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The Covariety of Saturated Numerical Semigroups with Fixed Frobenius Number

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Abstract: In this work, we show that if F is a positive integer, then $\mathrm{Sat}(F) = \{S \mid S$ is a saturated numerical semigroup with Frobenius number $F\}$ is a covariety. As a consequence, we present two algorithms: one that computes $\mathrm{Sat}(F)$, and another which computes all the elements of $\mathrm{Sat}(F)$ with a fixed genus. If $X \subseteq S \backslash \Delta(F)$ for some $S \in \mathrm{Sat}(F)$, then we see that there exists the least element of $\mathrm{Sat}(F)$ containing X. This element is denoted by $\mathrm{Sat}(F)[X]$. If $S \in \mathrm{Sat}(F)$, then we define the $\mathrm{Sat}(F)$-rank of S as the minimum of $\{\mathrm{cardinality}(X) \mid S = \mathrm{Sat}(F)[X]\}$. In this paper, we present an algorithm to compute all the elements of $\mathrm{Sat}(F)$ with a given $\mathrm{Sat}(F)$-rank.

Keywords: numerical semigroup; covariety; Frobenius number; genus; saturated numerical semigroup; algorithm

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ which is closed by sum $0 \in S$ and $\mathbb{N}\backslash S$ is finite. The set $\mathbb{N}\backslash S$ is known as the set of *gaps* of S and its cardinality, denoted by $\mathrm{g}(S)$, is the *genus* of S. The largest integer not belonging to S is known as the *Frobenius number* of S and it will be denoted by $\mathrm{F}(S)$.

Let A be a nonempty subset of $\mathbb{N}$. Then

$$\langle A \rangle = \left\{ \sum_{i=1}^{p} \alpha_i a_i \mid p \in \mathbb{N}, \{a_1, \cdots, a_p\} \subseteq A \text{ and } \{\alpha_1, \dots, \alpha_p\} \subset \mathbb{N} \right\}$$

is a numerical semigroup if and only if $\gcd(a_1, \dots, a_p) = 1$ and every numerical semigroup has this form (see [1], Lemma 2.1). The set A is called a *system of generators* of a numerical semigroup S if $S = \langle A \rangle$. In addition, if $S \neq \langle B \rangle$ for every $B \subsetneq A$, then we say that A is a *minimal system of generators* of S.

In [1], Corollary 2.8, it is proven that every numerical semigroup has a unique minimal system of generators which is also finite. We denote this by $\mathrm{msg}(S)$ for the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$ is called the *embedding dimension* of S and is denoted by $\mathrm{e}(S)$. Another invariant which we use in this work is the minimum of $S\backslash\{0\}$. It is called the *multiplicity* of S and it is denoted by $\mathrm{m}(S)$.

If S is a numerical semigroup S, the multiplicity, the genus, and the Frobenius number of S are three essential invariants in the theory of numerical semigroups (see for example [2,3] and the references given there). These invariants will be fundamental tools in this paper.

The Frobenius problem (see [3]) for numerical semigroups consists of obtaining formulas for calculating the Frobenius number and the genus of a numerical semigroup from its minimal system of generators. When the numerical semigroup has an embedding dimension of two, the problem has been solved by J. J. Sylvester (see [4]). However, if

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the numerical semigroup has an embedding dimension greater than or equal to three, the problem is still open.

To find a solution to the Frobenius problem, in [5] we study the set $\mathscr{A}(F) = \{S \mid S \text{ is a numerical semigroup with } \mathrm{F}(S) = F\}$, where $F \in \mathbb{N}\backslash\{0\}$. The generalization of $\mathscr{A}(F)$ as a family of numerical semigroups that verifies certain properties lead us to introduce the concept of covariety in [5]. That is, a *covariety* is a nonempty family $\mathscr{C}$ of numerical semigroups that fulfills the following conditions:

(1) $\mathscr{C}$ has a minimum, denoted by $\Delta(\mathscr{C}) = \min(\mathscr{C})$, with respect to set inclusion.
(2) If $\{S, T\} \subseteq \mathscr{C}$, then $S \cap T \in \mathscr{C}$.
(3) If $S \in \mathscr{C}$ and $S \neq \Delta(\mathscr{C})$, then $S\backslash\{\mathrm{m}(S)\} \in \mathscr{C}$.

This concept has allowed us to study common properties of some families of numerical semigroups. For instance, in [6] we have studied the set of all numerical semigroups which have the Arf property (see for example [2]) with a given Frobenius number, showing some algorithms to compute them.

In the semigroup literature, one can find a long list of works dedicated to the study of one-dimensional analytically irreducible domains via their value semigroup (see for instance [7–11]). One of the properties studied for this type of rings using this approach has been to be saturated. Saturated rings were introduced in three different ways by Zariski [12], Pham-Teissier [13], and Campillo [14]. These three definitions coincide for algebraically closed fields of characteristic zero. The characterization of saturated rings in terms of their value semigroups gave rise to the notion of saturated numerical semigroups (see [15,16]).

If $A \subseteq \mathbb{N}$ and $a \in A$, then we let $\mathrm{d}_A(a) = \gcd\{x \in A \mid x \leq a\}$. A numerical semigroup S is *saturated* if $s + \mathrm{d}_S(s) \in S$ for all $s \in S\backslash\{0\}$.

If $F \in \mathbb{N}\backslash\{0\}$, then we also let

$$\mathrm{Sat}(F) = \{S \mid S \text{ is a saturated numerical semigroup and } \mathrm{F}(S) = F\}.$$

The aim of this paper is to study the set $\mathrm{Sat}(F)$ by using the techniques of covarieties. This work is structured as follows. Section 2 is devoted to recalling some concepts and results which will be used in this work. Additionally, we show how we can compute some of them with the help of the `GAP` [17] package `numericalsgps` [18]. In Section 3, we show that $\mathrm{Sat}(F)$ is a covariety. This fact allows us to order the elements of $\mathrm{Sat}(F)$ making it a tree; consequently, we can show an algorithm that allows us to calculate all the elements belonging to $\mathrm{Sat}(F)$.

In Section 4, we show what the maximal elements of $\mathrm{Sat}(F)$ are. We compute the set $\{\mathrm{g}(S) \mid S \in \mathrm{Sat}(F)\}$ and we apply this result to give an algorithm which enables us to calculate all the elements of $\mathrm{Sat}(F)$ with a fixed genus.

Now a set X is called a $\mathrm{Sat}(F)$-*set*, if it verifies the following conditions:

(1) $X \cap \{0, F+1, \rightarrow\} = \emptyset$, where the symbol $\rightarrow$ means that every integer greater than $F+1$ belongs to the set.
(2) There exists $S \in \mathrm{Sat}(F)$ such that $X \subseteq S$.

In Section 5, we see that if X is a $\mathrm{Sat}(F)$-set, then there exists the least element of $\mathrm{Sat}(F)$ containing X. This element will be denoted by $\mathrm{Sat}(F)[X]$.

We say that X is a $\mathrm{Sat}(F)$-*system of generators of* S if $S = \mathrm{Sat}(F)[X]$. Additionally, we show that every element of $\mathrm{Sat}(F)$ admits a unique minimal $\mathrm{Sat}(F)$-system of generators.

The $\mathrm{Sat}(F)$-*rank* of an element of $\mathrm{Sat}(F)$ is the cardinality of its minimal $\mathrm{Sat}(F)$-system of generators. In Section 6, we present an algorithmic procedure to compute all the elements of $\mathrm{Sat}(F)$ with a given $\mathrm{Sat}(F)$-rank.

2. Preliminaries

In this section, we present some concepts and results which are necessary for understanding the work. In [1], Proposition 3.10 reveals the proof of the following result.

Proposition 1. *If S is a numerical semigroup, then* $\mathrm{e}(S) \leq \mathrm{m}(S)$.

We say that a numerical semigroup S has *maximal embedding dimension* (MED-*semigroup*) if $\mathrm{e}(S) = \mathrm{m}(S)$.

By applying the results of [1], Section 3, the next property arises.

Proposition 2. *Every saturated numerical semigroup is a* MED-*semigroup.*

An integer z is a *pseudo-Frobenius number* of a numerical semigroup S if $z \notin S$ and $z + s \in S$ for all $s \in S\backslash\{0\}$ (see [19]). The set formed by the pseudo-Frobenius numbers of S is denoted by $\mathrm{PF}(S)$. Its cardinality is an important invariant of S (see [2,20]) called the *type* of S, denoted by $\mathrm{t}(S)$.

For instance, let $S = \langle 7, 8, 9, 11, 13\rangle$, and if we want to calculate the set $\mathrm{PF}(S)$, then we use the following sentences:

```
gap> S := NumericalSemigroup(7,8,9,11,13);
<Numerical semigroup with 5 generators>
gap> PseudoFrobeniusOfNumericalSemigroup(S);
[ 6, 10, 12 ]
```

Let S be a numerical semigroup; we set $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \in S\}$. The elements of $\mathrm{SG}(S)$ will be called *special gaps* of S.

For instance, given the numerical semigroup $S = \langle 6, 7, 8, 10, 11\rangle$, if we want to calculate the set $\mathrm{SG}(S)$, then we use the following sentences:

```
gap> S := NumericalSemigroup(6,7,8,10,11);
<Numerical semigroup with 5 generators>
gap> SpecialGaps(S);
[ 4, 5, 9 ]
```

In [1], Proposition 4.33, the following result appears.

Proposition 3. *Let S be a numerical semigroup and $x \in \mathbb{N}\backslash S$. Then $x \in \mathrm{SG}(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

Let S be a numerical semigroup and $n \in S\backslash\{0\}$. The *Apéry set* of n in S (in honor of [21]) is defined as $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$.

For instance, to compute $\mathrm{Ap}(S, 8)$, with $S = \langle 8, 9, 11, 13\rangle$, we use the following sentences:

```
gap> S := NumericalSemigroup(8,9,11,13);
<Numerical semigroup with 4 generators>
gap> AperyList(S,8);
[ 0, 9, 18, 11, 20, 13, 22, 31 ]
```

The following result follows from [1], Lemma 2.4.

Proposition 4. *Let S be a numerical semigroup and $n \in S\backslash\{0\}$. Then $\mathrm{Ap}(S, n)$ is a set with cardinality n. Moreover, $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n, for all $i \in \{0, \ldots, n-1\}$.*

The following result characterizes MED-semigroups. The proof can be deduced from [1], Proposition 3.1.

Proposition 5. *Let S be a numerical semigroup. Then S is a* MED-*semigroup if and only if* $\mathrm{msg}(S) = (\mathrm{Ap}(S, \mathrm{m}(S))\backslash\{0\}) \cup \{\mathrm{m}(S)\}$.

Given that S is a numerical semigroup, we define an order relation on $\mathbb{Z}$ as follows: $x \leq_S y$ if $y - x \in S$. The following result appears in [19], Lemma 10.

Proposition 6. *If S is a numerical semigroup and $n \in S\backslash\{0\}$, then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S}\mathrm{Ap}(S,n)\}.$$

The next proposition has an easy proof.

Proposition 7. *Let S be a numerical semigroup and $n \in S\backslash\{0\}$ and $w \in \mathrm{Ap}(S,n)$. Then $w \in \mathrm{Maximals}_{\leq_S}\mathrm{Ap}(S,n)$ if and only if $w + w' \notin \mathrm{Ap}(S,n)$ for all $w' \in \mathrm{Ap}(S,n)\backslash\{0\}$.*

The following proposition has an immediate proof.

Proposition 8. *If S is a numerical semigroup and $S \neq \mathbb{N}$, then*

$$\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \notin \mathrm{PF}(S)\}.$$

Remark 1. *Observe that as a consequence of Propositions 6–8, if S is a numerical semigroup and we know the set $\mathrm{Ap}(S,n)$ for some $n \in S\backslash\{0\}$, then we can easily calculate the set $\mathrm{SG}(S)$.*

The following result is well known, as well as very easy to prove.

Proposition 9. *Let S and T be numerical semigroups and $x \in S$. Then the following hold:*

(1) $S \cap T$ *is a numerical semigroup and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.
(2) $S\backslash\{x\}$ *is a numerical semigroup if and only if* $x \in \mathrm{msg}(S)$.
(3) $\mathrm{m}(S) = \min(\mathrm{msg}(S))$.

The following result is Lemma 2.14 from [1].

Proposition 10. *If S is a numerical semigroup, then* $\frac{\mathrm{F}(S)+1}{2} \leq \mathrm{g}(S)$.

3. The Tree Associated to Sat(F)

Our first goal in this section is to show that given F, a positive integer, the set $\mathrm{Sat}(F) = \{S \mid S \text{ is a saturated numerical semigroup and } \mathrm{F}(S) = F\}$ is a covariety.

The next result can be found in [22], Proposition 5.

Lemma 1. *If S and T are saturated numerical semigroups, then $S \cap T$ is also a saturated numerical semigroup.*

The following result has an immediate proof.

Lemma 2. *Let F be a positive integer. Then the following properties are verified as follows:*

(1) *If $m \in \mathbb{N}$, then $\Delta(m) = \{0, m, \rightarrow\}$ is a saturated numerical semigroup.*
(2) $\Delta(F+1)$ *is the minimum of* $\mathrm{Sat}(F)$.
(3) *If S is a saturated numerical semigroup, then $S\backslash\{\mathrm{m}(S)\}$ is also a saturated numerical semigroup.*

By applying Proposition 9 and Lemmas 1 and 2, we can easily deduce the following fact.

Proposition 11. *If F is a positive integer, then* $\mathrm{Sat}(F)$ *is a covariety.*

A *graph* G is a pair (V,E) where V is a nonempty set and E is a subset of $\{(u,v) \in V \times V \mid u \neq v\}$. The elements of V and E are called *vertices* and *edges*, respectively. A *path* *of length n*, connecting the vertices x and y of G, is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

A graph G is *a tree* if there exists a vertex r (known as *the root* of G) such that for any other vertex x of G, there exists a unique path connecting x and r. If (u, v) is an edge of the tree G, we say that u is a *child* of v.

For a positive integer F we define the graph $\mathrm{G}(F)$ as follows:

- the set of vertices of $\mathrm{G}(F)$ is $\mathrm{Sat}(F)$;
- $(S, T) \in \mathrm{Sat}(F) \times \mathrm{Sat}(F)$ is an edge of $\mathrm{G}(F)$ if and only if $T = S \backslash \{\mathrm{m}(S)\}$.

By using [5], Propositions 2.6 and 11, we obtain the following result.

Proposition 12. *Let F be a positive integer. Then $\mathrm{G}(F)$ is a tree with root $\Delta(F+1)$.*

A tree can be built in a recurrent way starting from the root and joining, by using an edge, the vertices already built with their children. Therefore it is very necessary to characterize who a given vertex's children are in the tree $\mathrm{G}(F)$. This is the reason for introducing the following concepts and results.

The following result is deduced from Proposition 11 and [5], Proposition 2.9.

Proposition 13. *If $S \in \mathrm{Sat}(F)$, then the children of S in the tree $\mathrm{G}(F)$, is the set*

$$\{S \cup \{x\} \mid x \in \mathrm{SG}(S),\ x < \mathrm{m}(S) \text{ and } S \cup \{x\} \in \mathrm{Sat}(F)\}.$$

Let $S \in \mathrm{Sat}(F)$ and $x \in \mathrm{SG}(S)$ such that $x < \mathrm{m}(S)$ and $x \neq F$. The following result provides us an algorithm to decide if $S \cup \{x\}$ belongs to $\mathrm{Sat}(F)$.

Proposition 14. *Let $S \in \mathrm{Sat}(F)$, $x \in \mathrm{SG}(S)$ with $x < \mathrm{m}(S)$, and $x \neq F$. Then $S \cup \{x\} \in \mathrm{Sat}(F)$ if and only if $s + \mathrm{d}_{S\cup\{x\}}(s) \in S$ for every $s \in \{\mathrm{m}(S), \cdots, \mathrm{m}(S) + x\}$.*

Proof. *Necessity.* Trivial.
Sufficiency. We have to prove that if $s \in S$ and $s > \mathrm{m}(S) + x$, then $s + \mathrm{d}_{S\cup\{x\}}(s) \in S$. Hence, it is enough to show that $\mathrm{d}_S(s) = \mathrm{d}_{S\cup\{x\}}(s)$. But it is true because $\mathrm{d}_S(s) = \gcd\{\mathrm{m}(S), \cdots, \mathrm{m}(S) + x, \cdots, s\} = \gcd\{x, \mathrm{m}(S), \cdots, s\} = \mathrm{d}_{S\cup\{x\}}(s)$. □

Example 1. *It is clear that $S = \{0, 8, 10, 12, 14, 16, 18, \rightarrow\} \in \mathrm{Sat}(17)$ and $6 \in \mathrm{SG}(S)$. As*

$$\{8 + \mathrm{d}_{S\cup\{6\}}(8), 10 + \mathrm{d}_{S\cup\{6\}}(10), 12 + \mathrm{d}_{S\cup\{6\}}(12), 14 + \mathrm{d}_{S\cup\{6\}}(14)\} = \{8+2, 10+2, 12+2, 14+2\} \subseteq S,$$

by applying Proposition 14, we have that $S \cup \{6\} \in \mathrm{Sat}(17)$.

The next proposition is Proposition 4.6 of [6].

Proposition 15. *Let S be a numerical semigroup and $x \in \mathrm{SG}(S)$ such that $x < \mathrm{m}(S)$ and $S \cup \{x\}$ is a* MED*-semigroup. Then the following conditions hold.*

(1) For every $j \in \{1, \ldots, x-1\}$, there exists $a \in \mathrm{msg}(S)$ such that $a \equiv j \pmod{x}$.
(2) If $\lambda(j) = \min\{a \in \mathrm{msg}(S) \mid a \equiv j \pmod{x}\}$ for all $j \in \{1, \ldots, x-1\}$, then $\mathrm{msg}(S \cup \{x\}) = \{x, \lambda(1), \ldots, \lambda(x-1)\}$.

Remark 2. *Note that as a consequence of Propositions 2, 13, and 15, if $S \in \mathrm{Sat}(F)$ and if we know the set $\mathrm{msg}(S)$, then we can easily compute $\mathrm{msg}(T)$ for every child T of S in the tree $\mathrm{G}(F)$.*

Algorithm 1 Computation of $\mathrm{Sat}(F)$.

INPUT: *A positive integer F.*
OUTPUT: $\mathrm{Sat}(F)$.

(1) $\Delta = \langle F+1, \ldots, 2F+1 \rangle$, $\mathrm{Sat}(F) = \{\Delta\}$, *and* $B = \{\Delta\}$.
(2) *For every* $S \in B$, *compute* $\theta(S) = \{x \in \mathrm{SG}(S) \mid x < \mathrm{m}(S), x \neq F$, *and* $S \cup \{x\}$ *is a saturated numerical semigroup*$\}$ *(by using Proposition 5 and 14, Remark 1).*
(3) *If* $\bigcup_{S \in B} \theta(S) = \emptyset$, *then return* $\mathrm{Sat}(F)$.
(4) $C = \bigcup_{S \in B} \{S \cup \{x\} \mid x \in \theta(S)\}$.
(5) *For all* $S \in C$ *compute* $\mathrm{msg}(S)$ *by using Proposition 15.*
(6) $\mathrm{Sat}(F) = \mathrm{Sat}(F) \cup C$, $B = C$, *and go to Step* (2).

Next, we illustrate this algorithm with an example.

Example 2. *We calculate* $\mathrm{Sat}(7)$ *by using Algorithm 1.*

- $\Delta = \langle 8, 9, 10, 11, 12, 13, 14, 15 \rangle$, $\mathrm{Sat}(7) = \{\Delta\}$, *and* $B = \{\Delta\}$.
- *By Proposition 5, we know that* $\mathrm{Ap}(\Delta, 8) = \{0, 9, 10, 11, 12, 13, 14, 15\}$. *By using Remark 1, we have that* $\mathrm{SG}(\Delta) = \{4, 5, 6, 7\}$ *and by using Proposition 14,* $\theta(\Delta) = \{4, 5, 6\}$.
- $C = \{\Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}\}$ *and by applying Proposition 15, we have that* $\mathrm{msg}(\Delta \cup \{4\}) = \{4, 9, 10, 11\}$, $\mathrm{msg}(\Delta \cup \{5\}) = \{5, 8, 9, 11, 12\}$ *and* $\mathrm{msg}(\Delta \cup \{6\}) = \{6, 8, 9, 10, 11, 13\}$.
- $\mathrm{Sat}(7) = \{\Delta, \Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}\}$ *and* $B = \{\Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}\}$.
- $\mathrm{Ap}(\Delta \cup \{4\}, 4) = \{0, 9, 10, 11\}$, $\mathrm{Ap}(\Delta \cup \{5\}, 5) = \{0, 8, 9, 11, 12\}$ *and* $\mathrm{Ap}(\Delta \cup \{6\}, 6) = \{0, 8, 9, 10, 11, 13\}$. *Then* $\mathrm{SG}(\Delta \cup \{4\}) = \{5, 6, 7\}$, $\mathrm{SG}(\Delta \cup \{5\}) = \{4, 6, 7\}$ *and* $\mathrm{SG}(\Delta \cup \{6\}) = \{3, 4, 5, 7\}$. *Therefore,* $\theta(\Delta \cup \{4\}) = \emptyset = \theta(\Delta \cup \{5\})$ *and* $\theta(\Delta \cup \{6\}) = \{3, 4\}$.
- $C = \{\Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}\}$, $\mathrm{msg}(\Delta \cup \{3, 6\}) = \{3, 8, 10\}$ *and* $\mathrm{msg}(\Delta \cup \{4, 6\}) = \{4, 6, 9, 11\}$.
- $\mathrm{Sat}(7) = \{\Delta, \Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}\}$ *and* $B = \{\Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}\}$.
- $\mathrm{Ap}(\Delta \cup \{3, 6\}, 3) = \{0, 8, 10\}$ *and* $\mathrm{Ap}(\Delta \cup \{4, 6\}, 4) = \{0, 6, 9, 11\}$. *Then* $\mathrm{SG}(\Delta \cup \{3, 6\}) = \{5, 7\}$ *and* $\mathrm{SG}(\Delta \cup \{4, 6\}) = \{2, 5, 7\}$. *Therefore,* $\theta(\Delta \cup \{3, 6\}) = \emptyset$ *and* $\theta(\Delta \cup \{4, 6\}) = \{2\}$.
- $C = \{\Delta \cup \{2, 4, 6\}\}$ *and* $\mathrm{msg}(\Delta \cup \{2, 4, 6\}) = \{2, 9\}$.
- $\mathrm{Sat}(7) = \{\Delta, \Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}, \Delta \cup \{2, 4, 6\}\}$ *and* $B = \{\Delta \cup \{2, 4, 6\}\}$.
- $\mathrm{Ap}(\Delta \cup \{2, 4, 6\}, 2) = \{0, 9\}$. *Then* $\mathrm{SG}(\Delta \cup \{2, 4, 6\}) = \{7\}$ *and* $\theta(\Delta \cup \{2, 4, 6\}) = \emptyset$.
- *The algorithm returns*

$$\mathrm{Sat}(7) = \{\Delta, \Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}, \Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}, \Delta \cup \{2, 4, 6\}\}.$$

4. The Elements of $\mathrm{Sat}(F)$ with a Fixed Genus

Given positive integers F and g, let

$$\mathrm{Sat}(F, g) = \{S \in \mathrm{Sat}(F) \mid \mathrm{g}(S) = g\}.$$

From Proposition 10, the following result is deduced.

Lemma 3. *With the previous notation, if* $\mathrm{Sat}(F, g) \neq \emptyset$, *then* $\frac{F+1}{2} \leq g \leq F$.

Let S be a numerical semigroup; then the *associated sequence* to S is recursively defined as follows:

- $S_0 = S$,
- $S_{n+1} = S_n \backslash \{\mathrm{m}(S_n)\}$ for all $n \in \mathbb{N}$.

Let S be a numerical semigroup. We say that an element s of S is a *small element* of S if $s < \mathrm{F}(S)$. The set of small elements of S is denoted by $\mathrm{N}(S)$. The cardinality of $\mathrm{N}(S)$ is denoted by $\mathrm{n}(S)$.

Clearly, the set $\{0, \dots, \mathrm{F}(S)\}$ is the disjointed union of the sets $\mathrm{N}(S)$ and $\mathbb{N} \backslash S$. Hence, we have the following result.

Lemma 4. *If S is a numerical semigroup, then* $\mathrm{g}(S) + \mathrm{n}(S) = \mathrm{F}(S) + 1$.

Let S be a numerical semigroup and $\{S_n\}_{n \in \mathbb{N}}$ its associated sequence; then the set $\mathrm{Cad}(S) = \{S_0, S_1, \dots, S_{\mathrm{n}(S)-1}\}$ is called the *associated chain* to S. Note that $S_0 = S$ and $S_{\mathrm{n}(S)-1} = \Delta(\mathrm{F}(S)+1)$.

Observe that, from Proposition 11, we know that if $S \in \mathrm{Sat}(F)$, then $\mathrm{Cad}(S) \subseteq \mathrm{Sat}(F)$. Therefore, we can present the following result.

Lemma 5. *If $S \in \mathrm{Sat}(F)$, then $\mathrm{Sat}(F, g) \neq \emptyset$ for all $g \in \{\mathrm{g}(S), \cdots, F\}$.*

Our next aim is to determine the minimum element of the set $\{\mathrm{g}(S) \mid S \in \mathrm{Sat}(F)\}$. For this purpose we introduce the following notation. If $\{a, b\} \subseteq \mathbb{N}$, then we denote this by

$$\mathrm{T}(a, b) = \langle a \rangle \cup \{x \in \mathbb{N} \mid x \geq b\}.$$

For integers a and b, we say that a *divides* b if there exists an integer c such that $b = ca$, and we denote this by $a \mid b$. Otherwise, a *does not divide* b, and we denote this by $a \nmid b$.

The next lemma is [23], Lemma 2.3, which shows a characterization of saturated numerical semigroups.

Lemma 6. *Let S be a numerical semigroup. Then S is a saturated numerical semigroup if and only if there are positive integers $a_1, b_1, \cdots, a_n, b_n$ verifying the following properties:*

(1) $a_{i+1} \mid a_i$ *for all* $i \in \{1, \cdots, n-1\}$.
(2) $a_i < b_i < b_{i+1}$ *for all* $i \in \{1, \cdots, n-1\}$.
(3) $S = \mathrm{T}(a_1, b_1) \cap \cdots \cap \mathrm{T}(a_n, b_n)$.

The next lemma is an immediate consequence of Lemma 6.

Lemma 7. *If S is a maximal element of $\mathrm{Sat}(F)$, then $S = \mathrm{T}(a, F+1)$ for some $a \in \{1, \cdots, F\}$ such that $a \nmid F$.*

If n is a positive integer, then we denote $\mathrm{A}(n) = \{x \in \{1, \cdots, n\} \mid x \nmid n\}$ and $\mathrm{B}(n) = \{x \in \mathrm{A}(n) \mid x' \nmid x \text{ for all } x' \in \mathrm{A}(n) \backslash \{x\}\}$.

The following result is a consequence of Lemmas 6 and 7.

Theorem 1. *With the previous notation, S is a maximal element of $\mathrm{Sat}(F)$ if and only if $S = \mathrm{T}(x, F+1)$ for some $x \in \mathrm{B}(F)$.*

In the following example, we illustrate how the previous theorem works.

Example 3. *We are going to apply Theorem 1 to compute the maximal elements of $\mathrm{Sat}(30)$. As*

$$\mathrm{A}(30) = \{4, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\},$$

we obtain $\mathrm{B}(30) = \{4, 7, 9, 11, 13, 17, 19, 23, 25, 29\}$. Therefore, by applying Theorem 1, we have that the set formed by the maximal elements of $\mathrm{Sat}(30)$ is $\{\mathrm{T}(4, 31), \mathrm{T}(7, 31), \mathrm{T}(9, 31), \mathrm{T}(11, 31), \mathrm{T}(13, 31), \mathrm{T}(17, 31), \mathrm{T}(19, 31), \mathrm{T}(23, 31), \mathrm{T}(25, 31), \mathrm{T}(29, 31)\}$.

Let $q \in \mathbb{Q}$. Denote $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$. The following result is a consequence of Theorem 1.

Corollary 1. *If p is the least positive integer such that $p \nmid F$, then* $\min\{\mathrm{g}(S) \mid S \in \mathrm{Sat}(F)\} = F - \left\lfloor \dfrac{F}{p} \right\rfloor$.

By using this corollary, in the following example we calculate the minimum genus of the elements belonging to $\mathrm{Sat}(7)$, as well as the minimum genus of the elements of $\mathrm{Sat}(6)$.

Example 4. *We have that*

- *minimum*$\{\mathrm{g}(S) \mid S \in \mathrm{Sat}(7)\} = 7 - \left\lfloor \frac{7}{2} \right\rfloor = 7 - 3 = 4$. *Moreover,* $\mathrm{g}(\mathrm{T}(2,8)) = 4$.
- *minimum*$\{\mathrm{g}(S) \mid S \in \mathrm{Sat}(6)\} = 6 - \left\lfloor \frac{6}{4} \right\rfloor = 6 - 1 = 5$. *In addition,* $\mathrm{g}(\mathrm{T}(4,7)) = 5$.

We now have all the ingredients needed to present the following Algorithm 2.

Algorithm 2 Computation of $\mathrm{Sat}(F, g)$.

INPUT: *Two positive integers F and g such that $\frac{F+1}{2} \leq g \leq F$.*
OUTPUT: $\mathrm{Sat}(F, g)$.

(1) *Compute the smallest positive integer p such that $p \nmid F$.*
(2) *If $g < F - \left\lfloor \dfrac{F}{p} \right\rfloor$, then return $\emptyset$.*
(3) $\Delta = \langle F+1, \cdots, 2F+1 \rangle$, $H = \{\Delta\}$, $i = F$.
(4) *If $i = g$, then return H.*
(5) *For all $S \in H$, compute the set $\theta(S) = \{x \in \mathrm{SG}(S) \mid x < \mathrm{m}(S), x \neq F$ and $S \cup \{x\}$ is a saturated numerical semigroup$\}$.*
(6) $H = \bigcup_{S \in H} \{S \cup \{x\} \mid x \in \theta(S)\}$, $i = i - 1$ *and go to Step* (4).

Next we illustrate this algorithm with an example.

Example 5. *By using Algorithm 2, we are going to calculate the set* $\mathrm{Sat}(7, 5)$.

- 2 *is the smallest positive integer such that it does not divide* 7 *and* $7 - \left\lfloor \frac{7}{2} \right\rfloor = 7 - 3 = 4 < 5$, *therefore we can assert that* $\mathrm{Sat}(7, 5) \neq \emptyset$.
- $\Delta = \langle 8, 9, 10, 11, 12, 13, 14, 15 \rangle$, $H = \{\Delta\}$, $i = 7$.
- $\theta(\Delta) = \{4, 5, 6\}$.
- $H = \{\Delta \cup \{4\}, \Delta \cup \{5\}, \Delta \cup \{6\}\}$, $i = 6$.
- $\theta(\Delta \cup \{4\}) = \emptyset$, $\theta(\Delta \cup \{5\}) = \emptyset$ *and* $\theta(\Delta \cup \{6\}) = \{3, 4\}$.
- $H = \{\Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}\}$, $i = 5$.
- *The algorithm returns* $\{\Delta \cup \{3, 6\}, \Delta \cup \{4, 6\}\}$.

5. $\mathrm{Sat}(F)$-System of Generators

We will say that a set X is a $\mathrm{Sat}(F)$-*set* if it verifies the following conditions:

(1) $X \cap \Delta(F+1) = \emptyset$.
(2) There exists $S \in \mathrm{Sat}(F)$ such that $X \subseteq S$.

If X is a $\mathrm{Sat}(F)$-set, then the intersection of all elements of $\mathrm{Sat}(F)$ containing X will be denoted by $\mathrm{Sat}(F)[X]$. As $\mathrm{Sat}(F)$ is a finite set, by applying Proposition 11, we have that the intersection of elements of $\mathrm{Sat}(F)$ is again an element of $\mathrm{Sat}(F)$. Consequently we have the following result.

Proposition 16. *If X is a* $\mathrm{Sat}(F)$*-set, then* $\mathrm{Sat}(F)[X]$ *is the smallest element of* $\mathrm{Sat}(F)$ *containing X.*

If X is a $\mathrm{Sat}(F)$-set and $S = \mathrm{Sat}(F)[X]$, we will say that X is a $\mathrm{Sat}(F)$-*system of generators* of S. Moreover, if $S \neq \mathrm{Sat}(F)[Y]$ for all $Y \subsetneq X$, then X is called a *minimal* $\mathrm{Sat}(F)$-*system of generators* of S.

Our next aim in this section will be to prove that every element of $\mathrm{Sat}(F)$ has a unique minimal $\mathrm{Sat}(F)$-system of generators.

The following result appears in [22], Lemma 8.

Lemma 8. *Let S be a saturated numerical semigroup and $x \in S\backslash\{0\}$. Then the following conditions are equivalent.*

(1) *$S\backslash\{x\}$ is a saturated numerical semigroup.*
(2) *If $y \in S\backslash\{0\}$ and $y < x$, then $\mathrm{d}_S(y) \neq d_S(x)$.*

Lemma 9. *Let $S \in \mathrm{Sat}(F)$ and $s \in S$ such that $0 < s < F$ and $\mathrm{d}_S(s) \neq \mathrm{d}_S(s')$ for all $s' \in S$ with $0 < s' < s$. If X is a $\mathrm{Sat}(F)$-system of generators of S, then $s \in X$.*

Proof. By using Lemma 8, $S\backslash\{s\}$ is an element of $\mathrm{Sat}(F)$. If $s \notin X$, then $X \subseteq S\backslash\{s\}$ and, by applying Proposition 16, we have that $S = \mathrm{Sat}(F)[X] \subseteq S\backslash\{s\}$, which is absurd. □

The following result can be found in [22], Theorem 4.

Lemma 10. *Let $A \subseteq \mathbb{N}$ such that $0 \in A$ and $\gcd(A) = 1$. Then the following conditions are equivalent.*

(1) *A is a saturated numerical semigroup.*
(2) *$a + \mathrm{d}_A(a) \in A$ for all $a \in A$.*
(3) *$a + k \cdot \mathrm{d}_A(a) \in A$ for all $(a, k) \in A \times \mathbb{N}$.*

Lemma 11. *Let $S \in \mathrm{Sat}(F)$ and $X = \{x \in S\backslash\{0\} \mid \mathrm{d}_S(x) \neq \mathrm{d}_S(y)$ for all $y \in S$ with $y < x$ and $x < F\}$. Then $\mathrm{Sat}(F)[X] = S$.*

Proof. Let $T = \mathrm{Sat}(F)[X]$. As $X \subseteq S$, by applying Proposition 16, we have that $T \subseteq S$. Now we will show the reverse inclusion; that is, $S \subseteq T$. Assume that $X = \{x_1, \ldots, x_n\}$, $s \in S\backslash\{0\}$ and $x_1 < \cdots < x_k \leq s < x_{k+1} < \cdots < x_n$. Then $\mathrm{d}_S(s) = \mathrm{d}_S(x_k) = \mathrm{d}_T(x_k)$ and $s = x_k + a$ for some $a \in \mathbb{N}$. We deduce that $\mathrm{d}_S(x_k) \mid a$ and so $s = x_k + t \cdot \mathrm{d}_S(x_k)$ for some $t \in \mathbb{N}$. Consequently, by applying Lemma 10, $s = x_k + t \cdot \mathrm{d}_T(x_k) \in T$. □

The minimal $\mathrm{Sat}(F)$-system of generators is unique. This is the content of the following proposition.

Proposition 17. *If $S \in \mathrm{Sat}(F)$, then the unique minimal $\mathrm{Sat}(F)$-system of generators of S is the set*

$$\{x \in S\backslash\{0\} \mid x < F \text{ and } \mathrm{d}_S(x) \neq \mathrm{d}_S(y) \text{ for all } y \in S \text{ such that } y < x\}.$$

Proof. By Lemma 11, the set $X = \{x \in S\backslash\{0\} \mid x < F$ and $\mathrm{d}_S(x) \neq \mathrm{d}_S(y)$ for all $y \in S$ such that $y < x\}$ is a $\mathrm{Sat}(F)$-system of generators of S.

Let Y be a set such that $S = \mathrm{Sat}(F)[Y]$ with $Y \subseteq X$. Let $x \in X$. As Y is a $\mathrm{Sat}(F)$-system of generators of S, by Lemma 9, we have $x \in Y$ and therefore $X = Y$. □

Let $S \in \mathrm{Sat}(F)$; we denote by $\mathrm{Sat}(F)\mathrm{msg}(S)$ the minimal $\mathrm{Sat}(F)$-system of generators of S. The cardinality of $\mathrm{Sat}(F)\mathrm{msg}(S)$ is called the $\mathrm{Sat}(F)$-*rank* of S and it will be denoted by $\mathrm{Sat}(F)\text{-rank}\,(S)$. Let us illustrate these two concepts with an example.

Example 6. *It is clear that $S = \{0, 4, 8, 10, 12, 14, 16, 18, 20, 22, \rightarrow\} \in \mathrm{Sat}(21)$. By applying Proposition 17, we ascertain that $\mathrm{Sat}(21)\mathrm{msg}(S) = \{4, 10\}$. Therefore, $\mathrm{Sat}(21)\text{-rank}\,(S) = 2$.*

Lemma 12. *Let $n_1 < n_2 < \cdots < n_p < F$ be positive integers, $d = \gcd\{n_1, \cdots, n_p\}$ and $d \nmid F$. For every $i \in \{1, \cdots, p\}$, let $d_i = \gcd\{n_1, \cdots, n_i\}$, and for each $j \in \{1, \cdots, p-1\}$, let $k_j = \max\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\}$ and $k_p = \max\{k \in \mathbb{N} \mid n_p + kd_p < F\}$. Then* $\mathrm{Sat}(F)[\{n_1, \cdots, n_p\}] = \{0, n_1, n_1 + d_1, \cdots, n_1 + k_1d_1, n_2, n_2 + d_2, \cdots, n_2 + k_2d_2, \cdots, n_{p-1}, n_{p-1} + d_{p-1}, \cdots, n_{p-1} + k_{p-1}d_{p-1}, n_p, n_p + d_p, \cdots, n_p + k_pd_p, F + 1, \rightarrow\}$.

Proof. Let $S = \{0, n_1, n_1 + d_1, \cdots, n_1 + k_1d_1, n_2, n_2 + d_2, \cdots, n_2 + k_2d_2, \cdots, n_{p-1}, n_{p-1} + d_{p-1}, \cdots, n_{p-1} + k_{p-1}d_{p-1}, n_p, n_p + d_p, \cdots, n_p + k_pd_p, F + 1, \rightarrow\}$. By Lemma 10, $S \in \mathrm{Sat}(F)$. As $\{n_1, \cdots, n_p\} \subseteq S$, then by Proposition 16, we have $\mathrm{Sat}(F)[\{n_1, \cdots, n_p\}] \subseteq S$. By using similar reasoning to the proof of Lemma 11, we obtain the reverse inclusion. □

As a consequence of Proposition 17 and Lemma 12, we present a characterization of the minimal $\mathrm{Sat}(F)$-system of generators of $\mathrm{Sat}(F)[\{n_1, \cdots, n_p\}]$ in the following proposition.

Proposition 18. *Let $n_1 < n_2 < \cdots < n_p < F$ be positive integers, $d = \gcd\{n_1, \cdots, n_p\}$ and $d \nmid F$. Then $\{n_1, \cdots, n_p\}$ is the minimal $\mathrm{Sat}(F)$-system of generators of $\mathrm{Sat}(F)[\{n_1, \cdots, n_p\}]$ if and only if $\gcd\{n_1, \cdots, n_i\} \neq \gcd\{n_1, \cdots, n_{i+1}\}$ for all $i \in \{1, \cdots, p-1\}$.*

Example 7. *By applying Lemma 12, we deduce that* $\mathrm{Sat}(51)[\{8, 28, 42\}] = \{0, 8, 16, 24, 28, 32, 36, 40, 42, 44, 46, 48, 50, 52, \rightarrow\}$. *Moreover, as $\gcd\{8\} > \gcd\{8, 28\} > \gcd\{8, 28, 42\}$, by Proposition 18, we know that $\{8, 28, 42\}$ is the minimal $\mathrm{Sat}(51)$-system of generators of* $\mathrm{Sat}(51)[\{8, 28, 42\}]$.

The following result is a direct consequence of Proposition 17.

Lemma 13. *If $S \in \mathrm{Sat}(F)$ and $S \neq \Delta(F+1)$, then $\mathrm{m}(S) \in \mathrm{Sat}(F)\mathrm{msg}(S)$.*

Proposition 19. *If $S \in \mathrm{Sat}(F)$, then the following conditions are verified as follows:*

(1) $\mathrm{Sat}(F)\text{-rank}\,(S) \leq \mathrm{e}(S)$.
(2) $\mathrm{Sat}(F)\text{-rank}\,(S) = 0$ *if and only if* $S = \Delta(F+1)$.
(3) $\mathrm{Sat}(F)\text{-rank}\,(S) = 1$ *if and only if* $\mathrm{Sat}(F)\mathrm{msg}(S) = \{\mathrm{m}(S)\}$.

Proof. (1) By definition of $\mathrm{Sat}(F)$-rank of S, Lemma 8, and Propositions 9 and 17, we have $\mathrm{Sat}(F)\text{-rank}\,(S) = \#\mathrm{Sat}(F)\mathrm{msg}(S) \leq \#\mathrm{msg}(S) = \mathrm{e}(S)$, where $\#A$ means the cardinality of A.
(2) As $\Delta(F+1) = \{0, F+1, \rightarrow\}$, by Proposition 17, we obtain the assert.
(3) By applying Proposition 17, we obtain the result.
□

Corollary 2. *Under the standing notation, the following conditions are equivalent:*

(1) $S \in \mathrm{Sat}(F)$ *and* $\mathrm{Sat}(F)\text{-rank}\,(S) = 1$.
(2) *There exists $m \in \mathbb{N}$ such that $2 \leq m < F$, $m \nmid F$, and $S = T(m, F+1)$.*

Proof. *(1) implies (2).* If $S \in \mathrm{Sat}(F)$ and $\mathrm{Sat}(F)\text{-rank}\,(S) = 1$, then, by Proposition 19, $\mathrm{Sat}(F)\mathrm{msg}(S) = \{\mathrm{m}(S)\}$. By taking $m = \mathrm{m}(S)$, we have the assert.

(2) implies (1). If there exists $m \in \mathbb{N}$ such that $2 \leq m < F$, $m \nmid F$, and $S = \langle m \rangle \cup \{x \in \mathbb{N} \mid x \geq F+1\}$, the assert is trivially true.
□

6. $\mathrm{Sat}(F)$-Sequences

Given $k \in \mathbb{N}\backslash\{0\}$, a $\mathrm{Sat}(F)$-*sequence of length* k is a k-sequence of positive integers $(d_1, d_2, \ldots, d_k)$ such that $d_1 > d_2 > \cdots > d_k$, $d_{i+1} \mid d_i$ for all $i \in \{1, \cdots, k-1\}$ and $d_k \nmid F$.

Theorem 2. *If $(d_1, d_2, \ldots, d_p)$ is a* Sat(F)*-sequence and $t_1, t_2, \cdots, t_p$ are positive integers such that $t_1d_1 + \cdots + t_pd_p < F$ and* $\gcd\left\{\dfrac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ *for all $i \in \{1, \ldots, p-1\}$, then $\{d_1, t_1d_1 + t_2d_2, \cdots, t_1d_1 + t_2d_2 + \cdots + t_pd_p\}$ is the minimal* Sat(F)*-system of generators of an element of* Sat(F) *with* Sat(F)*-rank equal to p. Moreover, every minimal* Sat(F)*-system of generators of an element of* Sat(F) *with* Sat(F)*-rank equal to p, has this form.*

Proof. It is easy to see that $\gcd\{d_1, t_1d_1 + t_2d_2, \cdots, t_1d_1 + t_2d_2 + \cdots + t_id_i\} = d_i$ for all $i \in \{1, \cdots, p\}$. By applying Proposition 18, we obtain that $\{d_1, t_1d_1 + t_2d_2, \cdots, t_1d_1 + t_2d_2 + \cdots + t_pd_p\}$ is the minimal Sat(F)-system of generators of an element of Sat(F) with Sat(F)-rank equal to p.

Conversely, if $\{n_1 < n_2 < \cdots < n_p\}$ is the minimal Sat(F)-system of generators of an element of Sat(F) and $d_i = \gcd\{n_1, \cdots, n_i\}$ for all $i \in \{1, \cdots, p\}$, then by applying Proposition 18 and Lemma 12, we have that $(d_1, \ldots, d_p)$ is a Sat(F)-sequence. To conclude the proof, we will show that there are positive integers $t_1, \cdots, t_p$ such that $n_1 = d_1, n_2 = t_1d_1 + t_2d_2, \cdots, n_p = t_1d_1 + t_2d_2 + \cdots + t_pd_p$ and $\gcd\left\{\dfrac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ for all $i \in \{1, \ldots, p-1\}$. Let $t_1 = 1$ and $t_{i+1} = \frac{n_{i+1} - n_i}{d_{i+1}}$ for all $i \in \{1, \cdots, p-1\}$. Let us prove, by induction on i, that $n_i = t_1d_1 + \cdots + t_id_i$ for all $i \in \{2, \ldots, p\}$. For $i = 2$, the result is true since $t_1d_1 + t_2d_2 = 1 \cdot n_1 + \frac{n_2 - n_1}{d_2}d_2 = n_2$. As $n_{i+1} = n_i + t_{i+1}d_{i+1}$, by the induction hypothesis, we have $n_{i+1} = t_1d_1 + \cdots + t_id_i + t_{i+1}d_{i+1}$. To conclude the proof, it suffices to show that $\gcd\left\{\dfrac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ for all $i \in \{1, \ldots, p-1\}$. In fact, $d_{i+1} = \gcd\{n_1, \cdots, n_{i+1}\} = \gcd\{\gcd\{n_1, \cdots, n_i\}, n_{i+1}\} = \gcd\{d_i, t_1d_1 + \cdots + t_id_i + t_{i+1}d_{i+1}\} = \gcd\{d_i, t_{i+1}d_{i+1}\} = d_{i+1} \cdot \gcd\left\{\dfrac{d_i}{d_{i+1}}, t_{i+1}\right\}$. Therefore, $\gcd\left\{\dfrac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$. □

As a direct consequence of the previous theorem, we have the following result.

Corollary 3. *If $(d_1, d_2, \ldots, d_p)$ is a* Sat(F)*-sequence and $d_1 + d_2 + \cdots + d_p < F$, then $\{d_1, d_1 + d_2, \cdots, d_1 + d_2 + \cdots + d_p\}$ is a minimal* Sat(F)*-system of generators of an element of* Sat(F).

As a consequence of Theorem 2 and Corollary 3, if we want to compute all the elements belonging to Sat(F) with Sat(F)-rank equal to p, it will be enough to perform the following steps:

(1) To compute

$$\mathrm{L}(F, p) = \{(d_1, \ldots, d_p) \mid (d_1, \ldots, d_p) \text{ is a Sat}(F)\text{-sequence and } d_1 + \cdots + d_p < F\}.$$

(2) For every $(d_1, \ldots, d_p) \in \mathrm{L}(F, p)$, compute

$$\mathrm{C}(d_1, \ldots, d_p) = \{(t_1, \ldots, t_p) \in (\mathbb{N}\backslash\{0\})^p \mid t_1d_1 + \cdots + t_pd_p < F \text{ and}$$
$$\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1 \text{ for all } i \in \{1, \ldots, p-1\}\}.$$

A characterization of a Sat(F)-sequence appears in the following result.

Proposition 20. *If $\{a_1, a_2, \cdots, a_p\} \subseteq \mathbb{N}\backslash\{0, 1\}$ and $a_1 \nmid F$, then $(a_1a_2 \cdots a_p, a_1a_2 \cdots a_{p-1}, \cdots, a_1)$ is a* Sat(F)*-sequence of length p. Moreover, every* Sat(F)*-sequence of length p is of this form.*

Proof. If we take $d_{i+1} = a_1 \cdots a_{p-i}$ with $i \in \{0, \cdots, p-1\}$, the result follows trivially. Furthermore, by definition, every Sat(F)-sequence of length p has the above form. □

Corollary 4. *Let a be the smallest positive integer that does not divide F. Then* Sat(F) *contains at least one element of* Sat(F)*-rank equal to p if and only if $a(2^p - 1) < F$.*

Proof. By applying Theorem 2 and Corollary 3, we deduce that $\mathrm{Sat}(F)$ contains at least an element of $\mathrm{Sat}(F)$-rank equal to p if and only if $\mathrm{L}(F,p) \neq \emptyset$. By applying Proposition 20 now, we have that $\mathrm{L}(F,p) \neq \emptyset$ if and only if there exists $\{a_1, a_2, \cdots, a_p\} \subseteq \mathbb{N}\backslash\{0,1\}$ such that $a_1 \nmid F$ and $a_1 a_2 \cdots a_p + a_1 a_2 \cdots a_{p-1} + \cdots + a_1 < F$. To conclude the proof, it suffices to note that this is verified if and only if $a \cdot 2^{p-1} + a \cdot 2^{p-2} + \cdots + a < F$. By using the formula of the sum of a geometry progression, we obtain that $a \cdot 2^{p-1} + a \cdot 2^{p-2} + \cdots + a < F$ if and only if $a(2^p - 1) < F$. $\square$

Example 8. *We can assert, by using Corollary 4, that* $\mathrm{Sat}(18)$ *does not have elements with* $\mathrm{Sat}(F)$*-rank equal to* 3*, because* $4(2^3 - 1) > 18$.

We finish this work by showing an Algorithm 3 which allows us to compute the set $\mathrm{C}(d_1, \ldots, d_p)$ from $(d_1, \ldots, d_p) \in \mathrm{L}(F,p)$.

For the first time we note that to computing the set

$$\{(t_1, \ldots, t_p) \in (\mathbb{N}\backslash\{0\})^p \mid t_1 d_1 + \cdots + t_p d_p \leq F - 1\}$$

is equivalent to computing the set

$$\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid d_1 x_1 + \cdots + d_p x_p \leq F - 1 - (d_1 + \cdots + d_p)\}.$$

Additionally, observe that

$$\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid d_1 x_1 + \cdots + d_p x_p \leq F - 1 - (d_1 + \cdots + d_p)\} =$$

$$\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid d_1 x_1 + \cdots + d_p x_p = k \text{ for some } k \in \{0, \cdots, F - 1 - (d_1 + \cdots + d_p)\}.$$

If $(x_1, \ldots, x_p) \in \mathbb{N}^p$ and $d_1 x_1 + \cdots + d_p x_p = k$, then $d_p \mid k$. Hence, $k = a \cdot d_p$ and consequently, $\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid d_1 x_1 + \cdots + d_p x_p = k\} = \{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid \frac{d_1}{d_p} x_1 + \cdots + \frac{d_p}{d_p} x_p = a\}$.

Finally, observe that Algorithm 14 from [24] allows us to compute the set $\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid \frac{d_1}{d_p} x_1 + \cdots + \frac{d_p}{d_p} x_p = a\}$.

Algorithm 3 Computation of $\mathrm{C}(d_1, \ldots, d_p)$.

INPUT: $(d_1, \ldots, d_p) \in \mathrm{L}(F,p)$.
OUTPUT: $\mathrm{C}(d_1, \ldots, d_p)$.

(1) $\alpha = F - 1 - (d_1 + \cdots + d_p)$.
(2) *For all* $k \in \{0, \cdots, \lfloor \frac{\alpha}{d_k} \rfloor\}$*, by using Algorithm 14 from [24], compute* $D_k = \{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid \frac{d_1}{d_p} x_1 + \cdots + \frac{d_p}{d_p} x_p = k\}$.
(3) *For all* $k \in \{0, \cdots, \lfloor \frac{\alpha}{d_k} \rfloor\}$*, let* $E_k = \{(x_1 + 1, \ldots, x_p + 1) \mid (x_1, \ldots, x_p) \in D_k\}$.
(4) $A = \bigcup_{k=0}^{\lfloor \frac{\alpha}{d_k} \rfloor} E_k$.
(5) *Return* $\{(t_1, \cdots, t_p) \in A \mid \gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ *for all* $i \in \{1, \ldots, p-1\}\}$.

Thereby, given $(d_1, \ldots, d_p) \in \mathrm{L}(F,p)$, by using [24], Algorithm 14, the previous algorithm computes the set $\mathrm{C}(d_1, \ldots, d_p)$. Consequently, we have a procedure to compute all the elements belonging to $\mathrm{Sat}(F)$ with $\mathrm{Sat}(F)$-rank equal to p.

7. Conclusions

The fact that $\mathrm{Sat}(F)$ is a covariety has allowed us to present three algorithms:

(1) An algorithm which calculates all the elements of $\mathrm{Sat}(F)$.
(2) An algorithm to compute the elements belonging to $\mathrm{Sat}(F)$ with a fixed genus.
(3) An algorithm that calculates all the elements of $\mathrm{Sat}(F)$ with a fixed $\mathrm{Sat}(F)$-rank.

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