

Article

Modeling Directional Monotonicity in Sequence with Copulas

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Abstract: In this paper, we present the concept of being monotonic in sequence according to a specific direction for a collection of random variables. This concept broadens the existing notions of multivariate dependence, such as sequential left-tail and right-tail dependence. Furthermore, we explore connections with other multivariate dependence concepts, highlight key properties, and analyze the new concept within the framework of copulas. Several examples are provided to demonstrate our findings.

Keywords: copula; monotonic in sequence dependence; random variable

MSC: 60E15; 62H05

1. Introduction

There are several approaches to describing dependence relationships among random variables, and, as noted by Jogdeo [1], “this is one of the most widely studied objects in probability and statistics”. For a multivariate model, it is essential to analyze the type of dependence structure that it incorporates to determine its suitability for a specific application or dataset. This study focuses on positive and negative dependence, with positive dependence defined as any criterion that mathematically captures the tendency of components within an n -variate random vector to display concordant values [2]. According to Barlow and Proschan [3], concepts of (positive) dependence become notably more diverse and intricate in the multivariate context compared to the bivariate one.

Several generalizations of bivariate dependence notions to the multivariate setting have been explored in the literature (see, for example, [4,5]). In this paper, our objective is to extend some established multivariate dependence concepts—both positive and negative—such as orthant dependence and tail monotonicity to examine relationships with other dependence structures and to outline various properties.

Aggregation functions are crucial in numerous applications, including fuzzy set theory and fuzzy logic [6], among other fields. Copulas, which are multivariate distribution functions with uniform univariate margins on $[0, 1]$, represent a specific kind of conjunctive aggregation function. They are commonly applied in aggregation, as they ensure stability, meaning that minor input errors lead to minor output errors [7]. This paper investigates the new dependence concepts through the lens of copulas.

The paper is organized as follows. We begin with preliminaries on (multivariate) dependence properties in Section 2. Section 3 introduces the concept of monotonic in sequence random variables in a given direction, alongside properties and examples. Section 4 explores the concept through copulas, focusing first on the bivariate and trivariate cases, and then extending to the general case. Finally, conclusions are presented in Section 5.



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2. Preliminaries

In the following, we use the terms “increasing” (or “decreasing”) interchangeably with “nondecreasing” (or “nonincreasing”), unless specified otherwise. Additionally, a subset $A \subseteq \mathbb{R}^d$, with $d \geq 1$, is called an *increasing set* if its indicator function χ_A is increasing.

Let $n \geq 2$ be a natural number. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a nonempty set, $\mathcal{F}$ is a σ -algebra of subsets of Ω , and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a random vector from Ω to $\mathbb{R}^n$, composed of n random variables $X_i : \Omega \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n$. In this context, we assume that the random vector $\mathbf{X}$ is continuous.

We now summarize several established concepts of multivariate dependence.

Orthant dependence according to a direction is defined as follows [8]: Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a vector in $\mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. An n -variate random vector $\mathbf{X}$ (or its joint distribution function F) is said to be *orthant positive* (respectively, *orthant negative*) *dependently according to the direction* α , denoted $\text{PD}(\alpha)$ (respectively, $\text{ND}(\alpha)$), if

$$\mathbb{P}\left[\bigcap_{i=1}^n (\alpha_i X_i > x_i)\right] \geq \prod_{i=1}^n \mathbb{P}[\alpha_i X_i > x_i] \quad \text{for all } x_i \in \mathbb{R} \quad (1)$$

(or, respectively, with the inequality in (1) reversed).

For certain choices of the direction α , such as $\alpha = \mathbf{1} = (1, 1, \dots, 1)$ or $\alpha = -\mathbf{1} = (-1, -1, \dots, -1)$, we retrieve well-known dependence concepts, including positive quadrant dependence and positive upper orthant dependence (for further details, see [2,5,9–11]). Additional related concepts in multivariate total positivity by direction can be found in [12].

For a pair of random variables (X_1, X_2) , two bivariate positive dependence notions are introduced in [13]: left-tail decreasing (LTD) and right-tail increasing (RTI). Specifically, X_2 is said to be *left-tail decreasing* (or *right-tail increasing*) in X_1 if $\mathbb{P}[X_2 \leq x_2 | X_1 \leq x_1]$ (or $\mathbb{P}[X_2 > x_2 | X_1 > x_1]$) is a nonincreasing (or nondecreasing) function of x_1 for all x_2 —the negative dependence counterparts are defined by reversing the inequalities. For instance, in the LTD concept above, the probability of X_2 being less than or equal to any x_2 given that $X_1 \leq x_1$ increases as x_1 decreases, indicating positive dependence. Multivariate extensions of RTI and LTD are presented in [14,15]. A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be *left-tail decreasing in sequence* (LTDS) if

$$\mathbb{P}[X_i \leq x_i | X_1 \leq x_1, \dots, X_{i-1} \leq x_{i-1}]$$

is decreasing in $x_1, \dots, x_{i-1}$ for all x_i , $i \in \{2, \dots, n\}$; $\mathbf{X}$ is *right-tail increasing in sequence* (RTIS) if

$$\mathbb{P}[X_i > x_i | X_1 > x_1, \dots, X_{i-1} > x_{i-1}]$$

is increasing in $x_1, \dots, x_{i-1}$ for all x_i , $i \in \{2, \dots, n\}$. For the properties of these notions and their relationships to other multivariate dependence concepts, see [2,16].

In the following section, we generalize these multivariate dependence concepts according to a direction.

3. Monotonic in Sequence According to a Direction

In this section, we introduce the concepts of left-tail and right-tail dependence in sequence according to a direction for a set of random variables, generalizing the LTDS and RTIS concepts presented in Section 2. We also provide a characterization of these notions and examine several of their key properties.

3.1. Definition and Characterization

Definition 1. Let $X_1, X_2, \dots, X_n$ be n random variables, and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. The random variables $X_1, X_2, \dots, X_n$ are said to be *increasing*

(or decreasing) in sequence according to the direction α —denoted by $IS(\alpha)$ (or $DS(\alpha)$)—if, for any $x_i \in \mathbb{R}$,

$$\mathbb{P}[\alpha_i X_i > x_i \mid \alpha_1 X_1 > x_1, \dots, \alpha_{i-1} X_{i-1} > x_{i-1}]$$

is nondecreasing (or nonincreasing) in $x_1, x_2, \dots, x_{i-1} \in \mathbb{R}$ for all $i = 2, 3, \dots, n$.

From here, we focus primarily on the $IS(\alpha)$ concept. Parallel results apply to $DS(\alpha)$, so we omit them for brevity. Additionally, we refer to a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ —or its joint distribution function—as being $IS(\alpha)$.

The $IS(\alpha)$ concept allows for an analysis of how high or low values in one variable can directionally influence other variables in the sequence. This concept extends bivariate dependence to the multivariate case and provides a directional analysis that is not fully covered by previous approaches. To be precise, it implies that large values of the variables X_j , for indices $j \in J$, correspond to small values for variables X_i , where $j \in I \setminus J$, with $I = \{1, 2, \dots, n\}$ and $J = \{i \in I : \alpha_i = 1\}$. Consequently, if $\mathbf{X}$ is $IS(\alpha)$, then, for all $i = 2, 3, \dots, n$ and any $x_i \in \mathbb{R}$, we have that

$$\mathbb{P}[X_i > x_i \mid \cap_{j \in J_1} (X_j > x_j), \cap_{j \in I_1 \setminus J_1} (X_j < x_j)]$$

is nondecreasing (or nonincreasing) in $x_1, x_2, \dots, x_{i-1}$ if $\alpha_i = 1$ (or $\alpha_i = -1$), where $I_1 = \{1, 2, \dots, i-1\}$ and $J_1 = \{i \in I_1 : \alpha_i = 1\}$ (or $J_1 = \{i \in I_1 : \alpha_i = -1\}$). Furthermore, observe also that the $IS(\alpha)$ concept generalizes the LTDS and RTIS concepts introduced in Section 2: $IS(1)$ is RTIS and $IS(-1)$ is LTDS.

Next, we provide a useful characterization of the $IS(\alpha)$ concept, though a preliminary definition is needed.

Definition 2. Let $X_1, X_2, \dots, X_n$ be n random variables, and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. The random variables $X_1, X_2, \dots, X_n$ are said to be stochastically increasing (or decreasing) in sequence according to a direction $\alpha \in \mathbb{R}^n$ —denoted by $SIS(\alpha)$ (or $SDS(\alpha)$)—if

$$\mathbb{E}[f(\alpha_i X_i) \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] \quad (2)$$

is nondecreasing (or nonincreasing) in $x_1, x_2, \dots, x_{i-1}$ for any real-valued, non-decreasing function f and for each i , with $2 \leq i \leq n$. We will also say that the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ —or its joint distribution function—is $SIS(\alpha)$ (or $SDS(\alpha)$).

In the next result, we characterise the $IS(\alpha)$ concept in terms of $SIS(\alpha)$. Similar results can be formulated for $DS(\alpha)$ and $SDS(\alpha)$.

Theorem 1. A random vector is $IS(\alpha)$ if, and only if, it is $SIS(\alpha)$.

Proof. Suppose the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is $SIS(\alpha)$, and consider, for each i and any x_i , the function

$$f(\alpha_i X_i) = \begin{cases} 0, & \alpha_i X_i \leq x_i, \\ 1, & \alpha_i X_i > x_i. \end{cases}$$

Then, we have that

$$\mathbb{E}[f(\alpha_i X_i) \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] = \mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)]$$

is nondecreasing in $x_1, x_2, \dots, x_{i-1}$ and, hence, $\mathbf{X}$ is $IS(\alpha)$.

Conversely, if $\mathbf{X}$ is $IS(\alpha)$, then we have that

$$\mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] = \mathbb{E}[\chi_{\{\alpha_i X_i > x_i\}} \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)]$$

is nondecreasing in $x_1, x_2, \dots, x_{i-1}$. Thus, for any simple function f that is non-negative and nondecreasing, the expression in (2) remains nondecreasing in $x_1, x_2, \dots, x_{i-1}$. Utilizing the monotone convergence theorem confirms that this property holds for all nondecreasing (and non-negative) functions. Consequently, we conclude that $\mathbf{X}$ is $\text{SIS}(\alpha)$, thereby completing the proof. $\square$

3.2. Relationships with Other Multivariate Dependence Concepts

In this subsection, we explore the connections between the $\text{IS}(\alpha)$ concept and several established multivariate dependence notions in relation to a specific direction. The initial result demonstrates the link between the $\text{IS}(\alpha)$ and $\text{PD}(\alpha)$ concepts, as expressed in (1).

Proposition 1. *If the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is $\text{IS}(\alpha)$, then it is $\text{PD}(\alpha)$.*

Proof. Let $x_1, x_2, \dots, x_n \in \mathbb{R}$. Since $\mathbf{X}$ is $\text{IS}(\alpha)$ then, for every $2 \leq i \leq n$, we have

$$\mathbb{P}[\alpha_i X_i > x_i | \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] \geq \mathbb{P}[\alpha_i X_i > x_i | \cap_{j=1}^{i-1} (\alpha_j X_j > x'_j)]$$

whenever $x_j \geq x'_j$ for all $j = 1, 2, \dots, i-1$. Therefore,

$$\mathbb{P}[\alpha_i X_i > x_i | \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] \geq \mathbb{P}[\alpha_i X_i > x_i]$$

by letting $x'_j \rightarrow -\infty$ for $j = 1, 2, \dots, i-1$. Thus,

$$\begin{aligned} \mathbb{P}[\cap_{i=1}^n (\alpha_i X_i > x_i)] &= \mathbb{P}[\alpha_1 X_1 > x_1] \cdot \prod_{i=2}^n \mathbb{P}[\alpha_i X_i > x_i | \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] \\ &\geq \prod_{i=1}^n \mathbb{P}[\alpha_i X_i > x_i], \end{aligned}$$

i.e., $\mathbf{X}$ is $\text{PD}(\alpha)$. $\square$

For the forthcoming results, we review several multivariate dependence concepts based on a specific direction. For any vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, we define $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \dots, \min(x_n, y_n))$.

Definition 3 ([12]). Let $\mathbf{X}$ be an n -dimensional random vector with joint density function f , and $\alpha \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. The random vector $\mathbf{X}$ is referred to as being multivariate totally positive of order two according to the direction α —denoted by $\text{MTP}_2(\alpha)$ —if

$$f(\alpha(\mathbf{x} \vee \mathbf{y}))f(\alpha(\mathbf{x} \wedge \mathbf{y})) \geq f(\alpha\mathbf{x})f(\alpha\mathbf{y})$$

holds for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$.

It follows from Proposition 2 and Theorem 3 in [12] that an n -dimensional random vector $\mathbf{X}$ is $\text{MTP}_2(\alpha)$ if, and only if, $\alpha\mathbf{X}$ is $\text{MTP}_2(\mathbf{1})$ —or simply MTP_2 .

Definition 4 ([17]). Let $\mathbf{X}$ be an n -dimensional random vector, and $\alpha \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. The random vector $\mathbf{X}$ is said to be increasing (or decreasing) according to the direction α —denoted by $I(\alpha)$ (or $D(\alpha)$)—if

$$\mathbb{P}[\alpha_1 X_1 > x_1, \alpha_2 X_2 > x_2, \dots, \alpha_n X_n > x_n | \alpha_1 X_1 > x'_1, \alpha_2 X_2 > x'_2, \dots, \alpha_n X_n > x'_n]$$

is non-decreasing (or non-increasing) in $x'_1, x'_2, \dots, x'_n$ for all $x_1, x_2, \dots, x_n$.

It is important to note that the notion $I(\mathbf{1})$ (or $D(-\mathbf{1})$) generalizes the established concept of RCSI (or LCSD), as discussed in [18] for the bivariate case and [5] for the multivariate context.

If a random vector $\mathbf{X}$ is $\text{MTP}_2(\alpha)$, then it is known to be $\text{I}(\alpha)$ [17]. The next result illustrates the relationship between the concepts $\text{I}(\alpha)$ and $\text{IS}(\alpha)$.

Proposition 2. *If the random vector $\mathbf{X}$ is $\text{I}(\alpha)$, then it is $\text{IS}(\alpha)$.*

Proof. Let $x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n \in \mathbb{R}$ such that $x_i \leq x'_i$ for $1 \leq i \leq n$. Since $\mathbf{X}$ is $\text{I}(\alpha)$, for any $i, 2 \leq i \leq n$, we have

$$\begin{aligned} \mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] &= \mathbb{P}[\cap_{j=1}^n (\alpha_j X_j > t_j) \mid \cap_{j=1}^n (\alpha_j X_j > s_j)] \\ &\leq \mathbb{P}[\cap_{j=1}^n (\alpha_j X_j > t_j) \mid \cap_{j=1}^n (\alpha_j X_j > s'_j)] \\ &= \mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x'_j)] \end{aligned}$$

where

$$t_j = \begin{cases} x_i, & j = i, \\ -\infty, & j \neq i, \end{cases}$$

$$s_j = \begin{cases} x_j, & \text{for } j = 1, 2, \dots, i-1, \\ -\infty, & \text{for } j = i, \dots, n, \end{cases}$$

and

$$s'_j = \begin{cases} x'_j, & \text{for } j = 1, 2, \dots, i-1, \\ -\infty, & \text{for } j = i, \dots, n. \end{cases}$$

Therefore, $\mathbf{X}$ is $\text{IS}(\alpha)$, which concludes the proof. $\square$

It should be noted that the converse of Proposition 2 does not hold, as illustrated in Example 9 in Section 4.

From Proposition 2, and given that $\text{MTP}_2(\alpha)$ implies $\text{I}(\alpha)$, we can derive the following result.

Corollary 1. *If the random vector $\mathbf{X}$ is $\text{MTP}_2(\alpha)$, then it is $\text{IS}(\alpha)$.*

3.3. Properties

The following results are properties of the $\text{IS}(\alpha)$ families, which include results for independent random variables, subsets of the new concept $\text{IS}(\alpha)$, the concatenation of $\text{IS}(\alpha)$ random vectors, weak convergence, etc.

Proposition 3. *A set of independent random variables is $\text{IS}(\alpha)$ for any $\alpha \in \mathbb{R}^n$.*

Proof. Let $X_1, X_2, \dots, X_n$ be n independent random variables. For any $\alpha \in \mathbb{R}^n$, we have

$$\mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)] = \mathbb{P}[\alpha_i X_i > x_i]$$

for all $i = 2, \dots, n$ and any x_i , whence it is immediate that the random variables are $\text{IS}(\alpha)$. $\square$

Proposition 4. *Any subset of random variables that are $\text{IS}(\alpha)$ are also $\text{IS}(\alpha^*)$, where α^* is defined as the vector formed by omitting the components of α that correspond to the random variables not present in the subset.*

Proof. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a vector of n random vector that is $\text{IS}(\alpha)$, and let $\mathbf{X}_k = \{X_{i_1}, X_{i_2}, \dots, X_{i_k}\}$ represent a subset of $\mathbf{X}$. If x_j approaches $-\infty$ for each $j = 1, \dots, i-1$ such that $j \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_k\}$ in the expression

$$\mathbb{P}[\alpha_i X_i > x_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)]$$

for $i \in \{i_1, i_2, \dots, i_k\}$, then we find that

$$\mathbb{P}[\alpha_i X_i > x_i \mid \cap_{i_j < i} (\alpha_{i_j} X_{i_j} > x_{i_j})]$$

is nondecreasing in all x_{i_j} with $i_j < i$, for any x_i . Consequently, $\mathbf{X}_k$ is $\text{IS}(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$. $\square$

Proposition 5. If the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is $\text{IS}(\alpha)$, and $g_1, g_2, \dots, g_n$ are n strictly increasing real-valued functions, then the random vector $(g_1(X_1), g_2(X_2), \dots, g_n(X_n))$ also satisfies the property of being $\text{IS}(\alpha)$.

Proof. Let $y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n \in \mathbb{R}$ such that $y_i \leq y'_i$ for $i = 1, 2, \dots, n$. Since $\mathbf{X}$ is $\text{IS}(\alpha)$ and $\alpha_j g_j^{-1}(\alpha_j y_j) \leq \alpha_j g_j^{-1}(\alpha_j y'_j)$ for every $j = 1, 2, \dots, n$, we have

$$\begin{aligned} \mathbb{P}[\alpha_i g_i(X_i) > y_i \mid \cap_{j=1}^{i-1} (\alpha_j g_j(X_j) > y_j)] &= \mathbb{P}[\alpha_i X_i > \alpha_i g_i^{-1}(\alpha_i y_i) \mid \cap_{j=1}^{i-1} (\alpha_j X_j > \alpha_j g_j^{-1}(\alpha_j y_j))] \\ &\leq \mathbb{P}[\alpha_i X_i > \alpha_i g_i^{-1}(\alpha_i y_i) \mid \cap_{j=1}^{i-1} (\alpha_j X_j > \alpha_j g_j^{-1}(\alpha_j y'_j))] \\ &= \mathbb{P}[\alpha_i g_i(X_i) > y_i \mid \cap_{j=1}^{i-1} (\alpha_j g_j(X_j) > y'_j)], \end{aligned}$$

i.e., $(g_1(X_1), g_2(X_2), \dots, g_n(X_n))$ is $\text{IS}(\alpha)$. $\square$

For the subsequent result, given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m$, we define (α, β) to represent the concatenation, which is expressed as

$$(\alpha, \beta) = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m) \in \mathbb{R}^{n+m};$$

this definition similarly applies to random vectors.

Proposition 6. If $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is $\text{IS}(\alpha)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_m)$ is $\text{IS}(\beta)$, with $\mathbf{X}$ and $\mathbf{Y}$ being independent, then the combined random vector $(\mathbf{X}, \mathbf{Y})$ is $\text{IS}(\alpha, \beta)$.

Proof. Let $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ and $\gamma = (\alpha, \beta)$. Then, for any i , with $2 \leq i \leq n$, and any $z_i \in \mathbb{R}$, we have

$$\mathbb{P}[\gamma_i Z_i > z_i \mid \cap_{j=1}^{i-1} (\gamma_j Z_j > z_j)] = \mathbb{P}[\alpha_i X_i > z_i \mid \cap_{j=1}^{i-1} (\alpha_j X_j > z_j)],$$

and, thus, it is non-decreasing in $z_1, z_2, \dots, z_{i-1}$.

Consider now any i , with $n+1 \leq i \leq n+m$, and any $z_i \in \mathbb{R}$. Taking into account that $\mathbf{X}$ and $\mathbf{Y}$ are independent, we have

$$\mathbb{P}[\gamma_i Z_i > z_i \mid \cap_{j=1}^{i-1} (\gamma_j Z_j > z_j)] = \mathbb{P}[\beta_{i-n} Y_{i-n} > z_i \mid \cap_{j=1}^{i-n-1} (\beta_j Y_j > z_{n+j})],$$

which is non-decreasing in $z_1, z_2, \dots, z_{i-1}$. Therefore, $(\mathbf{X}, \mathbf{Y})$ is $\text{IS}(\alpha, \beta)$. $\square$

The following result pertains to a closure property of the $\text{IS}(\alpha)$ family of multivariate distributions, as well as the $\text{DS}(\alpha)$ family.

Proposition 7. The collection of $\text{IS}(\alpha)$ distribution functions is closed with respect to weak convergence.

Proof. Let $\{\mathbf{X}_m\}_{m \in \mathbb{N}}$ be a sequence of $\text{IS}(\alpha)$ n -variate random vectors such that H_m is the distribution function of $\mathbf{X}_m$ for each m , and let $\mathbf{X}$ be a vector of n random variables with joint distribution function H such that $H_m \xrightarrow{w} H$ as m tends to $+\infty$, where $\xrightarrow{w}$ denotes weak convergence. We prove that $\mathbf{X}$ is $\text{IS}(\alpha)$.

Given $m \geq 1$, consider the n -variate random vector $\mathbf{X}_m = (X_{1m}, X_{2m}, \dots, X_{nm})$, which, by hypothesis, is $\text{IS}(\alpha)$. From Theorem 1, $\mathbf{X}_m$ is $\text{SIS}(\alpha)$. Since $H_m \xrightarrow{w} H$ as m tends to $+\infty$, by using the Helly–Bray theorem (see, e.g., [19]), we have

$$\mathbb{E}\left[f(\alpha_i X_{im}) | \cap_{j=1}^{i-1} (\alpha_j X_{jm} > x_j)\right] \xrightarrow{w} \mathbb{E}\left[f(\alpha_i X_i) | \cap_{j=1}^{i-1} (\alpha_j X_j > x_j)\right]$$

as m tends to $+\infty$ for any real-valued and nondecreasing function f and every i , with $2 \leq i \leq n$, whence $\mathbf{X}$ is $\text{SIS}(\alpha)$. By using Theorem 1 again, we conclude that $\mathbf{X}$ is $\text{IS}(\alpha)$. $\square$

3.4. Examples

We provide several examples that demonstrate the applicability of the dependence concepts studied in this work.

Example 1. Consider the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ which follows a multivariate normal distribution represented as $N(\mu, \Sigma)$, where $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is the mean vector and Σ denotes the covariance matrix. Define $(r_{ij}) = \Sigma^{-1}$ such that $r_{ij} < 0$ for all pairs (i, j) , where $1 \leq i < j \leq n$; a similar analysis can be conducted for $r_{ij} > 0$. The probability density function (PDF) of $\mathbf{X}$ is given by

$$f(x_1, x_2, \dots, x_n) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n r_{ij} (x_i - \mu_i)(x_j - \mu_j)\right).$$

For each pair (i, j) with $1 \leq i < j \leq n$, we can rewrite

$$f(x_1, x_2, \dots, x_n) = f_1(x^{(i)}) f_2(x^{(j)}) \exp(-r_{ij} x_i x_j),$$

where $x^{(k)} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ for $k = i, j$, and f_1, f_2 are suitable functions.

Now, consider x_i, x_j, x'_i, x'_j such that $x_i \leq x'_i$ and $x_j \leq x'_j$, along with (α_i, α_j) , where $|\alpha_k| = 1$ for $k = i, j$. We can derive

$$\begin{aligned} & f(x_1, \dots, \alpha_i x_i, \dots, \alpha_j x_j, \dots, x_n) f(x_1, \dots, \alpha_i x'_i, \dots, \alpha_j x'_j, \dots, x_n) \\ & - f(x_1, \dots, \alpha_i x'_i, \dots, \alpha_j x_j, \dots, x_n) f(x_1, \dots, \alpha_i x_i, \dots, \alpha_j x'_j, \dots, x_n) \\ & = f_1(x^{(i)}) f_1(x^{(i')}) f_2(x^{(j)}) f_2(x^{(j')}) \\ & \quad \cdot [\exp(-r_{ij} \alpha_i \alpha_j (x_i x_j + x'_i x'_j)) - \exp(-r_{ij} \alpha_i \alpha_j (x'_i x_j + x_i x'_j))]. \end{aligned} \quad (3)$$

Since

$$\alpha_i \alpha_j (x_i x_j + x'_i x'_j - x'_i x_j - x_i x'_j) = \alpha_i \alpha_j [(x'_i - x_i)(x'_j - x_j)] \geq 0,$$

this holds true, provided that $\alpha_i \alpha_j > 0$. Thus, the expression in (3) is non-negative if and only if $\alpha_i \alpha_j > 0$. Consequently, for any vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ with $|\alpha_i| = 1$ for each $i = 1, 2, \dots, n$, the random vector $\mathbf{X}$ exhibits the property of $\text{MTP}_2(\alpha)$ if, and only if, $\alpha_i \alpha_j > 0$ for every selected pair (i, j) —refer to Theorem 3 in [12]. By Corollary 1, we conclude that $\mathbf{X}$ is $\text{IS}(\alpha)$ for both $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$.

Example 2. Consider the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ following a Dirichlet distribution denoted as $\text{Dir}(\gamma)$, where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n; \gamma_{n+1})$ and $\gamma_i > 0$ for all $i = 1, 2, \dots, n$, with $\gamma_{n+1} \geq 1$. The probability density function (PDF) for this distribution is expressed as

$$f(x_1, x_2, \dots, x_n) = \frac{\Gamma(\sum_{i=1}^{n+1} \gamma_i)}{\prod_{i=1}^{n+1} \Gamma(\gamma_i)} \prod_{i=1}^n x_i^{\gamma_i-1} \left(1 - \sum_{i=1}^n x_i\right)^{\gamma_{n+1}-1},$$

where $x_i \geq 0$ and $\sum_{i=1}^n x_i \leq 1$. For any chosen pair (i, j) with $1 \leq i < j \leq n$ and any real numbers x_i, x_j, x'_i, x'_j such that $x_i \leq x'_i$ and $x_j \leq x'_j$, we have the following relationship:

$$\begin{aligned} & f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_n) \\ & - f(x_1, \dots, x'_i, \dots, x_j, \dots, x_n) f(x_1, \dots, x_i, \dots, x'_j, \dots, x_n) \\ & = \left[\frac{\Gamma(\sum_{i=1}^{n+1} \gamma_i)}{\prod_{i=1}^{n+1} \Gamma(\gamma_i)} \right]^2 \prod_{\substack{k=1 \\ k \neq i, j}}^n x_k^{2(\gamma_k-1)} x_i^{\gamma_i-1} (x'_i)^{\gamma_i-1} x_j^{\gamma_j-1} (x'_j)^{\gamma_j-1} \\ & \cdot \left\{ \left[\left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x_i - x_j \right) \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x'_i - x'_j \right) \right]^{\gamma_{n+1}-1} \right. \\ & \left. - \left[\left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x'_i - x_j \right) \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x_i - x'_j \right) \right]^{\gamma_{n+1}-1} \right\}. \quad (4) \end{aligned}$$

Since we have

$$\begin{aligned} & \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x_i - x_j \right) \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x'_i - x'_j \right) \\ & - \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x'_i - x_j \right) \left(1 - \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k - x_i - x'_j \right) \\ & = x_i x'_j + x'_i x_j - x'_i x'_j - x_i x_j = -(x'_i - x_i)(x'_j - x_j) \leq 0, \end{aligned}$$

and given that $\gamma_{n+1} \geq 1$, it follows that (4) is non-positive. Thus, we conclude that the random vector $\mathbf{X}$ exhibits the $\text{MRR}_2(\mathbf{1})$ property, indicating that $\mathbf{X}$ is characterized by the multivariate reverse rule of order two, which is the negative counterpart to the concept defined in Definition 3 by reversing the inequality sign, according to the direction $\mathbf{1}$. Consequently, by applying the corresponding negative dependence framework, analogous to what is provided in Corollary 1 for the positive dependence framework, we deduce that $\mathbf{X}$ is $\text{DS}(\mathbf{1})$.

Example 3. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a random vector following a multinomial distribution characterized by parameters N (the number of trials) and $p = (p_1, p_2, \dots, p_n)$ (the probabilities of the events), with the constraints $p_i \geq 0$ for each $i = 1, 2, \dots, n$ and $0 < \sum_{i=1}^n p_i < 1$. The joint probability mass function (PMF) is given by

$$f(x_1, x_2, \dots, x_n) = \frac{N!}{\prod_{i=1}^n x_i! (N - \sum_{i=1}^n x_i)!} \prod_{i=1}^n p_i^{x_i} \left(1 - \sum_{i=1}^n p_i \right)^{N - \sum_{i=1}^n x_i},$$

where $\sum_{i=1}^n x_i \leq N$. It is notable that the multinomial distribution can be viewed as the conditional distribution of independent Poisson random variables, conditioned on their total. As established in Theorem 4.3 of [16] and Theorem 3 of [12], we find that the random vector $\mathbf{X}$ satisfies the $\text{MRR}_2(\mathbf{1})$ property. Consequently, we can conclude that $\mathbf{X}$ is also $\text{DS}(\mathbf{1})$.

Remark 1. It is worth noting that, by applying reasoning analogous to that outlined in Example 3, any random vector that follows a multivariate Hypergeometric distribution—essentially the conditional distribution of independent binomial random variables given their total—also qualifies as $\text{DS}(\mathbf{1})$.

Next, we provide an illustrative example showing the application of Proposition 7 related to weak convergence.

Example 4. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ represent a random vector with a joint distribution function defined as

$$H_\theta(\mathbf{x}) = \exp \left[- \left(\sum_{i=1}^n e^{-\theta x_i} \right)^{1/\theta} \right]$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\theta \geq 1$. This family of distribution functions serves as a multivariate generalization of Type B bivariate extreme-value distributions (see [20,21]). By invoking Theorem 2.11 in [22], which addresses log-convex functions [23], we can assert that the random vector $\mathbf{X}$ is $IS(-1)$. We consider the sequence of distribution functions $\{H_\theta\}_{\theta \in \mathbb{N}}$. As θ approaches infinity, we derive $H_\infty(\mathbf{x}) = \min(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$, where F_i are the one-dimensional marginals of H_θ for $i = 1, 2, \dots, n$. Hence, by virtue of Proposition 7, it follows that H_θ retains the $IS(-1)$ property too.

4. Monotonic in Sequence According to a Direction and Copulas

Copulas serve as an essential tool for analyzing the positive dependence characteristics of continuous random vectors. They encapsulate the dependence structure of the corresponding multivariate distribution function, are independent of the marginal distributions, and provide scale-invariant measures of dependence. Additionally, copulas act as a foundational element for constructing families of distributions (see [24]). In this section, our objective is to examine continuous n -copulas associated with random vectors that are $IS(\alpha)$. For simplicity, we focus on the bivariate and trivariate cases.

Let us recall some key concepts related to copulas. For $n \geq 2$, an n -dimensional copula (shortened to n -copula) is defined as the restriction to $[0, 1]^n$ of a continuous n -dimensional distribution function, where the univariate margins are uniformly distributed on $[0, 1]$. The significance of copulas in statistics is highlighted by the following theorem from Abe Sklar [25]: Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a random vector with joint distribution function F and one-dimensional marginal distributions $F_1, F_2, \dots, F_n$. There exists a unique n -copula C (which is determined on $\times_{i=1}^n \text{Range}(F_i)$) such that

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) \quad \text{for all } \mathbf{x} \in [-\infty, +\infty]^n.$$

Moreover, if $F_1, F_2, \dots, F_n$ are continuous, then C is uniquely defined. A comprehensive proof of this result can be found in [26]. Thus, copulas serve to connect joint distribution functions with their respective one-dimensional margins. For an overview of copulas, see [21,27], and references discussing positive dependence properties through copulas can be found in [5,21,22,28,29].

Let Π^n represent the n -copula for independent random variables (also known as the product n -copula), defined as $\Pi^n(\mathbf{u}) = \prod_{i=1}^n u_i$ for all $\mathbf{u} = (u_1, u_2, \dots, u_n) \in [0, 1]^n$.

For any n -copula C , the following inequality holds:

$$W^n(\mathbf{u}) = \max \left\{ 0, \sum_{i=1}^n u_i - n + 1 \right\} \leq C(\mathbf{u}) \leq \min\{u_1, u_2, \dots, u_n\} = M^n(\mathbf{u})$$

for all $\mathbf{u}$ in $[0, 1]^n$. While M^n is an n -copula for all $n \geq 2$, W^n qualifies as an n -copula only when $n = 2$.

Let $\mathbf{X}$ be a random vector with an associated n -copula C , and consider the random pair (X_i, X_j) corresponding to the components i and j (where $i < j$) of $\mathbf{X}$. Define C_{ij} as the (i, j) -margin of C :

$$C_{ij}(u_i, u_j) = C(1, \dots, 1, u_i, 1, \dots, 1, u_j, 1, \dots, 1),$$

for all $1 \leq i < j \leq n$, which represents the 2-copula corresponding to the random pair (X_i, X_j) . Additionally, $C_{1,\dots,i}$ denotes the i -copula $C(u_1, \dots, u_i, 1, \dots, 1)$.

For a random vector $(X_1, X_2, \dots, X_n)$ with associated n -copula C , the *survival n -copula* associated with C , denoted by $\hat{C}$, is defined as

$$\hat{C}(\mathbf{u}) = \mathbb{P}[X_1 \geq 1 - u_1, X_2 \geq 1 - u_2, \dots, X_n \geq 1 - u_n]$$

for all $\mathbf{u} \in [0, 1]^n$.

We begin by characterizing the concept of $IS(\alpha)$ in terms of n -copulas, specifically focusing on the bivariate and trivariate cases for clarity.

4.1. The General Case

To characterize the $IS(\alpha)$ concept in terms of n -copulas, we examine the flipping transformations of copulas. Recall that, for an n -copula C , the *flipping of C in the i -th coordinate* (referred to as the *i -flip of C*) is defined as the function $C^i: [0, 1]^n \rightarrow [0, 1]$ given by, for all $\mathbf{u} \in [0, 1]^n$,

$$C^i(\mathbf{u}) = C(u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n) - C(u_1, \dots, u_{i-1}, 1 - u_i, u_{i+1}, \dots, u_n)$$

(see [30]).

Moreover, the j -flipping transformation of the i -flip of C is denoted by C^{ij} , and is given, for all $\mathbf{u} \in [0, 1]^n$, by

$$\begin{aligned} C^{ij}(\mathbf{u}) &= C(u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n) \\ &\quad - C(u_1, \dots, u_{i-1}, 1 - u_i, u_{i+1}, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n) \\ &\quad - C(u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_{j-1}, 1 - u_j, u_{j+1}, \dots, u_n) \\ &\quad + C(u_1, \dots, u_{i-1}, 1 - u_i, u_{i+1}, \dots, u_{j-1}, 1 - u_j, u_{j+1}, \dots, u_n). \end{aligned}$$

Similarly, we denote by C^{ijk} the k -flipping transformation of the function C^{ij} , and so on. Now, the following characterization can be established.

Theorem 2. Let $X_1, X_2, \dots, X_n$ be n random variables with associated n -copula C , and $\alpha \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \dots, n$. Let $J = \{1, 2, \dots, n\}$, and $J_i = \{j \in J : j \leq i, \alpha_j = -1\}$, for $i = 1, 2, \dots, n$. Then, C is $IS(\alpha)$ if, and only if, for every $i = \{2, \dots, n\}$, the function G given by

$$G(u_1, \dots, u_i) = \frac{\hat{C}_{1,\dots,i}^{J_i}(u_1, \dots, u_i)}{\hat{C}_{1,\dots,i-1}^{J_{i-1}}(u_1, \dots, u_{i-1})}$$

is nonincreasing in all u_j for $j \in \{1, 2, \dots, i-1\} \setminus J_{i-1}$, and nondecreasing in all u_j for $j \in J_{i-1}$, for all u_i , where $\hat{C}_{1,\dots,i}^{J_i}$ denotes the function obtained from the survival copula $\hat{C}_{1,\dots,i}$ of $(X_1, \dots, X_i)$, after flipping consecutively in all indices j in J_i .

In Theorem 3 of [8], it is shown that a 3-copula C is $PD(\alpha)$ for every direction α if, and only if, $C = \Pi^3$. This result can be generalised to any dimension $n \geq 2$. Since every $IS(\alpha)$ n -copula is $PD(\alpha)$ for every α —recall Proposition 1—as a consequence of Proposition 3, the following result easily follows.

Corollary 2. An n -copula C is deemed $IS(\alpha)$ for every direction α if, and only if, C is equal to Π^n .

Now, we provide two examples of applications of Theorem 2.

Example 5. For all $n \geq 2$, the n -copula M^n is $IS(\alpha)$ only for $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$.

Example 6. Let $\{C_{n,\lambda}\}_{\lambda \in [-1,1]}$ be an n -dimensional generalization of the one-parameter Farlie–Gumbel–Morgenstern (FGM, for short) family of 2-copulas, which is given by

$$C_{n,\lambda}(\mathbf{u}) = \prod_{i=1}^n u_i \left[1 + \lambda \prod_{i=1}^n (1 - u_i) \right]$$

for all $\mathbf{u} \in [0,1]^n$ (see [21,27]). For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, let $J = \{i \in \{1, 2, \dots, n\} : \alpha_i = 1\}$. After some elementary operations, we have that $C_{n,\lambda}$ is $I(\alpha)$ for $\lambda \in [0,1]$ if $|J|$ —the cardinality of J —is even, and $C_{n,\lambda}$ is $I(\alpha)$ for $\lambda \in [-1,0]$ if $|J|$ is odd.

4.2. The Bivariate Case

We study now the bivariate case, which is a consequence of Theorem 2.

Corollary 3. Let (X, Y) be a pair of random variables with associated 2-copula C . Then, C is:

i. $IS(1,1)$ if, and only if,

$$\frac{1 - u - v + C(u, v)}{1 - u}$$

is increasing in u for all v ;

ii. $IS(1,-1)$ if, and only if,

$$\frac{v - C(u, v)}{1 - u}$$

is increasing in u for all v ;

iii. $IS(-1,1)$ if, and only if,

$$\frac{u - C(u, v)}{u}$$

is decreasing in u for all v ;

iv. $IS(-1,-1)$ if, and only if,

$$\frac{C(u, v)}{u}$$

is decreasing in u for all v .

We provide several examples for the bivariate case.

Example 7. The 2-copula W^2 is $IS(\alpha)$ for $\alpha = (-1,1)$ and $\alpha = (1,-1)$.

Example 8. Let $\{C_\delta\}_{\delta \in [-1,1]}$ be the Ali–Mikhail–Haq one-parameter family of 2-copulas [31] given by

$$C_\delta(u, v) = \frac{uv}{1 + \delta(1 - u)(1 - v)}$$

for all $(u, v) \in [0,1]^2$. It is easy to prove that C_δ is $IS(-1,1)$ and $IS(1,-1)$ for $\delta \in [0,1]$, and $IS(1,1)$ and $IS(-1,-1)$ for $\delta \in [-1,0]$.

The following example shows that the converse of Proposition 2 does not hold in general.

Example 9. Consider the 2-copula

$$C(u, v) = uv[1 + (1 - u)(1 - v)^2]$$

for all $(u, v) \in [0,1]^2$. C belongs to the family of 2-copulas studied in [32]. Then, we have that C is $IS(1,1)$. Moreover, we have that C is not $I(1,1)$.

4.3. The Trivariate Case

In the next result, we study the $IS(\alpha)$ concept in terms of 3-copulas.

Theorem 3. Let (X, Y, Z) be a random triple with associated 3-copula C . Then, C is:

i. $IS(1, 1, 1)$ if, and only if,

$$\frac{1 - u - v + C_{12}(u, v)}{1 - u}$$

is increasing in u for all v and

$$\frac{1 - u - v - w + C_{12}(u, v) + C_{13}(u, w) + C_{23}(v, w) - C(u, v, w)}{1 - u - v + C_{12}(u, v)}$$

is increasing in (u, v) for all w ;

ii. $IS(1, 1, -1)$ if, and only if,

$$\frac{1 - u - v + C_{12}(u, v)}{1 - u}$$

is increasing in u for all v and

$$\frac{w - C_{13}(u, w) - C_{23}(v, w) + C(u, v, w)}{1 - u - v + C_{12}(u, v)}$$

is increasing in (u, v) for all w ;

iii. $IS(1, -1, 1)$ if, and only if,

$$\frac{v - C_{12}(u, v)}{1 - u}$$

is increasing in u for all v and

$$\frac{v - C_{12}(u, v) - C_{23}(v, w) + C(u, v, w)}{v - C_{12}(u, v)}$$

is increasing in u and decreasing in v for all w ;

iv. $IS(-1, 1, 1)$ if, and only if,

$$\frac{u - C_{12}(u, v)}{u}$$

is decreasing in u for all v and

$$\frac{u - C_{12}(u, v) - C_{13}(u, w) + C(u, v, w)}{u - C_{12}(u, v)}$$

is decreasing in u and increasing in v for all w ;

v. $IS(1, -1, -1)$ if, and only if,

$$\frac{v - C_{12}(u, v)}{1 - u}$$

is increasing in u for all v and

$$\frac{C_{23}(v, w) - C(u, v, w)}{v - C_{12}(u, v)}$$

is increasing in u and decreasing in v for all w ;

vi. $IS(-1, -1, 1)$ if, and only if,

$$\frac{C_{12}(u, v)}{u}$$

is decreasing in u for all v and

$$\frac{C_{12}(u, v) - C(u, v, w)}{C_{12}(u, v)}$$

is decreasing in (u, v) for all w ;

vii. $IS(-1, 1, -1)$ if, and only if,

$$\frac{u - C_{12}(u, v)}{u}$$

is decreasing in u for all v and

$$\frac{C_{13}(u, w) - C(u, v, w)}{u - C_{12}(u, v)}$$

is decreasing in u and increasing in v for all w ;

viii. $IS(-1, -1, -1)$ if, and only if,

$$\frac{C_{12}(u, v)}{u}$$

is decreasing in u for all v and

$$\frac{C(u, v, w)}{C_{12}(u, v)}$$

is decreasing in (u, v) for all w .

We provide an example for the trivariate case.

Example 10. Let $C_{2,\lambda}$ be the one-parameter FGM family of 2-copulas—recall Example 6. Consider the 3-copula C_λ given by $C_\lambda(u, v, w) = wC_{2,\lambda}(u, v)$ for all $(u, v, w) \in [0, 1]^3$. Then, we have that C_λ is $IS(1, 1, 1)$, $IS(1, 1, -1)$, $IS(-1, -1, 1)$, and $IS(-1, -1, -1)$ for $\lambda \in [0, 1]$ and C_λ is $IS(1, -1, 1)$, $IS(-1, 1, 1)$, $IS(1, -1, -1)$, and $IS(-1, 1, -1)$ for $\lambda \in [-1, 0]$.

5. Conclusions

The monotonic in sequence according to a direction concept, denoted by $IS(\alpha)$, constitutes a significant advancement in multivariate dependence analysis because it allows the modeling of dependencies that are not easily captured by previous concepts, such as simple positive or negative dependence, extending several known multivariate dependence concepts. There is clear potential for applications in fields such as financial risk analysis, where directional dependence relationships could help model the correlation of extreme events between financial assets. Another relevant area could be biostatistics, where the progression of certain conditions or responses may be influenced by a sequence of biological or environmental variables with marked directional dependence. Sequence dependencies may also have implications in network analysis, such as neural or data networks, where the direction in which nodes are affected significantly impacts information or state propagation. This new concept seems to address the limitations of traditional dependence measures, especially in configurations where dependencies are asymmetric or directional.

We have established certain relationships with other multivariate dependence concepts, for instance, the implication from $I(\alpha)$ to $IS(\alpha)$ and, subsequently, from $IS(\alpha)$ to $PD(\alpha)$. Additionally, we have highlighted key properties and conducted an examination of this novel concept in terms of n -copulas—specifically for the bivariate and trivariate scenarios for a better understanding. Copulas play a fundamental role in this paper. Section 4 examines how copulas capture the multivariate dependence structure independently of the marginal distributions, making it easier to analyze the $IS(\alpha)$ concept in terms of directional properties.

Further exploration involving analogous extensions of well-known dependence concepts discussed in this paper, the investigation of associated orders—akin to the approach

taken in [33]—and defining new measures of association based on the concepts of dependence studied here are the subject of ongoing investigation.

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