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THE COVARIETY OF PERFECT NUMERICAL SEMIGROUPS WITH FIXED FROBENIUS NUMBER

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Abstract. Let S be a numerical semigroup. We say that $h \in \mathbb{N}\setminus S$ is an isolated gap of S if $\{h-1, h+1\} \subseteq S$. A numerical semigroup without isolated gaps is called a perfect numerical semigroup. Denote by $\mathrm{m}(S)$ the multiplicity of a numerical semigroup S. A covariety is a nonempty family $\mathscr{C}$ of numerical semigroups that fulfills the following conditions: there exists the minimum of $\mathscr{C}$, the intersection of two elements of $\mathscr{C}$ is again an element of $\mathscr{C}$, and $S\setminus\{\mathrm{m}(S)\} \in \mathscr{C}$ for all $S \in \mathscr{C}$ such that $S \neq \min(\mathscr{C})$. We prove that the set $\mathscr{P}(F) = \{S\colon S$ is a perfect numerical semigroup with Frobenius number $F\}$ is a covariety. Also, we describe three algorithms which compute: the set $\mathscr{P}(F)$, the maximal elements of $\mathscr{P}(F)$, and the elements of $\mathscr{P}(F)$ with a given genus. A Parf-semigroup (or Psat-semigroup) is a perfect numerical semigroup that in addition is an Arf numerical semigroup (or saturated numerical semigroup), respectively. We prove that the sets $\mathrm{Parf}(F) = \{S\colon S$ is a Parf-numerical semigroup with Frobenius number $F\}$ and $\mathrm{Psat}(F) = \{S\colon S$ is a Psat-numerical semigroup with Frobenius number $F\}$ are covarieties. As a consequence we present some algorithms to compute $\mathrm{Parf}(F)$ and $\mathrm{Psat}(F)$.

Keywords: perfect numerical semigroup; saturated numerical semigroup; Arf numerical semigroup; covariety; Frobenius number; genus; algorithm

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1. Introduction

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{z \in \mathbb{Z}\colon z \geqslant 0\}$. A *submonoid* of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$ which is closed under addition and contains the element 0. A *numerical*

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semigroup is a submonoid S of $(\mathbb{N},+)$ such that $\mathbb{N}\setminus S=\{x\in\mathbb{N}\colon\ x\notin S\}$ has finitely many elements.

If S is a numerical semigroup, then $\mathrm{m}(S)=\min(S\setminus\{0\})$, $\mathrm{F}(S)=\max\{z\in\mathbb{Z}\colon z\notin S\}$, and $\mathrm{g}(S)=\sharp(\mathbb{N}\setminus S)$ (where $\sharp X$ denotes the cardinality of a set X) are three important invariants of S, called the *multiplicity*, the *Frobenius number*, and the *genus* of S, respectively.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A. That is, $\langle A\rangle=\{\lambda_1a_1+\ldots+\lambda_na_n\colon\ n\in\mathbb{N}\setminus\{0\},\ \{a_1,\ldots,a_n\}\subseteq A$ and $\{\lambda_1,\ldots,\lambda_n\}\subseteq\mathbb{N}\}$. In [21], Lemma 2.1 it is shown that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A)=1$.

If M is a submonoid of $(\mathbb{N},+)$ and $M=\langle A\rangle$, then we say that A is a *system of generators* of M. Moreover, if $M\neq\langle B\rangle$ for all $B\subsetneq A$, then we say that A is a *minimal system of generators* of M. In [21], Corollary 2.8 it is shown that every submonoid of $(\mathbb{N},+)$ has a unique minimal system of generators, which in addition is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the *embedding dimension* of M and denoted by $e(M)$.

The Frobenius problem (see [17]) focuses on finding formulas to calculate the Frobenius number and the genus of a numerical semigroup from its minimal system of generators. The problem was solved in [22] for numerical semigroups with embedding dimension two. Nowadays, the problem is still open in the case of numerical semigroups with embedding dimension greater than or equal to three. Furthemore, in this case the problem of computing the Frobenius number of a general numerical semigroup becomes NP-hard, see [16].

Let S be a numerical semigroup. We say that $h\in\mathbb{N}\setminus S$ is an *isolated gap* of S if $\{h-1,h+1\}\subseteq S$. A numerical semigroup without isolated gaps is called a *perfect numerical semigroup*.

If $F\in\mathbb{N}\setminus\{0\}$, we put

$$\mathscr{P}(F)=\{S\colon\ S \text{ is a perfect numerical semigroup and } \mathrm{F}(S)=F\}.$$

The main aim of this work is to study the set $\mathscr{P}(F)$.

In order to collect common properties of some families of numerical semigroups, the concept of covariety was introduced in [13]. A *covariety* is a nonempty family $\mathscr{C}$ of numerical semigroups that fulfills the following conditions:

(1) There exists the minimum of $\mathscr{C}$ with respect to set inclusion.
(2) If $\{S,T\}\subseteq\mathscr{C}$, then $S\cap T\in\mathscr{C}$.
(3) If $S\in\mathscr{C}$ and $S\neq\min(\mathscr{C})$, then $S\setminus\{\mathrm{m}(S)\}\in\mathscr{C}$.

In this paper, by using the techniques of covarieties, we study the set $\mathscr{P}(F)$. The paper is structured as follows. In Section 2, we see that $\mathscr{P}(F)$ is a covariety and its elements can be ordered in a rooted tree. Additionally, we see how the children of an arbitrary vertex of this tree are. These results will be used in Section 3 to show three algorithms which compute: the set $\mathscr{P}(F)$, the maximal elements of $\mathscr{P}(F)$, and the elements of $\mathscr{P}(F)$ with a given genus.

We say that a set X is a $\mathscr{P}(F)$-*set* if it satisfies two conditions:

(1) $X \cap \min(\mathscr{P}(F)) = \emptyset$.

(2) There is $S \in \mathscr{P}(F)$ such that $X \subseteq S$.

If X is a $\mathscr{P}(F)$-set in Section 4 we prove that then there exists the least element of $\mathscr{P}(F)$ (with respect to set inclusion) containing X. This element will be denoted by $\mathscr{P}(F)[X]$ and we will say that X is a $\mathscr{P}(F)$-*system of generators*. It will be shown that the minimal $\mathscr{P}(F)$-system of generators, in general, is not unique. Given an element $S \in \mathscr{P}(F)$, we define the $\mathscr{P}(F)$-rank of S as

$$\mathscr{P}(F)\text{-}\operatorname{rank}(S) = \min\{\sharp X\colon\ X \text{ is a } \mathscr{P}(F)\text{-set and } \mathscr{P}(F)[X] = S\}.$$

We finish with Section 4, characterizing how the elements of $\mathscr{P}(F)$ with $\mathscr{P}(F)$-rank 0, 1 and 2 are.

In the semigroup literature one can find a long list of works dedicated to the study of one dimensional analytically irreducible domains via their value semigroup. One of the properties studied for this kind of rings using this approach is to have the Arf property and to be saturated, see [2], [4], [6], [9], [14], [15], [23], [24], and [25]. The characterization of Arf rings and saturated rings via their value semigroup gave rise to the notion of Arf semigroup and saturated numerical semigroup.

Following the notation introduced in [10], a Parf-semigroup (or Psat-semigroup) is a perfect numerical semigroup that in addition is Arf (or saturated, respectively). Put

$$\begin{aligned}
\operatorname{Arf}(F) &= \{S\colon\ S \text{ is an Arf numerical semigroup and } \mathrm{F}(S) = F\},\\
\operatorname{Sat}(F) &= \{S\colon\ S \text{ is a saturated numerical semigroup and } \mathrm{F}(S) = F\},\\
\operatorname{Parf}(F) &= \{S\colon\ S \text{ is a Parf-numerical semigroup and } \mathrm{F}(S) = F\}, \quad \text{and}\\
\operatorname{Psat}(F) &= \{S\colon\ S \text{ is a Psat-numerical semigroup and } \mathrm{F}(S) = F\}.
\end{aligned}$$

By [12] and [11], we know that $\operatorname{Arf}(F)$ and $\operatorname{Sat}(F)$ are covarieties and, additionally, $\min(\operatorname{Arf}(F)) = \min(\operatorname{Sat}(F)) = \min(\mathscr{P}(F))$. This fact will be used in Section 5 to prove that $\operatorname{Parf}(F)$ and $\operatorname{Psat}(F)$ are also covarieties. Moreover, we present some algorithms to compute all the elements of $\operatorname{Parf}(F)$ and $\operatorname{Psat}(F)$.

Throughout this paper, some examples are shown to illustrate the results proven. The computation of these examples has been performed by using the GAP, see [8] and package `numericalsgps`, see [5].

2. The covariety $\mathscr{P}(F)$ and its associated tree

Along whole this work, we suppose F is a positive integer greater than or equal to 2. Our first aim is to prove that $\mathscr{P}(F)$ is a covariety.

The following result has an immediate proof.

Lemma 2.1. *The numerical semigroup $\Delta(F) = \{0, F+1, \rightarrow\}$, where the symbol $\rightarrow$ means that every integer greater than $F+1$ belongs to the set, is the minimum of $\mathscr{P}(F)$.*

The next lemma is well known and it is very easy to prove.

Lemma 2.2. *Let S and T be numerical semigroups and $x \in S$. Then the following statements hold:*

(1) *$S \cap T$ is a numerical semigroup and $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.*

(2) *$S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

(3) *$\mathrm{m}(S) = \min(\mathrm{msg}(S))$.*

Next we describe a characterization of perfect numerical semigroups that appears in Proposition 1 of [10].

Lemma 2.3. *Let S be a numerical semigroup. The following conditions are equivalent.*

(1) *S is a perfect numerical semigroup.*

(2) *If $\{s, s+2\} \subseteq S$, then $s+1 \in S$.*

By applying Lemmas 2.2 and 2.3, we can easily deduce the following result.

Lemma 2.4. *If $\{S, T\} \subseteq \mathscr{P}(F)$, then $S \cap T \in \mathscr{P}(F)$.*

The following lemma is straightforward to prove.

Lemma 2.5. *Let $S \in \mathscr{P}(F)$ be such that $S \neq \Delta(F)$. Then $S \setminus \{\mathrm{m}(S)\} \in \mathscr{P}(F)$.*

As a consequence of Lemmas 2.1, 2.4 and 2.5, we have the following result.

Proposition 2.6. *Whith the above notation, $\mathscr{P}(F)$ is a covariety and $\Delta(F)$ is its minimum.*

A *graph* G is a pair (V, E), where V is a nonempty set and E is a subset of $\{(u, v) \in V \times V \colon u \neq v\}$. The elements of V and E are called *vertices* and *edges*, respectively.

A *path (of length n)* connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

A graph G is a *tree* if there exists a vertex r (known as the *root* of G) such that for any other vertex x of G there exists a unique path connecting x and r. If (u, v) is an edge of the tree G, we say that u is a *child* of v.

Define the graph $\mathrm{G}(\mathscr{P}(F))$ in the following way:

▷ the set of vertices of $\mathrm{G}(\mathscr{P}(F))$ is $\mathscr{P}(F)$,

▷ $(S, T) \in \mathscr{P}(F) \times \mathscr{P}(F)$ is an edge of $\mathrm{G}(\mathscr{P}(F))$ if and only if $T = S \setminus \{\mathrm{m}(S)\}$.

As a consequence of Proposition 2.6 and Proposition 2.3 of [13] we can assert that $\mathrm{G}(\mathscr{P}(F))$ is a rooted tree.

Proposition 2.7. *With the above notation, $\mathrm{G}(\mathscr{P}(F))$ is a tree with root $\Delta(F)$.*

A tree can be built recurrently starting from the root and connecting, through an edge, the vertices already built with their children. Hence, it is very interesting to characterize the children of an arbitrary vertex of the tree $\mathrm{G}(\mathscr{P}(F))$. For this reason, next we are going to introduce some concepts and results that are necessary to understand the work.

Following the terminology introduced in [19], an integer z is a *pseudo-Frobenius number* of S if $z \notin S$ and $z + s \in S$ for all $s \in S \setminus \{0\}$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is an important invariant of S (see [3] and [7]) called the *type* of S, denoted by $\mathrm{t}(S)$.

Given a numerical semigroup S, denote by $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S)\colon\ 2x \in S\}$. The elements of $\mathrm{SG}(S)$ are called the *special gaps* of S. The following result is Proposition 4.33 of [21].

Lemma 2.8. *Let S be a numerical semigroup and $x \in \mathbb{N} \setminus S$. Then $x \in \mathrm{SG}(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

As a consequence of Proposition 2.6 and Proposition 2.4 of [13], we have the following result.

Proposition 2.9. *If $S \in \mathscr{P}(F)$, then the set formed by the children of S in the tree $\mathrm{G}(\mathscr{P}(F))$ is $\{S \cup \{x\}\colon\ x \in \mathrm{SG}(S),\ x < \mathrm{m}(S)$ and $S \cup \{x\} \in \mathscr{P}(F)\}$.*

The proof of the following result is immediate.

Lemma 2.10. *Let $S \in \mathscr{P}(F)$, $x \in \mathrm{SG}(S)$ and $x < \mathrm{m}(S)$. Then $S \cup \{x\} \in \mathscr{P}(F)$ if and only if $x \notin \{2, \mathrm{m}(S) - 2, F\}$.*

The next result is a consequence of Proposition 2.9 and Lemma 2.10.

Proposition 2.11. *If $S \in \mathscr{P}(F)$, then the set formed by the children of S in the tree* $\mathrm{G}(\mathscr{P}(F))$ *is* $\{S \cup \{x\} \colon x \in \mathrm{SG}(S),\ x < \mathrm{m}(S)$ *and* $x \notin \{2, \mathrm{m}(S)-2, F\}\}$.

3. Three algorithms

Our goal in this section is to describe some algorithms which compute:

(1) The set $\mathscr{P}(F)$.

(2) The maximal elements of $\mathscr{P}(F)$.

(3) The elements of $\mathscr{P}(F)$ with a fixed genus.

Let S be a numerical semigroup and $n \in S \setminus \{0\}$. The Apry set of n in S (named so in honour of the author of [1]) is defined as $\mathrm{Ap}(S,n) = \{s \in S \colon s - n \notin S\}$.

The following result is deduced from Lemma 2.4 of [21].

Lemma 3.1. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then* $\mathrm{Ap}(S,n)$ *is a set with cardinality n. Moreover,* $\mathrm{Ap}(S,n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, *where $w(i)$ is the least element of S congruent with i modulo n for all $i \in \{0, \ldots, n-1\}$.*

Let S be a numerical semigroup. In Remark 1 of [13] it appears that if we know $\mathrm{Ap}(S,n)$ for some $n \in S \setminus \{0\}$, then we can easily compute $\mathrm{SG}(S)$. And in [13], Remark 2 it is shown that if $\mathrm{Ap}(S,n)$ is known for some $n \in S \setminus \{0\}$, then it is very easy to compute $\mathrm{Ap}(S \cup \{x\}, n)$ for every $x \in \mathrm{SG}(S)$.

Algorithm 3.2

Input: An integer F greater than or equal to 2.

Output: $\mathscr{P}(F)$.

(1) $\mathscr{P}(F) = \{\Delta(F)\}$, $B = \{\Delta(F)\}$ and $\mathrm{Ap}(\Delta(F), F+1) = \{0, F+2, \ldots, 2F+1\}$.

(2) For every $S \in B$ compute $\theta(S) = \{x \in \mathrm{SG}(S) \colon x < \mathrm{m}(S)$ and $x \notin \{2, \mathrm{m}(S)-2, F\}\}$.

(3) If $\bigcup_{S \in B} \theta(S) = \emptyset$, then return $\mathscr{P}(F)$.

(4) $C = \bigcup_{S \in B} \{S \cup \{x\} \colon x \in \theta(S)\}$.

(5) $\mathscr{P}(F) = \mathscr{P}(F) \cup C$, $B = C$, compute $\mathrm{Ap}(S, F+1)$ for every $S \in C$ and go to Step (2).

In the next example, we show how the previous algorithm works.

Example 3.3. We are going to compute $\mathscr{P}(7)$, by using Algorithm 3.2.

▷ $\mathscr{P}(7)=\{\Delta(7)\}$, $B=\{\Delta(7)\}$ and $\mathrm{Ap}(\Delta(7),8)=\{0,9,10,11,12,13,14,15\}$.
▷ $\theta(\Delta(7))=\{4,5\}$.
▷ $C=\{\Delta(7)\cup\{4\},\Delta(7)\cup\{5\}\}$.
▷ $\mathscr{P}(7)=\{\Delta(7),\Delta(7)\cup\{4\},\Delta(7)\cup\{5\}\}$, $B=\{\Delta(7)\cup\{4\},\Delta(7)\cup\{5\}\}$, $\mathrm{Ap}(\Delta(7)\cup\{4\},8)=\{0,4,9,10,11,13,14,15\}$ and $\mathrm{Ap}(\Delta(7)\cup\{5\},8)=\{0,5,9,10,11,12,14,15\}$.
▷ $\theta(\Delta(7)\cup\{4\})=\emptyset$, $\theta(\Delta(7)\cup\{5\})=\{4\}$.
▷ $C=\{\Delta(7)\cup\{4,5\}\}$.
▷ $\mathscr{P}(7)=\{\Delta(7),\Delta(7)\cup\{4\},\Delta(7)\cup\{5\},\Delta(7)\cup\{4,5\}\}$, $B=\{\Delta(7)\cup\{4,5\}\}$ and $\mathrm{Ap}(\Delta(7)\cup\{4,5\},8)=\{0,4,5,9,10,11,14,15\}$.
▷ $\theta(\Delta(7)\cup\{4,5\})=\emptyset$.
▷ Therefore, the algorithm returns

$$\mathscr{P}(7)=\{\Delta(7),\Delta(7)\cup\{4\},\Delta(7)\cup\{5\},\Delta(7)\cup\{4,5\}\}.$$

Denote by $\max(\mathscr{P}(F))$ the set of maximal elements of $\mathscr{P}(F)$. Our next aim is to present two algorithms which allow to compute $\max(\mathscr{P}(F))$. For this reason, we need to introduce some concepts and results.

If S is not a perfect numerical semigroup, then we denote by $\mathrm{h}(S)$ its maximum isolated gap. The following result appears in Proposition 25 of [10].

Lemma 3.4. *If S is not a perfect numerical semigroup, then $S\cup\{\mathrm{h}(S)\}$ is a numerical semigroup.*

As a consequence of the previous lemma, we have the following result.

Lemma 3.5. *If S is a numerical semigroup with the Frobenius number F and $F-1\notin S$, then there exists $T\in\mathscr{P}(F)$ such that $S\subseteq T$.*

In the next proposition we present a characterization of maximal elements of $\mathscr{P}(F)$.

Proposition 3.6. *If S is a numerical semigroup, then the following conditions are equivalent.*

(1) $S\in\max(\mathscr{P}(F))$.
(2) *S is maximal in the set $\{T\colon\ T$ is a numerical semigroup and $T\cap\{F,F-1\}=\emptyset\}$.*

P r o o f. (1) $\Rightarrow$ (2). We suppose that S is not maximal in the set $\{T\colon T$ is a numerical semigroup and $T \cap \{F, F-1\} = \emptyset\}$, then there exists a numerical semigroup T such that $T \cap \{F, F-1\} = \emptyset$ and $S \subsetneq T$. It is clear that $\mathrm{F}(T) = F$ and $F - 1 \notin T$. Therefore, by applying Lemma 3.5, there is $P \in \mathscr{P}(F)$ such that $T \subseteq P$. Thus, $S \subsetneq P$ and consequently, $S \notin \max(\mathscr{P}(F))$.

(2) $\Rightarrow$ (1) First we show that $S \in \mathscr{P}(F)$. Otherwise, by applying Lemma 3.5, there exists $T \in \mathscr{P}(F)$ such that $S \subsetneq T$. It is clear that $T \cap \{F, F-1\} = \emptyset$ and so S is not maximal in the set $\{T\colon T$ is a numerical semigroup and $T \cap \{F, F-1\} = \emptyset\}$.

Finally, $S \in \max(\mathscr{P}(F))$ because $\mathscr{P}(F) \subseteq \{T\colon T$ is a numerical semigroup and $T \cap \{F, F-1\} = \emptyset\}$. □

If $C \subseteq \mathbb{N} \setminus \{0\}$, then we put $\mathscr{L}(C) = \{S\colon S$ is a numerical semigroup and $S \cap C = \emptyset\}$. Denote by $\max(\mathscr{L}(C))$ the set formed by the maximal elements of $\mathscr{L}(C)$. Algorithm 1 from [20] allows to compute the set $\max(\mathscr{L}(C))$ from C. Therefore, by using Proposition 3.6, we can assert that we have an algorithm to obtain $\max(\mathscr{P}(F))$.

The following result is deduced from Lemma 4.35 of [21].

Lemma 3.7. *Let S and T be numerical semigroups such that $S \subsetneq T$. Then $\max(T \setminus S) \in \mathrm{SG}(S)$.*

In the following result we show another characterization of the elements of $\max(\mathscr{P}(F))$ by using the set of special gaps.

Proposition 3.8. *If S is a numerical semigroup, then the following conditions are equivalent.*

(1) $S \in \max(\mathscr{P}(F))$.

(2) $\mathrm{SG}(S) = \{F, F-1\}$.

P r o o f. (1) $\Rightarrow$ (2) If $S \in \mathscr{P}(F)$ then it is clear that $\{F, F-1\} \subseteq \mathrm{SG}(S)$. If $\mathrm{SG}(S) \neq \{F, F-1\}$, then there is $x \in \mathrm{SG}(S)$ such that $x \notin \{F, F-1\}$. Thus, $S \cup \{x\}$ is a numerical semigroup and $(S \cup \{x\}) \cap \{F, F-1\} = \emptyset$. By Proposition 3.6, we obtain that $S \notin \max(\mathscr{P}(F))$.

(2) $\Rightarrow$ (1) We see that S is a maximal numerical semigroup under the condition that $S \cap \{F, F-1\} = \emptyset$. Otherwise, there is a numerical semigroup T such that $T \cap \{F, F-1\} = \emptyset$ and $S \subsetneq T$. By Lemma 3.7, we know that $\max(T \setminus S) \in \mathrm{SG}(S) = \{F, F-1\}$, which is absurd. Then, by Proposition 3.6, we conclude that $S \in \max(\mathscr{P}(F))$. □

Algorithm 3.5 from [18] enables us to compute the set $\{S\colon S$ is a numerical semigroup and $\mathrm{SG}(S) = \{F, F-1\}\}$. Hence, by Proposition 3.8, we have another algorithm to obtain the set $\max(\mathscr{P}(S))$.

Next, we illustrate how the algorithm works.

Example 3.9. By using Algorithm 3.5 from [18], (see [18], Example 3.8), we have that $\{S\colon S$ is a numerical semigroup and $\mathrm{SG}(S) = \{10, 11\}\} = \{S_1 = \{0, 6, 7, 8, 9, 12, \rightarrow\}, S_2 = \{0, 4, 8, 9, 12, \rightarrow\}, S_3 = \{0, 3, 6, 9, 12, \rightarrow\}\}$. Consequently, by applying Proposition 3.8, we have that $\max(\mathscr{P}(11)) = \{S_1, S_2, S_3\}$.

Note 3.10. *In Example* 3.9, *observe that* $\mathrm{g}(S_1) = 7$ *and* $\mathrm{g}(S_2) = \mathrm{g}(S_3) = 8$. *Then we can assert that all the elements of* $\max(\mathscr{P}(11))$ *have not the same genus, in general.*

We can use the following GAP sentences, to obtain the previous results:

```
gap> S1:=NumericalSemigroup(6,7,8,9);
<Numerical semigroup with 4 generators>
gap> Genus(S1);
7
gap> S2:=NumericalSemigroup(4,9,14,15);
<Numerical semigroup with 4 generators>
gap> Genus(S2);
8
gap> S3:=NumericalSemigroup(3,13,14);
<Numerical semigroup with 3 generators>
gap> Genus(S3);
8
```

If we put $\beta(F) = \min\{\mathrm{g}(S)\colon S \in \max(\mathscr{P}(F))\}$, then we can state that $\beta(11) = 7$.

We end this section by giving an algorithm which computes all the elements of $\mathscr{P}(F)$ with a given genus.

Let S be a numerical semigroup. Define recursively the *sequence associated* to S in the following way: $S_0 = S$ and $S_{n+1} = S_n \setminus \{\mathrm{m}(S_n)\}$ for all $n \in \mathbb{N}$.

If S is a numerical semigroup, then we denote by $\mathrm{N}(S) = \{s \in S\colon s < \mathrm{F}(S)\}$ the set of *small elements* of S. Its cardinality is denoted by $\mathrm{n}(S)$. Note that $\mathrm{g}(S) + \mathrm{n}(S) = \mathrm{F}(S) + 1$.

If S is a numerical semigroup and $\{S_n\}_{n\in\mathbb{N}}$ is the sequence associated to S, then $\mathrm{Cad}(S) = \{S_0, S_1, \ldots, S_{\mathrm{n}(S)-1}\}$ is called the *chain associated* to S. It is clear that $S_{\mathrm{n}(S)-1} = \{0, \mathrm{F}(S) + 1, \rightarrow\}$.

Next, if we apply that $\mathrm{g}(S_{i+1}) = \mathrm{g}(S_i) + 1$ for all $i \in \{0, \ldots, \mathrm{n}(S) - 2\}$, we easily get the following result.

Proposition 3.11. *Under the notation introduced*

$$\{\mathrm{g}(S)\colon\ S \in \mathscr{P}(F)\} = \{x \in \mathbb{N}\colon\ \beta(F) \leqslant x \leqslant F\}.$$

We illustrate the previous proposition with an example.

Example 3.12. Following Note 3.10, where $\beta(11) = 7$, then, by applying Proposition 3.11, we can assert that $\{\mathrm{g}(S)\colon\ S \in \mathscr{P}(11)\} = \{7, 8, 9, 10, 11\}$.

We now have all the necessary tools to obtain the previously announced algorithm.

Algorithm 3.13

Input: Two positive integers, F and g.
Output: $\{S \in \mathscr{P}(F)\colon\ \mathrm{g}(S) = g\}$.

(1) If $g > F$, then return $\emptyset$.
(2) Compute $\beta(F)$.
(3) If $g < \beta(F)$, then return $\emptyset$.
(4) $H = \{\Delta(F)\}$, $i = F$.
(5) If $i = g$, then return H.
(6) For every $S \in H$ compute $\theta(S) = \{x \in \mathrm{SG}(S)\colon\ x < \mathrm{m}(S)$ and $x \notin \{2, \mathrm{m}(S) - 2, F\}\}$.
(7) $H = \bigcup_{S \in H} \{S \cup \{x\}\colon\ x \in \theta(S)\}$, $i = i - 1$, and go to Step (5).

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4. $\mathscr{P}(F)$-system of generators

We say that a set X is a $\mathscr{P}(F)$*-set* if $X \cap \Delta(F) = \emptyset$ and there is $S \in \mathscr{P}(F)$ such that $X \subseteq S$.

If X is a $\mathscr{P}(F)$-set, then we denote by $\mathscr{P}(F)[X]$ the intersection of all elements of $\mathscr{P}(F)$ containing X. As $\mathscr{P}(F)$ is a finite set, then by applying Proposition 2.6, the intersection of elements of $\mathscr{P}(F)$ is again an element of $\mathscr{P}(F)$. Therefore, we can state the following proposition.

Proposition 4.1. *Let X be a $\mathscr{P}(F)$-set. Then $\mathscr{P}(F)[X]$ is the smallest element of $\mathscr{P}(F)$ containing X.*

If X is a $\mathscr{P}(F)$-set and $S = \mathscr{P}(F)[X]$, then we say that X is a $\mathscr{P}(F)$*-system of generators* of S. Moreover, if $S \neq \mathscr{P}(F)[Y]$ for all $Y \subsetneq X$, then X is called a *minimal* $\mathscr{P}(F)$*-system of generators* of S.

Let S be a numerical semigroup. Then we put

$$\mathscr{R}(S) = \{x \in \mathrm{msg}(S)\colon\ x < \mathrm{F}(S) \text{ and } \{x-1, x+1\} \not\subseteq S\}$$
$$\cup \{x \in \mathrm{msg}(S)\colon\ \{x-1, x+1\} \subseteq S, x+1 \in \mathrm{msg}(S) \text{ and } x+1 < \mathrm{F}(S)\}.$$

Proposition 4.2. *Let $S \in \mathscr{P}(F)$. Then $\mathscr{R}(S)$ is a $\mathscr{P}(F)$-set and*

$$\mathscr{P}(F)[\mathscr{R}(S)] = S.$$

P r o o f. It is clear that $\mathscr{R}(S)$ is a $\mathscr{P}(F)$-set and $\mathscr{R}(S) \subseteq S$. Therefore, by using Proposition 4.1, we have $\mathscr{P}(F)[\mathscr{R}(S)] \subseteq S$.

Let $T = \mathscr{P}(F)[\mathscr{R}(S)]$ and we suppose that $T \subsetneq S$. Then, there is $a = \min(S \setminus T)$. Obviously $a \in \mathrm{msg}(S)$ and $a < F$. We distinguish two cases:

Case 1: If $\{a-1, a+1\} \not\subseteq S$, then by using Lemma 2.2, we deduce that $S \setminus \{a\} \in \mathscr{P}(F)$. As $T \subseteq S \setminus \{a\}$, then $\mathscr{R}(S) \subseteq S \setminus \{a\}$, which is absurd because $a \in \mathscr{R}(S)$.

Case 2: If $\{a-1, a+1\} \subseteq S$, then by the minimality of a, we have that $a-1 \in T$. As $a \notin T$ and $T \in \mathscr{P}(F)$, then $a+1 \notin T$. Thus, $a+1 \in \mathrm{msg}(S)$ and $a+1 < F$. Consequently, $S \setminus \{a, a+1\} \in \mathscr{P}(F)$ and $T \subseteq S \setminus \{a, a+1\}$. Then $\mathscr{R}(S) \subseteq S \setminus \{a, a+1\}$, which is absurd because $a \in \mathscr{R}(S)$. □

The following result is straightforward to prove.

Lemma 4.3. *If X and Y are $\mathscr{P}(F)$-sets such that $X \subseteq Y$, then $\mathscr{P}(F)[X] \subseteq \mathscr{P}(F)[Y]$.*

In the following result we present a characterization of a minimal $\mathscr{P}(F)$-system of generators.

Lemma 4.4. *Let X be a $\mathscr{P}(F)$-set and $S = \mathscr{P}(F)[X]$. Then X is a minimal $\mathscr{P}(F)$-system of generators of S if and only if $x \notin \mathscr{P}(F)[X \setminus \{x\}]$ for all $x \in X$.*

P r o o f. *Necessity.* If $x \in \mathscr{P}(F)[X \setminus \{x\}]$, then by Proposition 4.1 we have that every element of $\mathscr{P}(F)$ containing $X \setminus \{x\}$, contains X, too. Therefore, $\mathscr{P}(F)[X \setminus \{x\}] = \mathscr{P}(F)[X]$.

Sufficiency. If X is not a minimal $\mathscr{P}(F)$-system of generators of S, then there exists $Y \subsetneq X$ such that $\mathscr{P}(F)[Y] = S$. If $x \in X \setminus Y$, then by applying Lemma 4.3, we have that $x \in \mathscr{P}(F)[Y] \subseteq \mathscr{P}(F)[X \setminus \{x\}]$. □

In general, the minimal $\mathscr{P}(F)$-systems of generators are not unique. Moreover, they may not even have the same cardinality as we show in the following example.

Example 4.5. Let

$$S = \langle 10, 11, 12, 13, 14, 15, 16 \rangle = \{0, 10, 11, 12, 13, 14, 15, 16, 20, \rightarrow\}.$$

It is clear that $S \in \mathscr{P}(19)$. It is obvious that if $T \in \mathscr{P}(19)$ and $\{10, 12, 14, 16\} \subseteq T$, then $S \subseteq T$. Hence, $\mathscr{P}(19)[\{10, 12, 14, 16\}] = S$. Furthemore, it is easy to see that

$$\begin{aligned}
\mathscr{P}(19)[\{10, 12, 14\}] &= \{0, 10, 11, 12, 13, 14, 20, \rightarrow\}, \\
\mathscr{P}(19)[\{10, 12, 16\}] &= \{0, 10, 11, 12, 16, 20, \rightarrow\}, \\
\mathscr{P}(19)[\{10, 14, 16\}] &= \{0, 10, 14, 15, 16, 20, \rightarrow\} \quad \text{and} \\
\mathscr{P}(19)[\{12, 14, 16\}] &= \{0, 12, 13, 14, 15, 16, 20, \rightarrow\}.
\end{aligned}$$

Thus, by applying Lemma 4.4, we have that $\{10, 12, 14, 16\}$ is a minimal $\mathscr{P}(19)$-system of generators of S.

Reasoning in a similar way, the reader will have no difficulty in seeing that $\{10, 11, 13, 15, 16\}$ is also a minimal $\mathscr{P}(19)$-system of generators of S.

If $S \in \mathscr{P}(F)$, then the $\mathscr{P}(F)$-rank of S is defined as $\mathscr{P}(F)\text{-rank}(S) = \min\{\sharp X : X$ is a $\mathscr{P}(F)$-set and $\mathscr{P}(F)[X] = S\}$. By applying Propositions 6.1, 6.2 and 6.4; and Lemma 6.1 of [13], we obtain the following result.

Proposition 4.6. *If $S \in \mathscr{P}(F)$ then the following statements hold.*

(1) $\mathscr{P}(F)\text{-rank}(S) \leqslant e(S)$.

(2) $\mathscr{P}(F)\text{-rank}(S) = 0$ *if and only if* $S = \Delta(F)$.

(3) *If* $S \neq \Delta(F)$ *and* X *is a* $\mathscr{P}(F)$*-set such that* $\mathscr{P}(F)[X] = S$, *then* $\mathrm{m}(S) \in X$.

(4) $\mathscr{P}(F)\text{-rank}(S) = 1$ *if and only if* $S = \mathscr{P}(F)[\{\mathrm{m}(S)\}]$.

For integers a and b, we say that a *divides* b if there exists an integer c such that $b = ca$, and we denote this by $a \mid b$. Otherwise, a *does not divide* b, and we denote this by $a \nmid b$.

The following result has an easy proof.

Lemma 4.7. *If m is a positive integer such that $m < F$, $m \nmid F$ and $m \nmid (F-1)$, then $\{m\}$ is a $\mathscr{P}(F)$-set and $\mathscr{P}(F)[\{m\}] = \langle m \rangle \cup \{F+1, \rightarrow\}$.*

Proposition 4.8. *If m is a positive integer such that $m < F$, $m \nmid F$ and $m \nmid (F-1)$, then $S = \langle m \rangle \cup \{F+1, \rightarrow\} \in \mathscr{P}(F)$ and $\mathscr{P}(F)\text{-rank}(S) = 1$. Moreover, every element of $\mathscr{P}(F)$ with $\mathscr{P}(F)$-rank equal to 1 has this form.*

P r o o f. By Lemma 4.7, we know that $S \in \mathscr{P}(F)$ and by Proposition 4.6 we know that $\mathscr{P}(F)\text{-rank}(S) = 1$. If $T \in \mathscr{P}(F)$ with $\mathscr{P}(F)\text{-rank}(T) = 1$, then, Proposition 4.6 asserts that $\mathrm{m}(T) < F$ and $T = \mathscr{P}(F)[\{\mathrm{m}(T)\}]$. Clearly $\mathrm{m}(T) \nmid F$ and $\mathrm{m}(T) \nmid (F-1)$. Finally, by Lemma 4.7, we conclude that $T = \langle \mathrm{m}(T) \rangle \cup \{F+1, \rightarrow\}$. □

Our next goal is to characterize the elements of $\mathscr{P}(F)$ with $\mathscr{P}(F)$-rank equal to 2. For this purpose we introduce some concepts and results.

If S is a numerical semigroup, we recursively define the following sequence of numerical semigroups:

$$S_0 = S, \quad S_{n+1} = \begin{cases} S_n \cup \{\mathrm{h}(S_n)\} & \text{if } S_n \text{ is not perfect,} \\ S_n & \text{otherwise.} \end{cases}$$

The number of isolated gaps of S is denoted by $\mathrm{i}(S)$. The following result appears in Proposition 26 of [10].

Proposition 4.9. *If S is a numerical semigroup and $\{S_n\}_{n\in\mathbb{N}}$ is the sequence previously defined, then $S = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_{\mathrm{i}(S)}$. Moreover, $S_{\mathrm{i}(S)}$ is a perfect numerical semigroup and $\sharp(S_{k+1} \setminus S_k) = 1$ for all $k \in \{0, \ldots, \mathrm{i}(S) - 1\}$.*

The numerical semigroup $S_{\mathrm{i}(S)}$ is called a *perfect closure* of S and denoted by $\mathrm{P}(S)$. Note that $\mathrm{P}(S)$ is the least perfect numerical semigroup that contains S.

Lemma 4.10. *Let $S \in \mathscr{P}(F)$ and $a \in \mathrm{msg}(S)$ such that $\{a-1, a+1\} \not\subseteq S$ and $a < F$. If X is a $\mathscr{P}(F)$-set and $\mathscr{P}(F)[X] = S$, then $a \in X$.*

P r o o f. By Lemma 2.2, we deduce that $S \setminus \{a\} \in \mathscr{P}(F)$. If $a \notin X$, then $X \subseteq S \setminus \{a\}$. Therefore, by applying Proposition 4.1, we have that $\mathscr{P}(F)[X] \subseteq S \setminus \{a\}$. Consequently, $S \subseteq S \setminus \{a\}$, which is absurd. □

Now let us define the ratio of a numerical semigroup. This concept will be needed in the proof of the following proposition.

Let S be a numerical semigroup such that $S \neq \mathbb{N}$, the *ratio* of S is defined as $\mathrm{r(S)} = \min\{\mathrm{s} \in \mathrm{S} \colon \mathrm{m(S)} \nmid \mathrm{s}\}$. Note that $\mathrm{r(S)} = \min(\mathrm{msg(S)} \setminus \{\mathrm{m(S)}\})$.

Proposition 4.11. *Let m and r be positive integers such that $m < r < F$, $m \nmid r$, and $\langle m, r \rangle \cap \{F-1, F\} = \emptyset$. Then $\mathrm{P}(\langle m, r \rangle \cup \{F+1, \rightarrow\})$ is an element of $\mathscr{P}(F)$ with $\mathscr{P}(F)$-rank equal to 2. Moreover, every element of $\mathscr{P}(F)$ with $\mathscr{P}(F)$-rank equal to 2 has this form.*

Proof. If $T = \langle m, r\rangle \cup \{F+1, \rightarrow\}$, then T is a numerical semigroup with the Frobenius number F and $F-1 \notin T$. Thus, $\mathrm{P}(T) \in \mathscr{P}(F)$. As $\mathrm{P}(T) \neq \Delta(F)$ and $\mathrm{P}(T) \neq \langle m\rangle \cup \{F+1, \rightarrow\}$, then $\mathscr{P}(F)$-rank$(\mathrm{P}(T)) \geqslant 2$. Certainly $\mathrm{P}(T)$ is the least element of $\mathscr{P}(F)$ that contains $\{m, r\}$ Hence, $\mathrm{P}(T) = \mathscr{P}(F)[\{m, r\}]$ and so, $\mathscr{P}(F)$-rank$(\mathrm{P}(T)) \leqslant 2$. Consequently, $\mathscr{P}(F)$-rank$(\mathrm{P}(T)) = 2$.

Let $S \in \mathscr{P}(F)$ such that $\mathscr{P}(F)$-rank$(S) = 2$. Then there is a $\mathscr{P}(F)$-set, X, with cardinality 2 such that $\mathscr{P}(F)[X] = S$. By Proposition 4.6, we know that $\mathrm{m}(S) \in X$. As $\{\mathrm{r(S)}-1, \mathrm{r(S)}+1\} \nsubseteq \mathrm{S}$ since it cannot happen that $\mathrm{m}(S) \mid (\mathrm{r(S)}-1)$ and $\mathrm{m}(S) \mid (\mathrm{r(S)}+1)$, then by Lemma 4.10 we know that $\mathrm{r(S)} \in \mathrm{X}$. Therefore, $X = \{\mathrm{m}(S), \mathrm{r(S)}\}$. It is clear that $\mathrm{m}(S) < \mathrm{r(S)} < \mathrm{F}$, $\mathrm{m}(S) \nmid \mathrm{r(S)}$ and $\langle \mathrm{m}(S), \mathrm{r(S)}\rangle \cap \{\mathrm{F}, \mathrm{F}-1\} = \emptyset$.

Finally, as $S = \mathscr{P}(F)[X] = \mathscr{P}(F)[\{\mathrm{m}(S), \mathrm{r(S)}\}]$, then S is the least element of $\mathscr{P}(F)$ containing $\{\mathrm{m}(S), \mathrm{r(S)}\}$. We conclude that $S = \mathrm{P}(\langle \mathrm{m}(S), \mathrm{r(S)}\rangle \cup \{\mathrm{F}+1, \rightarrow\})$. □

Next we illustrate this proposition with an example.

Example 4.12. Let $m = 8$, $r = 11$ and $F = 26$. Then $8 < 11 < 26$, $8 \nmid 11$ and $\langle 8, 11\rangle \cap \{25, 26\} = \emptyset$. By applying Proposition 4.11, we have that $\mathrm{P}(\langle 8, 11\rangle \cup \{27, \rightarrow\})$ is an element of $\mathscr{P}(26)$ with $\mathscr{P}(26)$-rank equal to 2.

Finally, as $\langle 8, 11\rangle \cup \{27, \rightarrow\} = \{0, 8, 11, 16, 19, 22, 24, 27, \rightarrow\}$, then

$$\mathrm{P}\left(\langle 8, 11\rangle \cup \{27, \rightarrow\}\right) = \{0, 8, 11, 16, 19, 22, 23, 24, 27, \rightarrow\} = \langle 8, 11, 23, 28, 29\rangle.$$

5. The Arf or saturated elements in $\mathscr{P}(F)$

We say that a numerical semigroup S is an *Arf numerical semigroup* if $x+y-z \in S$ for all $\{x, y, z\} \subseteq S$ such that $x \geqslant y \geqslant z$. We put $\mathrm{Arf}(F) = \{S \colon S$ is an Arf numerical semigroup and $\mathrm{F}(S) = F\}$.

Let $A \subseteq \mathbb{N}$ and $a \in A \setminus \{0\}$. Set $\mathrm{d}_A(a) = \gcd\{x \in A \colon x \leqslant a\}$. A numerical semigroup is *saturated* if $s + \mathrm{d}_S(s) \in S$ for all $s \in S \setminus \{0\}$. Put $\mathrm{Sat}(F) = \{S \colon S$ is a saturated numerical semigroup and $\mathrm{F}(S) = F\}$.

In Lemma 3.31 of [21] relation is shown between the saturated numerical semigroups and the Arf numerical semigroups. That is the following result.

Proposition 5.1. *Every saturated numerical semigroup is an Arf numerical semigroup.*

By applying Proposition 2.7 of [11] and Proposition 11 of [12], we obtain the following result.

Proposition 5.2. *Under the notation introduced, we have that* $\mathrm{Arf}(F)$ *and* $\mathrm{Sat}(F)$ *are covarieties and* $\Delta(F)$ *is their minimum.*

Following the notation introduced in [10] a Parf-semigroup (or Psat-semigroup) is a perfect numerical semigroup which in addition is Arf (or saturated, respectively). Put $\mathrm{Parf}(F) = \{S\colon S$ is a Parf-semigroup and $\mathrm{F}(S) = F\}$ and $\mathrm{Psat}(F) = \{S\colon S$ is a Psat-semigroup and $\mathrm{F}(S) = F\}$. Our next aim in this section is to prove that $\mathrm{Parf}(F)$ and $\mathrm{Psat}(F)$ are covarieties.

In Lemma 5.1 of [13] the following result appears.

Lemma 5.3. *Let* $\{\mathscr{C}_i\}_{i\in I}$ *be a family of covarieties with* $\min(\mathscr{C}_i) = \Delta$ *for all* $i \in I$. *Then* $\bigcap\limits_{i\in I} \mathscr{C}_i$ *is a covariety with minimum* Δ.

By Proposition 2.6, Proposition 2.7 of [11] and Proposition 11 of [12], we know that $\mathscr{P}(F)$, $\mathrm{Arf}(F)$ and $\mathrm{Sat}(F)$ are covarieties with minimum $\Delta(F)$. Then by applying Lemma 5.3, we have the following result.

Proposition 5.4. *Under the notation introduced,* $\mathrm{Parf}(F)$ *and* $\mathrm{Psat}(F)$ *are covarieties with minimum* $\Delta(F)$.

Our next purpose is to present some algorithms to compute the covarieties $\mathrm{Parf}(F)$ and $\mathrm{Psat}(F)$. The following result appears in Proposition 2.4 of [13].

Lemma 5.5. *Let* $\mathscr{C}$ *be a covariety and* $S \in \mathscr{C}$. *Then the set formed by the children of* S *in the tree* $\mathrm{G}(\mathscr{C})$ *is* $\{S \cup \{x\}\colon x \in \mathrm{SG}(S),\ x < \mathrm{m}(S)$ *and* $S \cup \{x\} \in \mathscr{C}\}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two covarieties such that $\mathscr{B} \subseteq \mathscr{A}$ and $S \in \mathscr{B}$. Put $\alpha(\mathscr{A}, \mathscr{B}, S) = \min\{x \in \mathrm{SG}(S)\colon x < \mathrm{m}(S)$ and $S \cup \{x\} \in \mathscr{B}\}$. Depending of the existence of $\alpha(\mathscr{A}, \mathscr{B}, S)$, we define

$$\mathcal{L}(S) = \begin{cases} S \cup \{\alpha(\mathscr{A}, \mathscr{B}, S)\} & \text{if there is } \alpha(\mathscr{A}, \mathscr{B}, S), \\ S & \text{otherwise.} \end{cases}$$

Define the sequence $\widehat{S}_0 = S$ and $\widehat{S}_{n+1} = \mathcal{L}(\widehat{S}_n)$ for all $n \in \mathbb{N}$. Obviously, there exists $l(\mathscr{A}, \mathscr{B}, S) = \min\{x \in \mathbb{N}\colon \mathcal{L}(\widehat{S}_x) = \widehat{S}_x\}$.

Let S be a numerical semigroup. Then we define the *sequence associated* to S in the form $S_0 = S$ and $S_{n+1} = S_n \setminus \{\mathrm{m}(S_n)\}$.

Let $\mathscr{A}$ be a covariety, $S \in \mathscr{A}$ and $\{S_n\}_{n\in\mathbb{N}}$ the sequence associated to S. Then it is clear that there exists $\mathrm{C}(\mathscr{A}, S) = \min\{n \in \mathbb{N}\colon S_n = \min(\mathscr{A})\}$. Put $\mathrm{Cad}_{\mathscr{A}}(S) = \{S_0, S_1, \ldots, S_{\mathrm{C}(\mathscr{A},S)}\}$.

Proposition 5.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be covarieties such that $\mathscr{B} \subseteq \mathscr{A}$ and $\min(\mathscr{B}) = \min(\mathscr{A})$. If $\gamma = \{S \in \mathscr{B} \colon S$ has no children in the tree $\mathrm{G}(\mathscr{A})$ which belong to $\mathscr{B}\}$, then $\mathscr{B} = \bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathscr{A}}(S)$.*

Proof. As $\mathscr{B}$ is a covariety, $\gamma \subseteq \mathscr{B}$ and $\min(\mathscr{B}) = \min(\mathscr{A})$, then we easily deduce that $\bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathscr{A}}(S) \subseteq \mathscr{B}$. For the other inclusion, if $S \in \mathscr{B}$ and $\{\widehat{S}_n\}_{n\in\mathbb{N}}$ is the sequence defined previously, then $\widehat{S}_{l(\mathscr{A},\mathscr{B},S)} \in \gamma$ and $S \in \mathrm{Cad}_{\mathscr{A}}(\widehat{S}_{l(\mathscr{A},\mathscr{B},S)})$. □

As an immediate consequence of Proposition 5.6, we have the following result.

Corollary 5.7. *Under the previous notation:*

(1) *If $\gamma = \{S \in \mathrm{Parf}(F) \colon S$ has no children in the tree $\mathrm{G}(\mathscr{P}(F))$ which belong to $\mathrm{Arf}(F)\}$, then $\mathrm{Parf}(F) = \bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathscr{P}(F)}(S)$.*

(2) *If $\gamma = \{S \in \mathrm{Psat}(F) \colon S$ has no children in the tree $\mathrm{G}(\mathscr{P}(F))$ which belong to $\mathrm{Sat}(F)\}$, then $\mathrm{Psat}(F) = \bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathscr{P}(F)}(S)$.*

If $S \in \mathscr{P}(F)$, then Algorithm 1 of [11] allows us to determine if a child of S in the tree $\mathrm{G}(\mathscr{P}(F))$ is an element of $\mathrm{Arf}(F)$. Therefore, we have an algorithm to compute the set $\mathrm{Parf}(F)$. In a similar way, if $S \in \mathscr{P}(F)$, Proposition 14 of [12] allows to determine if a child of S in the tree $\mathrm{G}(\mathscr{P}(F))$ is an element of $\mathrm{Sat}(F)$. Therefore, we have an algorithm to compute the set $\mathrm{Psat}(F)$.

As a consequence of Proposition 5.6, we have the following result.

Corollary 5.8. *Under the notation introduced:*

(1) *If $\gamma = \{S \in \mathrm{Parf}(F) \colon S$ has no children in the tree $\mathrm{G}(\mathrm{Arf}(F))$ which belong to $\mathscr{P}(F)\}$, then $\mathrm{Parf}(F) = \bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathrm{Arf}(F)}(S)$.*

(2) *If $\gamma = \{S \in \mathrm{Psat}(F) \colon S$ has no children in the tree $\mathrm{G}(\mathrm{Sat}(F))$ which belong to $\mathscr{P}(F)\}$, then $\mathrm{Psat}(F) = \bigcup\limits_{S\in\gamma} \mathrm{Cad}_{\mathrm{Sat}(F)}(S)$.*

We have implemented the gap order `IsPerfectNumericalSemigroup`, which allows us to know whether a numerical semigroup is perfect. The input is the minimal system of generators of the numerical semigroup.

We will see an example to illustrate how this order is used. If we want to know whether the numerical semigroups $S_1 = \langle 2, 3\rangle$ and $S_2 = \langle 4, 5, 11\rangle$ are perfect numerical semigroups, we use the following orders, respectively:

```
gap> IsPerfectNumericalSemigroup([2,3]);
false
gap> IsPerfectNumericalSemigroup([4,5,11]);
true
```

Thererfore, by using this order and Algorithm 2 of [11] or Algorithm 1 of [12], we get easily an algorithm to compute the set $\mathrm{Parf}(F)$ or $\mathrm{Psat}(F)$, respectively.

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