

RESEARCH ARTICLE

Mean-field limit of non-exchangeable systems

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Funding information

NSF, Grant/Award Numbers: 1908739, 2049020, 2205694, 2219397; State Research Agency of the Spanish Ministry of Science and FEDER-EU, Grant/Award Number: PID2022-137228OB-I00; Consejería de Universidad, Investigación e Innovación, Grant/Award Number: QUAL21-011; Marie Skłodowska-Curie, Grant/Award Number: 101064402; Horizon 2020, Grant/Award Number: 865711

Abstract

This paper deals with the derivation of the mean-field limit for multi-agent systems on a large class of sparse graphs. More specifically, the case of non-exchangeable multi-agent systems consisting of non-identical agents is addressed. The analysis does not only involve PDEs and stochastic analysis but also graph theory through a new concept of limits of sparse graphs (extended graphons) that reflect the structure of the connectivities in the network and has critical effects on the collective dynamics. In this article some of the main restrictive hypothesis in the previous literature on the connectivities between the agents (dense graphs) and the cooperation between them (symmetric interactions) are removed.

Mathematics Subject Classification (MSC) 2020

35Q49, 35Q83, 35R02, 35Q70, 05C90, 60G09, 35R06, 35Q89, 35Q92, 49N80, 92B20, 65N75

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1 | INTRODUCTION

We derive in this article the mean-field limit for multi-agent systems posed on a large class of graphs. More precisely, we consider the dynamics of N agents, interacting pairwise through a complex map of connections between them. The information about this connectivity map is encompassed through a weighted, *a priori* non-symmetric graph G_N where each vertex represents an agent. Denoting by $X_i(t) \in \mathbb{R}^d$ the state of the i -th agent, and by K the interaction kernel, this leads to the system

$$\begin{cases} \frac{dX_i}{dt} = \sum_{j=1}^N w_{ij} K(X_i - X_j), \\ X_i(0) = X_i^0, \end{cases} \quad (1)$$

where the $w_{ij} = w_{ij}^N \neq w_{ji}^N$ are the weight of the edge (i, j) of the graph.

While we consider for simplicity deterministic systems like (1), our analysis would extend in a straightforward manner to stochastic multi-agent systems with additive noise such as

$$dX_i = \sum_{j=1}^N w_{ij} K(X_i - X_j) dt + \sigma_N dW_i, \quad (2)$$

where the W_i^t are independent Wiener processes.

This paper focuses on the analysis of the behavior of the system as $N \rightarrow \infty$, for smooth Lipschitz kernels K but with minimal assumptions on the connectivities w_{ij} . We only impose scalings that naturally extend the classical notion of mean-field scaling: First, the total interaction felt by an agent should be of order 1, that is,

$$\max_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| = O(1), \quad \max_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| = O(1), \quad \text{as } N \rightarrow \infty, \quad (3)$$

ensuring that complex dynamics emerge at time scales of order 1 as well. Secondly, no single agent is allowed to have a dominant role, that is,

$$\max_{1 \leq i, j \leq N} |w_{ij}| = o(1), \quad \text{as } N \rightarrow \infty, \quad (4)$$

which, together with (3), implies that a given agent is influenced by the average of a large number of other agents, so that we expect some concept of mean-field limit to persist.

System (1) is a canonical example of so-called non-exchangeable or non-interchangeable systems. This refers to the fact that (1) is not symmetric under permutation: If $(X_i(t))_{1 \leq i \leq N}$ solves (1) and $\sigma \in S_N$ is any permutation, then $(X_{\sigma(i)}(t))_{1 \leq i \leq N}$ is not in general a solution to (1). Instead the same permutation σ would need to be applied to the connectivities w_{ij} , thus modifying the system. Another way of formulating this is that agents are not identical because the effect of the j -th on the i -th agent may be very different from the effect on the k -th agent: One could have $w_{ij} \gg 1$ but $w_{ik} \approx 0$ for example.

Classically, studies of many-particle systems have often focused on identical agents and exchangeable systems which corresponds to choosing $w_{ij} = 1/N$ in (1). The question of the mean-field limits for exchangeable systems where all agents are identical is now rather well understood for Lipschitz interaction kernels, with even significant progress on singular kernels; see for example the reviews [42, 48, 50]. The case of non-exchangeable systems and non-identical agents had remained comparatively less explored but recently started to receive a lot of attention because of their critical importance for many applications and their connections to diverse areas of Mathematics. The analysis of non-exchangeable multi-agent systems indeed not only involves PDEs and stochastic analysis but also graph theory. On the other hand, multi-agent systems are also found in a large number of fields from their classical applications in Physics to the Bio-sciences, Social Sciences or Economics. In many of those settings, agents naturally appear as non-identical: The structure of the connectivities in the network has proved to have critical effects on the dynamics, see among many [61, 77–80, 91, 92], and the role of the network of interactions is often critical. We present one example of such application in Appendix A which is based on models for the dynamics of biological neurons for which we briefly refer for example to [24, 37, 44, 65, 75, 86, 93].

The main result of this paper is to incorporate the complex non-interchangeable graph structures defined by w_{ij} into a simple limiting Vlasov equation

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, dy, \zeta) \right) = 0, \quad (5)$$

through the extended graphon $w(\xi, d\zeta)$.

In (5), f describes the probability of finding an agent at position $x \in \mathbb{R}^d$, in a certain state of interaction with the network $\xi \in [0, 1]$, at time $t > 0$. This means that the role of an agent can be represented by adding an extra one-dimensional variable. In this way, we associate to the discrete connectivities w_{ij} a function w , or some more general object, which accounts for the effect of the architecture of the complex original system (1) as the number N of agents is large. This type of graphon-like representation had previously been explored in particular in [20–22, 54, 66, 67] or in [41, 53]. However those results required more stringent assumptions on the connectivities, so that for example [20–22, 54] only apply to dense graphs or symmetric interactions, whilst [41, 53] allow for some sparse graphs, all of them requiring an *a priori* knowledge on how weights are generated as to discretize a prescribed limiting object. Of course, this way of prescribing weights provides some additional convergence, which is essential in these results. The case of sparse random graphs such as the famous Erdős–Rényi graphs, has also been treated for example in [25, 27, 67]. In these references one finds the common assumption that $Np_N \rightarrow \infty$, where $p_N \in (0, 1)$ represents the probability of drawing an edge. This condition characterizes the generalized mean-field scaling (3)–(4) on this class of sparse random graphs. Some even sparser Erdős–Rényi graphs

with $Np_N = O(1)$ have also been handled in [59, 71], but this time they do not satisfy the generalized mean-field scaling (3)–(4) and lead instead to diffusive systems. While graphons are capable of handling limits of some sparse graphs [7, 8], this usually requires some sort of renormalization of w_{ij} .

One of the major contributions of the present article is to offer a unified framework to derive the mean-field limit without any assumptions, structural or not, outside of the extended mean-field scaling provided by the assumptions (3)–(4). This comes with considerable difficulties and impose a completely new strategy to obtain the limit equation.

1.1 | Mean-field limits for identical weights

Before going further in the technical aspects of our work, we recall here the main notions for the idea of classical mean-field limit in exchangeable systems, corresponding to the simple case $w_{ij} = \bar{w}/N$,

$$\begin{cases} \frac{dX_i}{dt} = \frac{\bar{w}}{N} \sum_{j=1}^N K(X_i - X_j), \\ X_i(0) = X_i^0. \end{cases} \quad (6)$$

A critical question for any multi-agent system is how to choose initial data for (6), since it is not feasible in almost any applications to individually identify or measure every single X_i^0 . Classical assumptions rely on the notion of molecular chaos by imposing that the X_i^0 be independent and identically distributed (*i.i.d.*) random variables (or almost *i.i.d.* in some appropriate sense) according to an initial probability distribution $f^0 \in \mathcal{P}(\mathbb{R}^d)$.

This leads to the key notion of propagation of chaos, which consists in proving that the $X_i(t)$ solving (6) still remain approximately *i.i.d.* (again in some appropriate measure). For exchangeable systems such as (6), propagation of chaos often then directly implies that the limit as $N \rightarrow \infty$ of the 1-particle distribution of the system solves the mean-field equation

$$\partial_t f(t, x) + \operatorname{div}_x \left(\bar{w} f(t, x) \int_{\mathbb{R}^d} K(x - y) f(t, dy) \right) = 0. \quad (7)$$

We refer to the seminal results in [13, 35, 70] and later [13, 25, 27, 35, 42, 48, 50, 53, 59, 67, 70, 71, 85, 93] for the derivation of (7) for $K \in W^{1,\infty}(\mathbb{R}^d)$. As mentioned earlier, the case of non Lipschitz kernels corresponds to many realistic applications, it is much more difficult and still poorly understood to some degree [13, 43, 49, 51, 81, 82, 85]. An extensive discussion would however carry us too far from the main goals of this article and we only briefly refer again to [42, 48, 50].

A useful notion for system (6) is the so-called empirical measure which is given by,

$$\mu^N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x),$$

for every $t \geq 0$. In the deterministic case, the empirical measure is itself already a solution in distributional sense to the mean-field equation (7), which offers a straightforward approach to deriving the mean-field limit as $N \rightarrow \infty$, for Lipschitz kernels K , through a stability analysis of

Equation (7) in the space of measures. Specifically, the empirical measure offers a very simple way to formulate and understand the mean-field limit in terms of the weak convergence of the random measure μ^N to the deterministic limit f . Classical results for Lipschitz kernels for example imply that

$$\|\mu_N - f\|_{W^{-1,1}} \leq \|\mu_N^0 - f\|_{W^{-1,1}} e^{t \bar{w} \|\nabla K\|_{L^\infty}}.$$

Some major differences are immediately apparent for non-exchangeable systems like (1). First of all, because agents are non-identical, it may not be realistic to assume that they are initially identically distributed. We will still assume that the X_i^0 are independent though as this is essential for the reduction in complexity leading to (7). Instead of propagation of chaos, we consequently have to deal with the weaker notion of *propagation of independence*.

It is possible to extend some of the classical approaches to prove this propagation of independence (as we do in the proof later). Unfortunately propagation of independence no longer directly implies the limit (7) for general non-exchangeable systems, creating major difficulties. The one exception occurs in the special case where the degree of each node of the graph is independent of i , that is,

$$\sum_{j=1}^N w_{ij} = \bar{w},$$

for all $i = 1, \dots, N$. In that setting, it is possible to still derive the classical mean-field limit (7); see [31] for *i.i.d* initial positions X_i^0 , and to [26] for the stability for this initial distribution. For general connectivities w_{ij} however, new ideas and new methods seem to be needed.

1.2 | The use of graphons for multi-agent systems

Graphons offer a non-parametric method of modeling and estimating large networks and are constructed as limits of sequences of dense graphs. There now exists a large literature dedicated to graphons for which we refer to the seminal [9–12, 62].

To give a rough idea of how graphons play a critical role in non-exchangeable systems, consider a sequences of dense graphs G_N with nodes indexed from 1 to N and adjacency matrix $(w_{ij}^N)_{1 \leq i, j \leq N}$ with the key assumption that

$$w_{ij}^N = \frac{\bar{w}_{ij}^N}{N} \quad \text{with} \quad \max_{1 \leq i, j \leq N} \bar{w}_{ij}^N = O(1), \quad \text{as } N \rightarrow \infty. \quad (8)$$

For a fixed N , it is straightforward to define the graphon over $[0, 1]^2$

$$w_N(\xi, \zeta) = \sum_{i, j=1}^N N w_{ij}^N \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta). \quad (9)$$

One directly observes that w_N is uniformly bounded in $L^\infty([0, 1]^2)$ but a fundamental property behind graphons is that w_N can be made compact in the appropriate distance after re-arrangements. More precisely, there exists a measure-preserving map ϕ_N on $[0, 1]$ such that

the rearranged object $w_N(\phi_N(\xi), \phi_N(\zeta))$ is compact in the norms of operators from $L^\infty([0, 1])$ to $L^1([0, 1])$; this norm is equivalent to the so-called *cut metric*, see [62]. Identifying w with its associated adjacency operator $\phi \in L^\infty([0, 1]) \mapsto w(\phi) = \int_0^1 \phi(\zeta) w(\cdot, \zeta) d\zeta \in L^1([0, 1])$, its operator norm $\|\cdot\|_{L^\infty \rightarrow L^1}$ is defined by

$$\|w\|_{L^\infty \rightarrow L^1} = \sup_{\|\phi\|_{L^\infty} \leq 1} \|w(\phi)\|_{L^1} = \sup_{\|\phi\|_{L^\infty}, \|\psi\|_{L^\infty} \leq 1} \int_{[0, 1]^2} \psi(\xi) \phi(\zeta) w(\xi, \zeta) d\xi d\zeta.$$

The previous compactness property of w_N in the operator norm under the assumption that w_N are uniformly bounded in $L^\infty([0, 1]^2)$ was initially established in [63] using a weak version of the Szemerédi Regularity Lemma [87] and a martingale argument.

This turns out to be a key property for the purpose of deriving the mean-field limit. For example, it provides a natural way to obtain weak stability for Equation (5). Namely, consider a sequence of weak solutions f_N to (5) with weights w_N , where w_N are uniformly bounded in $L^\infty([0, 1]^2)$ and f_N are uniformly bounded in $L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$, for all $t_* > 0$. Assume further that f is any other solution to (5) with weights w lying in the same regularity class as above. We refer to Proposition 4.11 for the precise well-posedness result of (5). Then, we can infer the stability estimate

$$\frac{d}{dt} \int_0^1 \int_{\mathbb{R}^d} |f_N - f| dx d\xi \leq C_1 \int_0^1 \int_{\mathbb{R}^d} |f_N - f| dx d\xi + C_2 \|w_N - w\|_{L^\infty \rightarrow L^1}, \quad (10)$$

where C_1, C_2 are constants depending on the norms of f , the uniform bound of the norm of w_N and the L^1 and Lipschitz norms of K . This estimate can easily be derived formally and how to obtain it rigorously goes beyond the limited technical scope of this introduction. We just mention that, since K is Lipschitz, (10) can also follow from the notion of solutions in Proposition 4.11 as long as f_N and f are smooth in t and x : if $f_N, f \in W_{t,x}^{1,\infty} L_{\xi}^\infty$ for example, then we can use any $\chi(f_N - f)$ for χ smooth as a test function on (5).

Consequently, using the above theory of dense graph limits, one can set the measure-preserving maps ϕ_N on $[0, 1]$ so that the re-arranged objects $w_N(\phi_N(\xi), \phi_N(\zeta))$ converge (up to subsequence) in the operator norm, and also set w to be the obtained limit. By doing so, the second term in the right hand side of (10) converges to zero and, assuming well-prepared initial data, we can pass to the limit as $N \rightarrow \infty$ in $f_N(t, x, \phi_N(\xi))$ and obtain a solution $f(t, x, \xi)$ to (5). As a consequence, we can determine the limit of the 1-particle distribution given by $\int_0^1 f_N(t, x, \xi) d\xi$:

$$\int_0^1 f_N(t, x, \xi) d\xi = \int_0^1 f_N(t, x, \phi_N(\xi)) d\xi \rightarrow \int_0^1 f(t, x, \xi) d\xi, \quad \text{as } N \rightarrow \infty.$$

We refer to [20–22, 54] in particular for a complete analysis that this brief sketch cannot do justice to. An alternative and more primitive approach to derive macroscopic limits for (1) was obtained in [66], again for graphons $w \in L^\infty([0, 1]^2)$ and Lipschitz kernel K , in terms of the graph-limit equation

$$\partial_t X(t, \xi) = \int_0^1 w(\xi, \zeta) K(X(t, \xi) - X(t, \zeta)) d\zeta. \quad (11)$$

Here, $X \in C^1(\mathbb{R}_+, L^\infty([0, 1]))$ can be interpreted as a time-evolving parametrization of the continuum of agents by a continuous index $\xi \in [0, 1]$. Whilst a large portion of the literature sticks to this simpler formulation, we remark that (11) corresponds to a special class of solutions to (5) having the form $f(t, x, \xi) = \delta_{X(t, \xi)}(x)$, but clearly not all solutions of (5) come from solutions to (11).

The critical issue to implement a similar stability estimate like (10) in our case is that our assumptions (3)–(4) of course cannot ensure that the weights w_{ij} satisfy (8). This concerns both dense and sparse graphs: sparse graph sequences in general will not satisfy (8), but also many dense graph sequences may fail to satisfy (8). As we noted above, one may still use graphons to characterize limits of some sparse graph sequences. For example, for sparse deterministic graphs [7, Theorem 2.8] shows that any C -upper L^p regular sequence of weighted graphs with $p > 1$ converges to some $w \in L^p([0, 1])$ in the cut distance. For sparse random graphs, [7, Theorem 2.14, Corollary 2.15] derives a general convergence result of the sequence of w -random sparse graphs generated by any $w \in L^1([0, 1])$ toward w itself in the cut distance. In both cases, a suitable renormalization on the graphon is needed. One fundamental advantage of the theory developed in this paper is that it applies to all sparse graphs which satisfy (3)–(4). However, our analysis and motivation are not limited to extend previous results to sparse graph sequences.

Various extensions of graphons have been proposed to handle less stringent assumptions than (8). We mention in particular the aforementioned extension to L^p graphons in [7, 8], the more general operator-based extension in [3] based on *graphops*, and the measure-theoretic extension in [55] based on *s-graphons*. Each of the three classes of continuum objects contain the family of finite graphs, and they can be endowed with a compact metric space structure under a certain topology, similarly to what it was done for graphons. We emphasize that the above assumption (3) represents a scaling of weights w_{ij} and only provides uniform bounds for w_N in the mixed spaces $L^\infty_\xi \mathcal{M}_\zeta = L^\infty_\xi([0, 1], \mathcal{M}_\zeta([0, 1]))$ and $L^\infty_\zeta \mathcal{M}_\xi = L^\infty_\zeta([0, 1], \mathcal{M}_\xi([0, 1]))$. The scaling of weights in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ corresponds to a continuous versions of the same discrete version (3) of the scaling and suggests defining a new class of continuum objects which we shall call *extended graphons*, see next section and Definition 4.6 below for further details. A similarly scaled extension was developed in [55], based on *digraph measures* or bounded everywhere defined families of measures, and is discussed more at length after Theorem 1.1. Closely related is also the scaling in [64], where the author considered limits of random graphs sampled from graphons not necessarily bounded but belonging to the mixed space $L^\infty_\xi([0, 1], L^2_\zeta([0, 1]))$. In contrast with classical graphons, our measure-valued definition of extended graphons reflects, in part, the sparsity of the network (as they are intimately related to graphops, s-graphons, and digraph measures), but also, the microscopic inhomogeneity of the network. A fundamental example of this lies in constant graphons, which are associated with homogeneous graphs. We refer to Section 2.5 for some examples sparse and inhomogeneous graph sequences which converge to extended graphons under our notion of convergence.

The goal of this paper is not to provide a complete classification of the existing theories of graph limits. However, the above classes of objects can be compared, and so has been done in more dedicated literature, for example, [3, 55] and also [41, 53]. Without any intention of comparing their topologies, the previous classes of objects are related as sketched in Figure 1, at least when restricted to symmetric weights. We note that extended graphons account for an intermediate degree of sparsity as they occupy an intermediate position between the class of graphons and graphops. This fact will become clearer later in Section 4, and more particularly in the operator representation of extended graphons in Lemma 4.7. However, we do emphasize the topology that we use on those extended graphons is very distinct from other known objects.

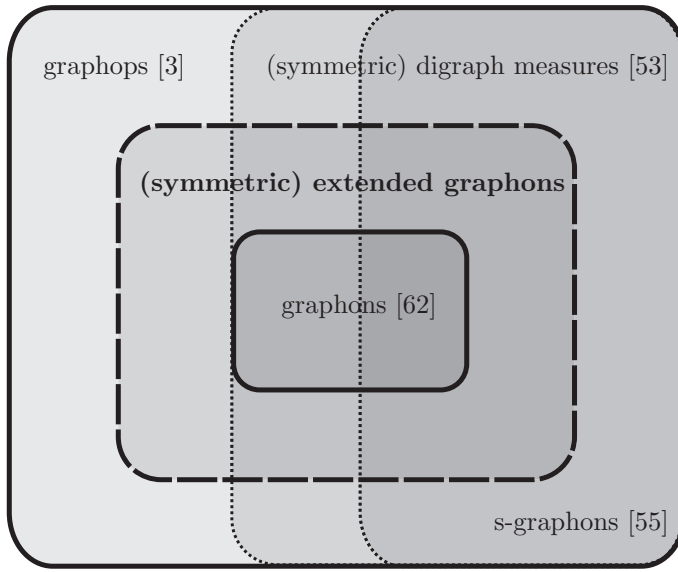


FIGURE 1 Comparison of the various graph limit theories.

1.3 | Our new result

The main contribution of this paper is to provide the mean-field limit with only assumptions (3)–(4). This first requires a careful definition of the space for limiting kernel which we call *extended graphons* and we denote by $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$; namely $w \in L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ iff

$$w(\xi, d\zeta) \in L^\infty([0, 1], \mathcal{M}([0, 1])), \quad w(d\xi, \zeta) \in L^\infty([0, 1], \mathcal{M}([0, 1])), \quad (12)$$

where we define the space $L^\infty([0, 1], \mathcal{M}([0, 1]))$ as the topological dual to the Banach space $L^1([0, 1], C([0, 1]))$, endowed with the corresponding weak-* topology.

We are now ready to state our main result,

Theorem 1.1. *Assume that the interaction kernel K belongs to $W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)$ and consider a sequence $(X_i(t))_{1 \leq i \leq N}$, solving the multi-agent system (1) for connection weights satisfying only (3)–(4). Assume moreover that the initial data X_i^0 are independent random variables, but not necessarily identically distributed, and denote their laws by f_i^0 . Finally assume that*

$$\sup_{N \in \mathbb{N}} \sup_{1 \leq i \leq N} \mathbb{E}[|X_i^0|^2] < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{1 \leq i \leq N} \|f_i^0\|_{W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)} < \infty.$$

Then, there exist $w \in L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ and $f \in L^\infty([0, t_] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$, for any $t_* > 0$, such that f solves (5) with w and, up to the extraction of a subsequence,*

$$\sup_{0 \leq t \leq T} \mathbb{E} W_1 \left(\int_0^1 f(t, \cdot, \xi) d\xi, \mu_N(t, \cdot) \right) \rightarrow 0,$$

as $N \rightarrow \infty$, where W_1 is the usual Wasserstein distance acting on the subset of $\mathcal{P}(\mathbb{R}^d)$ with finite first order moments, and $\mu^N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x)$ are the empirical measures associated to $(X_i(t))_{1 \leq i \leq N}$.

Compared to the existing literature, Theorem 1.1 is the only result so far capable of handling the mean-field limit where we only assume assumptions (3)–(4) on the connections w_{ij} without any other structural assumptions. As mentioned earlier in the introduction, when the graph has some specific structure, Erdős-Rényi for example, other approaches exist that only require the scaling (3)–(4) [25, 27, 59, 67, 71]. We do not obtain classical graphons since the limiting kernel w only belongs to $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, which is a natural space corresponding to the scaling in (3). As mentioned above, a space with a similar scaling is the space of digraph measures $B([0, 1], \mathcal{M}([0, 1]))$. This space was inspired in the s-graphons [55], specifically introduced to handle sparse graphs, and it was used in [53] for mean-field limits. There are a few key differences with the present analysis, as digraph measures $B([0, 1], \mathcal{M}([0, 1]))$ consists of everywhere defined in ξ bounded measures in ζ while our extended graphons in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ are only almost everywhere defined in ξ (or in ζ). The use of extended graphons offers several advantages when passing to the limit. In particular, in comparison [53] appears to require an improved continuous dependence in ξ of the digraph measures that further restricts the type of connectivities that can be used with respect to (3)–(4).

But our notion of extended graphons in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ does lead to some major issues, as it does not seem possible to pass to the limit in Equation (5) directly. That imposes a new strategy briefly outlined in the sketch of proof in the next subsection. Some of that technical difficulty can be glimpsed from the fact that it is not yet clear in which sense Equation (5) is posed with only $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$. The main issue is the non-linear term of course since f is only essentially bounded in ξ with no additional regularity, and only *a.e.* defined. More specifically, it is not immediately obvious how to make sense of terms like

$$\xi \in [0, 1] \mapsto \int_0^1 \phi(\zeta) w(\xi, d\zeta),$$

for $\phi \in L^\infty([0, 1])$. This is where the proper definition of $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ is needed and one reason why we require w to belong both to $L^\infty_\xi \mathcal{M}_\zeta$ and to $L^\infty_\zeta \mathcal{M}_\xi$ (which would be automatic if w was symmetric). A careful analysis then allows proving that the above function is correctly defined in $L^\infty([0, 1])$ with

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L^\infty_\xi} \leq \|w\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} \|\phi\|_{L^\infty},$$

as we prove later in Lemma 4.7. Note that the above also allow identifying any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ with a bounded linear operator $L^\infty([0, 1]) \rightarrow L^\infty([0, 1])$, and therefore this explains the above-mentioned relation in Figure 1 between extended graphons and graphops.

We also note that our proofs could imply an even more general formulation of Theorem 1.1. For example the assumptions $f_i^0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ would be enough, and it should even be possible to further relax some of the assumptions on the graphs with

$$\frac{1}{N} \sum_{i=1}^N \left(\sum_{j=1}^N |w_{ij}| \right)^p = O(1), \quad \frac{1}{N} \sum_{i=1}^N \sup_{1 \leq j \leq N} |w_{ij}|^{1/p} = o(1), \quad \text{as } N \rightarrow \infty,$$

for some $p > 1$, instead of (3)–(4).

As in classical mean-field limits, independence of initial data is the key to the reduction of complexity provided by Theorem 1.1. We assumed for simplicity that the X_i^0 were fully independent but it would be straightforward to relax this condition by having them approximately independent.

Theorem 1.1 obtains the limit up to the extraction of a subsequence in N , since it makes no assumptions on the convergence of the initial data and it then amounts to a compactness result. Nevertheless, it is possible to derive the convergence of the whole sequence but the formulation is trickier than usual because of the role played by the connectivities w_{ij} . The first necessary assumption is naturally the convergence of the full sequence of empirical measures: There exists $\bar{f}^0 \in W^{1,\infty}(\mathbb{R}^d)$ such that

$$\mathbb{E} W_1(\bar{f}^0, \mu_N^0) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (13)$$

Alternatively, since X_i^0 are independent, one may assume convergence of the 1-particle distribution

$$W_1\left(\bar{f}^0, \frac{1}{N} \sum_{i=1}^N f_i^0\right) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

However (13) is not enough and some additional convergence of the w_{ij} is also required.

Theorem 1.1 opens up new avenues to the analysis but still leaves some important questions unresolved. The main and obvious issue concerns the Lipschitz assumption on the kernel K . This is a critical issue for many applications which involve some form of singular interactions. Among several examples, we mention the integrate and fire system for biological neurons that we briefly describe in Appendix A. Integrate and fire models only very loosely fit within the multi-agent framework that we focus on in this paper so that extending the present analysis to integrate and fire model would be a highly non trivial extension.

The Lipschitz regularity of K plays a critical role in the first step of our analysis when propagating independence. It does not appear as essential in the next, more complex steps of our proofs. Replacing the classical trajectory methods to obtain propagation of independence could allow obtaining an equivalent of Theorem 1.1 with less stringent assumptions on K .

2 | NOTATIONS, BASIC DEFINITIONS, EXAMPLES AND SKETCH OF THE PROOF

2.1 | Notations

We denote by $\mathcal{M}(E)$ the space of finite Radon measures on E , where here the underlying space E could be \mathbb{R}^d , the interval $[0, 1]$, or also some product of those spaces like $\mathbb{R}^d \times [0, 1]$ or $[0, 1]^2$. The space $\mathcal{P}(E)$ denotes the subspace of probability measures on E , [89] that is measures $\mu \in \mathcal{M}(E)$ that are non-negative, with mass 1.

For a given measure μ , we denote by $\mu(x)$ the abstract notion and by $\mu(dx)$ when integrating against μ . For example, the Dirac mass at 0 is defined by $\mu(x) = \delta_0(x)$ and we write

$$\int_E \phi(x) \mu(dx) = \int_E \phi(x) \delta_0(dx) = \phi(0),$$

for all continuous function ϕ .

This notation offers a convenient way of representing marginals for measures of several variables. For example, if $\mu \in \mathcal{M}(E \times E)$, then $\nu_1(x) = \int_E \mu(x, dy)$ simply denotes the first marginal of μ , that is,

$$\nu_1(O) = \mu(O \times E),$$

for any measurable subset $O \subset E$, while $\nu_2(y) = \int_E \mu(dx, y)$ represents the second marginal of μ .

We recall that $\mathcal{M}(E)$ is the topological dual of $C_0(E)$, the space of continuous functions with vanishing limit at infinity (more precisely, $C_0(\mathbb{R}^d)$ for $E = \mathbb{R}^d$ and $C([0, 1])$ for $E = [0, 1]$). Unless otherwise specified, we always use the corresponding implied weak-* topology on $\mathcal{M}(E)$. Specifically, we have that $\mu_n \rightarrow \mu$ for the weak-* topology if, and only if,

$$\int_E \phi(x) \mu_n(dx) \rightarrow \int_E \phi(x) \mu(dx),$$

for all $\phi \in C_0(E)$. We also recall that, by the Banach–Alaoglu theorem, any ball of finite radius of $\mathcal{M}(E)$ is precompact for this topology. If E is compact, for example, $E = [0, 1]$, then $\mathcal{P}(E)$ is a compact metric space for the weak-* topology.

2.2 | The space $L^\infty([0, 1], \mathcal{M}([0, 1]))$

We define the space $L^\infty([0, 1], \mathcal{M}([0, 1]))$ as the topological dual of the Bochner space $L^1([0, 1], C([0, 1]))$. So defined $L^\infty([0, 1], \mathcal{M}([0, 1]))$ is not a Bochner space because $\mathcal{M}([0, 1])$ fails the Radon–Nikodym property [6, 33]. However, by a variant of the Riesz representation theorem operating in the absence of the Radon–Nikodym property [47] we have that the dual space $L^\infty([0, 1], \mathcal{M}([0, 1]))$ amounts to a weak-* Bochner space. Specifically, objects $w \in L^\infty([0, 1], \mathcal{M}([0, 1]))$ are alternatively identified with maps $\xi \in [0, 1] \mapsto w(\xi, \zeta) \in \mathcal{M}([0, 1])$ which are weakly-* measurables and such that there exists $C > 0$ such that

$$\left\| \int_0^1 \phi(\zeta) w(\cdot, d\zeta) \right\|_{L^\infty([0, 1])} \leq C \|\phi\|_{C([0, 1])},$$

for all $\phi \in C([0, 1])$. We refer to Definition 4.5 for further details. The best constant C above defines the norm on the weak-* Bochner space $L^\infty([0, 1], \mathcal{M}([0, 1]))$ and, by the above duality theorem, it is equivalent to the norm of $L^\infty([0, 1], \mathcal{M}([0, 1]))$ as a dual space. The corresponding duality reads

$$\langle w, \phi \rangle = \int_{[0, 1]^2} \phi(\xi, \zeta) w(\xi, d\zeta),$$

for all $w \in L^\infty([0, 1], \mathcal{M}([0, 1]))$ and all $\phi \in L^1([0, 1], C([0, 1]))$. Except otherwise specified, we always imbue $L^\infty([0, 1], \mathcal{M}([0, 1]))$ with its weak-* topology as a dual space. By the Banach–Alaoglu theorem, any ball of finite radius of $L^\infty([0, 1], \mathcal{M}([0, 1]))$ is again precompact in the weak-* topology.

We recall that $L^1([0, 1], C([0, 1]))$ is a separable, Banach space. Moreover, the following inclusions

$$C^\infty([0, 1]^2) \subset C([0, 1]^2) \subset L^1([0, 1], C([0, 1])),$$

are dense, which directly implies that $L^\infty([0, 1], \mathcal{M}([0, 1]))$ can be identified with a subspace of $\mathcal{M}([0, 1]^2)$ (and therefore of distributions $\mathcal{D}'([0, 1]^2)$). This leads to the inclusions:

$$L^\infty([0, 1], \mathcal{M}([0, 1])) \subset \mathcal{M}([0, 1]^2) \subset \mathcal{D}'([0, 1]^2).$$

Specifically, a distribution $w \in \mathcal{D}'([0, 1]^2)$ belongs to $L^\infty([0, 1], \mathcal{M}([0, 1]))$ if, and only if, there exists some $C > 0$ such that

$$\int_{E^2} \phi(\xi, \zeta) w(\xi, d\zeta) d\xi \leq C \|\phi\|_{L^1([0, 1], C_0([0, 1]))},$$

for all $\phi \in C^\infty([0, 1]^2)$, the best C being the norm of $L^\infty([0, 1], \mathcal{M}([0, 1]))$ as a dual space.

Since the weak-* topology on a dual space is metrizable when induced on balls, then we have that $w_n \rightarrow w$ for the weak-* topology of $L^\infty([0, 1], \mathcal{M}([0, 1]))$ if, and only if, $w_n \rightarrow w$ in the sense of distributions *and* $\sup_n \|w_n\|_{L^\infty([0, 1], \mathcal{M}([0, 1]))} < \infty$. This also directly implies that $C^\infty([0, 1]^2)$ functions are dense in $L^\infty([0, 1], \mathcal{M}([0, 1]))$, so that this space is separable for the weak-* topology.

Similarly, the space $L^\infty([0, 1], L^2([0, 1]))$ can be defined either through duality with $L^1([0, 1], L^2([0, 1]))$ or as a Bochner space. Since this time $L^2([0, 1])$ does verify the Radon-Nikodym property, the classical version of the Riesz representation theorem [33] ensure that both approaches coincide.

2.3 | The space $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ and examples

Depending on the order of variables, we can actually define two spaces $L^\infty_{\xi} \mathcal{M}_{\zeta} := L^\infty_{\xi}([0, 1], \mathcal{M}_{\zeta}([0, 1]))$ and $L^\infty_{\zeta} \mathcal{M}_{\xi} := L^\infty_{\zeta}([0, 1], \mathcal{M}_{\xi}([0, 1]))$. As above, both can be regarded as subspaces of distributions $\mathcal{D}'([0, 1]^2)$ through the dualities

$$\langle w_1, \phi \rangle := \int_{[0, 1]} \phi(\xi, \zeta) w_1(\xi, d\zeta), \quad \langle w_2, \phi \rangle := \int_{[0, 1]} \phi(\xi, \zeta) w_2(d\xi, \zeta),$$

for every $w_1 \in L^\infty_{\xi} \mathcal{M}_{\zeta}$, $w_2 \in L^\infty_{\zeta} \mathcal{M}_{\xi}$ and all $\phi \in C^\infty([0, 1]^2)$. As subspaces of distributions we can easily define the intersection $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ with the corresponding union of the two topologies. Equivalently, this could be defined as the topological dual of $L^1_{\xi} C_{\zeta} + L^1_{\zeta} C_{\xi}$, with the corresponding weak-* topology. The induced norm on $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ is simply

$$\|w\|_{L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}} = \max\{\|w\|_{L^\infty_{\xi} \mathcal{M}_{\zeta}}, \|w\|_{L^\infty_{\zeta} \mathcal{M}_{\xi}}\}.$$

We again have that $w_n \rightarrow w$ for the weak-* topology of $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ if, and only if, $w_n \rightarrow w$ in the sense of distributions and $\sup_n \|w_n\|_{L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}} < \infty$. As before $L^\infty_{\xi} \mathcal{M}_{\zeta} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ is also

separable and $C^\infty([0, 1])$ functions are dense for the weak-* topology. Again, any ball of finite radius of $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ is precompact for the same weak-* topology. The space $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ will also be introduced later in Definition 4.6 under the name of extended graphons, and will serve as a new graph limit theory.

An elementary but illuminating example of elements in $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ consists in considering just $w = \delta(\xi - \zeta)$. Obviously this kernel can be regarded in three different equivalent ways. First, as an element in $L_\xi^\infty \mathcal{M}_\zeta$, that is, a parametrized family $\xi \in [0, 1] \mapsto \delta_\xi(\zeta) \in \mathcal{M}([0, 1])$ of measures in ζ . Second, as an element in $L_\zeta^\infty \mathcal{M}_\xi$, that is, a parametrized family $\zeta \in [0, 1] \mapsto \delta_\zeta(\xi) \in \mathcal{M}([0, 1])$ of measures in ξ . Finally, as the measure in $\mathcal{M}([0, 1]^2)$ supported on the diagonal of $[0, 1]^2$. The definitions above provide a simple framework where we can simply switch between each interpretation as desired.

A more interesting example consists in considering $w = \delta(\Phi(\xi) - \zeta)$ where $\Phi : [0, 1] \rightarrow [0, 1]$ is any one-to-one, measure-preserving map. We build on this example to provide an illustration of a non-trivial limiting kernel for our main result. As above, this kernel can be seen in three different ways. First, it is clearly a parametrized family $\xi \in [0, 1] \mapsto \delta_{\Phi(\xi)}(\zeta) \in \mathcal{M}([0, 1])$ of measures in ζ . More precisely,

$$\int_{[0, 1]} \phi(\zeta) w(\xi, d\zeta) = \phi(\Phi(\xi))$$

for all $\phi \in C([0, 1])$, which yields obviously a measurable function in ξ , and therefore w defines an element in $L_\xi^\infty \mathcal{M}_\zeta$. Of course, the above embedding allows regarding w as a measure in $\mathcal{M}([0, 1]^2)$, that is,

$$\int_{[0, 1]} \varphi(\xi, \zeta) w(d\xi, d\zeta) = \int_{[0, 1]} \varphi(\xi, \Phi(\xi)) d\xi = \int_{[0, 1]} \varphi(\Phi^{-1}(\zeta), \zeta) d\zeta,$$

for all $\varphi \in C([0, 1]^2)$, where in the last equality we have made the change of variables $\zeta = \Phi(\xi)$ as Φ is one-to-one and measure-preserving. This allows regarding w also as a parametrized family $\zeta \in [0, 1] \mapsto \delta_{\Phi^{-1}(\zeta)}(\xi) \in \mathcal{M}([0, 1])$ of measures in ξ , that is,

$$\int_{[0, 1]} \phi(\xi) w(d\xi, \zeta) = \phi(\Phi^{-1}(\zeta)),$$

for all $\phi \in C([0, 1])$. Of course, they all represent the same object as for all $\varphi \in C([0, 1]^2)$ we have

$$\int_{[0, 1]^2} \varphi(\xi, \zeta) w(\xi, d\zeta) d\xi = \int_{[0, 1]^2} \varphi(\xi, \zeta) w(d\xi, d\zeta) = \int_{[0, 1]^2} \varphi(\xi, \zeta) w(d\xi, \zeta) d\zeta.$$

2.4 | Further comments on our extended graphons

There are compelling reasons why extended graphons $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ appear to be the correct scale for the type of complex structures of connectivities that we are interested in.

We first point out that our extended graphons occupy an intermediary place in the current hierarchy of graphon-like objects, see Figure 1. Obviously they are more general than classical

graphons since they are not necessarily bounded, that is, $w \notin L_{\xi, \zeta}^{\infty} = L^{\infty}([0, 1]^2)$ in general. However as mentioned above, the analysis will show later that any $w \in L_{\xi}^{\infty} \mathcal{M}_{\zeta} \cap L_{\zeta}^{\infty} \mathcal{M}_{\xi}$ can also be seen as the kernel of a bounded operator from $L^{\infty}([0, 1]) \rightarrow L^{\infty}([0, 1])$; instead of an operator $L^{\infty}([0, 1]) \rightarrow L^1([0, 1])$ for graphops. As such extended graphons appear as intermediary objects between classical graphons and other extensions such as graphops, s-graphons and digraph measures, see Figure 1.

However the topology that we consider in our analysis for this space $L_{\xi}^{\infty} \mathcal{M}_{\zeta} \cap L_{\zeta}^{\infty} \mathcal{M}_{\xi}$ is one of the major differences with respect to the existing literature. The proper convergence for our large-graph limit cannot hold in the classical cut distance, or any direct extensions such as the topology of bounded linear operators $L^{\infty}([0, 1]) \rightarrow L^{\infty}([0, 1])$, which we have defined earlier, to the best of our knowledge. This is clear from the stability estimate (10) for graphons, which inevitably breaks for unbounded graphons. Instead the key notion of convergence that we use is based on a new countable family of observables that mixes the connection kernel and the initial laws on each agent. Specifically, the observables can be defined as follows

$$\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})(0, x_1, \dots, x_{|T|}) = \frac{1}{N} \sum_{i_1, \dots, i_{|T|}=1}^N \prod_{(k, l) \in E(T)} w_{i_k i_l} \prod_{m \in V(T)} f_{i_m}^0(x_m), \quad (14)$$

and they are indexed by finite trees T , where f_i^0 denotes the initial law of the variable X_i^0 . Those observables have a corresponding representation at the limit when the discrete weights $(w_{ij})_{1 \leq i, j \leq N}$ and laws $f_i^0(x)$ are replaced by kernels $w(\xi, \zeta)$ and initial data $f^0(x, \xi)$ through the definition,

$$\tau(T, w, f)(0, x_1, \dots, x_{|T|}) = \int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \prod_{m \in V(T)} f^0(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}. \quad (15)$$

Those definitions allow defining precisely the convergence in Theorem 1.1: If we have the strong convergence in $L^2(\mathbb{R}^{d|T|})$ of the observables $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, w, f^0(\xi, x))$ as $N \rightarrow \infty$, for all trees T , then the dynamics of the X_i can effectively be represented at the limit by the solution f to (5),

$$\frac{1}{N} \sum_i \delta_{X_i(t)}(x) \rightarrow \int_0^1 f(t, x, d\xi).$$

As it can be seen in the sketch of proof below, the observables $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})$ indeed play a critical role in our analysis as they entirely control the dynamics.

Note that since $f_i^0(x)$ and $f^0(x, \xi)$ are normalized probability measures with respect to the variable x , then by integrating the observables (14)–(15) on the spacial variables we are led to the well-known notion of moments on a graph, that is,

$$\int_{\mathbb{R}^{d|T|}} \tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})(0, x_1, \dots, x_{|T|}) dx_1 \dots, dx_{|T|} = \tau(T, (w_{ij})_{1 \leq i, j \leq N}),$$

together with its continuum version

$$\int_{\mathbb{R}^{d|T|}} \tau(T, w, f)(0, x_1, \dots, x_{|T|}) dx_1 \dots, dx_{|T|} = \tau(T, w),$$

where $\tau(T, (w_{ij})_{1 \leq i, j \leq N})$ is the *homomorphism density* of the finite tree T in the finite graph $(w_{ij})_{1 \leq i, j \leq N}$, and $\tau(T, w)$ is its continuum version, which are also used in the classical theory of graphons. In particular, under the above-mentioned convergence of the observables, we also have $\tau(T, (w_{ij})_{1 \leq i, j \leq N}) \rightarrow \tau(T, w)$ as $N \rightarrow \infty$ for all finite tree T . In a first major difference to graphons though, when $w \in L^\infty_{\xi} \mathcal{M}_{\xi} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ is an extended graphon we can only define the associated observables $\tau(T, w, f)$ and the homomorphism density $\tau(T, w)$ for trees T and not any arbitrary graph.

A second critical difference is that our notion of observables completely entangles the kernel with the initial conditions. In particular our main theorem does not provide independent convergence of the extended graphons and the initial data. In other words (see the example below as well), we could have that for some initial data $f_i^0(x)$ and some connections w_{ij} , one has that $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, w, f^0)$ for some limiting $w(\xi, d\zeta)$ and $f^0(x, \xi)$. If one then considers different initial data $\tilde{f}_i^0(x)$ but for the same connections w_{ij} , there is no particular reason why we would be able to use the same limiting kernel: We may need to derive a different limiting kernel $\tilde{w}(\xi, d\zeta)$ together with a different limiting initial law $\tilde{f}^0(x, \xi)$ to maintain the convergence $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (\tilde{f}_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, \tilde{w}, \tilde{f}^0)$.

While our main theorem ensures that there always exists a limiting representation, it is not necessarily unique. We can for instance find different kernels $w(\xi, d\zeta) \neq \tilde{w}(\xi, d\zeta)$ and initial data $f^0(x, \xi) \neq \tilde{f}^0(x, \xi)$ such that $\tau(T, w, f^0) = \tau(T, \tilde{w}, \tilde{f}^0)$; see the example below again.

All of this makes the notion of convergence that is developed in the present paper very different from most of mean-field limit results based on graphons or graphon-like objects. Because our result applies to any connections with only the bounds (3)–(4) without any additional convergence assumptions, it is unclear whether it would even be possible to identify a specific topology on the connections w_{ij} or kernels w without mixing them with the initial data. Another fundamental question is whether our class $L^\infty_{\xi} \mathcal{M}_{\xi} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ is optimal for detecting the sparse connectivity and the microscopic inhomogeneity of networks, and again we believe that it will depend on the properties of the network and its connectivities to be studied. Note our main Theorem 1.1 operates up to the choice of a subsequence, that is, once a suitable subsequence of graphs is selected so that the observables $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})$ (and therefore the homomorphism densities $\tau(T, (w_{ij})_{1 \leq i, j \leq N})$) converge. Therefore, a natural question is to understand how big is the set of all possible accumulations points via this procedure. We want to clarify that any $w \in L^\infty_{\xi} \mathcal{M}_{\xi} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ can be recovered in this way, which suggests that the class of extended graphons is optimal for that purpose. More specifically, and as established in Lemma 4.12, for any $w \in L^\infty_{\xi} \mathcal{M}_{\xi} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$, there exists $w_N \in L^\infty_{\xi, \zeta}$ uniformly bounded in $L^\infty_{\xi} L^1_{\zeta} \cap L^\infty_{\zeta} L^1_{\xi}$, converging to w in $L^1_{\xi} H^{-1}_{\zeta} \cap L^1_{\zeta} H^{-1}_{\xi}$. This finding is subsequently utilized in Lemmas 5.3 and 5.4 to establish the convergence of $\tau(T, w_N)$ to $\tau(T, w)$. Certainly, if w_N is further uniformly bounded in $L^\infty_{\xi, \zeta}$ (which is compatible with weights satisfying the scaling (8)), then $\tau(T, w_N) \rightarrow \tau(T, w)$ for some $w \in L^\infty_{\xi, \zeta}$, thus retrieving the classical result of Lovász–Szegedy [62, 63].

Several other open questions concern the new system of observables that we introduce in (15). Those are conjectured to naturally extend the notion of marginals, and hierarchy of marginals to non-exchangeable systems. The structure of the hierarchy of observables (15) remains rather poorly understood however. A first example is the range of possible functions that one can reach through (15): What would be the conditions on a sequence $\alpha_T(x_1, \dots, x_{|T|})$ indexed by trees T so that we can find $w \in L^\infty_{\xi} \mathcal{M}_{\xi} \cap L^\infty_{\zeta} \mathcal{M}_{\xi}$ and $f \in L^\infty(\mathbb{R}^d \times [0, 1])$, such that

$$\tau(T, w, f) = \alpha(T), \quad \forall T?$$

2.5 | Example of convergence

To illustrate those remarks, we discuss in some depth a natural example of connection weights, based on a sparse graph. Choose $1 \ll M \ll N$ (with N multiple of M for simplicity), and separate $\{1, \dots, N\}$ into N/M subsets $E_1^N, \dots, E_{N/M}^N$, with size $|E_k^N| = M$ for each $k \leq N/M$. In other words, we separate our agents into N/M distinct subpopulations.

Introduce further a permutation $\phi \in S_{N/M}$ which lets us define

$$w_{ij} = \begin{cases} \frac{1}{M} & \text{if } i \in E_k^N \text{ and } j \in E_{\phi(k)}^N \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

This choice of connection has a very simple interpretation. Agents are divided in N/M distinct classes or layers E_k^N , $k = 1, \dots, N/M$. Each agent i in a class E_k^N is connected with exactly M other agents that all belong to the class $E_{\phi(k)}^N$.

We note that the adjacency matrix w_{ij} is sparse as we have exactly $N \cdot M$ non-zero entries among a total of N^2 . Because M is much smaller than N this does lead to a sparse graph of connections. Note also that those connections satisfy (3)–(4) as long as $M \gg 1$. In particular if $i \in E_k^N$

$$\sum_{j=1}^N |w_{ij}| = \sum_{j \in E_{\phi(k)}^N} \frac{1}{M} = \frac{|E_{\phi(k)}^N|}{M} = 1,$$

and also similarly $\sum_{i=1}^N |w_{ij}| = 1$ since ϕ is a permutation. We also observe that this weights satisfy the assumptions in [26]. Therefore, if the X_i^0 are initially *i.i.d.*, then we can derive the mean-field limit to the classical Vlasov equation, that is, we can choose as a limiting kernel $w = 1$. This is not possible in general if the X_i^0 are not *i.i.d.*, and this example will provide an excellent illustration that the choice of the limiting kernel may have to depend on the initial laws.

Let us introduce the usual graphon representation of w_N

$$w_N(\xi, \zeta) = \sum_{i,j=1}^N N w_{ij} \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta).$$

To derive a more explicit formula for w_N , it is useful to have a good labeling of the indices on each class E_k^N . Of course, for any $k = 1, \dots, N/M$, we can list the indices in E_k^N consecutively by

$$E_k^N = \{(\sigma(k) - 1)M + 1, \dots, \sigma(k)M\},$$

where $\sigma \in S_{N/M}$ is some permutation. Introduce now the approximation of the Dirac mass at scale M/N ,

$$\delta_{\frac{M}{N}}(\xi) = \frac{N}{M} \mathbb{1}_{\xi \in [0, \frac{M}{N})},$$

and let us extend the discrete permutation ϕ to the step function $\phi_N : [0, 1] \longrightarrow [0, 1]$ with

$$\phi_N(\xi) = (\phi(\sigma(k)) - 1) \frac{M}{N}, \quad \text{if } \xi \in \left[(\sigma(k) - 1) \frac{M}{N}, \sigma(k) \frac{M}{N} \right),$$

for all $k = 1, \dots, \frac{N}{M}$. We then have the simple expression

$$w_N(\xi, \zeta) = \delta_{\frac{M}{N}}(\zeta - \phi_N(\xi)).$$

Let us note immediately that the classical Lovasz-Szegedy theory based on graphons cannot apply here. First, we of course have that $\|w_N\|_{L^\infty} = \frac{N}{M} \rightarrow \infty$. Moreover, we cannot have convergence of w_N in the cut distance as it is not a Cauchy sequence for this topology. Consider w_N and $w_{N'}$ (with corresponding M and M') with $\frac{N}{M} \gg \frac{N'}{M'}$. Choose $\psi(\xi) = \mathbb{1}_{\xi \in [(\phi(\sigma(1))-1)\frac{M}{N}, \phi(\sigma(1))\frac{M}{N})}$ and note that

$$\int_0^1 w_N(\xi, \zeta) \psi(\zeta) d\zeta = \frac{N}{M} \mathbb{1}_{\xi \in [(\sigma(1)-1)\frac{M}{N}, \sigma(1)\frac{M}{N})}.$$

Therefore, we have

$$\left\| \int_0^1 w_N(\xi, \zeta) \psi(\zeta) d\zeta \right\|_{L^1} = 1.$$

However, using that $w_{N'} \in L_{\xi, \zeta}^\infty$ (and also any rearranged version) we also have

$$\left\| \int w_{N'}(\Phi(\xi), \Phi(\zeta)) \psi(\zeta) d\zeta \right\|_{L^1} \leq \|w_{N'}\|_{L^\infty} \|\psi\|_{L^1} = \frac{N'}{M'} \frac{M}{N},$$

for any measure-preserving map $\Phi : [0, 1] \rightarrow [0, 1]$. Altogether implies the control on the cut distance

$$\inf_{\Phi} \|w_N(\xi, \zeta) - w_{N'}(\Phi(\xi), \Phi(\zeta))\|_{L^\infty \rightarrow L^1} \geq 1 - \frac{N'}{M'} \frac{M}{N},$$

where the infimum ranges over all measure-preserving maps of $[0, 1]$. Of course, since $\frac{N}{M} \gg \frac{N'}{M'}$, the above lower bound that does not converge to 0 as $N' \rightarrow \infty$. Because w_N is purely deterministic, there is also no renormalization that would preserve the dynamics and provide convergence in the cut distance.

This expression of w_N of course amounts to a discretization of

$$\bar{w}_N(\xi, \zeta) = \delta_{\phi_N(\xi)}(\zeta),$$

where we observe that ϕ_N is a *a.e.* defined measure-preserving map on $[0, 1]$. At this point, it is tempting to conjecture that, after re-indexing such that ϕ_N is compact and extracting a subsequence such that $\phi_N \rightarrow \bar{\phi}$, the limiting dynamics should follow the limiting kernel $\bar{w}(\xi, \zeta) = \delta_{\bar{\phi}(\xi)}(\zeta)$. But this is not correct per se. Instead, in such an example, it is not possible to determine the limiting kernel without some further information on the laws $f_i^0(x)$ of the initial positions X_i^0 . To keep this example reasonably simple, we assume that we only have at most N/M different laws f_k with $k = 1, \dots, \frac{N}{M}$, and that the X_i^0 are actually *i.i.d.* for $i \in E_k^N$ according to the law $f_k(x)$.

To determine the limiting kernel, our analysis requires to find the limit of every observable defined in the previous subsection. Let us hence write explicitly some of them. First we perform

the obvious graphon extension of the initial law defined by

$$f_N(x, \xi) = \sum_{k=1}^{N/M} f_k(x) \mathbb{I}_{\xi \in [(\sigma(k)-1)\frac{M}{N}, \sigma(k)\frac{M}{N})}.$$

The first observable, corresponding to the unique tree $T_1 \in \mathcal{T}_1$ with just one vertex is simply

$$\tau(T_1, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) = \int_0^1 f_N(x, \xi) d\xi = \frac{M}{N} \sum_{k=1}^{N/M} f_k(x).$$

One can readily check that the observable for the unique tree $T \in \mathcal{T}_2$ with two vertices is

$$\tau(T_2, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) = \int_0^1 f_N(x_1, \xi) f_N(x_2, \phi_N(\xi)) d\xi.$$

Other observables are built in a similar manner. We provide below some more examples. For instance, the tree $T_{k+1} \in \mathcal{T}_{k+1}$ with one root and k leaves yields

$$\tau(T_{k+1}, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) = \int_0^1 f_N(x_1, \xi) \prod_{i=1}^k f_N(x_{i+1}, \phi_N(\xi)) d\xi.$$

The tree $T_{2k+1} \in \mathcal{T}_{2k+1}$ with one root connected to k vertices, each of them with one leaf, instead produces the observable

$$\tau(T_{2k+1}, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) = \int_0^1 f_N(x_1, \xi) \prod_{i=1}^k f_N(x_{2i}, \phi_N(\xi)) f_N(x_{2i+1}, \phi_N(\phi_N(\xi))) d\xi.$$

As one can readily see, a sufficient condition to obtain $\delta_{\bar{\phi}(\xi)}(\zeta)$ as a limiting kernel is to have both ϕ_N converges strongly to $\bar{\phi}$ and f_N converging strongly to some \bar{f} . We are allowed to reindex (or use a measure-preserving map) but it has to be the same map for both. We will in fact obtain later a lemma that guarantees that this is possible (and in fact up to a countable number of functions). However, we stress again that the limiting kernel is not unique in general. Hence, even when ϕ_N and f_N are jointly compact, we may have other, different, and simpler acceptable limiting kernels. A trivial example was mentioned at the beginning of this subsection: if f_N does not depend on ξ then we may take $\bar{w} = 1$.

We expect that the simplest limiting kernel depends on the decomposition in cycles of ϕ_N and how f_N behaves on each cycle. Performing a full analysis is well beyond the scope of this example though. For this reason, let us further simplify things by assuming that ϕ has exactly one cycle of full length with simply: $\phi(k) = k + 1$ for $k < \frac{N}{M}$, with $\phi(\frac{N}{M}) = 1$. Furthermore, let us assume that $\sigma = Id$, that dimension $d = 1$ and $f_k(x) = f(x - k\frac{M}{N})$ for some smooth f . With these choices, one can check that

$$f_N(x, \xi) = f(x - [\xi]_N), \quad \phi_N(\xi) = [\xi]_N$$

where $[\xi]_N = \frac{M}{N} \lceil \frac{N}{M} \xi \rceil$ and $\lceil \cdot \rceil$ is the ceiling function. Since f is smooth, we have that

$$f_N(x, \xi) = f(x - \xi) + O\left(\frac{M}{N}\right), \quad \phi_N(\xi) = \xi + O\left(\frac{M}{N}\right).$$

Hence, it is obvious that the limiting dynamics can be represented by choosing $\bar{w} = \delta(\xi - \zeta)$ and $\bar{f}(x, \xi) = f(x - \xi)$. But of course, even here, the representation is not unique: any composition by a measure-preserving map would still work.

However, we can prove in that special case that the limiting dynamics cannot be represented by some $\bar{w} \in L_{\xi, \zeta}^{\infty}$. Assume that there is one such \bar{w} and \bar{f} such that $\tau(T, \bar{w}, \bar{f}) = \bar{\tau}(T)$ for all tree T . To be specific, assume that \bar{w} obeys the mean-field scaling, that is,

$$\sup_{\xi \in [0, 1]} \int_0^1 |\bar{w}(\xi, \zeta)| d\zeta \leq 1, \quad \sup_{\zeta \in [0, 1]} \int_0^1 |\bar{w}(\xi, \zeta)| d\xi \leq 1.$$

Assume moreover that the function

$$x \in \mathbb{R} \mapsto \int_0^1 |f(x - \xi)|^2 d\xi,$$

does not take only a finitely many values. We can easily calculate the limiting observables for any tree T

$$\tau(T, w_N, f_N) \rightarrow \bar{\tau}(T) := \int_0^1 \prod_{i=1}^{|T|} f(x_i - \xi) d\xi.$$

This is of course a very remarkable formula as it depends only on $|T|$.

Consider the tree $T_3 \in \mathcal{T}_3$ with two leaves attached to its root. Since we have $\int_{\mathbb{R}} \bar{f}(x_1, \xi_1) dx = 1$ for all $\xi_1 \in [0, 1]$, then we can calculate

$$\begin{aligned} \int_{\mathbb{R}^2} \tau(T_3, \bar{w}, \bar{f})(x_1, x, x) dx_1 dx &= \int_{\mathbb{R}} \int_{[0, 1]^3} \bar{w}(\xi_1, \xi_2) \bar{w}(\xi_1, \xi_3) \bar{f}(x, \xi_2) \bar{f}(x, \xi_3) d\xi_1 d\xi_2 d\xi_3 dx \\ &= \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi, \end{aligned}$$

where $W : L^2(\mathbb{R} \times [0, 1]) \longrightarrow L^2(\mathbb{R} \times [0, 1])$ is the linear operator defined by

$$W(g)(x, \xi) := \int_0^1 \bar{w}(\xi, \zeta) g(x, \zeta) d\zeta.$$

Consider next the tree $T_4 \in \mathcal{T}_4$ where we add one leaf to T_3 connected to one leaf (say node 2) of T_3 (which is then no more a leaf of course). Calculate again,

$$\begin{aligned} \int_{\mathbb{R}^3} \tau(T_4, \bar{w}, \bar{f})(x_1, x_2, x, x) dx_1 dx_2 dx \\ &= \int_{\mathbb{R}} \int_{[0, 1]^4} \bar{w}(\xi_1, \xi_2) \bar{w}(\xi_2, \xi_4) \bar{w}(\xi_1, \xi_3) \bar{f}(x, \xi_3) \bar{f}(x, \xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dx \\ &= \int_{\mathbb{R}} \int_0^1 W(\bar{f})(x, \xi) W^2(\bar{f})(x, \xi) dx d\xi. \end{aligned}$$

Now observe that by the above shape $\tau(T, \bar{w}, \bar{f}) = \bar{\tau}(T)$ of the limiting observables, we have

$$\int_{\mathbb{R}^3} \tau(T_4, \bar{w}, \bar{f})(x_1, x_2, x, x) dx_1 dx_2 dx = \int_{\mathbb{R}} \bar{\tau}(T_3)(x_1, x, x) dx_1 dx,$$

that is,

$$\int_{\mathbb{R}} \int_0^1 W(\bar{f})(x, \xi) W^2(\bar{f})(x, \xi) dx d\xi = \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi.$$

Note that thanks to the mean-field scaling on \bar{w} , we have

$$\int_{\mathbb{R}} \int_0^1 |W^2(\bar{f})(x, \xi)|^2 dx d\xi \leq \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi,$$

since we have that by Cauchy–Schwartz

$$\begin{aligned} |W^2(\bar{f})(x, \xi)|^2 &= \left(\int_0^1 \bar{w}(\xi, \zeta) W(\bar{f})(x, \xi) d\zeta \right)^2 \leq \int_0^1 \bar{w}(\xi, \zeta) d\zeta \int_0^1 \bar{w}(\xi, \zeta) |W(\bar{f})(x, \xi)|^2 d\zeta \\ &\leq \int_0^1 \bar{w}(\xi, \zeta) |W(\bar{f})(x, \xi)|^2 d\zeta, \end{aligned}$$

from the mean-field scaling. Therefore as claimed,

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 |W^2(\bar{f})(x, \xi)|^2 dx d\xi &\leq \int_{\mathbb{R}} \int_0^1 \int_0^1 \bar{w}(\xi, \zeta) |W(\bar{f})(x, \xi)|^2 dx d\xi d\zeta \\ &\leq \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi, \end{aligned}$$

still from the mean-field scaling.

This equality, together with another Cauchy–Schwartz inequality, lead to

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 W(\bar{f})(x, \xi) W^2(\bar{f})(x, \xi) dx d\xi \\ \leq \|W(\bar{f})\|_{L^2} \|W^2(\bar{f})\|_{L^2} \leq \|W(\bar{f})\|_{L^2}^2 = \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi. \end{aligned}$$

The identity above corresponds to equality in the Cauchy–Schwartz inequality, which implies that $W(\bar{f})$ and $W^2(\bar{f})$ must be linearly dependent: there must exist $\lambda \in \mathbb{R}$ such that $W^2(\bar{f}) = \lambda W(\bar{f})$. Furthermore the last inequality above also implies that $\lambda = 1$.

Once this is proved, we can keep adding leaves and with the same method, we can prove that

$$W(\bar{f})^k(x, \xi) = W(W(\bar{f})^k)(x, \xi) = \int_0^1 \bar{w}(\xi, \zeta) W(\bar{f})^k(x, \zeta) d\zeta.$$

The density of polynomials in continuous functions shows that in fact

$$\chi(W(\bar{f})(x, \xi)) = W(\chi(W(\bar{f}))(x, \xi)) = \int_0^1 \bar{w}(\xi, \zeta) \chi(W(\bar{f})(x, \zeta)) d\zeta,$$

for any continuous function χ .

This unique fact implies an extremely simple structure on $W(\bar{f})$ and \bar{w} if $\bar{w} \in L_{\xi, \zeta}^\infty$. It is straightforward to show that $W(\bar{f})$ can only take a finite number of values: there exists level sets L_k such that

$$W(\bar{f})(x, \xi) = \sum_{k=1}^K \alpha_k \mathbb{1}_{(x, \xi) \in L_k}.$$

Moreover \bar{w} needs to be decomposed along, because we must have for all x and ξ that

$$\mathbb{1}_{(x, \xi) \in L_k} = \int_0^1 \bar{w}(\xi, \zeta) \mathbb{1}_{(x, \zeta) \in L_k} d\zeta. \quad (16)$$

Indeed, assume by contradiction that $W(\bar{f})$ takes infinitely many values. By taking appropriate functions χ , this implies that we have an infinite number (at least countable) of sets L_k such that for every k

$$\mathbb{1}_{(x, \xi) \in L_k} = \int_0^1 \bar{w}(\xi, \zeta) \mathbb{1}_{(x, \zeta) \in L_k} d\zeta.$$

However for any fixed x , we can only have a limited number of L_k such that there exists ξ with $(x, \xi) \in L_k$. Denote by $\bar{L}_k(x)$ the corresponding section: $\bar{L}_k(x) = \{\xi, (x, \xi) \in L_k\}$. Observe that

$$\int_0^1 \bar{w}(\xi, \zeta) \mathbb{1}_{(x, \zeta) \in L_k} d\zeta \leq \|\bar{w}\|_{L^\infty} |\bar{L}_k(x)|.$$

Therefore, if $|\bar{L}_k(x)| > 0$, then we need to have that $|\bar{L}_k(x)| \geq \|\bar{w}\|_{L^\infty}^{-1}$ so that at most $\|\bar{w}\|_{L^\infty}^{-1}$ number of \bar{L}_k sets can be of positive measure. Since we are on a compact support, a straightforward covering lemma then shows that we cannot have an infinite number of L_k sets.

Observe that, in addition, we have that

$$\int_{\mathbb{R}} \int_0^1 W^*(W(\bar{f}))(x, \xi) W(\bar{f})(x, \xi) dx d\xi = \int_{\mathbb{R}} \int_0^1 W(\bar{f})(x, \xi) W^2(\bar{f})(x, \xi) dx d\xi,$$

with W^* defined by the formal adjoint

$$W^*(g)(x, \xi) := \int_0^1 \bar{w}(\zeta, \xi) g(x, \zeta) d\zeta.$$

We hence have from the argument above that

$$\int_{\mathbb{R}} \int_0^1 W^*(W(\bar{f}))(x, \xi) W(\bar{f})(x, \xi) dx d\xi = \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi.$$

Again the same mean-field scaling implies that

$$\int_{\mathbb{R}} \int_0^1 |W^*(W(\bar{f}))(x, \xi)|^2 dx d\xi \leq \int_{\mathbb{R}} \int_0^1 |W(\bar{f})(x, \xi)|^2 dx d\xi,$$

so we again have a case of equality in Cauchy–Schwartz and that proves that $W^*(W(\bar{f})) = W(\bar{f})$, or

$$\mathbb{I}_{(x, \xi) \in L_k} = \int_0^1 \bar{w}(\xi, \zeta) \mathbb{I}_{(x, \zeta) \in L_k} d\zeta. \quad (17)$$

Of course, this shows that the mean-field scaling also has to be an equality

$$\int_0^1 \bar{w}(\xi, \zeta) d\zeta = 1, \quad \int_0^1 \bar{w}(\xi, \zeta) d\xi = 1.$$

For any x , we recall that $\bar{L}_k(x)$ is the corresponding section: $\bar{L}_k(x) = \{\xi, (x, \xi) \in L_k\}$. The identities (16)–(17) imply that if $\xi \in \bar{L}_k(x)$ then $\text{supp } \bar{w}(\xi, \cdot) \subset \bar{L}_k(x)$ and reciprocally $\text{supp } \bar{w}(\cdot, \zeta) \subset \bar{L}_k(x)$ if $\zeta \in \bar{L}_k(x)$.

This has some surprising consequences on the structure of the $\bar{L}_k(x)$. For example, if $M = \bar{L}_l(y) \setminus \bar{L}_k(x)$, then for each $\xi \in M$, since $\xi \in \bar{L}_l(y)$ but $\xi \notin \bar{L}_k(x)$

$$\int_0^1 \bar{w}(\xi, \zeta) \mathbb{I}_{\zeta \in M} d\zeta = \int_0^1 \bar{w}(\xi, \zeta) \mathbb{I}_{\zeta \in \bar{L}_l(y)} d\zeta - \int_0^1 \bar{w}(\xi, \zeta) \mathbb{I}_{\zeta \in \bar{L}_l(y) \cap \bar{L}_k(x)} d\zeta = 1 - 0 = 1.$$

Because $\bar{w} \in L_{\xi, \zeta}^\infty$, a similar calculation as for proving that there is a finite number of level sets implies that, if $\bar{L}_l(y) \cap \bar{L}_k(x) \neq \emptyset$

$$|\bar{L}_l(y) \setminus \bar{L}_k(x)| \geq \frac{1}{\|\bar{w}\|_{L^\infty}},$$

and of course, individually, we have also for any x and k that

$$|\bar{L}_k(x)| \geq \frac{1}{\|\bar{w}\|_{L^\infty}}.$$

Because the measure of $[0, 1]$ is finite, we can deduce that we can only have a finite number of different $\bar{L}_k(x)$: there exists $M_1, \dots, M_L \subset [0, 1]$ such that for any x and k , there exists l with $\bar{L}_k(x) = M_l$. This is a variant of the famous Vitali's covering lemma.

Note that for example

$$\int_0^1 |W(\bar{f})(x, \xi)|^2 d\xi = \sum_{k=1}^K \alpha_k^2 |\bar{L}_k(x)|.$$

Since $\bar{L}_k(x)$ has only a finite number of possibilities, this finally implies that the integral above can only take a finite number of values. But this integral is derived from $\tau(T_3, \bar{w}, \bar{f})$ and it has to

be equal to the corresponding quantity derived from $\bar{\tau}(T_3)$ which is

$$\int_0^1 |f(x - \xi)|^2 d\xi.$$

This provides the contradiction since we assumed that the above quantity takes an infinite number of values as a function in x .

This example shows that it can be very complicated to determine what are the possible limiting kernels even knowing explicitly the limiting observables (and even with simple formulas such as here). One major issue is that we have much less observables than for classical graphon theory where one can work with homomorphism densities $\tau(G, w_N)$ for any graph G and not only trees.

This means that several additional transforms can leave the observables invariant. As a matter of fact, one can for example change $\bar{w}(\xi, \zeta)$ into $\bar{w}(\Phi(\xi), \Phi(\zeta))$ and $\bar{f}(x, \xi)$ into $\bar{f}(x, \Phi(\xi))J(\xi)$, where J is the Jacobian of the transform Φ . In particular, we are not necessarily limited here to measure-preserving transforms Φ . This can lead to all sort of complicated phenomena, where it is sometimes possible to “split” Dirac masses for instance. It is one of the reasons why the analysis above is so intricate in spite of the simplicity of the choices.

2.6 | Sketch of the proof

We briefly explain here some of the main ideas and steps that are used in the proof of the main Theorem 1.1.

◊ *Step 1: Propagation of independence.*

This is done in Section 3 by introducing the independent system of coupled PDEs (20), restated here

$$\begin{cases} \partial_t \bar{f}_i + \operatorname{div}_x \left(\bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x - y) \bar{f}_j(t, dy) \right) = 0, \\ \bar{f}_i(t = 0) = f_i^0, \end{cases} \quad (18)$$

and it proves (cf. Proposition 3.2) that $\bar{f}_i(t, x)$ are correct approximations for the law of $X_i(t)$ under the assumption (4). As mentioned earlier, this relies on a straightforward extension to the classical trajectorial methods for mean-field limits, see [88].

For general connectivities w_{ij} , this is only the very beginning as it does not allow directly obtaining the limit of the corresponding 1-particle distribution given by

$$\frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x).$$

The one exception concerns the special case where $\sum_{j=1}^N w_{ij} = \bar{w}$ is independent of i . If moreover all f_i^0 are equal to \bar{f}^0 , then it can be seen that $\bar{f}_i = \bar{f}$ solves (18) provided that \bar{f} is a solution to the classical mean-field limit.

◊ *Step 2: Introducing extended graphons.*

The next step in our proof is to define our extended empirical graphons for a fixed N , which is still done through the equivalent of formula (9), namely

$$w_N(\xi, \zeta) = \sum_{i,j=1}^N N w_{ij} \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{I}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta),$$

$$f_N(t, x, \xi) = \sum_{i=1}^N \bar{f}_i(t, x) \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi).$$

It is straightforward to check that, since the \bar{f}_i solve (18), then w_N and f_N solve the limiting equation (5), namely we have

$$\partial_t f_N(t, x, \xi) + \operatorname{div}_x \left(f_N(t, x, \xi) \int_0^1 w_N(\xi, \zeta) \int_{\mathbb{R}^d} K(x-y) f_N(t, y, \zeta) dy d\zeta \right) = 0.$$

This is performed at the beginning of Section 4 together with the study of some basic properties of our extended graphons under the only scaling $w_N \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, which include the aforementioned

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^\infty} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^\infty}.$$

For a fixed $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, this allows for example obtaining the existence and uniqueness of a solution $f \in L^\infty([0, T] \times [0, 1], W^{1,\infty}(\mathbb{R}^d))$ to (5), see Proposition 4.11. However, we are unfortunately incapable of passing to the limit directly in the above equation for f_N to derive (5). The main obstruction is the lack of compactness in ξ on f_N combined with the very weak topology in $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$.

♦ *Step 3: Introducing and studying the new observables.*

The latter difficulty is what leads to the introduction of the observables, that can be defined at any time t through the formula (15) that we repeat here,

$$\tau(T, w_N, f_N)(t, x_1, \dots, x_{|T|}) = \int_{[0, 1]^{|T|}} \prod_{(k,l) \in E(T)} w_N(\xi_k, \xi_l) \prod_{m \in V(T)} f_N(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}.$$

Those observables include the 1-particle distribution which simply corresponds to choosing the tree $T = T_1$ with only one vertex:

$$\tau(T_1, w_N, f_N)(t, x) = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x).$$

The fact that the $\tau(T, w_N, f_N)$ are well defined follows from the basic analysis of our extended graphons in the previous point.

A critical point is that those observables solve an independent hierarchy of equations, which reads

$$\partial_t \tau(T, w_N, f_N) + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z) \tau(T + i, w_N, f_N)(t, x_1, \dots, x_{|T|}, z) dz \right) = 0,$$

where $T + i$ denotes the tree obtained from T by adding a leaf on the i -th vertex.

It turns out that we can pass to the limit in the above hierarchy and even obtain the uniqueness of solutions upon appropriate assumptions on the $\tau(T, w_N, f_N)$ as per Theorem 4.23. The analysis is performed in the second part of Section 4 and is one of the major contribution of the paper.

Independently to the proof in this paper, we mention the very recent [58] that obtains uniqueness on the classical exchangeable BBGKY or Vlasov hierarchies. The basic idea in that proof is reminiscent of the strategy that we develop here for Theorem 4.23, even if [58] uses relative entropies while we base our estimates on L^2 bounds.

◊ *Step 4: Identifying the limit.*

Theorem 4.23 allows passing to the limit in all $\tau(T, w_N, f_N)$, after extracting a sub-sequence, to some hierarchy of h_T . It remains to identify the h_T by finding w and f^0 such that if f solves the limiting equation (5) with w and f^0 as initial data then we have the representation

$$\tau(T, w, f) = h_T.$$

This is done in Section 5 in Theorem 5.1 and relies on a modified regularity lemma for graphons stated in Lemma. 5.7. This also requires the definition of the $\tau(T, w, f)$ in general with only $f \in L^\infty([0, T] \times [0, 1], W^{1,\infty}(\mathbb{R}^d))$ and $w \in L^\infty_\xi \mathcal{M}_\xi \cap L^\infty_\xi \mathcal{M}_\xi$, which is based on an abstract algebra that encompasses the sequence of operations on the tree that are performed to produce $\tau(T, w, f)$. Putting all estimates together finally allows to conclude.

As one can see, this does not prove the convergence of f_N to f in any direct sense. Instead it only shows the convergence of all $\tau(T, w_N, f_N)$ to the $\tau(T, w, f)$. It implies the claimed result by taking $T = T_1$ the trivial tree with only one node and no edge, which in particular yields

$$\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x) \rightarrow \int_0^1 f(t, x, \xi) d\xi.$$

3 | A GENERALIZED McKean SDE AND PROPAGATION OF INDEPENDENCE

The aim of this section is to show that independence of events is propagated in our system (1) in an appropriate sense. It will be the equivalent of the aforementioned classical concept of propagation of chaos (for uniform weights $w_{ij} = \frac{1}{N}$), but it is not and it requires a careful extension of the usual arguments. To that end, we shall propose an extension of the classical [88] McKean SDE associated with the particles system to our current case with non-uniform weights.

Lemma 3.1 (McKean SDE). *Consider the nonlinear system of SDE for $(\bar{X}_1, \dots, \bar{X}_N)$ given by*

$$\begin{cases} \frac{d\bar{X}_i}{dt} = \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy), \\ \bar{X}_i(t=0) = \bar{X}_i^0, \end{cases} \quad (19)$$

where $K \in W^{1,\infty}$ and we denote $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$. Then, for any random initial data $(\bar{X}_1^0, \dots, \bar{X}_N^0)$ such that $\mathbb{E}[|\bar{X}_i^0|] < \infty$ there is existence and uniqueness, trajectorial and in law of solutions of (19). In addition, if \bar{X}_i^0 are independent then $\bar{X}_i(t)$ are also independent for each $t \in \mathbb{R}_+$.

The proof is an elementary extension of the classical case with uniform weights, see [88, Theorem 1.1]. We notice that the independence of initial data is crucial in the previous result. However, \bar{X}_i^0 are not necessarily identically distributed, although we could have also assumed so. Unfortunately, notice that the presence of non-uniform weights makes particles non-exchangeable and there is a rupture of symmetry. Thus, the necessity for the use of the different laws \bar{f}_j in the SDE is justified. This was not the case for uniform weights, where symmetry is propagated and for each \bar{X}_i the McKean SDE can be closed in terms of its law \bar{f}_i itself.

Proposition 3.2 (Propagation of independence). *Let (X_1, \dots, X_N) be solution to (1) with $K \in W^{1,\infty}$ and consider the associated laws $f_i(t, \cdot) = \text{Law}(X_i(t))$. Assume that X_i^0 are independent random variables such that $\mathbb{E}[|X_i^0|^2] < \infty$, and the following uniform estimates hold*

$$\sup_{1 \leq i \leq N} \sqrt{\mathbb{E}|X_i^0|^2} \leq M, \quad \sup_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| \leq C,$$

for every $N \in \mathbb{N}$ and appropriate $M, C \in \mathbb{R}_+$. Consider the solution \bar{f}_i to the coupled PDE system

$$\begin{cases} \partial_t \bar{f}_i + \text{div}_x \left(\bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x-y) \bar{f}_j(t, dy) \right) = 0, \\ \bar{f}_i(0, x) = f_i(0, x). \end{cases} \quad (20)$$

Then, the following estimate holds

$$\sup_{1 \leq i \leq N} W_1(f_i(t, \cdot), \bar{f}_i(t, \cdot)) \leq C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \quad (21)$$

for every $t \in \mathbb{R}_+$, where W_1 is the 1-Wasserstein distance. We also have a direct control on the empirical measure of the system, namely

$$\mathbb{E} W_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}, \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot) \right) \leq \frac{\tilde{C} C_2(t)}{N^\theta} + C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \quad (22)$$

for every $t \in \mathbb{R}_+$ and appropriate constants $\tilde{C}, \theta > 0$ depending only on the dimension d . In addition, $C_1(t)$ and $C_2(t)$ depend on $M, C, \|K\|_{W^{1,\infty}}$ and t , and they can be made explicit by

$$C_1(t) := \sqrt{\frac{2}{C}} (e^{2Ct \|K\|_{W^{1,\infty}}} - 1),$$

$$C_2(t) := (2M^2 + 2C^2 \|K\|_{L^\infty} t^2)^{1/2}.$$

We notice that if weights relax uniformly to zero, that is, $\lim_{N \rightarrow \infty} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2} = 0$, then the above estimate (21) guarantees that the creation of correlation between particles of system (1) is weak and relax to zero as $N \rightarrow \infty$. Indeed, the relaxation rate extends the usual $N^{-1/2}$ of the classical propagation of chaos for uniform weights $w_{ij} = \frac{1}{N}$, and the laws f_i of each particle is approximated by the independent laws \bar{f}_i in the McKean SDE (19) in the mean field limit $N \rightarrow \infty$, see [88, Theorem 1.4].

Proof. Let $(\bar{X}_1, \dots, \bar{X}_N)$ be the unique solution to (19) with initial data $\bar{X}_i^0 = X_i^0$, according to Lemma 3.1. We note first that we can simply propagate the second moments,

$$\mathbb{E}|X_i|^2 \leq \mathbb{E}(|X_i^0| + C \|K\|_{L^\infty} t)^2 \leq C_2(t)^2.$$

In addition, let us denote the laws $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$ and consider the sub- σ -algebras generated by \bar{X}_i , that is, $\mathcal{F}_i(t) = \sigma(\bar{X}_i(t))$, for each $t \in \mathbb{R}_+$ and $i = 1, \dots, N$. Since the variables \bar{X}_i and \bar{X}_j are independent by Lemma 3.1, for any $j = 1, \dots, N$ with $j \neq i$, then we have

$$\int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy) = \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)],$$

where $\mathbb{E}_i = \mathbb{E}_i[\cdot | \mathcal{F}_i(t)]$ denotes the conditional expectation given $\mathcal{F}_i(t)$. Therefore, taking the difference in Equations (1) and (19) we obtain the SDE

$$\begin{aligned} \frac{d}{dt}(X_i - \bar{X}_i) &= \sum_{j=1}^N w_{ij} (K(X_i - X_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \\ &= \sum_{j=1}^N w_{ij} (K(X_i - X_j) - K(\bar{X}_i - \bar{X}_j)) + \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]), \end{aligned}$$

for every $t \in \mathbb{R}_+$. Hence, taking total expectation and using the Lipschitz continuity of K we have that

$$\frac{d}{dt} \mathbb{E}|X_i - \bar{X}_i| \leq 2C[K]_{\text{Lip}} \sup_{1 \leq j \leq N} \mathbb{E}|X_j - \bar{X}_j| + \mathbb{E} \left[\left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right| \right]. \quad (23)$$

We remark that all the expectations are finite by the assumption on X_i^0 . Our goal now is to derive a bound from (23). Notice that the key point is to control the relaxation of the second term of the right hand side as $N \rightarrow \infty$ in terms of weights w_{ij} . Such a step is reminiscent of the classical propagation of chaos with uniform weights $w_{ij} = \frac{1}{N}$, see [88, Theorem 1.4]. However, the presence of non-uniform weights makes the argument more subtle. By Jensen's inequality we shall control the second order moment instead. Indeed, expanding the square we obtain that

$$\begin{aligned} &\mathbb{E} \left[\left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right|^2 \right] \\ &= \sum_{j,k=1}^N w_{ij} w_{ik} \mathbb{E} \left[(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \cdot (K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right]. \end{aligned}$$

Since $K(0) = 0$, then only the terms with $j \neq i$ and $k \neq i$ persist in the above sum. We shall prove that all the terms with $j \neq k$ also disappear. This will be a consequence of the statistical independence of \bar{X}_i . Indeed, take $j \neq k$ in the sum and notice that $(\bar{X}_i, \bar{X}_j, \bar{X}_k)$ are independent. Therefore,

$$\begin{aligned} & \mathbb{E} \left[(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \cdot (K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right] \\ &= \mathbb{E} \left\{ \mathbb{E}_i \left[(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right] \mathbb{E}_i \left[(K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right] \right\} = 0. \end{aligned}$$

Here, we have used again the law of total expectation $\mathbb{E} = \mathbb{E} \mathbb{E}_i$ and the fact that the random variables $K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]$ and $K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]$ are conditionally independent given \mathcal{F}_i . This means that only the terms $j = k$ persist in the sum and

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right|^2 \right] &= 2 \sum_{j=1}^N w_{ij}^2 |K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]|^2 \\ &\leq 8C \|K\|_{L^\infty}^2 \sup_{1 \leq j \leq N} |w_{ij}|, \end{aligned}$$

for each $i = 1, \dots, N$ and $t \in \mathbb{R}_+$. Finally, putting everything together into (23), using Grönwall's inequality and noting that $\mathbb{E}[|X_i(0) - \bar{X}_i(0)|] = 0$ (by definition), we obtain the following estimate

$$\sup_{1 \leq i \leq N} \mathbb{E} |X_i - \bar{X}_i| \leq C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2},$$

for every $t \in \mathbb{R}_+$.

To conclude the proof, it only remains to check that $\bar{f}_i = \text{Law}(\bar{X}_i)$ are precisely the solution to the PDE system (20), and that the above error estimate for the processes implies the corresponding W_1 error bounds (21) and (22) for the laws. Let us assume the former and let us prove the later first. Since $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$, this directly implies (21), since

$$\begin{aligned} W_1(f_i(t, \cdot), \bar{f}_i(t, \cdot)) &= \sup_{\|\nabla \phi\|_{L^\infty} \leq 1} \int_{\mathbb{R}^d} \phi(x) (f_i - \bar{f}_i)(t, dx) \\ &= \sup_{\|\nabla \phi\|_{L^\infty} \leq 1} \mathbb{E}[\phi(X_i(t)) - \phi(\bar{X}_i(t))] \leq \mathbb{E} |X_i(t) - \bar{X}_i(t)|. \end{aligned}$$

Similarly, denoting the two empirical measures

$$\mu_N(t, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(dx), \quad \bar{\mu}_N(t, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}_i(t)}(dx),$$

we have that

$$\begin{aligned} \mathbb{E} W_1(\mu_N(t, \cdot), \bar{\mu}_N(t, \cdot)) &= \mathbb{E} \left[\sup_{\|\nabla \phi\|_{L^\infty} \leq 1} \frac{1}{N} \sum_{i=1}^N (\phi(X_i(t)) - \phi(\bar{X}_i(t))) \right] \\ &\leq \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |X_i(t) - \bar{X}_i(t)| \right] \leq \sup_{1 \leq i \leq N} \mathbb{E} |X_i(t) - \bar{X}_i(t)|. \end{aligned}$$

Furthermore, by the independence of the \bar{X}_i , we have, through a straightforward extension of the proofs in [34] or [5] (see also Theorem 1 in [38]) which consider the exchangeable case that for some $\tilde{C}, \theta > 0$,

$$\mathbb{E} W_1 \left(\bar{\mu}_N(t, \cdot), \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot) \right) \leq \frac{\tilde{C}}{N^\theta} \sup_{1 \leq i \leq N} \mathbb{E}[|X_i(t)|^2]^{1/2} \leq \frac{\tilde{C} C_2(t)}{N^\theta},$$

provided for example that the second moments $\mathbb{E}|\bar{X}_i|^2$ are bounded uniformly in N , which we proved earlier. Combining the above estimates and using the triangle inequality allows deducing (22).

Finally, for completeness we show that the laws $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$ verify (20). Set any $\varphi \in C_c^1(\mathbb{R}^{2d})$. Then, we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x_i) \bar{f}_i(t, x) dx = \frac{d}{dt} \mathbb{E}[\varphi(\bar{X}_i)] = \mathbb{E} \left[\nabla \varphi(\bar{X}_i) \cdot \frac{d\bar{X}_i}{dt} \right] = \sum_{j=1}^N w_{ij} \mathbb{E}[\nabla \varphi(\bar{X}_i) \cdot \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]].$$

We notice that $\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] = \int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy)$, by independence of \bar{X}_i and \bar{X}_j . Then, we have

$$\begin{aligned} \mathbb{E}[\nabla \varphi(\bar{X}_i) \cdot \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]] &= \mathbb{E} \left[\nabla \varphi(\bar{X}_i) \cdot \int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy) \right] \\ &= \int_{\mathbb{R}^{2d}} \nabla \varphi(x) \cdot K(x - y) \bar{f}_i(t, dx) \bar{f}_j(t, dy). \end{aligned}$$

Putting everything together and noticing that φ is arbitrary, we deduce that \bar{f}_i verifies (20) in weak form, for each $i = 1, \dots, N$. \square

The above result is only the beginning to prove our main result because deriving the system (20) is not enough to conclude. Indeed, so far we have only replaced a coupled system of ODEs by a couple system of PDEs. We do not even have any compactness with respect to N . There is one big exception:

Remark 3.3 (Exchangeable case). Assume that $\sum_{j=1}^N w_{ij} = \bar{w}$ and $\bar{f}_i^0 = \bar{f}^0$ for every $i = 1, \dots, N$ and some $\bar{w} \in \mathbb{R}$ and $\bar{f}^0 \in \mathcal{P}(\mathbb{R}^d)$. Then, there is usual propagation of chaos. Specifically, we have that $\bar{f}_i(t, x) = \bar{f}(t, x)$ for every $i = 1, \dots, N$, where \bar{f} solves the usual Vlasov equation:

$$\partial_t \bar{f} + \bar{w} \operatorname{div}_x \left(\bar{f}(t, x) \int_{\mathbb{R}^d} K(x - y) \bar{f}(t, dy) \right) = 0.$$

4 | A NEW HIERARCHY

To make some of the calculations in this section rigorous, let us introduce a variant of the original system (20) for the $\bar{f}_i(t, \cdot) := \text{Law}(\bar{X}_i(t))$ which includes an artificial diffusion term:

$$\partial_t \bar{f}_i + \operatorname{div}_x \left(\bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x - y) \bar{f}_j(t, dy) \right) = \nu \Delta_x \bar{f}_i. \quad (24)$$

With identical arguments as in Proposition 3.2 introducing a Wiener process in the McKean process (19) with variance $2\nu t$ we can deduce (24). More specifically, this can be done starting from system (1) with a noise dW_i , where the W_i are N independent Wiener processes

$$dX_i = \sum_{j=1}^N w_{ij} K(X_i - X_j) dt + \sqrt{2\nu} dW_i.$$

Of course (20) is just a special case of (24) with $\nu = 0$. As in the case without diffusion, in (24) there is an abrupt rupture of symmetry caused by the presence of potentially heterogeneous weights w_{ij} . Indeed, even for identical initial data \bar{f}_i^0 , the intrinsic dynamics instantaneously yields non-exchangeable distributions $\bar{f}_i(t)$ for any $t > 0$. This prevent us from reducing the system (24) of N coupled PDEs to a single Vlasov–McKean type PDEs in the usual way as in the classical setting with uniform weights. In addition, there is no realistic hope to identify the limit of each individual \bar{f}_i as $N \rightarrow \infty$. However, we may still be able to study the limit of appropriate averaged statistics. In this section, we shall introduce a natural family of observables that will allow determining a closed averaged description of (24) in terms of an infinite hierarchy of PDEs. For such a novel hierarchy we study uniqueness and stability under the presence of artificial diffusion ($\nu > 0$). Although an analogous result is still open for the hierarchy in the absence of artificial diffusion ($\nu = 0$), we shall prove that the stability can still be recovered for the full initial system (20) under appropriate regularity of the initial data.

4.1 | Hierarchy of observables indexed by trees

For simplicity of the presentation, in this part we shall restrict to the starting case (20) without artificial diffusion ($\nu = 0$). Associated with any solution of (20) we shall define an infinite family of observables indexed by the set of trees containing an averaged information of \bar{f}_i and w_{ij} . The role of trees is suggested by the construction and will become apparent in a moment. For clarity, we recall some necessary notation of graph theory that will be used throughout the paper.

Definition 4.1 (Graphs and trees).

- (i) A **(simple) graph** is a pair $G = (V, E)$ where V is a finite set (vertices) and $E \subseteq \{(i, j) \in V \times V : i \neq j\}$ is a subset (edges). A graph $G = (V, E)$ is called a **weighted graph** if it is endowed with a (weight) function $W : V \times V \rightarrow \mathbb{R}$ such that $E = \{(i, j) \in V \times V : W(i, j) \neq 0\}$. The amount of vertices (or order) will be denoted by $|G| := \#V$.
- (ii) A **tree** is a graph $T = (V, E)$ so that any couple of different vertices $i, j \in V$ are connected by exactly one path (concatenation of edges) and each vertex $i \in V$ cannot be connected with itself.
- (iii) We define the family \mathcal{T}_n of **labeled trees** of order n by the following recursive formula

$$\begin{aligned} \mathcal{T}_1 &:= \{T_1\}, \\ \mathcal{T}_{n+1} &:= \{T + i : T \in \mathcal{T}_n, i \in \{1, \dots, n\}\}, \quad n \in \mathbb{Z}_0^+, \end{aligned}$$

where T_1 is the only tree with one vertex $\{1\}$. Here, for any tree T with vertices $\{1, \dots, n\}$ and any such vertex i , we will denote by $T + i$ to the new tree with vertices $\{1, \dots, n + 1\}$

after adding the leaf $n + 1$ at node i . The family of all labeled trees of arbitrary order is then defined by $\mathcal{T} := \cup_{n=1}^{\infty} \mathcal{T}_n$. Note that, in particular, \mathcal{T} contains all possible trees modulo graph isomorphism.

Remark 4.2 (Directed graphs and directed trees). Our graphs $G = (V, E)$ are *a priori* assumed directed so that $(i, j) \in E$ and $(j, i) \in E$ do not in general represent the same edge. Since we work with labeled trees, those are imbued with a natural orientation starting from the root.

With this notation, we introduce our family of observables $\{\mathcal{O}_N^T : T \in \mathcal{T}\}$ as follows.

Definition 4.3 (Observables). Consider any solution $(\bar{f}_i)_{i=1,\dots,N}$ of (20) with weights $(w_{ij})_{i,j=1,\dots,N}$. Then, we define the family of time-dependent Radon measures:

$$\mathcal{O}_N^T(t, x_1, \dots, x_{|T|}) := \frac{1}{N} \sum_{i_1, \dots, i_{|T|}=1}^N \prod_{(k,l) \in E(T)} w_{ik} w_{il} \prod_{m \in V(T)} \bar{f}_{i_m}(t, x_m), \quad (25)$$

for any $t \geq 0$, every $(x_1, \dots, x_{|T|}) \in \mathbb{R}^{d|T|}$ and each tree $T \in \mathcal{T}$.

Let us note that $\mathcal{O}_N^T(t, \cdot) \in \mathcal{M}(\mathbb{R}^{d|T|})$ are finite Radon measures for each $t \geq 0$. Unfortunately, the heterogeneity of weights still implies that \mathcal{O}_N^T are not symmetric. In the next section we shall show that they indeed satisfy an appropriate hierarchy of PDEs containing an averaged information of (20). Just to illustrate the underlying idea, let us explore some examples. First, consider the only existing tree $T_1 \in \mathcal{T}_1$ of order 1 and note that (25) simply reduces to $\mathcal{O}_N^{T_1}(t, x) = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x)$. Summing in (20) over $i = 1, \dots, N$ yields

$$\partial_t \mathcal{O}_N^{T_1}(t, x) + \operatorname{div}_x \left(\int_{\mathbb{R}^d} K(x-y) \frac{1}{N} \sum_{i,j=1}^N w_{ij} \bar{f}_i(t, x) \bar{f}_j(t, y) dy \right) = 0, \quad (26)$$

which cannot be closed in terms of $\mathcal{O}_N^{T_1}$ only. In fact, setting the only existing tree $T_2 \in \mathcal{T}_2$ of order 2 allows identifying the new observable $\mathcal{O}_N^{T_2}(t, x, y) = \frac{1}{N} \sum_{i,j=1}^N w_{ij} \bar{f}_i(t, x) \bar{f}_j(t, y)$ inside the divergence of (26). In other words, (26) can be readily written as

$$\partial_t \mathcal{O}_N^{T_1}(t, x) + \operatorname{div}_x \left(\int_{\mathbb{R}^d} K(x-y) \mathcal{O}_N^{T_2}(t, x, y) dy \right) = 0. \quad (27)$$

Since (27) requires a priori knowledge of $\mathcal{O}_N^{T_2}$ we may seek for an equation of such a new observable. Indeed, from (20) we obtain again that

$$\begin{aligned} \partial_t \mathcal{O}_N^{T_2}(t, x, y) + \operatorname{div}_x \left(\int_{\mathbb{R}^d} K(x-z) \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{ik} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z) dz \right) \\ + \operatorname{div}_y \left(\int_{\mathbb{R}^d} K(y-z) \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{jk} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z) dz \right) = 0. \end{aligned} \quad (28)$$

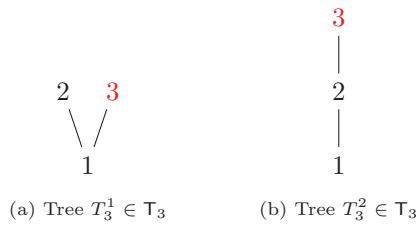


FIGURE 2 Trees in \mathcal{T}_3 .

Note that in this case two different divergence terms appear. Accordingly, note that \mathcal{T}_3 also contains two possible trees $T_3^1 = T_2 + 1$ and $T_3^2 = T_2 + 2$ which respectively arise by adding the new leaf 3 at either of the vertices 1 and 2 of $T_2 \in \mathcal{T}_2$ (see Figure 2)

By construction

$$\begin{aligned}\mathcal{O}_N^{T_3^1}(t, x, y, x) &= \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{ik} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z), \\ \mathcal{O}_N^{T_3^2}(t, x, y, x) &= \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{jk} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z).\end{aligned}$$

Again, we identify the above observables in the divergences of (28), so that the latter can be restated as

$$\partial_t \mathcal{O}_N^{T_2}(t, x, y) + \operatorname{div}_x \left(\int_{\mathbb{R}^d} K(x - z) \mathcal{O}_N^{T_3^1}(t, x, y, dz) \right) + \operatorname{div}_y \left(\int_{\mathbb{R}^d} K(y - z) \mathcal{O}_N^{T_3^2}(t, x, y, dz) \right) = 0. \quad (29)$$

It is now apparent that an analogous recursive argument would lead to similar equations like (27) and (29) involving observables indexed by all other higher order trees. In the following section we will make this argument rigorous and will recover the full hierarchy of equations for the observables \mathcal{O}_N^T with $T \in \mathcal{T}$. To do so, we shall rely on a more general formulation of the system (20) and the observables \mathcal{O}_N^T in (25) inspired in the treatment of graphons which is formally independent of the decomposition into agents.

4.2 | A graphon-like representation of the system and its hierarchy

Let us remark that in the limit $N \rightarrow \infty$ we must control two different processes simultaneously: a many-particles limit and a large-graph limit. Unfortunately, the explicit dependence on the discrete labels $i = 1, \dots, N$ of agents still persists both in the expression of the observables in Definition 4.3 and in the systems (20) and (24) with and without artificial diffusion. This is reminiscent of the theory of graphons (cf. [62, 63]). Specifically, graphons have proved a suitable tool to identify the limit of dense graph limits by replacing discrete indices by continuous ones. Accordingly, we expect to recover integrals as $N \rightarrow \infty$ that replace the various finite sums over particles' labels. However, for our graphs, we need to develop an appropriate extension of the theory of graphons that allows for the joint treatment of the large-graph limit for non necessarily dense graphs and the many-particles limit.

Definition 4.4 (Empirical graphons). Consider weights $(w_{ij})_{i,j=1,\dots,N} \subseteq \mathbb{R}$ and probability distributions $(\bar{f}_i)_{i=1,\dots,N} \subseteq \mathcal{P}(\mathbb{R}^d)$. Then, we define w_N and f_N as follows

$$\begin{aligned} w_N(\xi, \zeta) &:= \sum_{i,j=1}^N N w_{ij} \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta), \quad \xi, \zeta \in [0, 1], \\ f_N(x, \xi) &:= \sum_{i=1}^N \bar{f}_i(x) \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi), \quad x \in \mathbb{R}^d, \xi \in [0, 1]. \end{aligned} \quad (30)$$

Note that Definition 4.4 allows representing discrete weights $(w_{ij})_{i,j=1,\dots,N}$ as scalar functions of two continuous variables $\xi, \eta \in [0, 1]$. Similarly, distributions $(\bar{f}_i)_{i=1,\dots,N}$ can be represented as a family of probability measures in $\mathcal{P}(\mathbb{R}^d)$ parametrized by the continuous variable $\xi \in [0, 1]$. Using such a notation, we can rewrite the coupled system of PDEs (20) and (24) with and without artificial diffusion in a more compact form that forgets labels of the specific individuals. Specifically, we equivalently obtain

$$\begin{aligned} \partial_t f_N(t, x, \xi) + \operatorname{div}_x \left(f_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} w_N(\xi, \zeta) K(x-y) f_N(t, y, \zeta) dy d\zeta \right) &= 0, \\ \partial_t f_N(t, x, \xi) + \operatorname{div}_x \left(f_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} w_N(\xi, \zeta) K(x-y) f_N(t, y, \zeta) dy d\zeta \right) &= \nu \Delta_x f_N(t, x, \xi), \end{aligned} \quad (31)$$

$$(32)$$

in the sense of distributions. Our ultimate goal will be to identify the limit of the above systems as $N \rightarrow \infty$. For later use, we specify minimal assumptions on the space which will contain the limiting structures.

Definition 4.5. We define the weak-* Bochner space $L_\xi^\infty \mathcal{M}_\zeta$ as the topological dual of $L_\xi^1 C_\zeta$; namely those elements consist of the weak-* measurable essentially bounded maps $\xi \in [0, 1] \mapsto w(\xi, d\zeta) \in \mathcal{M}([0, 1])$ which make the following seminorm finite

$$\|w\|_{L_\xi^\infty \mathcal{M}_\zeta} := \sup_{\|\phi\|_{C([0, 1])} \leq 1} \operatorname{ess\,sup}_{\xi \in [0, 1]} \left| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right|.$$

Similarly, we define $L_\zeta^\infty \mathcal{M}_\xi$, or more generally $L_\xi^\infty B_\zeta^*$ for any dual Banach space B^* .

As usual, we will identify $L_\xi^\infty \mathcal{M}_\zeta$ with its Kolmogorov quotient under the above semi-norm $\|\cdot\|_{L_\xi^\infty \mathcal{M}_\zeta}$, which is a Banach space. Specifically, we shall identify elements $w_1 \approx w_2$ when for any $\phi \in C([0, 1])$ the scalar functions $\xi \in [0, 1] \mapsto \int_{\mathbb{R}^d} \phi(\zeta) w_i(\xi, d\zeta)$ are equal almost everywhere for $i = 1, 2$. As mentioned above, the main feature of such a Banach space is its duality representation $L_\xi^\infty \mathcal{M}_\zeta \equiv (L_\xi^1 C_\zeta)^*$, where $L_\xi^1 C_\zeta$ stands for the (strong) Bochner space, see [47]. Although this property is desirable, the weak-* Bochner spaces raise several analytical difficulties. For instance, strong-measurability of the representatives is not granted because $\mathcal{M}([0, 1])$ fails the Radon–Nikodym property. Fortunately, in our special case $C([0, 1])$ is separable so that, at least, $\xi \in [0, 1] \mapsto \|w(\xi, \cdot)\|_{\mathcal{M}([0, 1])}$ is measurable and essentially bounded. For the same reason, the above equivalence relation \approx reduces to the usual *a.e.* identity of functions. Therefore, a

simpler expression of the norm in the separable case is available:

$$\|w\|_{L^\infty_\xi \mathcal{M}_\zeta} = \operatorname{ess\,sup}_{\xi \in [0, 1]} \|w(\xi, \cdot)\|_{\mathcal{M}([0, 1])}.$$

Definition 4.6. We shall say that $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ is an (extended) graphon.

For our purpose we will need to integrate extended graphons against functions ϕ that are only in L^∞ . This leads to the straightforward issue as to whether $\xi \in [0, 1] \mapsto \int_0^1 \phi(\zeta) w(\xi, d\zeta)$ can be well defined if $\phi \in L^\infty([0, 1])$. In fact if we only had $w \in L^\infty_\xi \mathcal{M}_\zeta$, then the above integral would be highly sensitive to modifying $\phi(\zeta)$ over the atoms of $w(\xi, d\zeta)$, preventing any straightforward definition.

The key point for our investigations is that our extended graphons belongs to both $L^\infty_\xi \mathcal{M}_\zeta$ and $L^\infty_\zeta \mathcal{M}_\xi$. This yields two weakly-star measurable and bounded families of measures $\xi \in [0, 1] \mapsto w(\xi, d\zeta) \in \mathcal{M}([0, 1])$ and $\zeta \in [0, 1] \mapsto w(d\xi, \zeta) \in \mathcal{M}([0, 1])$ that induce the same bi-variate measure $w(\xi, d\zeta) d\xi = w(d\xi, \zeta) d\zeta \in \mathcal{M}([0, 1]^2)$. This implies that w is a transition kernel in the full sense and the following continuity of the associated linear operator on L^∞ holds true.

Lemma 4.7 (Operator representation of kernels). *Consider the following bounded bilinear operator*

$$\begin{aligned} (L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi) \times C_\zeta &\longrightarrow L^\infty_\xi, \\ (w, \phi) &\longmapsto \int_0^1 \phi(\zeta) w(\cdot, d\zeta). \end{aligned}$$

This operator extends into a bounded operator from $(L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi) \times L^\infty_\zeta$ to L^∞_ξ . Specifically, we have the following estimates

$$\begin{aligned} \left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L^\infty_\xi} &\leq \|w\|_{L^\infty_\xi \mathcal{M}_\zeta} \|\phi\|_{L^1}, \\ \left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L^\infty_\xi} &\leq \|w\|_{L^\infty_\xi \mathcal{M}_\zeta} \|\phi\|_{L^\infty}, \end{aligned} \tag{33}$$

for any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $\phi \in L^\infty$. Moreover, the following continuity property holds: if $w_n \xrightarrow{} w$ weakly-star in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, $\phi_n \rightarrow \phi$ strongly in L^1 and the ϕ_n are uniformly bounded in L^∞ , then*

$$\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta) \rightarrow \int_0^1 \phi(\zeta) w(\cdot, d\zeta), \tag{34}$$

weakly-star in L^∞ .

The proof of Lemma 4.7 relies on a careful but straightforward density argument, which we provide in Appendix B. Lemma 4.7 immediately implies the following version with test functions ϕ depending on x .

Lemma 4.8. *Consider the following bounded bilinear operator*

$$\begin{aligned} (L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times C_\zeta B_x^* &\longrightarrow L_\xi^\infty B_x^*, \\ (w, \phi) &\longmapsto \int_0^1 \phi(\cdot, \zeta) w(\cdot, d\zeta), \end{aligned}$$

where B_x^* stands for any Banach space. Then it is also bounded from $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times L_\zeta^\infty B_x^*$ to $L_\xi^\infty B_x^*$, and from $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times L_\zeta^\infty B_x^*$ (with $L_\zeta^\infty B_x^*$ endowed with the $L_\zeta^1 B_x^*$ norm) to $L_\xi^1 B_x^*$. Specifically, we have the following estimates

$$\begin{aligned} \left\| \int_0^1 \phi(x, \zeta) w(\xi, d\zeta) \right\|_{L_\xi^1 B_x^*} &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\phi\|_{L_\zeta^1 B_x^*}, \\ \left\| \int_0^1 \phi(x, \zeta) w(\xi, d\zeta) \right\|_{L_\xi^\infty B_x^*} &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\phi\|_{L_\zeta^\infty B_x^*}, \end{aligned} \quad (35)$$

for any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $\phi \in L_\zeta^\infty B_x^*$,

Note that w_N and f_N in Definition 4.4 must belong to the previous spaces. Then, the study of the limit of those structures as $N \rightarrow \infty$ and the identification of the limiting equations of (31) and (32) is in order. Namely, we expect to obtain $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $f \in L^\infty([0, t_*], L_\xi^\infty L_x^1)$ with $t_* > 0$ verifying either of the following equations (with or without artificial diffusion) in a weak sense:

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, y, \zeta) dy \right) = 0, \quad (36)$$

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, y, \zeta) dy \right) = \nu \Delta_x f(t, x, \xi). \quad (37)$$

In (36) and (37), we recognize a nonlinear drift, which is fibered as there is no transport on the variable ξ . For later use, we define formally the associated velocity field.

Definition 4.9 (Velocity field). Consider any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, $f \in L_{t,\xi}^\infty L_x^1$ and $K \in L^\infty$. Let us set $\phi_f(t, x, \zeta) := \int_{\mathbb{R}^d} K(x - y) f(t, y, \zeta) dy$ which is well defined in $L_{\xi,t,x}^\infty$ by the usual convolution estimates. We define the associated velocity field $\mathcal{V}[f]$ as follows,

$$\begin{aligned} \mathcal{V}_f(t, x, \xi) &:= \int_0^1 w(\xi, d\zeta) \phi_f(t, x, \zeta) = \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, \zeta, y) dy, \\ (\xi, x) &\in [0, 1] \times \mathbb{R}^d, \quad t \geq 0, \end{aligned}$$

which is well defined in $L_{t,\xi,x}^\infty$ by Lemma 4.8.

We may then define the weak solutions to (36)–(37) in the usual way.

Definition 4.10 (Weak solutions). Consider any pair consisting of a $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and any $f \in L^\infty([0, t_*], L_\xi^\infty L_x^1)$. We say that (w, f) is a weak solution of (37) for some $K \in L^\infty$ and for

any $\nu \geq 0$ if the following weak formulation is verified

$$\begin{aligned} & \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(t, x, \xi) f(t, x, \xi) dx d\xi dt \\ & + \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \nabla_x \varphi(t, x, \xi) \cdot \mathcal{V}_f(t, x, \xi) f(t, x, \xi) dx d\xi dt \\ & + \nu \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \Delta_x \varphi(t, x, \xi) f(t, x, \xi) dx d\xi dt = 0, \end{aligned} \quad (38)$$

for any $\varphi \in C_c^2((0, t_*) \times [0, 1] \times \mathbb{R}^d)$.

We have the following result about the existence of weak solutions.

Proposition 4.11. *Consider any $K \in L^1 \cap L^\infty$ with $\operatorname{div} K \in L^1$, any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and any $f^0 \in L^\infty([0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$. Then for any $\nu \geq 0$, there exists $f \in L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$ such that (w, f) is a weak solution to (37) (or (36) if $\nu = 0$) for every $t_* > 0$ as per Definition 4.10.*

The proof is given in Appendix B and follows a straightforward fixed point argument from classical a priori estimates for advection-diffusion equations. We also note that it would be possible to obtain an even more existence result with f only a measure; for example, through the pathwise approaches developed in [83, Chapter 10].

We finish this short analysis of the space $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ by noting the following density result

Lemma 4.12. *For any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, there exists $w_n \in L^\infty_{\xi,\zeta}$, uniformly bounded in $L^\infty_\xi L^1_\zeta \cap L^\infty_\zeta L^1_\xi$, converging to w in $L^1_\xi H^{-1}_\zeta \cap L^1_\zeta H^{-1}_\xi$.*

We also provide the proof of Lemma 4.12 in Appendix B.

Unfortunately we do not know how to pass to the limit directly in Equations (31) or (32), as we lack some compactness in the ξ variable either for w_N or for f_N . Indeed, from our uniform bounds, we could extract converging subsequences $w_N \xrightarrow{*} w$ and $f_N \xrightarrow{*} f$ respectively in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $L^\infty_{t,\xi}(L^1_x \cap W^{1,\infty}_x)$. But this gives only weak convergence on w_N and weak convergence in ξ on f_N whereas Lemma 4.7 for example requires some strong convergence on f_N . To further emphasize the point, the existence result that we just stated does rely on strong convergence of f through the contraction argument.

To bypass this issue, we will look at an analogous family of observables providing an averaged information of the system, since it is much easier to obtain compactness on those observables as we will see in the next section. Inspired in Definition 4.3 we set the following operator acting on abstract elements $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f \in L^\infty_\xi L^1_x$.

Definition 4.13 (τ operator). Consider any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and any $f \in L^\infty_\xi L^1_x$. Then, we (formally) define the operator

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) := \int_{[0, 1]^{|T|}} \prod_{(k,l) \in E(T)} w(\xi_k, \xi_l) \prod_{m \in V(T)} f(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}, \quad (39)$$

for every tree $T \in \mathcal{T}$. Time dependence can be included when f also depends on t .

This notation allows removing again the explicit discrete indices for the agents' labels in Definition 4.3 of the observables \mathcal{O}_N^T . Specifically, note that the latter enjoy the following more universal representation

$$\mathcal{O}_N^T = \tau(T, w_N, f_N), \quad T \in \mathcal{T}.$$

We also note that, at the present, it is not clear why $\tau(T, w, f)$ given by (39) is well defined with only $w \in L_\xi^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\xi$ and $f \in L_\xi^\infty L_x^1$. This requires more work and will be performed in the next section (cf. Definition 5.5). Instead, for the moment it is enough to observe that $\tau(T, w_N, f_N)$ is indeed well defined, and more generally, $\tau(T, w, f)$ is also well defined as soon as w has a density, that is, $w \in L_\xi^\infty L_\xi^1 \cap L_\xi^\infty L_\xi^1$. Moreover, we can then obtain uniform bounds in terms of only the $L_\xi^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\xi$ norms as follows.

Lemma 4.14 (Homomorphism densities). *Consider any $w \in L_\xi^\infty L_\xi^1 \cap L_\xi^\infty L_\xi^1$ and define*

$$\tau(T, w) := \int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) d\xi_1 \dots d\xi_{|T|}, \quad T \in \mathcal{T}. \quad (40)$$

Then, $\tau(T, w)$ is finite and the following relation holds true

$$|\tau(T, w)| \leq \|w\|_{L_\xi^\infty \mathcal{M}_\xi}^{|T|-1}. \quad (41)$$

Proof. We argue by induction. First, note the only tree $T_1 \in \mathcal{T}_1$ actually consists of only one vertex and no edges. Then, we trivially obtain that $\prod_{(k, l) \in E(T_1)} |w(\xi_k, \xi_l)| = 1$ so that $\tau(T_1, |w|) = 1$. By the induction hypothesis, let us assume now that we have

$$\tau(|T_n|, w) \leq \|w\|_{L_\xi^\infty \mathcal{M}_\xi}^{|T_n|-1},$$

for any $T_n \in \mathcal{T}_n$. Consider any $T_{n+1} \in \mathcal{T}_{n+1}$ and find a tree $T_n \in \mathcal{T}_n$ and a vertex $i \in \{1, \dots, n\}$ so that we can write $T_{n+1} = T_n + i$. Hence, we obtain

$$\begin{aligned} \tau(T_{n+1}, |w|) &= \int_{[0, 1]^{n+1}} \prod_{(k, l) \in E(T_{n+1})} |w(\xi_k, \xi_l)| d\xi_1 \dots d\xi_{n+1} \\ &= \int_{[0, 1]^n} \left(\int_0^1 |w(\xi_i, \xi_{n+1})| d\xi_{n+1} \right) \prod_{(k, l) \in E(T_n)} |w(\xi_k, \xi_l)| d\xi_1 \dots d\xi_n, \end{aligned}$$

where we have used that ξ_{n+1} does not appears in the above product over edges of T_n . Hence, by our hypothesis on w and by definition (40) of $\tau(T_n, |w|)$ we obtain

$$\tau(T_{n+1}, |w|) \leq \|w\|_{L_\xi^\infty \mathcal{M}_\xi} \tau(T_n, |w|).$$

We then conclude by applying the induction hypothesis to $\tau(T_n, |w|)$. □

We can obtain as a straightforward consequence of Lemma 4.14,

Lemma 4.15. Consider any $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$, any $f \in L_\xi^\infty L_x^1$ and $T \in \mathbb{T}$. Then $\tau(T, w, f) \in L^1(\mathbb{R}^{d|T|})$ and the following estimate is fulfilled

$$\|\tau(T, w, f)\|_{L^1(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty L_x^1}^{|T|}.$$

A similar result holds when the space $L_\xi^\infty L_x^1$ for f is replaced by any other function Banach space stable under tensorization and admitting a Minkowski integral inequality, but maintaining $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$. In particular, if $f \in L_\xi^\infty W_x^{k,p}$ for some $k \geq 0$ and $p \in [1, \infty]$, then $\tau(T, w, f) \in W^{k,p}(\mathbb{R}^{d|T|})$ and

$$\|\tau(T, w, f)\|_{W^{k,p}(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty W_x^{k,p}}^{|T|}.$$

Formula (40) for $\tau(T, w)$ in Lemma 4.15 is reminiscent of the usual extension of the homomorphism density for bounded graphons $w \in L^\infty([0, 1]^2)$ [62, 63]. However, our theory must account for eventually unbounded graphons so that a more delicate treatment is required. Namely, note that in (40) we have restricted to trees $T \in \mathbb{T}$ and graphons $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$. Indeed, in such a class of graphons, if T is replaced by a generic graph G , then the above proof breaks down due to the eventual presence of cycles in the graph. This leads to eventual infinite values of $\tau(G, w)$.

We note that we prove in Section 5 that the operator $\tau(T, w, f)$ can be rigorously defined for any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$. We refer more precisely to Definition 5.5 and in the present context to Lemma 5.3 which implies that if $w_n \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ converges to w in $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$ then $\tau(T, w, f)$ can be obtained as

$$\tau(T, w, f) = \lim_{n \rightarrow \infty} \tau(T, w_n, f).$$

Of course Lemma 4.12 exactly allows constructing an appropriate sequence of w_n . But deriving the limit of $\tau(T, w_n, f_n)$ as above requires the precise construction of an algebra describing the sequence of operations involved in obtaining $\tau(T, w, f)$. This algebra plays a key role in Section 5 and, for this reason, the corresponding results are stated there.

Assuming for the time being that type of property holds, it implies an immediate extension of Lemma 4.15 to more generic kernels.

Corollary 4.16. Consider any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, any $f \in L_\xi^\infty W_x^{k,p}$ for any $k \geq 0$, $p \in [1, \infty]$, and $T \in \mathbb{T}$. Then $\tau(T, w, f) \in W^{k,p}(\mathbb{R}^{d|T|})$

$$\|\tau(T, w, f)\|_{W^{k,p}(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty W_x^{k,p}}^{|T|}.$$

In the following result, we derive the full hierarchy of observables presented in the previous section under our graphon representation. Specifically, we now provide the system solved by $\tau(T, w, f)$ whenever (w, f) is a weak solution to either (36) or (37) in the sense of Definition 4.10.

Proposition 4.17. Consider any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $f \in L^\infty([0, t_*], L_\xi^\infty L_x^1)$ so that (w, f) is a weak solution to (37) in the sense of Definition 4.10 for some $K \in L^\infty$ and for any $\nu \geq 0$. Then,

$\tau(T, w, f)$ solves the generalized, linear, non-exchangeable Vlasov hierarchy

$$\partial_t \tau(T, w, f) + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z) \tau(T + i, w, f)(t, x_1, \dots, x_{|T|}, z) dz \right) = \nu \sum_{i=1}^{|T|} \Delta_{x_i} \tau(T, w, f), \quad (42)$$

for any tree $T \in \mathbb{T}$ in the sense of distributions.

Proof. First of all, we notice that from the properties of w and f we can guarantee that $\tau(T, w, f) \in L^\infty([0, t_*], L^1(\mathbb{R}^{d|T|}))$ thanks to Corollary 4.16. Since $K \in L^\infty$, then there is no issue to define the various terms in the equation and justify our various calculations in the distributional sense. For simplicity of the presentation, we avoid using weak formulations here but a straightforward adaptation yields the rigorous argument. Also, we shall shorten notation by denoting $\tau(T) \equiv \tau(T, w, f)$, for any $T \in \mathbb{T}$, when there is no confusion. First, we differentiate $\tau(T, w, f)$ in time

$$\partial_t \tau(T) = \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \partial_t f(t, x_n, \xi_n) \prod_{m \neq n} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}.$$

Using (37) on $f(t, x_n, \xi_n)$ yields

$$\begin{aligned} \partial_t \tau(T) &= - \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \operatorname{div}_{x_n} \left(\int_0^1 \int_{\mathbb{R}^d} K(x_n - y) w(\xi_n, \zeta) f(t, x_n, \xi_n) f(t, y, \zeta) dy d\zeta \right) \\ &\quad \times \prod_{m \neq n} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \\ &\quad + \nu \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \Delta_{x_n} f(t, x_n, \xi_n) \prod_{m \neq n} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \\ &= - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \left(\int_{[0, 1]^{|T|}} \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \int_0^1 \int_{\mathbb{R}^d} K(x_n - y) w(\xi_n, \zeta) f(t, x_n, \xi_n) f(t, y, \zeta) dy d\zeta \right. \\ &\quad \left. \times \prod_{m \neq n} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \right) + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T), \end{aligned}$$

where in the last line we have used that no other $f(t, x_m, \xi_m)$ depends on x_n as we exclude $m = n$ from the product. Hence, we find

$$\begin{aligned} \partial_t \tau(T) &= - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \int_{\mathbb{R}^d} K(x_n - z) \int_{[0, 1]^{|T|}} \int_0^1 \left(w(\xi_n, \zeta) \prod_{(k, l) \in E(T)} w(\xi_k, \xi_l) \right. \\ &\quad \left. \times f(t, z, \zeta) \prod_{m \neq n} f(t, x_m, \xi_m) \right) d\zeta d\xi_1 \dots d\xi_{|T|} dz + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T). \end{aligned}$$

We recall that for any $n = 1, \dots, |T|$ the tree $T + n$ contains exactly the same edges as T plus a new edge from the vertex n to the new vertex $|T| + 1$. Thus, recalling that we only consider edges

$(k, l) \in E(T + n)$ that run from the root to the leaves, we have that

$$\tau(T + n) = \int_{[0, 1]^{|T|+1}} w(\xi_n, \xi_{|T|+1}) \prod_{(k,l) \in E(T)} w(\xi_k, \xi_l) f(t, x_{|T|+1}, \xi_{|T|+1}) \prod_{m \in V(T)} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|+1}.$$

Changing variables $\xi_{|T|+1}$ with ζ and $x_{|T|+1}$ with z , we arrive at

$$\partial_t \tau(T) = - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \int_{\mathbb{R}^d} K(x_n - z) \tau(T + n)(x_1, \dots, x_{|T|}, z) dz + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T),$$

thus, concluding the proof. \square

Proposition 4.17 shows that we can study directly the propagation of the observables $\tau(T, w, f)$. Our next step is naturally to analyze the stability on the hierarchy with diffusion (42).

4.3 | Stability on the hierarchy with artificial diffusion

Along this section, we will study a stability property of generic solutions to the previous hierarchy of non-exchangeable Vlasov-type equations (42) which are not necessarily parametrized as $\tau(T, w, f)$ for a weak solution (w, f) of (37). Specifically, we shall consider any sequence $h = (h_T)_{T \in \mathbb{T}}$ with $h_T \in L^\infty([0, t_*], L^1(\mathbb{R}^{d|T|}))$ for any $T \in \mathbb{T}$ that solves the analogous hierarchy of non-exchangeable Vlasov equations

$$\partial_t h_T + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z) h_{T+i}(t, x_1, \dots, x_{|T|}, z) dz \right) = \nu \sum_{i=1}^{|T|} \Delta_{x_i} h_T, \quad (43)$$

in the sense of distributions, for some $K \in L^\infty$ and any $\nu > 0$.

We do need to emphasize that this section does not provide a well-posedness theory for (43). While we do obtain uniqueness on solutions with enough a priori estimates, this does not directly yield the existence of any solutions in that class. This does not impact the present paper as we only need the uniqueness result for the special type of solutions parametrized as $h_T = \tau(T, w, f)$ with (w, f) a weak solution of (37) (cf. Proposition 4.17).

We also emphasize that the presence of artificial diffusion is needed in the method of proof in this subsection and a similar result with $\nu = 0$ is still unknown. To analyze stability we shall define the following norms.

Definition 4.18 (Norm of the hierarchy). Consider any family $h = (h_T)_{T \in \mathbb{T}}$ such that $h_T \in L^2(\mathbb{R}^{d|T|})$ for every $T \in \mathbb{T}$. Then, we define the following norms:

$$\|h\|_\lambda = \sup_{T \in \mathbb{T}} \lambda^{|T|/2} \|h_T\|_{L^2(\mathbb{R}^{d|T|})}, \quad (44)$$

for any $\lambda > 0$.

The norm (44) is natural since, for any $f \in L^2(\mathbb{R}^d)$ and the special sequence $h = (h_T)_{T \in \mathbb{T}}$ given by $h_T = \tau(T, w, f)$ for any $T \in \mathbb{T}$, we obtain the following bound

$$\|h\|_\lambda \leq \sup_{T \in \mathbb{T}} \lambda^{|T|/2} \|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}}^{|T|-1} \|f\|_{L^2}^{|T|},$$

for any $\lambda > 0$ thanks to Corollary 4.16. Hence, by choosing $\lambda > 0$ small enough relative to the inverses of the preceding norms of w and f , that is, less than $\|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}}^{-2 \frac{|T|-1}{|T|}} \|f\|_{L^2}^{-2}$, we obtain a finite quantity. Indeed, it provides the following stability result for solutions of (43).

Theorem 4.19. *Consider any solution $h = (h_T)_{T \in \mathbb{T}}$ in the sense of distribution to (43) for some $\nu > 0$ and $K \in L^2$ such that $h_T \in L^\infty([0, t_*], (L^1 \cap L^2)(\mathbb{R}^{d|T|}))$ for any $T \in \mathbb{T}$. Assume that there exists some $\lambda > 0$ such that $C_\lambda := \sup_{t \in [0, t_*]} \|h(t, \cdot)\|_\lambda < \infty$. Then, for any $p > 1$ and any $\theta \in (0, 2^{-p'})$ there exists a constant $C_{p, \theta} \in \mathbb{R}_+$ such that*

$$\|h(t, \cdot)\|_{\theta\lambda} \leq C_\lambda C_{p, \theta} \exp \left(p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \frac{\|h^0\|_{\theta\lambda}}{C_\lambda} \right), \quad (45)$$

for any $t \in [0, t_*]$, where p' is the conjugate of p .

Remark 4.20. Taking $p > 1$ sufficiently large, note that any $\theta \in (0, \frac{1}{2})$ provides an estimate (45). However, such a constraint on the size of θ is clearly an artifact of the method of proof since a similar estimate can be obtained for any $\theta \in (0, 1)$ by interpolation. Namely, set any $p_0 > 1$ and any $\theta_0 \in (0, 2^{-p'_0})$ so that (45) is fulfilled. Consider any arbitrary $\theta \in [\frac{1}{2}, 1)$. Then, note that $\theta_0\lambda < \theta\lambda < \lambda$ and

$$\|h(t, \cdot)\|_{\theta\lambda} \leq \|h(t, \cdot)\|_{\theta_0\lambda}^\alpha \|h(t, \cdot)\|_\lambda^{1-\alpha},$$

for any $t \in [0, t_*]$, where we have set $\alpha := \log \theta / \log \theta_0$. Then, we obtain

$$\|h(t, \cdot)\|_{\theta\lambda} \leq C_\lambda C_{p_0, \theta_0}^\alpha \exp \left(\alpha p_0^{-\frac{\|K\|_{L^2}^2}{2\theta_0\lambda\nu} t} \log \frac{\|h^0\|_{\theta\lambda}}{C_\lambda} \right), \quad (46)$$

for any $t \in [0, t_*]$.

Remark 4.21. Thanks to the linearity of the hierarchy, Theorem 4.19 provides uniqueness among weak solutions $h = (h_T)_{T \in \mathbb{T}}$ to (43) for some $\nu > 0$, such that $h_T \in L^\infty([0, t_*], (L^1 \cap L^2)(\mathbb{R}^{d|T|}))$ for any $T \in \mathbb{T}$ and $\sup_{t \in [0, t_*]} \|h(t, \cdot)\|_\lambda < \infty$ for some $\lambda > 0$. Indeed, assume that $h^0 = 0$. Then, $\|h^0\|_{\theta\lambda} = 0$ for any $\theta \in (0, 1)$ so that (45) trivially implies that $\|h(t, \cdot)\|_{\theta\lambda} = 0$ (i.e., $h(t, \cdot) = 0$) at any later time $t \in (0, t_*]$. Unfortunately, it does not however provide a bounded or Markov semi-group for those norms and in fact semi-group like estimates blow-up in finite time as it will be apparent in the proof.

Proof of Theorem 4.19. We restrict to the special case $C_\lambda = 1$, all the other cases following by linearity of the hierarchy (43) and homogeneity of the norms (44). First, for a given $T \in \mathbb{T}$, notice

that (43) implies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{d|T|}} |h_T|^2 dx_1 \dots dx_{|T|} &= 2 \sum_{i=1}^{|T|} \int_{\mathbb{R}^{d(|T|+1)}} \nabla_{x_i} h_T K(x_i - z) h_{T+i} dx_1 \dots dx_{|T|} dz \\ &\quad - 2\nu \sum_{i=1}^{|T|} \int_{\mathbb{R}^{d|T|}} |\nabla_{x_i} h_T|^2 dx_1 \dots dx_{|T|}. \end{aligned}$$

To justify the above formal calculation, we note that

$$\partial_t h_T = \nu \Delta h_T - \operatorname{div} j_T,$$

where $j_T = \int_{\mathbb{R}^d} K(x_i - z) h_{T+i} dz$. From our assumption on K and h_{T+i} , we immediately have that $j_T \in L^\infty([0, t_*], L^2)$. Namely, we obtain

$$\|j_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})} \leq \|K\|_{L^2} \|h_{T+i}(t, \cdot)\|_{L^2(\mathbb{R}^{d(|T|+1)})}.$$

Standard properties of the heat kernel then prove that $h_T \in L^2([0, t_*], H^1(\mathbb{R}^{d|T|}))$.

By the Cauchy-Schwartz inequality and Young's inequality we further obtain

$$\frac{d}{dt} \|h_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})}^2 \leq \frac{\|K\|_{L^2}^2}{2\nu} \sum_{i=1}^{|T|} \|h_{T+i}(t, \cdot)\|_{L^2(\mathbb{R}^{d(|T|+1)})}^2.$$

For this reason, we introduce the family of intermediary norms

$$\|h(t, \cdot)\|_n := \sup_{|T|=n} \|h_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})},$$

for any tree order $n \in \mathbb{N}$, which readily satisfies

$$\frac{d}{dt} \|h(t, \cdot)\|_n^2 \leq n \frac{\|K\|_{L^2}^2}{2\nu} \|h(t, \cdot)\|_{n+1}^2.$$

By induction this estimate yields

$$\begin{aligned} \|h(t, \cdot)\|_n^2 &\leq \left(\frac{\|K\|_{L^2}^2}{2\nu} \right)^{m-n} \int_s^t \frac{(m-1)!(t-r)^{m-n-1}}{(n-1)!(m-n-1)!} \|h(r, \cdot)\|_m^2 dr \\ &\quad + \sum_{k=n}^{m-1} \left(\frac{\|K\|_{L^2}^2}{2\nu} \right)^{k-n} \frac{(k-1)!(t-s)^{k-n}}{(n-1)!(k-n)!} \|h(s, \cdot)\|_k^2. \end{aligned}$$

for any $m > n$. Using the assumption on h one has

$$\|h(r, \cdot)\|_m^2 \leq \lambda^{-m},$$

for any $r \in [s, t]$, recall that we have assumed that $C_\lambda = 1$. We then infer

$$\begin{aligned} \|h(t, \cdot)\|_n^2 &\leq \left(\frac{\|K\|_{L^2}^2}{2\nu} \right)^{m-n} \binom{m-1}{n-1} (t-s)^{m-n} \lambda^{-m} \\ &\quad + \sum_{k=n}^{m-1} \left(\frac{\|K\|_{L^2}^2}{2\nu} \right)^{k-n} \binom{k-1}{n-1} (t-s)^{k-n} \|h(s, \cdot)\|_k^2. \end{aligned} \quad (47)$$

Obviously, (47) can only provide smallness for short time intervals $(t-s) \sim \nu$, if used directly. As we advanced in Remark 4.21, we cannot bound a semi-group corresponding to the hierarchy for the norm $\|h(t, \cdot)\|_\lambda$ on all times. Instead we fix now the time step

$$\delta := \frac{2\theta\lambda\nu}{\|K\|_{L^2}^2},$$

and consider the associated sequence of discrete times $t_i = \delta i$ with $i = 1, \dots, N$. Applying (47) to $t = t_{i+1}$ and $s = t_i$, one obtains

$$\sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 \leq \theta^m \binom{m-1}{n-1} + \sum_{k=n}^{m-1} (\theta\lambda)^k \binom{k-1}{n-1} \|h(t_i, \cdot)\|_k^2.$$

We now interpolate the last factors

$$\begin{aligned} (\theta^{1/p}\lambda)^k \|h(t_i, \cdot)\|_k^2 &= ((\theta\lambda)^k \|h(t_i, \cdot)\|_k^2)^{1/p} (\lambda^k \|h(t_i, \cdot)\|_k^2)^{1/p'} \leq \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p} \|h(t_i, \cdot)\|_\lambda^{2/p'} \\ &\leq \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p}, \end{aligned}$$

where $1/p + 1/p' = 1$ and we have used the assumption $\|h(t_i, \cdot)\|_\lambda \leq 1$. This yields

$$\sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 \leq \theta^m \binom{m-1}{n-1} + \sum_{k=n}^{m-1} \theta^{k/p'} \binom{k-1}{n-1} \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p}.$$

Since $\sum_{n=1}^m \binom{m-1}{n-1} = 2^{m-1}$, we have that $\binom{m-1}{n-1} \leq 2^{m-1}$. Similarly, summing over k , we obtain

$$\begin{aligned} \sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 &\leq \frac{1}{2} (2\theta)^m + \frac{1}{2} \sum_{k=n}^{m-1} (2\theta^{1/p'})^k \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p} \leq \frac{1}{2} (2\theta)^m \\ &\quad + \frac{1}{2(1 - 2\theta^{1/p'})} \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p}, \end{aligned}$$

where we have used that $2\theta^{1/p'} < 1$ by hypothesis to guarantee the summability of the last factor. Indeed, since $2\theta < 1$ then taking limits $m \rightarrow \infty$ shows that

$$\sup_{t \in [t_i, t_{i+1}]} \|h(t, \cdot)\|_{\theta\lambda} \leq \left(\frac{1}{2(1 - 2\theta^{1/p'})} \right)^{1/2} \|h(t_i, \cdot)\|_{\theta\lambda}^{1/p}.$$

Then, for $t \in [t_i, t_{i+1}]$ and $C_p = \frac{1}{2(1-2\theta^{1/p'})}$, we have

$$\begin{aligned} \|h(t, \cdot)\|_{\theta\lambda} &\leq C_p^{\frac{1}{2}} \|h(t_i, \cdot)\|_{\theta\lambda}^{1/p} \leq C_p^{\frac{1}{2}} C_p^{\frac{2p}{2p}} \|h(t_{i-1}, \cdot)\|_{\theta\lambda}^{1/p^2} \leq C_p^{\frac{1}{2}} C_p^{\frac{2p}{2p}} C_p^{\frac{1}{2p^2}} \|h(t_{i-2}, \cdot)\|_{\theta\lambda}^{1/p^3} \\ &\leq \dots \leq C_p^{\frac{1}{2} \sum_{j=0}^{i-1} \frac{1}{p^j}} \|h(0, \cdot)\|_{\theta\lambda}^{1/p^i}. \end{aligned}$$

From this, by employing the fact that $\sum_{j=0}^{i-1} \frac{1}{p^j} = \frac{1}{1-1/p} = p'$ we find

$$\sup_{t \in [t_i, t_{i+1}]} \|h(t, \cdot)\|_{\theta\lambda} \leq \left(\frac{1}{2(1-2\theta^{1/p'})} \right)^{p'/2} \|h^0\|_{\theta\lambda}^{p'-i}.$$

For any $t \in [0, t_*)$ set i with $t \in [t_i, t_{i+1})$. Since $\|h^0\|_{\theta\lambda} \leq \|h^0\|_{\lambda} \leq 1$, the relation

$$i \leq \frac{t}{\delta} = \frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t$$

allows concluding the proof. \square

We emphasize that Theorem 4.19 only requires $K \in L^2$ instead of $K \in W^{1,\infty}$, but it only provides stability on the *viscous* hierarchy (43). Indeed, we do not know if any similar result could hold on the non-viscous hierarchy (i.e., $\nu = 0$). However Theorem 4.19 does imply a comparable stability result on the solutions f of the starting graphon equation (36) without diffusion, provided that some added regularity is available on the f or on K . The strategy is straightforward and consists in adding some artificial viscosity, which is performed in the next subsection.

4.4 | Stability on the system without artificial diffusion

Our stability result relies on stronger regularity for weak solutions (w, f) to (36). Before stating it, it is useful to observe that smoothness indeed propagates in time by a similar argument as in Proposition 4.11.

Lemma 4.22. *Consider any $K \in W^{1,1}$ with $\operatorname{div} K \in L^\infty$, any $w \in L^\infty_\xi \mathcal{M}_\xi \cap L^\infty_\xi \mathcal{M}_\xi$ and any that $f^0 \in L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x)$. Assume that (w, f) is a weak solution to (36). Then, $f \in L^\infty([0, t_*], L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x))$, for any $t_* < \infty$, and it satisfies*

$$\|f(t, \cdot, \cdot)\|_{L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x)} \leq C e^{e^{Ct}-1},$$

for some $C \in \mathbb{R}_+$ depending only on $\|w\|_{L^\infty_\xi \mathcal{M}_\xi}$, $\|K\|_{W^{1,1}}$, $\|\operatorname{div} K\|_{L^\infty}$ and $\|f^0\|_{L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x)}$.

Proof. The following estimates can be made rigorous through a regularization process, for example, adding viscosity as in the Equation (37), together with a classical iterative delay in time leading

to a linear system. We first observe that by integration by parts we obtain the *a priori* bounds

$$\begin{aligned} \frac{d}{dt} \|f(t, \cdot, \xi)\|_{L_x^p}^p &= -(p-1) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) f(t, x, \xi)^p f(t, y, \zeta) dx dy \\ &\leq (p-1) \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|f(t, \cdot, \xi)\|_{L_x^p}^p, \end{aligned}$$

for $t \in [0, t_*]$ and $\xi \in [0, 1]$, where above we have used again Lemma 4.8. Since $f(t, \cdot, \xi)$ preserves mass for each $\xi \in [0, 1]$, one has that $\|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} = \|f^0\|_{L_\xi^\infty L_x^1}$ for each $t \in [0, t_*]$ and this allows easily propagating any $L_\xi^\infty L_x^p$ bound of f by Gronwall's inequality, leading to the following estimate

$$\|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^p} \leq C_p e^{C_p t}.$$

Similarly, differentiating (20) with respect to x we now get

$$\begin{aligned} \partial_t \nabla_x f(t, x, \xi) + \operatorname{div}_x \left(\nabla_x f(t, x, \xi) \otimes \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) \\ + \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} \nabla K(x-y)^\top f(t, y, \zeta) dy \right) = 0. \end{aligned}$$

Again, by integrating by parts we easily get the following decomposition

$$\frac{d}{dt} \frac{1}{2} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2 = I_1 + I_2 + I_3,$$

where each term takes the form

$$\begin{aligned} I_1 &:= \frac{1}{2} \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) |\nabla_x f(t, x, \xi)|^2 f(t, y, \zeta) dx dy, \\ I_2 &:= - \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) \nabla_x f(t, x, \xi) \cdot \nabla_y f(t, y, \zeta) f(t, x, \xi) dx dy, \\ I_3 &:= - \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \nabla_x f(t, x, \xi)^\top \cdot \nabla K(x-y) \cdot \nabla_x f(t, x, \xi) f(t, y, \zeta) dx dy. \end{aligned}$$

By the hypothesis on K , we have

$$\begin{aligned} |I_1| &\leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \\ |I_2| &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \\ |I_3| &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\nabla K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \end{aligned}$$

where we have used again Lemma 4.8. Putting everything together, using the previous L^p estimates and the fact that $\|f(t, \cdot, \cdot)\|_{L^\infty_\xi L^1_x} = \|f^0\|_{L^\infty_\xi L^1_x}$ for all $t \in [0, t_*]$, one obtains

$$\frac{d}{dt} \|\nabla_x f(t, \cdot, \xi)\|_{L^2_x}^2 \leq C e^{Ct} \|\nabla_x f(t, \cdot, \xi)\|_{L^2_x}^2.$$

Therefore, we conclude by means of Gronwall's lemma. \square

With this additional regularity, in the context of the preceding subsections, and from Theorem 4.19 and Proposition 4.17, we may derive the following uniqueness result.

We do note that in the following theorem, we use the $\tau(T, w, f)$ for generic $w, \tilde{w} \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\xi \mathcal{M}_\xi$. While the operator τ is trivially well defined if $w, \tilde{w} \in L^\infty_\xi L^1_\zeta \cap L^\infty_\xi L^1_\xi$, we rigorously justify the definition of $\tau(T, w, f)$ for kernels in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\xi \mathcal{M}_\xi$ in the next Section 5 around Definition 5.5. We refer to the more extensive discussion just after Lemma 4.15.

Theorem 4.23. *Consider any couple of weak solution to (36) $f, \tilde{f} \in L^\infty([0, t_*], L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x))$ with associated $w, \tilde{w} \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\xi \mathcal{M}_\xi$, for some $K \in L^\infty \cap W^{1,1}$ with $\operatorname{div} K \in L^\infty$. Then, we have*

$$\left\| \int_0^1 (f - \tilde{f})(t, \cdot, \xi) d\xi \right\|_{L^2_x} \leq \frac{C}{(\log |\log \|\tau(\cdot, w, f^0) - \tau(\cdot, \tilde{w}, \tilde{f}^0)\|_\lambda|)_+^{1/2}},$$

for any $t \in [0, t_*]$, and some constants $C, \lambda > 0$, which only depend on t_* , the various norms of K , the norm of w, \tilde{w} in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\xi \mathcal{M}_\xi$ and the norm of the initial data f^0, \tilde{f}^0 in $L^\infty_\xi(L^1_x \cap L^\infty_x \cap H^1_x)$.

Remark 4.24. A direct L^p estimate would only provide the following trivial estimate

$$\|f - \tilde{f}\|_{L_t^\infty L^\infty_\xi L^2_x} \leq e^{Ct} \left(\|f^0 - \tilde{f}^0\|_{L^\infty_\xi L^2_x} + C \|w - \tilde{w}\|_{L^\infty_\xi \mathcal{M}_\zeta} \right).$$

Note that such estimate would require a very precise strong control on the difference $w - \tilde{w}$, which is not available to us. However, as we will see in the next section, we can obtain good estimates on the weaker objects $\tau(T, w, f^0) - \tau(T, \tilde{w}, \tilde{f}^0)$.

Remark 4.25. If we assume more regularity on K , for example $K \in W^{1,\infty}$, then it is possible to obtain stability with instead less regularity on f and \tilde{f} (typically $f^0 \in L^\infty_\xi L^\infty_x$, for example).

Before turning to the proof of Theorem 4.23, let us emphasize some of its main consequences. The key point is that it offers a direct way to obtain some form of compactness on our solutions. More specifically, we can extract strong convergence in L^2 on each of the $\tau(T, w_N, f_N^0)$ through a classical diagonal extraction process since each of them is bounded in $L^1 \cap W^{1,\infty}$ by Corollary 4.16. From Theorem 4.23, this implies that $\int_0^1 f_N(t, x, \xi) d\xi$ is strongly compact in L^2_x . However, we are still not able to identify the corresponding limit in terms of a solution to (36). This will require an appropriate extension of graphons that is fully developed in the next section.

Proof of Theorem 4.23. For some $\nu > 0$, let us consider the solution f^ν to the system (37) with artificial diffusion with the same initial data f^0 and same weights w as f . We define similarly \tilde{f}^ν

solution to (37) with the same initial data \tilde{f}^0 and weights \tilde{w} as \tilde{f} . The existence of both f^ν and \tilde{f}^ν is provided by Proposition 4.11.

The strategy of the proof is classical: First use the regularity of f (resp. \tilde{f}) to compare them with f^ν (resp. \tilde{f}^ν). Then use Theorem 4.19 to compare the observables from f^ν and \tilde{f}^ν . To compare f with f^ν , we first derive H^1 estimates for f^ν in the usual way

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2 &= -\nu \int_{\mathbb{R}^d} |\nabla_x f^\nu(t, x, \xi)|^2 dx \\ &\quad + \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} f^\nu(t, x, \xi) \nabla f^\nu(t, x, \xi) \cdot K(x - y) f^\nu(t, y, \zeta) dy dx. \end{aligned}$$

Integrating by parts in the second term yields

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2 &+ \nu \int_{\mathbb{R}^d} |\nabla_x f^\nu(t, x, \xi)|^2 dx \\ &= -\frac{1}{2} \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x - y) |f^\nu(t, x, \xi)|^2 f^\nu(t, y, \zeta) dy dx \\ &\leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2, \end{aligned}$$

where we have used Lemma 4.8 in the last step. Again, notice that $\|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} = \|f^0\|_{L_\xi^\infty L_x^1}$ for all $t \in [0, t_*]$. Using Gronwall's lemma we obtain

$$\|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^2 + \nu \int_0^t \int_{\mathbb{R}^d} |\nabla_x f^\nu(s, x, \xi)|^2 dx ds \leq \|f^0\|_{L_\xi^\infty L_x^2}^2 e^{\frac{t}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^1}}. \quad (48)$$

Now, observe that the difference $f^\nu - f$ satisfies the following equation

$$\begin{aligned} \partial_t(f^\nu - f)(t, x, \xi) &+ \operatorname{div}_x \left((f^\nu - f)(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f^\nu(t, y, \zeta) dy \right) \\ &+ \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) (f^\nu - f)(t, y, \zeta) dy \right) \\ &- \nu \Delta_x (f^\nu - f)(t, x, \xi) = \nu \Delta_x f(t, x, \xi). \end{aligned}$$

Therefore, similar arguments as above lead to analogous H^1 estimates

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2}^2 &+ \nu \int_{\mathbb{R}^d} |\nabla_x (f^\nu - f)(t, x, \xi)|^2 dx \\ &\leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \\ &\quad + \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|K\|_{L^2} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty H_x^1} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \\ &\quad + \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \\ &\quad + \nu \|\nabla_x (f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty H_x^1}. \end{aligned}$$

Using the additional regularity $f, \tilde{f} \in L^\infty([0, t_*], L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1))$ from Lemma 4.22, applying Young's inequality in the last term, and recalling that $f(0, \cdot, \cdot) = f^\nu(0, \cdot, \cdot)$ we obtain by Gronwall's lemma that

$$\|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \leq C(t) \sqrt{\nu}, \quad (49)$$

for some continuous and non-decreasing function $C = C(t) \in \mathbb{R}_+$ that only depends on the various norms of K , the norm of w in $L_\xi^\infty \mathcal{M}_\xi$ and the norm of the initial datum f^0 in $L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1)$. Note that a similar estimate can be obtained for $\tilde{f}^\nu - \tilde{f}$. The function C is actually a triple exponential.

By Proposition 4.17, the observables $\tau(T, w, f^\nu)$ and $\tau(T, \tilde{w}, \tilde{f}^\nu)$ with $T \in \mathbb{T}$ both solve the same hierarchy (43) (since w, \tilde{w} do not appear explicitly in that formulation). Therefore, $h = (h_T)_{T \in \mathbb{T}}$, with $h_T = \tau(T, w, f^\nu) - \tau(T, \tilde{w}, \tilde{f}^\nu)$, again solves (43) by linearity. In addition, Corollary 4.16 implies that

$$\begin{aligned} \|h_T(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{d|T|})} &\leq \|\tau(T, w, f^\nu(t, \cdot, \cdot))\|_{L^2(\mathbb{R}^{d|T|})} + \|\tau(T, \tilde{w}, \tilde{f}^\nu(t, \cdot, \cdot))\|_{L^2(\mathbb{R}^{d|T|})} \\ &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\xi}^{|T|-1} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^{|T|} + \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\xi}^{|T|-1} \|\tilde{f}^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^{|T|} \\ &\leq \frac{2}{w_{\min}} \left(w_{\max} \frac{\|f^0\|_{L_\xi^\infty L_x^2} + \|\tilde{f}^0\|_{L_\xi^\infty L_x^2}}{2} e^{\frac{t_*}{4} w_{\max} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^1}} \right)^{|T|}, \end{aligned}$$

where $w_{\min} = \min\{\|w\|_{L_\xi^\infty \mathcal{M}_\xi}, \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\xi}\}$ and $w_{\max} = \max\{\|w\|_{L_\xi^\infty \mathcal{M}_\xi}, \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\xi}\}$. Consequently, we obtain that $\sup_{t \in [0, t_*]} \|h(t, \cdot, \cdot)\|_\lambda < 1$, for any $\lambda > 0$ such that

$$\sqrt{\lambda} < \min \left\{ \frac{2}{w_{\max}}, \frac{w_{\min}}{w_{\max}} \right\} e^{\frac{-t_*}{4} w_{\max} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^1}} \frac{1}{(\|f^0\|_{L_\xi^\infty L_x^2} + \|\tilde{f}^0\|_{L_\xi^\infty L_x^2})}.$$

We can now apply Theorem 4.19 for this λ , any $p > 1$ and $\theta \in (0, 2^{-p'})$, and obtain

$$\sup_{t \in [0, t_*]} \|h(t, \cdot, \cdot)\|_{\theta\lambda} \leq C_{p,\theta} \exp \left(p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \|h^0\|_{\theta\lambda} \right),$$

for any $T \in \mathbb{T}$ and some constant $C_{p,\theta} > 0$ depending only on p and θ . In the special case of the trivial tree $T_1 \in \mathbb{T}_1$ with only one vertex, we obtain

$$\|h\|_{\theta\lambda} \geq \lambda \|h_{T_1}\|_{L^2} = \sqrt{\lambda} \|\tau(T_1, w, f^\nu) - \tau(T_1, \tilde{w}, \tilde{f}^\nu)\|_{L_x^2} = \sqrt{\lambda} \left\| \int_0^1 (f^\nu - \tilde{f}^\nu)(t, \cdot, \xi) d\xi \right\|_{L_x^2}.$$

Therefore, we deduce that

$$\left\| \int_0^1 (f^\nu - \tilde{f}^\nu)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C_{p,\theta}}{\sqrt{\lambda}} \exp \left(p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \|h^0\|_{\theta\lambda} \right), \quad (50)$$

for any $t \in [0, t_*]$. By (49) and Minkowski's integral inequality, we also have that

$$\left\| \int_0^1 (f^\nu - f)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \|f^\nu - f\|_{L_\xi^\infty L_x^2} \leq C(t) \sqrt{\nu},$$

and similarly for $\tilde{f}^\nu - \tilde{f}$, where the function $C = C(t)$ is given above. This combined with (50) immediately yields that

$$\left\| \int_0^1 (f - \tilde{f})(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C_{p,\theta}}{\sqrt{\lambda}} \exp \left(p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu}} t \log \|h^0\|_{\theta\lambda} \right) + C(t) \sqrt{\nu}.$$

Since the above is valid for any $\nu > 0$, it only remains to optimize in ν appropriately. For example, considering $p = e$ for simplicity and setting

$$\nu = \frac{\|K\|_{L^2}^2}{\theta\lambda} t (\log |\log \|h^0\|_{\theta\lambda}|)_+^{-1},$$

ends the proof. \square

5 | EXTENDING GRAPHONS

5.1 | Our goals

This section is centered on the process of taking limits on the $\tau(T, w_N, f_N)$ defined by the operator (39) for the pairs (w_N, f_N) given by Definition 4.4, at some fixed time $t \in \mathbb{R}_+$. Since the time variable plays no role in this part, we omit it from the calculations. The first aim of this section is to analyze the following question already raised in the previous section: If $\lim_{N \rightarrow \infty} \tau(T, w_N, f_N)$ exists for any $T \in \mathbb{T}$, can we find appropriate w and f such that we recover the representation

$$\tau(T, w, f) = \lim_{N \rightarrow \infty} \tau(T, w_N, f_N), \quad \forall T \in \mathbb{T}?$$

As we will see below, this can be rephrased as the question of how to take appropriate limits of w_N and f_N so that they allow passing to the limit in $\tau(T, w_N, f_N)$. The following result answers this question.

Theorem 5.1. *The definition of $\tau(T, w, f)$ can be uniquely extended for any $w \in L_\xi^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\zeta$ and $f \in L_\xi^\infty L_x^\infty$. Furthermore, consider any sequence $\{w_N\}_{N \in \mathbb{N}}$ and $\{f_N\}_{N \in \mathbb{N}}$ such that the following hypothesis hold true*

- (i) $\sup_{N \in \mathbb{N}} \sup_{\xi \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\zeta < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{\zeta \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\xi < \infty,$
- (ii) $\sup_{N \in \mathbb{N}} \|f_N\|_{L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})} < \infty.$

Then, there exists an extracted subsequence of N (that we still denote as N for simplicity) and $w \in L_\xi^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\zeta$ together with $f \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ such that

$$\tau(T, w_N, f_N) \rightarrow \tau(T, w, f) \quad \text{in } L_{loc}^p(\mathbb{R}^{d|T|}),$$

as $N \rightarrow \infty$, for each $T \in \mathbb{T}$ and any $1 \leq p < \infty$.

We note that Theorem 5.1 only imposes that the sequences $\{w_N\}_{N \in \mathbb{N}}$ and $\{f_N\}_{N \in \mathbb{N}}$ fulfill the above uniform estimates (i) and (ii). Those sequences can, hence, be more general than the empirical graphons in Definition 4.4 associated to discrete objects $(w_{ij})_{i,j=1,\dots,N}$ and $(\tilde{f}_i)_{i=1,\dots,N} \subseteq \mathcal{P}(\mathbb{R}^d)$. However, when restricting to empirical w_N and f_N , note that we have equivalently the following uniform bounds

$$(i') \quad \sup_{N \in \mathbb{N}} \max_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| < \infty, \quad \sup_{N \in \mathbb{N}} \max_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| < \infty,$$

$$(ii') \quad \sup_{N \in \mathbb{N}} \max_{1 \leq i \leq N} \|\tilde{f}_i\|_{W_x^{1,\infty}} < \infty.$$

Let us remark some critical differences of our generalized graphons with standard graphons:

- As discussed in Section 1, the notion of graphon arises naturally under the scaling condition (8), that is, $w_{ij} \lesssim 1/N$, since it leads to w_N uniformly bounded in $L_{\xi,\zeta}^\infty$. Compactness is then obtained by considering w_N as the kernel of an operator from L^1 to L^∞ . This corresponds to the natural topology on graphons characterized by the *cut metric*, which can be reframed in terms of the homomorphism densities $\tau(G, w)$ in Lemma 4.14 for general simple graphs G , see the pioneer works of Lovász and Lovász-Szegedy [62, 63].
- Instead, in this work, we focus on the generalized mean-field scaling given by (3)–(4). This leads to w_N which are not uniformly bounded in $L_{\xi,\zeta}^\infty$, but only in $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$. As we mentioned in Section 1, graphons could still be able to capture graph limits in the absence of the scaling $w_{ij} \lesssim 1/N$ by suitably renormalizing the weights, but these results do not appear to be able to handle such a general scaling assumption as we have here. Our scaling still allows to consider w_N as the kernel of an operator, for example, from L^∞ to L^∞ and from L^1 to L^1 as it has been depicted in Lemma 4.7. But the connection with the cut metric and the natural compactness that derives from it are lost. Instead Theorem 5.1 suggests an alternative natural topology on pairs (w, f) with $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $f \in L_\xi^\infty L_x^\infty$.
- One straightforward result that highlights these differences is the following: As we have seen in the previous section (cf. Lemma 4.15), we have

$$\|\tau(T, w, f)\|_{L^\infty} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f\|_{L_\xi^\infty L_x^\infty}^{|T|},$$

for any tree T and any $w \in L_\xi^\infty \mathcal{M}_\zeta$, while this estimate is obviously false if we consider $\tau(G, w)$ for general simple graphs G as it is done in the standard theory of graphons.

Of direct impact to our work here is the fact that we cannot easily define $\tau(T, w, f)$ through formula (39) in Definition 4.13 when we only have $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $f \in L_\xi^\infty L_x^\infty$. We explain how this can be done in the next subsection through the construction of an appropriate algebra. The rest of the section is devoted to the proof of Theorem 5.1 which is based on obtaining simultaneous compactness on all elements of the algebra.

5.2 | Defining $\tau(T, w, f)$ for unbounded w

The key to the definition of $\tau(T, w, f)$ (and later of the compactness argument) for only $f \in L_\xi^\infty (L_x^1 \cap L_x^\infty)$ and $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, is an iterative way to construct it from the leaves of the tree. This leads us to the following definition.

Definition 5.2 (A countable algebra). We will denote by \mathcal{T} the countable algebra of transforms over spaces of arbitrarily large dimensions which is built as follows: For each transform $F \in \mathcal{T}$ there exists $n \in \mathbb{N}$ (the rank of F) so that F maps each couple (w, f) into a scalar function $F(w, f)$ on $[0, 1] \times \mathbb{R}^{dn}$. The full algebra \mathcal{T} is obtained in a recursive way according to the following rules:

- (i) The elementary 1-rank transform $F_0 : (w, f) \mapsto f$ belongs to the algebra \mathcal{T} .
- (ii) Let $F_1 \in \mathcal{T}$ and $F_2 \in \mathcal{T}$ be n_1 -rank and n_2 -rank transforms respectively. Then, the following $(n_1 + n_2)$ -rank transform also belongs to \mathcal{T} :

$$F_1 \otimes F_2 : (w, f) \mapsto F_1(w, f)(\xi, x_1, \dots, x_{n_1}) F_2(w, f)(\xi, x_{n_1+1}, \dots, x_{n_1+n_2}).$$

- (iii) Let $F \in \mathcal{T}$ be a n -rank transform. Then, the following n -rank transform also belongs to \mathcal{T} :

$$F^* : (w, f) \mapsto \int_0^1 F(w, f)(\zeta, x_1, \dots, x_n) w(\xi, d\zeta).$$

For a given choice of (w, f) , we also denote by $M(w, f)$ the countable algebra consisting of functions over spaces of arbitrary large dimensions given by $F(w, f)$ for any transform $F \in \mathcal{T}$. Of course $M(w, f)$ can also be obtained directly by transposing the rules (i), (ii) and (iii) above:

- (i) We have $f(x_1, \xi) \in M(w, f)$.
- (ii) If $\phi(\xi, x_1, \dots, x_{n_1}), \psi(\xi, x_1, \dots, x_{n_2}) \in M(w, f)$, then we have:

$$\phi(\xi, x_1, \dots, x_{n_1}) \psi(\xi, x_{n_1+1}, \dots, x_{n_1+n_2}) \in M(w, f).$$

- (iii) If $\phi(\xi, x_1, \dots, x_n) \in M(w, f)$, then we have:

$$\int_0^1 \phi(\zeta, x_1, \dots, x_n) w(\xi, d\zeta) \in M(w, f).$$

The first point is to observe that $M(w, f)$ (and thus the algebra \mathcal{T}) is well defined with the only hypotheses $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$ and $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$.

Lemma 5.3. Consider any $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$ and any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$. Then all functions of $M(w, f)$ are well defined and belong to $L_\xi^\infty(L_{x_1, \dots, x_n}^1 \cap L_{x_1, \dots, x_n}^\infty)$, for some $n \in \mathbb{N}$. Moreover,

$$F(w_N, f_N) \rightarrow F(w, f) \quad \text{in} \quad L^1([0, 1] \times \mathbb{R}^{dn}),$$

an $N \rightarrow \infty$ for any fixed $F \in \mathcal{T}$, any sequence $\{f_N\}_{N \in \mathbb{N}}$ uniformly bounded in $L_\xi^\infty(L_x^1 \cap L_x^\infty)$ and converging to some f in $L_\xi^\infty(L_x^1 \cap L_x^\infty)$, and any sequence of $\{w_N\}_{N \in \mathbb{N}}$ uniformly bounded in $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and converging to w in $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$.

Proof. We of course use an induction argument based on the recursive rules above defining the countable algebra $M(w, f)$.

♦ *Step 1: Good definition of elements of $M(w, f)$.*

Obviously, for the elementary transform F_0 in rule (i) we have that $F_0(w, f) = f$ is well defined and belongs to $L^\infty([0, 1] \times \mathbb{R}^d)$ by hypothesis. The second rule (ii) also poses no issue. Indeed, let us set $F = F_1 \otimes F_2$ and assume $F_1, F_2 \in \mathcal{T}$ are transform with rank $n_1, n_2 \in \mathbb{N}$ for which we already now that $F_1(w, f)$ and $F_2(w, f)$ are well defined and belong to $L^\infty([0, 1] \times \mathbb{R}^{dn_1})$ and $L^\infty([0, 1] \times \mathbb{R}^{dn_2})$ respectively. Then, $F(w, f)$ is well defined and belongs to $L^\infty([0, 1] \times \mathbb{R}^{d(n_1+n_2)})$ as the product of two bounded functions. Finally, it only remains to check the third rule (iii). Consider any $F \in \mathcal{T}$ of rank $n \in \mathbb{N}$ for which we already now that $F(w, f)$ is well defined and belongs $L^\infty([0, 1] \times \mathbb{R}^{dn})$. Then, $F^*(w, f)$ is also well defined and belongs to $L^\infty([0, 1] \times \mathbb{R}^{dn})$ by the estimate (35)₂ in Lemma 4.8.

Moreover, since L^1 norms are also stable under tensor products, by the same argument we also have that if $f \in L^\infty_\xi(L^1_x \cap L^\infty_x)$ and $w \in L^\infty_\xi \mathcal{M}_\xi \cap L^\infty_\xi \mathcal{M}_\xi$, then all functions in $M(w, f)$ are well defined and belong to $L^\infty_\xi(L^1_{x_1, \dots, x_n} \cap L^\infty_{x_1, \dots, x_n})$ for some $n \in \mathbb{N}$.

♦ *Step 2: Convergence of elements of $M(w_N, f_N)$.*

First note that taking the elementary transform $F_0 \in \mathcal{T}$ in rule (i) we have that

$$F_0(w_N, f_N) = f_N \rightarrow f = F_0(w, f) \quad \text{in } L^1([0, 1] \times \mathbb{R}^d),$$

by the convergence hypothesis on $\{f_N\}_{N \in \mathbb{N}}$. Let us set $F = F_1 \otimes F_2$ as in the rule (ii) for a couple of transforms $F_1, F_2 \in \mathcal{T}$ and assume that we already now that $F_1(w_N, f_N)$ and $F_2(w_N, f_N)$ are both convergent in L^1 and uniformly bounded in L^∞ . Therefore, it is clear that

$$F(w_N, f_N) \rightarrow F(w, f) \quad \text{in } L^1([0, 1] \times \mathbb{R}^{d(n_1+n_2)}),$$

and it is uniformly bounded in L^∞ as the product of uniformly bounded sequences in L^∞ , each of them convergent in L^1 . Finally, consider a n -rank transform $F \in \mathcal{T}$ and assume that we already now that $F(w_N, f_N)$ converges to $F(w, f)$ in L^1 and is uniformly bounded in L^∞ . Our last goal is to show that $F^*(w_N, f_N)$ given by the rule (iii) is also convergent in L^1 and uniformly bounded in L^∞ . Note that Lemma 4.7 yields a partial answer in the weak topology of L^∞ though. We shall improve it here at the expense of the stronger convergence assumed on $\{w_N\}_{N \in \mathbb{N}}$. For simplicity of notation, we set $\phi_N := F(w_N, f_N)$, which by the induction hypothesis converges to $\phi := F(w, f)$ in L^1 and uniformly bounded in L^∞ . Then, we find that

$$F^*(w_N, f_N)(\xi, x_1, \dots, x_n) = \int_0^1 \phi_N(\zeta, x_1, \dots, x_n) w_N(\xi, d\zeta),$$

is uniformly bounded in L^∞ by estimate (35)₂ in Lemma 4.8 thanks to the uniform bound of $\{w_N\}_{N \in \mathbb{N}}$ in $L^\infty_\xi \mathcal{M}_\xi$ and of ϕ_N in L^∞ . Regarding convergence, we have

$$F^*(w_N, f_N) - F^*(w, f) = I_N^1 + I_N^2,$$

where each factor reads

$$\begin{aligned} I_N^1(\xi, x_1, \dots, x_n) &:= \int_0^1 (\phi_N - \phi)(\zeta, x_1, \dots, x_n) w_N(\xi, d\zeta), \\ I_N^2(\xi, x_1, \dots, x_n) &:= \int_0^1 \phi(\zeta, x_1, \dots, x_n) (w_N(\xi, d\zeta) - w(\xi, d\zeta)). \end{aligned}$$

for $N \in \mathbb{N}$. On the one hand, it is clear by estimate (35)₁ in Lemma 4.8 that we have

$$\|I_N^1\|_{L^1} \leq \|w_N\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi_N - \phi\|_{L^1},$$

and, therefore, $I_N^1 \rightarrow 0$ in L^1 by the convergence of ϕ_N to ϕ in L^1 and the uniform estimate of w_N in $L_\zeta^\infty \mathcal{M}_\xi$. On the other hand, consider any sequence $\{\phi_\varepsilon\}_{\varepsilon>0} \subset C_c^\infty([0, 1] \times \mathbb{R}^{dn})$ such that $\phi_\varepsilon \rightarrow \phi$ in L^1 as $\varepsilon \rightarrow 0$ and define

$$I_{N,\varepsilon}^2(\xi, x_1, \dots, x_n) := \int_0^1 \phi_\varepsilon(\zeta, x_1, \dots, x_n) (w_N(\xi, d\zeta) - w(\xi, d\zeta)),$$

for $N \in \mathbb{N}$ and $\varepsilon > 0$. Then, by (35)₁ in Lemma 4.8 we have

$$\begin{aligned} \|I_{N,\varepsilon}^2\|_{L^1} &\leq \|w_N\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi - \phi_\varepsilon\|_{L^1} + \|I_{N,\varepsilon}^2\|_{L^1} \\ &\leq \|w_N\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi - \phi_\varepsilon\|_{L^1} + \|\phi_\varepsilon\|_{L_{x_1, \dots, x_n}^1 H_\zeta^1} \|w_N - w\|_{L_\xi^1 H_\zeta^{-1}}, \end{aligned}$$

for every $N \in \mathbb{N}$ and $\varepsilon > 0$. Taking \limsup as $N \rightarrow \infty$ we have that

$$\limsup_{N \rightarrow \infty} \|I_N^2\|_{L^1} \leq \sup_{N \in \mathbb{N}} \|w_N\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi_\varepsilon - \phi\|_{L^1}.$$

Since $\varepsilon > 0$ is arbitrary, taking $\varepsilon \rightarrow 0$ and recalling that w_N is uniformly bounded in $L_\zeta^\infty \mathcal{M}_\xi$ and $\phi_\varepsilon \rightarrow \phi$ in L^1 allow concluding. \square

The critical reason for the introduction of the algebras $M(w, f)$ and \mathcal{T} is that they allow easily recovering all $\tau(T, w, f)$, at least for bounded w . Namely, for any tree $T \in \mathbb{T}$, there exists a transform $F \in \mathcal{T}$ so that $\tau(T, w, f)$ can be described in terms of the transform $F(w, f)$ of the pair (w, f) .

Lemma 5.4. *For any tree $T \in \mathbb{T}$ there exists a transform $F \in \mathcal{T}$ such that*

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_{|T|}) d\zeta,$$

for any $w \in L_{\xi, \zeta}^\infty$, and any $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$.

Proof. For any tree $T_n \in \mathbb{T}_n$, index the vertices so that the root has index 1. We define the modified τ operator as follows

$$\hat{\tau}(T_n, w, f)(\xi, x_1, \dots, x_n) := \int_{[0, 1]^n} \prod_{(k, l) \in E(T_n)} w(\xi_k, \xi_l) \prod_{m \in (T_n)} f(\xi_m, x_m) d\xi_2 \dots d\xi_n \Big|_{\xi_1 = \xi}.$$

Note that since $w \in L_{\xi, \zeta}^\infty$ and $f \in L_\xi^\infty L_x^\infty$ then all the computations above make sense and this will also apply in the manipulations below. We also note that we readily obtain the following representation

$$\tau(T, w, f) = \int_0^1 \hat{\tau}(T, w, f)(\xi, \cdot) d\xi,$$

for every tree $T \in \mathcal{T}$. Then, we prove the more precise result that there must exist $F \in \mathcal{T}$ such that $\hat{\tau}(T, w, f) = F(w, f)$. We proceed by induction on the size of the tree T .

Let $T_2 \in \mathcal{T}_2$ be the only tree with two vertices. Then, we have

$$\hat{\tau}(T_2, w, f)(\xi, x_1, x_2) = f(\xi, x_1) \int_0^1 w(\xi, \zeta) f(\zeta, x_2) d\zeta.$$

In other words, $\hat{\tau}(T_2, w, f) = F_2(w, f)$ with $F_2 := F_0^* \otimes F_0$. Of course, $F_0 \in \mathcal{T}$ by item (i) in Definition 5.2. Hence $F_0^* \in \mathcal{T}$ by item (iii), and finally F_2 so does by item (ii).

Assume by induction that the above holds for any tree in \mathcal{T}_n . We need to show that it is also verified for any tree in \mathcal{T}_{n+1} . Assume that the root is index by 1 and has degree k . Index the corresponding vertices connected to the root by $2, \dots, k+1$. Denote by T^2, \dots, T^{k+1} the subtrees starting from each vertex 2 to $k+1$. Then, we have that

$$\begin{aligned} \hat{\tau}(T_{n+1}, w, f) &= f(\xi, x_1) \int_{[0, 1]^k} \prod_{i=2}^{k+1} w(\xi, \xi_i) \hat{\tau}(T^i, w, f)(\xi_i) d\xi_i \\ &= f(\xi, x_1) \prod_{i=2}^{k+1} \int_{[0, 1]} w(\xi, \xi_i) \hat{\tau}(T^i, w, f)(\xi_i) d\xi_i. \end{aligned}$$

By our induction hypothesis, we have F_i such that $\hat{\tau}(T^i, w, f) = F_i(w, f)$. By rule (iii), for each i ,

$$\int_{[0, 1]} w(\xi, \xi_i) \hat{\tau}(T^i, w, f)(\xi_i) d\xi_i = \int_{[0, 1]} w(\xi, \xi_i) F_i(w, f)(\xi_i) d\xi_i = F_i^*$$

belongs to \mathcal{T} . Of course, $f(\xi, x_1) = F_0$ by rule (i). Hence finally $\hat{\tau}(T_{n+1}, w, f) = F_0 \otimes F_2^* \cdots \otimes F_{k+1}^*$ by rule (ii). \square

The combination of Lemmas 5.3 and 5.4 naturally lead to the following extended definition of $\tau(T, w, f)$.

Definition 5.5 (Extension of the τ operator). Consider a transform $F \in \mathcal{T}$, for any $T \in \mathcal{T}$, as provided by Lemma 5.4. We then define

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_{|T|}) d\zeta,$$

for any $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$ and any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$.

In Lemma 5.4 we built a special transform $F \in \mathcal{T}$ associated to each tree $T \in \mathcal{T}$. It relies on a recursive construction involving the operations (i), (ii) and (iii) in Definition 5.1, starting from the leaves of T and descending towards the root. As there could a priori be several transforms F that fit a given tree, the last point to check is that Definition 5.5 is independent of the particular choice of F .

Lemma 5.6. Consider any $F_1, F_2 \in \mathcal{T}$ such that

$$\int_0^1 F_1(w, f)(\zeta, \cdot) d\zeta = \int_0^1 F_2(w, f)(\zeta, \cdot) d\zeta, \quad (51)$$

for all $f \in L^\infty_\xi(L^1_x \cap L^\infty_x)$ and all $w \in L^\infty_{\xi,\zeta}$. Then, the equality (51) also holds for all $f \in L^\infty_\xi(L^1_x \cap L^\infty_x)$ and all $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$.

Proof. Define $w_N = K_N \star_{\xi,\zeta} w$ for some smooth convolution kernel K_N with $K_N \rightarrow \delta_0$, and for any $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$.

First of all $w_N \in L^\infty_{\xi,\zeta}$ since $w \in L^\infty_\xi \mathcal{M}_\zeta$. We also have that w_N converges to w in $L^1_\xi H^{-1}_\zeta \cap L^1_\zeta H^{-1}_\xi$.

Therefore, we first have that for w_N

$$\int_0^1 F_1(w_N, f)(\zeta, \cdot) d\zeta = \int_0^1 F_2(w_N, f)(\zeta, \cdot) d\zeta.$$

But, furthermore, by applying Lemma 5.3, we have that $F_1(w_N, f)$ and $F_2(w_N, f)$ converge strongly in L^1 to $F_1(w, f)$ and $F_2(w, f)$, respectively. Hence, we obtain the desired equality. \square

Being able to correctly define $\tau(T, w, f)$ is of course only the first and simplest step in the proof of Theorem 5.1. Passing to the limit in $\tau(T, w_N, f_N)$ is considerably more intricate and in particular we cannot apply Lemma 5.3 as of course we cannot have the required compactness on f_N and w_N from the assumptions of Theorem 5.1.

Instead we have to derive compactness through the clever use of measure-preserving transforms which are the object of the next subsections.

5.3 | An extended multilinear framework

We briefly explain in this subsection how the analysis and estimates presented here can also be applied to a larger framework of multilinear operators that extends on the transforms introduced earlier. We define a family of multilinear operators

$$M : (L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi)^k \times (L^\infty_\xi(L^1_x \cap L^\infty_x))^l \rightarrow L^\infty_\xi((L^1 \cap L^\infty(\mathbb{R}^{dn})))$$

$$(w_1, \dots, w_k) \times (f_1, \dots, f_l) \mapsto M(w_1, \dots, w_k, f_1, \dots, f_l)(\xi, x_1, \dots, x_n).$$

This family \mathcal{M} is constructed exactly as the algebra of transform before through the following rules

- (i) The map $M_0 : f \mapsto f$ belongs to \mathcal{M} .
- (ii) Let $M_1 \in \mathcal{M}$ and $M_2 \in \mathcal{M}$ be posed on $(L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi)^{k_1} \times (L^\infty_\xi(L^1_x \cap L^\infty_x))^{l_1}$ and $(L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi)^{k_2} \times (L^\infty_\xi(L^1_x \cap L^\infty_x))^{l_2}$. Then, the following multilinear operator also belongs to \mathcal{M} :

$$M_1 \otimes M_2 : (L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi)^{k_1+k_2} \times (L^\infty_\xi(L^1_x \cap L^\infty_x))^{l_1+l_2} \rightarrow L^\infty_\xi((L^1 \cap L^\infty(\mathbb{R}^{dn})))$$

$$(w_1, \dots, w_{k_1}, w_{k_1+1}, \dots, w_{k_1+k_2}) \times (f_1, \dots, f_{l_1}, f_{l_1+1}, \dots, f_{l_1+l_2})$$

$$\mapsto M_1(w_1, \dots, w_{k_1}, f_1, \dots, f_{l_1}) M_2(w_{k_1+1}, \dots, w_{k_1+k_2}, f_{l_1+1}, \dots, f_{l_1+l_2}).$$

(iii) Let $M \in \mathcal{M}$ be posed on $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi)^k \times (L_\xi^\infty (L_x^1 \cap L_x^\infty))^l$. Then, the following operator posed on $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi)^{k+1} \times (L_\xi^\infty (L_x^1 \cap L_x^\infty))^l$ also belongs to \mathcal{M} :

$$(w_1, \dots, w_k) \times (f_1, \dots, f_l) \mapsto \int_0^1 M(w_1, \dots, w_k, f_1, \dots, f_l)(\zeta, x_1, \dots, x_n) w_{k+1}(\xi, d\zeta).$$

As one can readily see, these exactly mimic our previous construction: the only difference is that we can actually take different kernels w_i and functions f_i at every step when building the family \mathcal{M} . This extended freedom does not seem to play a significant role in this paper but could be useful in the future.

As one would expect, there is a one-to-one correspondence between the $F \in \mathcal{T}$ and the $M \in \mathcal{M}$ with the straightforward relation

$$F(w, f) = M(w, \dots, w, f, \dots, f).$$

As we mentioned above, all estimates and properties of \mathcal{T} could be extended with the exact same arguments to \mathcal{M} .

5.4 | The key compactness lemma

A key tool to prove Theorem 5.1 is the following result that encompasses the inherent regularity on graphons.

Lemma 5.7. *Consider any sequence g_n in $L^\infty([0, 1])$ with $0 \leq g_n(x) \leq 2^{-n+1}$. Then, there exists $\Phi : [0, 1] \rightarrow [0, 1]$, a.e. injective, measure-preserving, such that the following uniform regularity estimate is verified*

$$\sup_{n \in \mathbb{N}} \int_0^1 |(g_n \circ \Phi)(\xi) - (g_n \circ \Phi)(\xi + h)| d\xi \leq 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for any $0 < |h| < 1$ and some universal constant C .

Remark 5.8. By a.e. injective, we mean that there exists a full measure subset $F' \subset [0, 1]$, $|[0, 1] \setminus F'| = 0$, such that $\Phi : F' \rightarrow [0, 1]$ injective.

We note that independently on the measure-preserving map Φ , by hypothesis we get that $g_n \circ \Phi \rightarrow 0$ in L^1 when $n \rightarrow \infty$. In particular, $g_n \circ \Phi$ is compact in L^1 . Therefore, by the Fréchet-Kolmogorov theorem we infer that the sequence $g_n \circ \Phi$ must be uniformly equicontinuous so that the left hand side in the above inequality always converges to zero when $h \rightarrow 0$. The novelty of the previous result is that we can choose a special measure-preserving map Φ so that such an equicontinuity condition is made explicit in terms of a modulus of continuity. As usual, we extend by zero for evaluations outside $[0, 1]$, namely $(g_n \circ \Phi)(\xi + h) = 0$, if $\xi + h \notin [0, 1]$.

Proof of Lemma 5.7. Our proof is performed in several steps and relies on a hierarchical construction associated to a suitable hierarchical decomposition of the interval $[0, 1]$. Throughout the proof we shall assume that $g_n > 0$ and that the g_n do not charge any point, i.e., $\{x \in [0, 1] :$

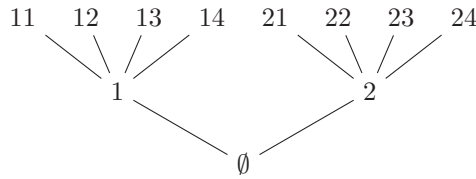


FIGURE 3 Words in W_0 , W_1 , and W_2 .

$g_n(x) = t\} = 0$ for every $t \in \mathbb{R}$ which will make decomposing $[0, 1]$ according to the level sets of the g_n easier. For this, we observe that if g_n charges some points (at most in a countable way), then there exists $g_{n,\varepsilon}$, for every $\varepsilon > 0$, that does not charge any point and with $\|g_n - g_{n,\varepsilon}\|_{L^\infty} < \varepsilon$, for all $n \in \mathbb{N}$. We may hence obtain the claimed regularity on each $g_{n,\varepsilon}$ independently of ε which implies the desired result on g_n .

◊ *Step 1: The hierarchical decomposition.*

We built a sequence of covering of $[0, 1]$ determined through a hierarchical decomposition. For simplicity of the construction, we use indices ranging over the following special sets of words with k letters

$$W_k := \{i_1 i_2 \cdots i_k : i_m \in \{1, \dots, 2^m\}, m = 1, \dots, k\},$$

for any $k \in \mathbb{Z}_+$ (see Figure 3). Note that $W_0 = \{\emptyset\}$ and for $k \geq 1$ we have that W_k contains all possible words with k letters, where the letter at position $m \in \{1, \dots, k\}$ is allowed to take values in $\{1, \dots, 2^m\}$. For any $i \in W_k$ we shall define $j_k(i) := \{i1, \dots, i2^{k+1}\}$ the subset of W_{k+1} obtained by juxtaposition of a letter at the end of the word i . If we denote $n_k := \#W_k$, then the following induction follows

$$W_{k+1} = \bigcup_{i \in W_k} j_k(i), \quad n_{k+1} = 2^{k+1} n_k. \quad (52)$$

Hence, we actually obtain that $n_k = \prod_{m=0}^k 2^m = 2^{\sum_{m=0}^k m} = 2^{\frac{k(k+1)}{2}}$.

For every level $k \in \mathbb{Z}_+$ we shall consider a covering of $[0, 1]$ into n_k sub-intervals of identical size as follows. At level 0 we set $I_0^\emptyset = [0, 1]$ and given a covering $[0, 1] = \bigcup_{i \in W_k} I_k^i$ at level k , we define the covering at level $k+1$ by decomposing each I_k^i with $i \in W_k$ into 2^{k+1} identical sub-intervals and labeling them in lexicographical order with indices in $j_k(i)$. For instance, at level 1 we obtain 2 sub-intervals $I_1^1 = [0, 1/2]$, $I_1^2 = [1/2, 1]$, and at level 2 we obtain 8 sub-intervals: 4 associated with I_1^1

$$I_2^{11} = [0, 1/8], I_2^{12} = [1/8, 1/4], I_2^{13} = [1/4, 3/8], I_2^{14} = [3/8, 1/2],$$

and other 4 associated with I_1^2

$$I_2^{21} = [1/2, 5/8], I_2^{22} = [5/8, 3/4], I_2^{23} = [3/4, 7/8], I_2^{24} = [7/8, 1].$$

We proceed in an analogous way at higher levels. Note that $I_{k+1}^j \subset I_k^i$ for every $j \in j_k(i)$ and $|I_k^i| = \frac{1}{n_k}$ since there are exactly n_k intervals I_k^i with $i \in W_k$.

If we had only one function g_n , the proof would be straightforward: we would just perform a decreasing (or increasing) re-arrangement to obtain a BV function. The main idea of the proof is

to try to mimic this by re-arranging the level sets of g_n to decrease their oscillations. Because there are a countable number of g_n , this is complicated. We introduce a specific decomposition of the g_n along hierarchical level sets through the O_k^i below. At each step, we want to cut the level set into two pieces that have same mass.

Now, we define new coverings $[0, 1] = \bigcup_{i \in W_k} O_k^i$ with measurable subsets $O_k^i \subset [0, 1]$ such that $O_{k+1}^j \subset O_k^i$ for every $j \in j_k(i)$ and $|O_k^i| = |I_k^i| = \frac{1}{n^k}$. Specifically, they will take the form

$$O_k^i := \{x \in [0, 1] : s_{k,m}^i < g_m(x) \leq t_{k,m}^i, m = 1, \dots, k\}, \quad (53)$$

for some values $0 \leq s_{k,m}^i < t_{k,m}^i \leq 2^{-m+1}$ with $m = 1, \dots, k$ to be determined below by recursion on k . Again, $O_0^\emptyset = [0, 1]$. We define O_1^1 and O_1^2 at level 1 as follows. Since $t \mapsto |\{x \in [0, 1] : g_1(x) \geq t\}|$ is continuous (because $|\{x \in [0, 1] : g_1(x) = t\}| = 0$) then there must exist some $\tilde{t} \in [0, 1]$ such that

$$|\{x \in [0, 1] : g_1(x) > \tilde{t}\}| = |\{x \in [0, 1] : g_1(x) \leq \tilde{t}\}| = \frac{1}{2}.$$

Therefore, we can set the values $s_{1,1}^1 = 0, t_{1,1}^1 = \tilde{t}$ for O_1^1 and $s_{1,1}^2 = \tilde{t}, t_{1,1}^2 = 1$ for O_1^2 so that

$$O_1^1 = \{x \in [0, 1] : s_{1,1}^1 < g_1(x) \leq t_{1,1}^1\} = \{x \in [0, 1] : 0 < g_1(x) \leq \tilde{t}\},$$

$$O_1^2 = \{x \in [0, 1] : s_{1,1}^2 < g_1(x) \leq t_{1,1}^2\} = \{x \in [0, 1] : \tilde{t} < g_1(x) \leq 1\}.$$

So defined note that $[0, 1] = O_1^1 \cup O_1^2$ thanks to the hypothesis $0 < g_1 \leq 1$, and $|O_1^1| = |O_1^2| = \frac{1}{2}$. Let us now assume that at level k the values $0 \leq s_{k,m}^i < t_{k,m}^i \leq 2^{-m+1}$ have been set so that the O_k^i given in (53) determine a covering $[0, 1] = \bigcup_{i \in W_k} O_k^i$ and $|O_k^i| = \frac{1}{n^k}$. Let us now proceed in the recursion by defining at level $k+1$ the values $0 \leq s_{k+1,m}^j < t_{k+1,m}^j \leq 2^{-m+1}$ which will determine the O_{k+1}^j .

For any fixed $i \in W_k$ we will built the O_{k+1}^j with $j \in j_k(i)$ by appropriately decomposing O_k^i into 2^{k+1} measurable pieces with identical measure. In this construction we will use words \tilde{W}_m containing m letters with only allowed values $\{1, 2\}$, that is,

$$\tilde{W}_m := \{i_1 i_2 \dots i_m : i_l \in \{1, 2\}, l = 1, \dots, m\},$$

for any $m = 0, \dots, k+1$. By a new recursion on m , we shall proceed by defining a family of auxiliary subsets \tilde{O}_m^α for $\alpha \in \tilde{W}_m$ and $m = 0, 1, \dots, k+1$ verifying $\tilde{O}_m^\alpha = \tilde{O}_{m+1}^{\alpha 1} \cup \tilde{O}_{m+1}^{\alpha 2}$ and $|\tilde{O}_{m+1}^{\alpha 1}| = |\tilde{O}_{m+1}^{\alpha 2}| = \frac{1}{2} |\tilde{O}_m^\alpha|$ starting at $\tilde{O}_0^\emptyset = O_k^i$. For $m = 1$ we note that $t \mapsto |\{x \in O_k^i : t < g_1(x) \leq t_{k,1}^i\}|$ is continuous. Then, there must exist some $\tilde{t}_0^\emptyset \in (s_{k,1}^i, t_{k,1}^i)$ such that

$$|\{x \in O_k^i : s_{k,1}^i < g_1(x) \leq \tilde{t}_0^\emptyset\}| = |\{x \in O_k^i : \tilde{t}_0^\emptyset < g_1(x) \leq t_{k,1}^i\}| = \frac{1}{2} |O_k^i|.$$

Then, we set

$$\tilde{O}_1^1 := \{x \in O_k^i : s_{k,1}^i < g_1(x) \leq \tilde{t}_0^\emptyset\},$$

$$\tilde{O}_1^2 := \{x \in O_k^i : \tilde{t}_0^\emptyset < g_1(x) \leq t_{k,1}^i\}.$$

Note that in \tilde{O}_1^1 and \tilde{O}_1^2 we have simply split the interval $[s_{k,1}^i, t_{k,1}^i]$ for the g_1 into $[s_{k,1}^i, \tilde{t}_0^\theta]$ and $[\tilde{t}_0^\theta, t_{k,1}^i]$ respectively to divide O_k^i into two pieces with exactly half the mass. Assume that \tilde{O}_m^α have been defined for $\alpha \in \tilde{W}_m$ and $m < k$ and let us define $\tilde{O}_{m+1}^{\alpha 1}$ and $\tilde{O}_{m+1}^{\alpha 2}$. The argument is similar: since $t \mapsto |\{x \in \tilde{O}_m^\alpha : t < g_{m+1}(x) \leq t_{k,m+1}^i\}|$ is continuous, there exists $\tilde{t}_m^\alpha \in (s_{k,m+1}^i, t_{k,m+1}^i)$ with

$$|\{x \in \tilde{O}_m^\alpha : s_{k,m+1}^i < g_{m+1}(x) \leq \tilde{t}_m^\alpha\}| = |\{x \in \tilde{O}_m^\alpha : \tilde{t}_m^\alpha < g_{m+1}(x) \leq t_{k,m+1}^i\}| = \frac{1}{2} |\tilde{O}_m^\alpha|.$$

Then, we define

$$\tilde{O}_{m+1}^{\alpha 1} = \{x \in \tilde{O}_m^\alpha : s_{k,m+1}^i < g_{m+1}(x) \leq \tilde{t}_m^\alpha\},$$

$$\tilde{O}_{m+1}^{\alpha 2} = \{x \in \tilde{O}_m^\alpha : \tilde{t}_m^\alpha < g_{m+1}(x) \leq t_{k,m+1}^i\}.$$

Again note that in $\tilde{O}_{m+1}^{\alpha 1}$ and $\tilde{O}_{m+1}^{\alpha 2}$ we have simply split the interval $[s_{k,m}^i, t_{k,m}^i]$ for the g_m into two pieces $[s_{k,m}^i, \tilde{t}_m^\alpha]$ and $[\tilde{t}_m^\alpha, t_{k,m}^i]$ respectively to divide \tilde{O}_m^α into pieces with exactly half the mass. Once the auxiliary family has been set up to $m = k$, note that we can repeat the above argument once more for each $\alpha \in \tilde{W}_k$ and find $\tilde{t}_k^\alpha \in (0, 2^{-k})$ (by the bound $0 < g_{k+1} \leq 2^{-k}$) such that the following subsets divide \tilde{O}_k^α into two halves with exact mass

$$\tilde{O}_{k+1}^{\alpha 1} := \{x \in \tilde{O}_k^\alpha : 0 < g_{k+1}(x) \leq \tilde{t}_k^\alpha\},$$

$$\tilde{O}_{k+1}^{\alpha 2} := \{x \in \tilde{O}_k^\alpha : \tilde{t}_k^\alpha < g_{k+1}(x) \leq 2^{-k}\}.$$

By construction the last level $\{\tilde{O}_{k+1}^\alpha : \alpha \in \tilde{W}_{k+1}\}$ contain exactly 2^{k+1} pieces with same mass so that $O_k^i = \bigcup_{\alpha \in \tilde{W}_{k+1}} \tilde{O}_{k+1}^\alpha$. We then define O_{k+1}^j with $j \in j_k(i)$ by labeling them in lexicographical order and we set the resulting values $0 \leq s_{k+1,m}^j < t_{k+1,m}^j \leq 2^{-m+1}$ accordingly.

From this construction, we easily infer that fixing any $m \leq k$ and any $i \in W_k$, then at least half of the new intervals $[s_{k+1,m}^j, t_{k+1,m}^j]$ with $j \in j_k(i)$ must have at most half the length of $[s_{k,m}^i, t_{k,m}^i]$, that is,

$$\#\left\{j \in j_k(i) : (t_{k+1,m}^j - s_{k+1,m}^j) > \frac{1}{2} (t_{k,m}^i - s_{k,m}^i)\right\} < 2^k. \quad (54)$$

We can even be more precise. Note that going from $[s_{k,m}^i, t_{k,m}^i]$ to the next level $k+1$, we introduced 2^{k+1} new intervals $[s_{k+1,m}^j, t_{k+1,m}^j]$. They correspond to either of the two pieces $[s_{k,m}^i, \tilde{t}_{m-1}^\alpha]$ or $[\tilde{t}_{m-1}^\alpha, t_{k,m}^i]$ for $\alpha \in \tilde{W}_{m-1}$ in which we split the initial interval. By construction note that each of the pieces is repeated in exactly 2^{k-m+1} of the O_{k+1}^j with $j \in j_k(i)$. Therefore, we obtain

$$\begin{aligned} & \frac{1}{2^{k+1}} \sum_{j \in j_k(i)} (t_{k+1,m}^j - s_{k+1,m}^j) \\ &= \frac{1}{2^{k+1}} 2^{k-m+1} \sum_{\alpha \in \tilde{W}_{m-1}} \left((t_{k,m}^i - \tilde{t}_{m-1}^\alpha) + (\tilde{t}_{m-1}^\alpha - s_{k,m}^i) \right) \\ &= \frac{1}{2^m} \sum_{\alpha \in \tilde{W}_{m-1}} (t_{k,m}^i - s_{k,m}^i) = \frac{1}{2} (t_{k,m}^i - s_{k,m}^i). \end{aligned}$$

Summing over $i \in W_k$, dividing by n_k and recalling (52) yields

$$\frac{1}{n_{k+1}} \sum_{j \in W_{k+1}} (t_{k+1,m}^j - s_{k+1,m}^j) = \frac{1}{2n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i).$$

for any $k \geq m$. By induction we then infer that

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \leq \frac{1}{2^{k-m}} \frac{1}{n_m} \sum_{i \in W_m} (t_{m,m}^i - s_{m,m}^i),$$

for any $k \geq m$. Notice that by construction and our assumption on the g_m we obtain that the intervals $(s_{m,m}^i, t_{m,m}^i]$ they all reduce to $(0, \frac{1}{2^{m-1}}]$. Therefore, we obtain

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \leq \frac{1}{2^k}, \quad (55)$$

for any $k \in \mathbb{N}$ and every $m \leq k$. Note that the fact that $0 < g_m \leq \frac{1}{2^m}$ has crucially used in the above cancellations, thus leading to a uniform in m bound.

♦ *Step 2: The measure-preserving map Φ .*

We begin by defining $\Psi : [0, 1] \rightarrow [0, 1]$, which will correspond to the inverse of $\Phi : [0, 1] \rightarrow [0, 1]$. Set any point $x \in [0, 1]$ and use the previous covering by the disjoint $\{O_k^i\}_{i \in W_k}$ to find a unique nested sequence $i_{k+1}(x) \in j_k(i_k(x))$ such that $x \in \bigcap_{k \in \mathbb{N}} O_k^{i_k(x)}$. Since $\{I_k^{i_k(x)}\}_{k \in \mathbb{N}}$ is a nested sequence of compact sets, by Cantor's intersection theorem it is evident that their intersection is a non-empty compact interval again. In fact, since $|I_k^{i_k(x)}| \rightarrow 0$ as $k \rightarrow \infty$ then $\bigcap_{k \in \mathbb{N}} I_k^{i_k(x)}$ must consist in a singleton $\{y\}$, which we use to define $\Psi(x) := y$.

Now we prove that Ψ is a measure-preserving map. Specifically, we shall prove that for every Borel $B \subset [0, 1]$ we have that $\Psi^{-1}(B)$ is measurable and $|\Psi^{-1}(B)| = |B|$. First, we argue for $B = I_k^i$ for any fixed $k \in \mathbb{N}$ and $i \in W_k$. Note that the inclusion $\Psi^{-1}(I_k^i) \supset O_k^i$ is clear by definition, but the converse does not necessarily hold. Indeed, we have that $\Psi^{-1}(I_k^i) = O_k^i \cup N_k^i$, where

$$N_k^i = \left(\bigcap_{m \in \mathbb{N}} O_m^{l_m} \right) \bigcup \left(\bigcap_{m \in \mathbb{N}} O_m^{r_m} \right).$$

Here, $l_{m+1} \in j_m(l_m)$ and $r_{m+1} \in j_m(r_m)$ describe the only two nested sequences of intervals $\{I_m^{l_m}\}_{m \in \mathbb{N}}$ and $\{I_m^{r_m}\}_{m \in \mathbb{N}}$ which stay always adjacent to (and respectively at the left or the right of) I_k^i for $m \geq k$. Obviously, if I_k^i is the first or last interval at level k in lexicographical order, then there is only one such sequence. In particular, $\Psi^{-1}(I_k^i)$ is measurable because so are O_k^i and N_k^i (as countable union and intersection of measurable sets). In fact, $N_k^i \subset [0, 1]$ is negligible (as the union of two negligible subsets), then $|\Psi^{-1}(I_k^i)| = |O_k^i| = |I_k^i|$. To extend the above property to general Borel sets B , we note that it is enough to verify the property for any semialgebra generating the Borel σ -algebra (cf. [90, Theorem 1.1]), for example, the algebra of intervals with endpoints in the dyadic rationals J :

$$\mathcal{A} := \{[a, b) : a, b \in J, 0 \leq a \leq b \leq 1\} \cup \{[a, 1] : a \in J, 0 \leq a < 1\}.$$

Set any such $B \in \mathcal{A}$, for instance $B = [a, b)$ for $a, b \in J$ and $0 \leq a < b \leq 1$ (the other case follows similarly). Then, there exists some (countable) subset I of indices (k, l) so that $[a, b) = \bigcup_{(k,l) \in I} I_k^i$ because the I_k^i are a basis of neighborhoods of $[0, 1]$. By appropriately removing eventual nested

elements in I to avoid redundant information, we may assume that all the involved I_k^i are not contained in each other so that they can only intersect at points J of their boundaries. Therefore, we obtain

$$\Psi^{-1}(B) = \bigcup_{(k,i) \in I} \Psi^{-1}(I_k^i) = \bigcup_{(k,i) \in I} (O_k^i \cup N_k^i).$$

so that $\Psi^{-1}(B)$ is measurable. Since N_k^i are negligible and all the involved O_k^i are disjoint then

$$|\Psi^{-1}(B)| = \sum_{(k,i) \in I} |O_k^i| = \sum_{(k,i) \in I} |I_k^i| = |B|,$$

where in the last equality we have used again that the I_k^i only intersect at most at points J of their boundaries and this has a null contribution to the sum.

Let us now study the injectivity and surjectivity *a.e.* of Ψ . Consider any point $y \in [0, 1] \setminus J$ and set unique nested sequence $i'_{k+1}(y) \in j_k(i'_k(y))$ so that $\{y\} = \bigcap_{k \in \mathbb{N}} I_k^{i'_k(y)}$. We can then consider the associated negligible measurable set $\bigcap_{k \in \mathbb{N}} O_k^{i'_k(y)}$. Notice that the *a.e.* surjectivity of Ψ amounts to proving that the latter intersection is non-empty for *a.e.* y , whilst the *a.e.* injectivity of Ψ amounts to proving that the intersection contains at most one point for *a.e.* y . For the injectivity, let us assume that g_1 is strictly decreasing and continuous (by adding it to the sequence $\{g_n\}_{n \in \mathbb{N}}$ if needed). Then note that

$$\bigcap_{k \in \mathbb{N}} O_k^{i'_k(y)} \subset \bigcap_{k \in \mathbb{N}} g_1^{-1}([s_{k,1}^{i'_k(y)}, t_{k,1}^{i'_k(y)}]) = g_1^{-1}\left(\bigcap_{k \in \mathbb{N}} [s_{k,1}^{i'_k(y)}, t_{k,1}^{i'_k(y)}]\right),$$

for each $y \in [0, 1] \setminus J$. Since the right hand side consists in an intersection of a nested sequence of compact intervals, then it reduces to a compact interval again. The condition for it to actually contain a single point is that $t_{k,1}^{i'_k(y)} - s_{k,1}^{i'_k(y)} \rightarrow 0$ when $k \rightarrow \infty$. Although it does not necessarily happen for each $y \in [0, 1] \setminus J$, we shall prove that it does happen for *a.e.* y . Our argument relies heavily on (54) or, its more quantitative version (55) above. Namely, for any $m \in \mathbb{N}$ let us define the sets

$$J_{k,m} := \left\{ i \in W_k : t_{k,m}^i - s_{k,m}^i \geq \frac{2^k}{3^k} \right\}. \quad (56)$$

for any level $k \geq m$. As it will be observed below, the choice of $\frac{2}{3}$ in $J_{k,m}$ is rather arbitrary and could be replaced by any fixed rate $r \in (\frac{1}{2}, 1)$. Therefore,

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \geq \frac{1}{n_k} \sum_{i \in J_{k,m}} (t_{k,m}^i - s_{k,m}^i) \geq \frac{2^k |J_{k,m}|}{3^k n_k}.$$

Hence, using (55) implies

$$|J_{k,m}| \leq \frac{3^k}{4^k} n_k, \quad (57)$$

for any $k \geq m \geq 1$. In particular, if one defines the poor sets

$$P_m := \{y \in [0, 1] \setminus J : t_{k,m}^{i'_k(y)} - s_{k,m}^{i'_k(y)} \not\rightarrow 0 \text{ when } k \rightarrow \infty\}, \quad (58)$$

for any $m \in \mathbb{N}$, then one has that

$$P_m \subset \bigcap_{k_0 \in \mathbb{N}} \bigcup_{k \geq k_0} \bigcup_{j \in J_{k,m}} I_k^j. \quad (59)$$

Since $|I_k^j| = \frac{1}{n_k}$, using (57) and (59) we find that $|P_m| = 0$ since

$$|P_m| \leq \sum_{k \geq k_0} \frac{3^k}{4^k},$$

for any $k_0 \in \mathbb{N}$. Define the associated global poor set

$$P := \bigcup_{m \in \mathbb{N}} P_m. \quad (60)$$

which is again negligible, and consider pair of full measure sets

$$F'_0 := [0, 1] \setminus (J \cup P), \quad F := \Psi^{-1}(F'_0).$$

Then, it is clear that $\Psi : F \rightarrow [0, 1]$ becomes an injective (measurable) measure-preserving map. Since we are dealing with the (completed) Lebesgue σ -algebra, which determines a standard probability space, then $\Psi(F)$ is measurable (cf. [28, Theorem 3-2]). Unfortunately, we only have the inclusion $\Psi(F) \subset F'_0$. Hence, we define the (eventually) smaller full measure set

$$F' := \Psi(F).$$

Therefore, $\Psi : F \rightarrow F'$ becomes indeed a bijective (measurable) measure-preserving map. Therefore, $\Psi^{-1} : F' \rightarrow F$ is also a measure-preserving map. Extending Ψ^{-1} to all $[0, 1]$ we obtain our measure-preserving map Φ . In particular, note that by construction Φ and Ψ are characterized by the fact that

$$\Phi(I_k^i \cap F') = O_k^i \cap F, \quad \Psi(O_k^i \cap F) = I_k^i \cap F', \quad (61)$$

for each $k \in \mathbb{N}$ and $i \in W_k$.

♦ *Step 3: L^1 modulus of continuity of $\{g_n \circ \Phi\}_{n \in \mathbb{N}}$.*

For simplicity of notation, we define $h_n := g_n \circ \Phi$ for any $n \in \mathbb{N}$. First, we notice that for any $y \in F' \subset F'_0$ the value $h_m(y)$ is determined by the sequence $t_{k,m}^{i'_k(y)}$ (or $s_{k,m}^{i'_k(y)}$), where $i'_{k+1}(y) \in j_k(i'_k(y))$ is defined as before as the unique nested sequence with $\{y\} = \bigcap_{k \in \mathbb{N}} I_k^{i'_k(y)}$. Specifically, we shall prove

$$h_m(y) = \lim_{k \rightarrow \infty} t_{k,m}^{i'_k(y)} = \lim_{k \rightarrow \infty} s_{k,m}^{i'_k(y)}, \quad (62)$$

for each $y \in F' \subset F'_0$. Indeed, by (61) and (53) note that

$$s_{k,m}^{i'_k(y)} < h_m(y) \leq t_{k,m}^{i'_k(y)},$$

for any $k \geq m$ and any $y \in F' \subset F'_0$. By the definitions (58) and (60) of P_m and P the nested intervals $[s_{k,m}^{i'_k(y)}, t_{k,m}^{i'_k(y)}]$ verify $t_{k,m}^{i'_k(y)} - s_{k,m}^{i'_k(y)} \rightarrow 0$ when $k \rightarrow \infty$ so that we conclude (62).

To finish the proof, we need to show that

$$\int_0^1 |h_m(y) - h_m(y + \tau)| dy \leq 2^{-C} \sqrt{\log \frac{1}{|\tau|}},$$

for any $0 < |\tau| < 1$ and some universal constant C . Let us fix any $0 < \tau < 1$ (a similar argument yields $-1 < \tau < 0$) and any level $k \in \mathbb{N}$ so that $\tau < \frac{1}{n_k}$. That is, the size τ of the shift is smaller than the length of the dyadic intervals I_k^i at the level k . Then, we can split each interval I_k^i as follows

$$I_k^i \cap F' = G_{k,\tau}^i \cup B_{k,\tau}^i,$$

for every $i \in W_k$, where $G_{k,\tau}^i$ consist of the points in I_k^i which stay in I_k^i up to a shift of size τ and $B_{k,\tau}^i$ are all the other points in I_k^i . Specifically,

$$G_{k,\tau}^i := \{y \in I_k^i \cap F' : y + \tau \in I_k^i \cap F'\},$$

$$B_{k,\tau}^i := \{y \in I_k^i \cap F' : y + \tau \notin I_k^i \cap F'\}.$$

Note that a point $y \in B_{k,\tau}^i$ must stay within τ of the right boundary of I_k^i . Therefore, we infer

$$|G_{k,\tau}^i| = \frac{1}{n_k} - \tau \leq \frac{1}{n_k}, \quad |B_{k,\tau}^i| = \tau. \quad (63)$$

Using such a decomposition, we can split the above integrals as follows

$$\begin{aligned} \int_0^1 |h_m(y) - h_m(y + \tau)| dy &= \sum_{i \in W_k} \int_{G_{k,\tau}^i} |h_m(y) - h_m(y + \tau)| dy \\ &\quad + \sum_{i \in W_k} \int_{B_{k,\tau}^i} |h_m(y) - h_m(y + \tau)| dy. \end{aligned} \quad (64)$$

For points $y \in G_{k,\tau}^i$, since $y, y + \tau \in I_k^i \cap F'$ then $\Phi(y), \Phi(y + \tau) \in O_k^i$ so that

$$|h_m(y) - h_m(y + \tau)| \leq t_{k,m}^i - s_{k,m}^i.$$

For all other points $y \in B_{k,\tau}^i$ we only have that

$$|h_m(y) - h_m(y + \tau)| \leq |h_m(y)| + |h_m(y + h)| \leq \frac{2}{2^{m-1}} \leq 2.$$

Using the above bounds along with (63) in (64), which exploit the decay of g_m , we obtain

$$\int_0^1 |h_m(y) - h_m(y + \tau)| dy \leq \frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) + \sum_{i \in W_k} 2\tau \leq \frac{1}{2^k} + 2\tau n_k. \quad (65)$$

Here, $k \in \mathbb{N}$ has only been set under the restriction $\tau < \frac{1}{n_k}$ but we could optimize our choice of k . Specifically, let us set $k = k_\tau$ more precisely so that $\frac{1}{n_{k+1}^2} < \tau \leq \frac{1}{n_k^2}$. Then, we obtain that

$$\frac{1}{2^k} + 2\tau n_k \leq \frac{3}{2^k} \leq 2^{-C} \sqrt{\log \frac{1}{\tau}},$$

for universal C , which by (65) yields our result. For the last inequality, note that by our choice of k in terms of τ we get $\frac{1}{\tau} \leq n_{k+1}^2 = 2^{(k+1)(k+2)} \leq 2^{6k^2}$ so that $\sqrt{\log \frac{1}{\tau}} \leq \sqrt{6 \log 2} k$. \square

We note that the dyadic character of the hierarchical construction in the proof of Lemma 5.7 required a specific geometric uniform decay $0 \leq g_n \leq 2^{-n+1}$ in order for the L^1 modulus of continuity of $g_n \circ \Phi$ to be independent of k . However, applying the above to the rescaled sequence

$$\tilde{g}_n := \frac{g_n + \|g_n\|_{L^\infty}}{2^n \|g_n\|_{L^\infty}},$$

allows considering a generic sequence g_n in $L^\infty([0, 1])$, and obtaining a n -dependent L^1 modulus of continuity (eventually growing with n) as follows.

Corollary 5.9. *Consider any sequence g_n in $L^\infty([0, 1])$. Then, there exists $\Phi : [0, 1] \rightarrow [0, 1]$, a.e. injective, measure-preserving, such that the following estimate is verified*

$$\int_0^1 |(g_n \circ \Phi)(\xi) - (g_n \circ \Phi)(\xi + h)| d\xi \leq 2^n \|g_n\|_{L^\infty} 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for any $0 < |h| < 1$, each $n \in \mathbb{N}$ and some universal constant C .

As a further remark, let us highlight the relationship of this lemma with the classical results of regularity on graphons. If w is a graphon, then the Lemma 5.7 implies the classical regularity lemma by Lovász-Szegedy [63], and in turn the famous regularity lemma on graphs by Szemerédi.

Lemma 5.10 Regularity lemma on graphons [63]. *If $w \in L^\infty([0, 1]^2)$ is symmetric with $0 \leq w \leq 1$, then there exists $\Phi : [0, 1] \rightarrow [0, 1]$, measure-preserving, such that in the cut-distance*

$$\delta_\square(w(\Phi(\cdot), \Phi(\cdot)), w(\Phi(\cdot + h), \Phi(\cdot))) \leq \frac{C}{\sqrt{\log \frac{1}{h}}},$$

where the cut-distance is given by

$$\delta_\square(w, \tilde{w}) = \sup_{\Phi, \tilde{\Phi}} \sup_{S, T} \left| \int_{S \times T} (w(\Phi(\xi), \Phi(\zeta)) - \tilde{w}(\tilde{\Phi}(\xi), \tilde{\Phi}(\zeta))) d\xi d\zeta \right|,$$

where the supremum is taken over any pair $\Phi, \tilde{\Phi} : [0, 1] \rightarrow [0, 1]$ of measure-preserving maps, and any pair $S, T \subset [0, 1]$ of measurable subsets.

We once again emphasize that in our case w is not necessarily symmetric nor a bounded function.

5.5 | Proof of Theorem 5.1

We are now ready to prove the main theorem of this section. By hypothesis (i) and (ii) in Theorem 5.1 we have that there exist $C_1, C_2 \in \mathbb{R}_+$ so that

$$\|w_N\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \leq C_1, \quad \|f_N\|_{L_\xi^\infty (W_x^{1,1} \cap W_x^{1,\infty})} \leq C_2,$$

for any $N \in \mathbb{N}$. For each fixed $T \in \mathbb{T}$, by Lemma 4.15, we have that $\tau(T, w_N, f_N)$ is uniformly bounded in $W^{1,1} \cap W^{1,\infty}$ and is hence locally compact in L_x^p for all $p < \infty$. By a standard diagonal extraction procedure, we may hence assume that for some subsequence of N (still denoted as N for simplicity)

$$\lim_{N \rightarrow \infty} \tau(T, w_N, f_N) \text{ exists in } L_{loc}^p, \quad \forall p < \infty, \quad \forall T \in \mathbb{T}.$$

Our goal is to identify the above limits for each $T \in \mathbb{T}$.

We also recall the above countable algebras $M(w_N, f_N)$ in Definition 5.2 endowed with the family of transforms \mathcal{T} according to the rules (i), (ii) and (iii). From Lemma 5.4, we already know that this algebra represents the hierarchy accurately. The rest of the proof is handled in several steps.

♦ *Step 1: Stability by re-arrangements.*

In order to apply our compactness Lemma 5.7 note that we have to perform appropriate re-arrangements in the ξ variable by measure-preserving maps. The following result show that the representation through the algebra $M(w_N, f_N)$ is stable under re-arrangements.

Lemma 5.11. *Consider any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, $f \in L_\xi^\infty (L_x^1 \cap L_x^\infty)$ and any a.e. injective measure-preserving map $\Phi : [0, 1] \rightarrow [0, 1]$. Define the re-arranged objects*

$$\tilde{w}(\xi, d\zeta) := \Phi_\#^{-1} w(\Phi(\xi), \cdot)(d\zeta), \quad \tilde{f}(x, \xi) := f(x, \Phi(\xi)). \quad (66)$$

Then we have that

$$F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F(w, f)(\Phi(\xi), x_1, \dots, x_m),$$

for any transform $F \in \mathcal{T}$, and for a.e. $\xi \in [0, 1]$ and each $x_1, \dots, x_m \in \mathbb{R}^d$, where Φ^{-1} is any a.e. defined left inverse of Φ . Moreover, $\tau(T, \tilde{w}, \tilde{f}) = \tau(T, w, f)$ for any tree T .

If $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$, then the re-arranged object \tilde{w} in (66) clearly belongs to $L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ and we have the straightforward formula

$$\tilde{w}(\xi, \zeta) = w(\Phi(\xi), \Phi(\zeta)).$$

For general $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, we have that \tilde{w} is defined as the pull-back of w through Φ . Since Φ is injective, the pull-back through Φ agrees with the push-forward through Φ^{-1} . We also emphasize that the definition is not sensitive to modifications of Φ^{-1} over the atoms of $w(\Phi(\xi), d\zeta)$ within a Lebesgue-negligible set, as we show in the following straightforward argument based on Lemma 4.7. Assume that Φ_1^{-1} and Φ_2^{-1} are two different choices of the a.e. left inverse of Φ (i.e.,

$\Phi_1^{-1} \circ \Phi = Id = \Phi_2^{-1} \circ \Phi$ a.e.) and define

$$\begin{aligned}\tilde{w}_1(\xi, d\xi) &:= \Phi_{1\#}^{-1} w(\Phi(\xi), \cdot)(d\xi), & \tilde{w}_2(\xi, d\xi) &:= \Phi_{2\#}^{-1} \tilde{w}(\Phi(\xi), \cdot)(d\xi), \\ \phi_1(\zeta) &:= \phi(\Phi_1^{-1}(\zeta)), & \phi_2(\zeta) &:= \phi(\Phi_2^{-1}(\zeta)),\end{aligned}$$

for any $\phi \in L^\infty([0, 1])$. Then, by estimate (33)₂ we have

$$\begin{aligned}\left\| \int_0^1 \phi(\zeta) (\tilde{w}_1(\xi, d\xi) - \tilde{w}_2(\xi, d\xi)) \right\|_{L_\xi^\infty} &= \left\| \int_0^1 (\phi(\Phi_1^{-1}(\zeta)) w(\Phi_1(\xi), d\xi) - \phi(\Phi_2^{-1}(\zeta)) \tilde{w}(\Phi_2(\xi), d\xi)) \right\|_{L_\xi^\infty} \\ &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\xi} \|\phi_1 - \phi_2\|_{L^\infty} = 0,\end{aligned}$$

where we have used that $\phi_1 = \phi_2$ a.e. by definition. Since $\phi \in L^\infty([0, 1])$ is arbitrary, we can restrict to $\phi \in C([0, 1])$ and by a straightforward separability argument we conclude that $\tilde{w}_1 = \tilde{w}_2$.

Proof of Lemma 5.11. The proof is done by induction on the number of operations performed by F . Of course, for the basic transformation F_0 in item (i) of Definition 5.2, the property is obvious since

$$F_0(\tilde{w}, \tilde{f})(\xi, x) = \tilde{f}(\xi, x) = f(\Phi(\xi), x) = F_0(w, f)(\Phi(\xi), x),$$

by definition (66) of the re-arrangements. Now consider any $F \in \mathcal{T}$ with at least one operation, either of the type (ii) or (iii) in Definition 5.2. On the one hand, let us first assume that $F = F_1 \otimes F_2$ is obtained through operations of type (ii) by taking the tensor product of some F_1 and F_2 . Since both the transforms F_1 and F_2 must contain at least one less operation each, by the induction hypothesis we have

$$F_i(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_i}) = F_i(w, f)(\Phi(\xi), x_1, \dots, x_{m_i}),$$

for $i = 1, 2$. Then, we obviously have

$$\begin{aligned}F(\tilde{f}, \tilde{w})(\xi, x_1, \dots, x_{m_1+m_2}) &= F_1(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_1}) F_2(\tilde{w}, \tilde{f})(\xi, x_{m_1+1}, \dots, x_{m_1+m_2}) \\ &= F_1(w, f)(\Phi(\xi), x_1, \dots, x_{m_1}) F_2(w, f)(\Phi(\xi), x_{m_1+1}, \dots, x_{m_1+m_2}) \\ &= F(w, f)(\Phi(\xi), x_1, \dots, x_{m_1+m_2}).\end{aligned}$$

On the other hand, assume now that the identity holds for some $F \in \mathcal{T}$ and let us consider F^* via an operation of (iii), that is,

$$F^*(w, f)(\xi, x_1, \dots, x_m) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_m) w(\xi, d\zeta).$$

Then, we have again that

$$\begin{aligned} F^*(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) &= \int_0^1 F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) \tilde{w}(\xi, d\zeta) \\ &= \int_0^1 F(w, f)(\Phi(\xi), x_1, \dots, x_m) \tilde{w}(\xi, d\zeta) \\ &= \int_0^1 F(w, f)(\xi, x_1, \dots, x_m) w(\Phi(\xi), d\zeta), \end{aligned}$$

where in the second line we have used that $F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F(f, w)(\Phi(\xi), x_1, \dots, x_m)$ and in the third line we have used the definition of \tilde{w} as a push-forward measure. Hence, we find

$$F^*(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F^*(w, f)(\Phi(\xi), x_1, \dots, x_m).$$

Finally, by Lemma 5.4, for any $T \in \mathbb{T}$ there exists $F \in \mathcal{T}$ such that

$$\begin{aligned} \tau(T, w, f)(x_1, \dots, x_{|T|}) &= \int_0^1 F(w, f)(\xi, x_1, \dots, x_{|T|}) d\xi, \\ \tau(T, \tilde{w}, \tilde{f})(x_1, \dots, x_{|T|}) &= \int_0^1 F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{|T|}) d\xi. \end{aligned}$$

Since Φ is measure-preserving, by the theorem of change of variables we obtain

$$\begin{aligned} \tau(T, \tilde{w}, \tilde{f}) &= \int_0^1 F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{|T|}) d\xi = \int_0^1 F(w, f)(\Phi(\xi), x_1, \dots, x_{|T|}) d\xi \\ &= \int_0^1 F(w, f)(\xi, x_1, \dots, x_{|T|}) d\xi = \tau(T, w, f), \end{aligned}$$

which concludes the proof. \square

◇ *Step 2: Obtaining compactness.*

Since \mathcal{T} is countable, we may label its elements with countable indices as follows $\mathcal{T} = \{F_k : k \in \mathbb{N}\}$, where we choose as the first element F_0 the elementary transform in rule (i) of Definition 5.2. Assume that F_k involves m_k variables x_1, \dots, x_{m_k} and denote $n_k \in \mathbb{N}$ the amount of operations of type (iii) involved in the transform F_k . Then, by Definition 5.2 of the algebra $M(w_N, f_N)$ and arguing like in Lemma 4.15 by induction on the number of operations in F_k implies

$$\|F_k(w_N, f_N)\|_{L^\infty_\xi(W^{1,1} \cap W^{1,\infty})} \leq \|w_N\|_{L^\infty_\xi L^1_\xi}^{n_k} \|f_N\|_{L^\infty_\xi(W^{1,1} \cap W^{1,\infty})}^{m_k} \leq C_1^{m_k} C_2^{n_k}, \quad (67)$$

for any $N \in \mathbb{N}$ and each $k \in \mathbb{N}$. We remark that by our hypothesis, the above estimate is independent on N . Our goal in this part is to use Lemma 5.7 (more specifically its Corollary 5.9) to obtain compactness with respect to N after suitable re-arrangements of the sequence $F_k(w_N, f_N)$ for any $k \in \mathbb{N}$. Unfortunately, $F_k(w_N, f_N)$ does not only depend on ξ but also on the extra variables x_1, \dots, x_{m_k} , so that it is not totally clear how compactness in the joint variables $(\xi, x_1, \dots, x_{m_k})$ arises from Corollary 5.9. Note though that dependency on x_1, \dots, x_{m_k} is actually smooth.

Lemma 5.12. *Under the assumptions in Theorem 5.1, there exist $\Phi_N : [0, 1] \rightarrow [0, 1]$ measure-preserving maps for each $N \in \mathbb{N}$ and $\phi_k(\xi, x_1, \dots, x_{m_k}) \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ for each $k \in \mathbb{N}$ so that we have*

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}) \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d_{m_k}}),$$

when $N \rightarrow \infty$ (up to a subsequence on N) for each $k \in \mathbb{N}$, and any $1 \leq p < \infty$.

Proof. For every $k \in \mathbb{N}$, let $D_k = \{(x_1^l, \dots, x_{m_k}^l) : l \in \mathbb{N}\} \subset \mathbb{R}^{d_{m_k}}$ be any countable dense subset (e.g., $D_k = \mathbb{Q}^{d_{m_k}}$), and define the functions $g_{k,l}^N(\xi) := F_k(w_N, f_N)(\xi, x_1^l, \dots, x_{m_k}^l)$ for each $\xi \in [0, 1]$ and any $k, l, N \in \mathbb{N}$. Let us fix N and apply Corollary 5.9 to the countable family $\{g_{k,l}^N\}_{k,l \in \mathbb{N}} \subset L^\infty([0, 1])$: that is the role of n in Corollary 5.9 is played by (k, l) . We obtain that for each $N \in \mathbb{N}$ there exists and a measure-preserving map $\Phi_N : [0, 1] \rightarrow [0, 1]$ so that the re-arrangements $\tilde{g}_{k,l}^N = g_{k,l}^N \circ \Phi_N$ fulfill the following estimate

$$\sup_{N \in \mathbb{N}} \int_0^1 |\tilde{g}_{k,l}^N(\xi + h) - \tilde{g}_{k,l}^N(\xi)| d\xi \leq C_{k,l} 2^{-C \sqrt{\log \frac{1}{|h|}}},$$

for any $0 < |h| < 1$, an universal constant $C \in \mathbb{R}_+$, and some constants $C_{k,l} \in \mathbb{R}_+$. We remark that the constants $C_{k,l}$ are indeed independent of N thanks to the above uniform bound (67). By the Fréchet–Kolmogorov theorem and using a diagonal extraction there exists some subsequence of N 's (which we still denote N for simplicity) and there exists $\tilde{g}_{k,l} \in L^1([0, 1])$ so that $\tilde{g}_{k,l}^N \rightarrow \tilde{g}_{k,l}$ in $L^1([0, 1])$ as $N \rightarrow \infty$ for any $k, l \in \mathbb{N}$. Our final step will be to lift the above convergence on D_k to all $\mathbb{R}^{d_{m_k}}$. To such an end, we define the functions $\tilde{g}_k : [0, 1] \times D_k \rightarrow \mathbb{R}$ by $\tilde{g}_k(\xi, x_1^l, \dots, x_{m_k}^l) := \tilde{g}_{k,l}(\xi)$. Since L^1 convergence implies convergence *a.e.* of an appropriate subsequence, then the uniform Lipschitz bounds on (67) imply

$$|\tilde{g}_k(\xi, x_1^l, \dots, x_{m_k}^l)| \leq C_1^{m_k} C_2^{n_k},$$

$$|\tilde{g}_k(\xi, x_1^{l_1}, \dots, x_{m_k}^{l_1}) - \tilde{g}_k(\xi, x_1^{l_2}, \dots, x_{m_k}^{l_2})| \leq C_1^{m_k} C_2^{n_k} |(x_1^{l_1}, \dots, x_{m_k}^{l_1}) - (x_1^{l_2}, \dots, x_{m_k}^{l_2})|,$$

for any $k, l, l_1, l_2 \in \mathbb{N}$ and *a.e.* $\xi \in [0, 1]$. Then, $\tilde{g}_k : [0, 1] \times D_k \rightarrow \mathbb{R}$ can be extended by continuity in a unique way into a function $\phi_k \in L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1, \infty}$ with $\|\phi_k\|_{L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1, \infty}} \leq C_1^{m_k} C_2^{n_k}$. From the uniform Lipschitz bounds on the $\tilde{g}_{k,l}^N$ and ϕ_k , we obtain that

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}),$$

as $N \rightarrow \infty$, for each $k \in \mathbb{N}$, *a.e.* $\xi \in [0, 1]$, and any $x_1, \dots, x_{m_k} \in \mathbb{R}^d$.

Note we also know that $F_k(w_N, f_N)(\cdot, \Phi_N(\cdot), \cdot, \dots, \cdot)$ and $\phi_k(\cdot, \cdot, \dots, \cdot)$ are bounded in L^∞ uniformly with respect to N thanks to the uniform bounds (67). Therefore, the above pointwise convergence can indeed be improved into L_{loc}^p convergence for any $1 \leq p < \infty$, on any compact set, through the dominated convergence theorem. Indeed, since $F_k(w_N, f_N)(\cdot, \Phi_N(\cdot), \cdot, \dots, \cdot)$ are also uniformly bounded in $L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1} \subseteq L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$ with respect to N by (67), then we can also take the subsequence of N 's so that it converges locally weakly-star in $L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$, thus guaranteeing that $\phi_k \in L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1}$. Note we also know that $F_k(w_N, f_N)(\cdot, \Phi_N(\cdot), \cdot, \dots, \cdot)$ and $\phi_k(\cdot, \cdot, \dots, \cdot)$ are bounded in L^∞ uniformly with respect to N thanks to the uniform bounds (67). Therefore, the above pointwise convergence can indeed be improved into L_{loc}^p convergence for any $1 \leq p < \infty$, on any compact set, through the dominated convergence theorem. Indeed,

since $F_k(w_N, f_N)(\cdot, \Phi_N(\cdot), \cdot, \dots, \cdot)$ are also uniformly bounded in $L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1} \subseteq L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$ with respect to N by (67), then we can also take the subsequence of N 's so that it converges locally weakly-star in $L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$, thus guaranteeing that $\phi_k \in L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1}$. \square

Remark 5.13. Lemma 5.12 could also be extended to time-dependent objects, but we had omitted this full generality of the proof in a first approach to avoid complicating the notation. More specifically, assume that $\{w_N\}_{N \in \mathbb{N}}$ and $\{f_N\}_{N \in \mathbb{N}}$ (now time-dependent) verify the following hypothesis

- (i) $\sup_{N \in \mathbb{N}} \sup_{\xi \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\zeta < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{\zeta \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\xi < \infty,$
- (ii) $\sup_{N \in \mathbb{N}} \|f_N\|_{L_\xi^\infty(W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty})} < \infty.$

Then, there exist $\Phi_N : [0, 1] \rightarrow [0, 1]$ measure-preserving maps for each $N \in \mathbb{N}$, the same Φ_N for all t , and there exist $\phi_k(t, \xi, x_1, \dots, x_{m_k}) \in L_\xi^\infty(W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty})$ for each $k \in \mathbb{N}$ so that we have

$$F_k(w_N, f_N)(t, \Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(t, \xi, x_1, \dots, x_{m_k}) \quad \text{in } L_{loc}^p([0, t^*] \times [0, 1] \times \mathbb{R}^{dm_k}),$$

when $N \rightarrow \infty$ (up to a subsequence on N) for each $k \in \mathbb{N}$, any $t^* > 0$ and any $1 \leq p < \infty$.

Indeed, the proof of the time-dependent version is identical to the one of Lemma 5.12 and simply considers the time variable t along with the space variables x_1, \dots, x_{m_k} that are already included. In the proof, instead of discretizing in space we can also discretize in time by setting for every $k \in \mathbb{N}$ a countable dense set $\mathcal{D}_k = \{(t^l, x_1^l, \dots, x_{m_k}^l) : l \in \mathbb{N}\} \subset [0, t_*] \times \mathbb{R}^{dm_k}$ and defining $g_{k,l}^N(\xi) := F_k(w_N, f_N)(t^l, \xi, x_1^l, \dots, x_{m_k}^l)$ for each $\xi \in [0, 1]$ and any $k, l, N \in \mathbb{N}$. Applying Corollary 5.9 to the countable family $\{g_{k,l}^N\}_{k,l \in \mathbb{N}} \subset L^\infty([0, 1])$ we obtain time-independent measure-preserving maps $\Phi_N : [0, 1] \rightarrow [0, 1]$ such that for all $k, l \in \mathbb{N}$ the rearranged $\tilde{g}_{k,l}^N := g_{k,l}^N \circ \Phi_N$ converge when $N \rightarrow \infty$ (up-to subsequence of N) toward some limiting $\tilde{g}_{k,l}$ in $L^1([0, 1])$. The rest of the proof follows the same train of thoughts in order to build the extension by continuity of this discrete (in time and space) limit to $\phi_k \in L_\xi^\infty(W_{t,x_1, \dots, x_{m_k}}^{1,\infty} \cap W_{t,x_1, \dots, x_{m_k}}^{1,1})$ so that the above convergence of the observables takes place in $L_{loc}^p([0, t_*] \times [0, 1] \times \mathbb{R}^{dm_k})$.

Before entering into the last step in the proof of Theorem 5.1 (namely, the identification of the limits ϕ_k in Lemma 5.12), let us first note that at this point we already have the necessary information to prove the following result.

Theorem 5.14. *Let (X_1, \dots, X_N) be solution to (1) with $K \in W^{1,1} \cap W^{1,\infty}$ and consider the associated laws $f_i(t, \cdot) := \text{Law}(X_i(t))$. Assume that X_i^0 are independent with $\mathbb{E}[|X_i^0|^2] < \infty$ and*

$$\sup_{1 \leq i \leq N} \sqrt{\mathbb{E}[|X_i^0|^2]} \leq M, \quad \sup_{1 \leq i \leq N} \|f_i^0\|_{W^{1,1} \cap W^{1,\infty}} \leq L, \quad \sup_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| \leq C, \quad \sup_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| \leq C,$$

for every $N \in \mathbb{N}$ and appropriate $M, L, C \in \mathbb{R}_+$. Then, there exists $\tilde{w}_N \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ and $\tilde{f}_N \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ for every $N \in \mathbb{N}$ satisfying the graphon-type Vlasov PDE (36), that is,

$$\partial_t \tilde{f}_N(t, x, \xi) + \text{div}_x \left(\tilde{f}_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} K(x-y) \tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, y, \zeta) dy d\zeta \right) = 0, \quad (68)$$

in the distributional sense. Moreover, for any finite time interval $[0, t_*]$ and each compact set $\Omega \subset \mathbb{R}^d$ we have that \tilde{w}_N and \tilde{f}_N verify

$$\begin{aligned} & \int_0^{t_*} \int_{\Omega} \int_0^1 \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) - \tilde{w}_N(\xi + h, \zeta)) \tilde{f}_N(t, x, \zeta) d\zeta \right| d\xi dx dt \leq \varepsilon(|h|), \\ & \sup_{t \in [0, t_*]} W_1 \left(\mathbb{E} \mu_N(t, \cdot), \int_0^1 \tilde{f}_N(t, \cdot, \xi) d\xi \right) \leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \end{aligned} \quad (69)$$

for any $h \in \mathbb{R}$, each $N \in \mathbb{N}$, some constant $\tilde{C} \in \mathbb{R}_+$ and some continuous function $\varepsilon(|h|)$ with $\varepsilon(0) = 0$. Both \tilde{C} and $\varepsilon(h)$ are independent on N : \tilde{C} only depends on C , $\|K\|_{W^{1,\infty}}$ and t_* , and $\varepsilon(|h|)$ only depends on C , L , t_* and Ω . Here, μ_N is the empirical measure $\mu_N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x)$.

Remark 5.15. By interpolation with Lemma 5.7, the function $\varepsilon(h)$ can be calculated explicitly by considering an explicit choice of a dense sequence of points $x_1^l, \dots, x_{m_k}^l$ in the argument above (by constructing a dyadic grid for example), as we briefly explain in the proof.

Proof of Theorem 5.14. First, by Proposition 3.2 we know that the laws f_i and the solution \tilde{f}_i to the coupled system (20) with the same initial data $\tilde{f}_i^0 = f_i^0$ satisfy the estimate (21), that is,

$$W_1(f_i(t, \cdot), \tilde{f}_i(t, \cdot)) \leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2},$$

for each $t \in [0, t_*]$, any $i = 1, \dots, N$, and the constant $\tilde{C} := \sqrt{2/C} (e^{2C t_* \|K\|_{W^{1,\infty}}} - 1)$. Let us set f_N and w_N according to the graphon-type representation in Definition 4.4 associated with \tilde{f}_i and w_{ij} . Therefore, noting that $\mathbb{E} \mu_N(t, \cdot) = \frac{1}{N} \sum_{i=1}^N f_i(t, \cdot)$ and $\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \tilde{f}_i(t, x, \xi)$ we infer

$$\sup_{t \in [0, t_*]} W_1 \left(\mathbb{E} \mu_N(t, \cdot), \int_0^1 f_N(t, \cdot, \xi) d\xi \right) \leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}. \quad (70)$$

Second, as already studied in Section 4, we recall that f_N solves the same transport equations as in (68) in the sense of distribution. Arguing like in the proof of Proposition 4.11 we can propagate the initial $L_{\xi}^{\infty}(W_x^{1,1} \cap W_x^{1,\infty})$ norms of the solution. This implies that on any bounded time interval $[0, t_*]$, we have that $f_N \in L_{\xi}^{\infty} W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty}$ and, in addition, $\|f_N\|_{L_{\xi}^{\infty} W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty}}$ is bounded uniformly with respect to N in terms of $\|K\|_{W^{1,\infty}}$, the constants L , C and t_* .

We note that Lemma 5.12 was initially proved for time-independent objects. However, as mentioned in Remark 5.12 since f_N have appropriate Lipschitz dependence in the joint variable (t, x) and the norms are bounded uniformly on N , then a similar result holds true for the time dependent objects. Specifically, there exists a measure-preserving map $\Phi_N : [0, 1] \rightarrow [0, 1]$ for every $N \in \mathbb{N}$ such that the sequence $\{F_k(w_N, f_N)(\Phi_N(\cdot), \cdot, \dots, \cdot)\}_{N \in \mathbb{N}}$ is compact in $L_{loc}^1([0, t_*] \times [0, 1] \times \mathbb{R}^{d m_k})$ for each $k \in \mathbb{N}$. For simplicity of notation, we denote again the re-arranged objects

$$\tilde{w}_N(\xi, \zeta) := w_N(\Phi_N(\xi), \Phi_N(\zeta)), \quad \tilde{f}_N(t, x, \xi) := f_N(t, x, \Phi_N(\xi)),$$

as given in (66). In the sequel, we shall restrict to the special transform $F_0^* \in \mathcal{T}$ consisting of only one operation of the type (iii) on the elementary transform F_0 of rule (i) in Definition 5.2.

Specifically, we choose $F_0^*(w_N, f_N)(t, \xi, x) = \int_0^1 w_N(\xi, \zeta) f_N(t, x, \zeta) d\zeta$. Indeed, by Lemma 5.11 we can reformulate it as

$$F_0^*(w_N, f_N)(t, \Phi_N(\xi), x) = F_0^*(\tilde{w}_N, \tilde{f}_N)(t, \xi, x) = \int_0^1 \tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, x, \zeta) d\zeta,$$

which is compact in $L_{loc}^1([0, t_*] \times [0, 1] \times \mathbb{R}^d)$ in particular. Then, by the Fréchet–Kolmogorov theorem, for every compact set $\Omega \subset \mathbb{R}^d$ there must exist then some N -independent continuous function $\varepsilon_\Omega(|h|)$ with $\varepsilon_\Omega(|0|) = 0$ such that

$$\int_0^1 \int_0^{t_*} \int_\Omega \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, x, \zeta) - \tilde{w}_N(\xi + h, \zeta) \tilde{f}_N(t + h, x + h v, \zeta)) d\zeta \right| d\xi dx dt \leq \varepsilon_\Omega(|h|),$$

for every $h \in \mathbb{R}$, each $v \in \mathbb{S}^{d-1}$ and each $N \in \mathbb{N}$.

Using the Lipschitz-continuity of f_N (thus \tilde{f}_N) on the joint variables (t, x) implies

$$\int_0^1 \int_0^{t_*} \int_\Omega \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) - \tilde{w}_N(\xi + h, \zeta)) \tilde{f}_N(t, x, \zeta) d\zeta \right| d\xi dx dt \leq \varepsilon_\Omega(|h|) + C t_* |\Omega| \|\tilde{f}_N\|_{L_\xi^\infty W_{t,x}^{1,\infty}} |h|,$$

for every $h \in \mathbb{R}$ and any $N \in \mathbb{N}$. Since f_N (thus \tilde{f}_N) are bounded in $L_\xi^\infty W_{t,x}^{1,\infty}$ uniformly in N , then the above estimate implies (69)₁. Finally, note that (68) and the above estimate (70) for f_N are stable under re-arrangements on f_N . Hence, we conclude that \tilde{f}_N also satisfies (68) and (69)₂ by a simple change of variables.

We observe here that a more intricate interpolation argument can show that ε depend only on the uniform bounds on w_N and f_N , and on Ω and could even be made explicit. As in the proof of Lemma 5.12, we construct a dense sequence $(t_l, x_l) \in [0, t_*] \times \Omega$, for example, through dyadic grid such that for some $\theta > 0$, we have that there exists $l \leq C_\Omega \varepsilon^{-\theta}$ with $|(t, x) - (t_l, x_l)| \leq \varepsilon$, for any (t, x) and any $\varepsilon > 0$. Now Corollary 5.9 implies that

$$\sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t_l, \Phi_N(\xi + h), x_l) - F_0^*(w_N, f_N)(t_l, \Phi_N(\xi), x_l)| d\xi \leq C_\Omega 2^l 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for some constant C_Ω . Thanks to the Lipschitz bound in t and x , we also have

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t, \Phi_N(\xi + h), x) - F_0^*(w_N, f_N)(t, \Phi_N(\xi), x)| d\xi \\ & \leq \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t_l, \Phi_N(\xi + h), x_l) - F_0^*(w_N, f_N)(t_l, \Phi_N(\xi), x_l)| d\xi \\ & \quad + C t_* |\Omega| \|\tilde{f}_N\|_{L_\xi^\infty W_{t,x}^{1,\infty}} |(t, x) - (t_l, x_l)|, \end{aligned}$$

for any (t, x) and any (t_l, x_l) . Recalling the property on the grid (t_l, x_l) , we find that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t, \Phi_N(\xi + h), x) - F_0^*(w_N, f_N)(t, \Phi_N(\xi), x)| d\xi \\ & \leq \inf_l \left(C_\Omega 2^l 2^{-C \sqrt{\log \frac{1}{|h|}}} + C t_* |\Omega| \|\tilde{f}_N\|_{L_\xi^\infty W_{t,x}^{1,\infty}} l^{1/\theta} \right), \end{aligned}$$

which yields an explicit $\varepsilon_\Omega(|h|)$, indeed depending only on Ω , t_* and $\|\tilde{f}_N\|_{L_\xi^\infty W_{t,x}^{1,\infty}}$. \square

◇ *Step 3: Identifying limits.*

We are finally ready to identify the limits obtained by the compactness result in Lemma 5.12. We recall that, according to such a result, there exists a measure-preserving map $\Phi_N : [0, 1] \rightarrow [0, 1]$ for every $N \in \mathbb{N}$ and $\phi_k(\xi, x_1, \dots, x_{m_k}) \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ for every $k \in \mathbb{N}$ so that for an appropriate subsequence in N (still denoted by N for simplicity) we obtain

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}) \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d m_k}),$$

as $N \rightarrow \infty$ for all $k \in \mathbb{N}$ and each $1 \leq p < \infty$. For simplicity of notation, we shall denote again the re-arranged objects by \tilde{f}_N and \tilde{w}_N as given in (66). Using the stability under re-arrangements in Lemma 5.11, the above convergence can then be simply reformulated as follows:

$$F_k(\tilde{w}_N, \tilde{f}_N) \rightarrow \phi_k \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d m_k}), \quad (71)$$

as $N \rightarrow \infty$ for all $k \in \mathbb{N}$ and each $1 \leq p < \infty$. As mentioned above, our goal here reduces to identifying ϕ_k as $F(\tilde{w}, \tilde{f})$ for some appropriate limiting objects \tilde{f} and \tilde{w} of the re-arranged objects \tilde{f}_N and \tilde{w}_N .

On the one hand, since \tilde{f}_N is the first element in the algebra $M(\tilde{w}_N, \tilde{f}_N)$, the above implies that there exists $\tilde{f} \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ (which is indeed the first of the above ϕ_k) so that we have

$$\tilde{f}_N \rightarrow \tilde{f} \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^d), \quad (72)$$

when $N \rightarrow \infty$ for each $1 \leq p < \infty$. On the other hand, by the uniform bound of w_N in item (i) of Theorem 5.1, we readily obtain that $\|\tilde{w}_N\|_{L_\xi^\infty L_\xi^1} \leq C$ and $\|\tilde{w}_N\|_{L_\xi^\infty L_\xi^1} \leq C$ for every $N \in \mathbb{N}$. Since we have (see [47])

$$L^\infty([0, 1], L^1(0, 1)) \subset L^\infty([0, 1], \mathcal{M}([0, 1])) = (L^1([0, 1], C([0, 1]))^*)^*,$$

then the Alaoglu–Bourbaki theorem claims that there exist a subsequence on N (again denoted by N for simplicity) and some $\tilde{w} \in L_\xi^\infty \mathcal{M}_\xi$ (which is in fact exchangeable because so are the \tilde{w}_N) so that

$$\tilde{w}_N \xrightarrow{*} \tilde{w} \quad \text{in } L_\xi^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\xi, \quad (73)$$

when $N \rightarrow \infty$. Our last step is to prove that indeed $\phi_k = F_k(\tilde{w}, \tilde{f})$ for every $k \in \mathbb{N}$. Again, we proceed by induction on the number of operations in F_k .

First, let us assume that F_k is built via an operation of type (ii) in Definition 5.2. Specifically, assume that $F_k = F_{k_1} \otimes F_{k_2}$. Since each F_{k_1} and F_{k_2} must contain at least one less operation that

F_k , then the induction hypothesis implies

$$F_{k_i}(\tilde{w}, \tilde{f}) = \phi_{k_i} = \lim_{N \rightarrow \infty} F_{k_i}(\tilde{w}_N, \tilde{f}_N),$$

for $i = 1, 2$. We recall that the above is a limit in L_{loc}^p for any $1 \leq p < \infty$. Therefore,

$$\begin{aligned} \phi_k(\xi, x_1, \dots, x_{m_k}) &= \lim_{N \rightarrow \infty} F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) \\ &= \lim_{N \rightarrow \infty} F_{k_1}(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_{k_1}}) \lim_{N \rightarrow \infty} F_{k_2}(\tilde{w}_N, \tilde{f}_N)(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= \phi_{k_1}(\xi, x_1, \dots, x_{m_{k_1}}) \phi_{k_2}(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= F_{k_1}(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_{k_1}}) F_{k_2}(\tilde{w}, \tilde{f})(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= F_k(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_k}). \end{aligned}$$

Second, let us assume that F_k is obtained through an operation of type (iii) in Definition 5.2. Specifically, we have there exists some transform $F_{k'} \in \mathcal{T}$ with $m_k = m'_{k'}$ such that $F_k = F_{k'}^*$, that is

$$F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) = \int_0^1 \tilde{w}_N(\xi, \zeta) F_{k'}(\tilde{w}_N, \tilde{f}_N)(\zeta, x_1, \dots, x_{m_k}) d\zeta, \quad (74)$$

holds. Since $F_{k'}$ contains at least one operation less than F_k , then the induction hypothesis shows that

$$F_{k'}(\tilde{w}, \tilde{f}) = \phi_{k'} = \lim_{N \rightarrow \infty} F_{k'}(\tilde{w}_N, \tilde{f}_N), \quad (75)$$

again in L_{loc}^p for any $1 \leq p < \infty$. Take any $\varphi \in C_c([0, 1] \times \mathbb{R}^{d m_k})$ and define

$$\begin{aligned} \psi_N(\xi, \zeta) &:= \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_{k'}(\tilde{w}_N, \tilde{f}_N)(\zeta, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k}, \\ \psi(\xi, \zeta) &:= \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_{k'}(\tilde{w}, \tilde{f})(\zeta, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k}. \end{aligned}$$

Then, by (75) it is clear that $\psi_N(\xi, \zeta) \rightarrow \psi(\xi, \zeta)$ in $L_\zeta^1 C_\xi$. Multiplying (74) by φ and integrating note that we can write

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k} d\xi \\ &= \int_0^1 \int_0^1 \tilde{w}_N(\xi, \zeta) \psi_N(\xi, \zeta) d\xi d\zeta. \end{aligned}$$

Using that $\psi_N(\xi, \zeta) \rightarrow \psi(\xi, \zeta)$ in $L_\zeta^1 C_\xi$ and the weak-* convergence of \tilde{w}_N in $L_\zeta^\infty \mathcal{M}_\xi$ as given in (73) imply that

$$\begin{aligned} &\lim_{N \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_k(\tilde{f}_N, \tilde{w}_N)(\xi, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k} d\xi \\ &= \int_0^1 \int_0^1 \tilde{w}(\xi, d\zeta) \psi(\xi, \zeta) d\xi, \end{aligned}$$

where in the last step we have used Remark 3.3. Undoing the definition of ψ we have proven (by arbitrariness of $\varphi \in C_c([0, 1] \times \mathbb{R}^{dm_k})$) that

$$F_k(\tilde{w}_N, \tilde{f}_N) \xrightarrow{*} F_k(\tilde{w}, \tilde{f}) \quad \text{in } \mathcal{M}([0, 1] \times \mathbb{R}^{dm_k}),$$

as $N \rightarrow \infty$. Since $F_k(\tilde{w}_N, \tilde{f}_N) \rightarrow \phi_k$ in L^p_{loc} by (71), then we conclude that $\phi_k = F_k(\tilde{w}, \tilde{f})$.

Finally, we conclude the identifications of the limits of $\tau(T, w_N, f_N)$ by using Lemmas 5.4, 5.11. Specifically, for any $T \in \mathbb{T}$, there exists some $F \in \mathcal{T}$ such that

$$\tau(T, w_N, f_N)(x_1, \dots, x_{|T|}) = \tau(T, \tilde{w}_N, \tilde{f}_N)(x_1, \dots, x_{|T|}) = \int_0^1 F(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{|T|}) d\xi.$$

We recall the above convergence of the $F(\tilde{f}_N, \tilde{w}_N)$ to $F(\tilde{f}, \tilde{w})$ in L^p_{loc} as $N \rightarrow \infty$. Then, we can pass to the limit in the right hand side of the above relation to achieve

$$\tau(T, w_N, f_N)(x_1, \dots, x_{|T|}) \rightarrow \int_0^1 F(\tilde{f}, \tilde{w})(\xi, x_1, \dots, x_{|T|}) d\xi = \tau(T, \tilde{w}, \tilde{f}),$$

as $N \rightarrow \infty$ in $L^p_{loc}(\mathbb{R}^{d|T|})$, for any $1 \leq p < \infty$ and each $T \in \mathbb{T}$, where in the last step we have used Definition 5.5 for the extension of the operator τ .

5.6 | An alternative formulation of the proof

We mention here a possible variant of the proof above, which consists in re-arranging the arguments in our proof along the following 3 lemmas (separated to make the discussion easier here).

Lemma 5.16. *For any sequence w_N uniformly bounded in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and any sequence f_N uniformly bounded in $L^\infty_\xi W^{1,\infty}_{t,x}$, there exists a measure-preserving map Φ_N such that $\tilde{f}_N(t, x, \xi) = f_N(t, x, \Phi_N(\xi))$ converges strongly in $L^1_{t,x,\xi}$, $\tilde{w}_N(\xi, \zeta) = w_N(\Phi_N(\xi), \Phi_N(\zeta))$ converges weakly to $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, and*

$$\int_0^1 (\tilde{w}_N(\xi, d\zeta) - w(\xi, d\zeta)) h(\zeta) \rightarrow 0$$

strongly in L^1_ξ for all h in a countable dense family of $C([0, 1])$.

The proof of Lemma 5.16 is a straightforward consequence of Lemma 5.7 and Corollary 5.9 for \tilde{w}_N . For f_N , it follows from a time-dependent version of Lemma 5.12 as mentioned in Remark 5.13. We remark here that the proof of Lemma 5.12 could equivalently be performed by using a basis of the function space.

The next step is as follows:

Lemma 5.17. *Consider a sequence \tilde{w}_N uniformly bounded in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$. Assume that $\tilde{w}_N(\xi, \zeta)$ converges weakly to $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, and*

$$\int_0^1 (\tilde{w}_N(\xi, d\zeta) - \tilde{w}(\xi, d\zeta))h(\zeta) \rightarrow 0$$

strongly in L^1_ξ for all h in a countable dense family of $C([0, 1])$. Then for any function $g_N(t, x_1, \dots, x_k, \xi)$ uniformly bounded in L^∞ and converging to g strongly in L^1 , we have that

$$\int_0^1 (\tilde{w}_N(\xi, d\zeta) - \tilde{w}(\xi, d\zeta))g_N(t, x_1, \dots, x_k, \zeta) \rightarrow 0$$

strongly in $L^1_{t, x_1, \dots, x_d, \xi}$.

Lemma 5.17 follows easily from a density argument, together with the estimate in Lemma 4.8 (notice that this is already implicitly used in step 3 of the proof of Theorem 5.1).

The final step is as follows

Lemma 5.18. *Consider any sequence \tilde{w}_N uniformly bounded in $L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and any sequence \tilde{f}_N uniformly bounded in $L^\infty_\xi W^{1, \infty}_{t, x}$, such that $\tilde{f}_N(t, x, \xi) = f_N(t, x, \Phi_N(\xi))$ converges strongly in $L^1_{t, x, \xi}$, $\tilde{w}_N(\xi, \zeta) = w_N(\Phi_N(\xi), \Phi_N(\zeta))$ converges weakly to $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$, and assume that for any function $g_N(t, x_1, \dots, x_k, \xi)$ uniformly bounded in L^∞ and converging to g strongly in L^1 , we have that*

$$\int_0^1 (\tilde{w}_N(\xi, d\zeta) - \tilde{w}(\xi, d\zeta))g_N(t, x_1, \dots, x_k, \zeta) \rightarrow 0$$

strongly in $L^1_{t, x_1, \dots, x_d, \xi}$. Then for all transform F (or equivalently all multilinear operators), we have that $F(\tilde{w}_N, \tilde{f}_N)$ converges strongly in L^1 to $F(w, f)$.

The proof of Lemma 5.18 is exactly the induction argument in step 3 of the proof of Theorem 5.1.

This approach skips Theorem 5.14, which is however interesting on itself. But it does have some advantage and in particular, it allows to disentangle more the limits of f_N and of w_N through Lemma 5.16.

6 | PROOF OF THE MAIN RESULT: THEOREM 1.1

We are now ready to perform the full proof of Theorem 1.1.

First of all, denote by f_i^0 the density of each X_i^0 . Then, we define $w_N(\xi, \zeta)$ and $f_N^0(x, \xi)$ according to equation (30) in Definition 4.4, which we recall

$$w_N(\xi, \zeta) = \sum_{i,j=1}^N N w_{ij} \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{I}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta), \quad \xi, \zeta \in [0, 1],$$

$$f_N^0(x, \xi) = \sum_{i=1}^N f_i^0(x) \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi), \quad x \in \mathbb{R}^d, \xi \in [0, 1].$$

From the uniform bounds on f_i^0 and w_{ij} , it is immediate to check that w_N and f_N^0 satisfy the assumptions (i) – (ii) of Theorem 5.1. We note as well that $\tau(T, w_N, f_N)$ is uniformly bounded in $W^{1,\infty}$, for any fixed T . Since trees are countable, by the usual diagonal extraction technique, we can obtain a subsequence (still denoted by N) such that $\tau(T, w_N, f_N)$ converges strongly in L^∞ , for every T . Hence, this subsequence also satisfies assumption (iii) of Theorem 5.1.

Applying now Theorem 5.1, we find $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ and $f^0 \in L_\xi^\infty (W_x^{1,1} \cap W_x^{1,\infty})$ such that

$$\tau(T, w, f^0) = \lim_{N \rightarrow \infty} \tau(T, w_N, f_N^0), \quad \forall T,$$

in any L_{loc}^p with $p < \infty$. Furthermore, we note that, for each T , $\tau(T, w, f^0)$ is uniformly bounded in $L_x^1 \cap L_x^\infty$. Fixing now any $R > 0$, a straightforward extension of Lemma 4.15 leads to

$$\|\tau(T, w, f_N^0)\|_{L^1(\mathbb{R}^d |T|) \setminus B(0,R)^{|T|}} \leq |T| \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f_N^0\|_{L_\xi^\infty L^1(\mathbb{R}^d)}^{|T|-1} \|f_N^0\|_{L_\xi^\infty L^1(\mathbb{R}^d \setminus B(0,R))}.$$

On the other hand, from the assumptions the main theorem, we have that

$$\int_{\mathbb{R}^d} |x|^2 f_i^0(x) dx \leq C,$$

for some constant C independent of N and i . By the definition of f_N^0 , this implies as well that

$$\int_{\mathbb{R}^d} |x|^2 f_N^0(x) dx \leq C.$$

This shows that

$$\|\tau(T, w, f_N^0)\|_{L^1(\mathbb{R}^d |T|) \setminus B(0,R)^{|T|}} \leq \frac{C}{R^2} |T| \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f_N^0\|_{L_\xi^\infty L^1(\mathbb{R}^d)}^{|T|-1}.$$

Combined with the local convergence in L^p , it implies that $\tau(T, w_N, f_N^0)$ converges strongly to $\tau(T, w_N, f^0)$ in every $L^p(\mathbb{R}^d |T|)$ and in particular in $L^2(\mathbb{R}^d |T|)$.

We further note that also by Corollary 4.16, we have that

$$\|\tau(T, w, f^0)\|_{L_x^2} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f^0\|_{L_\xi^\infty L_x^2}^{|T|}.$$

Consequently, there exists some $\lambda > 0$ small enough such that

$$\|\tau(\cdot, w, f^0) - \tau(\cdot, w_N, f_N^0)\|_\lambda \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (76)$$

By Proposition 4.11, we have existence of weak solution $f \in L_{t,\xi}^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ to (36) associated with w in the sense of Definition 4.10, namely

$$\begin{cases} \partial_t f(t, x, \xi) + \operatorname{div}_x \left(f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = 0, \\ f(0, \cdot, \cdot) = f^0. \end{cases}$$

Of course, f is the function that we will use to compare to the empirical measure, meaning that we have to prove that

$$\mathbb{E} W_1 \left(\int_0^1 f(t, \cdot, \xi) d\xi, \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)} \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (77)$$

First of all, we apply Proposition 3.2: Since $\sup_{1 \leq i, j \leq N} |w_{ij}| \rightarrow 0$ as $N \rightarrow \infty$, then this shows that the solutions \bar{f}_i to (20) satisfy that

$$\mathbb{E} W_1 \left(\frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot), \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)} \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (78)$$

Then, we define f_N as per (30), namely

$$f_N(t, x, \xi) = \sum_{i=1}^N \bar{f}_i(t, x) \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d, \xi \in [0, 1].$$

We recall that (w_N, f_N) is also a weak solution to (36), with $f_N \in L_{t,\xi}^\infty(W_x^{1,1} \cap W_x^{1,\infty})$. Therefore, both f_N and f satisfy the assumptions of Theorem 4.23. Consequently, using (76), we find

$$\left\| \int_0^1 (f - f_N)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C(t)}{(\log |\log \|\tau(\cdot, w_N, f_N^0) - \tau(\cdot, w, f^0)\|_\lambda|)_+^{1/2}} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Since the L^2 norm locally dominates the Wasserstein distance and we have propagation of any moments, this implies that

$$W_1 \left(\int_0^1 f_N(t, \cdot, \xi) d\xi, \int_0^1 f(t, \cdot, \xi) d\xi \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Recalling that $\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x)$, and combining this convergence with (78), finally we obtain (77), which therefore concludes the proof of Theorem 1.1.

ACKNOWLEDGMENTS

This work has been partially supported by the grants: DMS Grant 1908739, 2049020, 2205694 and 2219397 by the NSF (USA); by the State Research Agency of the Spanish Ministry of Science and FEDER-EU, project PID2022-137228OB-I00 (MICIU/AEI /10.13039/501100011033); by Modeling Nature Research Unit, Grant QUAL21-011 funded by Consejería de Universidad, Investigación e Innovación (Junta de Andalucía); by the European Union's Horizon Europe research and innovation program under the Marie Skłodowska-Curie grant agreement No 101064402, and by the European Union's Horizon 2020 research and innovation program (grant agreement No 865711).

REFERENCES

1. L. F. Abbott, *Lapicque's introduction of the integrate-and-fire model neuron* (1907), Brain Res. Bull. **50** (1999), no. 5-6, 303–304.
2. T. Aoki and T. Aoyagi, *Co-evolution of phases and connection strengths in a network of phase oscillators*, Phys. Rev. Lett. **102** (2009), 034101.
3. A. Backhausz and B. Szegedy, *Action convergence of operators and graphs*, Canad. J. Math. (2020), 1–50.
4. A. L. Barabási and R. Albert, *Emergence of scaling in random networks*, Science **286** (1999), 509–512.
5. E. Boissard, *Problèmes d'interaction discret-continu et distances de Wasserstein*, PhD thesis, Université de Toulouse III, 2011.
6. S. Bochner and A. E. Taylor, *Linear functionals on certain spaces of abstractly-valued functions*, Ann. of Math. **39** (1938), no. 4, 913–944.
7. C. Borgs, J. T. Chayes, H. Cohn, and Y. Zhao, *An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions*, Trans. Am. Math. Soc. **372** (2019), no. 5, 3019–3062.
8. C. Borgs, J. T. Chayes, H. Cohn, and Y. Zhao, *An L^p theory of sparse graph convergence II: LD convergence, quotients, and right convergence*, Ann. Probab. **46** (2018), no. 1, 337–396.
9. C. Borgs, J. T. Chayes, J. Kahn, and L. Lovász, *Left and right convergence of graphs with bounded degree*, Random Struct. Algorithms **42** (2013), 1–28.
10. C. Borgs, J. T. Chayes, L. Lovász, V. Sós, and K. Vesztegombi, *Counting graph homomorphisms*, M. Klazar, J. Kratochvíl, M. Loeb, J. Matoušek, R. Thomas, and P. Valtr (Eds.), Topics in discrete mathematics, Springer, Berlin, Heidelberg, 2006, pp. 315–371.
11. C. Borgs, J. T. Chayes, L. Lovász, V. Sós, and K. Vesztegombi, *Convergent graph sequences I: subgraph frequencies, metric properties, and testing*, Adv. Math. **219** (2008), 1801–1851.
12. C. Borgs, J. T. Chayes, L. Lovász, V. Sós, and K. Vesztegombi, *Convergent graph sequences II: multiway cuts and statistical physics*, Ann. Math. **176** (2012), 151–219.
13. W. Braun and K. Hepp, *The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles*, Comm. Math. Phys. **56** (1977), no. 2, 101–113.
14. N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons*, J. Comput. Neurosci. **8** (2000), no. 3, 183–208.
15. A. N. Burkitt, *A review of the integrate-and-fire neuron model: I. Homogeneous synaptic input*, Biol. Cybern. **95** (2006), no. 1, 1–19.
16. M. J. Cáceres, J. A. Carrillo, and B. Perthame, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, J. Math. Neurosci. **1** (2011), Art. 7, 33 pp.
17. M. J. Cáceres and B. Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, J. Theor. Biol. **350** (2014), 81–89.
18. J. A. Carrillo, M. d. M. Gonzalez, M. P. Gualdani, and M. E. Schonbek, *Classical solutions for a nonlinear Fokker-Planck equation arising in computational neuroscience*, Comm. Part. Differ. Equat. **38** (2013), no. 3, 385–409.
19. J. A. Carrillo, B. Perthame, D. Salort, and D. Smets, *Qualitative properties of solutions for the noisy integrate & fire model in computational neuroscience*, Nonlinearity **28** (2015), no. 9, 3365.
20. H. Chiba and G. S. Medvedev, *The mean field analysis for the Kuramoto model on graphs I. The mean field equation and transition point formulas*, Discrete Contin. Dyn. Syst. Ser. A **39** (2019), no. 1, 131–155.
21. H. Chiba and G. S. Medvedev, *The mean field analysis of the Kuramoto model on graphs II. Asymptotic stability of the incoherent state, center manifold reduction, and bifurcations*, Discrete Contin. Dyn. Syst. Ser. A **39** (2019), no. 7, 3897–3921.
22. H. Chiba, G. S. Medvedev, and M. S. Muzuhara, *Bifurcations in the Kuramoto model on graphs*, Chaos **28** (2018), 073109.
23. A. Compte, N. Brunel, P. S. Goldman-Rakic, and X.-J. Wang, *Synaptic mechanisms and network dynamics underlying spatial working memory in a cortical network model*, Cereb. Cortex **10** (2000), 910–923.
24. A. Compte, M. V. Sanchez-Vives, D. A. McCormick, and X.-J. Wang, *Cellular and network mechanisms of slow oscillatory activity (<1 Hz) and wave propagations in a cortical network model*, J. Neurophysiol. **89** (2003), 2707–2725.
25. F. Coppini, *Long time dynamics for interacting oscillators on graphs*, Ann. Appl. Probab. **32** (2022), no. 1.

26. F. Coppini, H. Dietert, and G. Giacomin, *A law of large numbers and large deviations for interacting diffusions on Erdős-Rényi graphs*, Stoch. Dyn. **20** (2020), 2050010.
27. F. Coppini, E. Luçon, and C. Poquet, *Central limit theorems for global and local empirical measures of diffusions on Erdős-Rényi graphs*, Electron. J. Probab. **28** (2023).
28. T. de La Rue, *Espaces de Lebesgue*, Séminaire de probabilités de Strasbourg, Tome **27** (1993), 15–21.
29. F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stoch. Process. Their Appl. **125** (2015), 2451–2492.
30. F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Global solvability of a networked integrate-and-fire model of McKean-Vlasov type*, Ann. Appl. Probab. **25** (2015), no. 4, 2096–2133.
31. S. Delattre, G. Giacomin, and E. Luçon, *A note on dynamical models on random graphs and Fokker-Planck equations*, J. Stat. Phys. **165** (2016), no. 4, 785–798.
32. A. De Masi, A. Galves, E. Löcherbach, and E. Presutti, *Hydrodynamic limit for interacting neurons*, J. Stat. Phys. **158** (2015), 866–902.
33. J. Diestel and J. J. Uhl Jr, *Vector Measures*, Mathematical Surveys and Monographs, vol. 15, American Mathematical Society, Providence, RI, 1977.
34. V. Dobrić and J. E. Yukich, *Asymptotics for transportation cost in high dimensions*, J. Theoret. Probab. **8** (1995), no. 1, 97–118.
35. R. L. Dobrushin, *Vlasov equations*, Funct. Anal. its Appl. **13** (1979), no. 2, 115–123.
36. R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical J. **1** (1961), 445–466.
37. F. Flandoli, E. Priola, and G. Zanco, *A mean-field model with discontinuous coefficients for neurons with spatial interaction*, Discrete Contin. Dyn. Syst. Ser. A **39** (2019), 3037–3067.
38. N. Fournier and A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure*, Probab. Theory Relat. Fields **162** (2015), 707–738.
39. N. Fournier and E. Löcherbach, *On a toy model of interacting neurons*, Ann. Inst. H. Poincaré Probab. Statist. **52** (2016), no. 4, 1844–1876.
40. W. Gerstner and W. M. Kistler, *Spiking neuron models: single neurons, populations, plasticity*, Cambridge University Press, Cambridge, 2002.
41. M. A. Gkogkas and C. Kuehn, *Graphop mean-field limits for Kuramoto-type models*, SIAM J. Appl. Dyn. Syst. **21** (2022), no. 1, 248–283.
42. F. Golse, *On the dynamics of large particle systems in the mean field limit macroscopic and large scale phenomena: coarse graining*, Mean Field Limits and Ergodicity, Springer, Berlin, 2016, pp. 1–144.
43. M. Hauray and P. E. Jabin, *Particles approximations of Vlasov equations with singular forces: propagation of chaos*, Ann. Sci. Ec. Norm. Super. **48** (2015), no. 4, 891–940.
44. D. O. Hebb, *The organization of behavior*, Wiley, New York, 1949.
45. A. L. Hodgkin and A. F. Huxley, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol. **117** (1952), 500–544.
46. B. K. Hulse, H. Haberkern, R. Franconville, D. B. Turner-Evans, S. Y. Takemura, T. Wolff, M. Noorman, M. Dreher, C. Dan, R. Parekh, A. M. Hermundstad, G. M. Rubin, and V. Jayaraman, *A connectome of the Drosophila central complex reveals network motifs suitable for flexible navigation and context-dependent action selection*, eLife **10** (2021), e66039.
47. A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge, vol. 48, Springer-Verlag, Berlin, Heidelberg, 1977.
48. P. E. Jabin, *A review of the mean field limits for Vlasov equations*, Kinet. Relat. Mod. **7** (2014), 661–711.
49. P. E. Jabin and Z. Wang, *Mean field limit and propagation of chaos for Vlasov systems with bounded forces*, J. Funct. Anal. **271** (2016), no. 12, 3588–3627.
50. P.-E. Jabin and Z. Wang, *Mean field limit for stochastic particle systems*, Active Particles, vol. 1, Theory, Models, Applications, Birkhauser-Springer, Boston, 2017, pp. 379–402.
51. P.-E. Jabin and Z. Wang, *Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels*, Invent. Math. **214** (2018), no. 1, 523–591.
52. P.-E. Jabin and D. Zhou, *The mean-field Limit of sparse networks of integrate and fire neurons*, Preprint arXiv:2309.04046.
53. C. Kuehn and C. Xu, *Vlasov equations on digraph measures*, J. Differ. Equ. **339** (2022), 261–349.

54. D. Kaliuzhnyi-Verbovetskyi and G. S. Medvedev, *The mean field equation for the Kuramoto model on graph sequences with non-Lipschitz limit*, SIAM J. Math. Anal. **50** no. 3, 2441–2465.
55. D. Kunszenti-Kovács, L. Lovász, and B. Szegedy, *Measures on the square as sparse graph limits*, J. Comb. Theory. Ser. B **138** (2019), 1–40.
56. Y. Kuramoto, *Chemical oscillations, waves and turbulence*, Springer-Verlag, Berlin, 1984.
57. Y. Kuramoto, *International symposium on mathematical problems in mathematical physics*, Lect. Notes Phys. **30** (1975), 420.
58. D. Lacker, *Hierarchies, entropy and quantitative propagation of chaos for mean field diffusions*, Probability and Mathematical Physics. **4** (2023), 377–432.
59. D. Lacker, K. Ramanan, and R. Wu, *Local weak convergence for sparse networks of interacting processes*, Ann. Appl. Probab. **33** (2023), no. 2.
60. C. Lancellotti, *On the Vlasov limit for systems of nonlinearly coupled oscillators without noise*, Transport Theor. Stat. Phys. **34** (2005), 523–535.
61. J. M. Levine, J. Bascompte, P. B. Adler, and S. Allesina, *Beyond pairwise coexistence: biodiversity maintenance in complex communities*, Nature **456** (2017), 56–64.
62. L. Lovász, *Large Networks and Graph Limits*, Amer. Math. Soc. **60** (2012), 475 pp.
63. L. Lovász and B. Szegedy, *Limits of dense graph sequences*, J. Comb. Theory. Ser. B **96** (2006), 933–957.
64. E. Luçon, *Quenched asymptotics for interacting diffusions on inhomogeneous random graphs*, Stoch. Process. Their Appl. **130** (2020), no. 11, 6783–6842.
65. M. Mattia and P. Del Giudice, *Efficient event-driven simulation of large networks of spiking neurons and dynamical synapses*, Neural Comput. **12** (2000), no. 10, 2305–2329.
66. G. Medvedev, *The nonlinear heat equation on dense graphs and graph limits*, SIAM J. Math. Anal. **46** (2014), no. 4, 883–898.
67. G. Medvedev, *The continuum limit of the Kuramoto model on sparse random graphs*, Commun. Math. Sci. **17** (2019), no. 4, 883–898.
68. J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE. **50** (1962), 2061–2070.
69. H. Neunzert, *An introduction to the nonlinear Boltzmann–Vlasov equation*, Kinetic Theories and the Boltzmann Equation, Lecture Notes in Mathematics, vol. 1048, Springer-Verlag, Berlin, Heidelberg, 1984.
70. H. Neunzert and J. Wick, *Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen*, R. Ansorge and W. Törnig (Eds.), Numerische Behandlung nichtlinearer Integrodifferential- und Differentialgleichungen, Lecture Notes in Mathematics, vol. 395, Springer, Berlin, Heidelberg, 1974, pp. 275–290.
71. R. I. Oliveira, G. H. Reis, and L. M. Stolerma, *Interacting diffusions on sparse graphs: hydrodynamics from local weak limits*, Electron. J. Probab. **25** (2020), 135.
72. A. Omurtag, B. W. Knight, and L. Sirovich, *On the simulation of large populations of neurons*, J. Comput. Neurosci. **8** (2000), no. 1, 51–63.
73. K. Pakdaman, B. Perthame, and D. Salort, *Dynamics of a structured neuron population*, Nonlinearity **23** (2010), no. 1, 55–75.
74. J. Park, D. Poyato, and J. Soler, *Filippov trajectories and clustering in the Kuramoto model with singular couplings*, J. Eur. Math. Soc. **23** (2021), 3193–3278.
75. E. Pastorelli, C. Capone, F. Simula, M. V. Sanchez-Vives, P. Del Giudice, M. Mattia, and P. S. Paolucci, *Scaling of a large-scale simulation of synchronous slow-wave and asynchronous awake-like activity of a cortical model with long-range interconnections*, Front. Syst. Neurosci. **13** (2019), 33.
76. B. Perthame and D. Salort, *On a voltage-conductance kinetic system for integrate & fire neural networks*, Kinet. Relat. Models **6** (2013), 841–864.
77. D. Pham and D. Karaboga, *Intelligent optimisation techniques: genetic algorithms, tabu search, simulated annealing and neural networks*, Springer Science & Business Media, London, 2012.
78. J. Pham, K. Pakdaman, J. Champagnat, and J.-F. Vibert, *Activity in sparsely connected excitatory neural networks: effect of connectivity*, Neural Netw. **11** (1998), no. 3, 415–434.
79. A. S. Pillai and V. K. Jirsa, *Symmetry breaking in space-time Hierarchies shapes brain dynamics and behavior*, Neuron **94** (2017), no. 5, 1010–1026.

80. J. W. Pillow, J. Shlens, L. Paninski, A. Sher, A. M. Litke, E. J. Chichilnisky, and E. P. Simoncelli, *Spatio-temporal correlations and visual signalling in a complete neuronal population*, Nature **454** (2008), 995–999.
81. D. Poyato, *Filippov flows and mean-field limits in the kinetic singular Kuramoto model*, Preprint arXiv:1903.01305.
82. D. Poyato and J. Soler, *Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models*, Math. Models Methods Appl. Sci. **27** (2017), no. 6, 1089–1152.
83. S. T. Rachev and L. Ruschendorf, *Mass transportation problems*, Springer-Verlag, 1998.
84. A. Renart, N. Brunel, and X. J. Wang, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, Computational neuroscience: A comprehensive approach, J. Feng (Ed.), CRC Press, Boca Raton, 2003, pp. 431–490.
85. H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, 2004.
86. O. Sporns, *Networks of the brain*, MIT Press, Cambridge, MA, 2010.
87. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 299–345.
88. A.-S. Sznitman, *Topics in propagation of chaos*, Ecole d'Été de Probabilités de Saint-Flour XIX -1989 P. L. Hennequin, (Ed.), Lecture Notes in Mathematics, vol. 1464, Springer, Berlin, Heidelberg, 1991, pp. 165–251.
89. V. S. Varadarajan, *On the convergence of sample probability distributions*, Sankhya **19** (1958), 23–26.
90. P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, vol. 79, Springer, New-York, 1982.
91. D. J. Watts and S. H. Strogatz, *Collective dynamics of small-world networks*, Nature **393** (1998), 440–442.
92. H. R. Wilson and J. D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophys. J. **12** (1972), 1–24.
93. M. M. Woodman and V. K. Jirsa, *Emergent dynamics from spiking neuron networks through symmetry breaking of connectivity*, PLoS ONE **8** (2013), no. 5, e64339.

APPENDIX A: MODELS OF BIOLOGICAL NEURONS

A.1 | Smooth neuron dynamics of the Fitzhugh–Nagumo or Hodgkin–Huxley type

Models of biological neurons offer a natural example which requires the more complex structure of the connectivities that we handle in this article. The purpose of this appendix is to give a very brief overview of some of the models, how they motivate our current study and what are the remaining open questions. There exists an extensive literature on the subject that we cannot do full justice to in this appendix. We first briefly refer to [40, 86], for a comprehensive point of view and to [23, 24] for the cortical network.

Typical models of neuron dynamics include the now famous Fitzhugh–Nagumo [36, 68], or Hodgkin–Huxley [45], which naturally fit in the framework of (1). In the case of Hodgkin–Huxley for example, one would take $X_i(t) = (V_i(t), n_i(t), m_i(t), h_i(t))$ where V_i is the membrane potential, (n_i, m_i) are connected to the activation for the potassium and sodium channels and h_i is connected to the inactivation for the Sodium channel. $(n_i(t), m_i(t), h_i(t))$ solve uncoupled ODE's that can be represented by adding some self-interaction in our model, for example

$$\begin{cases} \frac{d}{dt} n_i(t) = \alpha_n(V_i(t))(1 - n_i(t)) - \beta_n(V_i(t)) n_i(t), \\ \frac{d}{dt} m_i(t) = \alpha_m(V_i(t))(1 - m_i(t)) - \beta_m(V_i(t)) m_i(t), \\ \frac{d}{dt} h_i(t) = \alpha_h(V_i(t))(1 - h_i(t)) - \beta_h(V_i(t)) h_i(t), \end{cases} \quad (\text{A1})$$

for some given functions $\alpha_{n,m,h}$ and $\beta_{n,m,h}$.

The neurons are coupled through their membrane potentials solving

$$C_m \frac{dV_i}{dt} + g_K n_i^4 (V_i - V_K) + g_{Na} m_i^3 h_i (V_i - V_{Na}) + g_L (V_i - V_L) + \sum_{j \neq i} w_{ij} (V_i - V_j) = 0, \quad (\text{A2})$$

for given constants C_m , g_K , g_{Na} , g_L , V_K , V_{Na} , V_L .

The coupled system composed of (A1)–(A2) obviously fits within the general framework of systems like (1). Provided all coefficients are smooth, it also satisfies the assumptions of Theorem 1.1 so that all our results are immediately applicable.

The simulation of such large systems is challenging, and mean-field limits are again an attractive alternative; see [65, 72, 75, 84]. For identical weights $w_{ij} = 1/N$, the mean-field limit can be rigorously derived by classical methods and has proved useful in understanding some of the large scale behavior of such systems, see for example, [29, 30, 32, 37, 39] on the stochastic side or [16–19, 37, 73, 76] on the deterministic side.

Those studies do not account for the structure of connectivity in the neural network, whose effect has been shown to be critical, as in [78–80, 91–93]. Of course, a first practical modeling issue, when considering more complex networks, simply is that the connectivity map w_{ij} has long remain mysterious. Random weights w_{ij} are popular for this reason and naturally lead to random networks, see [4]. The corresponding graph of interactions is typically dense and some notion of mean-field limit, based on graphons, has been derived for example in [20–22] for smooth interaction kernels K , and actually for the so-called Kuramoto [56, 57] model without learning (see (A7) below with $\eta = 0$).

Those follow an interesting approach to the problem when the interacting network is described as a random graph, usually based on symmetric Erdős–Rényi type graphs. The family of Erdős–Rényi graphs is one example of a convergent family of random graphs. The limiting behavior of such families is determined by a symmetric measurable function on the unit square: a classical graphon. In the case of the Erdős–Rényi graphs, the limiting graphon is even a constant function. This also leads to investigating bifurcations in the Kuramoto models on a variety of graphs ranging from symmetric and random small-world and power law graphs. The graph structure plays a key role in the transition to synchronization in the model. In [22], Kuramoto models on Erdős–Rényi and Paley graphs have been proven to have the same critical value that defines synchronization, and the authors explain the onset of synchronization in the Kuramoto model on a broad class of dense and sparse random graphs by establishing an explicit link between the network structure and the onset of synchronization. Furthermore, the mean-field limits of the Kuramoto models on these graphs can be proved to coincide with that for the Kuramoto models on weighted complete graphs [20, 21].

However, from a scaling point of view, it is expected that the *graph of connection maps* w_{ij} should be *sparse* for applications in neurosciences. Mammalian brains contain between $10^8 - 10^{11}$ neurons ($86 \cdot 10^9$ actually in humans). Although, of course, the models generally apply only to subdomains or certain neural clusters in charge of some specific mission, the dimensions of these clusters is very large $N \gg 1$ and even these clusters tend to be correlated with others, which also implies very large dimensions. For example, in the human brain each neuron has on average 7000 synaptic connections to other neurons. This implies that synaptic weights must have a scale according to this number of the order of $\sup_{i,j} w_{ij} \sim 10^{-3} - 10^{-4} \ll 1$.

We also emphasize the importance of not imposing symmetry conditions to keep $w_{ij} \neq w_{ji}$ in general. In the neuronal framework, the breakdown in symmetry of network connectivity can be

correlated with a time scale hierarchy in some special situations. Small deviations from symmetric connectivity will cause small deviations from the invariant dynamics and introduce a temporal hierarchy, ensuring the consequence of an emergent separation on the time scale. Exploring network mechanisms due to asymmetric couplings to create more complex processes suggests that asymmetric coupling is not always a variation that needs to be averaged, since complex dynamics can be formed by simpler dynamics through a combination of asymmetric couplings. For example, stimulating a local part of neural tissue can trigger temporarily extended paths of motion by inciting the spread and sustained activation of motor networks. Asymmetries in connectivity can reshape the emerging dynamics of a neural network, see [79, 93].

The derivation of mean-field limits for neuron models with sparse, non-symmetric connection maps had remained fully open and is critical for biological applications. The question was made even more pressing by recent progress that have lead to the first fully detailed representation of the connections between neurons, for some simple animals such as the *Drosophila*, see for example [46]. The present article is the first to allow handling of arbitrary connection maps. However our investigations still leave a certain number of open questions which we briefly discuss below.

A.2 | Integrate and fire models

Many variants of models (A1)–(A2) have been studied to make the individual dynamics of a neuron more accurate. Those include adding noise to the system in various forms that often still fully compatible with our results.

However, so-called integrate and fire models are considered to better capture the dynamics of large networks; see for example [14, 15] or [1] for a historical perspective. Among many elaborate variants, a basic model is described here for simplicity. The state of each neuron is again represented by its membrane potential $V_i(t)$ which follows a simple SDE at almost all time,

$$dV_i(t) = b(V_i(t))dt + \sigma_N dW_i, \quad (\text{A3})$$

but with a probability to spike

$$\mathbb{P}(\text{spike between } t \text{ and } t + dt) \approx f(V(t))dt. \quad (\text{A4})$$

In the case of a spike at time t , V_i is reset at 0 while all connected neurons are activated or inhibited

$$V_i(t_+) = 0, \quad V_j(t_+) = V_j(t_-) + \alpha w_{ij} \quad \forall j \neq i. \quad (\text{A5})$$

There exist many variations around those models, for example to include more complex equations instead of (A3), or deterministic spiking when V_i reaches a given threshold. While (A3)–(A5) would formally fit in a modified version of (1), it includes *singular interactions* that prevent our results to apply. The nature of this singularity is however different from many of the other applications in this proposal, as it comes from the jumps in the V_i at spiking times, instead of unbounded interactions at $X_i = X_j$.

Still a full result on the derivation of integrate and fire models for general connectivities has been performed in [52], by building on the approach introduced here.

A.3 | Open problem: Learning mechanisms in neuron dynamics

Long-term perspectives also include understanding how to incorporate learning mechanisms into system (1). This is a key question for biological and artificial neuron dynamics as learning is of course a critical function of the network.

Learning can simply translate in having time dependent connection weights $w_{ij}(t)$ that evolve according to the dynamics of the agents themselves, see [2] for instance; this corresponds for example to the well-known Hebb rule that neurons wire together if they fire together. Considering again N neurons with activation X_i and time-dependent connectivity or synaptic weights $w_{ij}(t)$ between neuron i and j , a straightforward extension of (1) simply reads

$$\begin{cases} \frac{dX_i}{dt} = \sum_{j=1}^N w_{ij}(t) K(X_i, X_j) \quad (+\text{noise}), \\ \frac{dw_{ij}}{dt} = \eta (\Gamma(X_i, X_j) - w_{ij}) \quad (+\text{noise}), \\ X_i(0) = X_i^0, \quad w_{ij}(0) = w_{ij}^0, \end{cases} \quad (\text{A6})$$

for some plasticity function Γ , where η is regarded as the learning rate parameter.

How the mechanisms of learning can be reinforced or relaxed was specified in the neuronal context by Hebb [44]. Hebb proposed a theory of the adaptation of neurons in the brain during the learning process. The basic idea of his rule is based on the fact that neurons wire together if they fire together. Such a mechanism is based on the hypothesis that synchronous activation of cells (firing of neurons) leads to selectively pronounced increases in synaptic strength between these cells. From this process emerges the self-organized patterns. These postulates provide the neural basis for unsupervised complex learning of cognitive function in neural networks and may explain some processes that occur in the development of the nervous system.

The modeling is however made especially difficult in the present setting: changes in the $w_{ij}(t)$ need to be able to altering the structure of sparse networks while preserving bounds such as (3). If all w_{ij} are of the same order, which is typical of dense graphs, then system (A6) likely has a similar behavior to (1). However for sparse graphs, we expect the values of the w_{ij} to vary wildly and a straightforward system like (A6) just cannot correctly capture very different orders of magnitude in connectivities.

On the analytical side, weights w_{ij} that adapt in time can lead to stronger synchronization between agents, strengthening correlations between them and putting in doubt the propagation of independence. Useful hints can be derived from studies on the Kuramoto model [56, 57] of coupled oscillators that was briefly mentioned above. The system is posed on the frequencies θ_i and can read, with learning,

$$\begin{cases} \frac{d\theta_i}{dt} = \Omega_i + \frac{1}{N} \sum_{j=1}^N w_{ij} \sin(\theta_j - \theta_i), \\ \frac{dw_{ij}}{dt} = \eta (\Gamma(\theta_i, \theta_j) - w_{ij}) \quad (+\text{noise}), \\ \theta_i(0) = \theta_i^0, \quad w_{ij}(0) = w_{ij}^0, \end{cases} \quad (\text{A7})$$

where the Ω_i are the natural frequencies of oscillators.

For large η , the other variable in the model that is the choice of Γ takes on great relevance since, at the limit $\eta \rightarrow \infty$, we have that $w_{ij}(t) = \Gamma(\theta_i, \theta_j)$. If for example Γ is assumed to be $\Gamma(\theta_i, \theta_j) = \cos(\theta_i - \theta_j)$, the attraction between near oscillators is reinforced whereas repulsive interaction arises between apart phases. On the other hand if one takes $\Gamma(\theta_i, \theta_j) = C|\sin(\theta_i - \theta_j)|$, synchronization only slowly emerges due to the reduction in connectivities for nearby oscillators.

Hebbian-type dynamics can be considered for the Kuramoto model. This learning law ensures that the weight of the adaptive coupling increases if the phases of the oscillators are close. One possibility discussed in [74] is to assume that the Hebbian-like plasticity function Γ is given by

$$\Gamma(\theta, \theta') := \frac{1}{c_{\alpha, \zeta}^\alpha |\theta - \theta'|_0^{2\alpha}}, \quad (\text{A8})$$

where $|\theta - \theta'|_0$ is the orthodromic distance (to zero) over the unit circle and the parameter $c_{\alpha, \zeta} := 1 - \zeta^{-1/\alpha}$, $\zeta \in (0, 1]$, has been chosen so that whenever two phases θ_i and θ_j stay at orthodromic distance σ or larger, then the adaptive function Γ predicts a maximum degree of connectedness not larger than ζ between such oscillators. This is another way to create and amplify correlations between neurons.

However, at this stage, time-evolving connectivities do not fit within our framework. At least formally, note that when $t \gg 1$, weights behave as $w_{ij} \approx \Gamma(\theta_j, \theta_j)$. If Γ were bounded, it appears that our extended mean-field scaling should be satisfied (note that (A7) contains weights $\frac{w_{ij}}{N}$ instead of w_{ij}); but that does not apply anymore when using unbounded Γ as the Hebbian-like plasticity mentioned above.

APPENDIX B: TECHNICAL PROOFS FROM SUBSECTION 4.2

For the sake of completeness, we collect here the more technical proofs from subsection 4.2, starting with the proof of Lemma 4.7.

Proof of Lemma 4.7. Set any $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$, any $\phi \in L^\infty([0, 1])$ and any couple of sequences $\{w_n\}_{n \in \mathbb{N}}$ and $\{\phi_n\}_{n \in \mathbb{N}}$ as in the statement of the lemma. On the one hand, since $\phi \in L^1([0, 1])$, then

$$\int_0^1 \psi(\xi) \left(\int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi = \int_0^1 \phi(\zeta) \left(\int_0^1 \psi(\xi) w(d\xi, \zeta) \right) d\zeta \leq \|w\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^1} \|\psi\|_{L^\infty},$$

for any $\psi \in C([0, 1])$. This implies that $\int_0^1 \phi(\zeta) w(\cdot, d\zeta) \in \mathcal{M}([0, 1])$ and

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{\mathcal{M}_\xi} \leq \|w\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^1}. \quad (\text{B1})$$

Moreover, given any $\psi \in C([0, 1])$ we have that $\psi \otimes \phi_n \rightarrow \psi \otimes \phi$ strongly in $L_\xi^1 C_\xi$. Since in addition $w_n \xrightarrow{*} w$ weakly-star in $L_\zeta^\infty \mathcal{M}_\xi = (L_\xi^1 C_\xi)^*$, then we also obtain

$$\int_0^1 \psi(\xi) \left(\int_0^1 \phi_n(\zeta) w_n(\xi, d\zeta) \right) d\xi \rightarrow \int_0^1 \psi(\xi) \left(\int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi.$$

Due to the arbitrariness of ψ , the above amounts to

$$\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta) \xrightarrow{*} \int_0^1 \phi(\zeta) w(\cdot, d\zeta), \quad (\text{B2})$$

weakly-star in $\mathcal{M}([0, 1])$. On the other hand, set any $\psi \in C([0, 1])$ and another sequence $\{\tilde{\phi}_n\}_{n \in \mathbb{N}}$ as above consisting of continuous functions. It clearly exists by density of $C([0, 1])$ in $L^1([0, 1])$ and we can indeed assume that $\{\tilde{\phi}_n\}_{n \in \mathbb{N}} \in C([0, 1])$, $\tilde{\phi}_n \rightarrow \phi$ strongly in L^1 and $\|\tilde{\phi}_n\|_{L^\infty} \leq \|\phi\|_{L^\infty}$. Then,

$$\int_0^1 \psi(\xi) \left(\int_0^1 \tilde{\phi}_n(\zeta) w(\xi, d\zeta) \right) d\xi \leq \|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}} \|\psi\|_{L^1} \|\tilde{\phi}_n\|_{L^\infty} \leq \|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}} \|\psi\|_{L^1} \|\phi\|_{L^\infty}.$$

By the above weak-* convergence, we obtain

$$\int_0^1 \psi(\xi) \left(\int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi \leq \|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}} \|\psi\|_{L^1} \|\phi\|_{L^\infty},$$

for every $\psi \in C([0, 1])$. Noting that $C([0, 1])$ is dense in $L^1([0, 1])$, we find that the above Radon measure $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$ actually belongs to $L^1([0, 1])^* = L^\infty([0, 1])$ and

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L^\infty_{\xi}} \leq \|w\|_{L^\infty_{\xi} \mathcal{M}_{\xi}} \|\phi\|_{L^\infty},$$

which implies (33)₂. Using (B1) we also infer (33)₁ again because the Radon measure $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$ is absolutely continuous with respect to the Lebesgue measure and we already had an estimate in the total variation norm. To end the proof of (34) note that $\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta)$ are uniformly bounded in L^1 and L^∞ thanks to the already proven estimates (33). Since we already had that they converge weakly-star in $\mathcal{M}([0, 1])$ to $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$ by (B2), we conclude that they also converge weakly-star in L^∞ . \square

We now turn to the proof of Proposition 4.11.

Proof of Proposition 4.11. The argument relies on a direct Picard fixed point. Specifically, let us define

$$E := L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)), \quad E_R := \left\{ f \in E : \|f\|_{L^\infty_{t,\xi} W_x^{1,1} \cap W_x^{1,\infty}} < R \right\},$$

for any $R > 0$ to be determined later. It is straightforward to check that E_R is a closed subset of $L^1([0, t_*] \times [0, 1] \times \mathbb{R}^d)$ under the L^1 norm in all variables t , ξ and x . Then, endowed with the L^1 norm, E_R becomes a complete and convex metric space, which we shall use in the fixed point argument below.

Notice that the nonlinear PDE (36) can be reformulated as a fixed point equation in the usual way. Specifically, for any $f \in E$ let us set $\mathcal{L}f \in E$ to be the unique solution of the linear Cauchy

problem

$$\begin{cases} \partial_t \mathcal{L}f(t, x, \xi) + \operatorname{div}_x \left(\mathcal{L}f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = \nu \Delta_x \mathcal{L}f(t, x, \xi), \\ \mathcal{L}f(0, \cdot, \cdot) = f^0. \end{cases} \quad (\text{B3})$$

Hence, the fixed points of the operator \mathcal{L} amount to weak solutions of (36). The rest of the proof will focus on proving that the operator $\mathcal{L} : E \longrightarrow E$ is well defined, and we have that $\mathcal{L}(E_R) \subset E_R$ and it is contractive in the L^1 norm for appropriate $R > 0$.

◊ *Step 1: Well-posedness of (B3).*

We note that (B3) is a linear transport equation in $\mathcal{L}f$ parametrized by ξ and whose velocity field \mathcal{V}_f is given in Definition 4.9. Notice that integrating by parts we have

$$\nabla_x \mathcal{V}_f(t, x, \xi) = \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) \otimes \nabla_y f(t, y, \zeta) dy,$$

which is bounded uniformly in all variables by the regularity of $f \in E$. Therefore, the velocity field verifies $\mathcal{V}_f \in L^\infty([0, t_*] \times [0, 1], W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d))$. For $\nu = 0$ and for a.e. value $\xi \in [0, 1]$ we can associate a unique and well-defined flow solving the characteristic system

$$\begin{cases} \frac{dX_f}{dt}(t, s, x; \xi) = \mathcal{V}_f(t, X_f(t, s, x; \xi); \xi), \\ X_f(s, s, x; \xi) = x, \end{cases} \quad (\text{B4})$$

in the sense of Caratheodory. Hence, the classical results ensure that the unique solution of (B3) can be obtained for example by the method of characteristics, namely, $\mathcal{L}f(t, \cdot, \xi) = X_f(t, 0, \cdot; \xi)_\# f^0(\cdot, \xi)$ for $t \in [0, t_*]$ and a.e. $\xi \in [0, 1]$. In other words, we have that

$$\mathcal{L}f(t, x, \xi) = f_0(X_f(0, t, x; \xi)) J_f(0, t, x; \xi), \quad (\text{B5})$$

where $J_f(t, s, x; \xi) := \det(\nabla_x X_f(t, s, x; \xi))$ denotes the Jacobian associated to the flow.

For $\nu > 0$, we may instead use the heat kernel $G_\nu(t, x)$ and we can again classically obtain a solution through the mild formulation

$$\begin{aligned} \mathcal{L}f(t, x, \xi) &= G_\nu(t, \cdot) \star_x f^0(x, \xi) \\ &+ \int_0^t \int_{\mathbb{R}^d} G_\nu(t-s, x-y) \operatorname{div}_y \left(\mathcal{L}f(s, y, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(y-z) f(s, z, \zeta) dz \right) ds dy. \end{aligned} \quad (\text{B6})$$

◊ *Step 2: A priori estimates of (B3).*

We first recall classical estimates for advection and advection-diffusion equations. Consider any $v(t, x) \in L^\infty([0, t_*], W^{1,\infty}(\mathbb{R}^d))$, any weak solution u in L^1_{loc} to

$$\begin{cases} \partial_t u + \operatorname{div}_x(u v) = \nu \Delta_x u + R(t, x), \\ u(0, x) = u^0(x). \end{cases} \quad (\text{B7})$$

Then if $R \in L^1_{t,x}$, we have that $u \in L^\infty_t L^1_x$ with

$$\|u(t, \cdot)\|_{L^1_x} \leq \|u^0\|_{L^1_x} + \int_0^t \|R(s, \cdot)\|_{L^1_x} ds. \quad (\text{B8})$$

If $R \in L^1_t L^\infty_x$, then $u \in L^\infty_{t,x}$ with

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty_x} &\leq \|u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &+ \int_0^t \|R(s, \cdot)\|_{L^\infty_x} \exp\left((t-s) \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) ds. \end{aligned} \quad (\text{B9})$$

If additional derivatives are available on u^0 , v and R , Sobolev bounds are also propagated. Provided that v is C^1 , u is automatically a classical solution to (B7) so that

$$\partial_t \nabla_x u + \operatorname{div}(v \nabla_x u) + \operatorname{div}(\nabla_x v u) = v \Delta_x \nabla_x u + \nabla_x R.$$

In particular, we may apply estimates similar to the previous ones as $\nabla_x u$ solves (B7) with $\tilde{R} = \nabla_x R - \operatorname{div}(\nabla_x v u)$. We consequently find that, if $u^0 \in W^{1,1}_x$ and $R = 0$ for simplicity, then

$$\begin{aligned} \|\tilde{R}(t, \cdot)\|_{L^1_x} &\leq \|\operatorname{div} v(t, \cdot)\|_{W^{1,\infty}_x} \|u(t, \cdot)\|_{L^1_x} + \|v(t, \cdot)\|_{W^{1,\infty}_x} \|\nabla_x u(t, \cdot)\|_{L^1_x} \\ &\leq \|\operatorname{div} v(t, \cdot)\|_{W^{1,\infty}_x} \|u^0\|_{L^1_x} + \|v(t, \cdot)\|_{W^{1,\infty}_x} \|\nabla_x u(t, \cdot)\|_{L^1_x}. \end{aligned}$$

Hence we obtain by Gronwall's lemma that

$$\begin{aligned} \|\nabla_x u(t, \cdot)\|_{L^1_x} &\leq \|\nabla_x u^0\|_{L^1_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &+ \|u^0\|_{L^1} \int_0^t \|\operatorname{div} v(s, \cdot)\|_{W^{1,\infty}_x} \exp\left((t-s) \|v\|_{L^\infty_t W^{1,\infty}_x}\right) ds. \end{aligned} \quad (\text{B10})$$

Similarly if $u^0 \in W^{1,\infty}_x$ and $R = 0$, then

$$\begin{aligned} \|\tilde{R}(t, \cdot)\|_{L^\infty_x} &\leq \|\operatorname{div} v(t, \cdot)\|_{W^{1,\infty}_x} \|u(t, \cdot)\|_{L^\infty_x} + \|v(t, \cdot)\|_{W^{1,\infty}_x} \|\nabla_x u(t, \cdot)\|_{L^\infty_x} \\ &\leq \|\operatorname{div} v(t, \cdot)\|_{W^{1,\infty}_x} \|u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) + \|v(t, \cdot)\|_{W^{1,\infty}_x} \|\nabla_x u(t, \cdot)\|_{L^\infty_x}, \end{aligned}$$

and one has that, still by Gronwall's lemma,

$$\begin{aligned} \|\nabla_x u(t, \cdot)\|_{L^\infty_x} &\leq \|\nabla_x u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &+ \|u^0\|_{L^\infty} \int_0^t \|\operatorname{div} v(s, \cdot)\|_{W^{1,\infty}_x} \exp\left((t-s) (\|v\|_{L^\infty_t W^{1,\infty}_x} + \|\operatorname{div}_x v\|_{L^\infty_{t,x}})\right) ds. \end{aligned} \quad (\text{B11})$$

We may now propagate estimates to show that the solution $\mathcal{L}f$ lies in E , and indeed $\mathcal{L}(E_R) \subset E_R$ for an appropriate choice of $R > 0$. We note that Equation (B3) is immediately under the form (B7)

where ξ is only a parameter and

$$v_\xi(t, x) = V_f(t, x, \xi) = \int_0^1 \omega(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, y, \zeta) dy.$$

From here, since $f^0 \in L_\xi^\infty L_x^1$ we obtain that

$$\begin{aligned} \|\mathcal{L}f(t, \cdot, \xi)\|_{L_x^1} &= \|f^0(\cdot, \xi)\|_{L_x^1}, \\ \|\mathcal{L}f(t, \cdot, \xi)\|_{L_x^\infty} &\leq \|f^0(\cdot, \xi)\|_{L^\infty} \exp\left(t_* \|\operatorname{div}_x \mathcal{V}_f\|_{L_{t,x,\xi}^\infty}^{L_x^\infty}\right), \end{aligned} \quad (\text{B12})$$

for $t \in [0, t_*]$ and a.e. $\xi \in [0, 1]$.

From the definition of V_f , we also directly obtain that

$$\begin{aligned} \|\operatorname{div}_x V_f\|_{L_{t,x,\xi}^\infty}^{L_x^\infty} &\leq \|K\|_{L^\infty} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\nabla_x f\|_{L_{t,\xi}^\infty L_x^1} \\ \|V_f\|_{L_{t,\xi}^\infty W_x^{1,\infty}} &\leq \|K\|_{L^\infty} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\nabla_x f\|_{L_{t,\xi}^\infty L_x^1} \\ \|\operatorname{div} V_f\|_{L_{t,\xi}^\infty W_x^{1,\infty}} &\leq \|\operatorname{div} K\|_{L^1} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\nabla_x f\|_{L_{t,\xi}^\infty L_x^\infty}. \end{aligned} \quad (\text{B13})$$

Putting (B12), and (B13) together into (B10) and (B11) implies that

$$\begin{aligned} \|\mathcal{L}f\|_{L_{t,\xi}^\infty W_x^{1,1} \cap W_x^{1,\infty}} &\leq C_1 \|f^0\|_{L_\xi^\infty W_x^{1,1} \cap W^{1,\infty}} \\ &\quad \times \exp\left(C_2 t_* (\|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|f\|_{L_{t,\xi}^\infty W_x^{1,1} \cap W_x^{1,\infty}}\right), \end{aligned} \quad (\text{B14})$$

for some universal $C_1, C_2 \geq 1$. In particular, the above estimate show that $\mathcal{L}f \in E$ for every $f \in E$.

♦ *Step 3: Invariance of E_R and contractivity of \mathcal{L} .*

Set now any $R > C_1 \|f^0\|_{L_\xi^\infty W_x^{1,1} \cap W_x^{1,\infty}}$ and note that the above estimates also show that $\mathcal{L}(E_R) \subset E_R$ if the maximal time t_* is taken sufficiently small, namely,

$$0 < t_* \leq t_1(R) := \frac{1}{C_2 (\|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} R} \log\left(\frac{R}{C_1 \|f^0\|_{L_\xi^\infty W_x^{1,1} \cap W_x^{1,\infty}}}\right).$$

Our next goal is to show that \mathcal{L} is contractive under the L^1 norm. To do so, let us consider $f, g \in E_R$ and note that $f - g$ follows the following linear transport equation with a source term

$$\begin{cases} \partial_t(\mathcal{L}f - \mathcal{L}g) + \operatorname{div}_x(\mathcal{V}_g(\mathcal{L}f - \mathcal{L}g)) + \operatorname{div}_x((\mathcal{V}_f - \mathcal{V}_g)\mathcal{L}f) = \nu \Delta_x(\mathcal{L}f - \mathcal{L}g), \\ (\mathcal{L}f - \mathcal{L}g)(0, \cdot, \cdot) = 0. \end{cases}$$

Taking L^1 norms with respect to x and ξ yields

$$\begin{aligned} \|(\mathcal{L}f - \mathcal{L}g)(t, \cdot, \cdot)\|_{L_{x,\xi}^1} &\leq \|\operatorname{div}_x((\mathcal{V}_f - \mathcal{V}_g)\mathcal{L}f)\|_{L_{t,x,\xi}^1} \\ &\leq \|\operatorname{div}_x(\mathcal{V}_f - \mathcal{V}_g)\|_{L_{t,\xi}^\infty L_x^1} \|\mathcal{L}f\|_{L_{t,x,\xi}^\infty}^{L_x^\infty} + \|\mathcal{V}_f - \mathcal{V}_g\|_{L_{t,\xi}^\infty L_x^1} \|\nabla_x \mathcal{L}f\|_{L_{t,x,\xi}^\infty}^{L_x^\infty}, \end{aligned} \quad (\text{B15})$$

for $t \in [0, t_*]$. By the estimate (35)₁ in Lemma 4.8 we have

$$\begin{aligned} \|\mathcal{V}_f - \mathcal{V}_g\|_{L^1_{t,x,\xi}} &\leq \|K\|_{L^1} \|w\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} \|f - g\|_{L^1_{t,x,\xi}}, \\ \|\operatorname{div}_x(\mathcal{V}_f - \mathcal{V}_g)\|_{L^1_{t,x,\xi}} &\leq \|\operatorname{div} K\|_{L^1} \|w\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} \|f - g\|_{L^1_{t,x,\xi}}. \end{aligned} \quad (\text{B16})$$

Then, putting (B15) and (B16) together, and using that $\mathcal{L}f \in E_R$ by the above step, we have that

$$\|\mathcal{L}f - \mathcal{L}g\|_{L^1_{t,x,\xi}} \leq R t_* (\|K\|_{L^1} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} \|f - g\|_{L^1_{t,x,\xi}}. \quad (\text{B17})$$

Take t_* sufficiently small so that $t_* \leq t_1(R)$ as above and, in addition

$$0 < t_* < t_2(R) := \frac{1}{R(\|K\|_{L^1} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi}}.$$

Hence \mathcal{L} is contractive on E_R under the L^1 norm by the stability estimate (B17). By the Banach fixed point theorem there must exist a (unique) fixed point $f \in E_R$ of \mathcal{L} , leading to a weak solution of (36) in $[0, t_*]$. Note that t_* must be taken small enough so that $0 < t_* < \min\{t_1(R), t_2(R)\}$. However, restarting the argument from t_* allows extending this weak solution solution to any finite time interval as usual, since the above a priori estimate (B14) above does not blow up in time. \square

We finish this technical appendix with the proof of Lemma 4.12.

Proof of Lemma 4.12. As usual, one uses an approximation by convolution. Because w is defined on $[0, 1]^2$, we first extend w as a periodic function on with period 1 in both variables. Similarly we naturally consider any function on $[0, 1]$ as periodic, allowing to naturally use convolutions. Then, choose any convolution kernel $L_n \geq 0$ with $L_n \in C_c^\infty(\mathbb{R})$ for any fixed n and define $w_n = L_n \otimes L_n \star_{\xi,\zeta} w$. Since $w \in \mathcal{M}_{\xi,\zeta}$, we immediately note that $w_n \in L_{\xi,\zeta}^\infty$.

Next, for any $f \in L_{\xi,\zeta}^1$, we trivially have that $\|L_n \otimes L_n \star f\|_{L_{\xi,\zeta}^1} \leq \|f\|_{L_{\xi,\zeta}^1}$. By duality, we deduce that w_n is uniformly bounded in $L_{\xi,\zeta}^\infty \mathcal{M}_\zeta \cap L_{\xi,\zeta}^\infty \mathcal{M}_\xi$ and, hence, in $L_{\xi,\zeta}^\infty L_{\xi,\zeta}^1 \cap L_{\xi,\zeta}^\infty L_{\xi,\zeta}^1$, since $w_n \in L_{\xi,\zeta}^\infty$.

Consider now any ϕ_n converging weakly to ϕ in $H^1([0, 1])$, we have that

$$\int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta = \int_0^1 \tilde{L}_n \star_\zeta \phi_n(\zeta) w(\xi, \zeta) d\zeta,$$

with $\tilde{L}_n(\xi) = L_n(-\xi)$. By Lemma 4.7, for any $\psi \in L^\infty([0, 1])$, we deduce that

$$\begin{aligned} \int_0^1 \psi(\xi) \left(\int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi \\ \leq \|\psi\|_{L^\infty} \|w\|_{L_{\xi,\zeta}^\infty \mathcal{M}_\zeta \cap L_{\xi,\zeta}^\infty \mathcal{M}_\xi} \|\phi - \tilde{L}_n \star \phi_n\|_{L^1}. \end{aligned}$$

Since H^1 is compactly embedded in L^1 , this shows that

$$\left\| \int_0^1 \phi_n(\zeta) L_n \star_{\zeta} w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_{\xi}^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Still by Lemma 4.7 since $H^1 \subset L^{\infty}$, $\int_0^1 \phi(\zeta) w(\xi, d\zeta)$ is a fixed function in L^{∞} and consequently its convolution by L_n converges in L^1 (and in all L^p , $p < \infty$). Therefore

$$\begin{aligned} & \left\| \int_0^1 \phi_n(\zeta) w_n(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_{\xi}^1} \\ & \leq \left\| L_n \star_{\xi} \left(\int_0^1 \phi_n(\zeta) L_n \star_{\zeta} w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) \right\|_{L_{\xi}^1} \\ & \quad + \left\| L_n \star_{\xi} \int_0^1 \phi_n(\zeta) w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_{\xi}^1} \rightarrow 0, \end{aligned}$$

proving the convergence of w_n in $L_{\xi}^1 H_{\xi}^{-1}$. □