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Invariant Measures as Obstructions to Attractors in Dynamical Systems and Their Role in Nonholonomic Mechanics

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Abstract—We present some results on the absence of a wide class of invariant measures for dynamical systems possessing attractors. We then consider a generalization of the classical nonholonomic Suslov problem which shows how previous investigations of existence of invariant measures for nonholonomic systems should necessarily be extended beyond the class of measures with strictly positive C^1 densities if one wishes to determine dynamical obstructions to the presence of attractors.

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1. INTRODUCTION

It is a well-established idea that the existence of an invariant measure is an important feature of a dynamical system. However, the type of information that can be extracted from such an invariant measure depends on its properties. For instance, a measure concentrated at a point is invariant for a system if and only if the point is an equilibrium. It is clear that no additional information about the system can be extracted from the invariance of such measure. This paper focuses on the invariance of a specific class of measures having an almost everywhere positive density that results in the obstruction to the existence of attractors in the phase space.

Our motivation comes from nonholonomic mechanics. It is well known that this type of mechanical systems does not possess a symplectic Hamiltonian structure and, therefore, the existence of a smooth invariant volume form on the phase space is not guaranteed. Even though nonholonomic systems are energy-preserving, the absence of an invariant measure may lead to surprising phenomena like the reversal of the rattleback (see, e. g., [1, 3, 4] and references therein) which cannot occur in symplectic Hamiltonian systems.

Since Blackall's early work [2], a great number of references have analyzed the conditions of existence of smooth invariant measures for nonholonomic systems, e. g., [5, 7, 8, 10, 13, 15, 16, 20, 22]. A common point in these analyses is that the density of the invariant measure (with respect to a certain volume form) is assumed to be everywhere positive on the phase space. This assumption is used to work with the logarithm of the density function.

The point of this paper is to indicate that the condition that the density of the measure is everywhere positive is too strong if one is interested in determining obstructions to the existence

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of attractors in the phase space. For this reason, we relax this condition and allow the densities of measures in the class $\mathcal{A}$ that we consider to vanish on a set of measure zero. To illustrate the potential relevance of these measures in nonholonomic systems, we provide a concrete example, consisting of a generalization of the classical Suslov problem, which for certain parameter values possesses an invariant measure of type $\mathcal{A}$ whose density necessarily vanishes. In particular, the existence of this invariant measure cannot be detected with the techniques of the references indicated above.

The relevance of integral invariants which are not necessarily positive everywhere has been indicated before by Kozlov [17, Section 5]. The present paper complements his discussion in the context of obstructions to the existence of attractors. Moreover, we clarify their role in nonholonomic mechanics with our example.

The paper is organized as follows. We first introduce invariant measures of class $\mathcal{A}$ in Section 2 and prove that they are obstructions for the existence of attractors in Proposition 1. In Section 3 we study our example. After deriving the reduced equations of motion and their main properties, we discuss existence of invariant measures in Section 3.5. We finally give some concluding remarks in Section 4.

2. INVARIANT MEASURES AND ATTRACTORS IN DYNAMICAL SYSTEMS

In the Euclidean space $\mathbb{R}^N$ we consider measures μ of the type

$$\mu = M(x_1, \dots, x_N)\, dx_1 \cdots dx_N, \tag{2.1}$$

where the density function $M : \mathbb{R}^N \to \mathbb{R}$ is measurable, locally integrable and strictly positive almost everywhere in the sense of Lebesgue. This class of measures will be denoted by $\mathcal{A}$.

Assume now that $X : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 vector field such that the system of equations

$$\dot{x} = X(x) \tag{2.2}$$

defines a complete flow $\{\phi_t\}_{t\in\mathbb{R}}$ in $\mathbb{R}^N$. We say that the measure $\mu \in \mathcal{A}$ is invariant under the flow if

$$\mu\big(\phi_t(A)\big) = \mu(A),$$

for each $t \in \mathbb{R}$ and each Lebesgue measurable set $A \subset \mathbb{R}^N$. This definition makes sense since diffeomorphisms preserve the σ-algebra of Lebesgue measurable sets.

The book [19] contains an excellent introduction to the theory of invariant measures. Following the original terminology of Poincaré, and allowing a certain ambiguity, these measures are sometimes called *integral invariants*. It is well known that the partial differential equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\,(MX_i) = 0 \tag{2.3}$$

can be employed to characterize invariant measures with $M \in C^1(\mathbb{R}^N)$. See, e. g., [19, p. 429].

In contrast to some previous literature mentioned in the introduction, we are allowing sets of measure zero where the density can vanish. This is useful to enlarge the class of systems (2.2) with invariant measure. As a simple example, consider the linear system in $\mathbb{R}^2$,

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = 2x_2. \tag{2.4}$$

The measure defined by (2.1) with $M(x_1, x_2) = |x_1|^5 x_2^2$ is invariant. In fact, M is in $C^1(\mathbb{R}^2)$ and it is easy to check that it satisfies (2.3). Since M only vanishes on the coordinate axes we note that the measure belongs to the class $\mathcal{A}$. On the other hand, this linear system cannot admit an invariant measure μ of the form given in (2.1) with $M \in C^1(\mathbb{R}^2)$ and $M(x_1, x_2) > 0$ for every $(x_1, x_2) \in \mathbb{R}^2$. Equation (2.3) can be expressed as

$$-x_1 \frac{\partial M}{\partial x_1} + 2x_2 \frac{\partial M}{\partial x_2} + M = 0,$$

and it is clear that any solution must vanish at the origin.

Assume now that $K \subset \mathbb{R}^N$ is a nonempty compact set that is invariant under the flow; that is,

$$\phi_t(K) = K \qquad \text{for each } t \in \mathbb{R}.$$

We say that K is an *attractor* if there exists an open set $W \subset \mathbb{R}^N$ with $K \subset W$ and such that

$$\operatorname{dist}(\phi_t(x), K) \to 0 \quad \text{as} \quad t \to +\infty$$

for each $x \in W$.

It is intuitively clear that class $\mathcal{A}$ invariant measures cannot exist in the presence of an attractor. The following proposition formalizes this idea. In its proof, we use the following two properties of class $\mathcal{A}$ measures which follow directly from their definition: 1) the measure of any nonempty open set is positive; and 2) the measure of any bounded set is finite.

Proposition 1. *Assume that the flow associated to the system* (2.2) *has an attractor. Then there are no invariant measures in the class* $\mathcal{A}$.

Proof. By contradiction, assume that $\mu \in \mathcal{A}$ is invariant. Let W be an open and bounded set with $K \subset W$ and attracted by K. We fix a small number $r > 0$ such that $K_r \subset W$, where

$$K_r = \{x \in \mathbb{R}^N \,:\, \operatorname{dist}(x, K) \leqslant r\}. \tag{2.5}$$

Since $W \setminus K_r$ is a nonempty open set, we know that it has positive measure. Let us take $\delta > 0$ with

$$\delta < \mu(W \setminus K_r) = \mu(W) - \mu(K_r). \tag{2.6}$$

Now consider the sequence of measurable functions

$$\chi_{K_r} \circ \phi_{t_n} : W \to \mathbb{R},$$

where χ_{K_r} is the characteristic function of K_r and $\{t_n\}$ is a sequence with $t_n \to +\infty$. Since K attracts W, we obtain

$$\chi_{K_r} \circ \phi_{t_n}(x) \to 1 \qquad \text{as} \quad n \to +\infty,$$

for each $x \in W$. In principle, this convergence must be understood in a pointwise sense but we can apply Egoroff's theorem (see, e. g., [9]) to obtain uniform convergence in a suitable set. More precisely, we can find a measurable set $W_* \subset W$ such that $\mu(W \setminus W_*) < \delta$ and the above convergence is uniform in $x \in W_*$. Then, for n large enough, we know that $\phi_{t_n}(W_*)$ is contained in K_r. In particular, $\mu\big(\phi_{t_n}(W_*)\big) \leqslant \mu(K_r)$. Using the invariance of the measure, we find

$$\begin{aligned} \mu(W) &= \mu(W_*) + \mu(W \setminus W_*) \\ &= \mu\big(\phi_{t_n}(W_*)\big) + \mu(W \setminus W_*) \\ &< \mu(K_r) + \delta, \end{aligned}$$

which is incompatible with (2.6) since $\mu(K_r)$ and $\mu(W)$ are finite. □

Remark 1. The hypothesis that the region of attraction W contains the compact invariant set K may be weakened if K has zero measure. Specifically, the conclusion of the proposition remains valid under the following condition: *There exists a nonempty open set W with $\phi_t(x) \to K$ if $t \to +\infty$ for each $x \in W$.* Indeed, in this case one can obviously find $\delta, r > 0$ such that (2.6) holds with K_r defined by (2.5), and the rest of the proof applies unchanged. This can be used to conclude nonexistence of class $\mathcal{A}$ invariant measures in the presence of partial attractors. As an example, consider $N = 1$ and $\dot{x}_1 = \sin^2(x_1)$, $K = \{0\}$.

For the application we have in mind we need to replace $\mathbb{R}^N$ by a finite-dimensional manifold. This can be done without too much effort. Assume that $(\mathcal{M}, g)$ is a Riemannian manifold of dimension n and let μ_g be the associated measure. In local coordinates $(x_1, \ldots, x_N)$ the measure μ_g is defined by the formula

$$\sqrt{\det g(x_1, \ldots, x_N)}\, dx_1 \cdots dx_N.$$

Then we will consider the class $\mathcal{A}(\mathcal{M})$ of admissible measures

$$\mu = M\mu_g,$$

where $M : \mathcal{M} \to \mathbb{R}$ is a locally integrable function (with respect to μ_g) that is strictly positive almost everywhere in $\mathcal{M}$. The reader interested in measure theory will have no problems to prove that the class $\mathcal{A}(\mathcal{M})$ is independent of the metric g, however, this fact will not be used in this paper.

Given a vector field X on $\mathcal{M}$, we can consider the system (2.2) on the manifold and it will be assumed that the associated flow $\{\phi_t\}_{t\in\mathbb{R}}$ is complete on $\mathcal{M}$. The notion of attractor is defined as before and the proof of Proposition 1 is easily adapted[1)] to this setting.

Remark 2. If the density of an invariant measure of class $\mathcal{A}$ is of class C^1, then the zero-measure subset $Z \subset \mathcal{M}$ where it vanishes is invariant by the flow (see [17, Theorem 4]). Therefore, in this situation, the system possesses an invariant measure with positive C^1 density on the invariant full-measure subset $\mathcal{M} \setminus Z$. We remark that this last condition on its own is not sufficient to obstruct the existence of attractors on $\mathcal{M}$ since these may be contained in Z (it is not hard to build explicit examples). The existence of an invariant measure of class $\mathcal{A}$ contains more delicate information about the system which allows us to rule out the existence of attractors on the whole manifold $\mathcal{M}$.

3. NONHOLONOMIC EXAMPLE: A GENERALIZATION OF THE SUSLOV PROBLEM

In this section we introduce our example which is a generalization of the classical nonholonomic Suslov problem for the dynamics of a rigid body. Our generalization consists in the incorporation of an internal rotor, which adds a degree of freedom to the system, leading to a more delicate dependence of the dynamics on the system parameters. Our main point is to discuss existence of invariant measures, which we do in Section 3.5.

3.1. Description of the System

Consider a rigid body, which we refer to as the *carrier*, with a rotor in its inside. For simplicity, we suppose that the center of mass of the carrier coincides with that of the rotor and is fixed throughout the motion. Also, for simplicity, we assume that the axis of symmetry of the rotor is aligned with the smallest axis of inertia of the carrier, which we label as the third one. The angular velocity of the rotor is then $\Omega_r = \Omega + \dot{\theta} E_3$ where Ω is the angular velocity of the carrier, $E_3 = (0, 0, 1)$, and θ is the rotation angle of the rotor about its axis (see Fig. 1). All of the vectors given above, and all those appearing hereafter, are written with respect to a frame attached to the carrier which is centered at its center of mass and is aligned with its principal axes of inertia.

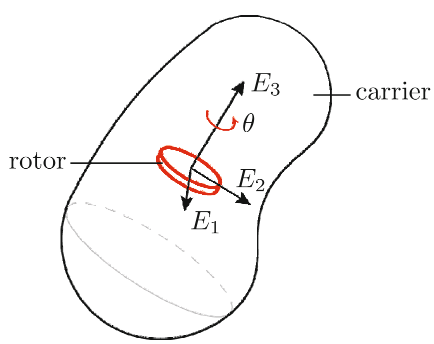


Fig. 1. Schematic representation of the carrier-rotor system.

[1)] the adaptation requires the replacement of bounded open sets in $\mathbb{R}^N$ by relatively compact open sets in the manifold $\mathcal{M}$.

The configuration space of the problem is the direct product Lie group $Q = SO(3) \times S^1$ where the elements of $SO(3)$ specify the orientation of the carrier relative to an inertial frame and the angle $\theta \in S^1$ gives the orientation of the rotor relative to the carrier as explained above. The kinetic energy Lagrangian is

$$L = \frac{1}{2}\langle \mathbb{I}_0 \Omega, \Omega \rangle + \frac{1}{2}\langle \mathbb{I}_r(\Omega + \dot{\theta}E_3), \Omega + \dot{\theta}E_3 \rangle,$$

where $\mathbb{I}_0 = \operatorname{diag}(I_1, I_2, I_3)$ is the inertia tensor of the carrier and $\mathbb{I}_r = \operatorname{diag}(K_1, K_1, K_3)$ is the inertia tensor of the rotor (we assume that $K_3 > 0$ and $K_1 \geqslant 0$). We assume that the principal moments of inertia of the carrier satisfy

$$0 < I_3 < I_2 < I_1. \tag{3.1}$$

We consider the evolution of the above Lagrangian system subject to a Suslov-type nonholonomic constraint imposed to the angular velocity Ω_r of the rotor. Specifically, we require that

$$\langle a, \Omega_r \rangle = \langle a, \Omega + \dot{\theta}E_3 \rangle = 0, \tag{3.2}$$

for a fixed nonzero vector $a \in \mathbb{R}^3$. Physically this means that the component of the (total) angular velocity of the rotor about an axis which is *fixed in the carrier* vanishes. This problem may be understood as a generalization of the classical Suslov problem [21][2)]. In the following, we will refer to a as the *vector of forbidden rotations* and denote its components by $a := (a_1, a_2, a_3)$. Figure 2 illustrates a realization of the system inspired by known realizations of the Suslov and Veselova problems (see, e. g., [6]).

The phase space of the system, consisting of all configurations and admissible velocities, is a 7-dimensional manifold diffeomorphic to $SO(3) \times S^1 \times \mathbb{R}^3$.

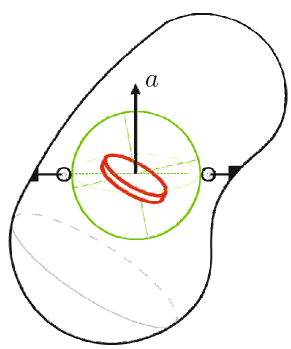


Fig. 2. Realization of the generalized Suslov problem. The rotor is enclosed in a spherical shell that rotates with it. The sphere is touched at antipodal points by a pair of parallel flat wheels (disks) that are attached to the carrier. The vector a of forbidden rotations is contained in the plane of the wheels and is perpendicular to the wheels' axis and is therefore fixed in the carrier frame. The effect of the wheels is to forbid rotations of the sphere, and hence also the rotor, about the a axis.

3.2. Equations of Motion

Since both the Lagrangian and the constraints are written solely in terms of Ω and $\dot{\theta}$, the constraint distribution and the Lagrangian are invariant under left multiplication on the direct product Lie group $SO(3) \times S^1$. Nonholonomic systems of this type, whose configuration space is a Lie group and possesses the invariance properties mentioned above, are called *LL-systems* (see,

[2)] We recall that the Suslov problem considers the motion of a rigid body, with no rotor, subject to the constraint $\langle a, \Omega \rangle = 0$.

e. g., [12, 14]). Following [11], the reduced equations of motion are termed *Euler – Poincaré – Suslov equations*. In our case these are given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \Omega}\right) &= \frac{\partial L}{\partial \Omega} \times \Omega + \zeta a, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) &= \zeta a_3, \end{aligned} \tag{3.3}$$

for a multiplier ζ that will be determined below.

If $a_3 = 0$, i. e., if the vector of forbidden rotations is perpendicular to the axis of the rotor, then the constraint does not involve $\dot\theta$, the second equation in (3.3) becomes a conservation law and the system may be shown to be equivalent to the Suslov problem with gyroscope studied before in [18]. In fact, the resulting system further simplifies to the classical Suslov problem since the axis of the gyroscope is perpendicular to the vector a of forbidden rotations (see [18]).

For the rest of the paper we will focus on the the case $a_3 \neq 0$, and assume, without loss of generality, that

$$a_3 = 1.$$

We now proceed to determine the multiplier ζ in terms of $\dot\Omega$. On the one hand, differentiation of the constraint (3.2) gives

$$\ddot\theta + \langle a, \dot\Omega\rangle = 0. \tag{3.4}$$

On the other hand, considering that $\frac{\partial L}{\partial \dot\theta} = K_3(\dot\theta + \Omega_3)$, where $\Omega := (\Omega_1, \Omega_2, \Omega_3)$, the second equation in (3.3) yields

$$\zeta = K_3(\ddot\theta + \dot\Omega_3). \tag{3.5}$$

Combining (3.4) and (3.5) leads to our desired expression for ζ:

$$\zeta = K_3(\dot\Omega_3 - \langle a, \dot\Omega\rangle).$$

We now use the above expression for ζ and the constraint (3.2) to rewrite the first equation in (3.3) as an autonomous system in 3 unknowns, $\Omega_1, \Omega_2, \Omega_3$, describing the evolution of the reduced system. Using that $\frac{\partial L}{\partial \Omega} = (\mathbb{I}_0 + \mathbb{I}_r)\Omega + K_3\dot\theta E_3$, we get

$$\frac{d}{dt}\big((\mathbb{I}_0 + \mathbb{I}_r)\Omega - K_3\langle a, \Omega\rangle E_3\big) = \big((\mathbb{I}_0 + \mathbb{I}_r)\Omega - K_3\langle a, \Omega\rangle E_3\big) \times \Omega + K_3\big(\dot\Omega_3 - \langle a, \dot\Omega\rangle\big)a. \tag{3.6}$$

The above equation may be written in a more appealing form by introducing the matrices[3)]

$$\begin{aligned} \tilde{\mathbb{I}} &:= \mathbb{I}_0 + \mathbb{I}_r - K_3 E_3 \otimes E_3 = \operatorname{diag}(I_1 + K_1, I_2 + K_1, I_3), \\ \mathbb{K}_a &:= \tilde{\mathbb{I}} + K_3\,(E_3 - a) \otimes (E_3 - a)\,, \\ \mathbb{B}_a &:= \tilde{\mathbb{I}} + K_3 E_3 \otimes (E_3 - a)\,. \end{aligned}$$

A calculation shows that (3.6) is equivalently written as the following system for the evolution of $\Omega \in \mathbb{R}^3$:

$$\frac{d}{dt}\,(\mathbb{K}_a\Omega) = (\mathbb{B}_a\Omega) \times \Omega. \tag{3.7}$$

For future reference we note that the explicit form of the matrices $\mathbb{K}_a$ and $\mathbb{B}_a$ is:

$$\mathbb{K}_a = \begin{pmatrix} \lambda_1 + a_1^2 K_3 & a_1 a_2 K_3 & 0 \\ a_1 a_2 K_3 & \lambda_2 + a_2^2 K_3 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \mathbb{B}_a = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ -a_1 K_3 & -a_2 K_3 & \lambda_3 \end{pmatrix},$$

3) For column vectors $b, c \in \mathbb{R}^3$ we denote by $b \otimes c$ the 3×3 matrix given by bc^T where ^{T} denotes transposition. It is clear that $b \otimes c$ has rank 1 if both b and c are nonzero.

with

$$\lambda_1 = I_1 + K_1, \qquad \lambda_2 = I_2 + K_1, \qquad \lambda_3 = I_3.$$

We also note that our assumption (3.1) implies

$$0 < \lambda_3 < \lambda_2 < \lambda_1. \tag{3.8}$$

The full (unreduced) equations of motion of the system on the 7-dimensional phase space consist of the reduced system (3.7) complemented with the reconstruction equations for the attitude matrix $g \in SO(3)$ of the carrier and the angle θ. These are given by the kinematical relations

$$\dot{g} = g\hat{\Omega}, \qquad \dot{\theta} = -\langle a, \Omega \rangle.$$

As usual, $\hat{\Omega}$ denotes the unique 3×3 skew-symmetric matrix such that $\hat{\Omega}v = \Omega \times v$ for all $v \in \mathbb{R}^3$.

3.3. Conservation of Energy

It is clear from (3.7) that the energy

$$E(\Omega) := \frac{1}{2}\langle \mathbb{K}_a \Omega, \Omega \rangle \tag{3.9}$$

is a first integral. We observe that $\mathbb{K}_a$ is symmetric and positive definite and, therefore, the level sets of E for the reduced system are ellipsoids, which we denote by

$$\mathcal{E}_\eta := \left\{ \Omega \in \mathbb{R}^3 \,:\, E(\Omega) = \eta > 0 \right\}.$$

Since Eqs. (3.7) are homogeneous quadratic in Ω, it is easy to show that if $t \mapsto \Omega(t)$ is a solution of (3.7), then so is $t \mapsto c\Omega(ct)$ for any $c \in \mathbb{R}$. In particular, since E is also homogeneous quadratic in Ω, it follows that the dynamics on the different constant energy ellipsoids $\mathcal{E}_\eta$, $\eta \neq 0$, is conjugated by a constant time rescaling.

3.4. Equilibria

The nonzero equilibrium points of (3.7) correspond to steady rotations of the carrier with constant angular velocity. We now classify these solutions and determine their stability on each constant energy ellipsoid $\mathcal{E}_\eta$.

It is clear from (3.7) that these equilibrium points are precisely the eigenvectors of $\mathbb{B}_a$. By our assumption (3.1), the matrix $\mathbb{B}_a$ has distinct (real) eigenvalues $\lambda_3 < \lambda_2 < \lambda_1$, so we conclude that the set of equilibria of (3.7) is composed by three lines V_i, $i = 1, 2, 3$, passing through the origin

$$\text{set of equilibria} = \bigcup_{i=1}^{3} V_i, \qquad V_i := \ker(\mathbb{B}_a - \lambda_i I).$$

Each constant energy ellipsoid $\mathcal{E}_\eta$, with $\eta > 0$, intersects the line V_i at two opposite points $\pm v_i$, so there is a total of 6 equilibrium points on $\mathcal{E}_\eta$ organized in 3 pairs $\pm v_i$, $i = 1, 2, 3$ (see Fig. 3). When analyzing their stability below it is useful to keep in mind that the dynamics near v_i coincides with that near $-v_i$ if the sense of time is reversed. This is a consequence of the reversibility of the dynamics of (3.7), i. e., if $t \mapsto \Omega(t)$ is a solution, so is $t \mapsto -\Omega(-t)$.

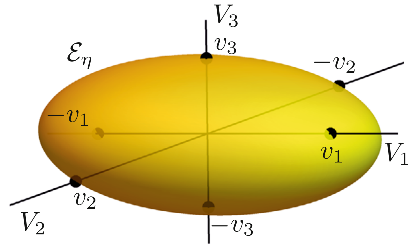


Fig. 3. The lines V_i and the 6 equilibrium points $\pm v_i$ on a constant energy ellipsoid $\mathcal{E}_\eta$.

3.4.1. Stability analysis

The linearization of (3.7) at the equilibrium point $0 \neq v_i \in V_i$, i. e., corresponding to the eigenvalue λ_i, is given by

$$\mathbb{K}_a \dot{\delta} = G_{v_i} \delta, \tag{3.10}$$

where G_{v_i} is the endomorphism of $\mathbb{R}^3$ defined by

$$G_{v_i}\delta = [(\mathbb{B}_a - \lambda_i I)\delta] \times v_i.$$

Since $G_{v_i} v_i = 0$, the linear system (3.10) has the line of equilibria $V_i = \langle v_i \rangle$. A more geometric insight into the linearized system may be obtained by recalling that the constant energy ellipsoids are invariant under the nonlinear flow (3.7). As a consequence, the tangent plane $T_{v_i}\mathcal{E}_\eta$ to the ellipsoid $\mathcal{E}_\eta$, $\eta = E(v_i)$, is invariant by the linearized flow (3.10). In other words, we can split $\mathbb{R}^3$ in the form

$$\mathbb{R}^3 = V_i \oplus W_i \quad \text{with} \quad W_i := T_{v_i}\mathcal{E}_\eta = \big(\nabla_\Omega E(v_i)\big)^\perp = (\mathbb{K}_a V_i)^\perp,$$

and

$$\mathbb{K}_a^{-1} G_{v_i}(W_i) \subseteq W_i, \qquad \mathbb{K}_a^{-1} G_{v_i} = 0 \text{ on } V_i.$$

As a consequence, the characteristic polynomial $\tilde{p}(z) := \det(zI - \mathbb{K}_a^{-1} G_{v_i})$ can be factorized as

$$\tilde{p}(z) = z(z^2 + \tilde{\alpha} z + \tilde{\beta}),$$

where the quadratic factor corresponds to the characteristic polynomial of the restricted endomorphism $\mathbb{K}_a^{-1} G_{v_i}\big|_{W_i} : W_i \to W_i$. The classical classification for linear equilibria in $\mathbb{R}^2$ in terms of the signs of the trace $-\tilde{\alpha}$ and the determinant $\tilde{\beta}$ may then be applied to determine the nature of the equilibrium point of the restriction of the flow to $\mathcal{E}_\eta$. These observations lead to the following.

Lemma 1. *Let $0 \neq v_i \in V_i$ be an equilibrium point of (3.7) with $\eta = E(v_i) > 0$. Define $p(z) := \det(z\mathbb{K}_a - G_{v_i})$. Then $p(z) = \det(\mathbb{K}_a) z^3 + \alpha z^2 + \beta z$ with $\alpha, \beta \in \mathbb{R}$ and:*

1) *if $\beta < 0$ then $\pm v_i$ are saddle points of the restricted flow to $\mathcal{E}_\eta$;*
2) *if $\beta > 0$ then the equilibrium points $\pm v_i$ of the restricted flow to $\mathcal{E}_\eta$ satisfy:*
   - (a) *they are both linear centers if $\alpha = 0$;*
   - (b) *v_i is a source and $-v_i$ a sink if $\alpha < 0$;*
   - (c) *v_i is a sink and $-v_i$ a source if $\alpha > 0$.*

Proof. We have $p(z) = \det(\mathbb{K}_a)\tilde{p}(z)$ and hence $p(z)$ may be written as indicated with $\alpha = \det(\mathbb{K}_a)\tilde{\alpha}$ and $\beta = \det(\mathbb{K}_a)\tilde{\beta}$. Considering that $\det(\mathbb{K}_a) > 0$, we conclude that the signs of $-\alpha$ and β, respectively, coincide with the signs of the trace and determinant of $\mathbb{K}_a^{-1} G_{v_i}\big|_{W_i} : W_i \to W_i$. The specific nature of the equilibrium point v_i described in items (i), (ii), follows, as was mentioned in the text above, from the standard classification of equilibria of linear systems in $\mathbb{R}^2$ in terms of the signs of the trace and the determinant of the linearization matrix. The conclusions about the behavior at $-v_i$ follow from the reversibility of the dynamics. □

Below we use this lemma to determine the linear stability of the equilibrium points of the restriction of the flow to a constant energy ellipsoid $\mathcal{E}_\eta$. For this we note that the matrix form of G_{v_i} is

$$G_{v_i} = (g_1|\, g_2|\, g_3)\,, \qquad g_j = (b_j - \lambda_i e_j) \times v_i, \quad j = 1, 2, 3,$$

where b_j, $j = 1, 2, 3$ are the columns of $\mathbb{B}_a$. Using the explicit expressions for the matrices G_{v_i} and $\mathbb{K}_a$, the coefficients α and β in the lemma can be conveniently calculated with the help of a symbolic algebra software and their sign can be determined considering (3.8). Finally, note that the stability properties determined at a specific equilibrium point $v_i \in V_i$ may be extended to the whole line V_i (excluding the origin) since, as mentioned above, the dynamics on the different ellipsoids $\mathcal{E}_\eta$ is conjugated by a constant time rescaling.

Linearization at $\pm v_1 \in V_1$. Taking v_1 as the column vector $(\lambda_3 - \lambda_1, 0, a_1 K_3)$, one finds

$$\beta = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\left(\left(a_1^2 K_3 + \lambda_1\right)(\lambda_1 - \lambda_3)^2 + a_1^2 K_3^2 \lambda_3\right) > 0,$$
$$\alpha = -a_2 K_3 \lambda_3 \left(a_1^2 K_3 (\lambda_1 - \lambda_2) + \lambda_1(\lambda_1 - \lambda_3)\right).$$

Therefore, using Lemma 1, we conclude that $\pm v_1$ are a source and a sink on the ellipsoid $\mathcal{E}_\eta$ if $a_2 \neq 0$ and instead two linear centers if $a_2 = 0$.

Linearization at $\pm v_2 \in V_2$. For the column vector $v_2 = (0, \lambda_3 - \lambda_2, a_2 K_3)$ one computes

$$\beta = -(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\left(\lambda_2(\lambda_2 - \lambda_3)^2 + a_2^2 K_3\left((\lambda_2 - \lambda_3)^2 + K_3\lambda_3\right)\right) < 0.$$

Hence, Lemma 1 implies that $\pm v_2$ are saddle points on $\mathcal{E}_\eta$ for all values of $a_1, a_2 \in \mathbb{R}$.

Linearization at $\pm v_3 \in V_3$. Finally, for the column vector $v_3 = (0, 0, 1)$ we have

$$\beta = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_3 > 0, \qquad \alpha = -a_1 a_2 K_3 (\lambda_1 - \lambda_2)\lambda_3.$$

By Lemma 1, we conclude that $\pm v_3$ are a source and a sink on the ellipsoid $\mathcal{E}_\eta$ if $a_1 \neq 0$ and $a_2 \neq 0$, and instead two linear centers if either a_1 or a_2 vanish.

Remark 3. It can be shown that the linear centers $\pm v_1$ on the plane W_1 occurring when $a_2 = 0$, and $\pm v_3$ on the plane W_3 occurring when $a_1 a_2 = 0$, are actually *nonlinear centers* on the corresponding constant energy ellipsoids. For instance, let us consider the equilibria $\pm v_3$ in the case $a_2 = 0$. In a neighborhood of $\pm v_3$ on the energy ellipsoid $\mathcal{E}_\eta$ ($\eta = E(v_3)$) we may write

$$\Omega_3 = \pm\frac{1}{\sqrt{\lambda_3}}\sqrt{2\eta - \lambda_2\Omega_2^2 - (\lambda_1 + a_1^2 K_3)\Omega_1^2},$$

with the equilibrium point $\pm v_3$ corresponding to $(\Omega_1, \Omega_2) = (0, 0)$. Then the system (3.7) restricted to $\mathcal{E}_\eta$ is locally equivalent to a planar system of the type

$$\dot\Omega_1 = P(\Omega_1, \Omega_2), \qquad \dot\Omega_2 = Q(\Omega_1, \Omega_2), \tag{3.11}$$

having an isolated equilibrium point at the origin which, by our previous analysis, is a linear center. As may be verified, this planar system possesses the symmetry

$$P(\Omega_1, -\Omega_2) = -P(\Omega_1, \Omega_2), \qquad Q(\Omega_1, -\Omega_2) = Q(\Omega_1, \Omega_2).$$

Therefore, a theorem by Poincaré (see, e. g., [19, Theorem 4.6571, p. 122]) implies that $(0, 0)$ is a nonlinear center. The above planar symmetry is inherited from the *reversibility* of the full 3D system with respect to the involution

$$\Sigma^{(2)} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \Sigma^{(2)}(\Omega_1, \Omega_2, \Omega_3) = (\Omega_1, -\Omega_2, \Omega_3),$$

when $a_2 = 0$ (i. e., if $t \mapsto \Omega(t)$ is a solution of (3.7), so is $t \mapsto \Sigma^{(2)}\Omega(-t)$). A similar approach may be used to analyze the equilibria $\pm v_3$ when $a_2 \neq 0$, $a_1 = 0$. This time the conclusion follows since the resulting planar system (3.11) possesses the symmetry

$$P(-\Omega_1, \Omega_2) = P(\Omega_1, \Omega_2), \qquad Q(-\Omega_1, \Omega_2) = -Q(\Omega_1, \Omega_2),$$

which is inherited from the reversibility of (3.7) with respect to

$$\Sigma^{(1)}: \mathbb{R}^3 \to \mathbb{R}^3, \qquad \Sigma^{(1)}(\Omega_1, \Omega_2, \Omega_3) = (-\Omega_1, \Omega_2, \Omega_3).$$

The proof that $\pm v_1$ are nonlinear centers on the constant energy ellipsoids when $a_2 = 0$ is analogous and ultimately follows from the reversibility of (3.7) with respect to $\Sigma^{(2)}$.

3.5. Existence of an Invariant Measure

We are now ready to discuss existence of an invariant measure which is the point that we wish to illustrate with our example.

We start by rewriting the reduced equations of motion (3.7) as $\dot{\Omega} = X(\Omega)$ where the vector field $X: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$X(\Omega) = \mathbb{K}_a^{-1}\left((\mathbb{B}_a\Omega) \times \Omega\right).$$

Given that X is homogeneous quadratic in Ω, it follows from Proposition 1 in Kozlov's paper [16] that a volume form $f(\Omega)\, d\Omega_1 d\Omega_2 d\Omega_3$ with *strictly positive* and C^1 density $f: \mathbb{R}^3 \to \mathbb{R}$ is invariant if and only if the standard Euclidean volume form $d\Omega_1 d\Omega_2 d\Omega_3$ is also invariant and f is a first integral. Therefore, necessary and sufficient conditions of existence of an invariant measure with a strictly positive C^1 density may be obtained by requiring that $\operatorname{div}(X)$ vanishes for all values of $\Omega \in \mathbb{R}^3$. A calculation, which is conveniently performed with the help of a symbolic algebra software, yields

$$\operatorname{div}(X)(\Omega) = \frac{\lambda_3 K_3}{\det(\mathbb{K}_a)}\left(-a_2\lambda_1\Omega_1 + a_1\lambda_2\Omega_2 + a_1 a_2(\lambda_1 - \lambda_2)\Omega_3\right).$$

Considering that $\lambda_j > 0$, $j = 1, 2, 3$, the above expression vanishes for all $\Omega \in \mathbb{R}^3$ if and only if $a_1 = a_2 = 0$. Therefore, we have the following.

Proposition 2. *The reduced equations of motion* (3.7) *possess an invariant volume form* $f(\Omega)\, d\Omega_1 d\Omega_2 d\Omega_3$ *with a strictly positive density function* $f: \mathbb{R}^3 \to \mathbb{R}$ *of class* C^1 *if and only if the vector of forbidden rotations is parallel to the axis* E_3 *of the rotor, i. e., if* $a_1 = a_2 = 0$.

Remark 4. The above proposition may also be proved by applying the criterium given in Theorem 2 in [16] (or any of its reformulations or generalizations, e. g., [10, 13, 22]).

Remark 5. If $a_1 = a_2 = 0$, then $\mathbb{K}_a = \mathbb{B}_a$, so the reduced Eqs. (3.7) become $\mathbb{K}_a\dot{\Omega} = (\mathbb{K}_a\Omega) \times \Omega$, which are Euler's equations for a free rigid body with inertia tensor $\mathbb{K}_a$. As is well known, the latter equations are Hamiltonian with respect to the Lie–Poisson structure on the coalgebra $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$. Therefore, in the terminology of nonholonomic systems, we say that the system is *Hamiltonizable* for $a_1 = a_2 = 0$.

Our example illustrates that a more delicate analysis of the dynamics, aimed at understanding obstructions to the existence of attractors in the phase space, should necessarily consider the invariance of a wider class of measures. Specifically, the following generalization of Proposition 2 holds (see Section 2 for the definition of measures of class $\mathcal{A}$).

Theorem 1. *The reduced equations of motion* (3.7) *possess an invariant measure of class* $\mathcal{A}$ *on* $\mathbb{R}^3$ *if and only if* $a_2 = 0$. *Namely, if and only if the vector of forbidden rotations lies on the plane generated by the largest and smallest axes of inertia of the carrier.*

Proof. If $a_2 \neq 0$ then, by the analysis of Section 3.4.1, the restricted dynamics to a constant energy ellipsoid $\mathcal{E}_\eta$, with $\eta > 0$, has a sink at one of the equilibrium points $\pm v_1$. Hence, the extension of Proposition 1 to manifolds rules out the existence of an invariant measure of class $\mathcal{A}$ for the restricted flow to $\mathcal{E}_\eta$ (see the comments at the end of Section 2). The same conclusion about the unrestricted system (3.7) on $\mathbb{R}^3$ can be obtained by applying the extension of Proposition 1 indicated in Remark 1. Indeed, an appropriate closed segment K of the line V_1 is a compact invariant set of measure zero which attracts a nonempty open subset $W \subset \mathbb{R}^3$. This shows that $a_2 = 0$ is a necessary condition for the existence of an invariant measure of class $\mathcal{A}$. To show that it is sufficient, we give an explicit formula for an invariant measure of class $\mathcal{A}$ when $a_2 = 0$. Let

$$R = \left(a_1^2 K_3^2 \lambda_3^2 + 4(\lambda_1 + a_1^2 K_3)\lambda_3(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\right)^{1/2},$$
$$\xi_\pm = \frac{a_1 K_3 \lambda_3 \pm R}{2(\lambda_1 - \lambda_2)(\lambda_1 + a_1^2 K_3)}, \qquad \gamma = \frac{R - a_1 K_3 \lambda_3}{R + a_1 K_3 \lambda_3},$$

and note that $\gamma > 0$ since $|a_1 K_3 \lambda_3| < R$. Choose $n \in \mathbb{N}$ odd large enough so that $M : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$M(\Omega) := (\Omega_1 - \xi_+ \Omega_3)^{n-1} \, |\Omega_1 - \xi_- \Omega_3|^{n\gamma - 1} \tag{3.12}$$

is of class C^1. A direct calculation, which is conveniently performed with a symbolic algebra software shows that, for $a_2 = 0$,

$$\operatorname{div}(MX)(\Omega) = 0, \quad \forall \Omega \in \mathbb{R}^3.$$

Considering that M is nonnegative and vanishes only along the planes

$$\pi_\pm : \Omega_1 - \xi_\pm \Omega_3 = 0,$$

which have measure zero, we conclude that the measure $M(\Omega)\, d\Omega_1 d\Omega_2 d\Omega_3$ is of class $\mathcal{A}$ and is invariant. □

We now make the following observations for the system in the case $a_2 = 0$:

1. The density M given in (3.12) only vanishes along the planes $\pi_\pm$ which can be shown to be invariant by the dynamics (in agreement with Remark 2). The intersection of these planes with a constant energy ellipsoid is comprised of the saddle points $\pm v_2$ together with four heteroclinic connections between them. In fact, we found the explicit form of M by writing the flow on the ellipsoid near the saddle point v_2 in coordinates suggested by the linearization of the stable and unstable manifolds and proceeding in analogy with example (2.4).

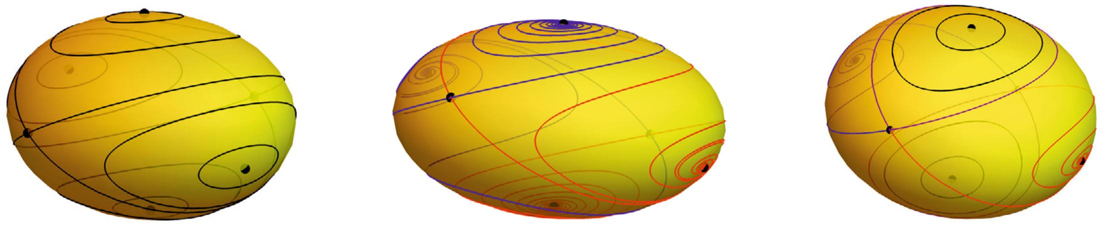
2. The liberty in the choice of n in the definition (3.12) of the density M comes from the fact that

$$F(\Omega) = (\Omega_1 - \xi_+ \Omega_3)\, |\Omega_1 - \xi_- \Omega_3|^\gamma$$

   is a first integral, i. e., this continuous function is constant along the orbits. (This last assertion is easy to check: F is constantly zero on the invariant plane π_- and satisfies $\langle \nabla F(\Omega), X(\Omega) \rangle = 0$ on its invariant complement). Therefore, M remains the density of an invariant measure when multiplied by arbitrary powers of F.

3. The intersections of the level sets of F with the constant energy ellipsoids are generically closed curves which are periodic orbits of the dynamics. The phase space on a fixed energy ellipsoid obtained numerically is illustrated in Fig. 4a. From the dynamical point of view, it seems undistinguishable from the phase flow of the Euler equations which occurs when a_1 also vanishes (see Remark 5). It is reasonable to expect that both systems are topologically equivalent.

For completeness, we also provide illustrations of the phase flow on a constant energy ellipsoid obtained numerically when $a_2 \neq 0$. The case $a_1 \neq 0$ is shown in Fig. 4b, where one can appreciate the attractive and repelling properties of the equilibria $\pm v_1$ and $\pm v_3$ which exclude the existence of an invariant measure of class $\mathcal{A}$. The case when $a_1 = 0$, shown in Fig. 4c, is more interesting. The equilibria $\pm v_1$ are again a source and a sink, but now $\pm v_3$ become nonlinear centers (see Remark 3). The numerics suggests the existence of a homoclinic orbit emanating from v_2 and another from $-v_2$ which enclose the period annulus of $\pm v_3$. It is reasonable to expect the existence of an invariant measure whose support is not the whole phase space but the invariant region occupied by the periodic orbits.



(a) $a_2 = 0$ (b) $a_1 \neq 0$, $a_2 \neq 0$ (c) $a_1 = 0$, $a_2 \neq 0$

Fig. 4. Phase flow on a constant energy ellipsoid for different values of a_1 and a_2. Attractors and sources are present unless $a_2 = 0$.

4. FINAL REMARKS

We have given a simple example of a nonholonomic system that, for certain values of the parameters, possesses an invariant measure of class $\mathcal{A}$ which is an obstruction to the existence of attractors. Our point is that the existence of such invariant measure cannot be detected with the methods developed in previous references treating the problem of existence of invariant volumes in nonholonomic mechanics (e. g., [7, 10, 13, 16, 20, 22]), since they are limited to the class of measures with strictly positive C^1 densities. Our example shows that the results of these references should be extended to a wider class of measures if one wishes to understand obstructions or mechanisms which lead to the existence of limit cycles (like those exhibited by the dynamics of the rattleback).

We note that the condition $a_2 = 0$, which leads to the existence of an invariant measure of class $\mathcal{A}$ in the example, also leads to the reversibility of the flow with respect to the involution $\Sigma^{(2)}$ mentioned in Remark 3. The relevance of this type of discrete symmetries as obstructions to the existence attractors in the phase space of nonholonomic systems had been indicated before (e. g., [22, Theorem 3.3], [5, Appendix]), and may be worth investigating further in connection with the existence of invariant measures in the class $\mathcal{A}$.

Finally, we indicate that the density (3.12), as well as that of the toy system (2.4), vanishes on a set comprised of equilibrium points and their invariant manifolds. This could be a general property shared by systems possessing an invariant measure of class $\mathcal{A}$ with vanishing density and could guide the search of examples.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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