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TOPOLOGY OF ATTRACTORS AND PERIODIC POINTS

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Abstract. The dynamics of a dissipative and area contracting planar homeomorphism were described in terms of the attractor. This was a subset of the plane defined as the maximal compact invariant set. We proved that the coexistence of two fixed points and an N-cycle produced some topological complexity: the attractor cannot be arcwise connected. The proofs were based on the theory of prime ends.

1. **Introduction.** In a remarkable paper [13], Levinson introduced the class of planar homeomorphisms which are dissipative and area-contracting. His main motivation came from nonconservative mechanics, but this class of maps is also relevant in other fields, such as population dynamics (see [23, 21] for more details).

A homeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called dissipative if the point of infinity is a repeller. The associated attractor $\mathcal{A} \subset \mathbb{R}^2$ is an invariant continuum composed of all the bounded orbits. The set $\mathcal{A}$ has zero measure if h is area-contracting. Understanding the topology of $\mathcal{A}$ is a crucial step in the study of the dynamics of h but this is not an easy task. Many examples have been constructed to show that the set $\mathcal{A}$, as well as the dynamics on it, can be very intricate (see [2, 27, 4, 9]).

In a recent paper [18] Nakajima used some clever and elementary arguments to prove that the attractor is not arcwise connected when h has at least two fixed points and one of them is an inverse saddle[1]. This result seems to be of a new type, showing that some dynamical assumptions (existence of certain fixed points) imply a certain complexity of the attractor (not arcwise connected). This line of research has been continued in [7], where the result in [18] is translated from the plane to the punctured sphere. In the present paper, we work in the plane and the goal will be to reach Nakajima's conclusion ($\mathcal{A}$ is not arcwise connected) without assuming the existence of an inverse saddle. Instead, we will assume that there exist at least two fixed points and one periodic point. The proofs are very different from those in [18, 7]. Our arguments are based on the theory of prime ends. The use of this theory in planar dynamics has a long tradition, starting with the work

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[1]We recall that a fixed point p of a diffeomorphism h is an inverse saddle if the eigenvalues of the Jacobian matrix $h'(p)$ satisfy $\lambda_1 < -1 < \lambda_2 < 0$. When $0 < \lambda_1 < 1 < \lambda_2$ the fixed point is a direct saddle.

of Cartwright and Littlewood in [5]. Incidentally we notice that these authors were motivated by the study of the attractor of the periodically forced Van der Pol oscillator. Later, Alligood and Yorke analyzed in [1] the connections between the rotation number of a local attractor and the existence of accessible periodic points (see also [20, 22, 11]). All these papers consider the complement of the attractor in the Riemann sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$, denoted as $\Omega = \mathbb{S}^2 \backslash \mathcal{A}$. This set is open, simply connected and invariant. The circle of prime ends is a fictitious boundary attached to Ω and the map h induces a homeomorphism h^* on this circle. We will follow along this line and show that there are obstructions for the dynamics of h^* when certain periodic points exist and $\mathcal{A}$ is arcwise connected. In consequence, $\mathcal{A}$ will not enjoy this topological property. The possible novelty of our paper is in the connection between Nakajima's ideas and prime ends. At this point it may be worth mentioning that this paper is an improvement on an unpublished manuscript. In our previous version the conclusions were weaker and an anonymous referee suggested applying Corollary 2 of [5]. The proof we present now is based on more elementary ideas but those comments certainly encouraged us to improve the paper.

An important aspect of the main result is the simplicity of the assumptions (two fixed points + one periodic orbit). This fact will allow us to obtain applications in the theory of periodic differential equations. This was the original motivation for the study of the attractors in [13].

The rest of the paper is organized in five sections. The class of maps under study and the main result are presented in Section 2. The next section provides a short introduction to the theory of prime ends. It follows Mather's approach and it is the main tool for the proof of the main result in Section 4. The last two sections are devoted to illustrate the applicability of our theorem. We show that some well-known results on bifurcation on subharmonic solutions can be combined with our result to produce new information on attractors. Two concrete situations are considered: a forced Duffing oscillator and the classical SIR model with periodic contact rate (see [6, 25]).

2. A class of planar homeomorphisms and main result.

A homeomorphism of the plane is a continuous and bijective map $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The class of all homeomorphisms will be denoted by $\mathcal{H}(\mathbb{R}^2)$. A map $h \in \mathcal{H}(\mathbb{R}^2)$ is called **dissipative** if there exists a closed ball $B \subset \mathbb{R}^2$ attracting all compact sets in a uniform sense. In other words, for each $p \in \mathbb{R}^2$,

$$\lim_{n\to+\infty} dist(h^n(p), B) = 0 \tag{2.1}$$

and this limit is uniform in $p \in K$ with K any compact subset of $\mathbb{R}^2$.

The attractor $\mathcal{A} \subset \mathbb{R}^2$ is defined as the maximal invariant and compact set. It satisfies the properties below,

- $\mathcal{A}$ is compact and $h(\mathcal{A}) = \mathcal{A}$,
- $\mathcal{A}$ contains any compact set $K \subset \mathbb{R}^2$ satisfying that $h(K) = K$.

With some work, it can be proved that $\mathcal{A}$ always exists and it is indeed a nonempty continuum (compact and connected set). Sometimes, it is useful to interpret $\mathcal{A}$ as the set of bounded orbits. More precisely,

$$\mathcal{A} = \{p \in \mathbb{R}^2 : \limsup_{|n|\longrightarrow+\infty} |h^n(p)| < \infty\}.$$

Notice that all orbits are bounded in the future ($n \longrightarrow +\infty$) and, therefore, $\mathcal{A}$ can be also described as the set of orbits bounded in the past ($n \longrightarrow -\infty$). The reader is referred to [13, 23, 8] for a detailed discussion on attractors and dissipative maps.

Levinson considered in Section 7 of [13] the class of dissipative homeomorphisms contracting Lebesgue's measure. In that case, the attractor has zero measure and, in particular, the interior is empty. We will consider the larger class $\mathcal{LH}(\mathbb{R}^2)$ composed of all dissipative homeomorphisms whose attractor has an empty interior in $\mathbb{R}^2$. The class of Levinson appears very often in the theory of nonlinear oscillations, but $\mathcal{LH}(\mathbb{R}^2)$ is more natural from a topological point of view. Notice that $\mathcal{LH}(\mathbb{R}^2)$ is invariant under conjugacy.

The next result is an almost direct consequence of the definition of $\mathcal{LH}(\mathbb{R}^2)$ and will be repeatedly applied throughout the paper.

Proposition 2.1. *If $h \in \mathcal{LH}(\mathbb{R}^2)$, then $\mathcal{A}$ does not contain Jordan curves.*

Proof. By contradiction, assume that $\Gamma \subset \mathbb{R}^2$ is a Jordan curve inside the attractor. Let $R_i(\Gamma)$ be the bounded component of $\mathbb{R}^2 \backslash \Gamma$. We will prove that $R_i(\Gamma)$ is also contained in $\mathcal{A}$, but this is impossible if $\mathcal{A}$ has an empty interior.
Since $\mathcal{A}$ is invariant, every iterate of the curve is also contained in $\mathcal{A}$, that is,

$$\bigcup_{n\in\mathbb{Z}} h^n(\Gamma) \subset \mathcal{A}.$$

Let B be a large ball containing $\mathcal{A}$. Then,

$$\bigcup_{n\in\mathbb{Z}} h^n(R_i(\Gamma)) = \bigcup_{n\in\mathbb{Z}} R_i(h^n(\Gamma)) \subset B$$

and all the orbits starting at $R_i(\Gamma)$ remain bounded. This implies that $R_i(\Gamma) \subset \mathcal{A}$. □

It is well-known that many continua in the plane have an intricate topology and they can be realized as attractors. A very interesting discussion on possible attractors for maps in $\mathcal{LH}(\mathbb{R}^2)$ can be found in Section 7 of [13]. The simplest possible attractor is a singleton, corresponding to a globally asymptotically stable fixed point. After this case, Levinson presented three figures of increasing complexity. In Figure 1 of [13], the attractor is an arc and two dynamics are considered: three fixed points or one fixed point and a 2-cycle. In Figure 2, the attractor is a triode with one fixed point and two 3-cycles. Finally, in Figure 3, the three branches of the triode wrap themselves infinitely many times around three arcs. The corresponding continuum is not arc-wise connected. The dynamics is also more involved and a 6-cycle appears. Another possibility, also mentioned by Levinson, is the Cantorian Sun. This continuum is obtained by drawing all the rays connecting the origin to the points of a Cantor set lying in $\mathbb{S}^1$. See Figure 1 of this paper. It was proved in [9] (see also [27]) that this set is the attractor of a map h in $\mathcal{LH}(\mathbb{R}^2)$. Moreover, the dynamics on the Cantor set is recurrent (non-periodic) and the only periodic point is the origin, $Fix(h) = Fix(h^n) = \{(0,0)\}$, for each $n \geq 1$. In this case, $\mathcal{A}$ is arc-wise connected but it is not locally connected.

After these examples, we are ready to present the main result.

Theorem 2.2. *Assume that $h \in \mathcal{LH}(\mathbb{R}^2)$ is orientation preserving and there are at least two fixed points and an M-cycle with $M \geq 2$. Then, $\mathcal{A}$ is not arc-wise connected.*

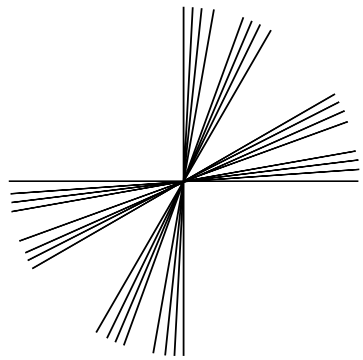

FIGURE 1. The third step in the construction of the Cantorian sun.

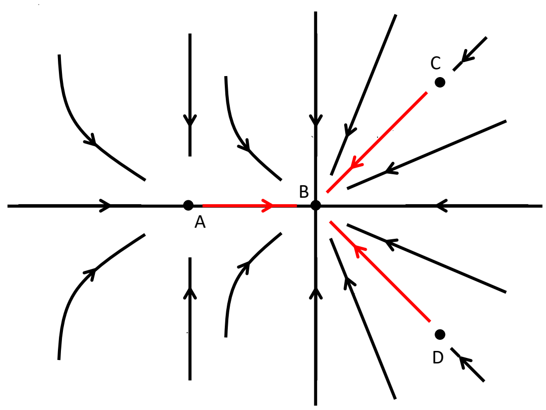

FIGURE 2. Phase portrait of the flow $\{\phi_t\}_{t\in\mathbb{R}}$

The proof of this result is postponed to Section 4.
Now we include some comments and examples:
Remarks:
1.- The theorem cannot be extended to orientation-reversing homeomorphisms when $M = 2$. We will construct an orientation-reversing map $h \in \mathcal{LH}(\mathbb{R}^2)$ having two fixed points, one 2-cycle, and the Y-set as attractor. We first construct a flow $\{\phi_t\}_{t\in\mathbb{R}}$ with four equilibria at the points $A = (-1, 0)$, $B = (0, 0)$, $C = (1, 1)$, $D = (1, -1)$ and the phase portrait illustrated in Figure 2. Also, we assume that the flow is symmetric with respect to the horizontal axis,

$$\phi_t \circ S = S \circ \phi_t, \qquad t \in \mathbb{R},$$

where $S(x, y) = (x, -y)$.

If we fix a time $T > 0$, the map $h_1 = \phi_T$ belongs to $\mathcal{LH}(\mathbb{R}^2)$ and the attractor is composed of the four equilibria and the heteroclinic orbits connecting them. This map is orientation-preserving, but $h = S \circ h_1$ is orientation reversing and has the

fixed points A, B and the 2-cycle $\{C, D\}$. The attractor of h_1 and h coincide because h_1 and S commute.
2.- The theorem does not extend to higher dimensions. The class $\mathcal{LH}(\mathbb{R}^3)$ can be defined in the obvious way. We consider $h \in \mathcal{LH}(\mathbb{R}^3)$ with

$$h(x, y, z) = (h_1(x, y), h_2(z)).$$

In addition, we assume that $h_1 \in \mathcal{LH}(\mathbb{R}^2)$ is orientation reversing and it has the attractor $\mathcal{A}_1 = [-1, 1] \times \{0\}$ and the three fixed points $(-1, 0)$, $(0, 0)$, and $(1, 0)$. The function $h_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is a decreasing homeomorphism having the attractor $\mathcal{A}_2 = [-1, 1]$, one fixed point at $z = 0$, and a 2-cycle at $z = \pm 1$. The product map h has three fixed points, a 2-cycle, it is orientation preserving, and the attractor

$$\mathcal{A} = (\mathcal{A}_1 \times \{0\}) \cup (\{(0, 0)\} \times \mathcal{A}_2)$$

is arc-wise connected.

3.- It would be nice to connect the theorem with Nakajima's result, but this is unclear. We do not know if there are maps $h \in \mathcal{LH}(\mathbb{R}^2)$ having an inverse saddle and no periodic orbits.

Given a map $h \in \mathcal{LH}(\mathbb{R}^2)$, the iterate h^n with $n \geq 2$ will also belong to $\mathcal{LH}(\mathbb{R}^2)$. Moreover, the attractor is the same. This observation allows us to apply the previous theorem to any positive power of h. As a consequence, we obtain:

Corollary 2.3. *The conclusion of Theorem 2.2 also holds under one of the assumptions below,*

- **i):** $h \in \mathcal{LH}(\mathbb{R}^2)$ *is orientation reversing and it has at least two fixed points and one M-cycle with $M \geq 3$.*
- **ii):** $h \in \mathcal{LH}(\mathbb{R}^2)$ *is orientation preserving and it has at least two cycles with minimal periods $M > N \geq 2$.*

In the first case, we apply the Theorem to h^2 and in the second to h^N. The third example in [13] is in the conditions of **ii)** with $M = 6$ and $N = 3$.

3. **Background on Prime Ends.** Let Ω be an open and simply connected subset of $\mathbb{S}^2$ such that $\infty \in \Omega$, $\Omega \neq \mathbb{S}^2$, and $\mathbb{S}^2 \setminus \Omega$ is not a singleton. It is well-known that Ω is always homeomorphic to the open unit disk $int(\mathbb{D})$, where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

However, the boundary of Ω in $\mathbb{S}^2$, denoted by $\partial_{\mathbb{S}^2}\Omega$, is not necessarily homeomorphic to $\partial\mathbb{D} = \mathbb{S}^1$. Caratheodory's theory of prime ends allows us to construct an abstract topological space Ω^* containing Ω, and such that the pairs (Ω^*, Ω) and $(\mathbb{D}, int(\mathbb{D}))$ are homeomorphic.

The space of prime ends is defined as

$$\mathbb{P} = \Omega^* \setminus \Omega.$$

The space Ω^* is not inside $\mathbb{S}^2$ but somehow describes the way in which Ω is embedded in $\mathbb{S}^2$. The following property reflects this fact.

Let $g : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be a homeomorphism with $g(\infty) = \infty$ and such that Ω is invariant under g, $g(\Omega) = \Omega$. Then, there exists another homeomorphism $g^* : \Omega^* \longrightarrow \Omega^*$ such that $g = g^*$ on Ω. After conjugation, the restriction of g^* to the space of prime ends, $g^* : \mathbb{P} \longrightarrow \mathbb{P}$, can be seen as a homeomorphism of $\mathbb{S}^1$.

An end-path is a continuous map $\gamma : [0, 1] \longrightarrow \mathbb{S}^2$ such that $\gamma(t) \in \Omega$ if $t \in [0, 1[$ and $\gamma(1) \notin \Omega$. End-paths can also be interpreted as paths in Ω^*. More precisely, it is possible to construct a continuous map $\gamma^* : [0,1] \longrightarrow \Omega^*$ such that $\gamma^* = \gamma$ on $[0,1)$ and $\gamma^*(1) \in \mathbb{P}$ (see Theorem 16 in [14]).

A prime end $\mathcal{P}$ is called to be accessible if $\mathcal{P} = \gamma^*(1)$ for some γ^* and $\xi = \gamma(1) \in \partial_{\mathbb{S}^2}\Omega$ is the **principal point** of $\mathcal{P}$. It can be proved that every accessible end has a unique principal point (Theorem 17.1 in [14]).

Given two end-paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow \mathbb{S}^2$, we say that they are homotopic if there is a continuous mapping $\Gamma : [0,1] \times [0,1] \longrightarrow \mathbb{S}^2$ such that $\Gamma(t,0) = \gamma_0(t)$, $\Gamma(t,1) = \gamma_1(t)$, $\Gamma(1,s) = \gamma_0(1) = \gamma_1(1)$, $\Gamma(t,s) \in \Omega$ if $t < 1$. This notion is relevant because two end-paths are homotopic if, and only if the corresponding accessible prime ends coincide, $\gamma_0^*(1) = \gamma_1^*(1)$ (Theorem 18 in [14]).

For a general domain Ω, not all prime ends are accessible. However, if the boundary of Ω is locally connected, then all prime ends are accessible (Theorem 20 in [14]). From now on we assume that $\partial_{\mathbb{S}^2}\Omega$ is locally connected. Under this assumption, the previous discussions are sufficient to construct the space $\mathbb{P}$ and the map g^* in concrete cases. Prime ends with the same principal point $\xi \in \partial_{\mathbb{S}^2}\Omega$ correspond to different ways of approaching ξ from Ω. Given $\mathcal{P} \in \mathbb{P}$, we can have an end-path γ such that $\gamma(1) = \zeta$ for a suitable point $\zeta \in \partial_{\mathbb{S}^2}\Omega$. To construct the topology of Ω^*, it is enough to define a sub-basis of neighborhoods for each $\mathcal{P} \in \mathbb{P}$. Given an end-path γ defining $\mathcal{P}$, we take an open ball B centered at the principal point $\zeta = \gamma(1)$. Let B_γ be the connected component of $B \cap \Omega$ such that $\gamma(t) \in B_\gamma$ if $t < 1$ and $1 - t$ sufficiently small. We define the neighborhood

$$U_B = B_\gamma \cup \{\lambda^*(1) : \lambda \in \Lambda\}$$

where Λ is the family of end-paths $\lambda : [0,1] \longrightarrow \mathbb{S}^2$ satisfying that $\lambda(t) \in B_\gamma$ if $t \in [0,1)$. After this definition, the following result is easily proved.

Lemma 3.1. *In the previous notations, assume that the boundary $\partial_{\mathbb{S}^2}\Omega$ is locally connected and consider the map*

$$\Pi : \mathbb{P} \longrightarrow \partial_{\mathbb{S}^2}\Omega, \quad \mathcal{P} \mapsto \zeta,$$

assigning the corresponding principal point ζ to each prime end $\mathcal{P}$. Then, Π is continuous.

Finally, we discuss how to induce homeomorphisms on $\mathbb{P}$. Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a homeomorphism such that Ω is invariant. Then, also $\partial_{\mathbb{S}^2}\Omega$ is invariant. Given an end-path γ defining $\mathcal{P}$, the composition $h \circ \gamma$ is another end-path inducing a prime end. The homeomorphism $h^* : \mathbb{P} \longrightarrow \mathbb{P}$ satisfies

$$h^*(\mathcal{P}) = (h \circ \gamma)^*(1).$$

The useful property $(h^*)^n = (h^n)^*$ holds for each $n \in \mathbb{Z}$.

As an example, we consider the domain $\Omega = \mathbb{S}^2 \backslash A$ where $A = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$. In this set, we distinguish the four end points a, b, c, d and the origin 0. For each point $\xi \in A \backslash \{a,b,c,d,0\}$, there are two prime ends whose principal point is ξ. Four prime ends have the origin as principal point. Finally, there is only one prime end for a, b, c, and d. Figure 3 illustrates this description.

Consider now the symmetry $h : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$, $h(z) = \bar{z}$. Then, $h(\Omega) = \Omega$ and $h^* : \mathbb{P} \longrightarrow \mathbb{P}$ satisfies $Fix(h^*) = \{a, c\}$, $Fix((h^2)^*) = \mathbb{P}$. In particular, $h(b) = d$, $h^*(O_1) = O_4$,...

4. **Proof of Theorem 2.2.**

We start with three preliminary results which will be used in the proof.

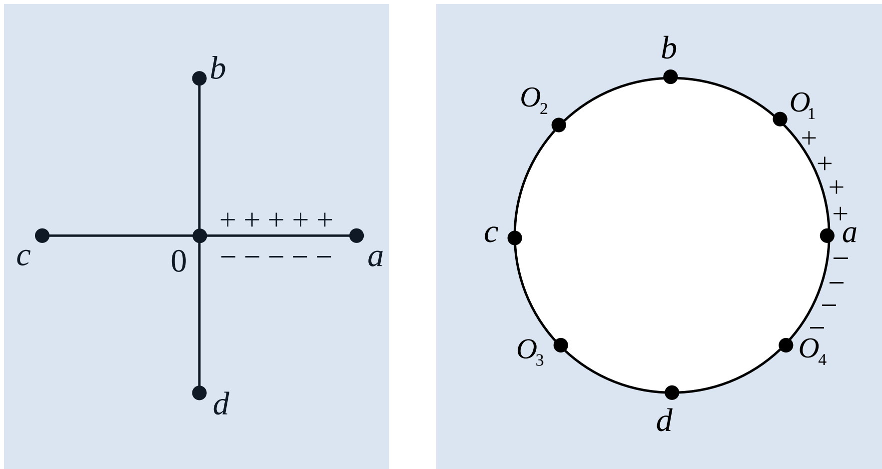

FIGURE 3. Description of the prime ends in $A = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$

Lemma 4.1. *Assume that K is a nonempty continuum in the plane and $h \in \mathcal{LH}(\mathbb{R}^2)$ is such that*

$$K \subset \mathcal{A} \quad \textit{and} \quad h(K) = K.$$

Then, $\Omega = \mathbb{S}^2 \backslash K$ is a simply connected domain with $\partial_{\mathbb{S}^2}\Omega = K$.

Proof. Let $\{\Omega_i\}_{i \in I}$ be the family of connected components of Ω. Since h is a homeomorphism and Ω is invariant, components must be mapped onto components. This means that $h(\Omega_i) = \Omega_{\sigma(i)}$, where σ is a permutation of I. Let Ω_∞ denote the component containing ∞. From $h(\infty) = \infty$, we deduce that $h(\Omega_\infty) = \Omega_\infty$. Also, the complement $\mathbb{S}^2 \backslash \Omega_\infty$ must be invariant under h. Then, all orbits lying in $\mathbb{S}^2 \backslash \Omega_\infty$ are bounded and this set is contained in the attractor. From $\mathbb{S}^2 \backslash \Omega_\infty \subset \mathcal{A}$, we deduce that $int_{\mathbb{S}^2}(\mathbb{S}^2 \backslash \Omega_\infty) = \emptyset$. This implies that $\Omega = \Omega_\infty$. Once we know that Ω is connected, it is easy to prove that it is also simply connected. Indeed, it is sufficient to observe that the complement $\mathbb{S}^2 \backslash \Omega = K$ is connected. Finally, we observe that Ω is dense in $\mathbb{S}^2$, and, in consequence, $\partial_{\mathbb{S}^2}\Omega = \mathbb{S}^2 \backslash \Omega = K$. □

The previous result can be applied to the case $K = \mathcal{A}$. In particular, we recover Proposition 2.1, and $\mathcal{A}$ cannot contain Jordan curves. From this fact we deduce a useful result.

Lemma 4.2. *Assume that $h \in \mathcal{LH}(\mathbb{R}^2)$ and $\gamma = \widehat{xy}$ is an arc contained in $\mathcal{A}$ such that the end points are fixed, that is, $x, y \in Fix(h)$. Then, $h(\gamma) = \gamma$.*

Proof. The set $\gamma \cup h(\gamma)$ is a closed loop contained in $\mathcal{A}$. This loop should contain a Jordan curve unless both arcs γ and $h(\gamma)$ coincide. To justify the previous reasoning, the reader can find it convenient to invoke Lemma 2.6 in [7]. □

Oriented arcs have a natural ordering, and this fact will be crucial for the following result.

Lemma 4.3. *Assume that $h \in \mathcal{LH}(\mathbb{R}^2)$, $\mathcal{A}$ is arc-wise connected, and $p \in \mathcal{A} \backslash Fix(h)$. Then, there exist a fixed point x and an arc $\Gamma = \widehat{px}$ contained in $\mathcal{A}$ so that*

$$\Gamma \cap Fix(h) = \{x\}.$$

Proof. Since h is dissipative, we know that it has at least one fixed point ζ. Let $\Gamma = \widehat{p\zeta}$ be an arc contained in $\mathcal{A}$. The arc Γ can contain many fixed points besides ζ, but we select the smallest fixed point x; that is, $h(z) \neq z$ if $z \in \widehat{px}$, $z \neq x$, and $h(x) = x$. Then, $\Gamma = \widehat{px}$ is the required arc. □

After these preliminary results, we are ready for the proof of the main result.

Proof of Theorem 2.2. By a contradiction argument we assume that $h \in \mathcal{LH}(\mathbb{R}^2)$ is orientation preserving, $\mathcal{A}$ is arc-wise connected, $\{z_0, ..., z_{N-1}\}$ is an N-cycle, and $Fix(h)$ contains at least two points.
Lemma 4.3 can be applied to find an arc $\Gamma_0 = \widehat{z_0 x}$ contained in $\mathcal{A}$ and such that $\Gamma_0 \cap Fix(h) = \{x\}$. Also, we select a second fixed point $y \neq x$ and find an arc $\gamma \subset \mathcal{A}$, $\gamma = \widehat{xy}$. We can apply Lemma 4.2 to deduce that $h(\gamma) = \gamma$. Define the set

$$K = \gamma \cup \Gamma_0 \cup h(\Gamma_0) \cup ... \cup h^{N-1}(\Gamma_0).$$

Then, K is invariant under h. Let us now recall that a Peano continuum is a locally connected continuum. A finite union of Peano continua with a common point is also a Peano continuum (see [19], page 88). In consequence, K is a Peano continuum.

In view of Lemma 4.1, the space of prime ends associated to $\Omega = \mathbb{S}^2 \backslash K$ can be considered. The boundary is locally connected and we know that each point of K is the principal point of some prime end. Let us take $\mathcal{P}_0 \in \mathbb{P}$ with $\Pi(\mathcal{P}_0) = z_0$. The sequence $\{\Pi \circ (h^*)^n(\mathcal{P}_0)\}_{n \in \mathbb{Z}}$ is periodic, namely, $\{..., z_0, z_1, ..., z_{N-1}, z_0, z_1, ..., z_{N-1} ...\}$. From the continuity of Π, we deduce that $\{(h^*)^n(\mathcal{P}_0)\}_{n \geq 0}$ cannot be a convergent sequence.

Let us now analyze the prime end $\mathcal{P}_y \in \mathbb{P}$ with $\Pi(\mathcal{P}_y) = y$. Indeed, this prime end is unique. To justify this assertion, observe that it is not restrictive to assume that the arc γ is a segment. This is possible because all arcs in the plane are tame (see [15]). Since x is the only fixed point lying in Γ_0, we deduce that $y \notin \Gamma_0$. In consequence,

$$y = h^k(y) \notin h^k(\Gamma_0), \quad 0 \leq k \leq N - 1.$$

This property allows us to find a disk D centered at y and having a small radius such that $D \cap K = D \cap \gamma$. The uniqueness of $\mathcal{P}_y$ together with the identity

$$\Pi(h^*(\mathcal{P}_y)) = h(\Pi(\mathcal{P}_y)) = h(y) = y$$

implies that $\mathcal{P}_y$ is fixed under h^*.
At this point we have found the required contradiction on the dynamics of h^*. This map can be seen as an orientation preserving homeomorphism of $\mathbb{S}^1$ having the fixed point $\mathcal{P}_y$ and the non-convergent orbit $\{(h^*)^n(\mathcal{P}_0)\}_{n \geq 0}$. It is well-known that such a map cannot exist. □

Remark 4.4. The above proof is a refined version of the proof we originally found. In the preprint version of our paper, we included the property

$$\gamma \cap h^k(\Gamma_0) = \{x\}, \ \ 0 \leq k \leq N - 1$$

as an intermediate step. Referee X has now observed that it is not necessary to check this property in order to complete the proof.

This referee has proposed other changes to make the proof more pictorial. We know that two arcs in $\mathcal{A}$ that are not disjoint will intersect in a point or in a subarc. This implies that K is a finite 1-D simplicial complex (a graph). In particular, K is locally connected. Graphs in the plane are tame and so any terminal vertex corresponds to a single prime end in $\mathbb{S}^2\backslash K$. This implies that $\mathcal{P}_y$ is unique.
We are in debt with Referee X for the clever remarks and the deep understanding of our work.

5. **The attractor of a forced oscillator.** We consider the equation

$$\ddot{x} + \mu_1\dot{x} + x^3 = \mu_2 F(t) \tag{5.1}$$

where $\mu_1 > 0$ and $\mu_2 \in \mathbb{R}$ are real parameters and $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and periodic function with period $T = 1$.

Given $(x_0, v_0) \in \mathbb{R}^2$, the solution satisfying $x(0) = x_0$, $\dot{x}(0) = v_0$ is denoted by $x(t; x_0, v_0)$. The dynamics of the equation (5.1) can be analyzed through the planar map

$$P : (x_0, v_0) \mapsto (x(T; x_0, v_0), \dot{x}(T; x_0, v_0)),$$

sometimes called the Poincaré map.

From now on, $\mathfrak{X}$ will denote the Banach space of 1-periodic and continuous functions $F : \mathbb{R} \longrightarrow \mathbb{R}$ endowed with the uniform norm

$$\|F\| = \max_{t\in\mathbb{R}} |F(t)|.$$

As a first step, we are going to prove that the map P belongs to the class of maps considered in the previous section.

Lemma 5.1. *For each $\mu_1 > 0$, $\mu_2 \in \mathbb{R}$, and $F \in \mathfrak{X}$, the map P belongs to $\mathcal{LH}(\mathbb{R}^2)$ and it is orientation preserving.*

Proof. To start, we show that P is a homeomorphism of $\mathbb{R}^2$. For this it is sufficient to show that all the solutions of (5.1) are globally defined (see [21] for more details). Assume that $x(t) = x(t; x_0, v_0)$ is a maximal solution with energy

$$E(t) = \frac{1}{2}\dot{x}(t)^2 + \frac{1}{4}x(t)^4.$$

After differentiation,

$$|\dot{E}| = | - \mu_1\dot{x}^2 + \mu_2 F\dot{x}| \leq \mu_1\dot{x}^2 + \frac{|\mu_2|}{2}(\dot{x}^2 + F^2)$$
$$\leq (2\mu_1 + |\mu_2|)E + \frac{|\mu_2|}{2}\|F\|^2.$$

We have found a linear differential inequality in E which can be solved. Then, it is easy to show that the pair $(x(t), \dot{x}(t))$ cannot blow up in finite time. Therefore, the maximal interval of this solution is $(-\infty, +\infty)$.

Once we know that $P \in \mathcal{H}(\mathbb{R}^2)$, we are going to prove that it is dissipative. This is equivalent to proving that there exists $C > 0$ such that for each solution $x(t)$ of (5.1), there exists a time $\tau > 0$ such that

$$|x(t)| + |\dot{x}(t)| \leq C$$

if $t \geq \tau$. This is a consequence of general results on dissipative systems (see [23]), but it is also possible to obtain a direct proof with the help of the modified energy

$$V(t) = \frac{1}{2}\dot{x}(t)^2 + \varepsilon x(t)\dot{x}(t) + \frac{1}{4}x(t)^4,$$

where $\varepsilon > 0$ is small enough.

To complete the proof of the lemma, it is sufficient to observe that P is area-contracting and orientation preserving. In fact, a well-known computation based on Liouville's formula shows that

$$0 < \det P'(x_0, v_0) = e^{-\mu_1 T} < 1 \tag{5.2}$$

for each $(x_0, v_0) \in \mathbb{R}^2$. □

Throughout this section, $\mathcal{A}_F$ will denote the attractor of the map P. Indeed, $\mathcal{A}_F$ will also depend upon the parameters μ_1 and μ_2, but we want to stress the functional dependence.

A possible criticism to this definition of attractor is that the map P is linked to the initial value problem at time $t_0 = 0$, and this is a rather arbitrary choice. We could instead select any other initial time $t_0 \in \mathbb{R}$. This leads to the family of maps

$$P_{t_0} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x(t_0), \dot{x}(t_0)) \mapsto (x(t_0 + T), \dot{x}(t_0 + T)).$$

Since they are conjugate, all of them belong to $\mathcal{LH}(\mathbb{R}^2)$ and the corresponing attractors $\mathcal{A}_F(t_0)$ are homeomorphic subsets of the plane.

Another alternative to define the attractor is to interpret equation (5.1) as an autonomous system on the 3-dimensional manifold $M = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$ with coordinates $(\bar{t}, x, v)$ and $\bar{t} = t + \mathbb{Z}$. The system

$$\dot{t} = 1, \quad \dot{x} = v, \quad \dot{v} = -\mu_1 v - x^3 + \mu_2 F(t)$$

is dissipative and the attractor $\mathcal{A}^* \subset M$ is precisely the union of all discrete attractors. Specifically,

$$\mathcal{A}^* = \{(\bar{t}, x_0, v_0) : 0 \le t < 1, (x_0, v_0) \in \mathcal{A}_F(t_0)\}.$$

The forward orbits $\{(\bar{t}, x(t), \dot{x}(t)) : t \ge 0\}$ are attracted as $t \to +\infty$ by $\mathcal{A}^*$. We thank referee Y for a remark that has motivated this discussion.

We are going to present a result on the topology of $\mathcal{A}_F$ that is valid for typical forcings F. This will be understood in the sense of category. We will say that a property holds for a generic F in $\mathfrak{X}$ if it holds for every $F \in \mathcal{G}$, where $\mathcal{G}$ is an open and dense subset of $\mathfrak{X}$.

Theorem 5.2. *For a generic F in $\mathfrak{X}$, there are numbers $\varepsilon > 0$ and $\eta > 0$ such that the attractor $\mathcal{A}_F$ is not arc-wise connected if*

$$0 < \mu_1 < \eta|\mu_2|, \quad |\mu_2| < \varepsilon.$$

The attractor in the red region is not arc-wise connected whereas it is a singleton on the green line (see Figure 4). Indeed, when $\mu_2 = 0$, the equation is autonomous and $x = 0$ is a globally asymptotically stable solution. When F is a constant function, the equation is again autonomous and the unique equilibrium $x = (\mu_2 F)^{\frac{1}{3}}$ attracts all solutions. Again, the attractor is a singleton. This shows that the conclusion of the theorem cannot be valid for all $F \in \mathfrak{X}$. Later we will be more precise on the nature of the set $\mathcal{G}$ where the conclusion of the theorem is valid.

To prove the theorem we will apply some well-known results on the existence of subharmonic solutions of a forced oscillator. We will work with the framework of page 372 in [6], which can be adapted to equation (5.1). It must be noticed that in [6] it is assumed that the equation is smooth in all variables and (5.1) is only

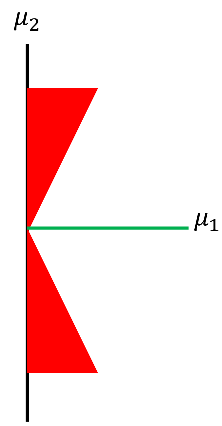

FIGURE 4.

continuous in t. This does not create essential differences because the Poincaré map is real analytic as a function of (x_0, v_0, μ_1, μ_2).

The first step will be to find a periodic solution of the autonomous equation ($\mu_1 = \mu_2 = 0$) with minimal period kT for each $k = 2, 3, \ldots$. Let $C(t)$ be the unique solution of

$$\ddot{x} + x^3 = 0 \tag{5.3}$$

with minimal period $T = 1$ and $C(0) > 0, \dot{C}(0) = 0$. This solution exists because $x = 0$ is a center and the minimal period of the closed orbits has the formula

$$\tau(A) = \frac{\kappa}{A}$$

where $\kappa > 0$ is a fixed constant and $A > 0$ is the amplitude of the oscillation. Actually, the nontrivial solutions of (5.3) are described by the family of two parameters

$$x(t) = \lambda C(\lambda t + \sigma) \tag{5.4}$$

with $\lambda > 0$ and $\sigma \in \mathbb{R}$. We refer to [16] for an expression of $C(t)$ in terms of special functions. Let us define $S(t) = \dot{C}(t)$ so that $t \mapsto (C(t), S(t))$ is a clockwise parameterization of a closed orbit of (5.3). For each $k = 2, 3, \ldots$, the solution

$$\varphi_k(t) = \frac{1}{k} C\left(\frac{t}{k}\right)$$

has minimal period $T = k$.

Differentiating the formula (5.4) with respect to t and λ, we observe that the variational equation

$$\ddot{y} + 3\varphi_k(t)^2 y = 0 \tag{5.5}$$

has a fundamental system composed of the functions

$$y_1(t) = \dot{\varphi}_k(t), \quad y_2(t) = \varphi_k(t) + t\dot{\varphi}_k(t).$$

In consequence, the only periodic solutions of (5.5) are of the type $c\dot{\varphi}_k(t)$ with $c \in \mathbb{R}$. This is the starting assumption in [6]. The other assumption is connected with the function

$$G_k(\alpha) = \frac{1}{k}\int_0^k \dot\varphi_k(t)F(t-\alpha)dt = \frac{1}{k}\int_0^k \dot\varphi_k(t+\alpha)F(t)dt.$$

This function is real analytic and we will impose the condition

$$G'_k(\alpha)^2 + G''_k(\alpha)^2 > 0 \tag{5.6}$$

for each $\alpha \in \mathbb{R}$. This means that G_k is a Morse function because all critical points ($G'_k(\alpha) = 0$) are nondegenerate ($G''_k(\alpha) \neq 0$). Under the assumption (5.6), the results in [6] imply the existence of $\varepsilon_k > 0$ and $\eta_k > 0$ such that the equation (5.1) has a solution with period kT if $0 < \mu_1 < \eta_k|\mu_2|$, $|\mu_2| < \varepsilon_k$. This solution is $\mathcal{C}^1$-close to $\varphi_k(t)$, and so it has no period of the type rT with $1 \leq r < k$ if ε_k is small enough.

Let us define $\mathcal{G}_k \subset \mathfrak{X}$ as the set of functions F in $\mathfrak{X}$ such that the corresponding function G_k satisfies the condition (5.6). We claim that the conclusion of the theorem holds if $F \in \mathcal{G}_{k_1} \cap \mathcal{G}_{k_2}$ with $k_1 < k_2$. To prove this, we will apply Corollary 2.3 **ii)**.

Let us expand the function $C(t)$ as a Fourier series

$$C(t) = \sum_{n\in\mathbb{Z}} C_n e^{2\pi n i t}$$

with $C_n = \overline{C}_{-n}$. We claim that the following result holds:

Lemma 5.3. *Assume that $k \geq 2$ is such that $C_k \neq 0$. Then, $\mathcal{G}_k$ is open and dense in $\mathfrak{X}$.*

Assuming by now that this result holds, we can easily complete the proof of the theorem. By direct substitution in (5.3), we observe that this equation does not admit solutions of the type $x(t) = \sum_{|n|\leq N} C_n e^{2\pi n i t}$ (trigonometric polynomials). In consequence, there exist infinitely many Fourier coefficients $C_n \neq 0$. Let us take $1 < k_1 < k_2$ such that $C_{k_1} \neq 0$ and $C_{k_2} \neq 0$. Then, according to Lemma 5.3, $\mathcal{G}_{k_1} \cap \mathcal{G}_{k_2}$ is open and dense. We can take $\mathcal{G} = \mathcal{G}_{k_1} \cap \mathcal{G}_{k_2}$ to complete the proof of Theorem 5.2. □

Proof of Lemma 5.3. Let $\mathfrak{X}_k^2$ be the class of functions $G : \mathbb{R} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$ and period k. It becomes a Banach space when it is endowed with the norm

$$\|G\|_2 = \max_{\alpha\in\mathbb{R}}[|G(\alpha)| + |G'(\alpha)| + |G''(\alpha)|].$$

The linear operator

$$\mathcal{L} : \mathfrak{X} \longrightarrow \mathfrak{X}_k^2$$
$$F \mapsto G_k$$

is continuous. The class of Morse functions is open in $\mathfrak{X}_k^2$. In consequence, $\mathcal{G}_k$ is open in $\mathfrak{X}$.

Next, we are going to compute $\mathcal{L}P$ for the choice $P(t) = \varepsilon\sin[2\pi(t-\tau)]$. From the definition,

$$\begin{aligned}\mathcal{L}P(\alpha) &= \frac{\varepsilon}{k}\int_0^k \dot\varphi_k(t)\sin[2\pi(t-\tau-\alpha)]dt \\ &= \frac{\varepsilon}{2ik^3}\int_0^k \dot C\left(\frac{t}{k}\right)(e^{2\pi i(t-\tau-\alpha)} - e^{-2\pi i(t-\tau-\alpha)})dt \\ &= -\frac{2\pi\varepsilon}{k}\mathcal{R}e(C_k e^{2\pi i(\tau+\alpha)}).\end{aligned} \tag{5.7}$$

If $C_k = |C_k|e^{i\omega}$,

$$\mathcal{L}P(\alpha) = -\frac{2\pi\varepsilon}{k}|C_k|\cos(2\pi(\tau+\alpha+\omega)).$$

Note that for simplicity in the notation, we do not include the dependence of ω with respect to k.

To prove that $\mathcal{G}_k$ is dense in $\mathfrak{X}$, we take an arbitrary function $F \in \mathfrak{X}$ with $G = \mathcal{L}F$. We will distinguish three cases which will cover all possibilities because G is a real analytic periodic function.

Case 1: $(G')^2 + \frac{1}{4\pi^2}(G'')^2$ is not a constant function.

We consider the perturbation $\widetilde{F}(t) = F(t) + \varepsilon \sin[2\pi(t-\tau)]$. We are going to prove that if $C_k \neq 0$ and $\varepsilon \neq 0$, then for all τ excepting a finite number, the function $\mathcal{L}\widetilde{F}$ is a Morse function.

The equations for a degenerate critical point of $\mathcal{L}\widetilde{F}$ are

$$\begin{aligned} G'(\alpha) + \frac{4\pi^2}{k}\varepsilon|C_k|\sin[2\pi(\tau+\alpha+\omega)] = 0 \\ \frac{1}{2\pi}G''(\alpha) + \frac{4\pi^2}{k}\varepsilon|C_k|\cos[2\pi(\tau+\alpha+\omega)] = 0. \end{aligned} \tag{5.8}$$

Equivalently,

$$R[2\pi(\alpha+\tau+\omega)] \begin{pmatrix} G'(\alpha) \\ \frac{1}{2\pi}G''(\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-4\pi^2\varepsilon}{k}|C_k| \end{pmatrix}, \tag{5.9}$$

where

$$R[\theta] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The previous system implies that

$$G'(\alpha)^2 + \frac{1}{4\pi^2}G''(\alpha)^2 = \frac{16\pi^4\varepsilon^2}{k^2}|C_k|^2.$$

From the analyticity of G and the assumption of case 1, we deduce that $\alpha \in [0,1)$ must belong to a finite set $\{\alpha_1, ..., \alpha_r\}$. For each $i = 1, ..., r$ and $\alpha = \alpha_i$, it is clear that (5.9) has a unique solution in $\tau \in [0,1)$, say, τ_i. After selecting some $\tau \neq \tau_i$, $i = 1, ..., r$, and $0 \leq \tau < 1$, we observe that $\widetilde{F}$ converges to F as $\varepsilon \longrightarrow 0$ and $\mathcal{L}\widetilde{F}$ is a Morse function.

Case 2: G is constant.

Taking $\widetilde{F}$ as in the previous case, we observe that $\mathcal{L}\widetilde{F}$ is a Morse function if $\varepsilon \neq 0$.

Case 3: $G'(\alpha) = c_1\cos(2\pi\alpha) + c_2\sin(2\pi\alpha)$, $c_1^2 + c_2^2 > 0$.

In this case, G is a Morse function. □

6. The attractor of the SIR model with periodic contact rate.

The classical SIR model with periodic contact rate [24, 25, 12] is given by the system of differential equations

$$\begin{cases} \dot{S} = \lambda - \mu S - \beta(t)SI \\ \dot{I} = \beta(t)SI - (\gamma+\mu)I \\ \dot{R} = \gamma I - \mu R \end{cases} \tag{6.1}$$

with all parameters strictly positive and $\beta : \mathbb{R} \longrightarrow [0, +\infty)$ a continuous and T-periodic function. Since $\dot{S} + \dot{I} + \dot{R} = \lambda - \mu(S+I+R)$, the planar system

$$\begin{cases} \dot{S} = \lambda - \mu S - \beta(t)SI \\ \dot{I} = \beta(t)SI - (\gamma+\mu)I \end{cases} \tag{6.2}$$

completely determines the dynamical behavior of (6.1). We emphasize that this system is meaningful only in $\mathbb{R}^2_+ = \{(S,I) : S \geq 0, I \geq 0\}$. Since the system (6.2) is defined on a proper subset of the plane, the dissipative structure has some subtleties. To describe them, we first observe that every solution of (6.2) is defined on a maximal interval of the type $[\alpha, +\infty)$ with $-\infty \leq \alpha < +\infty$. In fact, the vector field defined on (6.2) points inward or is tangent to the boundary of $\mathbb{R}^2_+$. More precisely,

$$X_1(t,0,I) > 0 \quad \text{and} \quad X_2(t,S,0) = 0$$

if $X(t,S,I) = (\lambda - \mu S - \beta(t)SI, \beta(t)SI - (\gamma+\mu)I)$. In addition, as long as the solutions stay in $\mathbb{R}^2_+$, they satisfy

$$\lambda - (\mu+\gamma)(S(t)+I(t)) \leq \frac{d}{dt}(S(t)+I(t)) \leq \lambda - \mu(S(t)+I(t)). \tag{6.3}$$

This differential inequality implies that the solutions cannot blow up in finite time.

Once we have analyzed the initial value problem, we can use (6.3) again to prove that

$$\limsup_{t \longrightarrow +\infty} [S(t)+I(t)] \leq \frac{\lambda}{\mu}$$

for each solution.

Let $\mathcal{B}$ be the family of the bounded solutions. This means that $\alpha = -\infty$ and

$$\sup_{t \in \mathbb{R}} [S(t)+I(t)] < \infty.$$

The corresponding section at time t is denoted by

$$\mathcal{A}_t = \{(S(t),I(t)) : (S,I) \in \mathcal{B}\}.$$

Standard arguments in the theory of dissipative systems can be employed to prove that $\mathcal{A}_t$ is a continuum contained in $\mathbb{R}^2_+$ with the properties

$$\mathcal{A}_{t_1} \cong \mathcal{A}_{t_2}, \;\; \mathcal{A}_{t+T} = \mathcal{A}_t, \mathcal{A}_t \cap \partial\mathbb{R}^2_+ = \{E\},$$

where $E = \left(\frac{\lambda}{\mu}, 0\right)$ is the semi-trivial equilibrium.

The periodic family $\{\mathcal{A}_t\}_{t \in \mathbb{R}}$ has the attracting property:

$$dist[(S(t),I(t)), \mathcal{A}_t] \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty$$

for every solution $(S(t),I(t))$ of (6.2).

In view of these facts, we can say that $\mathcal{A}_0$ is the attractor of the embedding

$$P_* : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+, \quad (S(0),I(0)) \mapsto (S(T),I(T)).$$

The main result of the section is concerned with the topology of $\mathcal{A}_0$.

Theorem 6.1. *Assume that*

$$\mathcal{R}_0 := \frac{\lambda}{\mu(\gamma+\mu)T} \int_0^T \beta(t)dt > 1. \tag{6.4}$$

If (6.2) admits a kT-periodic solution that is not T-periodic with $k \in \mathbb{N}$, then $\mathcal{A}_0$ is not arc-wise-connected.

Remark 6.2. The reader can consult [25, 12, 24] for sufficient conditions on the existence of subharmonics in (6.2). We emphasize that if (6.4) is not satisfied, then $(\frac{\lambda}{\mu}, 0)$ is globally asymptotically stable in (6.2); see Proposition 1 in [3].

The map P_* is not defined in the whole plane and so the results of Section 2 are not directly applicable. Our strategy will be to construct a map $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with the properties

$$P = P_* \quad \text{on } \mathbb{R}^2_+, \quad P \in \mathcal{LH}(\mathbb{R}^2), \quad \mathcal{A} = \mathcal{A}_0,$$

where $\mathcal{A}$ is the attractor of P as defined in Section 2.

To this end we consider the auxiliary system

$$\begin{cases} \dot{S} = \lambda - \mu S - \beta(t) S^+ I^+ \\ \dot{I} = \beta(t) S^+ I^+ - (\gamma + \mu) I \end{cases} \tag{6.5}$$

defined on the whole plane $\mathbb{R}^2$. We employ the notion $S^+ = \sup\{S, 0\}$ and $I^+ = \sup\{I, 0\}$.

This system is linear outside the first quadrant and all solutions are defined on the whole line. The associated Poincaré map is an orientation preserving homeomorphism of $\mathbb{R}^2$ extending P_*. The region $\{I \leq 0\}$ is invariant and all solutions lying there converge to the equilibrium E. The solutions starting at the second quadrant are unbounded in the past and eventually enter into the first quadrant. Therefore, the map P is dissipative and all bounded orbits lie in $\mathbb{R}^2_+$, and, in consequence, $\mathcal{A} = \mathcal{A}_0$.

At this moment it is not obvious that P belongs to $\mathcal{LH}(\mathbb{R}^2)$ because we do not know if $\mathcal{A}$ has an empty interior. It could seem reasonable to transform the system under the change of variable $L = \ln I$. In the variables (S, L), the system has negative divergence on $[0, +\infty) \times \mathbb{R}$ and the corresponding Poincaré map is area contracting. However, since $\mathcal{A}$ is a set touching the equilibrium E, the measure of $\mathcal{A} \backslash \{E\}$ could be infinite in the new coordinates.

Since the previous attempt does not work, we will need several preliminary results before concluding that $\mathcal{A} \subset \mathbb{R}^2_+$ has zero measure.

Lemma 6.3. *Assume that (6.4) holds. Then, there exists a compact set $B \subset [0, +\infty) \times (0, +\infty)$ so that $P(B) \subset B$, and for each $p \in [0, +\infty) \times (0, +\infty)$ there exist an integer $n_p \geq 1$ and a neighborhood U_p of p with*

$$P^n(U_p) \subset B$$

if $n \geq n_p$.

Proof. All topological notions will be understood with respect to the relative topology of $\mathbb{R}^2_+$. Using (6.3), the set

$$C = \{(S, I) \in \mathbb{R}^2_+ : S + I \leq \frac{\lambda}{\mu} + 1\}$$

is positively invariant under P. In fact,

$$P(C) \subset int_{\mathbb{R}^2_+}(C) \tag{6.6}$$

where $int_{\mathbb{R}^2_+}(C) = \{(S, I) \in \mathbb{R}^2_+ : S + I < \frac{\lambda}{\mu} + 1\}$ is the interior of C relative to $\mathbb{R}^2_+$. Moreover, for each $p \in \mathbb{R}^2_+$, there exists $n_1 \in \mathbb{N}$ so that

$$P^n(p) \in int_{\mathbb{R}^2_+}(C)$$

for all $n \geq n_1$.

On the other hand, condition (6.4) guarantees that the disease persists uniformly strongly (see Theorem 3.1 in [26]). Hence, there is a constant $\varepsilon > 0$ so that, for all $p = (S, I) \in \mathbb{R}^2_+$ with $I > 0$,

$$\liminf I_n > \varepsilon \tag{6.7}$$

with $\{(S_n, I_n)\} = \{P^n(p)\}$. Define the set

$$G = C \cap \{(S, I) \in \mathbb{R}^2_+ : I \geq \varepsilon\}.$$

From the previous properties we deduce that every forward orbit lying in $[0, +\infty) \times (0, +\infty)$ will enter the set $int_{\mathbb{R}^2} G$. In particular, due to the continuity of P, given $q \in G$, it is possible to find an open neighborhood W_q and an integer $n_q \geq 1$ such that

$$P^{n_q}(W_q) \subset G.$$

Since G is compact, we can take a family of points $q_1, ..., q_m \in G$ so that

$$G \subset \bigcup_{i=1}^{m} W_{q_i}.$$

Define $N = \max\{n_{q_i} : i = 1, ..., m\} + 1$ and

$$B = G \cup P(G) \cup ... \cup P^{N-1}(G).$$

We claim that B is positively invariant. In view of the definition of B, this will follow from the inclusion $P^N(G) \subset B$. To prove it, we pick any point $q \in G$, then $q \in W_{q_i}$ for some $i \in \{1, ..., m\}$. In consequence,

$$P^N(q) = P^{N-n_{q_i}}(P^{n_{q_i}}(q)) \in P^{N-n_{q_i}}(P^{n_{q_i}}(W_{q_i})) \subset P^{N-n_{q_i}}(G).$$

Once we know that B is positively invariant, we must prove that it also enjoys the attraction property. For each point $p \in [0, +\infty) \times (0, +\infty)$, we know that the forward orbit will enter into $int_{\mathbb{R}^2_+}(G)$. Then, there exist an integer $n_p \geq 1$ and a neighborhood U_p with

$$P^{n_p}(U_p) \subset G \subset B.$$

The positive invariance of B implies that $P^n(U_p) \subset B$ for each $n \geq n_p$. □

Our next step will be to construct a curve emanating from E and contained in $\mathcal{A}$. We want to construct it as one of the branches of the unstable manifold in E. There is an apparent obstruction because the system we have defined on the whole plane is not $\mathcal{C}^1$ and it is not possible to linearize around E. For this reason we go back to the original system (6.2) but now we assume that the phase space is the whole plane. The Poincaré map of this system is denoted by $\widehat{P} : \mathcal{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where $\mathcal{D}$ is an open set containing $\mathbb{R}^2_+$ and $P = \widehat{P}$ on $\mathbb{R}^2_+$. At the equilibrium E, the periodic linearized system is

$$\dot{\zeta} = -\mu\zeta - \frac{\lambda}{\mu}\beta(t)\eta, \quad \dot{\eta} = \frac{\lambda}{\mu}\beta(t)\eta - (\gamma + \mu)\eta$$

with Floquet multipliers $\mu_1 = e^{-\mu T}$, $\mu_2 = e^{(\mathcal{R}_0 - 1)(\gamma+\mu)T}$. These numbers are the eigenvalues of the Jacobian matrix $\widehat{P}'(E)$. When (6.4) holds, $0 < \mu_1 < 1 < \mu_2$ and E is a hyperbolic fixed point of $\widehat{P}$. The stable manifold is $W^s(E) = \mathbb{R} \times \{0\}$ and the unstable manifold can be split as

$$W^u(E) = W_+ \cup W_-$$

with $W_+ \subset \mathbb{R}^2_+$, $W_- \subset \{(S, I) : I \leq 0\}$, and $W_+ \cap W_- = \{E\}$. For our purposes, the branch W_- has no significance, but W_+ plays an important role in the dynamics of P. We list some useful properties:

$$P(W_+) = W_+,\ W_+ \subset \mathcal{A}_0,\ W_+ = \{p \in \mathbb{R}^2_+ : P^{-n}(p) \longrightarrow E \ as\ n \longrightarrow +\infty\}.$$

The set W_+ has zero measure. This is a consequence of the stable manifold theorem because W_+ can be described as $W_+ = \{\varphi(s) : s \in [0, +\infty)\}$ where $\varphi : [0, +\infty) \longrightarrow \mathbb{R}^2$ is a $\mathcal{C}^1$ map. Next, we consider the invariant splitting

$$\mathcal{A} = W_+ \cup (\mathcal{A} \backslash W_+)$$

and we are going to prove that also $\mathcal{A}\backslash W_+$ has zero measure. To prove this, we first observe that this set is contained in the compact set B given by Lemma 6.3. To prove $\mathcal{A}\backslash W_+ \subset B$, we take a point $q \in \mathcal{A}\backslash W_+$ and select a sequence of integers $\tau(n) \longrightarrow +\infty$ such that $P^{-\tau(n)}(q) \longrightarrow p$ with $p \neq E$. This is possible because $q \notin W_+$ and $\{P^{-n}(q)\}_{n\geq 0}$ is bounded. The point p belongs to the attractor and we know that $\mathcal{A} \cap \{(S, I) : I = 0\} = \{E\}$. Therefore, $p \in [0, +\infty) \times (0, +\infty)$, and Lemma 6.3 implies the existence of n_p and U_p with $P^n(U_p) \subset B$ if $n \geq n_p$. Let us select an integer n large enough so that $P^{-\tau(n)}(q) \in U_p$ and $\tau(n) > n_p$. Then, $P^{n_p-\tau(n)}(q) \in B$. Since B is positively invariant and $n_p - \tau(n) < 0$, we deduce that $q \in B$.

Once we know that $\mathcal{A}\backslash W_+ \subset B$, we can guarantee the existence of some $\delta > 0$ such that $I \geq \delta$ for each $(S, I) \in \mathcal{A}\backslash W_+$. Note that B is compact and $B \cap \{(S, I) : I = 0\} = \emptyset$.

It is time to go back to the change of variable $L = \ln I$ we mentioned before. In the variables $(S, L) \in [0, +\infty) \times \mathbb{R}$, the system (6.1) becomes

$$\begin{cases} \dot{S} = Y_1(t, S, L) = \lambda - \mu S - \beta(t) S e^L \\ \dot{L} = Y_2(t, S, L) = \beta(t) S - (\gamma + \mu). \end{cases} \tag{6.8}$$

The divergence of the vector field $Y(t, \cdot)$ is given by

$$divY(t, S, L) = \frac{\partial Y_1}{\partial S} + \frac{\partial Y_2}{\partial L} = -\mu - \beta(t) e^L < 0.$$

The Poincaré map in the new variables, denoted by $\widetilde{P} : [0, +\infty) \times \mathbb{R} \longrightarrow [0, +\infty) \times \mathbb{R}$, is area contracting. This means that

$$\mu(\widetilde{P}(A)) < \mu(A)$$

if $A \subset [0, +\infty) \times \mathbb{R}$ is a measurable set with $0 < \mu(A) < +\infty$ and μ is the Lebesgue measure in $\mathbb{R}^2$. Define

$$\mathcal{L} = \{(S, \ln I) : (S, I) \in \mathcal{A}\backslash W_+\}.$$

This set is bounded and measurable, in particular, $\mu(\mathcal{L}) < +\infty$. Since $\widetilde{P}(\mathcal{L}) = \mathcal{L}$, we must conclude that $\mathcal{L}$ has zero measure. Going back to the original variables, we conclude that $\mathcal{A}\backslash W_+$ has zero measure.

Once we know that $\mu(\mathcal{A}) = 0$, we can say that P is in the class $\mathcal{LH}(\mathbb{R}^2)$ and we are ready for the proof of Theorem 6.1. The disease-free equilibrium $(\frac{\lambda}{\mu}, 0)$ is always a fixed point of P. By assumptions, P has a k-cycle in $(0, +\infty)^2$ that is not a fixed point of P. On the other hand, the condition (6.4) guarantees the existence of a fixed point in $\{(S, I) : S > 0, I > 0\}$; see Theorem 1 in [10]. The conclusion now follows from Corollary 2.1 **i)**. □

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