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Approximation of the Fixed Point of the Product of Two Operators in Banach Algebras with Applications to Some Functional Equations

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Abstract: Making use of the Boyd-Wong fixed point theorem, we establish a new existence and uniqueness result and an approximation process of the fixed point for the product of two nonlinear operators in Banach algebras. This provides an adequate tool for deriving the existence and uniqueness of solutions of two interesting type of nonlinear functional equations in Banach algebras, as well as for developing an approximation method of their solutions. In addition, to illustrate the applicability of our results we give some numerical examples.

Keywords: Banach algebras; fixed point theory; functional equations; Schauder bases

MSC: 65J15; 47J25; 46B15; 32A65

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1. Introduction

Many phenomena in physics, chemistry, mechanics, electricity, and so as, can be formulated by using the following nonlinear differential equations with nonlocal initial condition of the form:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) & = g(t,x(t)), \quad t \in J := [0,\rho], \\ \qquad\qquad x(0) & = \mu(x), \end{cases} \tag{1}$$

where $\rho > 0$ is a real constant, $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}$ are supposed to be $\mathcal{D}$-Lipschitzian with respect to the second variable, and the operator $\mu : C(J) \to \mathbb{R}$ represents the nonlocal initial condition, see [1,2]. Here, $C(J)$ is the space of all continuous functions from J into $\mathbb{R}$ endowed with the norm $\|\cdot\|_\infty = \sup_{t\in J}\|x(t)\|$.

The nonlocal condition $x(0) = \mu(x)$ can be more descriptive in physics with better effect than the classical initial condition $x(0) = x_0$, (see, e.g., [2–5]). In the last case, i.e., $x(0) = x_0$, the problem (1) has been studied by Dhage [6] and O'Regan [7]. Therefore it is of interest to discuss and to approximate the solution of (1) with a nonlocal initial condition.

Similarly another class of nonlinear equations is used frequently to describe many phenomena in different fields of applied sciences such as physics, control theory, chemistry, biology, and so forth (see [8–11]). This class is generated by the nonlinear integral equations of the form:

$$x(t) = f(t,x(\sigma(t))) \cdot \left[q(t) + \int_0^{\eta(t)} K(t,s,x(\tau(s)))ds\right], \quad t \in J := [0,\rho], \tag{2}$$

where $\rho > 0$ is a real constant, $\sigma, \tau, \eta : J \to J$ and $q : J \to \mathbb{R}$ are supposed to be continuous, and the functions $f : J \times \mathbb{R} \to \mathbb{R}$, $K : J \times J \times \mathbb{R} \to \mathbb{R}$ are supposed to be $\mathcal{D}$-Lipschizian with respect to the second and the third variable, respectively.

Both, (1) and (2), can be interpreted as fixed point problems in which the equation involved is a nonlinear hybrid equation on a Banach algebra E of the type

$$x = A(x) \cdot B(x), \tag{3}$$

where A and B are nonlinear operators map a nonempty closed convex subset $\Omega \subset E$ into E.

A hybrid fixed point result to (3) was proved by Dhage in [12] and since then, several extensions and generalizations of this result have been achieved. See [13–15] and the references therein. These results can be used to achieves the existence of solutions. Although the explicit calculation of the fixed point is difficult in most cases, the previous cited results are regarded as one of the most powerful tools to give an approximation of the fixed point by a computational method and to develop numerical methods that allow us to approximate the solution of these equations.

In Banach spaces, several works deals with developing numerical techniques in order to approximate the solutions of integral and integro–differential equations, by using different methods such as the Chebyshev polynomial [16], the secant-like methods [17], using Schauder's basis [18,19], the parameterization method [20], the wavelet methods [21], a collocation method in combination with operational matrices of Berstein polynomials [22], the contraction principle and a suitable quadrature formula [23], the variational iteration method [24], etc.

Since the Banach algebras represents a practical framework for several equations such as (1) and (2), and in general (3), the purposes of this paper are twofold. Firstly, to present, under suitable conditions, a method to approximate the fixed point of a hybrid equation of type (3), by means of the product and composition of operators defined in a Banach algebra. Secondly, to set forth and apply the proposed method to obtain an approximation of the solutions of (1) and (2).

The structure of this work is as follows: in Section 2 we present some definitions and auxiliary results; in Section 3 we derive an approximation method for the fixed point of the hybrid Equation (3); in Sections 4 and 5, we apply our results to prove the existence and the uniqueness of solution of (1) and (2), we give an approximation method for these solutions and moreover, we establish some numerical examples to illustrate the applicability of our results. Finally, some conclusions are quoted in Section 6.

2. Analytical Tools

In this section, we provide some concepts and results that we will need in the following sections. The first analytical tool to be used comes from the theory of the fixed point. Let X be a Banach space with norm $\|\cdot\|$ and the zero element θ. We denote by $B(x,r)$ the closed ball centered at x with radius r. We write B_r to denote $B(\theta,r)$. For any bounded subset Ω of X, the symbol $\|\Omega\|$ denotes the norm of a set Ω, i.e., $\|\Omega\| = \sup\{\|x\|, x \in \Omega\}$.

Let us introduce the concept of $\mathcal{D}$-Lipschitzian mappings which will be used in the sequel.

Definition 1. *Let X be a Banach space. A mapping $A : X \longrightarrow X$ is said to be $\mathcal{D}$-Lipschitzian if*

$$\|Ax - Ay\| \leq \phi(\|x - y\|) \quad \forall x, y \in X$$

with $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a continuous nondecreasing function such that $\phi(0) = 0$. The mapping ϕ is called the $\mathcal{D}$-function associate to A. When $\phi(r) < r$ for $r > 0$, the mapping A is called a nonlinear contraction on X.

The class of $\mathcal{D}$-Lipschitzian mappings on X contains the class of Lipschitzian mapping on X, indeed if $\phi(r) = \alpha\, r$, for some $\alpha > 0$, then A is called Lipschitzian mapping with Lipschitz constant α or an α-Lipschitzian mapping. When $0 \leq \alpha < 1$, we say that A is a contraction.

The Banach fixed point theorem ensures that every contraction operator A on a complete metric space X has a unique fixed point $\tilde{x} \in X$, and, for every $x_0 \in X$, the sequence $\{A^n(x_0)\}_{n\in\mathbb{N}}$ converges to $\tilde{x}$. Much attention has been paid to Banach principle and it was generalized in different works (we quote, for instance, [25,26]). In [25], Boyd and Wong established the following result.

Theorem 1. *Let (X, d) be a complete metric space, and let $A : X \to X$ be a mapping satisfying*

$$d(A(x), A(y)) \leq \varphi(d(x, y)), \ \ \forall x, y \in X$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(r) < r$ if $r > 0$. Then A has a unique fixed point $\tilde{x} \in X$ and for any $x_0 \in X$, the sequence $\{A^n(x_0)\}_{n\in\mathbb{N}}$ converges to $\tilde{x}$.

On the other hand, Schauder bases will constitute the second essential tool. We recall that a Schauder basis in a Banach space E is a sequence $\{e_n\}_{n\in\mathbb{N}} \subset E$ such that for every $x \in E$, there is a unique sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$x = \sum_{n\geq 1} a_n e_n.$$

This notion produces the concept of the sequence of projections $P_n : E \to E$, defined by the formula

$$P_n\left(\sum_{k\geq 1} a_k e_k\right) = \sum_{k=1}^{n} a_k e_k,$$

and the sequence of coordinate functionals $e_n^* \in E^*$ defined as

$$e_n^*\left(\sum_{k\geq 1} a_k e_k\right) = a_n.$$

Moreover, in view of the Baire category Theorem [27], that for all $n \geq 1$, e_n^* and P_n are continuous. This yields, in particular, that

$$\lim_{n\to\infty} \|P_n(x) - x\| = 0.$$

3. Existence, Uniqueness and Approximation of a Fixed Point of the Product of Two Operators in Banach Algebras

Based on the Boyd-Wong Theorem, we establish the following fixed point result for the product of two nonlinear operators in Banach algebras.

Theorem 2. *Let X be a nonempty closed convex subset of a Banach algebra E. Let $A, B : X \to E$ be two operators satisfying the following conditions:*

(i) A and B are $\mathcal{D}$-lipschitzian with $\mathcal{D}$-functions φ and ψ respectively,
(ii) $A(X)$ and $B(X)$ are bounded,
(iii) $A(x) \cdot B(x) \in X$, for all $x \in X$.

Then, if $\|A(X)\|\psi(r) + \|B(X)\|\varphi(r) < r$ when $r > 0$, there is a unique point $\tilde{x} \in X$ such that $A(\tilde{x}) \cdot B(\tilde{x}) = \tilde{x}$. In addition, for each $x_0 \in X$, the sequence $\{(A \cdot B)^n(x_0)\}_{n\in\mathbb{N}}$ converges to $\tilde{x}$.

Proof. Let $x, y \in X$. we have

$$\|A(x) \cdot B(x) - A(y) \cdot B(y)\| \leq \|A(x) \cdot (B(x) - B(y))\| + \|(A(x) - A(y)) \cdot B(y)\| \leq$$

$$\|A(x)\|\,\|B(x)-B(y)\|+\|B(y)\|\,\|A(x)-A(y)\| \leq \|A(X)\|\,\psi(\|x-y\|)+\|B(X)\|\,\varphi(\|x-y\|).$$

This implies that $A \cdot B$ defines a nonlinear contraction with $\mathcal{D}$-function

$$\phi(r) = \|A(X)\|\,\psi(r)+\|B(X)\|\,\varphi(r),\ r>0.$$

Applying the cited Boyd-Wong's fixed point Theorem, we obtain the desired result. □

Boyd-Wong's fixed point Theorem expresses the fixed point of $A \cdot B$ as the limit of the sequence $\{(A\cdot B)^n(x_0)\}_{n\in\mathbb{N}}$ with $x_0 \in X$. If it is possible explicitly compute $(A\cdot B)^n(x_0)$, then for each n, the expression $(A\cdot B)^n(x_0)$ would be an approximation of the fixed point. But in the practice, this explicit calculation use to be not possible. For that, our aim is to propose another approximation of the fixed point which simple to calculate. We will need the following lemma.

Lemma 1. *Let X be a nonempty closed convex subset of a Banach algebra E. Let $A, B : X \to E$ be two $\mathcal{D}$-Lipschitzian operators with $\mathcal{D}$-functions φ and ψ, respectively, and $A \cdot B$ maps X into X. Moreover, suppose that*

$$\phi(r) < r,\ \ r>0.$$

Let $\tilde{x}$ be the unique fixed point of $A\cdot B$ and $x_0 \in X$. Let $\varepsilon > 0$, $m \in \mathbb{N}$, and $T_0, T_1, \ldots, T_m : E \to E$, with $T_0 \equiv I$, I being the identity operator on E, such that

$$\|\tilde{x}-(A\cdot B)^m(x_0)\| \leq \frac{\varepsilon}{2} \tag{4}$$

and

$$\sum_{p=1}^{m-1} \phi^{m-p}\big(\big\|(A\cdot B)\circ T_{p-1}\circ\ldots\circ T_1(x_0)-T_p\circ\ldots\circ T_1(x_0)\big\|\big)+ \\ \|(A\cdot B)\circ T_{m-1}\circ\ldots\circ T_1(x_0)-T_m\circ\ldots\circ T_1(x_0)\| \leq \frac{\varepsilon}{2}. \tag{5}$$

Then,

$$\|\tilde{x}-T_m\circ\ldots\circ T_1(x_0)\| \leq \varepsilon.$$

Proof. Arguing as in the proof of Theorem 2, it follows that $A \cdot B$ is a nonlinear contraction with $\mathcal{D}$-function ϕ, and by induction argument, it is easy to show that

$$\|(A\cdot B)^n(x)-(A\cdot B)^n(y)\| \leq \phi^n(\|x-y\|),\ x,y\in X. \tag{6}$$

By using the triangular inequality, we have

$$\begin{aligned}
&\|(A\cdot B)^m(x_0)-T_m\circ\ldots\circ T_1(x_0)\| \leq \\
&\qquad \Big\|(A\cdot B)^{m-1}\circ(A\cdot B)(x_0)-(A\cdot B)^{m-1}\circ T_1(x_0)\Big\| \\
&\qquad + \Big\|(A\cdot B)^{m-2}\circ(A\cdot B)\circ T_1(x_0)-(A\cdot B)^{m-2}\circ T_2\circ T_1(x_0)\Big\| + \cdots + \\
&\qquad + \|(A\cdot B)\circ(A\cdot B)\circ T_{m-2}\circ\ldots\circ T_1(x_0)-(A\cdot B)\circ T_{m-1}\circ\ldots\circ T_1(x_0)\| \\
&\qquad + \|(A\cdot B)\circ T_{m-1}\circ\ldots\circ T_1(x_0)-T_m\circ\ldots\circ T_1(x_0)\|.
\end{aligned}$$

Taking into account (6), we obtain

$$\begin{aligned}
&\|(A\cdot B)^m(x_0)-T_m\circ\ldots\circ T_1(x_0)\| \leq \\
&\qquad \sum_{p=1}^{m-1}\phi^{m-p}\big(\big\|(A\cdot B)\circ T_{p-1}\circ\ldots\circ T_1(x_0)-T_p\circ\ldots\circ T_1(x_0)\big\|\big) \\
&\qquad + \|(A\cdot B)\circ T_{m-1}\circ\ldots\circ T_1(x_0)-T_m\circ\ldots\circ T_1(x_0)\|.
\end{aligned}$$

This implies, by using the Triangular inequality again, that

$$\begin{aligned}\|\tilde{x} - T_m \circ \ldots \circ T_1(x_0)\| \leq & \\ & \sum_{p=1}^{m-1} \phi^{m-p}\big(\big\|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0) - T_p \circ \ldots \circ T_1(x_0)\big\|\big) \\ & + \|(A \cdot B) \circ T_{m-1} \circ \ldots \circ T_1(x_0) - T_m \circ \ldots \circ T_1(x_0)\| + \|\tilde{x} - (A \cdot B)^m(x_0)\| \leq \varepsilon. \quad (7)\end{aligned}$$

□

Taking into account the above lemma, observe that, under the previous hypotheses,

$$x^* = T_m \circ \ldots \circ T_1(x_0) \approx \tilde{x}$$

In order to get the approximation $x^* = T_m \circ \ldots \circ T_1(x_0)$ of the fixed point $\tilde{x}$, it is evident that, given $\varepsilon > 0$, by Theorem 2, condition (4) is satisfied for m sufficiently large. So, we are interested in building $T_1, T_2, \ldots, T_m$ satisfying (5), i. e. with the idea that

$$(A \cdot B)^m(x_0) \approx T_m \circ \ldots \circ T_1(x_0).$$

Schauder bases are the tool we will use next to build such operators. Concretely, for the case of problems (1) and (2), which can be written as a fixed point problem $x = A(x) \cdot B(x)$, where B is given by an integral operator, we will choice to approximate only the power terms of the operator B which is difficult to compute in general, unlike operator A which is easy to calculate and does not need to approximate their power terms. For this reason, we specifically propose the following scheme, in which we will construct $S_1, S_2, \cdots, S_m$:

$$\begin{array}{ccc} x_0 & & \\ \downarrow & & \\ (A \cdot B)(x_0) & \approx & T_1(x_0) = A(x_0) \cdot S_1(x_0) \\ \downarrow & & \downarrow \\ (A \cdot B)^2(x_0) & \approx & T_2 \circ T_1 x_0 = (A \cdot S_2) \circ T_1(x_0) \\ \vdots & \vdots & \vdots \\ & & \\ \vdots & \vdots & \vdots \\ \downarrow & & \downarrow \\ (A \cdot B)^m(x_0) & \approx & T_m \circ \ldots \circ T_1(x_0) = (A \cdot S_m) \circ T_{m-1} \circ \ldots \circ T_1(x_0) \approx \tilde{x} \end{array}$$

Remark 1. *The above scheme is constructed as follows. In the first term, we approximate $B(x_0)$ by $S_1(x_0)$, then we obtain $T_1(x_0) := A(x_0) \cdot S_1(x_0)$ as an approximation of the first term of the Picard iterate, $A(x_0) \cdot B(x_0)$. In the second term of our scheme, we approximate the second term of the Picard iterate, $(A \cdot B)^2(x_0) = A((A \cdot B)(x_0)) \cdot B((A \cdot B)(x_0))$. So we obtain the second term of our scheme by combining the first term $T_1(x_0)$, with an approximation of the operator B, which denoted by S_2, and consequently we obtain a second term of our scheme $T_2 \circ T_1(x_0) = (A \cdot S_2)(T_1(x_0))$ which approximate $(A \cdot B)^2(x_0)$.*

4. Nonlinear Differential Equations with Nonlocal Initial Condition

In this section we focus our attention in the nonlinear differential equation with nonlocal initial condition (1). This equation will be studied when the mappings $f, g : J \times \mathbb{R} \to \mathbb{R}$ are such that:

(i) The partial mappings $t \mapsto f(t,x)$, $t \mapsto g(t,x)$ are continuous and the mapping $\mu : C(J) \to \mathbb{R}$ is L_μ-Lipschitzian.

(ii) There exist $r > 0, \alpha, \gamma : J \to \mathbb{R}$ two continuous functions and $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ two nondecreasing, continuous functions such that

$$|f(t,x) - f(t,y)| \leq \alpha(t)\varphi(|x-y|), t \in J, \text{ and } x, y \in \mathbb{R} \text{ with } |x|, |y| \leq r,$$

and

$$|g(t,x) - g(t,y)| \leq \gamma(t)\psi(|x-y|), t \in J \text{ and } x, y \in \mathbb{R} \text{ with } |x|, |y| \leq r.$$

(iii) There is a constant $\delta > 0$ such that $\sup_{x \in \mathbb{R}, |x| \leq r} |f(0,x)|^{-1} \leq \delta$.

Throughout this section, Ω will denote the closed ball B_r of $C(J)$, where r is defined in the above assumption (ii). Observe that Ω is a non-empty, closed, convex and bounded subset of $C(J)$.

4.1. Existence and Uniqueness of Solutions

In this subsection, we prove the existence and the uniqueness of a solution to the functional differential problem (1).

Theorem 3. *Assume that the assumptions (i), (ii) and (iii) hold. If*

$$M_A M_B \leq r \text{ and}$$

$$M_A \delta L_\mu t + \left(M_A \delta^2 |\alpha(0)| M_\mu + M_B \|\alpha\|_\infty\right) \varphi(t) + M_A \|\gamma(\cdot)\|_{L^1} \psi(t) < t, \ \forall t > 0,$$

where $M_A = \|\alpha\|_\infty \varphi(r) + \|f(\cdot,0)\|_\infty$, $M_B = \delta M_\mu + \|\gamma\|_\infty \rho \psi(r) + \rho \|g(\cdot,0)\|_\infty$ *and* $M_\mu = (L_\mu r + |\mu(0)|)$, *then the nonlinear differential problem* (1) *has a unique solution in* Ω.

Proof. Notice that the problem of the existence of a solution to (1) can be formulated in the following fixed point problem $x = A(x) \cdot B(x)$, where A, B are given for $x \in C(J)$ by

$$\begin{aligned} (A(x))(t) &= f(t, x(t)) \\ (B(x))(t) &= \left[\frac{1}{f(0, x(0))}\mu(x) + \int_0^t g(s, x(s))ds\right], t \in J. \end{aligned} \tag{8}$$

Let $x \in \Omega$ and $t, t' \in J$. Since f is $\mathcal{D}$-lipschitzian with respect to the second variable and is continuous with respect to the first variable, then by using the inequality

$$|f(t, x(t)) - f(t', x(t'))| \leq |f(t, x(t)) - f(t', x(t))| + |f(t', x(t)) - f(t', x(t'))|,$$

we can show that A maps Ω into $C(J)$.

Now, let us claim that B maps Ω into $C(J)$. In fact, let $x \in \Omega$ and $t, t' \in J$ be arbitrary. Taking into account that $t \mapsto g(t, x)$ is a continuous mapping, it follows from assumption (ii) that

$$\begin{aligned} |(B(x))(t) - (B(x))(t')| \leq \int_{t'}^t |g(s, x(s)) - g(s, 0)|ds + (t - t')\|g(\cdot, 0)\|_\infty \leq \\ (t - t')(\|\gamma\|_\infty \psi(r) + \|g(\cdot, 0)\|_\infty). \end{aligned}$$

This proves the claim. Our strategy is to apply Theorem 2 to show the existence and the uniqueness of a fixed point for the product $A \cdot B$ in Ω which in turn is a continuous solution for problem (1).

For this purpose, we will claim, first, that A and B are $\mathcal{D}$-lipschitzian mappings on Ω. The claim regarding A is clear in view of assumption (ii), that is A is $\mathcal{D}$-lipschitzian with $\mathcal{D}$-function Φ such that

$$\Phi(t) = \|\alpha\|_\infty \varphi(t), t \in J.$$

We corroborate now the claim for B. Let $x, y \in \Omega$, and let $t \in J$. By using our assumptions, we obtain

$$\begin{aligned}
&|(B(x))(t) - (B(y))(t)| = \\
&\qquad \left| \frac{1}{f(0,x(0))}\mu(x) - \frac{1}{f(0,y(0))}\mu(y) + \int_0^t g(s,x(s)) - g(s,y(s))ds \right| \leq \\
&\qquad \frac{L_\mu}{|f(0,x(0))|}\|x-y\| + \frac{|\alpha(0)|}{|f(0,x(0))f(0,y(0))|}\big(L_\mu r + |\mu(0)|\big)\varphi(\|x-y\|) + \\
&\qquad\qquad \int_0^t |\gamma(s)|\psi(|x(s)-y(s)|)ds \leq \\
&\qquad \delta L_\mu \|x-y\| + \delta^2|\alpha(0)|\big(L_\mu r + |\mu(0)|\big)\varphi(\|x-y\|) + \|\gamma(\cdot)\|_{L^1}\psi(\|x-y\|).
\end{aligned}$$

Taking the supremum over t, we obtain that B is $\mathcal{D}$-lipschitzian with $\mathcal{D}$-function Ψ such that

$$\Psi(t) = \delta L_\mu t + \delta^2|\alpha(0)|\big(L_\mu r + |\mu(0)|\big)\varphi(t) + \|\gamma(\cdot)\|_{L^1}\psi(t), t \in J.$$

On the other hand, bearing in mind assumption (i), by using the above discussion we can see that $A(\Omega)$ and $B(\Omega)$ are bounded with bounds M_A and M_B respectively. Taking into account the estimate $M_A M_B \leq r$, we obtain that $A \cdot B$ maps Ω into Ω.
Since

$$\begin{aligned}
|(B(x))(t)| &\leq \left| \frac{1}{f(0,x(0))}\mu(x) \right| + \int_0^t |g(s,x(s))|ds \\
&\leq \delta(|\mu(x) - \mu(0)| + |\mu(0)|) + \int_0^t |g(s,x(s)) - g(s,0)|ds + \int_0^t |g(s,0)|ds \\
&\leq \delta(L_\mu\|x\| + |\mu(0)|) + \int_0^t |\gamma(s)|\psi(|x(s)|)ds + \int_0^t |g(s,0)|ds,
\end{aligned}$$

and using the fact that $|\gamma(s)|\psi(|x(s)|) \leq \|\gamma\|_\infty \psi(\|x\|) \leq \|\gamma\|_\infty \psi(r)$, we have that

$$\|B(x)\| \leq \delta(L_\mu\|x\| + |\mu(0)|) + \rho\|\gamma\|_\infty\psi(r) + \rho\|g(\cdot,0)\|_\infty = M_B.$$

On the other hand, $\|A(x)\| \leq M_A$ since

$$\begin{aligned}
&|(A(x))(t)| = |f(t,x(t))| \leq |f(t,x(t)) - f(t,0)| + |f(t,0)| \leq \\
&\qquad |\alpha(t)|\ \varphi(|x(t)|) + |f(t,0)| \leq \|\alpha\|_\infty\varphi(r) + \|f(\cdot,0)\|_\infty = M_A.
\end{aligned}$$

Taking into account that

$$\|(A\cdot B)(x) - (A\cdot B)(y)\| \leq \|A(x)\|\|B(x) - B(y)\| + \|B(y)\|\|A(x) - A(y)\|,$$

we can notice that $A \cdot B$ is a nonlinear contraction with $\mathcal{D}$-function $\Theta(\cdot) := M_A\Psi(\cdot) + M_B\Phi(\cdot)$, i.e.,

$$\|(A\cdot B)(x) - (A\cdot B)(y)\| \leq \Theta(\|x-y\|),\ x, y \in \Omega. \tag{9}$$

Now, applying Theorem 2, we infer that (1) has one and only one solution $\tilde{x}$ in Ω, and for each $x_0 \in \Omega$ we have

$$\lim_{n\to\infty}(A\cdot B)^n(x_0) = \tilde{x}. \tag{10}$$

□

In what follows we will assume that the hypotheses of the Theorem 3 are satisfied.

4.2. Numerical Method to Approximate the Solution

In this subsection we find a numerical approximation of the solution to the nonlinear Equation (1) using a Schauder basis $\{e_n\}_{n\geq 1}$ in $C(J)$ and the sequence of associated projections $\{P_n\}_{n\geq 1}$. Let $p \in \mathbb{N}$ and $n_p \in \mathbb{N}$. We consider

$$\begin{array}{rcl} S_p : C(J) & \longrightarrow & C(J) \\ x & \longrightarrow & S_p(x) \end{array}$$

defined as

$$S_p(x)(t) = \frac{1}{f(0, x(0))}\mu(x) + \int_0^t P_{n_p}(U_0(x))(s)ds,$$

where $U_0 : C(J) \longrightarrow C(J)$ is given by $U_0(x)(s) = g(s, x(s))$.

Now consider the operator $T_p : C(J) \longrightarrow C(J)$ such that for each $x \in C(J)$, $T_p(x)$ is defined by

$$T_p(x)(t) = A(x)(t)S_p(x)(t), \ \ t \in J, \tag{11}$$

with $A : C(J) \longrightarrow C(J)$, $A(x)(t) = f(t, x(t))$.

Remark 2. *For $p \geq 1$ and any $n_p \in \mathbb{N}$ that we use for defining T_p, the operator T_p maps Ω into Ω, since just keep in mind that for $x \in \Omega$, we have*

$$\left|T_p(x)(t)\right| = \left|A(x)(t)\left(\frac{1}{f(0, x(0))}\mu(x) + \int_0^t P_{n_p}(U_0(x))(s)ds\right)\right| \leq$$

$$|f(t, x(t))|\left(\delta|\mu(x)| + \int_0^t \left|P_{n_p}(U_0(x))(s)\right|ds\right),$$

and proceeding as in the above subsection and using the fact that P_{n_p} is a bounded linear operator on $C(J)$, we get

$$\left|T_p(x)(t)\right| \leq M_A\left[\delta|\mu(x)| + \rho\left\|P_{n_p}(U_0(x))\right\|\right] \leq$$

$$M_A\left[\delta(L_\mu r + |\mu(0)|) + \rho \sup_{s\in J}|g(s, x(s))|\right] \leq M_A M_B < r.$$

In particular, for $m \geq 1$, the operator $T_m \circ \ldots \circ T_1$ maps Ω into Ω.

Our goal is to prove that we can chose $n_1, n_2, \ldots \in \mathbb{N}$ in order that $T_1, T_2, \ldots$, which are defined above, can be used to approximate the solution of (1).

Theorem 4. *Let $\tilde{x}$ be the unique solution to the nonlinear problem (1). Let $x_0 \in \Omega$ and $\varepsilon > 0$, then there exist $m \in \mathbb{N}$ and $n_i \in \mathbb{N}$ to construct T_i for $i = 1, \ldots, m$, in such a way that*

$$\|\tilde{x} - T_m \circ \ldots \circ T_1(x_0)\| \leq \varepsilon.$$

Proof. Let $x_0 \in \Omega$ and $\varepsilon > 0$. By using (10), there is $m \in \mathbb{N}$ such that

$$\|(A \cdot B)^m(x_0) - \tilde{x}\| \leq \varepsilon/2.$$

For that m, and for $p \in \{1, \ldots, m\}$, we define $U_p : C(J) \to C(J)$ by

$$U_p(x)(s) := g(s, T_p \circ \ldots \circ T_1(x)(s)), \ s \in J, x \in C(J)$$

and $A_p : C(J) \to C(J)$ by

$$A_p(x)(s) := f\big(s, T_p \circ \ldots \circ T_1(x)(s)\big),\ s \in J, x \in C(J).$$

According to inequality (9), in view of (5) of Lemma 1, it suffices to show that

$$\sum_{p=1}^{m-1} \Theta^{m-p}\big(\big\|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0) - T_p \circ \ldots \circ T_1(x_0)\big\|\big) + \\ \|(A \cdot B) \circ T_{m-1} \circ \ldots \circ T_1(x_0) - T_m \circ \ldots \circ T_1(x_0)\| \le \varepsilon/2.$$

In view of (11), we have

$$\big|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t) - T_p \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t)\big| = \\ \big|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t) - (A \cdot S_p) \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t)\big| = \\ \big|A_{p-1}(x_0)(t)\big(B \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t) - S_p \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t)\big)\big|.$$

Taking into account Remark 2, we infer that $\|A_{p-1}(x)\|$ is bounded, and consequently we get

$$\big|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t) - T_p \circ T_{p-1} \circ \ldots \circ T_1(x_0)(t)\big| = \\ \left|A_{p-1}(x_0)(t)\left(\int_0^t g\big(s, T_{p-1} \circ \ldots \circ T_1(x_0)(s)\big)\, ds - \int_0^t P_{n_p}(U_{p-1}(x_0))(s)\, ds\right)\right| \le \\ \big|A_{p-1}(x_0)(t)\big| \int_0^t \left|\Big(P_{n_p}(U_{p-1}(x_0)) - U_{p-1}(x_0)\Big)(s)\right| ds \le \\ \rho\big\|A_{p-1}(x_0)\big\| \left\|P_{n_p}(U_{p-1})(x_0) - U_{p-1}(x_0)\right\|.$$

Taking the supremum over t, we get

$$\big\|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0) - T_p \circ T_{p-1} \circ \ldots \circ T_1(x_0)\big\| \le \\ \rho M_A \left\|P_{n_p}(U_{p-1})(x_0) - U_{p-1}(x_0)\right\|.$$

Since Θ is a nondecreasing continuous mapping, and taking into account the convergence of the projection operators associated to the Schauder basis, for all $1 \le p \le m$ we obtain

$$\Theta^{m-p}\Big(\rho M_A \left\|P_{n_p}(U_{p-1}(x_0)) - U_{p-1}(x_0)\right\|\Big) \le \varepsilon/2m,$$

for n_p sufficiently large. Consequently, we consider those $n_1, \ldots, n_m \in \mathbb{N}$ for defining T_1, $T_2, \ldots, T_m$ respectively, and we obtain

$$\sum_{p=1}^{m-1} \Theta^{m-p}\big(\big\|(A \cdot B) \circ T_{p-1} \circ \ldots \circ T_1(x_0) - T_p \circ \ldots \circ T_1(x_0)\big\|\big) + \\ \|(A \cdot B) \circ T_{m-1} \circ \ldots \circ T_1(x_0) - T_m \circ \ldots \circ T_1(x_0)\| \le \\ \sum_{p=1}^{m-1} \Theta^{m-p}\Big(\rho M_A \left\|P_{n_p}(U_{p-1}(x_0)) - U_{p-1}(x_0)\right\|\Big) + \rho M_A \left\|P_{n_m}(U_{m-1}(x_0)) - U_{m-1}(x_0)\right\| \le \varepsilon/2.$$

Now apply Lemma 1, in order to get $\|\tilde{x} - T_m \circ \ldots \circ T_1(x_0)\| < \varepsilon$. □

4.3. Numerical Experiments

This subsection is devoted to providing some examples and their numerical results to illustrate the theorems of the above sections. We will consider $J = [0,1]$ and the classical Faber-Schauder system in $C(J)$ where the nodes are the naturally ordered dyadic numbers (see Table 1 in [18] and [28,29] for details). In following examples, we will denote $x^* = T_m \circ \ldots \circ T_1(x_0)$ with $m = 4$ and $n_1 = \cdots = n_m = l$ with $l = 9$ or $l = 33$.

Example 1. *Consider the nonlinear differential equation with a nonlocal initial condition*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) = ae^{-x(t)}, \ \ t \in J, \\ \\ x(0) = b\left(\sup\limits_{t\in J}|x(t)| + \dfrac{3}{4}\right), \end{cases} \tag{12}$$

where $0 < a < 1/\log(2)$ *and* $f(t,x) = \dfrac{b}{1+ae^{-b}t}$.

Let us define the mappings $g : J \times \mathbb{R} \to \mathbb{R}$ *and* $\mu : C(J) \to \mathbb{R}$ *by*

$$g(t,x) = ae^{-x}, \ t \in J, x \in \mathbb{R}$$

and

$$\mu(u) = b\left(\sup_{t\in J}|u(t)| + 3/4\right), \ u \in C(J).$$

Let R *be small enough such that* $a(\log(2) + R) < 1$. *Let* $x, y \in [-R, R]$, *by an elementary calculus we can show that the functions* f *and* g *satisfy the condition (ii), with* $\alpha(t) = \varphi(t) = 0$, $\gamma(t) = ae^R(1 - e^{-t})$, *and* $\psi(t) = t$.
On the other hand, we have that μ *is Lipschizian with a Lipschiz constant* $L_\mu = b$, *and*

$$\sup_{x,|x|\le R} [f(0,x)]^{-1} \le \delta = \frac{1}{b}.$$

Applying Theorem 3, we obtain that (12) has a unique solution in $B_R = \{x \in C(J); \|x\| \le R\}$ *with* $R = 3/4$, *when* a *is small enough. In fact the solution is* $\tilde{x}(t) = b$. *We apply the numerical method for* $a = 0.1$, $b = \frac{1}{4}$ *and the initial* $x_0(t) = \frac{1}{4}\left(\sqrt{bt} + 1\right)$. *Table 1 collects the obtained results.*

Table 1. Numerical results for (12) with initial $x_0(t) = \frac{1}{4}\left(\sqrt{bt} + 1\right)$.

t	$\tilde{x}(t)$	$x^*(t)$ with $l = 9$	$x^*(t)$ with $l = 33$
0.1	0.25	0.2526360625738145	0.2506238401703868
0.2	0.25	0.2512245431325148	0.2506151528771704
0.3	0.25	0.2510208953229317	0.2506066551064274
0.4	0.25	0.2510087458298449	0.2505983412941664
0.5	0.25	0.2509968386936278	0.2505902060799007
0.6	0.25	0.2509851672563384	0.2505822442972077
0.7	0.25	0.2509737250885047	0.2505744509661791
0.8	0.25	0.2509625059364119	0.2505668212861210
0.9	0.25	0.2509515037642987	0.2505593506272617
1	0.25	0.2509407127451644	0.2505520345235613
$\|x^* - \tilde{x}\|_\infty$		2.86369×10^{-3}	1.0862×10^{-3}

Example 2. *Consider the nonlinear differential equation with a nonlocal initial condition*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) = a(x(t))^2, \ \ t \in J, \\ \\ x(0) = 1/(4b) \sup\limits_{t \in J} |x(t)|^2, \end{cases} \tag{13}$$

where a, b are positive constants such that $ab^2 < 3$ and $f(t,x) = \dfrac{b(t+1)}{1 + \frac{ab^2}{3}(x^3/b^3 - 1)}$.

Let us define the mappings $g : J \times \mathbb{R} \to \mathbb{R}$ and $\mu : C(J) \to \mathbb{R}$ by

$$g(t,x) = ax^2, \ t \in J, x \in \mathbb{R} \text{ and } \mu(u) = 1/(4b) \sup_{t \in J} |u(t)|^2, \ u \in C(J).$$

Let $R > 0$ such that $2b \le R$ and $\frac{a}{3b}(b^3 + R^3) < 1$. Let $x, y \in [-R, R]$. By an elementary calculus we can show that f and g satisfy the condition (ii) with $\alpha(t) = \dfrac{a(t+1)R^2}{\left(1 - \frac{a}{3b}(R^3 + b^3)\right)^2}$, $\gamma(t) = 2aR$, and $\varphi(t) = \psi(t) = t$.
On the other hand, we have that

$$|\mu(u) - \mu(v)| \ \ \le \ \ \frac{R}{2b}\|u - v\|.$$

Consequently, μ is Lipschizian with a Lipschiz constant $L_\mu = \frac{R}{2b}$. It is easy to prove that

$$\sup_{x \in \mathbb{R}, |x| \le R} [f(0,x)]^{-1} \le \delta = aR^3/(3b^2) + 1/b.$$

Now, applying Theorem 3, in order to obtain that (13), with a is small enough, has a unique solution in B_R with $R = 1/2$. We can check that the solution is $\tilde{x}(t) = b(t+1)$. Table 2 shows the numerical results of the proposed method for $a = 0.05$, $b = 1/4$ and $x_0(t) = \frac{1}{2}t$.

Table 2. Numerical results for (13) with initial $x_0(t) = \frac{1}{2}t$.

t	$\tilde{x}(t)$	$x^*(t)$ with $l = 9$	$x^*(t)$ with $l = 33$
0.1	0.275	0.2715154513364088	0.2714532970472882
0.2	0.3	0.2961167353030552	0.2961332465465061
0.3	0.325	0.3207837845940706	0.3208140511167786
0.4	0.35	0.3454635279153586	0.3454958547548318
0.5	0.375	0.3701445199310059	0.3701788114857308
0.6	0.40	0.3948268789541488	0.3948630864085328
0.7	0.425	0.4195107187398104	0.4195488540144761
0.8	0.45	0.4441962543294659	0.4442362958308083
0.9	0.475	0.4688837174935067	0.4689256009587782
1	0.5	0.4935733558651244	0.4936169655580174
$\|x^* - \tilde{x}\|_\infty$		6.42664×10^{-3}	6.38303×10^{-3}

5. Nonlinear Integral Equations

This section deals with the nonlinear integral Equation (2). More precisely, we prove the existence and the uniqueness of a solution to Equation (2) under the hypothesis that the mappings $f : J \times \mathbb{R} \to \mathbb{R}$ and $K : J \times J \times \mathbb{R} \to \mathbb{R}$ are such that:

(i) The partial mappings $t \mapsto f(t,x)$ and $(t,s) \mapsto K(t,s,x)$ are continuous.

(ii) There exist $r > 0$, $\gamma : J \times J \to \mathbb{R}$, $\alpha : J \to \mathbb{R}$ two continuous functions and $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ two nondecreasing continuous functions such that

$$|f(t,x) - f(t,y)| \le \alpha(t)\varphi(|x-y|), t \in J, \text{ and } x, y \in \mathbb{R} \text{ with } |x|, |y| \le r,$$

and

$$|K(t,s,x) - K(t,s,y)| \le \gamma(t,s)\psi(|x-y|), t, s \in J \text{ and } x, y \in \mathbb{R} \text{ with } |x|, |y| \le r.$$

Throughout this section, Ω will denote the closed ball B_r of $C(J)$, where r is defined in the above assumption (ii).

5.1. Existence and Uniqueness of Solutions

To allow the abstract formulation of Equation (2), we define the following operators on $C(J)$ by

$$\begin{aligned} (Ax)(t) &= f(t, x(\sigma(t))), \\ (Bx)(t) &= \left[q(t) + \int_0^{\eta(t)} K(t,s,x(\tau(s)))ds \right], t \in J. \end{aligned} \tag{14}$$

First, we will establish the following result which shows the existence and uniqueness of a solution.

Theorem 5. *Assume that the assumptions (i) and (ii) hold. If*

$$M_A M_B \le r \text{ and } M_A \rho \|\gamma\|_\infty \psi(t) + M_B \|\alpha\|_\infty \varphi(t) < t, \ \forall t > 0,$$

where

$$M_A = \|\alpha\|_\infty \varphi(r) + \|f(\cdot,\theta)\|_\infty \text{ and } M_B = \|q(\cdot)\|_\infty + \rho(\|K(\cdot,\cdot,0)\|_\infty + \|\gamma\|_\infty \psi(r)),$$

then the nonlinear integral Equation (2) has a unique solution in Ω.

Proof. By using similar arguments to those in the above section, we can show that A and B define $\mathcal{D}$-lipschitzian mappings from Ω into $C(J)$, with $\mathcal{D}$-functions $\|\alpha\|_\infty \varphi$ and $\rho\|\gamma\|_\infty \psi$, respectively. Also it is easy to see that $A(\Omega)$ and $B(\Omega)$ are bounded with bounds, respectively, M_A and M_B. Taking into account our assumptions, we deduce that $A \cdot B$ maps Ω into Ω.
Notice that $A \cdot B$ defines a nonlinear contraction with $\mathcal{D}$-function

$$\Theta(t) := \rho\|\gamma\|_\infty M_A \psi(t) + \|\alpha\|_\infty M_B \varphi(t), t \ge 0, \text{ i.e.,}$$

$$\|(A \cdot B)(x) - (A \cdot B)(y)\| \le \Theta(\|x-y\|), \ x, y \in \Omega. \tag{15}$$

Now, an application of Theorem 2 yields that (2) has one and only one solution $\tilde{x}$ in Ω, and for each $x_0 \in \Omega$ we have

$$\lim_{n \to \infty} (A \cdot B)^n (x_0) = \tilde{x}. \tag{16}$$

□

5.2. A Numerical Method to Approximate the Solution

Now we consider a Schauder basis $\{e_n\}_{n\geq 1}$ in $C(J \times J)$ and the sequence of associated projections $\{P_n\}_{n\geq 1}$. Let $p \in \mathbb{N}$, $n_p \in \mathbb{N}$ and consider

$$\begin{cases} S_p : C(J) & \longrightarrow C(J) \\ \quad\quad x & \longrightarrow S_p(x)(t) = q(t) + \int_0^{\eta(t)} P_{n_p}(U_0(x))(t,s)ds, \end{cases}$$

where $U_0 : C(J) \longrightarrow C(J \times J)$ is defined as $U_0(x)(t,s) = K(t,s,x(\tau(s)))$. Also, we consider the operator $T_p : C(J) \longrightarrow C(J)$, which assigns for all $x \in C(J)$ the valued $T_p(x) \in C(J)$ such that

$$T_p(x)(t) = A(x)(t)S_{n_p}(x)(t), t \in J,$$

where $A : C(J) \longrightarrow C(J)$ is defined as $A(x)(t) = f(t, x(\sigma(t)))$.

Remark 3. *Since for $p \geq 1$,*

$$\big|T_p(x)(t)\big| = \left|A(x)(t)\left(q(t) + \int_0^{\eta(t)} P_{n_p}(U_0(x))(t,s)ds\right)\right| \leq$$
$$|f(t,x(\sigma(t)))| \left(|q(t)| + \int_0^{\eta(t)} \left|P_{n_p}(U_0(x))(t,s)\right|ds\right),$$

proceeding essentially as in the above section and using the fact that P_{n_p} is a bounded linear operator on $C(J \times J)$, we get

$$\big|T_p(x)(t)\big| \leq M_A\left(|q(t)| + \rho\left\|P_{n_p}(U_0(x))\right\|\right) \leq$$
$$M_A\left(\|q\|_\infty + \rho \sup_{t,s\in J}|K(t,s,x(\tau(s)))|\right) \leq M_A M_B.$$

Accordingly, under the hypotheses of the Theorem 5, the mapping T_p maps Ω into Ω. In particular, for $m \geq 1$, the operator $T_m \circ \ldots \circ T_1$ maps Ω into Ω.

Analogously as we did in the previous section, the following result allow us to justify it is possible to choose $n_1, n_2, \ldots$ in order that $T_1, T_2, \ldots$ can be used to approximate the unique solution to Equation (2).

Theorem 6. *Let $\tilde{x}$ be the unique solution to the nonlinear Equation (2). Let $x_0 \in \Omega$ and $\varepsilon > 0$, then there exists $m \in \mathbb{N}$ and $n_i \in \mathbb{N}$ to construct T_i for $i = 1, \ldots, m$, such that*

$$\|\tilde{x} - T_m \circ \ldots \circ T_1(x_0)\| \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$, by using (16), there is $m \in \mathbb{N}$ such that

$$\|(A \cdot B)^m(x_0) - \tilde{x}\| \leq \varepsilon/2.$$

For that m, and for $p \in \{1, \ldots, m\}$, we define $U_p : C(J) \to C(J \times J)$ by

$$U_p(x)(t,s) := K(t,s,T_p \circ \ldots \circ T_1(x)(s)),\ t,s \in J, x \in C(J)$$

and $A_p : C(J) \to C(J)$ by

$$A_p(x)(s) := f\big(s, T_p \circ \ldots \circ T_1(x)(s)\big),\ s \in J, x \in C(J).$$

Proceeding essentially, as in the Theorem 4, and taking into account (15) together with Remark 3 the desired thesis can be proved. □

5.3. Numerical Experiments

This section is devoted to give some numerical examples to illustrate the previous results using the usual Schauder basis in $C([0,1]^2)$ with the well know square ordering (see Table 1 in [18] and [28,29]). In each example, we will denote $x^* = T_m \circ \ldots \circ T_1(x_0)$ for $m = 4$ and $n_1 = \cdots = n_m = l^2$ with $l = 9$ or $l = 33$.

Example 3. *Consider the nonlinear integral equation*

$$x(t) = a(t+1)\left[\frac{b}{a} - \frac{b^2}{3}\left((t+1)^3 - 1\right) + \int_0^t (x(s))^2 ds\right], \quad t \in J. \tag{17}$$

Now we consider the mappings $q : J \to J$, $f : J \times \mathbb{R} \to \mathbb{R}$ and $K : J \times J \times \mathbb{R} \to \mathbb{R}$ such that $q(t) = b/a - \frac{b^2}{3}\left((t+1)^3 - 1\right)$, $f(t,x) = a(t+1)$ and $K(t,s,x) = x^2$. Let $R > 0$ and let $x, y \in [-R,R]$. We have that

$$|K(t,s,x) - K(t,s,y)| \le \gamma(t,s)\psi(|x-y|),$$

where $\gamma(t,s) = 2R$, and $\psi(t) = t$. An application of Theorem 5, yields that (17) has a unique solution in B_R, with $R = 3$. In fact the solution is $\tilde{x}(t) = b(t+1)$.

Using the proposed method with $a = 0.1$, $b = 0.1$ and $x_0(t) = t^2$, we obtain Table 3.

Table 3. Numerical results for the (17).

t	$\tilde{x}(t)$	$x^*(t)$ with $l = 9$	$x^*(t)$ with $l = 33$
0.1	0.11	0.1099446333333333	0.1099595576568532
0.2	0.12	0.1198179180577049	0.1199472782251611
0.3	0.13	0.1297511699020331	0.1299327014013851
0.4	0.14	0.1396866403161547	0.1399156114644378
0.5	0.15	0.1496116012197044	0.1498957849652041
0.6	0.16	0.1595251486759711	0.1598729913214837
0.7	0.17	0.1694262809122463	0.1698469898893412
0.8	0.18	0.1793140741901599	0.1798175262525480
0.9	0.19	0.1891875688779072	0.1897843325246908
1	0.2	0.1990457618518603	0.1997471266515799
$\|x^* - \tilde{x}\|_\infty$		9.544238×10^{-4}	2.52873×10^{-4}

Example 4. *Consider the nonlinear differential equation*

$$x(t) = \left(ae^{-x(t)} + b\right)\left[\frac{t}{ae^{-t} + b} + \frac{1}{1-c}\log(\cos(1-c)t) + \int_0^t \tan((1-c)x(s))ds\right]. \tag{18}$$

Similarly to that above, (18) can be written as a fixed point problem with the same notations in (14). Let $R > 0$ and let $x, y \in [-R,R]$. By an elementary calculus we can show that the functions f and g satisfy the condition (ii), with $\alpha(t) = ae^R$, $\gamma(t) = (1 + \tan^2(1-c)R)$, and $\varphi(t) = (1 - e^{-t})$ and $\psi(t) = \tan(1-c)t$.

Apply Theorem 5, (18), with a small enough and $c = 1 - a$, has a unique solution in B_R with $R = 3$, in fact the solution is $\tilde{x}(t) = t$. We obtain the results given in Table 4 for $a = 0.01, b = 1, R = 3$, and $x_0(t) = \sin(t)$.

Table 4. Numerical results for (18) with initial $x_0(t) = \sin(t)$.

t	$\tilde{x}(t)$	$x^*(t)$ with $l = 9$	$x^*(t)$ with $l = 33$
0.1	0.1	0.0999495927525812	0.0999734131829520
0.2	0.2	0.1998269806205324	0.1999419676240642
0.3	0.3	0.2997014781005956	0.2999105694862292
0.4	0.4	0.3995761128223367	0.3998792008487213
0.5	0.5	0.4994508163308592	0.4998478468962116
0.6	0.6	0.5993255387084228	0.5998164954408373
0.7	0.7	0.6992002390137386	0.6997851365136741
0.8	0.8	0.7990748839377436	0.7997537620153589
0.9	0.9	0.8989494465775325	0.8997223654190059
1	1	0.9988239054111422	0.9996909415162489
$\Vert x^* - \tilde{x}\Vert_\infty$		1.17609×10^{-3}	3.09058×10^{-4}

Example 5. *Consider the problem (2) with*

$$\begin{aligned} f(t,x) &= at\left[(b+t)^2 + \frac{t}{(t+1)}\int_0^t \left(1 - e^{-(t+1)(as+1)}\right)ds\right]^{-1}, \\ K(t,s,x) &= \int_0^{x+1} e^{-(t+1)u}du, \\ q(t) &= (b+t)^2. \end{aligned} \tag{19}$$

Let $0 < R < 1$ and let $x, y \in [-R, R]$. By an elementary calculus, we can show that f and g satisfy the condition (ii), with $\alpha(t) = \varphi(t) = 0$, $\psi(t) = \int_0^{2t} e^{-s}ds$, and $\gamma(t,s) = \frac{1}{t+1}e^{(t+1)(R-1)}$.

Taking $a = 0.1, b = 1$, and applying Theorem 5, the problem has a unique solution in $B_R = \{x \in C([0,1]); \Vert x\Vert \le R\}$, in fact the solution is $\tilde{x}(t) = at$. We obtain the results given in Table 5.

Table 5. Numerical results for (19) with initial $x_0(t) = 1/2cos(10\pi t)$.

t	$\tilde{x}(t)$	$x^*(t)$ with $l = 9$	$x^*(t)$ with $l = 33$
0.1	0.01	0.0098078897681979	0.0098501736202539
0.2	0.02	0.0191334693414161	0.0197640067592651
0.3	0.03	0.0288588703908235	0.0297138485291223
0.4	0.04	0.0387456185368957	0.0396854768250511
0.5	0.05	0.0486866179763731	0.0496708731179798
0.6	0.06	0.0586657967463166	0.0596654694199951
0.7	0.06	0.0686685394448633	0.0696660302996126
0.8	0.08	0.0786865051341015	0.0796705375310556
0.9	0.09	0.0887140587924687	0.0896776281114000
1	0.09	0.0987473453913395	0.0996863636633998
$\Vert x^* - \tilde{x}\Vert_\infty$		1.33705×10^{-3}	3.34982×10^{-4}

6. Conclusions

In this paper we have presented a numerical method, based on the use of Schauder's bases, to solve hybrid nonlinear equations in Banach algebras. To do this, we have used

Boyd-Wong's theorem to establish the existence and uniqueness of a fixed point for the product of two nonlinear operators in Banach algebra (Theorem 2). The method is applied to a wide class of nonlinear hybrid equations such as the ones we have illustrated by means of several numerical examples.

The possibility of applying this process or a similar idea to other types of hybrid equations or systems of such equations is open and we hope to discuss this in the near future.

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