\mathbb{N}}$ is also bounded.

Theorem 3.3. *With the previous notation, if* $m \in \mathbb{N}$, $h = \max_{j=0,\ldots,m-1}\Delta T_{n_{j+1}}$ *with* $n_{j+1} \geq 2$ *and* $M = \sup\{M_m\}_{m\in\mathbb{N}}$, *then*

$$\|x - \tilde{y}_m\| \leq \|g\|\frac{\gamma^{m+1}}{1-\gamma} + 8Mh\sum_{j=0}^{m-1}\gamma^{m-1-j}. \tag{3.14}$$

Proof. The triangle inequality gives $\|x-\tilde{y}_m\| \leq \|x-y_m\| + \|y_m - \tilde{y}_m\|$. Because of the inequality (3.6), we have

$$\|x - y_m\| \leq \|g\|\frac{\gamma^{m+1}}{1-\gamma}. \tag{3.15}$$

For the second one, by an inductive argument, we can show that

$$\|y_m - \tilde{y}_m\| \leq \sum_{j=0}^{m-1}\gamma^{m-1-j}\|L\tilde{y}_j - v_{j+1}\|. \tag{3.16}$$

Indeed, for $m = 1$, the result is clearly true

$$\|y_1 - \tilde{y}_1\| = \|g + Lg - g - v_1\| = \|Lg - v_1\| = \|L\tilde{y}_0 - v_1\|. \tag{3.17}$$

Suppose it holds for $m-1$, that is, $\|y_{m-1} - \tilde{y}_{m-1}\| \leq \sum_{j=0}^{m-2}\gamma^{(m-1)-1-j}\|L\tilde{y}_j - v_{j+1}\|$, and we prove for m the following:

$$\begin{aligned}
\|y_m - \tilde{y}_m\| &= \|(g + Ly_{m-1}) - \tilde{y}_m\| = \|Ly_{m-1} - v_m\| \leq \|Ly_{m-1} - L\tilde{y}_{m-1}\| + \|L\tilde{y}_{m-1} - v_m\| \\
&\leq \|L\|\|y_{m-1} - \tilde{y}_{m-1}\| + \|L\tilde{y}_{m-1} - v_m\| \\
&\leq \|L\|\sum_{j=0}^{m-2}\gamma^{(m-1)-1-j}\|L\tilde{y}_j - v_{j+1}\| + \|L\tilde{y}_{m-1} - v_m\| \\
&\leq \gamma\sum_{j=0}^{m-2}\gamma^{(m-1)-1-j}\|L\tilde{y}_j - v_{j+1}\| + \|L\tilde{y}_{m-1} - v_m\| \\
&= \sum_{j=0}^{m-1}\gamma^{m-1-j}\|L\tilde{y}_j - v_{j+1}\|.
\end{aligned} \tag{3.18}$$

Then

$$\|x - \tilde{y}_m\| \leq \|g\|\frac{\gamma^{m+1}}{1-\gamma} + \sum_{j=0}^{m-1}\gamma^{m-1-j}\|L\tilde{y}_j - v_{j+1}\|. \tag{3.19}$$

For $j = 0, 1, \ldots, m-1$,

$$\begin{aligned}
\left|\left(L\tilde{y}_j - v_{j+1}\right)(t)\right| &= \left|\int_0^t f(u)\tilde{y}_j(u)du + \int_0^t\int_0^1 k_1(u,s)\tilde{y}_j(s)ds\,du\right.\\
&\quad + \int_0^t\int_0^u k_2(u,s)\tilde{y}_j(s)ds\,du - \int_0^t P_{n_{j+1}}\left(f(u)\tilde{y}_j(u)\right)du\\
&\quad \left.- \int_0^t\int_0^1 Q_{n_{j+1}^2}\left(k_1(u,s)\tilde{y}_j(s)\right)ds\,du - \int_0^t\int_0^u Q_{n_{j+1}^2}\left(k_2(u,s)\tilde{y}_j(s)\right)dsdu\right|\\
&\leq \int_0^t\left|f(u)\left(\tilde{y}_j(u)\right) - P_{n_{j+1}}\left(f(u)\tilde{y}_j(u)\right)\right|du\\
&\quad + \int_0^t\int_0^1\left|k_1(u,s)\tilde{y}_j(s) - Q_{n_{j+1}^2}\left(k_1(u,s)\tilde{y}_j(s)\right)\right|ds\,du\\
&\quad + \int_0^t\int_0^u\left|k_2(u,s)\tilde{y}_j(s) - Q_{n_{j+1}^2}\left(k_2(u,s)\tilde{y}_j(s)\right)\right|ds\,du\\
&\leq \left\|f\tilde{y}_j - P_{n_{j+1}}\left(f\tilde{y}_j\right)\right\|t + \left\|k_1\tilde{y}_j - Q_{n_{j+1}^2}\left(k_1\tilde{y}_j\right)\right\|t + \left\|k_2\left(\tilde{y}_j\right) - Q_{n_{j+1}^2}\left(k_2\tilde{y}_j\right)\right\|\frac{t^2}{2}\\
&\leq \left\|f\tilde{y}_j - P_{n_{j+1}}\left(f\tilde{y}_j\right)\right\| + \left\|k_1\tilde{y}_j - Q_{n_{j+1}^2}\left(k_1\tilde{y}_j\right)\right\| + \left\|k_2\tilde{y}_j - Q_{n_{j+1}^2}\left(k_2\tilde{y}_j\right)\right\|\frac{1}{2}.
\end{aligned} \tag{3.20}$$

And thus, applying (2.7), we obtain the following bound:

$$\left\|L\tilde{y}_j - v_{j+1}\right\| \leq 2M_jh + 4M_jh + 4M_jh\frac{1}{2} = 8M_jh \leq 8Mh. \tag{3.21}$$

The proof is complete in view of the triangular inequality and of (3.6) and (3.16). □

Remark 3.4. Under the hypothesis of Theorem 3.3, for all $\varepsilon > 0$, we can find $m \in \mathbb{N}$ and positive integers $n_1, \ldots, n_m$ such that $\|x - \tilde{y}_m\| < \varepsilon$. Furthermore, since $\gamma < 1$ and the points of partition can be chosen such that h is as small as we desire, $\|x - \tilde{y}_m\|$ is as small as we desire, and (3.14) also provides a quota of the error committed when we approach x by $\tilde{y}_m$.

Remark 3.5. It follows from Theorem 3.3 that the proposed method with the chosen Schauder bases has order of convergence one. This choice has been done by simplicity in the exhibition of the results, and it has allowed us to obtain satisfactory numerical results as we show in the following section. Nevertheless, by integrating the considered basis in $\mathcal{C}([0,1])$ (and adding the one constant function), we obtain new bases of $\mathcal{C}([0,1])$ and $\mathcal{C}([0,1]^2)$. Considering these bases, the order of convergence is 2. In general, integrating $n+1$ times, we would obtain order of convergence n.

4. Some Numerical Example

We now turn our attention to show two numerical performance results. For each example, we have fixed the subset $\{t_i\}_{i\geq 1}$ chosen for constructing the Schauder basis $\{b_n\}_{n\geq 1}$ in $C([0,1])$ and $\{B_n\}_{n\geq 1}$ in $C([0,1]^2)$, specifically, $t_1 = 0$, $t_2 = 1$; and for $n \in \mathbb{N}\cup\{0\}$, $t_{i+1} = (2k+1)/2^{n+1}$ if $i = 2^n + k + 1$ where $0 \leq k < 2^n$ are integers. To define the sequence $\{\widetilde{y}_m\}$, we take $n_j = i$ (for all $j \geq 1$). In addition we include, a table exhibiting, for $i = 9, 17$, and 33, the absolute errors committed for certain representative points of $[0,1]$ when we approximate the exact solution $x(t)$ by means of the iteration $\widetilde{y}_m(t)$ where m is shown in the table. The algorithms associated with the numerical method were performed using Mathematica 7. We have checked that when more points are used, the accuracy improves significantly, unlike the number of iterations.

In order to provide some details, we synthesize the steps of the programmed algorithm:

(1) we introduce the nodes; we construct the base $\{b_n\}$ and the base $\{B_n\}$,

(2) we calculate g (if it is not possible to explicitly arrive at g, we can apply a quadrature method),

(3) we define $v_0(t) = 0$ and $\widetilde{y}_0(t) = g(t) + v_0(t)$,

(4) we calculate $f\widetilde{y}_0$ and $P_{n_1}(f\widetilde{y}_0)$ using the base $\{b_n\}$ and $k_1\widetilde{y}_0$, $k_2\widetilde{y}_0$, $Q_{n_1^2}(k_1\widetilde{y}_0)$, and $Q_{n_1^2}(k_2\widetilde{y}_0)$ using the base $\{B_n\}$. Note that we calculate the projections by integrals of piecewise univariate and bivariate polynomials of degrees 1 and 2 and the calculation of the coefficients of such polynomials just requires linear combinations of several evaluations of the basic functions at adequate points. We do not have to solve systems of algebraic equations,

(5) we calculate the expression $\widetilde{y}_1$ using Definition (3.8),

(6) repeat the process.

Example 4.1. For the first example, we consider the following linear Fredholm-Volterra integro-differential equation with the exact solution $x(t) = t^2$. Its numerical results are given in Table 1 and Figure 1:

$$x'(t) = \frac{1}{120}\left(10 + 180t - 15t^3 - 6t^5\right)t + \frac{1}{8}tx(t) + \int_0^1 (2 - 3(s+t) + 6st)x(s)\,ds + \int_0^1 \frac{st}{5}x(s)\,ds \quad \left((t,s) \in [0,1]^2\right), \tag{4.1}$$

$$x(0) = 0.$$

Table 1: Absolute errors for Example 4.1.

t	$i = 9$ $\lvert\tilde{y}_4(t) - x(t)\rvert$	$i = 17$ $\lvert\tilde{y}_4(t) - x(t)\rvert$	$i = 33$ $\lvert\tilde{y}_4(t) - x(t)\rvert$
0.125	$7.63E-4$	$1.84E-4$	$4.04E-5$
0.250	$1.25E-3$	$3.02E-4$	$6.42E-5$
0.375	$1.47E-3$	$3.52E-4$	$7.12E-5$
0.5	$1.42E-3$	$3.35E-4$	$6.11E-5$
0.625	$1.10E-3$	$2.47E-4$	$3.36E-5$
0.750	$4.84E-4$	$8.88E-5$	$1.15E-5$
0.875	$4.30E-4$	$1.44E-4$	$7.47E-5$
1	$1.65E-3$	$4.54E-4$	$1.56E-4$

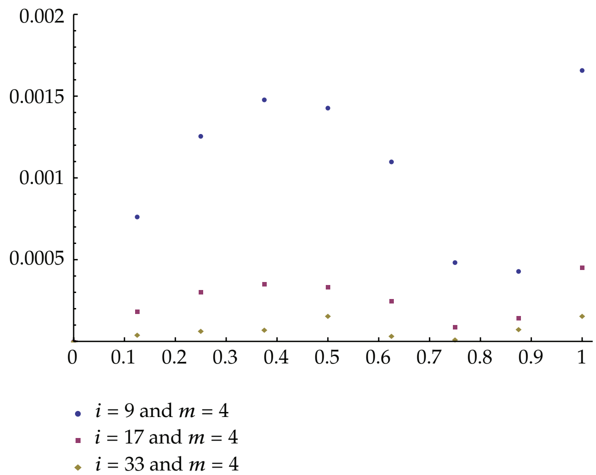

Figure 1: The plot of absolute errors by our method for Example 4.1.

Example 4.2. For the second example, we consider the following linear Fredholm-Volterra integro-differential equation with the exact solution $x(t) = t$. Its numerical results are given in Table 2 and Figure 2:

$$x'(t) = 1 - \frac{1}{4\pi} - \frac{1}{8}t^2\cos(t) - \frac{1}{3}t\sin(t) + \frac{\sin(t)}{3}x(t) + \frac{1}{4}\int_0^1 \sin(\pi s)x(s)ds + \frac{1}{4}\int_0^1 \cos(t)x(s)ds \quad \left((t,s) \in [0,1]^2\right), \qquad (4.2)$$

$$x(0) = 0.$$

5. Conclusions

An efficient approach easy to implement is proposed to solve the linear Fredholm-Volterra integro-differential equations. The approximating functions are the sum of integrals of piecewise bivariate polynomials of degree 2, and the calculation of the coefficients of such

Table 2: Absolute errors for Example 4.2.

t	$i = 9$ $\lvert\tilde{y}_6(t) - x(t)\rvert$	$i = 17$ $\lvert\tilde{y}_6(t) - x(t)\rvert$	$i = 33$ $\lvert\tilde{y}_6(t) - x(t)\rvert$
0.125	$2.35E-5$	$5.90E-6$	$1.51E-6$
0.250	$5.15E-5$	$1.29E-5$	$3.32E-6$
0.375	$8.83E-5$	$2.21E-5$	$5.68E-6$
0.5	$1.38E-4$	$3.46E-5$	$8.90E-6$
0.625	$2.05E-4$	$5.15E-5$	$1.32E-5$
0.750	$2.94E-4$	$7.37E-5$	$1.89E-5$
0.875	$4.07E-4$	$1.02E-4$	$2.63E-5$
1	$5.48E-4$	$1.37E-4$	$3.56E-5$

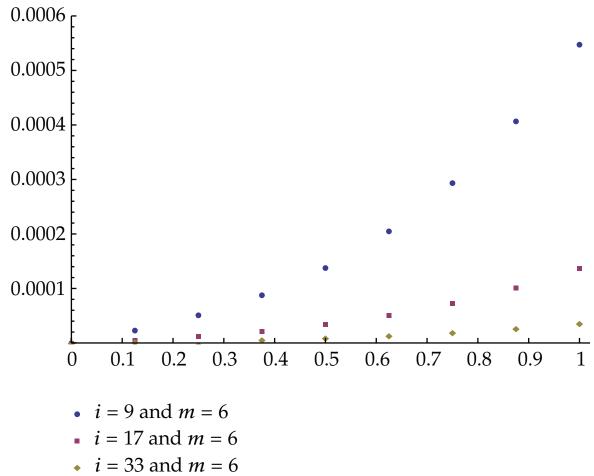

Figure 2: The plot of absolute errors by our method for Example 4.2.

polynomials just requires linear combinations of several evaluations of the basic functions at adequate points. The approach leads to an approximate solution of the integro-differential equation, which can be expressed explicitly in simple closed form, and which can be effectively computed using symbolic computing codes on any personal computer.

Acknowledgments

This research is partially supported by Junta de Andalucía Grant FQM359 and the ETSIE of the University of Granada.

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