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Approximation of Bivariate Functions by Generalized Wendland Radial Basis Functions

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Abstract: In this work, we deal with two approximation problems in a finite-dimensional generalized Wendland space of compactly supported radial basis functions. Namely, we present an interpolation method and a smoothing variational method in this space. Next, the theory of the presented method is justified by proving the corresponding convergence result. Likewise, to illustrate this method, some graphical and numerical examples are presented in $\mathbb{R}^2$, and a comparison with another work is analyzed.

Keywords: radial basis functions; generalized Wendland radial basis functions; interpolation; smoothing; variational methods

MSC: 65D05; 65D07; 65D10

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1. Introduction

Frequently, positive kernels reproducing Hilbert spaces of continuous functions appear in some applications, and they are presented as radial basis functions (RBFs),

$$\Psi(x,y) = \psi(\langle x-y\rangle_n), \quad \forall\, x,y \in \mathbb{R}^n,$$

where $\langle \cdot \rangle_n$ denotes the Euclidean norm in $\mathbb{R}^n$, and $\psi : [0,+\infty) \to \mathbb{R}$ is a given smooth univariate function.

The Wendland functions [1] yield compactly supported and differentiable functions in $\mathbb{R}^n$ that reproduce kernels of Hilbert spaces isomorphic to the Sobolev space $H^{n/2+k+1/2}(\mathbb{R}^n)$. Thus, when the dimension n is even, the order of this Sobolev space is not an integer.

Robert Schaback [2] extends the classical Wendland functions to the missing Wendland functions that reproduce kernels of Hilbert spaces isomorphic to the Sobolev spaces of integer order in even dimensions. Moreover, they have compact support. In this context, in [3], Schaback and Wendland used compactly supported radial basis functions in order to solve some partial differential equations.

In [4], Argáez, Hafstein, and Giesl provided a numerical code in C++ in order to calculate explicitly the Wendland function with any given parameters. Previously, in [2], Schaback and Zhu, in [5], used instead a code written in MAPLE. In [6], Chen and other authors proposed a study of a surrogate model assisted by an evolutionary algorithm for high-dimensional expensive optimization problems also using this type of radial basis functions. Saberi et al. in [7] provided the required formulas in one dimension for the Riemann Liouville fractional derivative of five kinds of RBFs, including the Powers, Gaussian, Multiquadric, Matérn, and Thin-plate splines. After, they also considered the discretization of the fractional diffusion equation with the RBF collocation method.

Radial basis function (RBF) approximations have been used for some time to interpolate data on a sphere. In this context, Fornberg and Piret, in [8], extended the earlier works for computations in three aspects: firstly, tested with a large number of different types of radial functions; secondly, calculated in a stable way for e-values all the way down to the parameter equal to zero; thirdly, results presented at both short and long times, in order to contrast time scales appropriate for weather and for climate modeling, respectively.

In [9], Rosenfeld and Dixon developed a pseudo spectral method for the estimation of the fractional Laplacian function using the approach by RBF interpolation.

Buhmann and Jager, in [10], presented the connections of the monotonicity properties and the strict positive definiteness of vectorial functions. They studied a technique to construct positive definite functions from multiple monotone functions.

Chernih et al., in [11], also demonstrated that with an appropriate rescaling of the variables, both the original and the missing Wendland functions converge uniformly to Gaussian, as the smoothness parameter tends to infinity.

To better understand the objective of this work, we believe that we should cite a brief history of the theory of the approximation problem using variational spline functions. The theory of the approach using variational splines was introduced by Attéia [12], based on the D^m-splines functions, after Duchon [13] developed the idea, using the technique of the minimization of quadratic functionals. We enriched this generic idea by minimizing various types of quadratic functionals, first in Hilbert spaces and secondly in a finite element space, such as in [14] by Kouibia et al. We studied some interpolation and smoothing methods for constructing free-form curves and surfaces from a given Lagrangian and/or Hermite data set. These methods consist of the minimization of a certain quadratic functional in a Sobolev space.

In [15], Kouibia et al. presented an approximation method from a given scattered data set, by minimizing a quadratic functional in a parametric finite element space. In [16], Kouibia and collaborators considered the same problem from a given noisy data set; meanwhile, in [17], they studied these problems in a bicubic spline functional space, and the optimal solution was obtained by a suitable optimization of some parameters that appear in the minimization functional.

In recent years, some of the authors of this article started to work on some problems of approximation using the Wendland radial basis functions. Recent publications include, for example, [18], where González et al. proposed an approximation method for solving second-kind Volterra integral equation systems by radial basis functions. Recently, in [19], Noorizadegan and Schaback introduced the evaluation condition number by a novel assessment of conditioning in radial basis function methods.

In this work, we deal with the smoothing problem in a finite-dimensional generalized Wendland functions space; formulating the problem of smoothing variational splines by generalized Wendland functions, we show how to compute, in practice, the solution of such a problem, and the method is justified by proving the corresponding convergence result. In order to illustrate the method, some graphical and numerical examples are presented in $\mathbb{R}^2$, and a comparison with another work is analyzed.

The remainder of this manuscript is organized as follows. In Section 2, we present some notations and preliminaries that are necessary to formulate the problem. Section 3 is devoted to studying the generalized Wendland compactly supported radial basis functions, while Section 4 is dedicated to developing the problem of the smoothing variational splines by generalized Wendland functions. In the last section, we finish this article by illustrating some numerical and graphical examples and presenting a comparison with another work.

2. Notations and Preliminaries

Given an open convex bounded set $\Omega \subset \mathbb{R}^2$, let $H^s(\Omega)$ be the usual Sobolev space of order s equipped with the semi-inner products given by

$$(u, v)_\ell = \sum_{|\alpha|=\ell} \int_\Omega u^\alpha(\boldsymbol{x}) v^\alpha(\boldsymbol{x}) d\boldsymbol{x}, \quad \ell = 0, \ldots, s,$$

for any $u, v \in H^s(\Omega)$; the corresponding semi-norms

$$|u|_\ell = (u,u)_\ell^{\frac{1}{2}}, \quad \ell = 0, \dots, s;$$

the inner product $((u,v))_s = \sum_{\ell=0}^{s} (u,v)_\ell$; and the corresponding norm $\|u\|_s = ((u,u))_s^{\frac{1}{2}}$.

Let $\mathbb{R}^{n,k}$ be the space of real matrices with n rows and k columns, equipped with the inner product

$$\langle A, B\rangle_{n,k} = \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}b_{ij}, \quad \forall\, A = (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le k}},\, B = (b_{ij})_{\substack{1\le i\le n\\ 1\le j\le k}} \in \mathbb{R}^{n,k},$$

and the corresponding norm $\langle A\rangle_{n,k} = \langle A, A\rangle_{n,k}^{\frac{1}{2}}$.

3. Generalized Wendland Compactly Supported Radial Basis Functions

Definition 1. *Let there be $\psi : [0, +\infty) \to \mathbb{R}$ as a continuous function, a set $\Omega \subset \mathbb{R}^2$, and a finite set $T_N = \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N\}$ of points of Ω; the linear space generated by the functions set*

$$S_N = \{\psi(\langle \cdot - \boldsymbol{\xi}_1\rangle_2), \dots, \psi(\langle \cdot - \boldsymbol{\xi}_N\rangle_2)\} \tag{1}$$

is called the radial basis functions space relative to the function ψ and the centers set T_N, where $< \cdot, \cdot >_2$ is the Euclidean inner product in $\mathbb{R}^2$.

Definition 2. *Consider a function $u \in C(\Omega)$ and the radial basis function $s_{u,T_N} \in S_N$ given by*

$$s_{u,T_N}(\boldsymbol{x}) = \sum_{i=1}^{N} c_i \psi(\langle \boldsymbol{x} - \boldsymbol{\xi}_i\rangle_2), \quad \boldsymbol{x} \in \Omega, \tag{2}$$

where $c_1, \dots, c_N \in \mathbb{R}$ are determined by the interpolating conditions

$$s_{u,T_N}(\boldsymbol{\xi}_i) = u(\boldsymbol{\xi}_i), \quad 1 \le i \le N. \tag{3}$$

Then, s_{u,T_N}, if it exists, is called the interpolation RBF of u in S_N (relative to ψ and T_N).

Remark 1. *The interpolation RBF s_{u,T_N} exists, and it is unique if and only if*

$$\det((\psi(\langle \boldsymbol{\xi}_i - \boldsymbol{\xi}_j\rangle_2))_{1\le i,j\le N}) \neq 0.$$

Robert Schaback in [2] considered the integral operator

$$I_\alpha(f)(t) = \int_t^\infty f(s)\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}ds,$$

for all $\alpha > 0$, $t \ge 0$.

Consider the truncated power functions for all $\mu > 0$.

$$a_\mu(s) = (1 - \sqrt{2s})_+^\mu.$$

Since the I_α operators preserve compact supports and are applicable to a_μ for all $\alpha, \mu > 0$, we can define $a_{\mu,\alpha} = I_\alpha(a_\mu)$.

Definition 3. *We call generalized Wendland functions to $\Psi_{\mu,\alpha}$ given by*

$$\Psi_{\mu,\alpha}(r) = a_{\mu,\alpha}(\frac{r^2}{2}), \quad \forall\, \alpha, \mu > 0,$$

which are well defined and supported in $[0,1]$.

Remark 2. *Taking into account the above definition, we have*

$$\Psi_{\mu,\alpha}(t) = \int_t^1 s(1-s)^{\mu} \frac{(s^2-t^2)^{\alpha-1}}{\Gamma(\alpha)2^{\alpha-1}} ds, \quad \forall t \in [0,1].$$

In [2], the author deduces an algorithm for constructing the generalized Wendland functions for even dimensions $2m$ in the following way (Table 1):

$$\Psi_{2m,(2\ell-1)/2}(r) = r^{2\ell} p_{m,\ell}(r^2) L(r) + q_{m,\ell}(r^2) S(r), \quad r \in [0,1],$$

for any integers $m, \ell \geq 0$, with

$$L(r) = \log\left(\frac{r}{1+\sqrt{1-r^2}}\right), \quad S(r) = \sqrt{1-r^2},$$

and $p_{m,\ell}, q_{m,\ell}$ as two associated polynomials of degree $m-1$ and $m-1+\ell$, respectively.

Table 1. Some generalized Wendland functions in even dimensions.

$\Psi_{2,1/2}(r) = \frac{\sqrt{2}}{3\sqrt{\pi}}(3r^2L(r) + (2r^2+1)S(r)),$
$\Psi_{2,3/2}(r) = -\frac{\sqrt{2}}{60\sqrt{\pi}}(15r^4L(r) + (8r^4+9r^2-2)S(r)),$
$\Psi_{2,5/2}(r) = \frac{\sqrt{2}}{2520\sqrt{\pi}}(105r^6L(r) + (48r^6+87r^4-38r^2+8)S(r)),$
$\Psi_{4,1/2}(r) = \frac{\sqrt{2}}{30\sqrt{\pi}}((45r^4+60r^2)L(r) + (16r^4+83r^2+6)S(r)),$
$\Psi_{4,3/2}(r) = -\frac{\sqrt{2}}{420\sqrt{\pi}}((105r^6+210r^4)L(r) + (32r^6+247r^4+40r^2-4)S(r)).$

Theorem 1. *Let there be $\Omega \subset \mathbb{R}^2$, $T_N = \{\xi_1, \ldots, \xi_N\} \subset \Omega$ as a centers set, and $n, k \in \mathbb{N}$. Let s_{f,T_N} be the interpolation RBF of $f \in H^{k+2}(\Omega)$ relative to T_N from $\Psi_{k+2,k+1/2} = \Psi_{\alpha+3/2,\alpha}$, with $\alpha = k + 1/2$.*

Let

$$h = \sup_{x\in\Omega} \min_{1\leq i\leq N} \langle x - \xi_i \rangle_2$$

be the fill distance of T_N in Ω, where $< \cdot >_2$ denotes the Euclidean norm in $\mathbb{R}^2$.

Then,

$$|f - s_{f,T_N}|_j \leq Ch^{k+2-j}\|f\|_{k+2}, \quad \forall\, j = 0, \ldots, k+2, \tag{4}$$

where C is independent of f.

Proof. Applying ([20], Proposition 3.2) for $\alpha = 0$, $s = 0$, and $\tau = k+2$, it is verified that $k+2 > \alpha+1$; thus, there exists a real constant $C > 0$, independent of f, such that

$$\|f - s_{f,T_N}\|_0 \leq Ch^{k+2}\|f\|_{k+2}. \tag{5}$$

From Madych-Nelson ([21], Theorem 6), it is verified that

$$(\Psi_{k+2,k+1/2}(\langle \cdot - \xi_j \rangle_2), s_{f,T_N}) = s_{f,T_N}(\xi_j),$$

and

$$(\Psi_{k+2,k+1/2}(\langle \cdot - \boldsymbol{\xi}_j \rangle_2), f) = f(\boldsymbol{\xi}_j) = s_{f,T_N}(\boldsymbol{\xi}_j),$$

where $(\cdot, \cdot)$ denotes the inner product in the dual space of S_N.

Then, $(\Psi_{k+2,k+1/2}(\langle \cdot - \boldsymbol{\xi}_j \rangle_2), s_{f,T_N} - f) = 0$, for all $i = 1, \ldots, N$, and we have that $s_{f,T_N} - f$ is orthogonal to S_N.

Thus, for any $s \in S_N$, it is verified that $((s_{f,T_N} - s, s_{f,T_N} - f))_{k+2} = 0$, and we obtain that

$$\|s - f\|_{k+2}^2 = \|s - s_{f,T_N} + s_{f,T_N} - f\|_{k+2} = \|s - s_{f,T_N}\|_{k+2}^2 + \|s_{f,T_N} - f\|_{k+2}^2.$$

Hence, we have

$$\|s_{f,T_N} - f\|_{k+2}^2 \le \|s - f\|_{k+2}^2,$$

and taking $s = 0$, we conclude that

$$\|s_{f,T_N} - f\|_{k+2} \le \|f\|_{k+2}. \tag{6}$$

From (5), (6), and Jiayin ([22], Lemma 3.3.3), we can affirm that there exists $C > 0$, independent of f, such that

$$\|f - s_{f,T_N}\|_j \le C h^{k+2-j} \|f\|_{k+2}, \quad \forall j = 0, \ldots, k+2.$$

Then, there exists $C > 0$, independent of f, such that

$$|f - s_{f,T_N}|_j \le C h^{k+2-j} \|f\|_{k+2}, \quad \forall j = 0, \ldots, k+2,$$

and (4) holds. $\square$

4. Smoothing Variational Splines by Generalized Wendland Functions

Given a function $f \in H^{k+2}(\Omega)$ with $k \ge 0$ and a finite set of points $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} \subset \Omega$, we consider the functional $\theta : H^{k+1}(\Omega) \to \mathbb{R}^n$ given by

$$\theta v = (v(\boldsymbol{a}_i))_{1 \le i \le n} \in \mathbb{R}^n,$$

and for any $\varepsilon > 0$, let Γ be the functional defined on $H^{k+2}(\Omega)$ by

$$\Gamma(v) = \langle \theta v - \theta f \rangle_n^2 + \varepsilon |v|_{k+2}^2.$$

Remark 3. *The first term of $\Gamma(v)$ indicates how well v approaches f in a least discrete square sense. The second term represents a classical smoothness measure weighted by the parameter ε.*

Let S_N be the radial basis functions space relative to the function $\psi_{k+2,k+\frac{1}{2}}$ and the centers set T_N, and consider the following minimization problem: find $\sigma \in S_N$ such that

$$\forall v \in S_N, \quad \Gamma(\sigma) \le \Gamma(v). \tag{7}$$

Suppose that A is a $\mathbb{P}_{k+1}(\Omega)$-unisolvent set; that is,

$$\ker \theta \cap \mathbb{P}_{k+1}(\Omega) = \{0\}, \tag{8}$$

and suppose that

$$\sup_{\boldsymbol{x} \in \Omega} \min_{\boldsymbol{a} \in A} \langle \boldsymbol{x} - \boldsymbol{a} \rangle_2 = o\left(\frac{1}{n}\right), \quad n \to +\infty. \tag{9}$$

Theorem 2. *Problem (7) has a unique solution, called the smoothing variational spline in S_N associated with A, θf, and ε, which is the unique solution of the following variational problem: find $\sigma_n \in S_N$, such that*

$$\forall\, v \in S_N, \quad \langle \theta\sigma_n, \theta v\rangle_n + \varepsilon(\sigma_n, v)_{k+2} = \langle \theta f, \theta v\rangle_n. \tag{10}$$

Proof. From (8), we have that the bilinear application $\eta : H^{k+2}(\Omega) \times H^{k+2}(\Omega) \to \mathbb{R}$, given by

$$\eta(u, v) = 2(\langle \theta u, \theta v\rangle_n + \varepsilon(u, v)_{k+2}),$$

is continuous and $H^{k+2}(\Omega)$-elliptic. Applying the Lax–Milgram Lemma ([23], Theorem 3.8.2) for η and the continuous linear application $\ell : H^{k+2}(\Omega) \to \mathbb{R}$ given by $\ell(v) = 2\langle \theta f, \theta v\rangle_n$, there exists $\sigma_n \in S_N$, such that

$$\forall\, v \in S_N, \quad \eta(\sigma_n, v) = \ell(v),$$

and (10) holds. Moreover, σ_n minimizes the functional $\varphi(v) = \frac{1}{2}\eta(\sigma_n, v) - \ell(v) = \Gamma(v) - \langle \theta f\rangle_n^2$; thus, σ_n is the solution to Problem (7). $\square$

To compute the solution function σ_n, for $i = 1, \ldots, N$, let $w_i \in S_N$ be the function

$$w_i(\boldsymbol{\xi}) = \psi_{k+2,k+\frac{1}{2}}(\langle \boldsymbol{\xi} - \boldsymbol{\xi}_i\rangle_2), \quad \forall\, \boldsymbol{\xi} \in \Omega;$$

then, $\sigma_n = \sum_{i=1}^{N} c_i w_i$. Applying Theorem 2, we obtain that $\boldsymbol{c} = (c_1, \ldots, c_N)^\top \in \mathbb{R}^N$ is the solution to the linear system

$$(\mathcal{A}\mathcal{A}^\top + \varepsilon\mathcal{R})\mathbf{c} = \mathcal{A}\theta f,$$

where its coefficients are given as follows:

$$\mathcal{A} = (\theta w_i)_{1 \le i \le N} \in \mathbb{R}^{N,n},$$

and

$$\mathcal{R} = ((w_i, w_j)_{k+2})_{1 \le i,j \le N}.$$

Now, we prove that the smoothing variational spline σ_n converges to the function f under suitable hypotheses.

Theorem 3. *Suppose the hypotheses (8) and (9) hold and that*

$$\varepsilon = o(1), \quad n \to +\infty, \tag{11}$$

and

$$\frac{n^2 h^{2k+4}}{\varepsilon} = o(1), \quad n \to +\infty. \tag{12}$$

Then, one has

$$\lim_{n \to +\infty} \|\sigma_n - f\|_{k+2} = 0.$$

Proof. Let s_{f,T_N} be the interpolation RBF of f relative to T_N from $\psi_{k+2,k+1/2}$; then, $\Gamma(\sigma_n) \le \Gamma(s_{f,T_n})$, and one has

$$\langle \theta\sigma_n - \theta f\rangle_n^2 + \varepsilon|\sigma_n|_{k+2}^2 \le \langle \theta s_{f,T_N} - \theta f\rangle_n^2 + \varepsilon|s_{f,T_N}|_{k+2}^2. \tag{13}$$

From (4), there exists $C > 0$, such that

$$|s_{f,T_N}|_{k+2}^2 \le C\|f\|_{k+2}^2, \tag{14}$$

and

$$\langle \theta f - \theta s_{f,T_N} \rangle_n^2 \leq n^2 C h^{2k+4} \| f \|_{k+2}^2. \tag{15}$$

Thus, from (13)–(15), we have that

$$|\sigma_n|_{k+2}^2 \leq \frac{1}{\varepsilon} \langle \theta f - \theta s_{f,T_N} \rangle_n^2 + |s_{f,T_N}|_{k+2}^2 \leq (\frac{n^2 h^{2k+4}}{\varepsilon} + 1) C \| f \|_{k+2}^2,$$

and from (12), we conclude that there exists $C_1 > 0$ and $n_1 \in \mathbb{N}$, such that

$$|\sigma_n|_{k+2}^2 \leq C_1, \quad \forall\, n \geq n_1. \tag{16}$$

Moreover, from (13)–(15), we have that

$$\langle \theta \sigma_n - \theta f \rangle_n^2 \leq (n^2 h^{2k+4} + \varepsilon) C \| f |_{k+2}^2,$$

and from (11) and (12), there exists $C_2 > 0$ and $n_2 \in \mathbb{N}$, such that

$$\langle \theta \sigma_n - \theta f \rangle_n \leq C_2, \quad \forall\, n \geq n_2. \tag{17}$$

From (16) and (17), we can deduce that there exists a real constant $C > 0$ and $n_0 \in \mathbb{N}$, such that

$$\| \sigma_n \|_{k+2} \leq C, \quad n \geq n_0,$$

which means that the family $(\sigma_n)_{n \geq n_0}$ is bounded in S_N. It follows that there exists a subsequence $(\sigma_{n_l})_{l \in \mathbb{N}}$ with $\lim\limits_{l \to +\infty} n_l = +\infty$ and an element $f^* \in H^{k+2}(\Omega)$, such that

$$\sigma_{n_l} \text{ converges weakly to } f^* \text{ in } H^{k+2}(\Omega).$$

Finally, reasoning as in the points (3), (4), and (5) of the proof of ([24], Theorem VI-3.2), we obtain the result. □

5. Numerical and Graphical Examples

To show the effectiveness of the method, we computed two relative error estimations given by

$$E_I = \sqrt{\frac{\sum\limits_{i=1}^{5000} (s_{f,T_N}(a_i) - f(a_i))^2}{\sum\limits_{i=1}^{5000} f(a_i)^2}}, \quad E_S = \sqrt{\frac{\sum\limits_{i=1}^{5000} (\sigma_n(a_i) - f(a_i))^2}{\sum\limits_{i=1}^{5000} f(a_i)^2}},$$

with $\{a_1, \ldots, a_{5000}\} \subset I$ as five thousand distinct random points, which are some approximations of the relative error of s_{f,T_N} and σ_n, respectively, with respect to f in $L^2(I)$.

From Theorems 1 and 3, these relative error estimations E_I and E_S tend to 0 as n tends to $+\infty$, under adequate conditions.

Consider the Franke function (see [25]), given by

$$\begin{aligned} f(x,y) = \;& 0.75 e^{-\frac{1}{10}(9x+1)^2 - \frac{1}{49}(9y+1)^2} - 0.2 e^{-\left((9x-7)^2 + (9y-4)^2\right)} \\ & + 0.5 e^{-\frac{1}{4}\left((9x-3)^2 + (9y-7)^2\right)} + 0.75 e^{-\frac{1}{4}\left((9x-2)^2 + (9y-2)^2\right)}, \end{aligned}$$

for any $(x,y) \in \Omega = (0,1) \times (0,1)$.

Moreover, the discrete space that we use to calculate the approximated solution σ_n is the RBFs space constructed from the generalized Wendland function $\Psi_{2,1/2}$ and the centers set

$$T_N = \left\{ \left(\frac{i}{r-1}, \frac{j}{r-1} \right) \; i,j = 0, \ldots, r-1 \right\},$$

with $N = r^2$.

Table 2 shows the relative error estimation E_S with $r = 10$ ($N = \dim S_N = 100$) and $n = 1000$ for different values of ε; this specific parameter is introduced to avoid any oscillation. In this case, $E_I = 5.8496 \times 10^{-3}$. We observe that there exists an optimum value of ε that could be estimated minimizing E_S.

Table 2. Computed relative error estimation E_S with $r = 10$ and $n = 1000$ for different values of ε. $E_I = 5.8496 \times 10^{-3}$.

ε	E_S
10^{-1}	3.4819×10^{-2}
10^{-2}	8.5638×10^{-2}
10^{-3}	6.1827×10^{-3}
10^{-4}	5.2971×10^{-3}
10^{-5}	7.2971×10^{-3}
10^{-6}	3.1421×10^{-3}
10^{-7}	3.7435×10^{-3}
10^{-8}	3.3564×10^{-3}
10^{-9}	2.9671×10^{-3}
10^{-10}	3.1934×10^{-3}
10^{-11}	3.2573×10^{-3}
10^{-12}	3.0629×10^{-3}
10^{-13}	3.0529×10^{-3}
10^{-14}	3.0917×10^{-3}
10^{-15}	3.0927×10^{-3}

Table 3 shows the relative error estimation E_S with $r = 10$ ($N = \dim S_N = 100$) and $\varepsilon = 10^{-9}$ for different values of n. In this case, $E_I = 5.8496 \times 10^{-3}$. We observe that E_S decreases when n increases, and it seems that it tends to stabilize.

Table 3. Computed relative error estimation E_S with $r = 10$ and $\varepsilon = 10^{-9}$ for different values of n. $E_I = 5.8496 \times 10^{-3}$.

n	E_S
100	5.7736×10^{-3}
500	3.5199×10^{-3}
1000	3.0276×10^{-3}
2500	2.9316×10^{-3}
5000	2.7236×10^{-3}

Table 4 shows the relative error estimations E_I and E_S with $n = 1000$ and $\varepsilon = 10^{-9}$ for different values of r. We observe that E_I and E_S decrease when r increases.

Table 4. Computed relative error estimations E_I and E_S with $n = 1000$ and $\varepsilon = 10^{-9}$ for different values of r.

r	E_I	E_S
5	5.9404×10^{-2}	4.4007×10^{-2}
7	3.1667×10^{-2}	2.5946×10^{-2}
10	5.8496×10^{-3}	3.0276×10^{-3}
12	3.9227×10^{-3}	1.8972×10^{-3}

Figure 1 shows the graphs of the function f, and Figure 2 shows the interpolation RBF s_{f,T_N} and the smoothing variational spline σ_n for $r = 10$, $n = 1000$, and $\varepsilon = 10^{-9}$, from left to right. We obtained that $E_I = 5.8496 \times 10^{-3}$ and $E_S = 3.0276 \times 10^{-3}$.

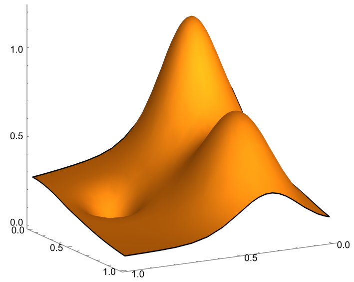

Figure 1. Graph of the function f.

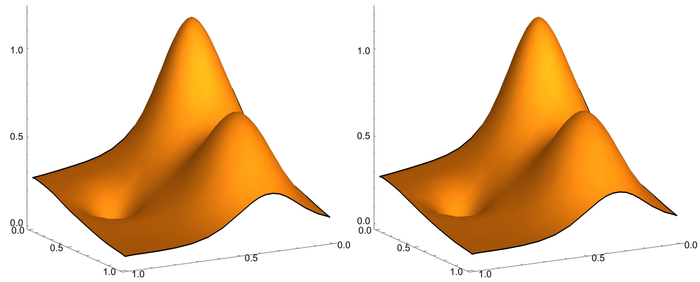

Figure 2. Graphs of the interpolation RBF s_{f,T_N} and the smoothing variational spline σ_n for $r = 10$, $n = 1000$, and $\varepsilon = 10^{-9}$, from left to right.

6. Conclusions

While the method developed in this work is known, the use of the generalized Wendland compactly supported RBFs in this context is totally new. In fact, the question that one can ask is why use these functions? The answer is that the time cost of programming these functions is quite reduced, if we compare it, for example, to the variational splines mentioned in the references [14–17]. Moreover, the order of the degree of approximation, represented with the calculation of the estimate of the interpolation error E_I and the smoothing error E_S, with 500–1000 approximation points are of an order between 1.8972×10^{-3} and 3.0276×10^{-3} in most cases, as shown in Tables 2–4, while in Table 2 subsection 5.2.2 of [14], the degree of approximation with 900 points of approximation is $8.8\ldots \times 10^{-3}$. All this shows the improvement and the effectiveness of the approximation method studied in this manuscript.

As a subject for another manuscript in the future or as an open topic, we think it is possible to extend the study to higher dimensions.

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