Induced operators on bounded lattices*

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Abstract

In this paper we show a methodology for designing operators on spaces of lattice-valued mappings. More precisely, from a family of operators on a bounded lattice L and mappings from a set X to itself, we may construct an operator, that we call the induced operator, on the lattice of set mappings from X to L. Furthermore, if X is also a bounded lattice, under suitable conditions preserving the orders on L and X, the induced operator belongs to the lattice of monotone mappings from X to L. The procedure is quite simple, versatile and allows to obtain plenty of different examples in a wide range of lattices. In particular, by appropriate choices of X and L, it can be applied to the most important types of fuzzy sets. The relation with some properties associated to popular types of operators is studied. Hence, we show that, under certain conditions, aggregation operators, implications, negations, overlap functions and others are preserved by the induction process.

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1. Introduction

The fusion of information into a single output object that gathers, or represents, the original input is a ubiquitous problem in several fields of knowledge as mathematics, economics or biology, among others. For a finite set of values, the simplest and oldest solution is to consider the mean. But modern engineering problems, as decision making problems, require something more sophisticated. For instance, the reader may consult the approaches based on the Choquet integral like TOPSIS and TODIM for dynamic and heterogeneous decision making with criteria interaction [30], influenced/disturbed multi-expert decision making [40] and group decision-making based on complex spherical fuzzy and other types of aggregations [1]. Another examples can be found in fuzzy rule-based classification problems (e.g., [32]), image processing [33], deep learnig [3] or computational brain [25]. As the reader may see, the problem is quite heterogeneous and the measures standing for the information could not be simply represented by real values but also by more complex structures.

Mathematically, the formalism to deal with this topic is covered by the notion of operator. Although it may admit wider definitions, an operator is simply a mapping $L_1 \times L_2 \times \cdots \times L_n \to L$, where $L, L_1, L_2, \ldots, L_n$ are the spaces containing the types of objects under consideration. Obviously, this allows to cover other semantically different, although formally similar, problems. For instance, it also models an action between several objects. Well-known examples are the set of connectives that forms a logical system or the arithmetic operations associated to certain domain. Normally, the problems handled by an operator are determined by the properties it satisfies. Thereby, one may find in the literature some efforts for developing effective aggregation operators, or ways of construction of them, aiming to solve practical problems requiring fusion procedures. For instance, as mentioned above, Choquet-based approaches have been widely applied to real problems. Actually, weaker forms of monotonicity are sufficient to solve

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several kinds of problems, see [13, 31]. In [23] aggregation operators and OWA's for interval fuzzy sets are studied. More generally, in [7], a basic theory for operators on n-dimensional intervals is introduced. Later, in [19], admissible orders, and a method for constructing them, are considered for these class of fuzzy objects prior OWA operators are defined. In the field of type-2 fuzzy sets, the definition of aggregation operator can be found in [44], where, by using the Zadeh's extension principle, aggregation operators on (type-1) fuzzy sets are extended to aggregation operators on type-2 fuzzy sets. This is generalized to arbitrary bounded posets in [18] providing more aggregation operators on $[0,1]^{[0,1]}$, the membership values of type-2 fuzzy sets. In the context of approximate reasoning, the extension of the classical logical connectives to appropriate models on a fuzzy ambient leads to some kind of operators as t-norms, negations or implications. This is done, actually, by Zadeh in his seminal paper [47] and, later, for extended fuzzy sets, in which the truth values belong to a lattice related to [0,1]. See, for instance, [27] for t-norms on type-2 fuzzy sets, [6, 12, 16] for negations and t-norms on interval fuzzy sets, [10, 49] for implications on some extended fuzzy sets. For classification tasks, we may consider the concept of overlap function [15], aiming to measure the degree of overlap between classes in a fuzzy classification problem with two classes. For interval-valued overlap functions the reader may consult the reference [4] and, for overlap and grouping functions on lattices, the references [35, 36]. All these operators, in classical or in extended fuzzy sets, yield a wide and extensive variety of practical applications in computer sciences-related areas .

The aim of this paper is to provide a method, that we call induction, for designing operators on spaces of lattice-valued mappings. This continues and generalizes the studies made by these authors in [29] for constructing induced t-norms, t-conorms and negations from families of such operators on the lattice chosen as codomain. The method described here can be applied to arbitrary operators with no additional property. Nevertheless, since monotonicity and boundary conditions are well-inherited from the input family to the resultant operator, it can be properly used for designing aggregation operators. Analogously, by appropriate conditions, it can be applied to design other important types of operators as the ones describe above. Fixed a bounded lattice L and a set X, the procedure input is a family Γ of m-ary operators on L indexed in X, and m mappings Σ from X to X. We then construct an m-ary operator $\Phi_{\Sigma,\Gamma}$, the induced operator, on $\mathrm{Map}(X, L)$, the lattice of set mappings from X to L. One may see that a potentially large class of different operators on $\mathrm{Map}(X, L)$ can be considered by modifying Γ and/or Σ. Hence, in practice, for a specific problem, a designer has many options available for choosing suitable input parameters in such a way the induced operator is the most appropriate for the requirements of the problem. Despite the abstract description, our approach can be applied to many frameworks simply by varying the lattice L and the set X. In particular, to the most popular types of extended fuzzy sets. For instance, if $X = L = [0,1]$ we may deal with operators on $[0,1]^{[0,1]}$, the set of membership degrees of type-2 fuzzy sets. Actually, if X is a lattice as well, we specify the theory so that the induced operator is defined on $\mathrm{Hom}(X, L)$, the lattice of monotone mappings, covering more application frameworks. For example, if $L = [0,1]$ and $X = \mathbb{2}$, the two element Boolean algebra, $\mathrm{Hom}(X, L)$ is the set of intervals on $[0,1]$, the membership degree of the well-known interval fuzzy sets. In spite of the theoretical description, the method is simple and can be applied with minimal mathematical requirements.

The paper is structured as follows: Section 2 provides a short reminder of the mathematical concepts needed to develop the theory: the concepts of poset and lattice. Additionally, in order to illustrate the wide range of applicability of our methods, we describe some feasible spaces of mappings, $\mathrm{Map}(X, L)$ or $\mathrm{Hom}(X, L)$, by choosing specific lattices L and sets X. In particular, it is shown that the membership degrees associated to common fuzzy objects (as interval, set-valued or type-2 fuzzy sets) can be recovered by a suitable choice of L and X. Section 3 deals with the induction process and shows how a family of operators on the base lattice L can define an induced operator on $\mathrm{Map}(X, L)$ or on $\mathrm{Hom}(X, L)$. We also prove that the class of aggregation operators is well-preserved under the process of induction. Section 4 analyzes which operators on $\mathrm{Map}(X, L)$ are representable, that is to say, which can be constructed by the induction process. In Section 5 we show how some common properties are inherited from operators on L to the induced operator on $\mathrm{Map}(X, L)$. In Section 6 we illustrate the theory by describing some examples and analyzing other types of operators. Finally, in Section 7 we give our conclusions.

2. Background and some ambits of application

For the convenience, we firstly fix the notation and recall the basic notions concerning the theory developed throughout this paper. All along this work X denotes a set. A binary relation $\leq$ on X (that is, a subset of $X \times X$) is said to be a *partial order* if and only if it is *reflexive*, *antisymmetric* and *transitive*. A *partially ordered set* (a *poset*, for short) is then a pair $(X, \leq)$, where X is a set and $\leq$ is partial order on X. In general, if the context is clear, we simply denote it by X.

A poset $(L, \leq)$ is called a *lattice* if, for any $a, b \in L$, there exist a *least upper bound*, or *supremum*, of a and b, and a *greatest lower bound*, or *infimum*, of them. We recall that a supremum of $a, b \in L$ is an element $c \in L$ verifying that $a, b \leq c$ and, if $c' \in L$ with $a, b \leq c'$, then $c \leq c'$. Dually, we define the infimum of $a, b \in L$. We may, equivalently, say that a lattice is a set L with two operations $\wedge$ and $\vee$, called infimum and supremum, respectively, verifying the following properties:

i) *associativity*: for any $a, b, c \in L$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$.

ii) *commutativity*: for any $a, b \in L$, $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

iii) *idempotency*: for any $a \in L$, $a \wedge a = a$ and $a \vee a = a$.

iv) *absorption law*: for any $a, b \in L$, $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

Actually, the equivalence between both approaches is given by the relations $a \leq b \Longleftrightarrow a \vee b = b \Longleftrightarrow a \wedge b = a$. Again, when the context is clear enough, we do not mention the operators $\wedge$ and $\vee$, and we simply denote by L the lattice $(L, \wedge, \vee)$.

A lattice $(L, \wedge, \vee)$ is said to be *bounded* if there exist a *maximum* and a *minimum* of L. That is, two elements in L, usually denoted by 1 and 0, respectively, such that, for every $a \in L$, $0 \leq a \leq 1$ or, equivalently, $a \wedge 0 = 0$, $a \wedge 1 = a$, $a \vee 0 = a$ and $a \vee 1 = 1$.

Given a non empty set X and a lattice L, we shall denote by $\mathrm{Map}(X, L)$ the class of all set mappings from X to L. In general, $\mathrm{Map}(X, L)$ inherits a poset structure $\leq_M$ from the one on L, say $\leq_L$, by the following rule:

$$f \leq_M g \text{ if and only if } f(x) \leq_L g(x) \text{ for all } x \in X,$$

for a given pair of mappings $f, g \in \mathrm{Map}(X, L)$. Actually, $\mathrm{Map}(X, L)$ is also a lattice with operators $\wedge_M$ and $\vee_M$, given by

$$(f \wedge_M g)(x) = f(x) \wedge_L g(x) \text{ and } (f \vee_M g)(x) = f(x) \vee_L g(x),$$

for any $f, g \in \mathrm{Map}(X, L)$ and $x \in X$, where $\wedge_L$ and $\vee_L$ provide the lattice structure on L.

Whenever X has also a poset structure, we may consider the set $\mathrm{Hom}(X, L)$ of all monotone set mappings from X to L, a subset of $\mathrm{Map}(X, L)$. Analogously to the latter discussion, $\mathrm{Hom}(X, L)$ can be endowed with a poset and a lattice structure.

Given a bounded lattice L and a positive integer m, we denote by $O_m(L)$ the set of all m-ary operators on L, that is, $O_m(L) = \mathrm{Map}(L^m, L)$, the class of all set mappings from L^m to L. Hence, all along the paper, we shall consider the sets $O_m(L)$, $\mathrm{Map}(X, L)$, $\mathrm{Hom}(X, L)$, $O_m(\mathrm{Map}(X, L))$ and $O_m(\mathrm{Hom}(X, L))$ with the corresponding inherited lattice structure from L. Our primary aim in this paper is to construct useful m-ary operators on $\mathrm{Map}(X, L)$ and $\mathrm{Hom}(X, L)$ from operators on L by a simple procedure. Let us first illustrate some frameworks covered by the theory described in the following sections.

2.1. Products

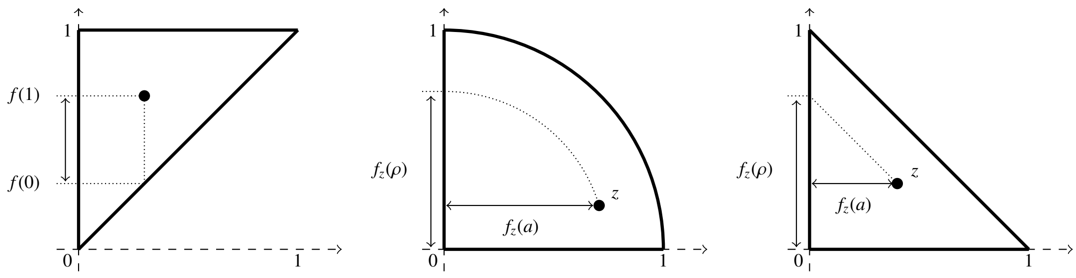
Let $\mathbb{2}$ denote the set with two elements. Then $\mathrm{Map}(\mathbb{2}, L)$ is the set $L^2 = L \times L$ with the standard lattice product structure, a particular case of the ones treated in [20] in the context of t-norms. Whenever $L = [0, 1]$, this covers the lattice of membership values for the so-called Neutrosophic Sets (NSs) in [42] by Smaradache. Actually, by a suitable bijection, it also covers the membership values for the notion of Bipolar Valued Fuzzy Sets (BVFSs) given in [46] by Zhang. In general, for any positive integer n, if $\mathbb{n} = \{1, 2, \ldots, n\}$ is the set with n elements, then $\mathrm{Map}(\mathbb{n}, L)$ is the set $L^n = \underbrace{L \times \cdots \times L}_{n\text{-times}}$ endowed with the product lattice structure.

2.2. *Intervals*

If we consider the standard Boolean lattice structure on $\mathbb{2}$, $\mathrm{Hom}(\mathbb{2}, L)$ can be identified with the set of all L-intervals

$$\mathbb{I} = \{[a, b] \text{ such that } a \leq b \text{ with } a, b \in L\},$$

where $[a, b] = \{s \in L \text{ such that } a \leq s \leq b\}$. Clearly, when $L = [0, 1]$, this yields the space of membership degrees of the well-known Interval-Valued Fuzzy Sets (IVFSs), the upper triangle described in Figure 1(a). The bijection maps any $f \in \mathrm{Hom}(\mathbb{2}, [0, 1])$ to the interval $[f(0), f(1)] \in \mathbb{I}$.



(a) IVFSs (b) PFSs (c) AIFSs

Figure 1: Space of membership values of some kind of fuzzy sets

Additionally, we may recover the membership values of other exotic kind of fuzzy sets. For instance, we recall from [45] that a Pythagorean Fuzzy Set (PFS) over a universe U is a mapping $A : U \to D([0, 1])$, where

$$D([0, 1]) = \{(x, y) \in [0, 1]^2 \text{ with } x^2 + y^2 \leq 1\},$$

a quarter of the unit disc, see Figure 1(b). Consider then $X = \{a, \rho\}$ with the order $a \leq \rho$ and $L = [0, 1]$. Hence,

$$\mathrm{Hom}(X, L) \cong D([0, 1]).$$

Indeed, the bijection is given as follows: for each $z = (x, y) \in D([0, 1])$, the associated mapping $f_z : X \to L$ is defined as $f_z(a) = x$ and $f_z(\rho) = \sqrt{x^2 + y^2}$.

Something similar can be done when considering Atanassov Intuitionistic Fuzzy Sets (AIFSs). An AIFS over a universe set U is a mapping

$$A : U \to S([0, 1]) = \{(x, y) \in [0, 1]^2 \text{ with } x + y \leq 1\},$$

where $A(x) = (\nu(x), \mu(x))$ with $\nu(x)$ being the membership degree of x and $\mu(x)$ its non-membership degree. The lattice structure on $S([0, 1])$ is provided by the partial order defined as $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \geq d$. Therefore $(0, 1)$ is the minimum of $S([0, 1])$ and $(1, 0)$, the maximum. Hence, $S([0, 1]) \cong \mathrm{Hom}(\{a, \rho\}, [0, 1])$. In this case, the bijection is given as follows: for each $z = (x, y) \in S([0, 1])$, the associated mapping $f_z : X \to L$ is defined as $f_z(a) = x$ and $f_z(\rho) = x + y$, see Figure 1(c).

In general, if $\mathbb{n} = \{1, 2, \ldots, n\}$ is endowed with a chain lattice structure, as for instance $1 \leq 2 \leq \cdots \leq n$, $\mathrm{Hom}(\mathbb{n}, L)$ is the set $L_n(L)$ of n-dimensional L-intervals, that is to say,

$$L_n(L) = \{(s_1, s_2, \ldots, s_n) \in L^n \text{ such that } s_1 \leq s_2 \leq \cdots \leq s_n\}.$$

Again, if $L = [0, 1]$, we construct the space of membership values of the so-called n-Dimensional Fuzzy Sets (nDFS).

2.3. *Type-n fuzzy sets*

In the 70's Zadeh asserted that a major issue for handling fuzzy set theory is the establishment of the membership degree of each element. Therefore, it is suggested to link soft object to them, meaning the uncertainty of computing such degrees [48]. This motivates the introduction of Type-2 Fuzzy Set (T2FS), as a fuzzy set whose membership degree is a (type-1) fuzzy set on [0,1]. Consequently, it takes values in $\mathrm{Map}([0,1],[0,1])$. In general, for an arbitrary lattice L, a type-2 L-fuzzy set takes values in $\mathrm{Map}(L, L)$, the L-fuzzy sets on L.

Although, as far we know, no application has been developed using them, we may define recursively higher types of FSs. Hence, a type-$(n+1)$ L-fuzzy set is a FS whose membership degree is a type-n L-fuzzy set (TnFS). That is to say, if L is the universe of discourse, it is a function in $\mathrm{Map}(L, \mathrm{T}n\mathrm{FS})$.

2.4. Set-valued fuzzy sets

An obvious generalization of IVFSs consists in allowing to choose an arbitrary subset of $[0,1]$ as membership degree, yielding the notion of Set-Valued Fuzzy Set (SVFS). Therefore, a SVFS on a universe of discourse U assigns to each element in U a non-empty set from the powerset $\mathcal{P}([0,1])$. In general, for an arbitrary set X, note that

$$\mathcal{P}(X) \cong \mathrm{Map}(X, \mathbb{2}),$$

then, in this case, $\mathcal{P}(X)$ inherits the lattice structure from $\mathbb{2}$.

2.5. Necessary-and-possible fuzzy sets

Following [2], a Necessary-and-Possible Hesitant Fuzzy Set (NaPHFS) over a universe X is a pair $H = (h_n, h_p)$ of hesitant fuzzy sets over X such that the inclusion $h_n(x) \subseteq h_p(x)$ holds for any $x \in X$. Obviously, the notion can be trivially extended by considering non-finite sets yielding what may be called Necessary-and-Possible Fuzzy Set (NaPFS) over X. By the latter examples, we may conclude that the membership degree of NaPFS's is given by the lattice

$$\mathrm{Hom}(\mathbb{2}, \mathcal{P}(X)) \cong \mathrm{Hom}(\mathbb{2}, \mathrm{Map}(X, \mathbb{2})).$$

2.6. Other example

Let us consider the finite lattice L given by the Hasse diagram of Figure 2.

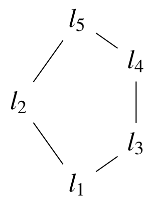


Figure 2: Hasse diagram of L

Hence, $\mathrm{Hom}(L,[0,1])$ is the set

$$D = \left\{ (x_1, x_2, x_3, x_4, x_5) \in [0,1]^5 \mid \substack{x_1 \le x_2 \le x_5 \\ x_1 \le x_3 \le x_4 \le x_5} \right\}.$$

Therefore, as showed in the next section, we may define operators on D from operators on $[0,1]$.

3. Inducing operators

In this section we deal with the main purpose of the paper, that is, to describe a construction method for operators on $\mathrm{Map}(X, L)$ from a family of operators on L. We also prove that (pre)aggregation operators are preserved under this method.

Let us consider a family of m-ary operators on L indexed in a set X, $\Gamma = \{O_x\}_{x \in X}$. This actually can be seen as the element in $\mathrm{Map}(X, O_m(L))$ mapping each $x \in X$ to the corresponding operator O_x. For convenience, we work simultaneously with both points of view. Let also $\Sigma = (\sigma_1, \ldots, \sigma_m)$ be an m-tuple, where each $\sigma_i : X \to X$ is an arbitrary set mapping. We may then define an m-ary operator

$$\mathbb{O}_{\Sigma,\Gamma} : \mathrm{Map}(X, L)^m \to \mathrm{Map}(X, L)$$

as

$$\mathbb{O}_{\Sigma,\Gamma}(f_1, \ldots, f_m)(x) = O_x(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \tag{1}$$

for any $(f_1, \ldots, f_m) \in \mathrm{Map}(X, L)^m$ and $x \in X$. That is to say, we may define a mapping

$$\Phi_\Sigma : \mathrm{Map}(X, O_m(L)) \to O_m(\mathrm{Map}(X, L))$$

given by $\Phi_\Sigma(\Gamma) = \mathbb{O}_{\Sigma,\Gamma}$ for any family $\Gamma \in \mathrm{Map}(X, O_m(L))$. In other words, it assigns a family of operators on L indexed in X to an operator on $\mathrm{Map}(X, L)$. We shall say that $\Phi_\Sigma(\Gamma)$ is Σ-induced (or simply induced, if the context is clear) by the family $\Gamma \in \mathrm{Map}(X, O_m(L))$. All along the paper, we shall denote the composition of mappings f and σ by using the symbol $\circ$, or simply by the juxtaposition $f\sigma$.

Remark 1. *This construction can be stated in a more general, although more intricate, form. Let us suppose that $L_1, \cdots, L_m, L$ are lattices and $X_1, \ldots, X_m, X$ are sets. Suppose also that there exist a family $\Gamma = \{O_x\}_{x \in X}$ of mappings $O_x : \prod_{i=1}^m L_i \to L$ for any $x \in X$ and an m-tuple $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \prod_{i=1}^m \mathrm{Map}(X, X_i)$. Hence, we may define a mapping*

$$\mathbb{O}_{\Sigma,\Gamma} : \prod_{i=1}^m \mathrm{Map}(X_i, L_i) \to \mathrm{Map}(X, L)$$

as

$$\mathbb{O}_{\Sigma,\Gamma}(f_1, \ldots, f_m)(x) = O_x(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x)))$$

for any $(f_1, \ldots, f_m) \in \prod_{i=1}^m \mathrm{Map}(X_i, L_i)$ and $x \in X$. All the claims in this section can be proved for this extended form by a straightforward reformulation. Nevertheless, in order to improve the readability of the text, we shall use the restricted version.

Remark 2. *Observe that, despite its abstract formulation, the technique is easy to apply. An eventual user only has to follow the next steps:*

- *Choose suitable L and X so that the lattice under consideration is the class of mappings from X to L.*
- *Select a family Γ of known operators on L*
- *Select m mappings from X to X.*

Then the induced operator is defined as in (1).

We recall that an m-ary operator O is monotone whenever $O \in \mathrm{Hom}(L^m, L)$, where L^m is endowed with the product lattice structure, that is to say, $O(l_1, \ldots, l_m) \leq O(t_1, \ldots, t_m)$ if $l_i \leq t_i$ for any $i = 1, \ldots, m$.

Proposition 3. *Under the above notation, let us suppose that the family $\Gamma = \{O_x\}_{x \in X}$ consists of monotone m-ary operators on L. Then $\Phi_\Sigma(\Gamma)$ is a monotone m-ary operator on $\mathrm{Map}(X, L)$.*

Proof. Let $\mathbb{O} = \Phi_\Sigma(\Gamma)$, $(f_1, \ldots, f_m), (g_1, \ldots, g_m) \in \mathrm{Map}(X, L)^m$ with $f_i \leq g_i$ for any $i = 1, \ldots, m$ and $x \in X$. Then $f_i(\sigma_i(x)) \leq g_i(\sigma_i(x))$ for any $i = 1, \ldots, m$, and then

$$\begin{aligned} \mathbb{O}(f_1, \ldots, f_m)(x) &= O_x((f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \\ &\overset{*}{\leq} O_x((g_1(\sigma_1(x)), \ldots, g_m(\sigma_m(x))) \\ &= \mathbb{O}(g_1, \ldots, g_m)(x), \end{aligned}$$

where $*$ follows from the monotonicity of operators O_x. Hence, $\mathbb{O}(f_1, \ldots, f_m) \leq \mathbb{O}(g_1, \ldots, g_m)$.

We may see that, in the lattice $\mathrm{Map}(X, L)$, the upper and lower bound are the constant mappings 1 and 0 that we denote by C_1 and C_0, respectively. In general, for an element $l \in L$, we shall denote by $C_l \in \mathrm{Map}(X, L)$ the constant mapping with the value l.

Proposition 4. *Under the above notation, given $a_1, \ldots, a_m, b \in L$, if $O_x(a_1, \ldots, a_m) = b$ for any $x \in X$, then $\Phi_\Sigma(\Gamma)(C_{a_1}, \ldots, C_{a_m}) = C_b$. In particular,*

1. *If $O_x(1, \ldots, 1) = 1$ for any $x \in X$, then $\Phi_\Sigma(\Gamma)(C_1, \ldots, C_1) = C_1$.*
2. *If $O_x(0, \ldots, 0) = 0$ for any $x \in X$, then $\Phi_\Sigma(\Gamma)(C_0, \ldots, C_0) = C_0$.*

Proof. Let $x \in X$, then

$$\begin{aligned}\Phi_\Sigma(\Gamma)(C_{a_1}, \ldots, C_{a_m})(x) &= O_x(C_{a_1}(\sigma_1(x)), \ldots, C_{a_m}(\sigma_m(x)))\\ &= O_x(a_1, \ldots, a_m)\\ &= b\\ &= C_b(x),\end{aligned}$$

thus $\Phi_\Sigma(\Gamma)(C_{a_1}, \ldots, C_{a_m}) = C_b$.

Theorem 5. *Let L be a bounded lattice, X a set and $\Gamma = \{O_x\}_{x\in X}$ a family of m-ary aggregation operators on L indexed in X. Then, for each m-tuple $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Map}(X, X)^m$, the m-ary operator $\Phi_\Sigma(\Gamma)$ on $\mathrm{Map}(X, L)$ is an aggregation operator.*

Proof. It follows directly from Propositions 3 and 4.

Example 6. *Let us consider $X = [0, 1]$ and $L = [0, 1]$, therefore $\mathrm{Map}(X, L) = [0, 1]^{[0,1]} = \mathrm{FS}([0, 1])$ the membership values of type-2 fuzzy sets, i. e., the class of (type-1) fuzzy sets in $[0, 1]$. Consider as well the family of ternary aggregation operators $\Gamma = \{O_t\}_{t\in[0,1]}$ on $[0, 1]$ given by*

$$O_t(a, b, c) = \begin{cases} a, & \text{if } t \le \frac{1}{2},\\ \frac{a+b+c}{3}, & \text{if } t > \frac{1}{2},\end{cases}$$

for any $t, a, b, c \in [0, 1]$. Set also the tuple $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the mappings $\sigma_i : [0, 1] \to [0, 1]$ with $i = 1, 2, 3$ are defined as $\sigma_1(t) = t$, $\sigma_2(t) = 1 - t$ and $\sigma_3(t) = t^2$ for any $t \in [0, 1]$. Hence, we may construct the induced ternary operator on $FS([0, 1])$, $\mathbb{O} : \mathrm{FS}([0, 1])^3 \to \mathrm{FS}([0, 1])$, provided by

$$\begin{aligned}\mathbb{O}(f_1, f_2, f_3)(t) &= O_t(f_1\sigma_1(t), f_2\sigma_2(t), f_3\sigma_3(t))\\ &= O_t(f_1(t), f_2(1 - t), f_3(t^2))\\ &= \begin{cases} f_1(t) & \text{if } t \le \frac{1}{2},\\ \frac{f_1(t)+f_2(1-t)+f_3(t^2)}{3}, & \text{if } t > \frac{1}{2}.\end{cases}\end{aligned}$$

for any mappings $f_1, f_2, f_3 \in M$ and any $t \in [0, 1]$. By Theorem 5, $\mathbb{O}$ is a ternary aggregation operator on $\mathrm{FS}([0, 1])$. Hence, from aggregation operators on $[0, 1]$, we may construct aggregation operators on type-1 fuzzy sets. In general, from aggregation operators on type-n fuzzy sets, we may induce aggregation operators on type-$(n + 1)$ fuzzy sets.

Let us suppose, additionally, that X is also a bounded lattice. Hence we may consider the space of monotone mappings $\mathrm{Hom}(X, L)$, a subspace of $\mathrm{Map}(X, L)$, and study m-ary operators on this space. We say that a family $\{O_x\}_{x\in X}$ of m-ary operators on L indexed in X is monotone if the family preserves the order on X, that is, it verifies that $O_x \le O_y$ whenever $x \le y$. It should not be confused with asserting that an operator O_x is monotone, meaning that $O_x(l_1, \ldots, l_m) \le O_x(s_1, \ldots, s_m)$ whenever $l_i \le s_i$ for any $i = 1, \ldots, m$. For brevity, we shall denote by $O_m^{\nearrow}(L)$ the set of m-ary monotone operators on an arbitrary bounded lattice L.

Theorem 7. *Let $\Gamma = \{O_x\}_{x\in X} \in \mathrm{Map}(X, O_m^{\nearrow}(L))$ be a family of m-ary monotone operators on L, $\Sigma \in \mathrm{Hom}(X, X)^m$ and $\Phi_\Sigma(\Gamma)$ the Σ-induced operator. Then the following assertions are equivalent:*

i) Γ *is monotone, i.e.* $\Gamma \in \mathrm{Hom}(X, O_m^{\nearrow}(L))$.

ii) $\Phi_\Sigma(\Gamma)$ *is an m-ary monotone operator on* $\mathrm{Hom}(X, L)$.

Proof. *i)* $\Rightarrow$ *ii)*. Given $(f_1, \ldots, f_n) \in \mathrm{Hom}(X, L)^m$, we prove that the mapping $\Phi_\Sigma(\Gamma)(f_1, \ldots, f_n)$ is in $\mathrm{Hom}(X, L)$. Indeed, let $x, y \in X$ with $x \leq y$. Hence $f_i(\sigma_i(x)) \leq f_i(\sigma_i(y))$ for any $i = 1, \ldots, n$. Thus

$$\begin{aligned} \Phi_\Sigma(\Gamma)(f_1, \ldots, f_n)(x) &= O_x(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \\ &\overset{*}{\leq} O_y(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \\ &\overset{\dagger}{\leq} O_y(f_1(\sigma_1(y)), \ldots, f_m(\sigma_m(y))) \\ &= \Phi_\Sigma(\Gamma)(f_1, \ldots, f_m)(y), \end{aligned}$$

where $*$ follows from the monotonicity of Γ and $\dagger$ from the monotonicity of O_y. Now, let $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathrm{Hom}(X, L)$ with $f_i \leq g_i$ for any $i = 1, \ldots, m$. Hence, for any $x \in X$,

$$\Phi_\Sigma(\Gamma)(f_1, \ldots, f_m)(x) = O_x(f_1\sigma_1(x), \ldots, f_m\sigma_m(x)) \leq O_x(g_1\sigma_1(x), \ldots, g_m\sigma_m(x)) = \Phi_\Sigma(\Gamma)(g_1, \ldots, g_m)(x),$$

thus $\Phi_\Sigma(\Gamma)(f_1, \ldots, f_m) \leq \Phi_\Sigma(\Gamma)(g_1, \ldots, g_m)$ and then $\Phi_\Sigma(\Gamma) \in O_m^{\nearrow}(\mathrm{Hom}(X, L))$.

ii) $\Rightarrow$ *i)*. Let $x, y \in X$ with $x \leq y$ and $l_1, \ldots, l_m \in L$. Hence

$$\begin{aligned} O_x(l_1, \ldots, l_m) &= O_x(C_{l_1}(\sigma_1(x)), \ldots, C_{l_m}(\sigma_m(x))) \\ &= \Phi_\Sigma(\Gamma)(C_{l_1}, \ldots, C_{l_m})(x) \\ &\leq \Phi_\Sigma(\Gamma)(C_{l_1}, \ldots, C_{l_m})(y) \\ &= O_y(C_{l_1}(\sigma_1(y)), \ldots, C_{l_m}(\sigma_m(y))) \\ &= O_y(l_1, \ldots, l_m). \end{aligned}$$

Thus Γ is monotone.

Therefore, under these conditions, the restriction

$$\Phi_{\Sigma_{|\mathrm{Hom}(X, O_m^{\nearrow}(L))}} : \mathrm{Hom}(X, O_m^{\nearrow}(L)) \to O_m^{\nearrow}(\mathrm{Hom}(X, L)), \tag{2}$$

is well-defined. For simplicity, we shall also denote it by Φ_Σ.

Example 8. *The condition of* Σ *being a tuple of monotone mappings is necessary. Suppose, for instance, that* σ_1 *is not monotone, i. e., there exist* $x, y \in X$ *with* $x \leq y$ *and* $\sigma_1(x) > \sigma_1(y)$. *Consider the constant family of m-ary operators* $\Gamma = \{O\}_{x \in X}$ *where* $O(l_1, \ldots, l_m) = \min\{l_1, \ldots, l_m\}$ *for all* $l_1, \ldots, l_m \in L$. *Let us denote by* $\mathbb{O}$ *the induced operator.*

Fix the monotone mappings C_1 *and* $f : L \to L$ *given by*

$$f(z) = \begin{cases} 0, & \text{if } z < \sigma_1(x) \\ 1 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \mathbb{O}(f, C_1, \ldots, C_1)(x) &= O_x(f(\sigma_1(x)), 1, \ldots, 1) \\ &= \min\{1, 1, \ldots, 1\} \\ &= 1, \end{aligned}$$

whilst,

$$\begin{aligned} \mathbb{O}(f, C_1, \ldots, C_1)(y) &= O_y(f(\sigma_1(y)), 1, \ldots, 1) \\ &= \min\{0, 1, \ldots, 1\} \\ &= 0. \end{aligned}$$

Thus $\mathbb{O}(f, C_1, \ldots, C_1) \notin \mathrm{Hom}(X, L)$.

Contrary to that, from a tuple formed by non monotone mappings we may obtain an operator on $\mathrm{Hom}(X, L)$ *(obviously, from a monotone family of operators). Set now* $m = 2$, $X = \{0, 1\}$, $L = [0, 1]$, $\Sigma = (n, \mathrm{Id}_X)$, *where* $n(0) = 1$

and $n(1) = 0$, and $O_0(a,b) = \min\{a,b\}$ and $O_1(a,b) = \max\{a,b\}$. Hence $\Phi_\Sigma(\Gamma) = \mathbb{O}$ is an operator on $\mathrm{Hom}(X,L)$*. Indeed, if $f, g \in \mathrm{Hom}(X,L)$, that is, $f(0) \leq f(1)$ and $g(0) \leq g(1)$,*

$$\mathbb{O}(f,g)(0) = O_0(f(1), g(0)) = \min\{f(1), g(0)\} \text{ and } \mathbb{O}(f,g)(1) = O_0(f(0), g(1)) = \max\{f(0), g(1)\}.$$

Since $\min\{f(1), g(0)\} \leq g(0) \leq g(1) \leq \max\{f(0), g(1)\}$, we find that $\mathbb{O}(f,g) \in \mathrm{Hom}(X,L)$.

Analogously to Theorem 5, families of aggregation operators yield aggregation operators on $\mathrm{Hom}(X,L)$.

Theorem 9. *Let L and X be a bounded lattice, Γ a family of m-ary aggregation operators on L indexed in X preserving the order on X. Then, for each tuple $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Hom}(X,X)^m$, the m-ary operator $\Phi_\Sigma(\Gamma)$ on $\mathrm{Hom}(X,L)$ is an aggregation operator.*

Proof. It follows directly from Proposition 4 and Theorem 7.

Example 10. *Let us consider $X = \{1,2,3\}$, the set with three ordered elements ($1 < 2 < 3$), and $L = [0,1]$. Hence $\mathrm{Hom}(X,L) = L_3([0,1])$, the set of 3-intervals on $[0,1]$. Fix the family of aggregation operators on $[0,1]$, $\Gamma = \{O_1, O_2, O_3\}$, constructed from the p-norms,*

$$O_t(a,b,c) = \begin{cases} \frac{a+b+c}{3}, & \text{if } t = 1,\\ \left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}, & \text{if } t = 2,\\ \max(a,b,c), & \text{if } t = 3, \end{cases}$$

for any $a,b,c \in [0,1]$. Clearly $O_1 \leq O_2 \leq O_3$, thus Γ is a monotone family of aggregation operators. Let $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Hom}(X,X)$ be constant mappings given by $\sigma_1(t) = 1$, $\sigma_2(t) = 2$ and $\sigma_3(t) = 3$ for any $t = 1,2,3$. Hence, $\mathbb{O} = \Phi_\Sigma(\Gamma)$ is a ternary aggregation operator on $L_3([0,1])$. Concretely, for any $a = (a_1,a_2,a_3)$, $b = (b_1,b_2,b_3)$, $c = (c_1,c_2,c_3) \in L_3([0,1])$.

$$\mathbb{O}(a,b,c) = \left(\frac{a_1+b_2+c_3}{3}, \left(\frac{a_1^2+b_2^2+c_3^2}{3}\right)^{\frac{1}{2}}, \max(a_1,b_2,c_3)\right).$$

Observe that a different choice of Σ provides a different induced operator. Actually, it could provide a different image for Φ_Σ, see Figure 3 below. For instance, consider the tuple $\Sigma' = (\mathrm{Id}_X, \mathrm{Id}_X, \mathrm{Id}_X)$. Hence, $\overline{\mathbb{O}} = \Phi_{\Sigma'}(\Gamma)$ is defined as

$$\overline{\mathbb{O}}(a,b,c) = \left(\frac{a_1+b_1+c_1}{3}, \left(\frac{a_2^2+b_2^2+c_2^2}{3}\right)^{\frac{1}{2}}, \max(a_3,b_3,c_3)\right).$$

for any $a = (a_1,a_2,a_3)$, $b = (b_1,b_2,b_3)$, $c = (c_1,c_2,c_3) \in L_3([0,1])$.

Under certain conditions, our theory remains valid for pre-aggregation operators. We recall from [31] that an m-ary operator $O : [0,1]^m \to [0,1]$ is said to be a pre-aggregation operator if verifies

a) the boundary conditions, $O(1,\ldots,1) = 1$ and $O(0,\ldots,0) = 0$,

b) and the property of being r-increasing for a non zero vector $r = (r_1,\ldots,r_m) \in [0,1]^m$, that is,

$$O((x_1,\ldots,x_m) + c(r_1,\ldots,r_m)) \geq O(x_1,\ldots,x_m)$$

for any $(x_1,\ldots,x_m) \in [0,1]^m$ and any $c > 0$ such that $(x_1,\ldots,x_m) + c(r_1,\ldots,r_m) \in [0,1]^m$.

Let now X be a set, we may consider two operations on $\mathrm{Map}(X,\mathbb{R})$ inherited from the sum and product on real numbers. Namely, for any $f, g \in \mathrm{Map}(X,\mathbb{R})$ and $c \in [0,1]$, the mappings $f+g$ and cf are defined as $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$, respectively, for all $x \in X$. This allows us to define the directional monotonicity in $\mathrm{Map}(X,[0,1])$.

Definition 11. *Given a non zero m-tuple* $r = (r_1, \dots, r_m) \in \mathrm{Map}(X, \mathbb{R})^m$*, an operator* $\mathbb{O} : \mathrm{Map}(X, [0,1])^m \to \mathrm{Map}(X, [0,1])$ *is r-increasing if*

$$\mathbb{O}((f_1, \dots, f_m) + c(r_1, \dots, r_m)) \geq \mathbb{O}(f_1, \dots, f_m)$$

for any $(f_1, \dots, f_m) \in \mathrm{Map}(X, [0,1])^m$ *and any* $c > 0$ *such that* $f_i + cr_i \in \mathrm{Map}(X, [0,1])$ *for all* $i = 1, \dots, m$*. Of course, we understand here that* $c(r_1, \dots, r_m) = (cr_1, \dots, cr_m)$*.*

Consequently, the definition of pre-aggregation operator on $\mathrm{Map}(X, [0,1])$ can be stated as follows.

Definition 12. *An m-ary pre-aggregation operator* $\mathbb{O}$ *on* $\mathrm{Map}(X, [0,1])$ *is an m-ary operator satisfying that* $\mathbb{O}(C_1, \dots, C_1) = C_1$*,* $O(C_0, \dots, C_0) = C_0$ *and the property of being r-increasing for some non zero vector* $r \in \mathrm{Map}(X, [0,1])^m$*.*

Additionally, if X is also a bounded lattice, we may state a similar definition for operators on $\mathrm{Hom}(X, [0,1])$. Indeed, we say that an m-ary operator $\mathbb{O}$ on $\mathrm{Hom}(X, [0,1])$ is a pre-aggregation operator if $\mathbb{O}(C_1, \dots, C_1) = C_1$, $O(C_0, \dots, C_0) = C_0$ and there exists a non zero vector $r = (r_1, \dots, r_m) \in \mathrm{Map}(X, [0,1])^m$ such that

$$\mathbb{O}((f_1, \dots, f_m) + c(r_1, \dots, r_m)) \geq \mathbb{O}(f_1, \dots, f_m)$$

for any $(f_1, \dots, f_m) \in \mathrm{Hom}(X, [0,1])^m$ and any $c > 0$ such that $f_i + cr_i \in \mathrm{Hom}(X, [0,1])$ for all $i = 1, \dots, m$. Observe that, when $X = \mathbb{2}$, this definition coincides with [41, Definition 10].

Theorem 13. *Let X be a set,* $\Gamma = \{O_x\}_{x \in X}$ *a family of m-ary operators on* $[0,1]$ *and* $\Sigma = (\sigma_1, \dots, \sigma_m) \in \mathrm{Map}(X, X)^m$*. Suppose that, for each* $x \in X$*,* O_x *is an* r^x*-increasing pre-aggregation operator for a vector* $r^x = (r_1^x, \dots, r_m^x) \in [0,1]^m$ *verifying that* $r_i^{\sigma_i(x)} = r_i^x$ *for all* $x \in X$ *and* $i = 1, \dots, m$*. Then* $\Phi_\Sigma(\Gamma)$ *is an m-ary* $(r_1, \dots, r_m)$*-increasing pre-aggregation operator on* $\mathrm{Map}(X, [0,1])$*, where* $r_i(x) = r_i^x$ *for all* $x \in X$ *and any* $i = 1, \dots, m$*.*

If, additionally, X is a bounded lattice, Γ *preserves the order on X and* $\Sigma \in \mathrm{Hom}(X, X)^m$*, the same result holds for operators on* $\mathrm{Hom}(X, [0,1])$*.*

Proof. By the former results, it is sufficient to show that $\Phi_\Sigma(\Gamma)$ is $(r_1, \dots, r_m)$-increasing. Let $f_1, \dots, f_m \in \mathrm{Map}(X, [0,1])$ and $c > 0$ such that $f_i(x) + cr_i^x \in [0,1]$ for all $x \in X$ and all $i = 1, \dots, m$, then, for all $x \in X$,

$$\begin{aligned}
\Phi_\Sigma(\Gamma)((f_1, \dots, f_m) + c(r_1, \dots, r_m))(x) &= O_x((f_1 + cr_1)(\sigma_1(x)), \dots, (f_m + cr_m)(\sigma_m(x))) \\
&= O_x(f_1(\sigma_1(x)) + cr_1(\sigma_1(x)), \dots, f_m(\sigma_m(x)) + cr_m(\sigma_m(x))) \\
&= O_x(f_1(\sigma_1(x)) + cr_1^{\sigma_1(x)}, \dots, f_m(\sigma_m(x)) + cr_m^{\sigma_1(x)}) \\
&= O_x(f_1(\sigma_1(x)) + cr_1^x, \dots, f_m(\sigma_m(x)) + cr_m^x) \\
&\geq O_x(f_1(\sigma_1(x)), \dots, f_m(\sigma_m(x))) \\
&= \Phi_\Sigma(\Gamma)(f_1, \dots, f_m)(x).
\end{aligned}$$

Then $\Phi_\Sigma(\Gamma)((f_1, \dots, f_m) + c(r_1, \dots, r_m)) \geq \Phi_\Sigma(\Gamma)(f_1, \dots, f_m)$.

Example 14. *Let us consider the family* $\Gamma = \{L_\lambda\}_{\lambda \in [0,1]}$ *of binary pre-aggregation operators on* $[0,1]$ *formed by the weighted Lehmer means, i.e., for any* $\lambda \in [0,1]$ *and* $x, y \in [0,1]$*,*

$$L_\lambda(x, y) = \frac{x^2 \lambda + y^2 (1 - \lambda)}{x\lambda + y(1 - \lambda)}$$

with the assumption $0/0 = 0$*. Let us consider* Σ *formed by the identity mappings, hence* $\mathbb{L} = \Phi_\Sigma(\Gamma)$ *is a binary pre-aggregation operator on* $[0,1]^{[0,1]}$*. Namely,*

$$\mathbb{L}(f, g)(\lambda) = L_\lambda(f(\lambda), g(\lambda)) = \frac{f(\lambda)^2 \lambda + g(\lambda)^2 (1 - \lambda)}{f(\lambda)\lambda + g(\lambda)(1 - \lambda)}$$

for any $f, g \in [0,1]^{[0,1]}$ *and* $\lambda \in [0,1]$*. Observe that, for any* $\lambda \in [0,1]$*,* L_λ *is* $(1 - \lambda, \lambda)$*-increasing [31], and then* $\mathbb{L}$ *is* (r_1, r_2)*-increasing, where* $r_1(t) = 1 - t$ *and* $r_2(t) = t$ *for any* $t \in [0,1]$*.*

4. Representable operators

At this point, a natural question is to determine which operators on $\mathrm{Map}(X, L)$ can be constructed in this way. Let us firstly analyze the aforementioned mappings Φ_Σ.

Proposition 15. *Let L be a bounded lattice, X a set and $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Map}(X, X)^m$. The mapping Φ_Σ is injective. Actually, a left inverse is given by*

$$\Lambda : \mathcal{O}_m(\mathrm{Map}(X, L)) \to \mathrm{Map}(X, \mathcal{O}_m(L))$$

$$\mathbb{O} \longmapsto \Lambda(\mathbb{O}) : X \to \mathcal{O}_m(L),$$

where $\Lambda(\mathbb{O})(x)(a_1, \ldots, a_m) = \mathbb{O}(C_{a_1}, \ldots, C_{a_m})(x)$ for any $x \in X$ and any $a_1, \ldots, a_m \in L$.

Proof. Let $\Gamma = \{O_x\}_{x\in X} \neq \{O'_x\}_{x\in X} = \Gamma'$ be two different families of operators, then there exist $(a_1, \ldots, a_m) \in L^m$ and $x \in X$ such that $O_x(a_1, \ldots, a_m) \neq O'_x(a_1, \ldots, a_m)$. Hence

$$\begin{aligned}\Phi_\Sigma(\Gamma)(C_{a_1}, \ldots, C_{a_m})(x) &= O_x(a_1, \ldots, a_m)\\ &\neq O'_x(a_1, \ldots, a_m)\\ &= \Phi_\Sigma(\Gamma')(C_{a_1}, \ldots, C_{a_m})(x).\end{aligned}$$

Then $\Phi_\Sigma(\Gamma) \neq \Phi_\Sigma(\Gamma')$ and therefore Φ_Σ is injective. The mapping Λ is left inverse to Φ_Σ. Indeed,

$$\begin{aligned}\Lambda(\Phi_\Sigma(\Gamma))(x)(a_1, \ldots, a_m) &= \Phi_\Sigma(\Gamma)(C_{a_1}, \ldots, C_{a_m})(x)\\ &= O_x(C_{a_1}(\sigma_1(x)), \ldots, C_{a_m}(\sigma_m(x)))\\ &= O_x(a_1, \ldots, a_m)\\ &= \Gamma(x)(a_1, \ldots, a_m),\end{aligned}$$

thus $\Lambda \circ \Phi_\Sigma = \mathrm{Id}_{\mathrm{Map}(X, \mathcal{O}_m(L))}$, the identity mapping.

Remark 16. *Observe that the mapping Λ is left inverse to Φ_Σ independently of Σ. For different choices of Σ, the mappings Φ_Σ may provide different "copies" of $\mathrm{Map}(X, \mathcal{O}_m(L))$ in $\mathcal{O}_m(\mathrm{Map}(X, L))$, see Figure 3.*

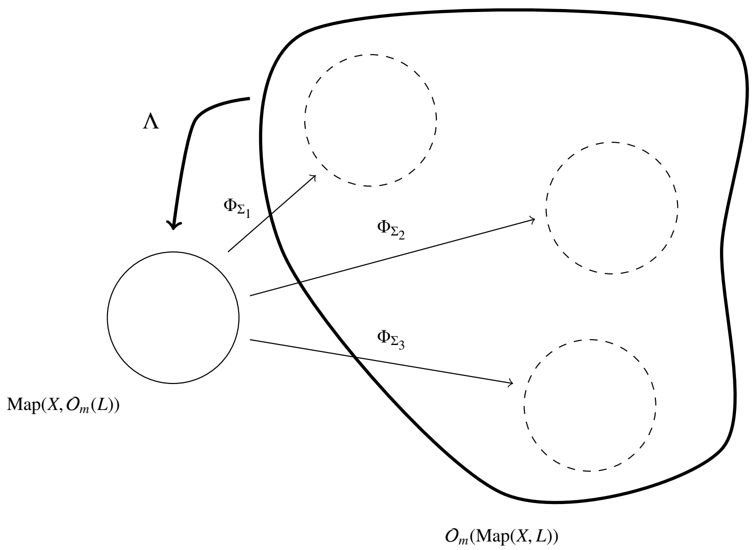


Figure 3: Mappings Λ and Φ_Σ

The same result can be stated whenever X is a lattice and we deal with operators on $\mathrm{Hom}(X, L)$.

Proposition 17. *Let L and X be bounded lattices and $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Hom}(X, X)^m$. The mapping*

$$\Lambda : O_m^{\nearrow}(\mathrm{Hom}(X, L)) \to \mathrm{Hom}(X, O_m^{\nearrow}(L))$$

defined as in Proposition 15 is left inverse of the mapping Φ_Σ described in (2).

Proof. Following the proof of Proposition 15, it only remains to prove that Λ is well-defined. Indeed, let $\mathbb{O} \in O_m^{\nearrow}(\mathrm{Hom}(X, L))$ and $x, y \in X$ with $x \leq y$. Hence, for any $a_1, \ldots, a_m, b_1, \ldots, b_m \in L$ with $a_i \leq b_i$ for any $i = 1, \ldots, m$,

$$\Lambda(\mathbb{O})(x)(a_1, \ldots, a_m) = \mathbb{O}(C_{a_1}, \ldots, C_{a_m})(x) \leq \mathbb{O}(C_{b_1}, \ldots, C_{b_m})(x) = \Lambda(\mathbb{O})(x)(b_1, \ldots, b_m),$$

since $\mathbb{O} \in O_m^{\nearrow}(\mathrm{Hom}(X, L))$, then $\Lambda(\mathbb{O})(x) \in O_m^{\nearrow}(L)$. On the other hand,

$$\Lambda(\mathbb{O})(x)(a_1, \ldots, a_m) = \mathbb{O}(C_{a_1}, \ldots, C_{a_m})(x) \leq \mathbb{O}(C_{a_1}, \ldots, C_{a_m})(y) = \Lambda(\mathbb{O})(y)(a_1, \ldots, a_m).$$

Thus $\Lambda(\mathbb{O})(x) \leq \Lambda(\mathbb{O})(y)$, and then $\Lambda(\mathbb{O}) \in \mathrm{Hom}(X, O_m^{\nearrow}(L))$.

Definition 18. *Given an m-tuple $\Sigma \in \mathrm{Map}(X, X)^m$, an m-ary operator $\mathbb{O} \in O_m(\mathrm{Map}(X, L))$ is said to be Σ-representable whenever $\mathbb{O} \in \mathrm{Im}\,\Phi_\Sigma$, the image of Φ_Σ or, equivalently, $\Phi_\Sigma \circ \Lambda(\mathbb{O}) = \mathbb{O}$.*

Theorem 19. *Let $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Map}(X, X)^m$ and $\mathbb{O} \in O_m(\mathrm{Map}(X, L))$ be an m-ary operator. The following conditions are equivalent:*

i) *$\mathbb{O}$ is Σ-representable,*

ii) *for each m-tuple $(f_1, \ldots, f_m) \in \mathrm{Map}(X, L)^m$ and each $x \in X$,*

$$\mathbb{O}(f_1, \ldots, f_m)(x) = \mathbb{O}(C_{f_1(\sigma_1(x))}, \ldots, C_{f_m(\sigma_m(x))})(x),$$

iii) *for each m-tuples $(f_1, \ldots, f_m), (g_1, \ldots, g_m) \in \mathrm{Map}(X, L)^m$ and each $x \in X$, if $f_i(\sigma_i(x)) = g_i(\sigma_i(x))$ for any $i = 1, \ldots, m$ then*

$$\mathbb{O}(f_1, \ldots, f_m)(x) = \mathbb{O}(g_1, \ldots, g_m)(x).$$

Proof. *i)* $\Rightarrow$ *ii)* If $\mathbb{O}$ is Σ-representable, there exists a family of operators on L, $\Gamma = \{O_x\}_{x \in X}$, such that

$$\begin{aligned} \mathbb{O}(f_1, \ldots, f_m)(x) &= O_x(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \\ &= O_x(C_{f_1(\sigma_1(x))}(\sigma_1(x)), \ldots, C_{f_m(\sigma_m(x))}(\sigma_m(x))) \\ &= \mathbb{O}(C_{f_1(\sigma_1(x))}, \ldots, C_{f_m(\sigma_m(x))})(x). \end{aligned}$$

ii) $\Rightarrow$ *iii)* If $f_i(\sigma_i(x)) = g_i(\sigma_i(x))$ then $C_{f_i(\sigma_i(x))} = C_{g_i(\sigma_i(x))}$ and the result follows.

iii) $\Rightarrow$ *i)* Let us denote $\widehat{\mathbb{O}} = \Phi_\Sigma(\Lambda(\mathbb{O}))$. Then

$$\begin{aligned} \widehat{\mathbb{O}}(f_1, \ldots, f_m)(x) &= \Phi_\Sigma(\Lambda(\mathbb{O}))(f_1, \ldots, f_m)(x) \\ &= (\Lambda(\mathbb{O}))_x(f_1(\sigma_1(x)), \ldots, f_m(\sigma_m(x))) \\ &= \mathbb{O}(C_{f_1(\sigma_1(x))}, \ldots, C_{f_m(\sigma_m(x))})(x) \\ &= \mathbb{O}(f_1, \ldots, f_m)(x), \end{aligned}$$

since $f_i(\sigma_i(x)) = C_{f_i(\sigma_i(x))}(\sigma_i(x))$ for all $i = 1, \ldots, m$.

Remark 20. *By virtue of Remark 16, given an m-ary operator $\mathbb{O}$ on $\mathrm{Map}(X, L)$, the family $\Lambda(\mathbb{O})$ is the only one which can eventually provide the representability of $\mathbb{O}$.*

Remark 21. *In view of Proposition 17, analogous versions of Definition 18 and Theorem 19 for operators on $\mathrm{Hom}(X, L)$ can be formulated almost word for word.*

Example 22. *Unfortunately, not all operators on* Map(X, L) *are representable. We may see this by an easy combinatorial exercise. Consider $X = L = \{0, 1\}$, hence the number of binary operators on L is 16.* Map(X, L) *is given by four elements, then there exist 4^{16} binary operators on* Map(X, L)*. Now, there exist 4^2 different choices for Σ and 16^2 different families of operators on L indexed in X, thus there exist, at most, $16^2 4^2 = 4^6$ induced operators. That is, most of the operators on* Map(X, L) *are not representable.*

Example 23. *An operator can be Σ_1-representable and Σ_2-representable for two different tuples Σ_1 and Σ_2. Set $m = 2$, $X = \{0, 1\}$ and $L = [0, 1]$. Let us consider the following family of operators $\Gamma = \{O_0, O_1\}$:*

- *$O_0(a, b) = \frac{a+b}{2}$ for any $a, b \in [0, 1]$, the mean, and*
- *$O_1(a, b) = 1$ for any $a, b \in [0, 1]$, the constant operator.*

Set $\Sigma_1 = (\mathrm{Id}_X, \mathrm{Id}_X)$ and $\Sigma_2 = (\mathrm{Id}_X, C_0)$. Hence, given $f, g \in$ Map(X, L)

$$\Phi_{\Sigma_1}(\Gamma)(f, g)(0) = O_0(f(0), g(0)) = \frac{f(0) + g(0)}{2}, \text{ and}$$

$$\Phi_{\Sigma_1}(\Gamma)(f, g)(1) = O_0(f(1), g(1)) = 1.$$

On the other hand,

$$\Phi_{\Sigma_2}(\Gamma)(f, g)(0) = O_0(f(0), g(0)) = \frac{f(0) + g(0)}{2}, \text{ and}$$

$$\Phi_{\Sigma_2}(\Gamma)(f, g)(1) = O_0(f(0), g(0)) = 1$$

Therefore, for each $f, g \in$ Map(X, L)*, $\Phi_{\Sigma_1}(\Gamma)(f, g) = \Phi_{\Sigma_2}(\Gamma)(f, g)$, and thus $\Phi_{\Sigma_1}(\Gamma) = \Phi_{\Sigma_2}(\Gamma)$. It is clear that $O_0 \leq O_1$, thus this example is also valid for operators on* Hom(X, L)*.*

5. Lifting properties

In this section, we focus on examining how some properties of operators are preserved by the induction process. Although we will not mention it explicitly, all the statements here are also valid when working over monotone families of operators, yielding monotone operators on Hom(X, L). We shall follow the same notation of the former sections.

5.1. Commutative/Symmetric operators

In what follows, let us denote by $\mathcal{S}_m$ the set of all permutations with m elements. Let us recall that an m-ary operator O on a bounded lattice L is said to be commutative (or symmetric) if, for each $l_1, \ldots, l_m \in L$, $O(l_1, \ldots, l_m) = O(l_{\tau(1)}, \ldots, l_{\tau(m)})$ for any permutation $\tau \in \mathcal{S}_m$.

Proposition 24. *Let $\Gamma = \{O_x\}_{x \in X}$ be a family of m-ary operators on L and $\Sigma = (\sigma_1, \ldots, \sigma_m) \in$* Map$(X, X)^m$*. The induced operator $\mathbb{O} = \Phi_\Sigma(\Gamma)$ is commutative if and only if*

i) O_x is commutative for any $x \in X$, and

ii) $\mathbb{O}$ is Σ^τ-induced for any $\tau \in \mathcal{S}_m$, that is to say, $\Phi_\Sigma(\Gamma) = \Phi_{\Sigma^\tau}(\Gamma)$ for any $\tau \in \mathcal{S}_m$, where $\Sigma^\tau = (\sigma_{\tau(1)}, \ldots, \sigma_{\tau(m)})$.

Proof. Suppose first that $\mathbb{O} = \Phi_\Sigma(\Gamma)$ is commutative. Let us prove *i*). For any $x \in X$, $l_1, \ldots, l_m \in L$ and $\tau \in \mathcal{S}_m$,

$$\begin{aligned} O_x(l_1, \ldots, l_m) &= O_x(C_{l_1}(\sigma_1(x)), \ldots, C_{l_m}(\sigma_m(x))) \\ &= \mathbb{O}(C_{l_1}, \ldots, C_{l_m})(x) \\ &= \mathbb{O}(C_{l_{\tau(1)}}, \ldots, C_{l_{\tau(m)}})(x) \\ &= O_x(C_{l_{\tau(1)}}(\sigma_1(x)), \ldots, C_{l_{\tau(m)}}(\sigma_m(x))) \\ &= O_x(l_{\tau(1)}, \ldots, l_{\tau(m)}). \end{aligned}$$

Thus O_x is commutative for any $x \in X$. Now, we prove *ii*). Let $\tau \in \mathcal{S}_m$, $f_1, \ldots, f_m \in \mathrm{Map}(X, L)$ and $x \in X$, hence

$$\begin{aligned} \mathbb{O}(f_1, \ldots, f_m)(x) &\overset{\dagger}{=} \mathbb{O}(f_{\tau^{-1}(1)}, \ldots, f_{\tau^{-1}(m)})(x) \\ &= O_x(f_{\tau^{-1}(1)}\sigma_1(x), \ldots, f_{\tau^{-1}(m)}\sigma_m(x)) \\ &\overset{*}{=} O_x(f_1\sigma_{\tau(1)}(x), \ldots, f_m\sigma_{\tau(m)}(x)) \\ &= \Phi_{\Sigma^\tau}(\Gamma)(f_1, \ldots, f_m)(x), \end{aligned}$$

where $\dagger$ and $*$ follow from the commutativity of $\mathbb{O}$ and O_x, respectively.

Conversely, let $\tau \in \mathcal{S}_m$, $f_1, \ldots, f_m \in \mathrm{Map}(X, L)$ and $x \in X$,

$$\begin{aligned} \mathbb{O}(f_1, \ldots, f_m)(x) &\overset{ii)}{=} \Phi_{\Sigma^{\tau^{-1}}}(\Gamma)(f_1, \ldots, f_m)(x) \\ &= O_x(f_1\sigma_{\tau^{-1}(1)}(x), \ldots, f_m\sigma_{\tau^{-1}(m)}(x)) \\ &\overset{i)}{=} O_x(f_{\tau(1)}\sigma_1(x), \ldots, f_{\tau(1)}\sigma_m(x)) \\ &= \mathbb{O}(f_{\tau(1)}, \ldots, f_{\tau(m)})(x), \end{aligned}$$

thus $\mathbb{O}(f_1, \ldots, f_m) = \mathbb{O}(f_{\tau(1)}, \ldots, f_{\tau(m)})$. Hence $\mathbb{O}$ is commutative.

Example 25. *Let $X = \{1, 2, 3\}$, $L = [0, 1]$ and $\Gamma = \{O_1, O_2, O_3\}$ be the family of operators given by*

- $O_1(a, b) = \max(a, b)$ *for any* $a, b \in [0, 1]$,
- $O_2(a, b) = \frac{a+b}{2}$ *for any* $a, b \in [0, 1]$,
- $O_3(a, b) = 1$ *for any* $a, b \in [0, 1]$,

and $\Sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 = \mathrm{Id}_X$ and $\sigma_2(1) = 1$, $\sigma_2(2) = 2$ and $\sigma_2(3) = 1$. Clearly, the operators in Γ are commutative. Let $\mathbb{O} = \Phi_\Sigma(\Gamma)$ and $\overline{\mathbb{O}} = \Phi_{\overline{\Sigma}}(\Gamma)$, where $\overline{\Sigma} = (\sigma_2, \sigma_1)$. Hence, for any $f, g \in \mathrm{Map}(X, L)$,

- $\mathbb{O}(f, g)(1) = O_1(f(1), g(1)) = \max(f(1), g(1)) = \overline{\mathbb{O}}(f, g)(1)$,
- $\mathbb{O}(f, g)(2) = O_2(f(2), g(2)) = \frac{f(2)+g(2)}{2} = \overline{\mathbb{O}}(f, g)(2)$,
- $\mathbb{O}(f, g)(3) = O_3(f(3), g(1)) = 1 = O_3(f(1), g(3)) = \overline{\mathbb{O}}(f, g)(3)$,

i. e., $\mathbb{O} = \overline{\mathbb{O}}$. By Proposition 24, the Σ-induced operator $\mathbb{O}$ is commutative.

Corollary 26. *Let $\Gamma = \{O_x\}_{x \in X}$ be a family of m-ary operators on L and $\Sigma = (\sigma, \ldots, \sigma) \in \mathrm{Map}(X, X)^m$, a constant m-tuple. Hence, $\Phi_\Sigma(\Gamma)$ is commutative if and only if O_x is commutative for any $x \in X$.*

Proof. Since $\Sigma = \Sigma^\tau$ for any $\tau \in \mathcal{S}_m$, the result follows immediately from Proposition 24.

5.2. Neutral element

An element $e \in L$ is said to be neutral for an operator $O \in O_m(L)$ if $O(e, \ldots, e, f, e, \ldots, e) = f$ for any $f \in L$, where f stands at any position $i \in \{1, \ldots, m\}$.

We shall need the following technical lemma.

Lemma 27. *Let L be a bounded lattice, X a set and $\sigma : X \to X$ a mapping such that, for each $f \in Map(X, L)$, $f \circ \sigma \leq f$. Then $\sigma = \mathrm{Id}_X$.*

Proof. Let $x \in X$ with $\sigma(x) \neq x$, consider the mapping $f_{\sigma(x)} : X \to L$ given by

$$f_{\sigma(x)}(y) = \begin{cases} 1, & \text{if } y = \sigma(x), \\ 0, & \text{otherwise.} \end{cases}$$

Hence $f_{\sigma(x)}(\sigma(x)) = 1 > 0 = f_{\sigma(x)}(x)$, yielding a contradiction.

Proposition 28. *Let $\Gamma = \{O_x\}_{x\in X}$ be a family of m-ary operators on L and $\Sigma = (\sigma_1, \ldots, \sigma_m) \in \mathrm{Map}(X, X)^m$. For $e \in L$, the mapping C_e is neutral for $\Phi_\Sigma(\Gamma)$ if and only if e is neutral for O_x for any $x \in X$ and $\sigma_1 = \ldots = \sigma_m = \mathrm{Id}_X$.*

Proof. Let us suppose that C_e is neutral for $\mathbb{O} = \Phi_\Sigma(\Gamma)$. Fix $i \in \{1, \ldots, m\}$. For simplicity we assume $i = 1$. Let $l \in L$ and $x \in X$, hence

$$\begin{aligned} O_x(l, e, \ldots, e) &= O_x(C_l\sigma_1(x), C_e\sigma_2(x) \ldots, C_e\sigma_m(x)) \\ &= \mathbb{O}(C_l, C_e \ldots, C_e)(x) \\ &= C_l(x) \\ &= l. \end{aligned}$$

Now, for any $x \in X$,

$$\begin{aligned} f(x) &= \mathbb{O}(f, C_e, \ldots, C_e)(x) \\ &= O_x(f\sigma_1(x), C_e\sigma_2(x), \ldots, C_e\sigma_m(x)) \\ &= O_x(f\sigma_1(x), e, \ldots, e) \\ &= f\sigma_1(x). \end{aligned}$$

By Lemma 27, σ_1 is the identity.

Conversely, let $f \in \mathrm{Map}(X, L)$,

$$\begin{aligned} \mathbb{O}(C_e, \ldots, C_e, f, C_e \ldots, C_e)(x) &= O_x(C_e(x), \ldots, C_e(x), f(x), C_e(x), \ldots, C_e(x)) \\ &= O_x(e, \ldots, e, f(x), e, \ldots, e) \\ &= f(x), \end{aligned}$$

for any $x \in X$, therefore $\mathbb{O}(C_e, \ldots, C_e, f, C_e \ldots, C_e) = f$, and C_e is a neutral element for $\mathbb{O}$.

5.3. Associative operators

An m-ary operator O on an arbitrary lattice L is said to be associative whenever, for any $l_1, \ldots, l_{2m-1} \in L$,

$$\begin{aligned} O(O(l_1, \ldots, l_m), l_{m+1}, \ldots, l_{2m-1}) &= O(l_1, O(l_2, \ldots, l_{m+1}), l_{m+2}, \ldots, l_{2m-1}) \\ &= \cdots \\ &= O(l_1, \ldots, l_{m-1}, O(l_m, l_{m+1}, \ldots, l_{2m-1})) \end{aligned}$$

Proposition 29. *Let $\Gamma = \{O_x\}_{x\in X}$ be a family of m-ary operators on L and $\Sigma = (\sigma, \ldots, \sigma) \in \mathrm{Map}(X, X)^m$ verifying that $\sigma^2 = \sigma$ and $O_x = O_{\sigma(x)}$ for all $x \in X$. The operator $\Phi_\Sigma(\Gamma)$ is associative if and only if O_x is associative for each $x \in X$.*

Proof. Let us denote $\mathbb{O} = \Phi_\Sigma(\Gamma)$. Suppose that $\mathbb{O}$ is associative. Hence, for each $x \in X$,

$$\begin{aligned} O_x(O_x(l_1, \ldots, l_m), l_{m+1}, \ldots, l_{2m-1}) &= O_x(\mathbb{O}(C_{l_1}, \ldots, C_{l_m})(\sigma(x)), l_{m+1}, \ldots, l_{2m-1}) \\ &= O_x(\mathbb{O}(C_{l_1}, \ldots, C_{l_m})(\sigma(x)), C_{l_{m+1}}(\sigma(x)), \ldots, C_{l_{2m-1}}(\sigma(x))) \\ &= \mathbb{O}(\mathbb{O}(C_{l_1}, \ldots, C_{l_m}), C_{l_{m+1}}, \ldots, C_{l_{2m-1}})(x) \\ &= \mathbb{O}(C_{l_1}, \ldots, C_{l_{m-1}}, \mathbb{O}(C_{l_m}, \ldots, C_{l_{2m-1}}))(x) \\ &= O_x(C_{l_1}(\sigma(x)), \ldots, C_{l_{m-1}}(\sigma(x)), \mathbb{O}(C_{l_m}, \ldots, C_{l_{2m-1}})(\sigma(x))) \\ &\overset{\dagger}{=} O_x(C_{l_1}(\sigma(x)), \ldots, C_{l_{m-1}}(\sigma(x)), O_x(C_{l_m}(\sigma(x)), \ldots, C_{l_{2m-1}}(\sigma(x))) \\ &= O_x(l_1, \ldots, l_{m-1}, O_x(l_m, \ldots, l_{2m-1})), \end{aligned}$$

where † follows from the equality $\sigma^2 = \sigma$. The other equalities can be proved similarly, thus O_x is associative. Conversely, given $f_1, \ldots, f_{2m-1} \in \mathrm{Map}(X, L)$ and $x \in X$,

$$\begin{aligned} \mathbb{O}(\mathbb{O}(f_1, \ldots, f_m), f_{m+1}, \ldots, f_{2m-1}))(x) &= O_x(\mathbb{O}(f_1, \ldots, f_m)(\sigma(x)), f_{m+1}(\sigma(x)), \ldots, f_{2m-1}(\sigma(x)))) \\ &\overset{\dagger}{=} O_x(O_x(f_1(\sigma(x)), \ldots, f_m(\sigma(x))), f_{m+1}(\sigma(x)), \ldots, f_{2m-1}(\sigma(x)))) \\ &= O_x(f_1(\sigma(x)), \ldots, f_{m-1}(\sigma(x)), O_x(f_m(\sigma(x)), \ldots, f_{2m-1}(\sigma(x)))) \\ &\overset{\ddagger}{=} O_x(f_1(\sigma(x)), \ldots, f_{m-1}(\sigma(x)), \mathbb{O}(f_m, \ldots, f_{2m-1})(\sigma(x))) \\ &= \mathbb{O}(f_1, \ldots, f_{m-1}, O(f_m, \ldots, f_{2m-1}))(x), \end{aligned}$$

where † and ‡ follow from the hypothesis $\sigma^2 = \sigma$ and $O_x = O_{\sigma(x)}$ for any $x \in X$. Thus

$$\mathbb{O}(\mathbb{O}(f_1, \ldots, f_m), f_{m+1}, \ldots, f_{2m-1})) = \mathbb{O}(f_1, \ldots, f_{m-1}, \mathbb{O}(f_m, \ldots, f_{2m-1})).$$

The other equalities are proved similarly.

In sight of Propositions 24, 28 and 29, we find the following result, which asserts that [29, Proposition 2] cannot be extended for $\Sigma \neq (\mathrm{Id}_X, \mathrm{Id}_X)$. For results about induced t-norms, t-conorms and negations the reader may consult [29] and the references therein.

Theorem 30. *Let $\Gamma = \{T_x\}_x$ be a family of binary operators on L indexed in X and $\Sigma \in \mathrm{Map}(X, L)^2$. Hence $\Phi_\Sigma(\Gamma)$ is a t-norm (respect. a t-conorm) on $\mathrm{Map}(X, L)$ if and only if T_x is a t-norm (respect. a t-conorm) for any $x \in X$ and $\Sigma = (\mathrm{Id}_X, \mathrm{Id}_X)$.*

Example 31. *Let $X = [0, 1]$, $L = [0, 1]$ and $\Sigma = (\mathrm{Id}_X, \mathrm{Id}_X)$, hence $\mathrm{Map}(X, L) = [0, 1]^{[0,1]}$. Consider the family of t-norms on $[0, 1]$ given by $\Gamma = \{T^{MT}_\lambda\}_{\lambda\in[0,1]}$ (the family of Mayor-Torrens t-norms), where*

$$T^{MT}_\lambda(x, y) = \begin{cases} \max\{x + y - \lambda, 0\}, & \lambda \in]0, 1],\ x, y \in [0, \lambda], \\ \min\{x, y\}, & \textit{otherwise.} \end{cases}$$

Hence $\Phi_\Sigma(\Gamma) = \mathbb{T}$ is defined as

$$\mathbb{T}(f, g)(x) = \begin{cases} \max\{f(x) + g(x) - x, 0\}, \textit{if } x \in]0, 1],\ f(x), g(x) \in [0, x], \\ \min\{f(x), g(x)\},\ \textit{otherwise.} \end{cases}$$

for any $f, g \in [0, 1]^{[0,1]}$ and $x \in [0, 1]$. By Theorem 30, $\mathbb{T}$ is a t-norm on $[0, 1]^{[0,1]}$.

5.4. Idempotency

An element $z \in L$ is said to be idempotent for an m-ary operator O if $O(z, \ldots, z) = z$.

Proposition 32. *Let $\Gamma = \{O_x\}_{x\in X}$ be a family of m-ary operators on L and $\Sigma \in \mathrm{Map}(X, X)^m$. An element $z \in L$ is idempotent for O_x for all $x \in X$ if and only if C_z is idempotent for $\Phi_\Sigma(\Gamma)$.*

Proof. C_z is idempotent for $\Phi_\Sigma(\Gamma)$, if and only if $\Phi_\Sigma(\Gamma)(C_z, \ldots, C_z) = C_z$, if and only if $\Phi_\Sigma(\Gamma)(C_z, \ldots, C_z)(x) = C_z(x)$ for any $x \in X$, if and only if $O_x(C_z\sigma_1(x), \ldots, C_z\sigma_m(x)) = z$ for any $x \in X$, if and only if $O_x(z, \ldots, z) = z$ for any $x \in X$.

5.5. Conjunctive and disjunctive operators

We recall that an m-ary operator O on L is said to be conjunctive, or to have the property of downward reinforcement, if $O(l_1, \ldots, l_m) \leq l_1 \wedge \ldots \wedge l_m$ for any $l_1, \ldots, l_m \in L$. O is said to be disjunctive, or to have the property of upward reinforcement, if $O(l_1, \ldots, l_m) \geq l_1 \vee \ldots \vee l_m$ for any $l_1, \ldots, l_m \in L$. O is called averaging if $l_1 \wedge \ldots \wedge l_m \leq O(l_1, \ldots, l_m) \leq l_1 \vee \ldots \vee l_m$ for any $l_1, \ldots, l_m \in L$.

Proposition 33. *Let $\Gamma = \{O_x\}_{x\in X}$ be a family of m-ary operators on L and $\Sigma = (\mathrm{Id}_X, \ldots, \mathrm{Id}_X) \in \mathrm{Map}(X, X)^m$. Hence $\Phi_\Sigma(\Gamma)$ is conjunctive (disjunctive, averaging) if and only if O_x is conjunctive (disjunctive, averaging) for any $x \in X$.*

Proof. It follows easily from the definition of induced operator.

5.6. Migrativity

A binary operator $O : [0, 1]^2 \to [0, 1]$ is said to be migrative [14] if $O(\alpha x, y) = O(x, \alpha y)$ for all $x, y, \alpha \in [0, 1]$. An extension for interval-valued functions can be found in [8]. For a bounded lattice L, by an obvious generalization of the definition given in [37], an m-ary operator $O : L^m \to L$ is said to be $\mathcal{A}$-migrative, where $\mathcal{A} = \{A_1, \ldots, A_m\}$ with $A_i : L \times L \to L$ for any $i = 1, \ldots, m$, if and only if

$$O(A_1(\alpha, a_1), a_2, \ldots, a_m) = O(a_1, A_2(\alpha, a_2), \ldots, a_m) = \cdots = O(a_1, a_2, \ldots, A_m(\alpha, a_m))$$

for any $\alpha, a_1, \ldots, a_m \in L$.

Proposition 34. *Let $\Gamma = \{O_x\}_{x\in X}$ be a family of m-ary operators on L and $\Sigma = (\mathrm{Id}_X, \ldots, \mathrm{Id}_X) \in \mathrm{Map}(X,X)^m$. Let $\mathcal{A}^x = \{A_1^x, \ldots, A_m^x\}$ be a set of binary operators on L for each $x \in X$, and let $\mathcal{A} = \{\mathbb{A}_1, \ldots, \mathbb{A}_m\}$, where, for each $i = 1, \ldots, m$, $\mathbb{A}_i : \mathrm{Map}(X,L)^2 \to \mathrm{Map}(X,L)$ is induced from the family $\{A_i^x\}_{x\in X}$ and the pair $(\mathrm{Id}_X, \mathrm{Id}_X)$. Hence $\Phi_\Sigma(\Gamma)$ is $\mathcal{A}$-migrative if and only if O_x is $\mathcal{A}^x$-migrative for any $x \in X$.*

Proof. Denote $\mathbb{O} = \Phi_\Sigma(\Gamma)$. Let us assume that O_x is $\mathcal{A}^x$-migrative for any $x \in X$. Hence, for each $x \in X$,

$$\begin{aligned}
\mathbb{O}(\mathbb{A}_1(f, g_1), g_2, \ldots, g_m)(x) &= O_x(\mathbb{A}_1(f, g_1)(x), g_2(x), \ldots, g_m(x)) \\
&= O_x(A_1^x(f(x), g_1(x)), g_2(x), \ldots, g_m(x)) \\
&= O_x(g_1(x), g_2(x), \ldots, A_m^x(f(x), g_m(x))) \\
&= O_x(g_1(x), g_2(x), \ldots, \mathbb{A}_m(f, g_m)(x)) \\
&= \mathbb{O}(g_1, g_2, \ldots, \mathbb{A}_m(f, g_m))(x).
\end{aligned}$$

Hence $\mathbb{O}(\mathbb{A}_1(f, g_1), g_2, \ldots, g_m) = \mathbb{O}(g_1, g_2, \ldots, \mathbb{A}_m(f, g_m))$ for any $f, g_1, \ldots, g_m \in \mathrm{Map}(X,L)$. The other equalities can be proved similarly, and then $\mathbb{O}$ is $\mathcal{A}$-migrative. Conversely, let $\alpha, a_1, \ldots, a_m \in L$ and $x \in X$,

$$\begin{aligned}
O_x(A_1^x(\alpha, a_1), a_2, \ldots, a_m) &= O_x(A_1^x(C_\alpha(x), C_{a_1}(x)), C_{a_2}(x), \ldots, C_{a_m}(x)) \\
&= O_x(\mathbb{A}_1(C_\alpha, C_{a_1})(x), C_{a_2}(x), \ldots, C_{a_m}(x)) \\
&= \mathbb{O}(\mathbb{A}_1(C_\alpha, C_{a_1}), C_{a_2}, \ldots, C_{a_m})(x) \\
&= \mathbb{O}(C_{a_1}, C_{a_2}, \ldots, \mathbb{A}_m(C_\alpha, C_{a_m}))(x) \\
&= O_x(C_{a_1}(x), C_{a_2}(x), \ldots, \mathbb{A}_m(C_\alpha, C_{a_m})(x)) \\
&= O_x(C_{a_1}(x), C_{a_2}(x), \ldots, A_m^x(C_\alpha(x), C_{a_m}(x))) \\
&= O_x(a_1, a_2, \ldots, A_m^x(\alpha, a_m)).
\end{aligned}$$

The other equalities can be proved similarly. Then O_x is $\mathcal{A}^x$-migrative.

Remark 35. *Observe that, if $\Gamma = \{O_t : [0,1]^2 \to [0,1]\}_{t\in[0,1]}$ is a family of migrative operators on $[0,1]$, in the original sense of [14], for any $t \in [0,1]$, the induced operator $\mathbb{O} = \Phi_{(\mathrm{Id}_{[0,1]}, \mathrm{Id}_{[0,1]})}(\Gamma)$ verifies that $\mathbb{O}(\alpha f, g) = \mathbb{O}(f, \alpha g)$ for any $\alpha, f, g \in [0,1]^{[0,1]}$.*

5.7. Abstract homogeneity

We recall from [40] that, given a mapping $g : [0,1]^2 \to [0,1]$ and an automorphism $\psi : [0,1] \to [0,1]$, a mapping $F : [0,1]^m \to [0,1]$ is said to be abstract homogeneous with respect to g and ψ, or (g,ψ)-homogeneous for short, if

$$F(g(\lambda, x_1), g(\lambda, x_2), \ldots, g(\lambda, x_m)) = g(\psi(\lambda), F(x_1, x_2, \ldots, x_m))$$

for any $\lambda, x_1, \ldots, x_m \in [0,1]$. Obviously, for an arbitrary bounded lattice L, abstract homogeneity can be defined verbatim.

Proposition 36. *Let $\Psi = \{\psi_x\}_{x\in X}$ be a family of automorphisms of L, $G = \{g_x\}_{x\in X}$ a family of binary operators on L and $\Gamma = \{F_x\}_{x\in X}$ be a family of m-ary operators on L. Let $\psi = \Phi_{\mathrm{Id}_X}(\Psi)$, $g = \Phi_{(\mathrm{Id}_X, \mathrm{Id}_X)}(G)$ and $F = \Phi_\Sigma(\Gamma)$, where $\Sigma = (\mathrm{Id}_X, \ldots, \mathrm{Id}_X)$, the corresponding induced operators on $\mathrm{Map}(X,L)$. Hence F is (g,ψ)-homogeneous if and only if F_x is (g_x, ψ_x)-homogeneous for any $x \in X$.*

Proof. Firstly, we must prove that ψ is an automorphism of $\mathrm{Map}(X,L)$. For any $f_1, f_2 \in \mathrm{Map}(X,L)$:

- For any $x \in X$, $\psi(f_1 \wedge f_2)(x) = \psi_x(f_1(x) \wedge f_2(x)) = \psi_x(f_1(x)) \wedge \psi_x(f_2(x)) = \psi(f_1)(x) \wedge \psi(f_2)(x) = (\psi(f_1) \wedge \psi(f_2))(x)$. Then $\psi(f_1 \wedge f_2) = \psi(f_1) \wedge \psi(f_2)$.
- Analogously we may prove that $\psi(f_1 \vee f_2) = \psi(f_1) \vee \psi(f_2)$.
- ψ is injective, since $\psi(f_1) = \psi(f_2)$, if and only if $\psi(f_1)(x) = \psi(f_2)(x)$ for all $x \in X$, if and only if $\psi_x(f_1(x)) = \psi_x(f_2(x))$ for all $x \in X$, if and only if $f_1(x) = f_2(x)$ for all $x \in X$, if and only if $f_1 = f_2$.
- ψ is surjective. Let $h \in \mathrm{Map}(X,L)$, consider $f \in \mathrm{Map}(X,L)$ given by $f(x) = \psi_x^{-1}(h(x))$ for each $x \in X$, hence $\psi(f)(x) = \psi_x(f(x)) = \psi_x(\psi_x^{-1}(h(x))) = h(x)$ for all $x \in X$, then $\psi(f) = h$.

Now, suppose F_x is (g_x, ψ_x)-homogeneous for any $x \in X$. Then, for any $\lambda, f_1, \ldots, f_m \in \text{Map}(X, L)$ and $x \in X$,

$$\begin{aligned} F(g(\lambda, f_1), \ldots, g(\lambda, f_m))(x) &= F_x(g(\lambda, f_1)(x), \ldots, g(\lambda, f_m)(x)) \\ &= F_x(g_x(\lambda(x), f_1(x)), \ldots, g_x(\lambda(x), f_m(x))) \\ &= g_x(\psi_x(\lambda(x)), F_x(f_1(x), \ldots, f_m(x))) \\ &= g_x(\psi(\lambda)(x), F(f_1, \ldots, f_m)(x)) \\ &= g(\psi(\lambda), F(f_1, \ldots, f_m))(x). \end{aligned}$$

Hence $F(g(\lambda, f_1), \ldots, g(\lambda, f_m)) = g(\psi(\lambda), F(f_1, \ldots, f_m))$, and thus F is (g, ψ)-homogeneous. Conversely, let $b, a_1, \ldots, a_m \in L$ and $x \in X$,

$$\begin{aligned} F_x(g_x(b, a_1), \ldots, g_x(b, a_m)) &= F_x(g_x(C_b(x), C_{a_1}(x)), \ldots, g_x(C_b(x), C_{a_m}(x))) \\ &= F_x(g(C_b, C_{a_1})(x), \ldots, g(C_b, C_{a_m})(x)) \\ &= F(g(C_b, C_{a_1}), \ldots, g(C_b, C_{a_m}))(x) \\ &= g(\psi(C_b), F(C_{a_1}, \ldots, C_{a_m}))(x) \\ &= g_x(\psi(C_b)(x), F(C_{a_1}, \ldots, C_{a_m})(x)) \\ &= g_x(\psi_x(C_b(x)), F_x(C_{a_1}(x), \ldots, C_{a_m}(x))) \\ &= g_x(\psi_x(b), F_x(a_1, \ldots, a_m)). \end{aligned}$$

Hence F_x is (g_x, ψ_x)-homogeneous.

6. Examples

In this final section we show some examples that illustrate the theory developed in the previous ones.

6.1. Takáč operator

According to Zadeh's extension principle, Takáč [44] extends an n-ary operator O on $[0,1]$ to an n-ary operator $\mathbb{O}$ on $[0,1]^{[0,1]}$ as

$$\mathbb{O}(f_1, \ldots, f_n)(x) = sup\{f_1(y_1) \wedge \cdots \wedge f_n(y_n) : O(y_1, \ldots, y_n) = x\},$$

for any $f_1, \ldots, f_n \in [0,1]^{[0,1]}$ and any $x \in [0,1]$. The operator $\mathbb{O}$ is well-defined whenever O is surjective. For instance, we may consider O as the arithmetic mean, that is surjective, and mappings $C_{a_i} : [0,1] \to [0,1]$ for some values a_i for $i = 1, \ldots, n$. Hence

$$\mathbb{O}(C_{a_1}, \ldots, C_{a_n})(x) = sup\{C_{a_1}(y_1) \wedge \cdots \wedge C_{a_n}(y_n) : O(y_1, \ldots, y_n) = x\} = a_1 \wedge \cdots \wedge a_n$$

for any $x \in [0,1]$. Thus the operator $\mathbb{O}$ does not always preserve the intrinsic nature of the operator O (that is to say, calculate the mean). Nevertheless, we may consider the tuple $\Sigma = (Id_{[0,1]}, \ldots, Id_{[0,1]})$ and the constant family $\Gamma = \{O\}_{t \in [0,1]}$. Hence, the induced operator $\hat{\mathbb{O}} = \Phi_\Sigma(\Gamma)$ on $[0,1]^{[0,1]}$ is given by

$$\hat{\mathbb{O}}(f_1, \ldots, f_n)(x) = \frac{\sum_{i=1}^n f_i(x)}{n}$$

for any $f_1, \ldots, f_n \in [0,1]^{[0,1]}$ and any $x \in [0,1]$.

6.2. Partially unknown data

We may use the induction process for working with mappings whose images are not entirely known. For example, in the context of type-2 fuzzy sets, let $X = L = [0,1]$ and consider a family of m-ary aggregation operators $O_t : [0,1]^m \to [0,1]$ for each $t \in [0,1]$. Suppose that by an appropriate binning algorithm, we find the frontier points

$$a_0 = 0 < a_1 < \cdots < a_{n-1} < a_n = 1.$$

We may then define the piece-wise mapping $\sigma : [0,1] \to [0,1]$ as $\sigma(x) = a_i$ if $x \in [a_{i-1}, a_i)$ for $i = 1, \ldots, n$ and $\sigma(1) = 1$, see Figure 4, although there are plenty of other options.

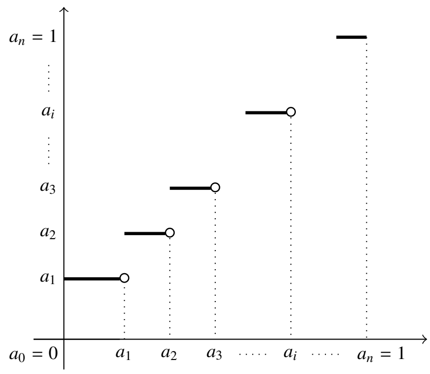


Figure 4: Mapping σ

Set $\Sigma = (\sigma, \dots, \sigma)$. Hence the family $\Gamma = \{O_t\}_{t\in[0,1]}$ and Σ induce a representable m-ary aggregation operator $\mathbb{O} : \left([0,1]^{[0,1]}\right)^m \to [0,1]^{[0,1]}$ on $[0,1]^{[0,1]}$. Then $\mathbb{O}(f_1, \dots, f_m) = \mathbb{O}(g_1, \dots, g_m)$ under the condition $f_i(a_j) = g_i(a_j)$ for each $i, j = 1, \dots, n$, that is, for each $f \in [0,1]^{[0,1]}$, we have to consider solely the values at the chosen points $f(a_0), f(a_1), \dots, f(a_n)$. This could be convenient whenever the analytic expression of f is unknown and we merely have experimental data. We may then aggregate partially known input mappings. If some mapping f_i is completely known, we simply may set, for instance, $\sigma_i = \mathrm{Id}_{[0,1]}$ in order to consider the whole information provided by the mapping.

6.3. Approximate reasoning

Implication operators play a prominent role in fuzzy logic since they are used for handling the inference rules in approximate reasoning. Formally, an implication on a lattice L is a mapping $I : L^2 \to L$ verifying $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$. Then we may use Proposition 4 to induce implications on spaces of mappings $\mathrm{Map}(X, L)$, for some set X. Namely, if $\Gamma = \{O_x\}_{x\in X}$ is a family of implications on L and $\Sigma = (\sigma_1, \sigma_2)$, then the induced operator $\Phi_\Sigma(\Gamma)$ is an implication on $\mathrm{Map}(X, L)$.

Stronger concepts of implication can be considered if the implication mapping I verifies additional properties. For instance, we may consider the properties:

1. If $x \le z$ then $I(x, y) \ge I(z, y)$ for any $y \in L$.
2. If $x \le z$ then $I(y, x) \le I(y, z)$ for any $y \in L$.
3. $I(1, x) = x$ for any $x \in L$.
4. $I(x, I(y, z)) = I(y, I(x, z))$ for any $x, y, z \in L$.
5. $I(x, y) = I(x, I(x, y))$ for any $x, y \in L$.
6. $I(x, N(x)) = N(x)$ for any $x \in L$ and any involutive negation N on L.
7. $N(x) = I(x, 0)$ is an involutive negation on L.
8. $I(x, 1) = 1$ for any $x \in L$.
9. $I(x, y) \ge y$ for any $x, y \in L$.
10. $I(x, y) = I(N(y), N(x))$ for any $x, y \in L$ and any involutive negation N on L.
11. $I(0, x) = 1$ for any $x \in L$.

Table 1: Conditions on Σ for preserving the properties

Property	Requirements on Σ
1.	none
2.	none
3.	$\sigma_2 = Id_X$
4.	$\sigma_2 = Id_X$
5.	$\sigma_2 = Id_X$
6.	$\sigma_2 = Id_X$, $\sigma_1 = \sigma$, N is σ-representable
7.	$\sigma_1 = Id_X$
8.	none
9.	$\sigma_2 = Id_X$
10.	$\sigma_2 = Id_X$, $\sigma_1 = \sigma$, N is σ-representable
11.	none

Then one might wonder whether the induced implication on $\mathrm{Map}(X, L)$ inherits any of these properties from the inducing family of implications on L. In Table 1, we sum up some requirements on $\Sigma = (\sigma_1, \sigma_2)$ for preserving these properties. For instance, we may prove the property 5. Indeed, assume that the implications of the family $\Gamma = \{I_x\}_{x\in X}$ satisfy $I_x(a, b) = I_x(a, I_x(a, b))$ for any $a, b \in L$, and that $\sigma_2 = Id_X$. Hence, for any $f, g \in \mathrm{Map}(X, L)$,

$$I(f, I(f, g))(x) = I_x(f\sigma_1(x), I(f, g)(x)) = I_x(f\sigma_1(x), I_x(f\sigma_1(x), g(x))) = I_x(f\sigma_1(x), g(x)) = I(f, g)(x),$$

for any $x \in X$, thus $I(f, g) = I(f, I(f, g))$. With respect to the properties 6 and 10, we also need the involutive negation N is σ-representable for some decreasing mapping $\sigma : X \to X$, according to the results showed in [29]. We recall from [29] that a negation N on $\mathrm{Map}(X, L)$ is said to be σ-representable, where $\sigma : X \to X$ is a decreasing mapping, if there exists a family $\{N_x\}_{x\in X}$ of negations on L such that $N(f)(x) = N_x(f(\sigma(x)))$ for any $f \in \mathrm{Map}(X, L)$ and any $x \in X$.

Regarding the induction of implications on $\mathrm{Hom}(X, L)$, Theorem 7 cannot be applied, since implications are not monotone. Nevertheless, something can be done if we assume that they hold the properties 1 and 2. In such a case, these operators are monotone considering the opposite order in the first component. Hence, we may state the following result, which can be proved similarly to Theorem 7 following the comments in Remark 1.

Theorem 37. *Let $\Gamma = \{I_x\}_{x\in X}$ be a family of implications on L verifying the properties 1 and 2. Let $\Sigma = (\sigma_1, \sigma_2) \in$ $\mathrm{Map}(X, L)$, where σ_1 is decreasing and σ_2 is increasing. Then $\Phi_\Sigma(\Gamma)$ is an implication on $\mathrm{Hom}(X, L)$ verifying the properties 1 and 2 if and only if Γ is monotone.*

Proof. Following the notation in Remark 1, fix $L_1 = L^{\mathrm{op}}$, $L_2 = L$, $X_1 = X^{\mathrm{op}}$ and $X_2 = X$, where op denotes the opposite order. Hence, for any monotone mappings $\sigma_1 : X \to X^{\mathrm{op}}$ (decreasing) and $\sigma_2 : X \to X$ (increasing),

$$\Phi_\Sigma(\Gamma) : \mathrm{Hom}(X^{\mathrm{op}}, L^{\mathrm{op}}) \times \mathrm{Hom}(X, L) \to \mathrm{Hom}(X, L)$$

verifies the properties 1 and 2. But $\mathrm{Hom}(X^{\mathrm{op}}, L^{\mathrm{op}}) = \mathrm{Hom}(X, L)$, and then the statement follows.

Under the conditions of Theorem 37, the properties 3-6 and 8-11 are preserved as described in Table 1, which agrees with, and extends, [10, Theorem 17]. There is no suitable condition in order to lift the property 7 whenever working on $\mathrm{Hom}(X, L)$. This is due to the fact that a mapping N defined as $N(f) = I(f, C_0)$ for any $f \in \mathrm{Hom}(X, L)$ is not necessarily well-defined, since $N(f)$ could be non decreasing. This fact is exemplified in [10, Remark 18] for implications on interval fuzzy sets. The reader may also compare with [49, Section 6] for results about implications on n-dimensional fuzzy sets.

Something similar can be said about negations. Induced negations are studied in [29]. When working over set mappings, induced negations can be constructed successfully in the standard way. Nevertheless, for negations on $\mathrm{Hom}(X, L)$, Theorem 7 cannot be applied, since negations are not monotone. They are anti-monotone. Clearly, the problem can solved by considering Remark 1. A negation on L is then a mapping $n : L^{\mathrm{op}} \to L$, where L^{op} denotes the same lattice L endowed with the opposite order. Now, given a lattice X, a family of negations $\{n_x\}_{x\in X}$ on L and a mapping $\sigma : X \to X^{\mathrm{op}}$ (a decreasing map!), we may induce a negation

$$N : \mathrm{Hom}(X^{\mathrm{op}}, L^{\mathrm{op}}) = \mathrm{Hom}(X, L) \to \mathrm{Hom}(X, L),$$

which coincides with the study made in [29]. The reader may also compare with [6] and [11].

6.4. Overlap in a classification task

The notion of overlap function over the unit interval comes from [15], aiming to measure the degree of overlap between classes in a fuzzy classification problem with two classes. This degree can be interpreted as the representation of the lack of knowledge of an expert in determining if an object belongs to one of these classes. Several extensions to wider contexts have been considered in the literature. For instance, for classification problems with m different classes in [26], for interval-valued fuzzy sets in [8], or for complete lattice in [35] and [36]. As an extension of this topic, the so-called general overlap and quasi-overlap functions are defined in [22] and [9], respectively. Dually, gruping functions [17] are supposed to yield the degree up to which the combination of classes is supported.

Here we propose a description of these notions for arbitrary bounded lattices and prove that these concepts can also be induced to spaces of mappings and, consequently, the construction methodology described in this paper is available for them. In order to improve the readability of the statement of Definition 38, we have enclosed both, the notions of overlap and gruping function into a general description using an arbitrary aggregation operator. We recall that a non empty subset D of L is called upward (resp. downward) directed if each pair of elements in D has an upper (resp. lower) bound in D.

Definition 38. *Let L be a bounded lattice, $O : L^m \to L$ an m-ary operator and $A : L^m \to L$ an aggregation operator. Consider the following properties:*

- $\mathbf{O}_1$. *O is symmetric.*
- $\mathbf{O}_2$. *$O(a_1, \dots, a_m) = 0$ if and only if $A(a_1, \dots, a_m) = 0$.*
- $\mathbf{D}_2$. *If $A(a_1, \dots, a_m) = 0$ then $O(a_1, \dots, a_m) = 0$.*
- $\mathbf{O}_3$. *$O(a_1, \dots, a_m) = 1$ if and only if $A(a_1, \dots, a_m) = 1$.*
- $\mathbf{D}_3$. *If $A(a_1, \dots, a_m) = 1$ then $O(a_1, \dots, a_m) = 1$.*
- $\mathbf{O}_4$. *O is monotone in each component.*
- $\mathbf{O}_5$. *For each component $i = 1, \dots, m$,*

$$O(a_1, \dots, a_{i-1}, \bigvee_{j\in\Lambda} d_j, a_{i+1}, \dots, a_m) = \bigvee_{j\in\Lambda} O(a_1, \dots, a_{i-1}, d_j, a_{i+1}, \dots, a_m)$$

  for any $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m \in L$ and any upward directed set $\{d_j\}_{j\in\Lambda} \subset L$.
- $\mathbf{O}_6$. *For each component $i = 1, \dots, m$,*

$$O(a_1, \dots, a_{i-1}, \bigwedge_{j\in\Lambda} d_j, a_{i+1}, \dots, a_m) = \bigwedge_{j\in\Lambda} O(a_1, \dots, a_{i-1}, d_j, a_{i+1}, \dots, a_m)$$

  for any $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m \in L$ and any downward directed set $\{d_j\}_{j\in\Lambda} \subset L$.

Then:

a) Suppose that $A(a_1, a_2, \dots, a_m) = T(a_1, T(a_2, \dots, T(a_{m-1}, a_m)))$ for any $a_1, \dots, a_m \in L$ for some t-norm T on L. The operator O is called T-overlap if it satisfies the set of properties $\{\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \mathbf{O}_5, \mathbf{O}_6\}$, general T-overlap if it satisfies the set of properties $\{\mathbf{O}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{O}_4, \mathbf{O}_5, \mathbf{O}_6\}$, and quasi T-overlap if it satisfies the set of properties $\{\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$.

b) *Dually, suppose that* $A(a_1, a_2, \ldots, a_m) = S(a_1, S(a_2, \ldots, S(a_{m-1}, a_m)))$ *for any* $a_1, \ldots, a_m \in L$ *for some t-conorm* S *on* L. *The operator* O *is called* S*-gruping if it satisfies the set of properties* $\{\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \mathbf{O}_5, \mathbf{O}_6\}$, *general* S*-gruping if it satisfies the set of properties* $\{\mathbf{O}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{O}_4, \mathbf{O}_5, \mathbf{O}_6\}$, *and quasi* S*-gruing if it satisfies the set of properties* $\{\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$.

Remark 39. a) *The definition of* T*-overlap operator provided in Definition 38 does not generalize the ones given in [35] and [36] for complete lattices when* $m = 2$. *There,* $\mathbf{O}_2$ *is stated as: "*$O(a_1, a_2) = 0$ if and only if $a_1 = 0$ or $a_2 = 0$*". Nevertheless, for some lattices, there is no t-norm verifying this property. For instance, let* $L = \{0, 1, a, b\}$ *be the lattice described by the Hasse diagram in Figure 5. A t-norm* T *on* L *must verify that* $T(a, b) = 0$.

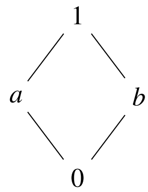


Figure 5: Hasse diagram of the lattice L

b) *The properties* $\mathbf{O}_5$ *and* $\mathbf{O}_6$ *reflect the continuity of* O *on each component. In this case, the notion of continuity is the standard for arbitrary lattices: the Scott-continuity. Hence,* $\mathbf{O}_5$ *and* $\mathbf{O}_6$ *mean that* O *is Scott-continuous for* L *and the dual of* L.

c) *As in the classic case, the concepts of* T*-overlap and* S*-gruping function are dual one to each other if there exists a De Morgan triple* (S, T, N) *for some involutive negation* N *on* L.

Theorem 40. *Let* L *be a bounded lattice,* X *a set,* $\Lambda = \{A_x\}_{x \in X}$ *a family of* m*-ary aggregation operators on* L *and* $\Gamma = \{O_x\}_{x \in X}$ *a family of* m*-ary operators on* L. *Denote* $\mathbb{A} = \Phi_\Sigma(\Lambda)$ *and* $\mathbb{O} = \Phi_\Sigma(\Gamma)$, *where* $\Sigma = (\mathrm{Id}_X, \ldots, \mathrm{Id}_X) \in \mathrm{Map}(X, L)^m$. *If, for all* $x \in X$, O_x *verifies any of the properties* $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{D}_2$, $\mathbf{O}_3$, $\mathbf{D}_3$, $\mathbf{O}_4$, $\mathbf{O}_5$, $\mathbf{O}_6$ *(with respect to* A_x*, when needed), then* $\mathbb{O}$ *verifies the same property (with respect to* $\mathbb{A}$*, when needed). Additionally, if* X *is a bounded lattice and* Λ *and* Γ *are family of monotone operators preserving the order on* X*, the same result holds for operators on* $\mathrm{Hom}(X, L)$.

Proof. The proof follows the same guidelines of similar statements proven all along the paper. Simply observe that, if $\{f_i\}_{i \in \Lambda}$ is an upward (resp. downward) directed subset of $\mathrm{Map}(X, L)$, hence, for each $x \in X$, $\{f_i(x)\}_{i \in \Lambda}$ is an upward (resp. downward) directed subset of L.

Corollary 41. *Under the conditions of Theorem 40, if* O_x *is* A_x*-overlap (general* A_x*-overlap, quasi* A_x*-overlap,* A_x*-gruping, general* A_x*-gruping, quasi* A_x*-gruping) for all* $x \in X$ *then* $\mathbb{O}$ *is* $\mathbb{A}$*-overlap (general* $\mathbb{A}$*-overlap, quasi* $\mathbb{A}$*-overlap,* $\mathbb{A}$*-gruping, general* $\mathbb{A}$*-gruping, quasi* $\mathbb{A}$*-gruping).*

Example 42. *Let us provide an example of usage. Fix a positive integer* $n > 1$, *and consider* T *the product t-norm in* $[0, 1]$ *and the family of* T*-overlap operators* $O_p : [0, 1] \times [0, 1] \to [0, 1]$ *given by*

$$O_p(x, y) = x^{n-p+1} y^{n-p+1}$$

for any $x, y \in [0, 1]$ *and* $p = 1, \ldots, n$. *Set* $X = \{1, \ldots, n\}$ *with the order* $1 \leq 2 \leq \cdots \leq n$. *Clearly* $O_p \leq O_q$ *if* $p \leq q$, *i. e., the family of operators preserves the order on* X. *We may then induce an operator* $\mathbb{O}$ *on* $\mathrm{Hom}(X, [0, 1]) = L_n([0, 1])$, *the set of* n*-dimensional intervals in* $[0, 1]$. *Concretely,*

$$\mathbb{O}(f, g)(p) = O_p(f(p), g(p)) = f(p)^{n-p+1} g(p)^{n-p+1},$$

for any $f, g \in \mathrm{Hom}(X, [0,1])$ *and* $p \in \{1, \ldots, n\}$. *For instance, set* $n = 3$. *Since each element in* $L_3([0,1])$ *can be seen as a* 3*-tuple of increasing values, if we consider* $f = (0.1, 0.3, 0.6)$ *and* $g = (0, 0.6, 0.9)$, $\mathbb{O}(f, g) = (0, 0.0324, 0.54)$.

Now, by Corollary 41, the induced operator $\mathbb{O}$ *is* $\mathbb{T}$*-overlap on* $L_n([0,1])$, *where*

$$\mathbb{T}(f, g)(p) = T(f(p), g(p)) = f(p)g(p)$$

for any $f, g \in \mathrm{Hom}(X, [0,1])$ *and* $p \in \{1, \ldots, n\}$.

6.5. *Multi-Expert Decision Making*

A direct application of aggregation operators (or, pre-aggregation operators [40]) is to deal with the aggregation and exploitation phases in a multi-expert decision making (MEDM) problem [38]. We may then consider the induction process for designing these aggregation operators. We recall that a MEDM problem can be summarized as follows. There exist a set of alternatives $A = \{a_1, \ldots, a_p\}$ and a set of experts $E = \{e_1, \ldots, e_m\}$. The experts provide their preferences about the alternatives, and these preferences are represented by m matrices, one for each expert,

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & \cdots & p^i_{1p} \\ p^i_{21} & p^i_{22} & \cdots & p^i_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ p^i_{p1} & p^i_{p2} & \cdots & p^i_{pp} \end{pmatrix}$$

for $i = 1, \ldots, m$. For any i, j, k, the value p^i_{jk} expresses the degree of preference of the alternative a_j over the alternative a_k given by the expert e_i. Hence, in general, the elements of the diagonal are not considered. The selection of an alternative consists of two phases:

1. The aggregation phase, where the preference matrices are joined into a collective preference matrix.
2. The exploitation phase, where a given method is applied to the collective preference matrix to obtain a selection of alternatives.

The degrees of preference are usually represented by numbers in $[0,1]$, but we may make use of the induction technique to develop operators for handling degrees in other lattices, as, for instance, intervals or mappings on $[0,1]$. Firstly, the aggregation phase needs to define a mapping

$$\mathbb{G} : \mathcal{M}_{p\times p}(R)^m \to \mathcal{M}_{p\times p}(R),$$

where $\mathcal{M}_{p\times p}(R)$ is the set of $p \times p$-matrices with coefficients in the lattice R, in order to join together the set of preferences of the experts into a single matrix P^c. This can be obtained simply by extending component-wise an aggregation operator $G : R^m \to R$, i. e., the (i, j)-component of $\mathbb{G}(P^1, \ldots, P^m)$ is $G(p^1_{ij}, \ldots, p^m_{ij})$ for any $i, j = 1, \ldots, p$. Observe that, in this way, $\mathbb{G}$ is the induced operator of the family $\{G\}_{x\in X}$, where $X = \{1, \ldots, p^2\}$, and Σ is the tuple with identity mappings. Nevertheless, as we have seen, the induction process can provide another alternatives. On the other hand, if the lattice R is $\mathrm{Map}(X, L)$ for some set X and lattice L, or $\mathrm{Hom}(X, L)$ when appropriate, the aggregation operator A may be also designed by induction from operators on L.

Once the preference matrices are collected into the matrix P^c, in the exploitation phase, a method is applied to P^c in order to yield a selection of alternatives. This can be done, for instance, by applying another aggregation operator H to the rows of P^c, which can also be designed by induction. Then, to each alternative x_i with $i = 1, \ldots, p$, we associate an element $r_i = H(p^c_{i1}, \ldots, p^c_{ip}) \in R$. Obviously, if the set $\{r_1, \ldots, r_p\}$ is totally ordered (as it always happens when $R = [0,1]$), the alternatives can be ordered according to the elements r_i. If not, a method for ranking them must be applied. Let us illustrate these comments with a numeric example.

Example 43. *Consider a MEDM problem with three alternatives* a_1, a_2, a_3 *and four experts* e_1, e_2, e_3, e_4. *These experts provide preference degrees that belong to the lattice* $\mathbb{I} = \mathrm{Hom}(\mathbb{2}, [0,1])$ *of closed intervals in* $[0,1]$. *The preference matrices provided by these experts are displayed in Figure 6.*

P^1	1	2	3
1	−	[0.2, 0.3]	[0.4, 0.6]
2	[0.5, 0.7]	−	[0.8, 0.9]
3	[0.2, 0.5]	[0.3, 0.6]	−

P^2	1	2	3
1	−	[0.4, 0.6]	[0.2, 0.3]
2	[0.1, 0.3]	−	[0.7, 0.9]
3	[0.5, 0.8]	[0, 0]	−

P^3	1	2	3
1	−	[0.5, 0.5]	[0.1, 0.2]
2	[0.5, 0.5]	−	[0.5, 0.7]
3	[0.8, 0.9]	[0.4, 0.6]	−

P^4	1	2	3
1	−	[0, 0.3]	[0.4, 0.6]
2	[0.8, 1]	−	[0.5, 0.6]
3	[0.2, 0.5]	[0.4, 0.5]	−

Figure 6: Preference matrices expressed by the experts.

In order to compute the collective preference matrix P^c we consider the aggregation operators on $[0,1]$ *given by* $G_0(x_1,x_2,x_3,x_4) = \min(x_1,x_2,x_3,x_4)$ *and* $G_1(x_1,x_2,x_3,x_4) = \max(x_1,x_2,x_3,x_4)$ *for any* $x_1,x_2,x_3,x_4 \in [0,1]$. *Obviously* $G_0 \leq G_1$, *and then we may consider the induced operator on closed interval* $G : \mathbb{I}^4 \to \mathbb{I}$ *given by*

$$G([x_1,y_1],[x_2,y_2],[x_3,y_3],[x_4,y_4]) = [\min(x_1,x_2,x_3,x_4), \max(y_1,y_2,y_3,y_4)]$$

for any intervals $[x_1,y_1],[x_2,y_2],[x_3,y_3],[x_4,y_4] \in \mathbb{I}$. *Suppose that, for some reason, we need to apply some penalty to the opinion of experts* e_1 *and* e_2 *about the preferences of* a_2 *over the other options, then analogously, we may consider the operator*

$$G'([x_1,y_1],[x_2,y_2],[x_3,y_3],[x_4,y_4]) = [\min(x_1^2,x_2^2,x_3,x_4), \max(y_1^2,y_2^2,y_3,y_4)]$$

for any intervals $[x_1,y_1],[x_2,y_2],[x_3,y_3],[x_4,y_4] \in \mathbb{I}$. *Hence we set* $\mathbb{G} : \mathcal{M}_3(\mathbb{I})^4 \to \mathcal{M}_3(\mathbb{I})$ *as the operator given by*

$$\mathbb{G}(P^1,P^2,P^3,P^4)_{ij} = \begin{cases} G'(P^1_{ij},P^2_{ij},P^3_{ij},P^4_{ij}) & \text{if } (i,j)=(2,1) \text{ or } (i,j)=(2,3), \\ G(P^1_{ij},P^2_{ij},P^3_{ij},P^4_{ij}) & \text{otherwise,} \end{cases}$$

for any matrices P_1,P_2,P_3,P_4. *Thus* $P^c = \mathbb{G}(P^1,P^2,P^3,P^4)$ *is given by*

P^c	1	2	3
1	−	[0, 0.6]	[0.1, 0.6]
2	[0.01, 1]	−	[0.49, 0.81]
3	[0.2, 0.9]	[0, 0.6]	−

Now, consider the aggregation operator $M : [0,1]^2 \to [0,1]$ *given by the arithmetic mean* $M(x,y) = \frac{x+y}{2}$ *for any* $x,y \in [0,1]$, *which yields the induced aggregation operator* $\mathbb{M} : \mathbb{I}^2 \to \mathbb{I}$ *defined as*

$$\mathbb{M}([x_1,y_1],[x_2,y_2]) = \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$$

for any $[x_1,y_1],[x_2,y_2] \in \mathbb{I}$. *Thus the elements* r_1, r_2 *and* r_3 *associated to the alternatives are the following:*

$$r_1 = \mathbb{M}([0,0.6],[0.1,0.6]) = [0.05,0.6],$$
$$r_2 = \mathbb{M}([0.01,1],[0.49,0.81]) = [0.25,0.9],$$
$$r_3 = \mathbb{M}([0.2,0.9],[0,0.6]) = [0.1,0.75].$$

Hence $r_1 \leq r_3 \leq r_2$ *and, consequently,* $a_1 \leq a_3 \leq a_2$.

7. Conclusions

In this work we have provided a method, that we have called induction, for extending operators to classes of lattice-valued mappings. Explicitly, for a lattice L and a set X, given a family of operators on L and set mappings from X to X, we build an operator on $\mathrm{Map}(X,L)$, or on $\mathrm{Hom}(X,L)$ whenever X is also a lattice. We have shown this methodology can be applied to a wide range of frameworks simply by varying L and X. In particular, it covers the best known classes of extended fuzzy sets. Nevertheless, the theory is described for arbitrary lattice L and set X in order to cover also the case of a future potentially useful lattice. Despite the abstract mathematical machinery, the method is simple and ready to use with minimal mathematical knowledge. We have also studied the preservation of certain properties under the induction process. Then we have proved that important types of operator are preserve under induction. In particular, families of aggregation operators on L yield aggregation operators on the lattice-valued mappings space. Analogously, implications, negations, overlap functions and others are well-preserved. In this sense, our study generalizes some results in the literature dealing with these kind of operators in the context of extended fuzzy sets. The method offers high flexibility in order to design ad-hoc operators for solving a specific problem under consideration, then, hopefully, it could be exploited in practical applications.

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