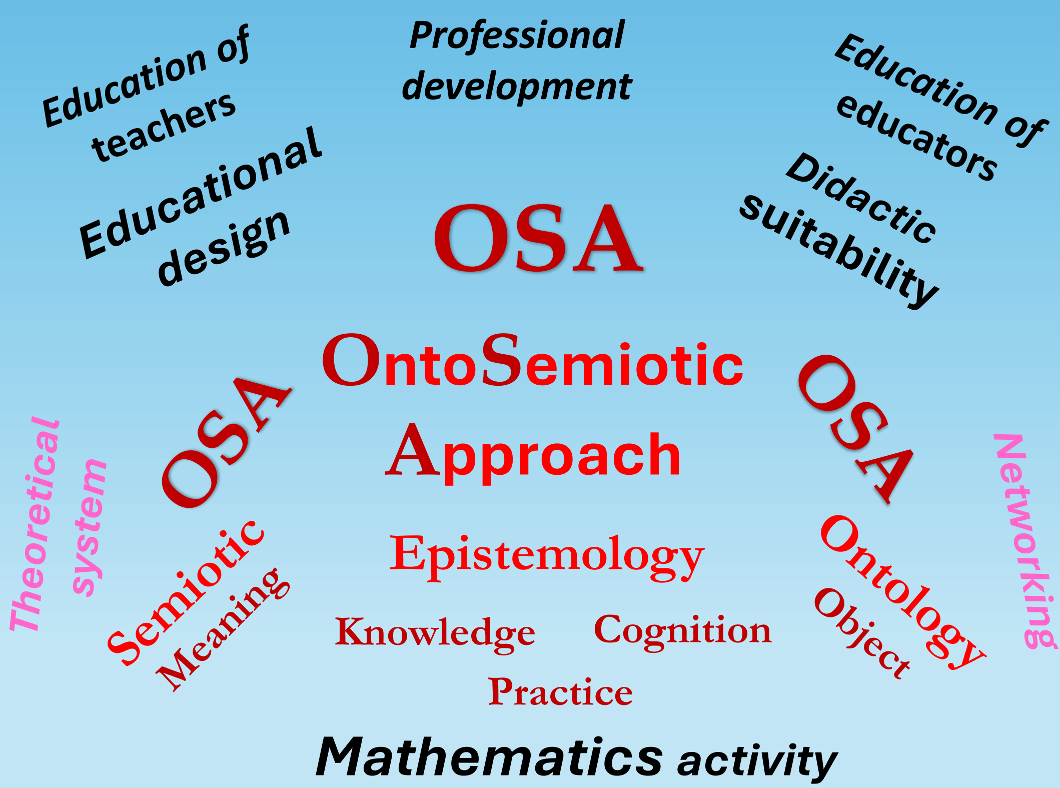
Juan D. Godino

Ontosemiotic Approach in Mathematics Education

Foundations, Tools, and Applications



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Juan D. Godino

Universidad de Granada

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Preface

The necessity and utility of writing this book arise from the vast number of studies based on the Ontosemiotic Approach (OSA) to mathematical knowledge and instruction published since the early 1990s. As can be seen in the web repository available at the University of Granada, http://enfoqueontosemiotic.ugr.es, following the publication of the article Institutional and Personal Meaning of Mathematical Objects (Godino & Batanero, 1994) in *Recherches en Didactique des Mathématiques*, the number of doctoral theses and research articles using the conceptual and methodological tools of the OSA has increased substantially. These theoretical tools have grown and refined over time and have been applied to didactic research related to various mathematical contents and educational levels. The successive developments of the OSA have encompassed the problems involved in the processes of mathematics education, including ontological, semiotic, and epistemological issues specific to mathematics education and those related to teaching and learning in different contexts and educational levels. As a result, a theoretical system was developed that coherently articulates the various dimensions involved in research on the teaching and learning of mathematics and the education of mathematics teachers.

Thirty years after publishing the first works on the OSA, it was necessary to undertake a review and systematization to facilitate its dissemination, which was carried out in multiple postgraduate courses, conferences, and research projects. This book aims to present the modules or theories that form the OSA theoretical system, the basic assumptions that support them, their articulation, their connections with other theories, and examples of applying the conceptual and methodological tools.

The OSA provides a system of notions, principles, and methodological tools to study and understand the nature of mathematical activity, mathematical knowledge, and teaching and learning processes. A technological (prescriptive) component —comprising a system of criteria or norms to optimize the design, implementation, and evaluation of educational-instructional processes— and a model of teachers' professional development complement the scientific (descriptive, explanatory, and predictive) component of mathematics education.

The OSA theoretical system presented in this book comprises five articulated theories:

1. *Ontosemiotic Theory of Mathematical Activity*: This theory develops an anthropological and pragmatist view of mathematics, that is, as a human activity centered on problem-solving. This anthropological view of mathematics is complemented and articulated with two other conceptions: mathematics as a system of objects and processes and mathematics as a system of signs.
2. *Ontosemiotic Theory of Meaning and Mathematical Cognition*: This theory develops a global view of the meaning of mathematical objects, articulating realistic and pragmatic assumptions as the basis of mathematical cognition from an individual (personal) and social (institutional) perspective.
3. *Theory of Educational Design in Mathematics*: This theory develops assumptions and theoretical tools for the description and design of teaching and learning processes in mathematics based on the specific theory of mathematical activity and the meaning of objects proposed by the OSA.

4. *Theory of Didactic Suitability*: This theory develops a system of criteria for the local optimization of the design, implementation, and evaluation of educational-instructional processes in mathematics based on OSA assumptions and constructs. These criteria consider the epistemic, ecological, mediational, interactional, cognitive, and affective facets of teaching and learning processes.
5. *Theory of Professional Development for Teachers*: This theory develops a model of knowledge and competencies for mathematics teachers that considers the facets, components, and sub-components of the educational processes involved in the activities of foundation, design, planning, and evaluation. It also includes a system of principles or criteria for the efficiency of teacher education programs.

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Chapter 1

Background and presentation

Introduction

Since the emergence of Didactics of mathematics in the 1970s, the research community has been developing various theories to describe and explain phenomena related to the processes of teaching and learning mathematics, with the aim of contributing to the improvement of these processes. The complexity of these phenomena, the different factors that must be considered, and the influence of the diverse cultural contexts in which theories are generated explain their profusion and the appearance of diverse dilemmas or controversies among them. This situation generates communication difficulties and efficient use of the knowledge produced by the research activity.

In this book, we present the theoretical framework of the Ontosemiotic Approach (OSA) in mathematics education (Godino and Batanero, 1994; Godino, 2002; Godino et al., 2007; Font et al., 2013; Godino et al., 2020), whose objective is to address in an articulated way the problems of grounding, design, implementation, and evaluation of the processes of teaching and learning mathematics. The strategy of clarification, comparison, hybridization and modular construction of theories, from an ontological and semiotic approach, lies at the foundation of OSA. We assume the relevance and potential usefulness of advancing toward the construction of a theoretical system that allows us to address in an articulated way the epistemological, ontological, semiotic, cognitive, and educational problems involved in the teaching and learning of mathematics.

In this first chapter, we motivate the construction of the OSA by responding to various dilemmas and contradictions between the different research paradigms and theories used in mathematics education. We also present chapters in which we describe the assumptions adopted and the theoretical tools developed to address the foundational issues of mathematics education as a field of research and their application to educational design and teacher education.

In Section 1.1, we analyze the tension between theory and practice observed in different ways of understanding mathematics education: as a field of basic scientific research or as a field of technological research and practical action. In Section 1.2, we describe various divergent theoretical positions on ontological and epistemological issues in mathematics, i.e., the nature of mathematical objects, the emergence and development of mathematical knowledge, and the role of languages and systems of representation. In Section 1.3, we describe various dilemmas in conceptualizing mathematical learning, constructivism, enactivism, cultural psychology, and discursive learning. Different ways of conceptualizing mathematics teaching, either student-centered or teacher-centered, the role of inquiry, and the transmission of knowledge are described in Section 1.4. Regarding the assessment of student learning and educational-instructional processes, various positions and views are summarized in Section 1.5. In Section 1.6, we identify some dilemmas among various approaches to mathematics teacher education that motivate the development of a specific model based on the OSA. Finally, in Section 1.7, we set out the structure of the book in six chapters, in addition to this introductory chapter. We present the OSA as a theoretical system consisting of five partial theories: ontosemiotic theory of mathematical activity and emergent objects; ontosemiotic theory of mathematical meaning and cognition; theory of instructional-educational design; theory of didactic suitability; and theory of teacher professional development. The last three theories are based on ontological and semiotic assumptions and tools described in Chapters 2 and 3.

1.1. Dilemmas in conceptualizing mathematics education

The term education has a broader use than didactics; thus, we can distinguish between mathematics education and didactics. However, in the Anglo-Saxon world the expression "mathematics education" is used to describe the area of knowledge that in continental Europe is called "didactics of mathematics". In this book, we use both expressions interchangeably, although as Steiner (1985) states, mathematics education, in addition to naming the scientific discipline, can also refer to the interactive social system comprising theory, development, and practice.

Philosophical reflection on the nature of mathematics education as a field of knowledge is essential to adequately orient the research because it conditions the formulation of the central questions of the research.

> Is mathematics education a discipline, a field of research, an interdisciplinary area, a field of extra disciplinary applications, or something else? Is it a branch of applied mathematics or a special part of educational theory? Is it a science, a social, artistic, or humanistic science, or none or all of them? (Kilpatrick, 2008) What is its relationship with other disciplines, such as philosophy, mathematics, sociology, psychology, linguistics, anthropology, etc. (Ernest, 2018, p. 22)?

Among the authors who have reflected on the nature of mathematics education, Steiner (1985) and Brousseau (1989) stand out, in an essay with the significant title, The Tower of Babel. Faced with the extreme complexity of problems related to mathematics education, Steiner (1985, p. 11) indicated two extreme reactions:

- Authors who affirm that mathematics education cannot become a field with a scientific foundation and that mathematics teaching is essentially an art.
- Those who think that the existence of mathematics education as a science is possible and reduce the complexity of its problems by selecting only a partial aspect of them (for example, the analysis of the content to be taught, the construction of the curriculum, the improvement of teaching methods, the development of student skills, classroom interaction, etc.), to which they attribute a special weight within the whole, giving rise to different definitions and visions of mathematics education.

Brousseau (1989) expresses himself similarly, although in his case he speaks of didactic of mathematics. The first conception of the didactics of mathematics identifies it as the art of teaching, that is, the set of means and procedures that tend to make mathematics known. In addition, he distinguished two conceptions of a scientific nature: an applied multidisciplinary one and an autonomous one (described by Brousseau himself as fundamental or mathematical). As a hinge between these two visions, he also distinguished a technonicist conception, in which didactics would be the set of teaching techniques, that is, the invention, description, study, production, and control of new means for teaching: curriculum, objectives, means of evaluation, materials, manuals, software, etc.

In the multidisciplinary conception, which would emerge with the second trend pointed out by Steiner, didactics appears as a label to designate the teachings necessary for the technical and professional training of teachers. Steiner (1990) identified various disciplines related to mathematics education, such as mathematics, epistemology and philosophy of mathematics, history of mathematics, psychology, sociology, or pedagogy. The activity of theorizing or grounding is seen by Steiner as a component of mathematics education, an academic field and a domain of interaction between research, development and practice.

Lesh and Sriramn (2010) also reflect on the nature of mathematics education as a field of research and ask the following questions: Should mathematics educators view themselves as applied educational psychologists, cognitive psychologists, or applied social scientists? Should they see themselves as similar to scientists in the field of physics, or other pure sciences, or rather as engineers or other design-oriented technicians whose research draws on multiple practical and disciplinary perspectives and whose work is driven by the need to solve real problems and also by the need to develop relevant theories? These authors consider mathematics education in the latter sense, that is, as a science oriented to the design of processes and resources to improve the processes of teaching and learning mathematics.

There is some controversy between those who emphasize the science aspect of mathematics education (Gascón and Nicolás, 2017), whose objective is to understand educational phenomena, and those who consider education as a form of sociotechnology (Bunge, 1998) and emphasize the component of intervention on practice for its improvement.

At the OSA, we consider it necessary to articulate a vision that recognizes the complementarity of the scientific and technological components of didactics. This means that, on the one hand, theoretical problems of ontological, epistemological and semiotic clarification of mathematical knowledge must be addressed, insofar as such problems are related to teaching and learning processes (scientific, descriptive, explanatory and predictive component). On the other hand, it is necessary to intervene in these processes to make them as suitable as possible (technological-prescriptive component). It is understood that description, explanation and prediction are the aims of scientific activity, while prescription and assessment are the main objectives of technological activity, although the latter also includes elements of research applied to the resolution of specific problems. We therefore assume an expanded conception of didactics as related to processes of teaching and learning, to mathematical knowledge and practice (genesis, development, dissemination, transposition and use), and the optimization of these processes in educational contexts.

In this book, mathematics education is considered, in addition to being a field or area of knowledge, as a system of activities carried out by individual subjects or teams in communities that are interested in problems related to the foundation of research, the dissemination of knowledge and the practice of mathematics education. The application of the notion of activity system in the sense of Cultural-Historical Activity Theory (CHAT) (Engeström, 1987; Engeström and Sannino, 2021; Roth and Lee, 2007) allows us to describe and understand mathematics education as a whole composed of several subsystems. For this purpose, we differentiate between several partial activities: foundation, design, implementation, evaluation, and teacher professional development. Identifying the different elements of each partial activity and their relationships can help to recognize controversial positions and progress in the elaboration of a modular and inclusive theoretical system (Ruthven, 2014) that addresses the complexity of mathematics education activity. The CHAT notion of contradiction, which includes dilemmas, tensions, and conflicts between elements of the activity (Núñez, 2009) or between related activities, clarifies the reasons for changing systems and identifying unresolved contradictions that need to be addressed in new developments. The idea of dilemma (controversies, contradictions) is useful for motivating the construction of the OSA as a theoretical system that seeks to address them, in some cases through a hybridization strategy of existing theories, and in others by recognizing the complementarity and coordinated use of various theories.

1.2. Dilemmas in conceptualizing mathematical knowledge and its emergence

The foundation of mathematics education activity requires problematizing the nature of mathematical knowledge: how can we approach the study of numbers, for example, if we do not clearly understand what numbers are and what it means to understand them? Tackling this question requires including ontological (nature and types of objects), epistemological (how knowledge arises and evolves), and semiotic (diversity and role of signs) questions about mathematics in mathematics education research.

Several theoretical frameworks have been addressing this issue by considering the interconnections between mathematical activity and mathematics education activity, often focusing on partial aspects and adopting different epistemological, ontological, semiotic, and cognitive positions. Improving the coherence and effectiveness of mathematics education research requires confronting and articulating this diversity of approaches.

The study of literature on the theoretical foundations of mathematics education allows us to identify the following tensions:

- Mathematics as human activity versus mathematics as a system of objects.
- Platonism (mathematical objects as pre-existing entities) and nominalism (mathematical objects reduced to names or symbols).
- Relationships between personal (cognitive) and institutional (epistemic) dimensions of knowledge.
- Links between internal (cognitive schemas, conceptions) and external (languages and material artifacts) representations of knowledge.
- Meaning as a mental reference to words, symbols, or their uses.

– Connections between professional mathematical knowledge, school knowledge, and processes of transposition or elementaritation.

These dilemmas are discussed in greater detail below.

1.2.1. The ontology and epistemology of mathematics

The ontology and epistemology of mathematics are two interrelated philosophical branches. Ontology is concerned with the nature of what we are studying in mathematics, i.e., mathematical objects, while epistemology focuses on how we understand those objects. The ontology of mathematics raises central questions about the nature of mathematical objects and their relation to the physical world and language:

– What kind of existence do numbers, functions, and geometric figures have?
– What relationships exist among mathematical objects, the physical world, and languages?
– Is mathematics universal, or does it depend on cultures and people's activity?

Regarding the nature of mathematical objects, Platonic realism has been the dominant philosophical trend. This philosophical position views mathematical objects as existing independently of the physical world in an ideal realm (Linnebo, 2009). Conceptualists and nominalists defend contrary positions. Conceptualism holds that mathematical objects are mental entities, i.e., they exist only in the human mind. In this sense, numbers, geometric figures, functions, etc., are mental constructs that have no existence independent of the mind that creates them. Nominalism, on the other hand, holds that mathematical objects are not real entities but simply names or labels for sets of physical objects (Bueno, 2020). In this sense, numbers are nothing but names for the quantities of objects, geometric figures are nothing but names for sets of points, and functions are nothing but names for relations between sets. Sociological and anthropological positions in philosophy of mathematics (Bloor, 1983) postulate that mathematical objects emerge from social and cultural practices; mathematics is not a product of the individual mind but the result of the interaction of people with each other and with their environment.

Consequently, regarding questions of the epistemology of mathematics—the nature of mathematical knowledge, its foundation, justification, and the ways of coming to know and learn—there are various approaches that are not always compatible and can also be identified in mathematics education. We find in it a series of epistemological controversies:

> including the subjective-objective character of mathematical knowledge; its role in knowledge of the social and cultural context; the transfer of knowledge and learning from one social context to another; the relationships between language and knowledge; and the tensions between the main tenets of constructivism, sociocultural views, interactionism, and French didactics, from an epistemological perspective. (Ernest, 2018, p. 27)

1.2.2. Semiotics of mathematics

Because mathematical objects cannot be apprehended directly through the senses, their ontological status, communication and learning require the use of signs, such as specific terms, symbols, diagrams or graphs. Consequently, semiotics, as the study or doctrine of signs, that is, the systematic investigation of their nature, properties and types, is receiving great attention in mathematics education research. "Semiotics has been a fruitful theoretical lens used by researchers interested in various mathematics education issues in recent decades" (Presmeg, 2014, 539).

The central questions in semiotic mathematics are as follows:

- What mathematical signs, such as numbers, geometric diagrams, and algebraic symbols, refer to?

- How do various representations (words, symbols, graphs) relate to each other and mathematical objects?
- How do students interpret and understand mathematical symbols and expressions?
- How does culture and context influence mathematical semiosis, that is, how are mathematical signs interpreted and given meaning in different cultural contexts?

There are several theories to address these issues, and they take different positions on the use of language. In realist or referential theories of the meaning of words and symbols (such as those defended by Frege or Carnap), linguistic expressions have a relation of attribution with certain entities (objects, attributes, facts). In pragmatic theories, such as the one defended by Wittgenstein (1953), the meaning of linguistic expressions depends on the language games in which they are used; the meaning of abstract objects must be inferred from their use. In Chapter 3 we develop the ontosemiotic theory of meaning and mathematical cognition, where we propose a complementary view between referential and pragmatic theories of meaning.

1.2.3. Internal and external representations

Research on mathematics education has highlighted the importance of using multiple representations in teaching and learning processes and the complexity of related factors. As Font et al. (2010) pointed out, one of the central open questions raised using representations is the nature and diversity of both the objects that play the role of representation and the objects represented. The large number of publications on the topic of representations (Cobb et al., 2000; Goldin, 1998; Goldin, 2020; Janvier, 1987) demonstrates their importance for mathematics education and at the same time their complexity. The reason for this interest is to be sought in the fact that talking about representation is equivalent to talking about knowledge, meaning, understanding, modeling, etc. Undoubtedly, these notions constitute the central core, not only of mathematics, but also of epistemology, psychology, and other sciences and technologies that deal with human cognition, its nature, origin, and development. This diversity of disciplines interested in representation is the reason for the diversity of approaches and ways of conceptualizing representation, not without debate.

> The complexity of the subject, the ambiguity of representations, and their importance lie in the mathematical objects that one attempts to represent, their diversity, and nature. Talking about representation (meaning and understanding) necessarily implies talking about mathematical knowledge, and therefore, about mathematical activity, its cultural and cognitive "productions" and those related to the world around us. (Font et al., 2010, p. 59-60)

1.3. Dilemmas in conceptualizing learning

Substantial discussions exist among radical constructivist, social constructivist, enactivist, and sociocultural learning theories. "These are primarily epistemological differences, although proponents and critics of the various theories also incorporate ontological, ethical, social, and methodological analyses and reasoning into their arguments" (Ernest, 2018, p. 28).

The various constructivisms share the construction metaphor, according to which they describe the subject's understanding as the construction of mental structures. They recognize that knowing is active, individual, and personal and that it is based on previously constructed knowledge. The construction metaphor is contained in the first principle of constructivism, as expressed by von Glasersfeld (1989, p. 182): "knowledge is not passively received by the cognitive subject but actively constructed".

1.3.1. Radical constructivism

In the radical version of constructivism, a second principle is added: "the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of an ontological reality" (von Glasersfeld, 1989, p. 182). Overall, radical constructivism is neutral in its ontology, making no assumptions about the existence of the world behind the subjective domain of experience. "The epistemology is decidedly fallibilist, skeptical, and anti-objectivist" (Ernest, 1994, p. 6). The fact that no ultimate possible true knowledge exists about the state of affairs in the world or about domains such as mathematics is a consequence of the second principle, which is proper for epistemological relativity. As its name indicates, learning theory is radically constructivist; individuals construct all knowledge based on their cognitive processes in dialog with their experiential world.

1.3.2. Social constructivism

The social version of constructivism considers the individual subject and the social domain as inextricably interconnected (Ernest, 1994). People are shaped by their interactions with others (as well as by their individual processes). Certainly, the underlying metaphor corresponds to that of people in conversation, encompassing meaningful linguistic and extralinguistic interactions. The mind is seen as part of a broader context, the 'social construction of meaning. Similarly, the social constructivist model of the world corresponds to a socially constructed world that creates (and is constrained by) the shared experience of the underlying physical reality.

In short, the social constructivist research paradigm adopts a modified relativistic ontology (there is an external world supporting appearances to which we have shared access, but we do not have secure knowledge of it). It is based on a fallibilist epistemology that regards conventional knowledge as that which is lived and socially accepted. The associated learning theory is constructivist (in the sense shared by sociologists such as Schutz, Berger and

Luckman, as well as constructivists), with an emphasis on the essential and constitutive nature of language and social interaction (Ernest, 1994). Piagetian constructivism emphasizes internal cognitive processes at the expense of social interaction in the construction of knowledge by learners. However, constructivism must accommodate the complementarity between individual constructions and social interactions.

1.3.3. Enactivism

Enactivism has become a learning theory of importance for researchers in mathematics education. According to this theory of cognition, "the individual is not a mere observer of the world but is bodily immersed in the world and is shaped, cognitively and as a complete physical organism, by his or her interaction with the world" (Ernest, 2010, p. 42). Another source of enactivism is found in the theory on the bodily basis of thought, through the role of metaphors, following the works of Lakoff and Johnson (1980) and Johnson (1987). According to these authors, all human understanding, including meaning, imagination, and reason, is based on the schemas of bodily movement and its perception. These schemas are extended using metaphors, which provide the basis for all human understanding, thinking, and communication. In the book by Lakoff and Núñez (2000), this idea was developed and applied to mathematics.

1.3.4. Discursive learning

In the research literature, the use of cognitive notions, such as mental schemes, conceptions, or cognitive conflicts, predominates, but the progressive introduction of others is observed, such as activity, interaction patterns, or communication failure (Kieran et al., 2001). Learning, conceived as a personal acquisition, is being complemented by a new vision as a process of participation in a collective activity. What is important is not the change in the individual learner but the change in the ways in which individuals communicate with others. The new research framework begins to be designated as discursive or communicational due to the emphasis that research attributes to language and communication, which is one of the various possible implementations of the sociocultural approach linked to the Vygotsky school of thought and the philosophy of Wittgenstein. This approach proposes a vision of human thought as essentially social in its origins and that is complexly dependent on historical, cultural, and situational factors in a complex way. According to Sfard (2001), the communication approach to cognition is based on the theoretical principle that "communication should not be considered as a mere aid to thinking but almost as equivalent to thinking itself" (p. 13). Thinking is conceived as a special case of communication activity, and “mathematical learning means mastering discourse that is recognized as mathematical by expert interlocutors” (Kieran et al. 2001, p. 5). Learning is conceived in terms of discourse, activity, culture, and practice, and its development focuses on interpersonal interactions. In the communication or discursive approach, the dichotomy between thought and language practically disappears; language ceases to be a mere "window of the mind", that is, a secondary activity of thought that expresses something already available. Although thought and language must be considered two different entities, "both must be understood basically as aspects of the same phenomenon, without either of them being prior to the other" (Sfard, 2001, p. 27).

1.4. Teaching dilemmas

Teaching in different disciplinary areas, particularly mathematics, raises various dilemmas about which teachers must make decisions:

- Individualization *vs*. standardization: Adapting teaching to the needs and interests of each student, respecting their learning pace, or ensuring that all students progress at the same time, using a uniform curriculum and assessment.

- Theory *vs*. Practice: Emphasizing the understanding of mathematical concepts and principles or prioritizing the application of knowledge to real situations and problem solving.
- Rigor *vs*. Creativity: Prioritize accuracy and academic rigor when teaching mathematics or encourage creativity and exploration in students.

Next, we develop two other dilemmas about which we find strong controversies: student-centered or teacher-centered teaching.

1.4.1. Student-centered teaching

The family of inquiry-based instructional theories; "Inquiry-Based Education (IBE), "Inquiry-Based Learning" (IBL), and "Problem-Based Learning" (PBL), designate theoretical models of instruction developed from various curricular disciplines. In them, a key role is attributed to the resolution of "authentic" problems, under a constructivist approach. In some applications to the field of mathematics education, it is assumed that students can construct knowledge following the work guidelines of mathematical and scientific professionals. Mathematicians face non-routine problems, explore, seek information, make conjectures, justify, and communicate their results to the scientific community. Mathematics instruction should follow similar guidelines.

In these theories, the use of problem situations (applications to everyday life, to other fields of knowledge, or internal problems to the discipline itself) is considered essential so that students can make sense of the conceptual structures that make up mathematics or science as cultural reality. The formulation of "rich" problem situations that require analysis and reflection on the mathematical structure involved, their solutions, and communication are key to developing students' mathematical competence. This is the main objective of the tradition called "problem solving" (Schoenfeld, 1992), whose emphasis is on the identification of heuristics and metacognitive strategies. It is also a key focus for other theoretical models such as Realistic Mathematical Education (Freudenthal, 1973; 1991) and the Theory of Didactic Situations (Brousseau, 2002).

English and Sriraman (2010) reported various reflections on and evaluations of the effectiveness of research on problem solving, concluding on its limited presence in school practice. These authors assert that "Unfortunately, there is a lack of studies that address conceptual development based on problem solving in interaction with the development of problem-solving competencies" (English and Sriraman, 2010, p. 267).

1.4.2. Teacher-centered teaching

We consider models based on the transmission of knowledge in various forms of educational intervention in which direct and explicit instruction takes precedence. "When dealing with new information, learners should be shown what to do and how to do it" (Kirschner et al., 2006, p. 79). The use of worked examples is a characteristic feature of strongly guided instruction, while the discovery of a solution to a problem in an information-rich environment is, similarly, the epitome of minimally guided discovery learning.

The uncritical adoption of constructivist pedagogical models may be motivated by the observation of the large amount of knowledge and skills that a subject learns through discovery or immersion in a context, particularly the concepts of everyday life. However, Sweller et al. (2007) stated that

> There is no reason to assume or empirical evidence to support the notion that constructivist teaching procedures based on the way humans acquire biologically primary information will be effective in acquiring the biologically secondary information required by citizens of an intellectually

> advanced society. This information requires direct and explicit instruction (p. 121).

This position agrees with the thesis maintained by Vygotsky, that scientific concepts do not develop in the same way as everyday concepts (Vygotsky, 1934). Sweller et al. (2007) considered that providing students with a completely solved example of a problem or task and information regarding the process used to reach the solution is necessary for the design of suitable learning tasks. These authors assert that empirical research on this problem over the past half century on this problem provides overwhelmingly clear evidence that minimal guidance during instruction is significantly less effective and efficient than guidance specifically designed to support the cognitive processing necessary for learning. Alfieri et al. (2011) obtained similar results in their meta-analysis. According to Kischner et al. (2006),

> We are skilled in an area because our long-term memory contains enormous amounts of information related to the area. This information allows us to quickly recognize the characteristics of a situation and tells us, often unconsciously, what to do and when to do it (p. 76).

1.5. Dilemmas in evaluating educational-instructional processes

Wheeler (1993) raised the problem of evaluating mathematical knowledge from an epistemological perspective. If we need to assess students' mathematical knowledge for a multiplicity of purposes, the first question that must be elucidated is the nature of the knowledge itself. The reason this author gives seems obvious: "How can we evaluate what we do not know?" (p. 87).

There are tensions between formative and summative assessment in the evaluation of learning at local (internal to the classroom) and global (external) levels (Stufflebeam et al., 2002). Summative evaluation requires the development of objective measurement instruments that allow comparisons between groups, schools and countries to be made when making decisions at a macro level. This evaluation reduces complexity in that it dispenses with contextual details, which can be essential from an educational perspective.

The use of standardized tests to assess students' mathematical learning has become widespread in many countries. This means imposing pressure on schools and teachers to ensure that students receive high scores in these exams, implying a series of widely documented threats. According to Ralston, the following holds:

> Three of the worst problems are teaching to the test, the emphasis on routine mathematics at the expense of advanced topics and problem solving, and the inordinate amount of time preparing for these tests, which not only pushes out important mathematics classrooms but often involves less attention to science, history, and the arts in general. (Ralston, 2006, p. 1651)

The vision of learning and teaching mathematics has substantially changed with the incorporation of different approaches to content, activities, and modes of interaction. This expanded vision must mean changes in how learning outcomes are evaluated. A complex vision of mathematics, learning and teaching requires new approaches to evaluation, especially formative evaluation carried out by teachers to intervene based on the organization of teaching.

Niss (1993) identified and discussed crucial issues regarding the evaluation of student learning (assessment) in mathematics education and the different positions of mathematics educators. He recognized that evaluation involves many profound and difficult theoretical and practical problems, which have a strong impact on the evaluated subjects. Niss also recognizes that what is not evaluated in education becomes invisible or unimportant. He concludes by formulating the following general dilemma regarding assessment in mathematics education: "How can we assess the essential components of mathematical knowledge, understanding, thinking, creativity, problem solving, and general ability without seriously distorting them?" (Niss, 1993, p. 27).

1.6. Dilemmas in mathematics teacher education

In the field of research on mathematics teacher education and thinking (Blömeke and Kaiser, 2017; Chapman, 2020; Ponte and Chapman, 2016; Wood, 2008), we found various theoretical models that describe the types of knowledge that teachers must put into play to promote student learning. These models are necessary to organize initial or ongoing training programs and evaluate their effectiveness. Although there has been a consensus that teachers should master corresponding disciplinary content, no similar agreement has been reached on how such mastery can be achieved. It is usually recognized that disciplinary knowledge is not sufficient to ensure professional competence, and other psychological knowledge is necessary (how students learn, what their characteristic difficulties and errors are, their emotions and attitudes, etc.). Teachers should also be able to organize teaching, design meaningful learning tasks, use appropriate resources, and understand the factors that condition educational-instructional processes.

Shulman's (1986) pioneered drawing attention to the specificity of content knowledge in teaching. He introduced the construct "pedagogical content knowledge" (PCK), which is widely accepted as relevant to teacher training. The PCK has been interpreted and adapted to mathematics by various authors (Scheiner et al., 2019). The notion of "mathematical knowledge for teaching" elaborated in various articles by Ball and collaborators (Ball, 2000; Ball et al., 2001), assumes and develops Shulman's ideas from the observation of teachers' work in the classroom. However, as Graeber and Tirosh (2008, p. 124) stated, "The fact that many researchers do not offer a precise and shared description of PCK (pedagogical content knowledge) but rather attempt to characterize it with lists or examples is an indication that the concept is still poorly defined." Silverman and Thompson (2008) found similar limitations in the notion of MKT (mathematical knowledge for teaching):

Although mathematical knowledge for teaching has begun to gain attention as an important concept in the teacher education research community, there is limited understanding of what it is, how it can be recognized, and how it can be developed in the minds of teachers. teachers (p. 499).

The search for meaning in mathematics teachers' knowledge continues to be an important topic in this field of research (Scheiner et al., 2019). From our point of view, the models of "mathematical knowledge for teaching" developed from research on mathematics education include general categories. We consider it useful to have models that allow a more detailed analysis of each type of knowledge that comes into play in effective mathematics teaching. This would guide the design of training actions and the development of instruments for evaluating the mathematics teacher's knowledge.

International studies (Even and Ball, 2009) have concluded that mathematics teacher training should be closely linked to teaching practice. They suggest the following three main issues that could benefit from stronger and more systematic international connections in improving teacher training and professional development. The first is the need to focus teacher training on practice and the problem of doing so effectively (Ball and Even, 2009, p. 255), articulating teachers' personal views from their practical experience with approaches derived from research (Potari, 2013). The other two problem areas that Ball and Even mention are the problem of training teacher educators and the development of valid evaluation instruments for teacher learning. To address these problems, it is necessary to develop theoretical models that consider the specificity and complexity of facets and components that intervene in the educational-instructional processes, both referring to the mathematical content and the didactic-mathematical content that teachers —and consequently teacher educators— must know and master.

1.7. Emergence and development of OSA

At the beginning of the 90s and in the context of a course on "Theory of Mathematics Education" in a doctoral program at the University of Granada, we became aware of the need to clarify fundamental notions of the area to study cognitive phenomena that were described with different constructs: knowledge, conceptions, concepts, schemes, operational invariants, meanings, praxeologies, etc. The recognition of the disparity of said constructs, formulated in theories such as the Theory of Didactic Situations (Brousseau, 2002); Conceptual Fields (Vergnaud, 1990), Registers of Semiotic Representation (Duval, 1995), and Anthropological Theory of the Didactic (Chevallard, 1992), motivated the first works of the OSA.

The problem posed in the first stage in the development of the OSA was the clarification of the notion of meaning of a mathematical object and its relationship with other constructs such as concept, conception, and understanding (Godino and Batanero, 1994). The distinction between personal and institutional aspects in relation to meaning was essential to articulate epistemological and cognitive approaches to mathematics education.

Considering that the epistemic-cognitive problem cannot be separated from the ontological and semiotic problem, in the second stage (starting in 1998), the theoretical framework was expanded (Godino, 2002) to describe the mathematical activity and the communication processes. In this extension, we advance the development of a specific ontology and semiotics to study the interpretation processes of mathematical sign systems put into play in didactic interaction. The development of a theory of mathematical knowledge on anthropological (Wittgenstein, 1953), pragmatist (Peirce, 1931-58) and semiotic (Hjelmslev, 1943) bases provided elements of articulation between theories of learning and teaching of mathematics.

In later stages we apply the OSA to develop tools for the analysis and design of educational-instructional processes (Godino et al. 2006), including work on the normative and meta-normative dimensions (D'Amore et al., 2007; Godino et al. , 2009;), the development of a didactic suitability tool (Godino et al., 2006; Godino, 2013) that includes a system of criteria for comprehensive evaluation of teaching and learning processes and a model of knowledge and competencies for mathematics teachers (Godino, 2009; Godino et al., 2017).

The OSA system of theoretical tools has been applied in didactic research on specific mathematical contents: arithmetic, algebra, geometry, and statistics. The OSA web repository, available at https://enfoqueontosemiotico.ugr.es, includes the main publications made on these topics, as well as the collection of more than 100 doctoral theses that have used OSA as a theoretical framework of reference. The development and application of the OSA have been carried out within the framework of various research projects and postgraduate programs at different universities.

The construction of the OSA can be considered a version of expansive learning. Engeström (1987) proposed that learning, within the framework of CHAT, involves not only the assimilation of existing knowledge but also the creation of new knowledge and practices. The collective subject formed by people interested in the development of OSA uses tools developed in other activity systems (theories). However, through collaborative and socially mediated actions, this approach seeks to expand the original activities and generate new concepts, tools and ways of approaching research in mathematics education. Rather than uncritically adopting existing theories, the OSA collective subject actively seeks to contribute to the transformation of learning environments.

1.8. Book structure

As we have seen previously, mathematics education issues of diverse nature must be addressed: epistemological, semiotic, educational design, evaluation, and teacher education. This led the OSA to develop five theories (Figure 1.1.) to address the specific problems that arise in different activities that constitute mathematics education. We understand a theory as a system of tools (concepts, principles and methods) that are used to answer a set of questions specific to a field of inquiry.

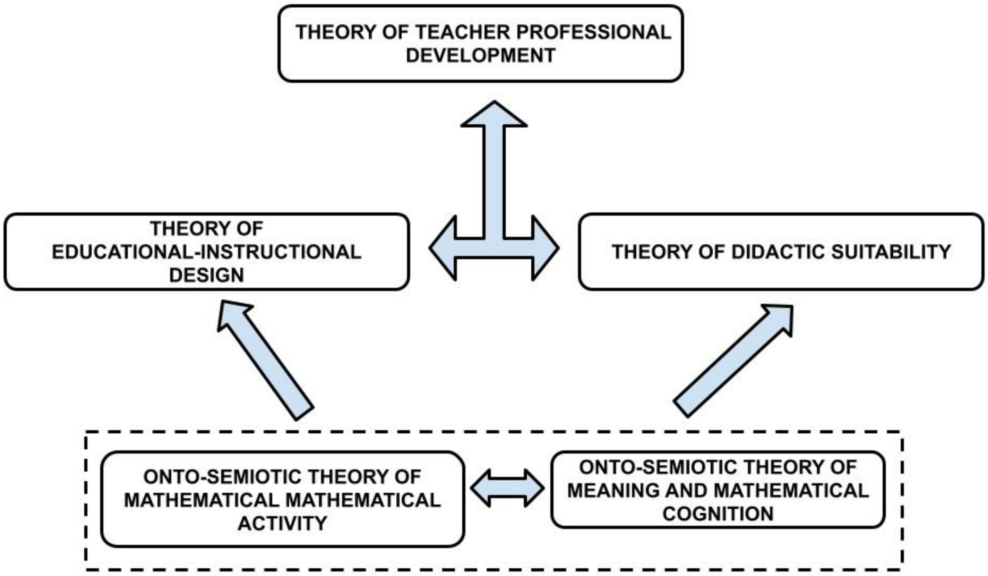

Figure 1.1. OSA theoretical system

Next, we synthesize the main characteristics of the five theories described in detail in chapters 2 to 6, as well as their articulation in the OSA theoretical system, which we present in chapter 7. In each chapter, we present the specific background, assumptions, and theoretical tools developed, their concordances, and complementarities with other theories, as well as examples of their application. The content of each chapter is based on previously published articles in journals with the collaboration of various authors. We will cite these documentary sources when appropriate. However, overcoming the space limitations inherent to scientific articles, which allows the writing of a book, makes it possible to expand the description of the background, assumptions, and constructs and, above all, present a global and articulated vision of the partial theories that make up the OSA theoretical system.

Chapter 2: Onto-semiotic theory of mathematical activity

The ontosemiotic theory of mathematical activity provides assumptions and theoretical tools for the analysis of mathematical activity, both professional and school, as well as the objects that intervene and emerge from this activity. It provides its own vision of the emergence of mathematical knowledge adapted to the educational context, with transdisciplinary features when addressing dilemmas in epistemological and ontological theories involved in mathematics education. This vision complements the formal-logical perspective, typical of the contexts of creation and justification of mathematical knowledge, with the empiricist-factual conception linked to the contexts of application. The postulates of this theory are as follows:

- Mathematics is a human activity that involves solving certain types of problems.
- Mathematical practices can be idiosyncratic to individuals or shared within institutions.
- Problem solving is carried out through the articulation of practice sequences.
- Various kinds of objects that fulfill different roles intervene in mathematical practices: instrumental/representational; regulative (setting rules on practices); explanatory; and justification.

Chapter 3: Onto-semiotic theory of meaning and mathematical cognition

The ontosemiotic theory of meaning and mathematical cognition develops a global vision of the meaning of mathematical objects as the basis of mathematical cognition from individual (personal) and social (institutional) perspectives. Meaning is the content of any semiotic function, understood as a relationship between two objects (functives), one functioning as an expression (signifier) and the other as content (signified), related according to a criterion or rule of correspondence (interpretant). Functives can be elements of the various languages used in mathematical practice and other types of objects of the OSA ontology (concepts, propositions, procedures, arguments), including the practice systems themselves. In this way, the theory articulates realist (referential) and pragmatic (operational) assumptions about meaning. The semiotic function construct serves as a basis for defining the knowledge and understanding of mathematics in terms of the webs of semiotic functions that a subject (person or institution) can establish between the objects involved in the practices required for problem solving.

Chapter 4: Theory of educational-instructional design in mathematics

The theory of educational-instructional design in mathematics provides assumptions and theoretical tools for designing teaching and learning processes in mathematics based on specific theories about mathematical activity and the meaning of emerging objects proposed by the OSA. This includes a model of the structure and dynamics of educational processes that considers the various facets and components that characterize these processes. It proposes a model of categories of norms and metanorms that explains didactic phenomena and provides guidelines for the optimization of educational processes. The configuration and didactic trajectory constructs allow for detailed (descriptive and explanatory) analyses of the design and implementation of educational processes. Complemented with the postulate of ontosemiotic complexity of content and the didactic suitability construct, which is developed in Chapter 5, they allow for the elaboration of a mixed didactic model to address the dilemma between constructivist (inquiry) and objectivist (transmissive) models to optimize mathematical learning.

Chapter 5: Theory of didactic suitability

Develops a system of criteria for the local optimization of the design, implementation, and evaluation of educational-instructional processes in mathematics, based on the assumptions and constructs of OSA. Criteria (value judgments) are expressed on the preferred didactic actions carried out in the different facets and components that structure educational processes (epistemic, ecological, mediational, interactional, cognitive and affective).

Chapter 6: Theory of teacher professional development

Develops a model of didactic-mathematical knowledge and competencies for mathematics teachers based on the structure of the educational-instructional processes and criteria of didactic suitability. It also develops a system of principles or criteria of suitability for training programs and actions for mathematics teachers, considering the facets, components, and subcomponents of the educational-instructional processes, as well as the foundation, design, planning, and evaluation activities of such processes. The system of criteria is formulated in terms of value judgements, that is, actions that the teacher and teacher educator should carry out to optimize the training processes, incorporating a system of knowledge, dispositions, and competencies of the trainer involved in the actions.

Chapter 7: OSA theoretical system

The OSA theoretical system for mathematics education is summarized here, and it attempts to provide constructs, principles, and methodological tools to study and understand the nature of mathematical activity, mathematical knowledge, as well as the teaching and learning processes of mathematics. This scientific component (descriptive, explanatory and predictive) on mathematics education is complemented by another technological (prescriptive) component formed by a system of criteria or standards to optimize the design, implementation and evaluation of educational-instructional processes and a professional development model. teacher.

Each chapter includes a list of bibliographic references to facilitate independent reading.

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Chapter 2

Ontosemiotic theory of mathematical activity

Introduction

Philosophical reflection on the foundations of mathematics education as a scientific and technological discipline is essential to guide research because it conditions formulating questions to be addressed in this area and the design of instructional models and resources. Likewise, to understand and optimize the processes of teaching and learning mathematics, it is necessary to investigate epistemological questions about mathematics, as proposed by Fundamental Didactics (Gascón, 1998), as well as ontological, semiotic, cognitive, sociological, and other questions. Clarifying mathematics, both in the contexts of formal uses of creation and justification of mathematical knowledge, as well as applied to solving scientific, technological and everyday life problems, is essential for mathematics education. However, this clarification is insufficient because the study of the processes of learning and dissemination of mathematics requires consideration of psychological, pedagogical, sociological, and other aspects. It is necessary to adopt a transdisciplinary perspective in mathematics education (Arboledas & Castrillón, 2007; Steiner, 1985).

The diversity of theories, as well as the dilemmas and contradictions among them, constitute the background against which OSA has generated a new vision of mathematical knowledge that can be adapted to the educational context with transdisciplinary features. In OSA, it is assumed that to understand and intervene in an informed manner in educational-instructional processes is necessary to address empirical problems that are proper to the psychology and pedagogy of mathematics, such as How do we learn mathematical ideas and how can we help to learn them? However, these questions must be addressed in an integrated way with other philosophical questions, such as What is the nature of mathematical objects and how do they differ from material objects? How do mathematical objects exist? What mathematical truth is it? What is mathematical proof? In summary, what is mathematical knowledge and how it arises (Leng et al., 2007; Shapiro, 2004).

In this chapter, we describe the theoretical tools developed in OSA to address the analysis of mathematical activity and the objects that intervene and emerge in such activity, highlighting currents in the philosophy of mathematics and other disciplines on which they rely. In Section 2.1, we introduce the construct of educational mathematics to distinguish, without separating, pure and applied mathematics when studying mathematics education. Recognizing the specific characteristics of educational mathematics as an ecological variety of mathematics, in which formal reasoning coexists symbiotically with empirical-intuitive reasoning, is important for understanding learning processes and designing informed educational interventions. Next, in Section 2.2, we synthesize the principal schools of philosophy of mathematics on which we will project an ontosemiotic approach to mathematical activity and emergent objects. In Section 2.3, we show that the construct configuration of practices, objects, and processes (ontosemiotic configuration) allows us to articulate coherently the basic elements of a philosophy of educational mathematics, interwoven with psychology and sociology. The ontosemiotic configuration condenses the vision proposed by OSA in mathematics as an activity (Section 2.4), a system of objects and processes (Section 2.5), and a system of signs (Section 2.6). The processes of idealization, generalization, and objectification that we address in Section 2.7 are used to characterize mathematical abstraction in the OSA framework (Section 2.8). In Section 2.9, we describe the concordances and complementarities of the ontosemiotic theory of mathematical activity with other theoretical frameworks used in mathematics education. Then, in Section 2.10, we describe two examples of applying the theory: an ontosemiotic approach to visualization in mathematics education and a model of algebraic reasoning levels. In Section 2.11, we include a synthesis of the ontosemiotic theory of mathematical activity following the (adapted) guide for describing theories in social and behavioral sciences proposed by Michie et al. (2014).

2.1. Characterization of educational mathematics[1]

To approach the problems of teaching and learning mathematics in a well-founded manner, it is essential to clarify the specific characteristics of pure and applied mathematics, as well as the relationships between them. This analysis revealed the emergence of an ecological variety of mathematics (Godino, 1994), which we designate educational mathematics. It is necessary to distinguish between the formal (theoretical) and the factual (empirical) dimensions of educational mathematics, which does not imply considering them as separate but recognizing that they maintain close symbiotic relationships when we are interested in the processes of knowledge generation and learning. Therefore, the meaning we attribute to "educational mathematics" differs substantially from its use in some mathematics education communities, where it is considered a synonym of mathematics education or didactics of mathematics, as expressed by Cantoral and Farfán (2003): "Educational mathematics is then a discipline of knowledge whose origin dates back to the second half of the twentieth century and that, in general terms, we could say deals with the study of didactic phenomena linked to mathematical knowledge" (p. 29).

[1] The content of this section 2.1 and the following 2.2 is based on the article by Godino (2023).

We now clarify the characteristics of pure and applied mathematics, relying mainly on Bunge (1985). We can describe contemporary pure mathematics, also designated as abstract, formal, or axiomatic (Marquis, 2014), as an investigation by theoretical means of problems about conceptual systems or their parts to find patterns that satisfy such objects, a finding that must be justified only by rigorous demonstration. In mathematics, as a formal science, symbols and constructs are involved but not empirical or factual objects (facts, things, properties of things and events). Applied mathematics deals with problems in factual science, technology, and humanities, with the help of constructs that belong to pure mathematics. Applied mathematics is then distinguished from pure mathematics as follows:

- Origin of the problems, which are extra mathematical in the first case and internal in the second.
- Ultimate referents, which are real with applied mathematics and constructs in the other case, and
- The goal is to help non-mathematical disciplines and advance pure mathematics.

A problem belongs to formal mathematics when its solution requires formal (i.e., non-empirical) proofs or refutations. Applied mathematics uses not only formal constructs and models but also artifacts and empirical constructs. For these two contexts, Echeverría (2007) added the context of education and dissemination of knowledge as a field of reflection in the philosophy of science because it constitutes a fundamental component of scientific activity, taken in its entirety.

In the educational context, the study of problems in both the extra-mathematical and intra-mathematical worlds is addressed, even at the early school level. For example, learning natural numbers begins with counting problems, where a number is assigned to a cardinal set of perceivable objects.

However, this requires the simultaneous learning of a mathematical structure, the sequence of number words and symbols, and the principles of counting, which constitute the first interconnection between formal mathematics and applied mathematics. Applied problems involve factual objects and empirical verifications, which must be differentiated from formal constructs and the conventional rules by which they are operated and justified.

In educational mathematics, we study not only propositions of reason, that is, constructs (conceptual objects, such as numbers or triangles), which correspond to pure mathematics, but also propositions of fact that they refer to concrete (real, material) things, such as the sizes or dimensions of triangular-shaped things. In mathematics teaching, students should be very careful not to confuse mathematical objects with their material or symbolic representations. This is not important in mathematical applications or pure mathematics, which only consider abstract entities. In addition, the justification procedures in educational mathematics are different because not only logical and deductive procedures are used, but also analogy, metaphor, induction, and plausible reasoning (English, 1997). Special care should be taken when distinguishing between empirical justifications and deductions from definitions and postulates.

In summary, pure mathematics is an activity that aims at creating mathematical models to address the solution of increasingly general problems, for which constructs and theories with progressive levels of abstraction and formalization are developed. The objective of applied mathematics is to solve specific problems in empirical, technological, and social sciences by applying mathematical models. The object of educational mathematics is the study of the dialectical relationships between pure and applied mathematics, between the processes of creation and application of mathematical knowledge, as they should be the object of teaching and learning. Consequently, educational mathematics should not only study the process of abstraction (progressive generalization, synthesis and formalization), but also the inverse process of interpretation (analysis, particularization, and concretion), as well as the dialectical relationships between them.

2.2. Philosophies of mathematics

The philosophies of mathematics that have emerged in the last twenty-five centuries address issues such as the following:

- Ontology: questions about the ontological status of mathematical objects.
- Semantics: questions about meaning, reference, and truth in mathematics.
- Epistemology: questions about the nature and sources of mathematical knowledge.
- Methodology: questions for justification (in particular, proof) and application.

No doubt these questions are essential and characteristic of the philosophy of pure and applied mathematics and educational mathematics, although in this case, they are intertwined with other questions concerning learning and teaching in different educational contexts and levels. Table 2.1 summarizes the typical principles of five widely recognized philosophies of mathematics (Bunge, 1985, p. 120).

Mathematical Platonism can be defined as the conjunction of the following three theses: (a) existence: mathematical objects exist, and mathematical sentences and theories provide true descriptions of such objects; (b) abstraction: mathematical objects are abstract, i.e., non-spatial and non-temporal entities; and (c) independence: mathematical objects are independent of intelligent agents and their language, thinking, and practices.

Furthermore, according to Platonists, abstract objects are totally non-physical, non-mental, and non-causal (Linnebo, 2009).

Table 2.1. Principles of five philosophies of mathematics

Philosophy	**Math objects**	**Mode of introduction**	**Meaning**	**Truth**	**Math knowledge**	**Math activity**
Platonism	Self-existing ideal and eternal	Discovery	Non-contradiction	Formal	A priori and conceptual	Deductive
Nominalism	Symbols	Convention	Nil	Convention	Nil	Formal manipulation of symbols
Intuitionism	Mental cons-tructions	Invention	Reducibility to positive integers	Reducibility to numerical computation	A priori and intuitive	Intuitive and rational
Empiricism	Mental	Discovery	Reference to experience	Empirical	Empirical	Trial and error, rational and empirical
Conceptualist and fictionist materialism	Fictions (classes of brain processes)	Invention and discovery	Conceptual reference and contextual sense	Formal	A priori and conceptual	Abstraction, generalization, formal manipulation, trial and error, analogy, induction and deduction

Nominalist positions attempt to explain mathematics and its applications without assuming a mathematical ontology (Burgess & Rosen, 1997). They argue that numbers, points, functions, sets, etc., should not be considered abstract entities, separate from concrete objects. There is not a single program of nominalist reinterpretation or reconstruction of mathematics, but there are several, as nominalism is a diffuse set of positions, and its different supporters prefer quite different strategies and methods. A variation on this theme that has played an important role in the history of mathematics is formalism, which holds that the essence of mathematics is the following rules without requiring them to make sense. "Mathematics is similar to a game like chess, in which the characters written on a piece of paper play the role of pieces that must be moved. The only thing that matters for achieving mathematics is that the rules have been followed correctly" (Shapiro, 2005, p. 16).

Intuitionism, a revisionist movement of the foundations of mathematics, maintains that mathematics and its objects must be humanly graspable. It has three facets: mathematical, formal logic, and philosophical (Posy, 2020). From a philosophical perspective, it is considered that mathematical objects are not abstract entities that exist independently of the human mind but rather mental constructions that arise from intuition and experience. From the semantic aspect, it is considered that only intuitive mathematical ideas—particularly those ultimately reducible to the intuition of the sequence of natural numbers—are significant. Mathematical knowledge is obtained through intellectual intuition rather than sensory experience or pure reason. Anything that is counterintuitive (for example, non-computable numbers and actual infinity) is not really known, so it is not part of mathematics. From a methodological viewpoint, only constructive concepts and proofs are admissible.

Empirical realism shares with Platonism the view that mathematics consists of the description of objects that exist independently of people and the language used to represent them. However, instead of placing such objects beyond space and time, empirical realism places them within a spatiotemporal world. Mathematical ideas are mental objects that reflect or summarize experiences. Mathematical knowledge can be obtained inductively, like any other. Regarding methodology, it considers that the ultimate test of mathematical propositions is human experience, even if it is indirect—for example, through the experimental test of scientific theories in which mathematics intervenes. The primary perspectives are physicalism, holistic empiricism, and radical empiricism (Font et al., 2013).

Naturalism in philosophy of mathematics (Kitcher, 1984; Maddy, 1997) shares some features with empiricist positions, although it adopts a broader perspective. Although both naturalism and empiricism recognize the importance of historical and cultural factors in the development of mathematics, naturalism tends to emphasize the interaction between human activity and the natural and cultural world in which mathematics is developed. Naturalism emphasizes that mathematics is a human activity that is subject to the limitations and perspectives of individuals and communities of practices. While empiricism may share this perspective to some extent, its primary emphasis is on empirical experience as a source of knowledge.

Naturalistic perspectives are considered by Ernest (1998) as a "maverick tradition" in philosophy of mathematics that offer a foundation to accommodate the social and historical factors involved in mathematics education. He argued that the philosophy of mathematics must consider the social construction of an individual mathematician and his creativity to account for mathematical knowledge naturalistically. Likewise, he highlights the negative consequences of Platonism, mathematical realism, and foundationalist and absolutist positions on mathematics education.

Table 2.1 includes the fifth philosophy of mathematics that Bunge (1985) calls "conceptualist and fictionist materialism," which characterizes his position on the subject. Although each of the classical philosophies of mathematics has its advantages, none adequately covers all aspects of mathematical research: "posing and reformulating problems, using theories or hypotheses to solve them, proving theorems, inventing axioms, definitions and algorithms, calculate, compare constructions, make mathematical considerations, etc. — all using intuition, analogy, induction, and deduction" (Bunge, 1985, p. 131). A philosophy of mathematics that is consistent with its general philosophical system includes, among others, the following characteristics:

- Recognizes the purely conceptual nature of mathematical objects and methods, while acknowledging their empirical or intuitive origins in some of them.

- Mathematical constructs are impersonal and universal and are the products of brain processes.
- Considers the differences between formal and factual propositions and between mathematical demonstration and empirical validation.
- Does not require the introduction of either mythical objects, such as self-existing Platonic ideas, or nonrational faculties, such as intuition (except as a heuristic aid).

In addition to the classical philosophies listed in Table 2.1, other relevant contributions to the philosophy of educational mathematics are also elaborated, such as the philosophical positions of Wittgenstein and Lakatos.

Wittgenstein (1956) dealt above all with the issues of learning, understanding, invention, and using elementary mathematical ideas. Wittgenstein's philosophy of mathematics lies at the opposite end of the Platonic-idealist currents and of psychological approaches. He posed the challenge of overcoming dominant Platonism and, therefore, of stopping discussing mathematical objects as ideal entities that are discovered and of considering mathematical propositions as descriptions of the properties of such objects. He proposes an alternative vision: mathematical propositions must be seen as instruments and rules for the transformation of empirical propositions. For example, theorems of geometry are rules for framing descriptions of the shapes and sizes of objects, their spatial relationships, and making inferences about them. Wittgenstein's view of mathematical language as a tool is also relevant to educational mathematics. He argued that we should consider words as tools and should clarify their use in our language games. For example, we must not lose sight of the fact that number words are instruments for counting and measuring and that the foundations of elementary arithmetic (i.e., mastery of natural number series) are based on counting training.

Lakatos' ideas about mathematics (Lakatos, 1976) are summarized in the following theses (Bunge, 1985). First, mathematical research is not essentially different from scientific research because it also involves the formulation of conjectures and the search for counterexamples. Second, since we often start from inaccurate concepts and errors can be made when proving theorems, a fallibilist epistemology of mathematics must be adopted. Third, formalism does not accurately represent the real work of a mathematician, which involves non-deductive procedures. In Bunge's opinion, these three theses are reasonable, but they do not constitute a philosophy of mathematics. On the one hand, Lakatos does not express clear ideas about the nature of mathematical objects: he is more interested in history than in the ontology or semantics of mathematics. Like any other person trying to solve a problem, a professional mathematician must use analogy and induction and try to find the correct solution, even when using material tools. However, the logic of mathematical discovery and the heuristic procedures in problem solving that Lakatos describes provide important elements for the philosophy of educational mathematics. Progressive mathematical growth, from both a cognitive and historical-cultural point of view, does not have to be linked to a deductivist style but rather follows the steps of the heuristics described in the book *Proofs and refutations*. However, this does not mean that pure mathematics, as a specific epistemological variety, is not fundamentally different from the factual sciences.

In mathematics education, we find authors who address issues specific to the philosophy of educational mathematics. Such is the case of Sfard (2000; 2008) when she analyzes the relationships between symbols and mathematical objects. The problem she addresses, expressed in semiotic terms, is: "Mathematical symbols refer to something, but to what? ... What is the ontological status of these entities? Where do they come from? How can we access them (or build them)?" (p. 43). Sfard rejects the conception that proposes signs and meanings as independent entities and adopts the view of psychologists such as Vygotsky and semioticians such as Peirce that signs (language in general) have a constitutive role in the objects of thought and are not merely representational. The central thesis defended by Sfard is as follows:

> mathematical discourse and its objects are mutually constitutive: discursive activity, including the continuous production of symbols, creates the need for mathematical objects; and it is the mathematical objects (or rather the use of symbols mediated by the objects) that influence the discourse and take it in new directions. (Sfard, 2000, p. 47)

In the study by Font et al. (2013), we argue that the way mathematics is taught in schools leads students to develop, even if implicitly, realistic views of mathematical objects. This view assumes that mathematical statements are descriptions of reality and that the mathematical objects described by these statements are part of this reality.

> In the teaching process, this “reality” to which mathematical objects belong is situated in an intermediate point between what, in philosophy of mathematics, are called Platonic and empiricist positions, although depending on the teaching process considered, a clear preference for one or the other of these two points of view can be observed, for example, in contextualized teaching or in realist mathematics (Font et al., 2013, p. 99).

The analysis in this chapter of the relationships between pure and applied mathematics provides a complementary explanation for this educational phenomenon. Mathematics teachers and educators, in general, do not discriminate the substantial differences between applied and formal mathematics, and there is a need for educational mathematics to identify the conflicts and obstacles that are generated in learning processes when they do not consider these differences.

2.3. The ontosemiotic configuration as a tool for the analysis of mathematical activity[2]

The dilemmas existing in various philosophical and psychological theories on the nature and origin of mathematical knowledge motivated the elaboration of a framework that would help us understand and act informedly on educational-instructional processes. We considered it necessary to elaborate a theory of mathematical activity and emergent objects that would serve as a basis for a theory of meaning and mathematical cognition. We started this project by publishing the article "Personal and institutional meaning of mathematical objects" (Godino & Batanero, 1994), which is the starting point of the OSA.

Sharing the perspective of fundamental didactics (Gascón, 1998), OSA considers it necessary to problematize the type of mathematics that is studied in educational systems. It assumes that educational mathematics must adopt a specific vision of mathematics that can be adapted to learning and teaching. This vision must complement the formal logical vision, proper to the contexts of creation and justification of mathematical knowledge, with an empiricist-factual vision linked to the contexts of application. It is essential to distinguish between pure or formal mathematics, applied mathematics, and educational mathematics, which result from ecological processes of adaptation of other mathematics to different educational settings and levels. It is necessary to elaborate a philosophy of educational mathematics that addresses the epistemological (emergence and development of mathematical knowledge), ontological (nature and types of mathematical objects), and semiotic (syntactic, semantic and pragmatic) problems specific to this variety of mathematics. The educational setting must also articulate the philosophical problems of mathematics through questions related to the

[2] The content of sections 2.3 to 2.7 is based on the articles by Godino and Batanero (1994), Godino (2002), Godino, Batanero, and Font (2007), and Font, Godino, and Gallardo (2013).

cognitive processes involved in learning, which occur in historical and cultural contexts that condition and support them.

In the following sections of this chapter, we describe the assumptions and theoretical constructs developed in OSA to describe mathematical activity and the emergent objects of such activity. These assumptions underlie the theory of the meaning of mathematical objects and the model of institutional and personal cognition that we present later in Chapter 3.

The theoretical constructs elaborated in the OSA that address central issues in philosophy, psychology, and sociology of educational mathematics are as follows:

- Mathematical practices.
- Mathematical objects and processes.
- Contextual attributes of practices and objects.

These theoretical constructs are articulated in the tool ontosemiotic configuration of practices, objects, and processes (Figure 2.1), the central part of which shows, besides practices, the six types of mathematical objects considered primary in the OSA ontology:

- *Problems* (intra or extra mathematical).
- *Languages* (terms, expressions, notations, graphics) in various registers (written, oral, gestural, etc.).
- *Concepts* (introduced by definitions or descriptions, such as, line, point, number, mean, function).
- *Propositions* (statements about concepts).
- *Procedures* (algorithms, operations, calculation techniques).
- *Arguments* (statements used to explain and validate propositions and procedures, deductive or otherwise).

The primary objects can be viewed from five pairs of viewpoints or dualities (contextual attributes); thus, each causes 10 types of secondary objects:

– *Personal* (relating to individual subjects); *institutional* (shared in an institution or community of practice).
– *Ostensive objects* (material, perceptible); *non-ostensive objects* (abstract, ideal, immaterial).
– *Extensive objects* (particular); *intensive objects* (general).
– *Signifier* or *signified* (antecedent or consequent of a semiotic function).
– *Unitary* (objects considered globally as a whole); *systemic* (considered as systems formed by structured components).

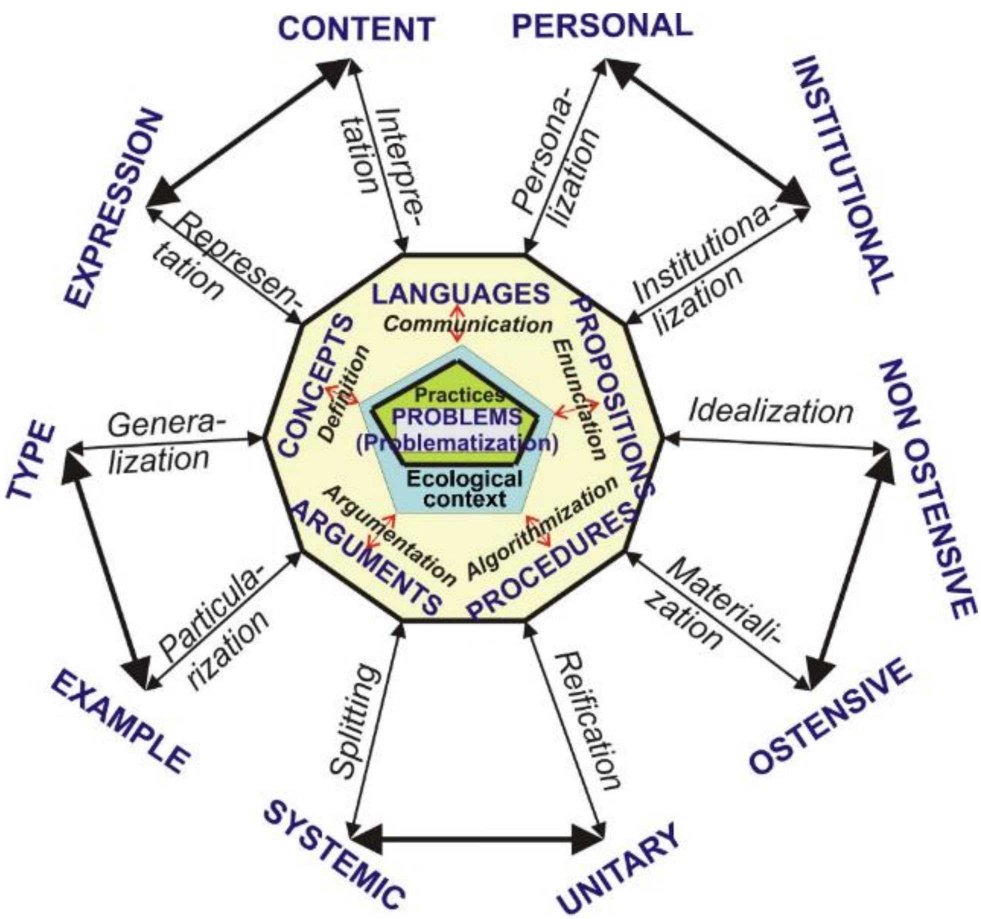

Figure 2.1. Ontosemiotic configuration of practices, objects, and processes

Both primary and secondary objects (derived from the application of dualities) can be considered from the process-product perspective: an object is an emergent (product) of sequences of practices (process). This view provides criteria for distinguishing primary and secondary mathematical processes. There are processes of problematization, definition, enunciation, argumentation, particularization-generalization, representation-signification, etc.

The different elements of the diagram synthesize OSA assumptions about the central role of problem solving in mathematical activity, types of objects and processes involved, and complementary points of view from which practices, objects, and processes can be viewed. These assumptions and elements reflect the position of the OSA on some dilemmas and controversies about the foundations of mathematics education related to the nature of objects, their emergence, meaning, and mathematical knowledge.

In the following sections, we explain in more detail the ontosemiotic configuration tool that synthesizes the vision proposed by the OSA of mathematics as human activity, a system of objects and processes, and a system of signs. Including the ostensive–non-ostensive, extensive–intensive, unitary–systemic dualities and the corresponding processes of idealization–materialization, particularization–generalization, and unitarization–decomposition allow us to develop an ontosemiotic interpretation of mathematical abstraction (section 2.8).

2.4. Mathematics as an activity

As shown in Figure 2.1, the activity of people to solve problems in specific ecological contexts (physical, biological, and social) is considered the central element in the construction of mathematical knowledge. This way of approaching the epistemological problem of the genesis of knowledge is made operational in the OSA with the notion of mathematical practice understood as “any action or expression (verbal, graphic, etc.) carried out by someone to solve mathematical problems, communicate to others the solution obtained, validate it or generalize it to other contexts and problems” (Godino & Batanero, 1994, p. 334). To solve a problem, the subject performs an organized sequence of various types of operative and discursive practices, intending to provide an answer to the given problem. The epistemological question of how mathematics emerges and develops is answered, therefore, by assuming an anthropological[3] (Wittgenstein, 1953; 1976) and pragmatist (Peirce, 1958) vision of mathematics.

The same type of problem can be solved with systems of practices that depend on the institutional contexts in which they occur, for example, within communities of mathematics professionals, of people who develop or apply new mathematical knowledge, and in diverse educational contexts. The relativity of practices regarding the institutional and temporal context adds a sociological and historical dimension to the epistemology assumed by the OSA.

> An institution comprises people who are involved in the same problematic situations. Mutual commitment to the same problem entails the realization of shared social practices, which are also linked to the institution to which they contribute (Godino & Batanero, 1994, p. 336).

Problems, which are the origin or motive of mathematical activity, can be extra-mathematical, therefore involving material things, objects, and facts, or intra-mathematical, in which objects of reason, which are non-material, intervene. In educational mathematics, especially at the first level, the starting point is extra-mathematical problems related to the environment and daily life; therefore, the objects that intervene in the practices can be material artifacts and abstractions, both empirical and formal or theoretical.

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[3] The Anthropological Theory of the Didactic developed by Chevallard also proposes a vision of mathematics as a human activity (Chevallard, 1992; 2019).

From an educational point of view, it is important to postulate that the mathematical activity performed to learn mathematics differs from the activity of mathematics professionals through which new knowledge is constructed. In the first case, the learner reconstructs or reinvents knowledge, which already has a historical-cultural existence; in the second case, new postulates are invented, and new relationships derived from previously developed knowledge are discovered.

The personal-institutional duality

The articulation of the epistemic and cognitive facets of mathematical knowledge is achieved in OSA by assigning mathematical practices a double character: personal (individual) or institutional (social). Mathematical practices can be idiosyncratic for individuals or shared within an institution or community of practices. There are no institutions without people, nor are people separated from the various institutions of which they are part (family, school, etc.). The distinction between personal and institutional practices allows the researcher to notice the dialectical relationships between them. On the one hand, people are subject to shared modes of action within the institutions in which they belong. Conversely, institutions are open to the initiative and creativity of their members. With this postulate, the cognitive (psychological) dimension is articulated with the epistemological and sociological dimensions of mathematical knowledge. Mathematics, besides the logical-formal dimension, has another factual dimension that accounts for the processes of creation of mathematical objects, emerging from practices, not as existing Platonic ideals that are discovered. From a personal perspective, mathematical objects have a mental/neuronal existence, while from an institutional point of view, their mode of existence is cultural.

The performance of practices by an individual subject occurs within an ecological context (institutional frameworks or communities of practices) that supports and conditions their performance and, therefore, the appropriation of knowledge by the subject. Therefore, the following Vygotsky postulate of the relationship between ontogenesis and phylogenesis (i.e., between thought and culture) is assumed:

> Any function in the cultural development of the child appears twice or on two levels. It first appears at the social and then psychological levels. It first appears among people as an inter-psychological category and then within child as an intra-psychological category. (Vygotsky, 1978, p. 57)

2.5. Mathematics as a system of objects and processes

Mathematics cannot be understood simply as an activity performed by individuals but as a system of culturally shared objects that emerge from such an activity. Therefore, the ontological problem must be addressed, i.e., to clarify what a mathematical object is, what types of objects are involved in mathematical activity, what is the way of being of mathematical objects, and in what sense mathematics speaks of objects (Parson, 2008). In OSA, mathematical practices, i.e., the actions performed by people in certain types of problem situations, are the origin and *raison d'être* of mathematical abstractions, ideas, or objects (Godino & Batanero, 1994). It is postulated that a mathematical object is any material or immaterial entity that intervenes in mathematical practice, supports, or regulates its realization. This is a metaphorical use of the term object, since a mathematical concept is usually conceived as an ideal or abstract entity, and not as something tangible, such as a rock, a drawing, or a manipulative artifact. This general idea of the object, consistent with that proposed in symbolic interactionism (Blumer, 1969; Cobb & Bauersfeld, 1995), is useful when complemented with a typology of mathematical objects that considers their different roles in mathematical activity.

We understand an ostensive object as a thing, in Bunge's sense, i.e., a real object or entity that exists independently and does not depend on human perception. These things have an objective existence and are not limited by subjective interpretation. A non-ostensive object is a construct, a conceptual entity created by the human mind to represent phenomena or aspects of the world. Constructs are products of cognitive and communicative activity; they are used as tools to understand, organize, and explain experiences. The distinction between constructs and things is essential because it highlights the difference between brain representations, regardless of nature or objective reality, and provides a clear and systematic way in approaching the understanding of the world.

> We pretend that there are constructs, i.e., creations of the human mind that we must distinguish not only from things (e.g., words) but also from individual brain processes. (But we do not assume that constructs exist independently of brain processes.) (Bunge, 2011, p. 154)

In the mathematical world furniture proposed by the OSA to describe mathematical activity, we say that there are ostensive and non-ostensive objects, or what is the same, things and constructs. Words and symbols are things; concepts, for example, of numbers or functions, are constructs. However, what is the relationship between things and mathematical constructs? In Figure 2.1, we show that the duality of ostensive-non-ostensive applies to all primary objects involved in mathematical activity (problems, languages, concepts-definitions, propositions, procedures, and arguments). What is meant by this and what implications does it have?

The number 5, as a construct, does not have an existence independent of the things we use for its expression and manipulation. Sfard (2008) asserted that following Wittgenstein, mathematical constructs are intra-discursive entities created to facilitate thought and action about the world. Through processes of reification and alienation, constructs gain their own life. They become detached from things and from people's actions in relation to them and become part of a virtual, fictitious, metaphorical reality. Thus, we have the numbers 5, 10, 17, ..., and hence the constructs of natural, integer, rational, real, and complex numbers. However, we should not lose sight of the fact that all these constructs come from things, words, symbols, actions of people in situations of the world of things, discourse, and the virtual world formed by previously constructed constructs, which pose new problems of organization and development to increase the efficiency of mathematical work.

Sfard finds it useful to speak of "mathematical object" (i.e., constructs). "My main reason is the hope that this special notion, with its deep metaphorical roots, will help us to understand the evolving connection between mathematical discourses and discourses about material reality" (Sfard, 2008, p. 163).

In addition to concepts such as numbers and functions, other constructs are involved in mathematical activity, such as propositions and statements that can be true or false. For example, 2+3=5 is a true proposition but 4x5=21 is false. To justify that a proposition must be accepted as true requires the elaboration of a valid argument, usually constituted by a routine or procedure of calculation or logical inference.

The three types of argumentations proposed by Peirce (1931-58) (abduction, induction and deduction) play a crucial role in the ontosemiotic model of mathematical activity analysis. In this modeling, a pragmatic theory of argumentation is adopted because it is essential to consider context (justification, discovery, application, education), audience, and communicative goals when constructing and evaluating arguments. Demonstrating mathematical propositions in a professional/academic context typically requires deductive arguments. However, mathematical proofs must involve various arguments in educational settings. Using inductive proofs may be justified at certain moments or educational levels, although always maintaining the perspective of logical rationality, toward which the development of mathematical thinking is directed. The educational context must also attend to the process of mathematical creation, which gives abduction a relevant place in the formation of students.

The propositions, procedures, and arguments involved in mathematical practices are non-ostensive objects. They refer to other constructs, not to properties of things, although they are linked to words and symbols for representation and processing. These representations constitute their ostensive facet.

The six types of primary entities postulated in Figure 2.1 (problems, languages, concept-definitions, propositions, procedures and arguments) extend the traditional distinction between conceptual and procedural entities, considering them insufficient to describe the intervening and emergent objects of mathematical activity. Problems are the origin or *raison d'être* of mathematical activity; language represents the remaining entities and serves as an instrument for operative practices; and arguments justify procedures and propositions that relate concepts to each other. Concepts (number, fraction, derivative, etc.) as components of ontosemiotic configurations are conceived as entities introduced by definitions, a vision different from that proposed by Vergnaud (1990) as a triplet formed by situations, operative invariants, and representations. In OSA, the ontosemiotic configuration construct assumes the notion of a concept as a system. Configurations are organized into more complex entities, such as conceptual systems and theories.

The constitution of objects and relationships, both in their personal and institutional facets, occurs over time through mathematical processes that are interpreted as sequences of practices. The emerging mathematical objects constitute the codified synthesis of such processes. A sequence of practices to solve a type of problem (how to crack a nut with two stones or solve linear equations)[4] is coded as doing the same thing when applied to cases with a

[4] Examples taken from Radford (2015).

certain similarity. It is not just naming diverse things in the same way; it is also synthesizing, guiding, and regulating the way actions should be performed to solve a problem. The final synthesis, through mathematical activity, is produced using specific practices that we call normative. These practices are the product of collective work that codifies efficient approaches to specific tasks. The object of knowledge, the construct in its cultural version, becomes a rule (definition, proposition, procedure, argument) that synthesizes what should be done to approach the solution of a problem, which can be extra-mathematical or intra-mathematical.

The interpretation of mathematical processes as sequences of practices in correspondence with mathematical objects provides criteria for categorizing them. The constitution of linguistic objects, problems, concepts, propositions, procedures, and arguments occurs through the primary mathematical processes of communication, problematization, definition, enunciation, elaboration of procedures (algorithmization, routinization, etc.), and argumentation. Problem solving and modeling should be considered rather as a mega-process because they involve the articulation of primary processes (establishment of connections between objects and generalization of procedures, propositions, and justifications)[5].

The personal-institutional duality also applies to objects and processes. If the systems of practices are shared within an institution (community of practices, culture), the emerging objects are institutional objects, whereas if such systems correspond to a person, they are personal objects. Personal objects include cognitive constructs such as conceptions, schemes, and internal representations although interpreted in pragmatic and discursive terms.

2.6. Mathematics as a system of signs

[5] See Font and Rubio (2017) for an analysis of the notion of process in OSA.

Educational mathematics must address the semiotic-cognitive problem by asking questions like these: What is it to know and understand a mathematical object? What does an object mean for a subject at a particular time and under certain circumstances? These questions are addressed in OSA, considering that mathematical activity and the processes of construction and use of mathematical objects are essentially relational. Different objects are not conceived as isolated entities but are placed in relation to each other. For example, between symbol 2 and the concept of number 2, as well as between the concept of natural number and the system of practices from which this mathematical object emerges, a relationship is established that OSA, following Eco (1991), calls a *semiotic function*. The semiotic function is understood as the correspondence between an antecedent object (expression/signifier) and a consequent object (content/meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. In this way, as developed in Chapter 3, the triadic conception of sign according to Peirce (1931-58) is assumed.

The construct semiotic function, included in Figure 2.1 as the expression-content duality, makes it possible to account for any use given to meaning: meaning is the content of a semiotic function (Godino et al., 2021). In OSA, it is assumed that any entity that participates in a process of semiosis, interpretation, or a language game is an object that can play the role of an expression (signifier), content (signified), or interpreter (rule that relates expression and content). The systems of operative and discursive practices themselves are objects that can be components of semiotic functions. The systemic pragmatic meaning construct of a concept (of any object) can thus be taken as the system of operative and discursive practices performed by a person (personal meaning) or within an institution (institutional meaning) to solve a type of mathematical problem.

As seen more extensively in Chapter 3, the construct semiotic function makes it possible to describe mathematical knowledge in a detailed and operational way as the set of relations (or connections) that a subject (person or institution) establishes between mathematical objects and practices. To speak of knowledge is equivalent to speaking of the content of one (or many) semiotic functions, resulting in a variety of types of knowledge that correspond to the diversity of semiotic functions that can be established between types of practices and objects. Since the systems of practices at play in problem solving are relative to individuals and communities of practice (institutions), pragmatic meanings and, thus, knowledge are relative. However, it is possible to reconstruct the global or holistic meaning of an object by systematically exploring its context and the systems of practices involved in its solution. This holistic meaning is used as an epistemological and cognitive reference model for the partial meanings or senses that an object may adopt (Godino et al., 2021). The constructs of institutional and personal meaning allow us to interpret understanding in terms of the progressive coupling of the subject's personal meanings with the institutional reference meanings (Godino & Batanero, 1994).

The OSA semiotic cognitive approach assumes that the objects placed in correspondence in semiotic functions (functives) are not only ostensive linguistic objects (words, symbols, expressions, diagrams, etc.), but that definitions, propositions, procedures, arguments, and even problems can also be antecedents of semiotic functions. For example, it makes sense, and it is necessary to ask about the meaning of the concept of number, as well as the meaning of propositions, procedures, arguments, situations, and representations involved in numerical practices. Functives in semiotic functions can also be unitary or systemic entities, particular or general, material or immaterial, personal or institutional. A variety of meanings are thus generated, and therefore, of knowledge and understanding, which orients and supports the realization of ontosemiotic analyses of mathematical activity at the macro and micro levels, both from the socio-epistemic (institutional) and cognitive (personal) perspectives (Godino et al., 2021). Thus, cognition, understood in OSA semiotic terms, is not only pragmatist but also empiricist and rationalist. Action is a source of knowledge; however, perception and reason are also sources.

The primary objects of ontosemiotic configurations (Figure 2.1) include language in its different registers (oral, written, gestural, iconic, symbolic) and the corresponding interpersonal and intrapersonal communication processes. Language intervenes in discursive practices to express the remaining primary entities (problems, definitions, propositions, procedures, and arguments) and acts as an instrument for the realization of operative practices. With language, things are said and done; therefore, it has representational and operational valence. Definitions, propositions, and procedures are understood as grammatical rules of languages used to describe problems and support the realization of argumentative practices that justify propositions and procedures. They are intra-discursive entities that do not have an existence independent of language; they emerge from operative and discursive practices. Their metaphorical existence is postulated as objects to distinguish between a rule and its various linguistic formulations, and consequently to discern between the world of thought and culture from the ostensive reality of sounds, images, or material artifacts.

2.7. Idealization, generalization, and unitarization

To account for the processes of idealization, generalization, and unitarization (and their duals, materialization, particularization, decomposition), three pairs of contextual attributes have been introduced into the OSA ontology from which primary practices and objects can be considered: ostensive- non-ostensive (material, immaterial), extensive-intensive (particular-general; exemplar-type), and systemic-unitary (process-object) (Figure 2.1). These constructs allow us to describe the types of abstraction (empirical and formal) that come into play in mathematical activity, as well as the objects that intervene and emerge in these processes. Likewise, they help to understand the interplay between pure and applied mathematics, between constructs and things, which is necessary in educational mathematics because, in the learning processes, at least at earlier levels, it is necessary to start from the tangible reality to access the virtual reality of formal mathematics.

2.7.1. Ostensive-non-ostensive duality

In OSA, an ostensive object is any perceptible object that can be shown directly to others. Symbols, notations, gestures, graphic representations, and material artifacts have that character; they are real or concrete objects. Concepts, propositions, procedures, and arguments are constructs, discursive creations of human activity, i.e., non-ostensive objects; they depend on subjects, their actions, and real artifacts for their existence. Non-ostensive objects can be mental objects (when they intervene in personal practices) or institutional objects (when they intervene in shared practices). However, the interpersonal communication of non-ostensive objects requires that they be materialized through ostensive representations. In both cases, non-ostensive objects regulate mathematical activity, whereas their ostensive representation supports or facilitates the performance of the said work. The distinction between ostensive and non-ostensive objects depends on the language game in which they are played. Ostensive objects can also be thought of, imagined by a subject, or implicit in mathematical discourse (for example, the multiplication sign in algebraic notation). In these cases, they participate as non-ostensive objects. This duality allows us to account for the dual processes of idealization (creation of non-ostensive objects) and materialization (creation of ostensive objects that materialize non-ostensive objects) in mathematical activity. For example, the number five is materialized by displaying the five fingers of the hand as an application. The number five as the class of all equipotent sets for the five fingers of a hand is an idealization.

2.7.2. Unitary-systemic duality

In some circumstances, mathematical objects participate as unitary entities (which are assumed to be previously known), whereas in others, they intervene as systems that must be decomposed for analysis. “The same object can be considered either an individual, a set (or a specific collection). There is nothing definitive about being an individual” (Bunge, 2011, p. 145). For example, in the study of addition and subtraction, in the last levels of primary education, the decimal number system (tens, hundreds, etc.) is considered a known unitary entity that does not need to be deployed in more elementary entities. These same objects in the first course must be considered systemically for learning. Both ontosemiotic configurations (in their socio-epistemic or cognitive version) and the primary objects that compose them can be considered from unitary or systemic perspectives, depending on the language game in which they participate. In the first case, unitarization (synthesis) processes occur, and in the second case, decomposition (analysis) of the system into its components.

The unitary-systemic duality allows us to reformulate the “naïve” view that “there is the same mathematical object (e.g., the arithmetic mean) with different representations”. As Rondero and Font stated,

> A complex system of practices that allows problem solving exists, in which the mathematical object “arithmetic mean” does not appear directly. Instead, what appears are representations of the arithmetic mean, different definitions of the arithmetic mean, its propositions and properties, procedures and techniques applied to the arithmetic mean, and arguments about it. In other words, throughout history, different epistemic configurations have been generated for the study of the arithmetic mean,

> some of which have served to generalize the preexisting ones. (Rondero & Font, 2015, p. 33)

The unitarization process is related to the emergence of new mathematical objects. We assert that regulative mathematical objects (definitions, propositions, procedures, arguments) emerge from the systems of operative and discursive practices. It can also be said that they are operative and discursive invariants. In any case, it starts with structured sets of actions or other objects (systemic entity), which, for reasons of operational or discursive efficiency, form a new unit. We have been calling this process in OSA the unitarization process, the formation of a new object as a unitary entity, which is why it can also be called the objectification process. In some circumstances or language games, the reverse process of considering a systemic entity in terms of its components occurs, which in OSA is called the decomposition process.

Interpreting the unitarization process in terms of Sfard's (2008) discursive theory of thought allows us to obtain an operational and coherent breakdown of the unitary-systemic duality. Sfard (2008) identified three mechanisms (discursive devices) that produce the emergence of new objects (non-ostensive or constructs): assimilation (saming), encapsulation, and reification.

- Assimilation (saming), that is, giving a common name to things that, although apparently unrelated, can be seen in certain contexts as equivalent (this happens, for example, when the term quadratic function is introduced to simultaneously refer to things as different as the expression x^2, a certain curve called a parabola, the set of numbers paired with their squares, etc.).
- Encapsulation, that is, replacing talk about separate objects with talk about a single entity (this occurs when several objects are referred to collectively as a single set; for example, when it is claimed that many ordered pairs of elements constitute a function).

– Reification, that is, converting discourse about a mathematical process into discourse about an object (this is the case, for example, when we replace “When I add 5 to 7, I get 12” with “the sum of 5 and 7 is 12“).

Once a new noun is introduced in one or more of these three ways, alienation from the new object gradually occurs: the noun ends up being used in impersonal narratives, implying that its referent exists independently of discourse. The discursive construct thus created becomes the object of mathematical explorations through which new mathematical narratives emerge. Alienation, as a complementary aspect of reification, ends the process of objectification, i.e., the emergence of the object by being completely dissociated from the actor.

With the last traces of people’s agency carefully erased, even the most common arithmetic propositions, such as the phrase “two plus three equals five” convey the message of the mind-independent existence of the mathematical object. Once reified and put into impersonal phrases, numbers have a “life of their own” (Sfard, 2008, p. 50).

The descriptions of the processes of assimilation, encapsulation, and reification demonstrate that, in all three cases, the process of unitarization occurs, thus constituting a breakdown of the process.

2.7.3. Extensive-intensive duality

A characteristic feature of mathematical activity is the generalization of problems, solution procedures, definitions, propositions, and justifications. Solutions are organized and justified in progressively more general structures. However, in instructional processes, the study begins with particular models of these general structures, although they gradually become more generalized as the study advances. The analysis of mathematical activity therefore requires consideration of both processes: particularization and generalization, and the objects involved in these processes. The generalization process involves finding or conjecturing a pattern or regularity in similar cases, whereas particularization involves generating or presenting individual examples that follow a pattern.

In OSA, the contextual attribute extensive-intensive (exemplar-type) has been introduced, applicable to primary practices and objects, to analyze the dialectic between particularization and generalization. Depending on the situation being worked on, an object can be an exemplar (extensive) if it intervenes by itself or a type (intensive) if it intervenes as a representative of a broader class.

> An extensive object is used as a particular case (a specific example, e.g., the function y = 2x + 1) of a more general class (e.g., the family of functions y = mx + n), which is an intensive object. The terms extensive and intensive are suggested by two ways of defining a set: by extension (an extensive is one of the members of the set) and by intension (all the elements are considered at once). By extensive, we mean a particularized (individualized) object, and by intensive, a class or set of objects. (Font et al., 2008, p. 169)

For example, Font and Contreras (2008) conducted a microscopic analysis of the objects, processes, and semiotic functions involved in defining the concept of the derivative of a function. The application of the ostensive-non-ostensive, extensive-intensive, expression-content duality enables the explanation of semiotic conflicts posed by the dialectic between the particular and the general in mathematics education.

2.8. Abstraction processes and abstract objects in OSA

In a first approximation, we can say that the ostensive-non-ostensive duality and its associated processes of materialization and idealization account for the concrete (ostensive) and abstract (ideal) objects usually considered in everyday language. However, the analysis of mathematical activity, from both a professional and educational point of view, requires a deeper understanding of the nature of abstraction and emerging abstract objects, as well as the reverse process of interpretation. For this reason, the

OSA proposes to complement the ostensive-non-ostensive duality with the unitary-systemic and exemplar-type duality. Thus, the abstract mathematical object is not only an ideal entity (non-ostensive) but also a generality that can be considered a unitary whole or a system, depending on the language game in which it participates. Because unitary objects are symbolically represented to intervene in new systems of practices, the abstraction process also involves the expression-content duality and the processes of representation and meaning.

Mathematical abstraction involves various facets, includes different components, and occurs at different levels (Figure 2.2). It can be seen as forms of the subjective act of knowing and as a characteristic of knowledge as a historical and objective product of collective activity (personal and institutional dimension). Sinaceur asserts from a philosophical perspective, "Mathematical abstraction is a multifaceted and multilevel process that leads to a sophisticated and branched hierarchy of mathematical concepts and operations". (Sinaceur, 2014, p. 100)

Abstraction is a gradable characteristic of objects, a matter of more or less, rather than presence or absence. The gradation of abstraction begins with the direct categorization of perceptive objects and continues to rise to increasingly higher levels of abstract objects. Concept F can be more abstract (intensive) than concept G, which can be abstract but less abstract, more concrete (extensive) than F.

For example, in the epistemic analysis of the concept of function, one must identify, besides the definitions that have been used, the different elements represented in Figure 2.1, which are mobilized to respond to the problems in which the function object participates determinately, although it may be implicit in the early stages of its emergence. Each configuration is interpreted as a partial pragmatic meaning of a function object that reflects and synthesizes the mathematical activity performed to solve specific problems in certain contexts or historical stages. The evolution of the concept implies a sequence of progressively more abstract configurations through which definitions, procedures, properties, and arguments are generalized, moving from the use of ordinary, tabular, and graphical language to alphanumeric language and from arithmetic to algebraic and analytical calculation.

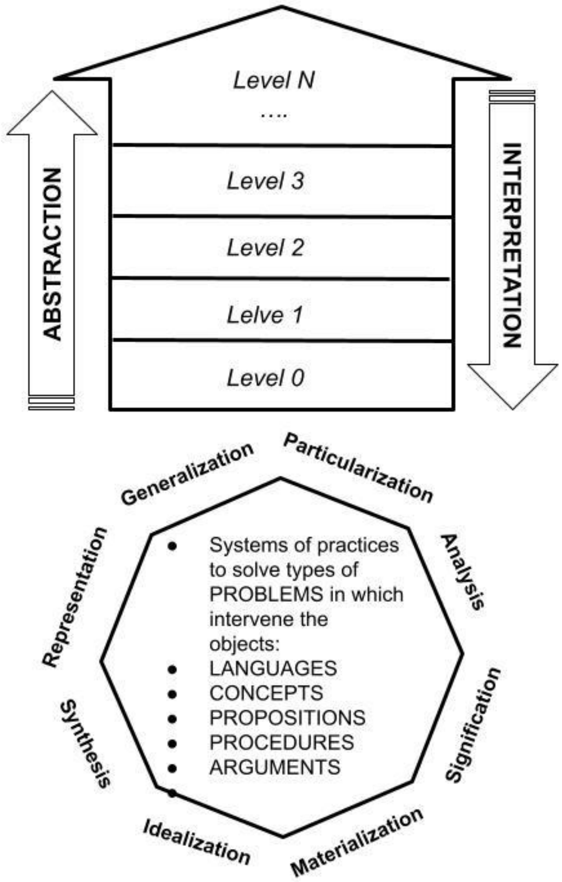

Figure 2.2. Facets, components, and levels of abstraction in mathematics

The creation of progressively more abstract mathematical objects is based on the natural tendency of mathematicians and scientists to organize their experiences with the help of unifying patterns and structures to produce useful generalizations. The construct of abstraction cannot be avoided in the analysis of mathematical activity from both a philosophical and educational perspective. Attempting to organize learning processes by imposing highly abstract mathematics from the outset without considering prior levels and particularization and contextualization processes is doomed to failure.

Next, we analyze the statement and solution of a sequence of problems using the ontosemiotic configuration tool, identifying the abstraction processes that occur and assigning degrees of intension (or generality) to the emerging objects. The different levels of abstraction depend on the degree of generality. There are no separate levels of abstraction and degrees or layers of generalization; instead, the levels of abstraction are inherited from the degrees of generality of the objects.

Examples of abstraction processes and their levels

Problem 1: Concrete number and abstract number (first level of abstraction)

Statement: How many apples are there in the figure? Are there more or fewer cherries than apples? Justify your answer.

Solution:

We say there are three apples and three cherries. There are the same number of apples as cherries. I counted the number of apples by matching each apple with the series of number words one, two, three, etc., and the last word was three apples. The same for the cherries. Therefore, there are the same number of apples as cherries.

Ontosemiotic analysis:

In the situation posed as problem 1, and the operative and discursive practices accompanying it as a solution, various types of objects and relationships are involved. The green, yellow, and red apples are shown. Through empirical abstraction, color and size are ignored, and it is said that there are three apples. Their position in the figure is also ignored. The same applies to the cherries.

The solution mentions the series of number words one, two, three, etc., suggesting that it is an unlimited series. The first three terms of this series are used for counting, a procedure in which each number word is matched one-to-one with each material (represented) object while respecting the principles of counting. Three apples and three cherries correspond to "concrete numbers."

The ostensive, perceptible nature of the objects being counted leads us to propose that the level of abstraction of the concepts involved is 0 (apples, cherries, three apples, three cherries). The transition from level 0 of generality to level 1 occurs when empirical abstraction (i.e., ignoring certain attributes and focusing on others) occurs.

Conversely, the nature of the series of number words, one, two, three, etc. (which can be replaced by other words, one, two, three, etc., or symbols 1, 2, 3, ...), the numeral system, is entirely different. Any of these systems refers to a set of natural numbers whose nature essentially differs from empirical objects, although some materialization is required for mental processing or interpersonal communication.

The number 3 is neither more nor less than the one preceded by 2 and 1 (and, if applicable, 0), followed by 4, 5, etc. Or, more precisely, it is an object preceded by two (or three) objects in a pre-established order and followed by infinitely many others, also ordered, such that any two elements defined as "contiguous" will always be so. Any object can play the role of 3; any object can be the third element in some progression (arbitrarily pre-established). What is peculiar to 3 is that it defines that role—not by being a paradigm of any object that plays it but by representing the relationship that any third member of a progression has with the rest of the progression (Godino et al., 2009, p. 42).

The concept of number arises from a formal or theoretical abstraction specific to mathematics that is fixed through a system of axioms. In OSA, natural numbers are abstract objects with a degree 1 of intension or generality because degree 0 corresponds to perceptible objects. This is so even for a particular number, e.g., 3, which is an abstract, albeit particular, object. The duality particular-general (extensive-intensive) is relative to the language game in which the object participates.

Problem 2 (Unlimited sequences of numbers through recurrence laws)

Starting from the unlimited sequence of natural numbers, 1, 2, 3, ..., $x_n = x_{(n-1)+1}$. In the sequence of even numbers, what is the even number corresponding to position 10? In the generic position *n*?

Solution: The even number corresponding to position 10 is 20 (2, 4, 6, 8, 10, ... 20). The even number *y* corresponding to position *n* is *y=2n* because for any natural number, its image in the correspondence can be obtained by multiplying it by 2.

Ontosemiotic analysis:

The unlimited sequence of natural numbers, considered as a unitary whole, is a new object at level 2 of abstraction. The transition from a finite collection to an infinite one, with its corresponding formation rule (intensive object), is a process that produces new, higher-level abstract objects. The formation rule is to add 1 to the previous one (recurrence rule).

Instead of adding 1, we can consider adding 2, 3, etc. (multiples of 2, 3, ...). In this way, we obtain unlimited collections of multiples of 2, 3, and so on. Each collection is a new abstract object given by the function *y=2n*; *y=3n*; etc. Each of these unlimited collections, defined by their respective functional rules, is a new abstract object of level 2, similar to the sequence of natural numbers. Consequently, the function defined in this case between the set *N* of naturals, with an image also in *N*, and the formula *y=2n* is a mathematical object with abstraction level 2 (intensive object of the 2^{nd} kind).

We can generate an object with a higher level of abstraction (generality) by considering the collection of collections of multiples of any number *a*. This collection is given by the following parametric rule: $y=an,\ a \in N,\ n \in N$. It seems reasonable to assign this object level 3 of abstraction.

The function defined between arbitrary numerical sets is an intensive object at level 4: $y = f(x), x \in C, f(x) \in C'$, where C and C' are arbitrary numerical sets.

The function defined between any sets is an intensive object at level 5: $(y \sim f(x), A, B), x \in A, f(x) \in B$, where A and B denote any sets.

It is observed that new ideal (non-ostensive) objects can be generated horizontally without increasing the level of abstraction. For example, finite collections of natural numbers (1, 2, 3, 4, 5) or multiples of a finite number (pairs, trios, ...) are abstract objects but not more abstract (general) than the objects that constitute them.

2.9. Theoretical frameworks related to the ontosemiotic theory of mathematical activity

The foundations of mathematics education described in Sections 2.4 to 2.7 are being used to develop tools that address issues related to the design, implementation, and evaluation of mathematical instruction processes. To consider issues related to the analysis of implementing instructional processes, the didactic configuration tool was developed (Godino et al., 2006). The theory of didactic suitability (Godino et al., 2023) examines issues concerning the evaluation of instructional processes and teacher education. All these tools are based on the ontosemiotic modeling of mathematical activity, the emerging objects described in this chapter, and the meaning and mathematical cognition developed in Chapter 3.

In this section, we examine the concordances and complementarities of the Theory of mathematical activity and emerging objects with the Discursive theory of thinking (Sfard), the Objectification theory (Radford), and related theoretical frameworks.

2.9.1. The discursive theory of thinking

The commognition theory (an acronym combining communication and cognition) developed by Sfard (2008) models individual thinking in discursive terms, drawing on Wittgenstein's postulates of language, thought, and meaning as use. Sfard challenges Platonic-idealist and mentalist views of thinking in general and mathematical thinking in particular. Mathematical objects should not be considered entities to be discovered or inaccessible mental entities; propositions are not descriptions of the properties of such objects. Instead, she proposes that understanding mathematical objects can be viewed as discursive entities emerging from interpersonal or self-communication. Sfard's central thesis is that mathematical discourse and its objects are mutually constitutive:

> Discursive activity, including the continuous production of symbols, creates the need for mathematical objects and mathematical objects (or better, the use of symbols mediated by the objects) in turn influence discourse and guide it in new directions (Sfard, 2000, p. 47).

Sfard's motivation to develop a discursive theory of thinking stems from the controversies surrounding the objectification of discourses intended to investigate cognitive processes. In these processes, it is as if there is a reality (in the mind) that these processes describe. However, the abstract, artificial nature of the objects being discussed, which are not effectively existing and can thus be disposed of, goes unnoticed. When describing what a subject does and says, we do not need to use constructs such as cognitive schemas, conceptions, intentions, and meanings, which are intangible and invisible.

Sfard identified various dilemmas and dangers in using the object metaphor to describe thought and its use in mathematics education. The reification and alienation of processes through which the ideal or abstract object is constructed, essential for human thought and communication, obscure its origin in discursive practices. The development of the commognition model is a proposal to keep the object metaphor alive while avoiding its pitfalls, i.e., the confusion it introduces into the study of knowledge, understanding, and learning.

The OSA Theory of mathematical activity addresses this issue but adds Peirce's pragmatic dimension to Wittgenstein's linguistic perspective and Vygotsky's discursive perspective. The strategy followed in the OSA to avoid the idealist or mentalist view of non-ostensive mathematical objects is to associate as the meaning of these objects the system of operative and discursive practices for solving problems from which such objects originate. Language, words, symbols, and discursive practices are considered primary entities, but situational problems and the operative and normative practices involved in mathematical activity are also given a central role.

The personal-institutional duality for practices, objects, and processes proposed by the OSA allows for a broader view of cognition than commognition, as cognition can be understood not only as an individual reality but also as a community and historical-cultural reality. It also accounts for the interdependent relationships between individual and community discourse in which the subject participates (i.e., the process of progressive coupling, through dialogical participation, of personal meanings with intended institutional ones. Additionally, the exemplar-type duality, with its recursive nature, accounts for processes of generalization and particularization and successive levels of abstraction, i.e., the creation of progressively more complex objects.

2.9.2. Theory of objectification

Radford (2008; 2015) developed a specific view on teaching and learning that considers not only the knowledge at play but also the formation of students as human subjects. This political-conceptual position is known as the Theory of Objectification (TO), which posits the goal of mathematics education as a political, social, historical, and cultural effort aimed at creating ethical and reflective individuals who critically engage in historically and culturally constituted mathematical practices. An essential principle of objectification theory is the idea of labor or work, as given by Hegel, Marx, Leóntiev, and dialectical materialism. Through labor or work, individuals continuously develop and transform, encountering the systems of ideas of culture (scientific, legal, artistic ideas, etc.).

TO assumes that thought is primarily an active reflection of the world mediated by artifacts, the body (through perception, gestures, movements, etc.), language, signs, etc. The notion of an object is essential in TO because "Thinking, indeed, is thinking about something. Thinking and that something which is the object of thought are intertwined and inseparable" (Radford, 2015, p.130). Knowledge objects are sociohistorical and cultural entities, not mental entities.

In fact, they are the result of social labor and are produced through it. In more precise terms, knowledge objects are a culturally and historically codified synthesis of doing, thinking, and relating to others and the world (Radford, 2015, p. 134).

Radford distinguishes between knowledge objects, culturally codified forms of doing and thinking, and concepts, which are particular realizations a subject makes of cultural objects through the process of objectification through school learning. "Learning arises from the sensual and conceptual awareness that results from the realization of the knowledge object (e.g., cracking nuts, solving linear equations) in its concrete realization or individualization" (Radford, 2015, p. 139). The concept is constituted from what has truly become an object of awareness for students during their joint work with teachers: the sensual and real forms of thinking and doing as encountered and known by the students.

> Objectification is the social, co-transformative, and sensual process of meaning-making through which students gradually become critically

> familiar with historically constituted cultural meanings and forms of thinking and acting (p. 139).

Through objectification processes embedded in the activity, cultural knowledge objects can become objects of awareness and thought.

In TO, as in OSA, certain epistemological and ontological principles about mathematical knowledge and learning are assumed, and these principles are shared by sociocultural approaches:

> p1: knowledge is historically generated during an individual's mathematical activity.
>
> p2: the production of knowledge is not responsive to adaptive piloting but is immersed in cultural forms of thought intertwined with a symbolic and material reality that provides the basis for interpreting, understanding and transforming the world of individuals and the concepts and ideas formed from them (Radford, 2018, pp. 4066-4067).

The postulate of considering mathematics as both a human activity and work with a social component underlies the emergence of knowledge objects or constructs (abstract or general entities) in TO and OSA. The process of objectification is equivalent in cognitive and educational terms to the process of students' personalization of institutional/cultural meanings, as proposed by OSA. The consideration of conceptual objects in their unitary version as socially agreed rules referring to how languages and artifacts should be used helps to understand the two sources of learning proposed by TO: contact with the material world, the world of cultural artifacts surrounding us (objects, instruments, etc.), and social interaction. Socially agreed rules for the use of artifacts must be learned.

The epistemological model proposed by OSA is broadly consistent with that corresponding to TO. Both theories share similar anthropological assumptions about mathematical activity and emerging sociocultural processes and products. However, OSA explicitly incorporates the basic elements of the linguistic turn introduced by Wittgenstein in the philosophy of mathematics and the contributions of Peircean semiotics to explain mathematical communication and interpretation processes. Anthropological and sociocultural changes in the way of conceiving mathematics assumed by both theories have not led to the neglect of the cognitive dimension, i.e., the role of the subject who learns mathematics and forms as a person. For this reason, OSA introduces, alongside a model of institutional cognition, another model of individual cognition, constructed on its same pragmatic, anthropological, and semiotic bases.

A notable difference between the two theories lies in the unequal development of the institutional/cultural, and personal dimensions of mathematical knowledge. TO preferentially focused on the cognitive (personal) dimension and learning processes, whereas OSA developed tools that address both institutional and personal knowledge dimensions. The ontosemiotic configuration tool (in its dual epistemic and cognitive versions) allows for a detailed analysis of mathematical activity and the objects involved, which are not reduced to conceptual or abstract objects. Recognizing the complex web of objects and processes involved in problem solving is an explanatory factor for learning and teaching difficulties and a necessary step for the proper management of educational-instructional processes. An *a priori* analysis of solutions to activities and the recognition of practices, objects, and processes is an epistemic or institutional analysis that refers to an epistemic or ideal subject. This a priori analysis helps understand and manage individual subject learning processes. This learning can be described through the ontosemiotic configuration tool in its cognitive version and applied to the students' responses and dialogs.

2.9.3. Other related frameworks

Various authors have developed constructs and theories to respond to the epistemological, ontological, and semiotic cognitive problems described in this chapter, which are specific to educational mathematics. The concordances and complementarities of the OSA model on these issues have been addressed in previous studies. In particular, the OSA is confronted with the Anthropological Theory of Didactic (Chevallard, 1992) by D'Amore and Godino (2007), the APOS theory (Dubinsky & McDonald, 2001) by Font et al. (2015), and the Theory of semiotic representation registers (Duval, 1995) by Godino et al. (2016). Manolino et al. (2023) compared OSA with the Theory of Semiotic Bundles (Arzarello et al., 2009).

2.10. Examples of applying the ontosemiotic theory of mathematical activity

The analysis of practices, objects, and processes that characterize mathematical activity has been applied in various studies as a component of didactic analysis. For instance, Rondero and Font (2015) used ontosemiotic configurations to study the complexity of the arithmetic mean. They develop an integrated view of the articulations of objects and configurations (levels of generalization, metaphorical projections, and networks of semiotic functions) through which a unitary vision of the arithmetic mean can be constructed. Molina et al. (2019) identified ontosemiotic configurations in processes of argumentation by analogy, complementing the analysis of argumentation according to Toulmin's model. This model highlights and describes the abductive and analogical arguments produced by students, while the ontosemiotic configuration tool concretizes how the mathematical objects activated in practices were articulated by the associated argumentative processes. Burgos et al. (2021) performed a microscopic analysis of two presentations of the definite integral, one informal/intuitive and the other formal, aimed at explicitly recognizing the objects involved and emerging in the corresponding mathematical practices. They also identify the processes (interpretation/signification, representation, argumentation, generalization, etc.) involved in these practices, considering that such analyses help explain learning difficulties and support informed decision-making regarding the teaching of the definite integral.

In the following sections, we include a broader synthesis of two works that use the ontosemiotic theory of mathematical activity, one characterizing visualization in mathematics education and the other developing a model of algebraic reasoning levels.

2.10.1. An ontosemiotic approach to visualization in mathematics education

Visualization has received much attention as a research topic in mathematics education (Bishop, 1989; Clements, 2014; Rivera, 2011), especially in geometry, where it has focused on evaluating individuals' processes and capacities to perform tasks requiring the mental "seeing" or "imagining" of spatial geometric objects. Interest in this topic is also seen from the perspective of the mathematician's own work, addressing problem-solving, conjecture formulation, and in areas other than geometry (Guzmán, 1996). Presmeg (2006), among others, posed the following questions related to visualization in mathematics education: "How can visualization be used to promote mathematical abstraction and generalization? ... What is the structure and components of a general visualization theory for mathematics education?". (Presmeg, 2006, p. 227)

In the study by Godino et al. (2012), we developed an analysis model on the nature and components of visualization and its relationship with other processes involved in mathematical activity, using OSA to understand the network of mathematical objects involved in visualization processes. In this study, we consider that a key aspect of developing a theory of visualization must include the study of its relationships with other modes of ostensive expression (analytical or sequential languages) and, above all, its relationship with non-ostensive mathematical objects (whether mental, formal, or ideal).

The analysis of mathematical activity and objects and processes initially focuses on the practices performed by individuals engaged in solving specific mathematical problem-situations. This visualization analysis allows us to distinguish between visual, non-visual, symbolic, and analytical practices. We focus on the linguistic objects and visual artifacts involved in a practice as they engage with visual perception. Although symbolic representations (natural language or formal languages) are visible, we do not consider such inscriptions visual but as analytical or sentential.

In Godino et al. (2012), visualization is analyzed first from the perspective of the primary objects involved, i.e., the problem situations (tasks), linguistic and material elements, concepts, propositions, procedures, and arguments. Visual objects usually participate in mathematical practices along with other non-visual (analytical or other) objects. Visualization in mathematics is not limited to seeing; it encompasses interpretation, action, and relationships. Second, visualization is analyzed by applying contextual dualities or modalities from which previously identified visual objects can be considered. Distinctions are made between personal (cognitive) and institutional (socio-epistemic) visual objects; particular (extensive) and general (intensive) objects; ostensive (material) and non-ostensive (mental, ideal, immaterial) objects; unitary (used as a whole) and systemic (formed by a system of structured elements) objects. Finally, visual objects are considered antecedents or consequents of semiotic functions (expression and content duality).

The developed visualization model is applied to the analysis of two problems: 1) algebraic and visual demonstration that the sum of the first n odd numbers is n^2; 2) demonstration that the sum of the interior angles of a triangle is a straight angle. The analysis focuses on the network of visual and non-visual objects and the relationships established between them, i.e., the semiotic system they form. In summary, it reveals the knowledge applied to problem solving and the synergy established between visual and analytical objects.

The application of the ostensive-non-ostensive duality to different primary mathematical objects (problems, languages, concepts, propositions, procedures, and arguments) provides a new perspective on the role of visualization in mathematical practice. Initially, Peirce's distinction of sign types is assumed to differentiate between visual languages, which are characterized by indices, icons, and diagrams, and analytical languages, based on the use of symbols. Subsequently, visual problems/tasks are distinguished from non-visual or analytical ones; the former refers to situations involving objects from the sensible world (physical bodies, spatial relationships, and visual representations), while the latter essentially involve logical, numerical, and analytical entities. These distinctions also apply to other primary entities (rules and justifications).

It is concluded that visualization permeates all branches of mathematics (not just geometry) in coordination with other forms of expression, particularly analytical/sequential languages. It is also present at various levels of mathematical study, from elementary to higher education or professional work. However, the role of visualization in mathematical work, whether professional or educational, is complex, as it is often intertwined with the use of symbolic inscriptions that, although "seen", have purely conventional significance. The problem is relevant even when visualization refers to the use of visual objects, which interact not only with symbolic inscriptions but also with the network of conceptual, procedural, propositional, and argumentative objects involved in the corresponding ontosemiotic configurations.

2.10.2. Development of an algebraic reasoning level model

As we will see in Chapters 3 to 6, the ontosemiotic theory of mathematical activity underpins the remaining theories that make up OSA: theory of meanings and mathematical cognition, theory of educational-instructional design, theory of didactic suitability, and theory of teacher professional development. Assumptions about mathematical activity, types of mathematical objects, and processes have enabled the development of a model of levels of elementary algebraic reasoning (Godino et al., 2014; 2015), which we refer to below as an example of the application of the ontosemiotic configuration tool. In addition to the degree of generality of the objects involved in mathematical practices, the languages used (natural, gestural, symbolic) and types of calculations performed with the represented objects are also considered. Thus, the model of levels of abstraction described in section 2.8 is expanded to algebraic reasoning.

A mathematical practice is algebraic if it involves certain types of objects and processes usually considered "algebraic" in the literature. The following are types of algebraic objects:

1) Binary relations—equivalence or order—and their respective properties (reflexive, transitive, and symmetric or antisymmetric). These relations are used to define new mathematical concepts.
2) Operations and their properties are performed on elements of various sets of different objects (numbers, geometric transformations, etc.). Algebraic calculations are characterized by applying various properties, such as associative, commutative, distributive, existence of identity elements, and inverses. Other concepts like equation, inequality, and unknowns, as well as procedures like elimination, transposition of terms, factorization, and expansion of terms, may also be involved.
3) Functions. It is necessary to consider the different functions and algebra associated with them, i.e., operations and their properties. The various objects involved (functions, variables, formulas, parameters, etc.) and their different representations (tabular, graphical, formulaic, analytical) must be distinguished.

4) Structures, their types, and properties (semigroup, monoid, semimodule, group, module, ring, field, vector space, etc.), characteristic of higher or abstract algebra.

In algebraic practice, particularization-generalization processes are especially important because generalization is a characteristic feature of algebraic reasoning (Carraher et al., 2008; Cooper & Warren, 2008; Mason & Pimm, 1984). Thus, to analyze levels of algebraization in mathematical activity, it is useful to focus on objects resulting from generalization and the dual process of particularization. A generalization process results in a type of mathematical object called an intensive object in OSA, which is essentially a rule that generates the class, type, or generality involved. Through the inverse process of particularization, extensive objects (i.e., particular objects) can be obtained. An intensive object can be considered a rule that generates the elements comprising a collection or set, whether finite or infinite. A finite collection of enumerated data is not considered intensive until a criterion or rule is applied to delimit the constituent elements of the set. Then, the set becomes something new, different from its constituent elements, emerging as a unitary entity from the system. Therefore, besides generalizing the set, a unitarization process is also possible.

The new unitary entity must be made ostensive or materialized through a name, icon, gesture, or symbol so that it can participate in other practices, processes, and operations. The ostensive object that materializes a unitary object emerging from generalization is another object that refers to the new intensive entity and thus involves a representation process accompanying generalization and materialization. Finally, the symbol detaches from the referents it represents/substitutes becoming an object to be acted upon. These symbol-objects form new sets on which operations, properties, and structures are defined, i.e., they are syntactically, analytically, or formally operated.

The triple process of recognizing or inferring generality, unitarization, and materialization allows us to define two primary levels of algebraic thinking that are distinguishable from a more advanced level, where the intensive object is a new entity represented by an alphanumeric language. We refer to Godino et al. (2014), who described the criteria for discriminating among these three levels of algebraization in elementary mathematics education. The use and treatment of parameters are the criteria used to delineate higher levels of algebraization because they are linked to families of equations and functions, thus implying new "layers" or degrees of generality (Radford, 2011). The first encounter with parameters is associated with a fourth level of algebraization, and performing combined treatments of parameters and variables is associated with a fifth level. Studying specific algebraic structures led to the recognition of the sixth level of algebraization in mathematical activity (Godino et al., 2015).

2.11. Synthesis of the ontosemiotic theory of mathematical activity

To summarize what was stated in the chapter, Table 2.2 includes a synthesis of the theory of mathematical activity presented in this chapter, following the (adapted) guide for the description of theories proposed by Michie et al. (2014) for the field of social and behavioral sciences.

Table 2.2. Synthesis of the ontosemiotic theory of mathematical activity

Elements	**Description**
Summary. What is the theory about and what are its main propositions?	The ontosemiotic theory of mathematical activity provides theoretical assumptions and tools for analyzing both professional and educational mathematical activities, as well as the objects involved and emerging from them. It offers a unique perspective on the emergence of mathematical knowledge tailored to the educational context, with transdisciplinary traits, by addressing dilemmas in epistemological and ontological theories involved in mathematics education. This perspective complements the logical-formal view, which is typical of the contexts of creation and justification of mathematical

	knowledge, with the empiricist-factual view linked to the contexts of application. The postulates of this theory are as follows: – Mathematics is a human activity that involves solving certain classes of problem situations. – Mathematical practices can be idiosyncratic to individuals or shared within institutions. – Problem-solving involves articulating sequences of practices. – Various classes of objects play different roles in mathematical practices: instrumental/representational, regulatory (fixation of rules on practices), explanatory, and justificative.
Scope/Objective. What phenomena does the theory explain?	The objective of this theory is to understand the nature of mathematics, its relationship with human activity, the various types of objects and processes emerging from this activity, and the relational nature of mathematical knowledge, both professional and educational.
Justification. Why is this theory necessary and how does it improve on previous theories?	The theory emerges by considering the diversity of philosophical approaches to the nature of mathematics and the contradictions and dilemmas among different theories and approaches. It is deemed possible and necessary to develop a coherent epistemological and ontological model of mathematics that serves as a foundation for mathematics education.
Hypotheses. What specific hypotheses does the proposed theory propose, and how do they differ from other theories?	It starts with the postulate that mathematical objects emerge from human activity when solving types of problems (anthropological postulate). Mathematical objects can be categorized according to the role they play (functional entities). Practices and objects can be viewed from five pairs of contextual dualities: personal-institutional, expression-content, ostensive-non-ostensive, particular-general, and unitary-systemic.
Constructs. What elements constitute the theory?	The theoretical constructs that constitute this theory are: mathematical practices, mathematical objects and processes, and contextual attributes of practices and objects. These theoretical constructs are articulated in the ontosemiotic configuration tool, which coordinates three complementary perspectives on mathematics: human activity, a system of objects, and a system of signs.
Relations. How are the elements of the theory related to each other?	The construct of the ontosemiotic configuration of practices, objects, and processes, including five pairs of dualities or contextual attributes from which these elements can be considered, reflects the relationships among the various components of the theory.
Origin. On which theories is it based, and how?	From an epistemological standpoint, the theory is based on Wittgenstein's anthropological perspective on mathematics, viewing mathematics as a human activity and attributing a conventional and regulatory nature to mathematical concepts

	and propositions. It is also based on Peirce's pragmatic approach to semiotics and Vygotsky's historical-cultural perspective on cognition.
Similarity. Which theories are most similar to this theory?	In mathematics education, the theory shares some assumptions with the Anthropological Theory in Didactics (Chevallard), the Theory of Objectification (Radford), and the Theory of Communication and Cognition (Sfard).
Complementarity. With which theories can it be complemented?	This theory includes the principles and methodological tools necessary to underpin the educational-instructional processes of mathematics in the epistemological and ontological dimensions. It should be complemented with an explicit theory of meaning and mathematical cognition (developed in Chapter 3) and with theories of educational design and professional development of teachers (Chapters 4 and 5).
Operationalization. How are the constructs measured or identified?	The constructs of the theory are unmeasurable traits. These are descriptive categories of the different types of practices, objects, and processes involved in mathematical activity. The institutional genesis of mathematical knowledge is investigated through 1) the identification and categorization of problem situations that require a response; 2) the description of the sequences of practices involved in the resolution; 3) the identification of the objects involved and their relationships.
Uses. What can the theory be used for?	This theory provides a foundation for the educational-instructional processes of mathematics. This approach allows for detailed analyses of mathematical activity from both personal (cognitive) and institutional (cultural) perspectives and allows understanding of the complexity of objects and processes involved in problem-solving. This theory is used as the foundation for the theory of meaning and mathematical cognition described in Chapter 3 and educational theories included in Chapters 4, 5, and 6.

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Chapter 3

Ontosemiotic theory of meaning and mathematical cognition

Introduction

The term “meaning”, closely linked to “understanding”, is insistently used in mathematical education research and practice because it is crucial for students to understand the meaning of mathematical terms, expressions, and representations, i.e., to understand what mathematical language refers to in its various registers. The importance of semiosis in mathematics education lies in its use of signs, which are omnipresent across all branches of mathematics. This is necessarily the case because the objects of mathematics are ideal in nature; to represent and work with them, it is necessary to employ sign vehicles, which are not the mathematical objects themselves but represent them in some manner (Presmeg et al., 2018).

The terms meaning and sense are persistently used in curricula related to understanding mathematics. In Principles and Standards (NCTM, 2000), the standard “understand the meanings of operations and how they relate to one another” was included in all grades from P-K2 to 9-12. It was related to the meaning of concepts and operations such as numbers, numerals, fractions, equal signs, addition, multiplication; the meanings and uses of variables, equations, inequalities, and relations; the meaning of equivalent forms of expressions, similarity, etc. The notion of sense also played an important role in the NCTM (2000), where it was used synonymously with meaning in expressions such as “Develop a sense of whole numbers”; “Make sense of mathematical ideas"; "Mathematics should make sense to students," etc. The Spanish mathematics curriculum (MEFP, 2022) is also structured around the concept of mathematical sense and is organized into two dimensions: cognitive and affective. Senses are understood as a set of skills related to mastery of numerical, metric, geometric, algebraic, stochastic, and socio-affective content.

An author who considers the idea of meaning as fundamental to mathematics education is Sierpinska (1990), who intimately relates it to understanding:

> Understanding the concept will then be conceived as the act of grasping this meaning. This act will probably be an act of generalization and synthesis of meanings related to particular elements of the "structure" of the concept (the "structure" being the net of senses of the sentences we have considered). These particular meanings also have to be grasped in acts of understanding (Sierpinska, 1990, p. 27).

However, the term "meaning" "is one of the most ambiguous and controversial in the theory of language" (Ullmann, 1962, p. 62). For example, Speaks (2014, p. 1) suggests that: "The term 'theory of meaning' has figured, one way or another, in a great many philosophical disputes over the last century. Unfortunately, this term has also been used to mean a great many different things". In their classic text *The Meaning of Meaning*, Ogden and Richards (1923) compiled 17 definitions of meaning to which new uses, whether implicit or explicit, have since been added, thus increasing ambiguity. In the case of mathematics education, Pimm (1995) also notes the lack of clarity in using the terms understanding and meaning: "What we variously understand by 'understanding' and mean by 'meaning' is far from obvious or clear, despite these being two central terms in any discussion of the learning and teaching of mathematics at whatever level" (Pimm, 1995, p. 3).

The complexity of semantic linguistic problems increases with mathematics because of the variety of semiotic registers (ordinary language, oral and written, specific symbols, graphs and tables, material objects, etc.) used in mathematical practice. We are not only interested in analyzing the meaning of mathematical linguistic elements but also in the various objects involved in the mathematical practices that individuals engage in when solving problem situations (languages, concepts, procedures, propositions, arguments). These objects require competent interpretation and use by teachers and researchers when they are concerned with teaching and learning. The question remains whether it is possible to develop a specific theory of meaning for mathematics education. This theory should consider both realistic/referential and pragmatic/operational positions on meaning and serve as a basis for addressing the epistemological, semiotic, cognitive, and sociocultural problems involved in the processes of teaching and learning mathematics.

The aim of this chapter is to systematize and delve into the characteristics of the theory of meaning proposed by OSA and its use in developing a theory of mathematical knowledge. We also describe the general theories of meaning in linguistics, semiotics, and philosophy that underpin OSA, particularly those of Hjelmslev (1943), Peirce (1931-58), and Wittgenstein (1953; 1956), as well as the agreements and complementarities with three semiotic models that have some impact on mathematics education: Frege (1891; 1892), Vergnaud (1990; 2009), and Steinbring (1997; 2006). In this way, we provide an initial response to the problem of clarifying and comparing the semiotic theories used in mathematics education.

In Section 3.1, we present a synthesis of general theories on meaning, with an emphasis on three authors: Hjelmslev, Peirce, and Wittgenstein, who were selected because the ontosemiotic perspective on meaning takes basic notions and assumptions from these authors. In particular, OSA interprets and adopts Hjelmslev's notion of semiotic function, Peirce's semiotic triad, the pragmatic maxim and Wittgenstein's notions of meaning as a use, language game, and form of life. In Section 3.2, we describe three theories with a strong impact on mathematics education: Frege introduces a key distinction between sense and reference; Vergnaud proposes a cognitive interpretation of meaning, and Steinbring emphasizes an epistemological interpretation of meaning. In Section 3.3, we present the ontosemiotic theory of meaning as a holistic approach to issues of meaning and sense, as well as to those of object and sign. Based on an anthropological (Wittgenstein) and pragmatist (Peirce) conception of mathematical activity developed in Chapter 2 and adopting the linguistic construct of semiotic function (Hjelmslev), we develop a semiotics that consider referential, operational, cognitive, and cultural theories of meaning. In Section 3.4, we introduce the notion of individual and social knowledge based on ontosemiotics, and in Section 3.5, we develop the ecology of meanings as a framework for studying the adaptations and transformations of mathematics in educational contexts. In Section 3.6, we present an example of articulating the notion of pragmatic meaning with the ontosemiotic configuration tool. The identification of agreements and complementarities among semiotic theories is discussed in Section 3.7. To demonstrate the utility of the ontosemiotic analysis of mathematical cognition, in Section 3.8, we include examples of its application to the study of natural numbers as cultural and personal objects and a synthesis of the meanings of the concept of function. The OSA framework was used to develop a model for analyzing the affective dimension in mathematical education and its relationships with the cognitive and epistemic dimensions; in Section 3.9, we include a summary of this model on affectivity. We conclude the chapter by synthesizing the ontosemiotic theory of meaning and mathematical cognition, responding to the questions proposed by Michie et al. (2014) in their model for analyzing theories in the social and behavioral sciences.

3.1. Theories of meaning[6]

There are two schools of thought addressing the issue of meaning from different perspectives: the analytical or referential tendency, which attempts to capture the essence of meaning by identifying its main components, and the operational or pragmatic tendency, which studies words in action and is less concerned with what meaning means than with how it operates and how the resources of expression and communication are used.

3.1.1. Realist or analytical theories of meaning

According to Kutschera (1975), theories of meaning can be grouped into two categories: realist and pragmatic. Realist (or referential) theories view meaning as a conventional relationship between signs and concrete or ideal entities that exist independently of linguistic signs, assuming conceptual realism. According to this conception, “the meaning of a linguistic expression does not depend on its use in specific situations, but it is governed by its meaning, being possible a sharp division between semantics and pragmatics” (Kutschera, 1975, p. 34). A word becomes meaningful when an object (concept or proposition) is assigned as its meaning. Thus, there are entities that are not concrete but are always given objectively before words that constitute their meanings.

Authors who attribute to linguistic expressions only a semantic function present the simplest form of realist semantics, which comprises designating (by conventions) certain entities. Therefore, in realist theories (such as those advocated by Frege, Carnap, or appearing in Wittgenstein’s Tractatus), linguistic expressions have an attributive relationship with certain entities (objects, attributes, facts). The semantic function of an expression comprises only a conventional relationship (denoted as a nominal relationship).

[6] The contents of sections 3.1, 3.2, 3.3, and 3.6 are based on the article by Godino et al. (2022).

3.1.2. Operational or pragmatic theories of meaning

The two basic ideas of the operational or pragmatic category of theories of meaning are as follows:

- The meaning of linguistic expressions depends on the context in which they are used.
- It is not possible to scientifically, empirically, and inter-subjectively observe abstract entities—such as concepts or propositions—which are implicitly admitted in realist theories. The only thing accessible to observation in these cases in a scientific investigation of language is linguistic usage. It follows from this usage that the meaning of abstract objects must be inferred.

The operational approach has the merit of defining meaning in contextual terms, i.e., in purely empirical terms, without resorting to vague, intangible, and subjective mental states or processes. Wittgenstein (1953), in his *Philosophical Investigations*, openly defended a pragmatic or operational conception of meaning. In his formulation, a word becomes meaningful by performing a certain function in a language game, and it is used in this game for a specific purpose. Thus, for a word to be meaningful, there does not need to be something that is the meaning of that word, in the sense of realist theories.

For some authors, the realist and operational views of meaning are irreconcilable. However, Ullmann (1962) suggested that pragmatic theories (which he calls operational or contextual) are a valid and necessary complement to realist theories (which he calls referential):

> The researcher should begin by gathering an adequate sample of contexts and then deal them with an open spirit, by allowing meaning or meanings to emerge from the contexts themselves. Once this phase has been completed, he can safely move into the "referential" phase and attempt to formulate the meaning, or meanings thus identified. (Ullmann, 1962, pp. 76-77)

Ullmann's observation is fundamental and supports the meaning model proposed by OSA (described in Section 3.3), where meaning is conceptualized foremost, pragmatically, as it relates to problem-solving practices and contexts of use. However, these practices involve words, symbols, and various representations that refer to other objects and systems; a referential type of meaning is also involved.

3.1.3. Semiotics and philosophy of language

Because mathematical objects cannot be directly apprehended through the senses, their ontological status requires the use of signs such as symbols and diagrams. Semiotics, a systematic study of nature, properties, and sign types, has received significant attention in mathematics education research. "Semiotics has been a fruitful theoretical lens used by researchers investigating diverse issues in mathematics education in recent decades" (Presmeg, 2014, p. 539). In this study, we have been particularly interested in the language theory of the Danish linguist Hjelmslev (1943), since it can be useful for describing mathematical activity and the cognitive processes involved in the production and communication of mathematical knowledge.

The description and analysis of mathematical instruction processes require to transcribe the participants' linguistic manifestations and the events that occur in didactic interaction into textual form. To perform their work, the didactics researcher has at their disposal the instructional planning texts, transcripts of class developments, interviews, and written responses to assessment tests. Ultimately, the analysis will be primarily applied to texts that record the participants' mathematical activities.

Starting from the text as data, Hjelmslev's linguistic theory attempts to show the path to a self-consistent and exhaustive description of it through its analysis, whose basic principle is that

> both the object under examination and its parts have existence only by virtue of these dependences; the whole of the object under examination can be defined only by their sum total; and each of its parts can be defined only by the dependences joining it to other coordinated parts, to the whole, and to

> its parts of the next degree, and by the sum of the dependences that these parts of the next degree contract with each other .(Hjelmslev 1943, p. 23)

A key notion in Hjelmslev's language theory is that of function, conceived as the dependency between the text and its components and between these components themselves. He calls *functives* the terminals of a function, which are any objects that have a function with others. This notion of function is halfway between the logical-mathematical and etymological dimensions and is more formally close to the former but not identical to it.

> We shall be able to say that an entity within the text (or within the system) has certain functions, and thereby think, first of all, with approximation to the logical-mathematical meaning, that the entity has dependences with other entities, such that certain entities premise others—and secondly, with approximation to the etymological meaning, that the entity functions in a definite way, fulfils a definite role, assumes a definite "position" in the chain. (Hjelmslev, 1943, p. 34)

The sign function

For Hjelmslev, language is a system of signs, and a sign (or sign expression) is characterized primarily by being a sign of something else, which is hence attributed a functional character. "A 'sign', in contradistinction to a non-sign, is the bearer of a meaning" (Hjelmslev, 1943, p. 43). "Any entity, and thus also any sign, is defined relatively, not absolutely, and only by its place in the context" (Hjelmslev, 1943, p. 45).

Among the dependencies that can be identified between parts of a text, those where one part designates or denotes another stand out; the first (expression plane) functions or represents the second (content plane), and it points to content outside the expression. This function is what Hjelmslev designates as the sign function and which Eco (2000, p. 83) presents as a semiotic function.[7]

[7] A sign is always constituted by one (or more) elements of a PLANE OF EXPRESSION conventionally placed in correlation with one (or more) elements of a PLANE OF CONTENT […].

3.1.4. Pragmatism and Peirce's semiotics

Charles Sanders Peirce (1839-1914) wrote a substantial number of works on diverse topics related to philosophy, mathematics, and semiotics, among other disciplines, which have been receiving special attention in recent years across various fields. In this section, we include some ideas that we consider of special interest because they have been employed as theoretical frameworks in several studies on mathematics education (Campos, 2010; Otte, 2006; Sáenz-Ludlow & Kadunz, 2016).

Pragmatism

Pragmatism is a philosophical movement that emerged in the United States at the end of the 19th century. William James and Charles S. Peirce were the main proponents of this doctrine, which is characterized by the quest for practical consequences of thought. Pragmatism places the criterion of truth in the effectiveness and value of thought in life. For this movement, understanding the practical use of a concept is more important than its conceptual definition. For pragmatists, the relevance of data arises from the interaction between intelligent organisms and the environment, leading to the rejection of invariable meanings and absolute truths: ideas, for pragmatism, are only provisional and may change based on future investigations. By establishing the meaning of things based on their consequences, pragmatism is often associated with practicality and utility, depending on the context.

Peirce's orientation toward pragmatism (who preferred to term his position 'pragmaticism' to avoid certain interpretations of pragmatism) was not the investigation of what signs mean within social life, but how a generic individual uses signs to form new ideas and concepts and reach the truth.

A semiotic function is realized when two functions (expression and content) enter into mutual correlation. (Eco, 2000, pp. 83-84)

"His theory of pragmaticism (that is, the logic of abduction) is the basis of his semiotics. For this reason, Peircean semiotics moves close to the realms of logic, without being reduced solely to it" (Radford, 2006, p. 9).

In his work *How to Make Your Ideas Clear*?, he defends his pragmaticist idea of clearly understanding concepts. The pragmatic maxim is a logical statement that he proposed as a normative recommendation or regulative principle on the optimal way to "achieve clarity in apprehension." Peirce stated the pragmatic maxim in various ways over the years. One that seems more comprehensible is the following:

> 402. It appears, then, that the rule for attaining the third grade of clearness of apprehension is as follows: Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object. (Peirce, 1931-58, CP 5.402)

According to Burch (2014, p. 8), when Peirce suggests that the complete meaning of a conception comprises the entire set of its practical effects, he means that a significant conception must have some kind of "effective experiential value", it must somehow be related to some kind of collection of possible empirical observations under specifiable conditions.

The sign notion

The development of the theory of signs, or semiotics, was fundamental in Peirce's intellectual life, with three stages distinguishable from 1860 to 1910, during which the notion of the sign and its different types were progressively enriched, although the basic structure of signs and the process of signification remained largely the same (Atkin, 2010).

For Peirce, the world of appearances—the world as we perceive and experience it through our senses and immediate experiences—is entirely constituted of signs, which refer to qualities, relations, events, states, regularities, habits, laws, etc., that have meanings or interpretations. A sign is a term in a triad of terms that are indissolubly connected by an essential triadic relation, which Peirce calls the sign relation. In Peirce's 1897 definition of a sign: "something that stands in place of something to someone in some sense or capacity" (CP 2.228), three basic elements are explicitly present: the sign, the object, and the interpretant.

The *sign* itself (also called the *representamen*) is a term that is usually used to represent or signify something. The *object* is what is ordinarily understood as the thing meant or represented by the sign, that for which the sign is a sign of. The *interpretant*[8] is the understanding we reach of some relation between the sign and the object, such as the translation or development of the original sign (Atkin, 2010). According to Peirce's definition of the sign relation, the interpretant must itself be a sign, and a sign, moreover, of the same object that is (or was) represented by the original sign. The interpretant is a second signifier of an object that openly has a mental status. However, this second sign must itself have an interpretant, which, in turn, is a new, third sign of the original object, and again is one with an openly mental status. And so on. Thus, if an object has a sign, then a sequence of signs of the same object exists. Therefore, for anything in the world of appearances, since it is a sign, an infinite sequence of mental interpretants of an object begins.

Peirce's pragmatic maxim is interpreted and adopted by the OSA (Section 3.3) when this framework proposes to conceive the meaning of a mathematical concept in terms of the systems of operative and discursive practices performed by a person (or institution) to respond to a type of problem situations. The OSA also interprets the notion of semiotic function from Hjelmslev and articulates this idea with Peirce's semiotic triad and the process of unlimited semiosis.

[8] Sáenz-Ludlow and Kadunz (2016, p. 3) represented the triadic sign with the word SIGN (in capital letters) to distinguish it from the representamen or sign-vehicle component. They indicate that understanding the process of meaning construction involves understanding the active role of the interpreting Person in the reconstruction of the real Object of a SIGN, based on the keys and indications provided by the sign vehicles, which only indicate certain aspects of the real Object.

3.1.5. Language games and forms of life: Language as a tool

The realistic conception of the meaning of words is based on treating each significant word as a name, an idea that informs much of the reflection on the philosophy of mathematics and psychology. Mathematical expressions such as '0', '-2', √(-1), 'aleph-null', or even '+', 'x^4', 'e^x', are taken as names of entities, and the question, what do they mean?, is reduced to what do they stand for?. (Baker & Hacker, 1985)

Wittgenstein (1953; 1956) argued that we should consider words as tools and clarify their uses in our language games. For example, numerical words are instruments for counting, ordering, and measuring, and the foundations of elementary arithmetic, that is, the mastery of the series of natural numbers are based on counting training.

The notions of "language game" and "forms of life" are main concepts in Wittgenstein's philosophy. Since the meaning of words is conceived as their use in various contexts, a sense of "language game" must be sought through Wittgenstein's use of the expression. For example, the communicative interaction established between master builder A, who requests materials from his assistant B, is a language game. The communicative processes through which children learn their mother tongue are another example. In paragraph 23 of his *Philosophical Investigations*, Wittgenstein develops this idea with new examples:

> 23. ... Here the term "language-game" is meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life.
>
> Review the multiplicity of language-games in the following examples, and in others:
>
> Giving orders, and obeying them—
>
> Describing the appearance of an object, or giving its measurements-
>
> Constructing an object from a description (a drawing)—
>
> Reporting an event—

> Speculating about an event—
> Forming and testing a hypothesis—
> Presenting the results of an experiment in tables and diagrams— ...

As Marrades (2014) explains, the expression "form of life" always appears in connection with language, more specifically, in particular language games. Moreover, in most examples, the notion of a form of life is characterized as a mode of acting that underlies the use of language. According to this author, recourse to this notion arises in the domain of conceptual language-understanding problems. Understanding the meaning of an expression requires not only appealing to the rules governing its use but also viewing that use by reference to a broader existential structure of which the language game is a part:

> More specifically, a way of life designates, for Wittgenstein, a factual framework of relations between linguistic behavior, non-linguistic behavior and situations in the world, within which a language game develops. [...] Life forms are always social forms of life, social practices. (Marrades, 2014, p. 146)

The constructs, form of life and language game, are incorporated into the OSA notion of institution (Section 3.3). The members of an institution or community of practice share certain types of problems and specific ways of addressing them, as well as habits, norms, material and linguistic resources, i.e., members of the institution share forms of life and language games. This basic postulate is accepted by any sociocultural approach to knowledge, particularly mathematical knowledge.

In section 3.3, we analyze how the OSA relies on the notions of sign function (Hjelmslev) and mathematical practice, operationalizing Wittgenstein's anthropological view of mathematics and its relativity to language games and forms of life. Furthermore, the OSA interprets Peirce's semiotic triad in terms of the function or correspondence between two terms, antecedent and consequent, connected by a criterion or rule of correspondence. Peirce's pragmatic maxim is also translated into the OSA in terms of systems of operative and discursive practices, used to propose a conceptualization of the pragmatic meaning of mathematical objects, as opposed to mentalistic or idealistic views of concepts.

3.1.6. Cognitive semiotics

Cognitive semiotics is an emerging field of research that aims

> ...integrating methods and theories developed in the disciplines of cognitive science with methods and theories developed in semiotics and the humanities, with the ultimate aim of providing new insights into the realm of human signification and its manifestation in cultural practices (Zlatev, 2012, p. 2).

Cognitive semiotics did not emerge until the mid-1990s, when Daddesio (1995) established a project to demonstrate both the feasibility and utility of a cognitive approach to semiosis, thus laying the foundation for a cognitive theory of symbols. Classical authors such as Piaget (1962) and Vygotsky (1962, 1978) have already addressed this issue, but new concepts, research methods, and a wealth of data have made it a very fruitful area (Zlatev, 2012, p. 4). Paulucci (2021) identified three principles or dimensions underpinning cognitive semiotics:

1) *Radical Enactivism.* From an enactive standpoint, all cognition, perception, or thought results from the interaction of a living organism with its environment. From the perspective of cognitive semiotics, this environment is not "natural", but a semiotic environment filled with objects, norms, habits, institutions, and artifacts that shape our minds. Another source of enactivism is the theory of the embodied basis of thought through the role of metaphors (Lakoff & Johnson, 1980; Johnson, 1987). According to these authors, all human understanding, including meaning, imagination, and reason, is based on schemas of bodily movements and perception. These schemas are extended using metaphors, which provide the basis for understanding,

thinking, and human communication. Lakoff and Núñez (2000) developed and applied these ideas to mathematics.

2) *Pragmatism*. Cognition does not serve to construct a true representation of the world; rather, it serves as a means of effectively acting in the world. To achieve this, it is necessary to construct versions of the world that bring about and do not represent it. Meaning is identified through habits and the creation of sense. Pragmatism places the criterion of truth in the efficacy and value of thought. Therefore, it opposes the philosophy that holds that human concepts represent the real meaning of things. For pragmatists, the relevance of data arises from the interactions between intelligent organisms and the environment. This leads to the rejection of invariable meanings and absolute truths: ideas, for pragmatism, are only provisional and can change based on future research.
3) *Material Engagement Theory*. Material Engagement Theory (MET) (Malafouris, 2013) considers the role of technical objects and material artifacts as constitutive of cognition. Artifacts are part of the environment where some cognitive functions are delegated that would not be possible to perform within the biological head or body. Texts, languages, and semiotic systems constitute the scaffolding that allows humans to know the world and represent the background of our perception of the environment.

This line of thought posits that the mind is embodied, extended, and distributed rather than being tied to the brain or being entirely "in the head". This shift in perspective raises significant questions about the relationship between cognition and material culture, posing great challenges for philosophy, cognitive science, archeology, and anthropology. Malafouris (2013) proposed an interdisciplinary analytical framework to investigate how things have become cognitive extensions of the human body. His Material Engagement Theory adds materiality—the world of things, artifacts, and material signs—to the cognitive equation.

3.1.7. Cultural semiotics

Cultural semiotics examines meaning within the framework of social and cultural life. It specifically focuses on the systemic and contextual relationships through which meaning is conferred. Lotman (1984/2005; 1990) was the first to speak of "semiotics of culture", focusing on the systematic cultural circle implicated in every text. Despite starting from textual semiotics, he immediately adopted a new perspective: essentially, the analysis of texts is subordinate to the identification of cultural processing and transmission on a general scale, and each text is a place where many codes intersect, forming new relationships and structures. For Lotman, the smallest functioning mechanism, the unit of semiosis, is not the language in isolation but the entire semiotic space of the culture in question. This space he calls the *semiosphere*: the semiotic space necessary for the existence and functioning of languages; in a sense, the semiosphere has a prior existence and is in constant interaction with languages.

> The semiosphere is marked by its heterogeneity. The languages which fill up the semiotic space are various, and they relate to each other along a spectrum which runs from complete mutual translatability to just as complete mutual untranslatability. Heterogeneity is defined both by the diversity of elements and their different functions. (Lotman, 1990, p. 125)

Eco (2000) adopted a cultural perspective on semiotics, i.e., semiotics whose deep roots and meaning lie in the interrogation and analysis of cultural systems.

> In culture every entity can become a semiotic phenomenon. The laws of signification are the laws of culture. For this reason, culture allows a continuous process of communicative exchanges, in so far as it subsists as a system of systems of signification. Culture can be studied completely under a semiotic profile. (Eco, 1976, p. 28.)

For Eco, meaning must be conceived as a cultural unit, always a matter of public and intersubjective negotiation. The semiotic universe consists not of signs but of cultural units, entities that absorb and reflect the influence of the culture in which they are found. Signs are not the entries in a rigid system of content organization (a dictionary) but the nodes of a network of meanings that can be traversed in multiple directions according to the inferences and interpretative connections chosen: a semiotic universe that takes the form of an encyclopedia.

3.2. Theories of meaning in mathematics education

Clarifying notions such as meaning and sense is a topic of interest in mathematics education and is approached from various perspectives. In this section, we briefly describe three semiotic theories specifically oriented toward mathematical knowledge: Frege's logical-semantic theory, Vergnaud's cognitive perspective, and Steinbring's epistemological approach. Frege, a classic author, distinguishes between sense and reference, which serves as the starting point for Steinbring's epistemological triangle, a model explicitly developed from a mathematics education standpoint. Vergnaud represents theories of meaning from a constructivist psychological perspective. These three theories share an interest in connecting the issue of the meaning of terms and expressions with the ontological problem regarding the nature of mathematical concepts, which is a central issue in OSA. In Section 3.6, we analyze some agreements and complementarities between these semiotic theories and OSA.

3.2.1. Sense and reference in Frege

Different triangular models have been proposed to deal with the problem of relationships between symbols and meanings. One such model was introduced by Frege (1892) in his work *On Sense and Reference*:

> It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the referent of the sign, also what I would like to call the sense

> of the sign, wherein the mode of presentation is contained. (Frege, 1892, p. 210)

For instance, let *a*, *b*, and *c* be segments connecting the vertices of a triangle to the midpoints of opposite sides. The intersection point of *a* and *b* is the same as those of *b* and *c* (centroid). Hence, we have different designations for the same point, and these names ("intersection point of *a* and *b*" and "intersection point of *b* and *c*") simultaneously indicate the mode of presentation; this is why the proposition contains effective knowledge. Both expressions have the same reference but differ in sense.

> To each sign corresponds a given sense and to this sense, in turn, a specific reference, while a reference (to an object) neither is linked to only one sign, nor receives a single sense. The same sense has different expressions in different languages, and it may happen that an expression makes sense but has not a reference. For example, the expression "the series that converges more slowly" possess a meaning but has no reference, since for each convergent series, another series that converges more slowly can be found. Therefore, by grasping a sense, one is not sure there is a reference. (Frege, 1892, p. 87)

Frege argued that it is necessary to distinguish between the reference and the sense of a sign regarding the representation associated with them; representation is internal for each subject. If the reference of a sign is a perceptible object, then its representation is an image derived from memories of sensory impressions and activities, both internal and external, that one has exercised. Representations are subjective: a representation from one person is not another's.

> The sense of an expression was supposed to consist in the way in which we determined its reference: but now it appears that, often, there is no one favored way to determine the reference of an expression, but that different people may determine it in different ways, and even that what is taken at one time as an acceptable means of determining it may later be dropped as not agreeing with the others. If so, then what is objective about the employment

> of an expression, what is shared by all the speakers of the language, is after all its reference. (Dummett, 1973, p. 102)

Initially, the theory of sense and reference was developed for proper names:

> The referent of a proper name is the object itself which we designate by its means; the conception, which we thereby have, is wholly subjective; in between lies the sense, which is indeed no longer subjective like the conception, but is yet not the object itself. (Frege, 1892, p. 213)

Frege further extends the theory of sense and reference to assertoric sentences, statements that affirm a judgment as true or false, and to common names or concepts. "Every declarative sentence concerned with the referents of its words is therefore to be regarded as a proper name, and its referent, if it exists, is either the true or the false" (Frege, 1892, p. 216).

Frege distinguishes between object and concept. The notion of concept in logic, which is the point of view that interests Frege, is closely related to that of function, for which he proposes the definition, "A function of *x* was taken to be a mathematical expression containing *x*, a formula containing the letter *x*" (Frege, 1891, p. 138).

Concerning the notion of an object, Frege asserts, "An object is anything that is not a function, so that an expression for it does not contain any empty place" (Frege, 1891, p. 147).

The Frege's logical-semantic model distinguishes whether a sign refers to an object or to a concept, under a certain modality or meaning (sign, sense, reference). This is a first step in acknowledging that a concept admits a plurality of possible interpretations, uses, or partial meanings. There is only one object/concept, but this object can be seen from different perspectives; for example, the centroid can be linked to the medians *a*, *b* of a triangle or to *b* and *c*.

While Frege's philosophy of mathematics is undoubtedly realist-Platonist in assuming that a mathematical object has its own independent existence, his theory of the sense and reference of signs, words, and expressions opens a window to the relativism of psychological and anthropological positions. A word designates or refers to an object or concept, but it is always accompanied by a thought, sense, or specific way of seeing the object or concept in the context in which communication occurs. These senses are considered inter-subjectively and, consequently, raise the problem of identifying and characterizing the possible universe of senses attributable to the object.

3.2.2. The conceptual triplet by Vergnaud

Vergnaud (1982) considers that it is a scientific challenge to promote the study of learning and teaching mathematics as a specific well-defined field, not reducible to mathematics, psychology, linguistics, sociology, or other sciences. This requires the analysis of different mathematical contents in their specificity, and the empirical study of their teaching and learning, considering both the long-term growth of knowledge in children and adolescents and the short-term change in conceptions in the face of new situations they encounter. To this end, he developed the theory of *conceptual fields* in which he proposes a definition of concepts that is useful for studying the evolutionary development of mathematical knowledge. He believed that a concept cannot be reduced to its definition, at least if one is interested in its learning and teaching (Vergnaud, 1990, p. 133).

Through the situations and problems resolved, a concept acquires meaning for the child. Vergnaud's study of the development and functioning of a concept, in learning or during its use, leads him to consider it necessary to distinguish three planes or components, the triplet (S, I, G), as constituents of a concept C, where:

S: set of situations that give meaning to the concept (reference).

I: set of invariants on which the operability of the schemes rests (meaning).

G: set of linguistic and non-linguistic forms that symbolically represent the concept, its properties, situations, and processing procedures (signifier).

In general, there is no bijection between signifier and meaning, nor between invariant and situation. Therefore, one cannot reduce meaning either to the signifier or to the situation. The notion of sense is understood as a relationship of the subject to situations and the signifier. “More precisely, the schemes evoked by the individual subject in a situation or by a signifier are what constitute the subject’s meaning of this situation or signifier” (Vergnaud, 1990, p. 158)

For example, the sense of addition for a subject is the set of schemes that can be put into action to deal with the situations that the subject comes to confront, which involve the idea of addition; it is also the set of schemes that can be deployed to operate on the symbols (numeric, algebraic, graphic, or linguistic) that represent addition.

Vergnaud (1982; 1990) goes beyond Frege in problematizing the mathematical concept by addressing the problem of learning and teaching: the concept itself is a complex and systemic entity formed by the interaction between three types of objects: systems of representation, problematic situations, and operative invariants.

3.2.3. The epistemological triangle

Steinbring (1997; 2006) interprets Frege’s triangle and the triangle proposed by Ogden and Richards (1923), adopting an epistemological perspective aimed at understanding the processes of interpretation, communication, and meaning construction that occur in mathematics education. The epistemological triangle he proposes includes three elements (Figure 3.1): the sign or symbol, the object or context of reference, and the concept. The latter is understood as an ideal or abstract mathematical concept.

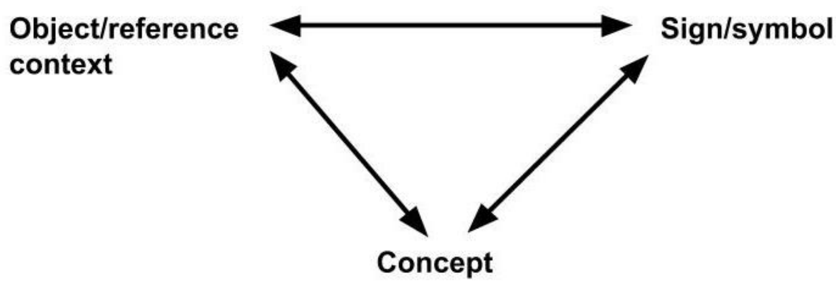


Figure 3.1. Epistemological triangle (Steinbring, 2006, p. 135)

Through the epistemological triangle, a semiotic (representational) mediation is modeled, where the links between the vertices of the epistemological triangle are not explicitly and invariably defined but form a balanced system in which they mutually support each other. In the ongoing development of knowledge, interpretations of sign systems and their accompanying reference contexts will change (Steinbring, 1997, p. 52).

Steinbring attributes these two functions to mathematical signs:

(1) A semiotic function: the role of the mathematical sign as "something which stands for something else".

(2) An epistemological function: the role of the mathematical sign in the frame of the epistemological constitution of mathematical knowledge. (Steinbring, 2006, p. 134)

To understand Steinbring's semiotic-epistemological model of mathematical knowledge, it is necessary to clarify the nature of the vertices of the triangle. It is assumed that "The true mathematical object, that is the mathematical concept, may not be identified with its representations" (Steinbring, 2006, p. 137). However, what mathematical concepts are? What are the objects/contexts of reference?

The application of the epistemological triangle to the concept of probability (Figure 3.2) allows us to understand the characteristics of this theoretical model of mathematical knowledge.

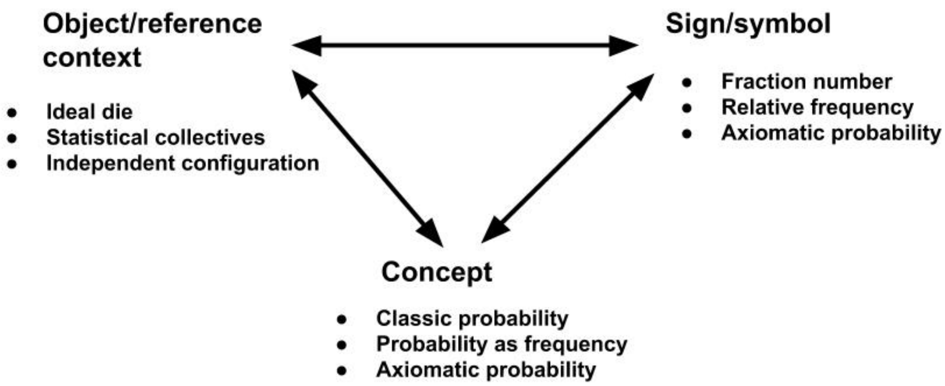


Figure 3.2. Application of the epistemological triangle to the concept of probability (Steinbring, 1997, p. 53)

Various expressions are included in the sign/symbol category (fractional number, relative frequency, axioms). Within the category of objects/context of reference, problematic situations where probability is applied are included, such as determining whether a die is biased (ideal), collecting data sets to determine probability, and calculating probabilities in configurations of independent events. The concept category includes various meanings or senses of probability: classical probability, frequency-based probability, and axiomatic probability.

Thus, although not explicitly mentioned, the concept of probability is assumed to have different meanings depending on the contexts or problem situations in which it intervenes, and such situations and meanings involve different systems of representation. The mention of axiomatic probability (Figure 3.2) is very vague. It possibly refers to the symbolic expression of axioms because the axioms themselves are not representations but properties of probability that link it to other mathematical objects, such as the union or intersection of events.

> The epistemological triangle is a model for making the invisible mathematical knowledge accessible with regard to its structural character, for describing its particularities and also for analyzing interactive processes

> of constructing mathematical knowledge – thus invisible relations that are embodied in exemplary contexts and activities. (Steinbring, 2006, p. 144)

Steinbring's epistemological triangle implicitly suggests that the concept, the sign/symbols of reference, and the objects/configuration of reference include a variety of general structures (various constructions of probability, natural number, etc.). The reciprocal relationships of conceptual structures with systems of representation and the different contexts and situations of use must be considered to organize and explain the generation of mathematical knowledge (i.e., the epistemology of the concept). This model adopts a systemic perspective for the structures of concepts, systems of symbols and contexts. While Frege attributes multiple senses to mathematical concepts, Steinbring relates them to diverse symbolic systems and contexts.

3.3. The ontosemiotic theory of meaning

Within the OSA framework, meaning and its relationship with the notions of practice and object play a central role. A practice is "any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems" (Godino & Batanero, 1998, p. 182).

> In our conception, it is the fact that certain types of practices are performed within certain institutions, which determines the progressive emergence of "mathematical objects" and that the "meaning" of these objects is intimately linked to the problems and activity carried out for their resolution, not being able to reduce the meaning of the object to its mere mathematical definition. (Godino & Batanero, 1994, p. 331)

Although the initial OSA aimed to develop a theoretical model that addressed the issue of the meaning of mathematical concepts, in successive developments, this aim was expanded and applied to any type of object involved in mathematical practices. The epistemological, cognitive, and instructional problems that mathematics education must address should first tackle the ontological problem, i.e., clarifying the nature and types of mathematical objects whose teaching and learning are intended.

In a preliminary approach, meaning is what a word, symbol, or any other form of expression refers to and is emitted by a person in a communicative act with another person or with oneself in a specific context. However, with words and symbols, things are not only mentioned or represented but also performed; that is, they are involved in operative practices. With words and symbols, operations and calculations are performed to produce new objects. For example, with the numerical symbols 2, 3, and the word 'sum', following certain agreed rules, the result 5 is produced, as well as a new mathematical object, the proposition that 2 + 3 equals 5, which is accepted as true.

Thus, the following question arises: what role, in addition to representation, does a word, symbol, or expression play in a specific operative practice? This is a central problem that must be addressed by a holistic theory of meaning that considers both referential and operational use to respond to the meaning of expressions, which refer to concepts (ideal, abstract objects) and any other type of object or do not refer to any object at all.

In this section, the use of meaning in OSA and its relationship with the notions of practice and mathematical objects are explained. We contextualize the explanation with an example of a demonstration of the elementary arithmetic proposition 2 + 3 = 5 in Figure 3.3. We accept that practices 1) to 7) are carried out by an epistemic subject who shares the language game and the form of life of people who know Peano's axiomatic.

Proposition: 2+3=5
Demonstration:

1) The symbols, 2, 3, and 5 represent natural numbers.
2) The natural numbers are a set of symbols satisfying Peano's axioms. In particular, there is a first element, 1, and a following (successor) function, s:N→N, injective, is defined. In such a set, the sum + is defined recursively as follows: n+1=s(n); n+s(m)=s(n+m).
3) In the sequence, 2 is the successor of 1, 2=s(1)=1+1, 3 is the successor of 2, 3=s(2)=2+1, and 5 is the successor of 4, which is the next to 3, 5=s(4)=s(s(s(3)).
4) Sign = indicates the equivalence of two expressions.
5) The expression 2+3 represents the sum of the natural numbers 2 and 3.
6) Considering the definition of the sum of natural numbers and successors, we obtain the following:
   2+3=2+s(2) = s(2+2)=s(2+s(1))=s(s(2+1))=s(s(3))=s(4)=5.
7) Therefore, the expressions 2+3 and 5 are equivalent.

Figure 3.3. Demonstration of the elementary arithmetic proposition (2+3=5)

3.3.1. Practices, objects, and meanings

In the statement of the proposition, 2 + 3 = 5, the symbols 2, 3, and 5 refer to the natural numbers 2, 3, and 5, respectively; + refers to the arithmetic operation of addition, and the symbol = means that the result of adding 2 and 3 matches the number 5. By interpreting these symbols, we follow rules agreed upon in mathematical culture. Thus, if we understand the numbers and symbols for addition and equality in that way, we must necessarily accept that “two plus three equal five”.

From a conceptualist-idealist perspective of mathematics, in the expression 2 + 3 = 5, besides the visible or audible material signs or objects, other non-visible immaterial objects, usually considered as concepts, are involved. In this case, these are the concepts of numbers 2, 3, and 5, addition, and equality. To understand the justification for the truth of proposition 2 + 3 = 5, it is necessary to explain what is meant by a natural number, in particular, the concepts of 2, 3, 5, addition, and equality, or equivalently, what meaning should be attributed to these concepts.

To avoid the idealist trap of Platonism, as warned by Wittgenstein, OSA assumes a pragmatist interpretation of such entities when discussing concepts and the meaning of concepts (or any type of mathematical object). For this purpose, Godino and Batanero (1994, p. 341) introduced the following definitions of meaning:

> *DEFINITION 8:* The meaning of an institutional object O_I is the system of institutional practices linked to the problem field from which O_I emerges at a given time.
>
> *DEFINITION 9:* The meaning of a personal object O_p is the system of personal practices that a person p carries out to solve the problem field from which the object O_p emerges at any given time.

The example of proposition 2 + 3 = 5 clarifies the scope of definitions 8 and 9 and, therefore, the OSA pragmatic assumptions. The statement O: 2 + 3 = 5 is a propositional mathematical object that requires justification within the specific language game of Peano's axiomatic system. Faced with the problem situation of demonstrating O, the epistemic subject who solves the problem responds to the question, "what does O mean"? In OSA, the pragmatic meaning of O is the system of practices 1) to 7). The problem can be posed to a student who will likely provide a different response. The pragmatic meaning of O for students (personal meaning) refers to the system of operative and discursive practices performed to verify the truth of the proposition.

Practices 1) to 7) (Figure 3.3) together constitute the argument that justifies proposition 2 + 3 = 5, in which, besides linguistic and conceptual entities, a procedural entity is involved: the technique of recursively applying the definition of the sum of natural numbers. Through discursive and operative practices, rules that fix the meaning of concepts and procedures are evoked, concluding with normative-discursive practice 7): Therefore, expressions 2 + 3 and 5 are equivalent.

In mathematical activity, concepts, propositions, and procedures can participate as unitary entities, described through a definition or statement that fixes the rule of use of such an object: for example, the definition of a natural number given in practice 2) of the demonstration (Figure 3.3). However, we know that other definitions of natural numbers can be found using different axiomatic systems or depending on different contexts or institutional frameworks in which the numbers are used. Each of these definitions engages different operative and discursive practices involving other objects and thus implies a different pragmatic meaning.

As described in Chapter 2, the term "object" is used broadly to refer to any entity that intervenes in mathematical practice and can be identified as a unit. The use of the term object is metaphorical because mathematical concepts are usually conceived as an ideal or abstract entity, not as something tangible, like a stone, a drawing, or a manipulative. This general idea of an object, consistent with symbolic interactionism (Blumer, 1969; Cobb & Bauersfeld, 1995), is useful when considering a typology of mathematical objects, considering their different roles and nature in mathematical activity.

Symbols, external material representations, and manipulatives are involved in school and professional mathematical activity and are considered mathematical objects because they are used in mathematical practices. The concepts of number, fraction, derivative, etc., are mathematical objects with a nature and function that differ from ostensive representations; they are non-ostensive, mental objects (when involved in personal or individual practices) or institutional objects (when involved in shared sociocultural practices). In both cases, they regulate mathematical activity, whereas their ostensive representations support or facilitate the performance of that activity.

Each type of object can be considered from different perspectives (see Figure 3.4)[9]. An object can be considered from a personal (individual subject) or institutional (social, shared) perspective and thus has a dual nature: mental/cognitive and cultural/epistemic. The personal-institutional duality applies to practices, objects, and meanings, allowing us to describe semiosis processes (expression-content duality) from cognitive and cultural perspectives.

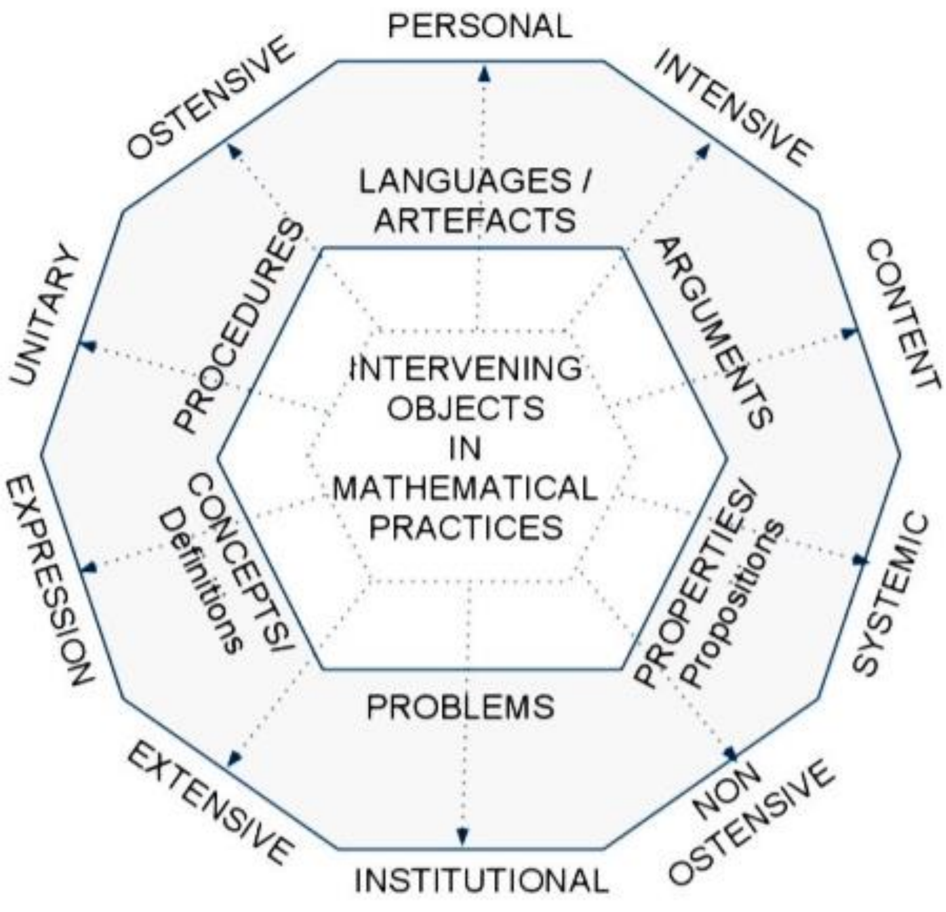


Figure 3.4. Objects and processes involved in mathematical practices (Font et al., 2013, p. 117)

[9] Complementary version of the ontosemiotic configuration described in Chapter 2.

There are no objects without practices nor practices without objects. Concepts, propositions, and procedures, in their unitary versions, are understood, as Wittgenstein proposes, as grammatical rules of the languages used in operative and discursive practices to describe our worlds and act upon the problem situations they present to us. Additionally, from OSA, mathematical objects are viewed from a systemic perspective, identifying and articulating their various partial meanings. Similarly, when semiotic analysis is conducted on the individual subjects' practices when facing problems involving a specific object (numbers, probability, etc.), various personal meanings can be identified.

3.3.2. Use and intentionality of practices

In operational theories of meaning, words, symbols, and expressions do not necessarily refer to or stand in place of other things but are used to do something with them. For example, numerals are instruments for counting, ordering, and measuring, and statements about numbers play the role of rules for the use of such words. According to this, 2+3=5 is not a property that establishes a relation between conceptual entities, as it could happen with the expression "lions are carnivores", but it is a rule about how the symbols 2, 3, 5, +, and = should be used; that is, whenever you have the expression 2+3, you can substitute it with 5, and vice versa.

The justification of mathematical propositions is accomplished through a sequence of operative and discursive practices (such as those shown in Figure 3.3) that have a determined intentionality. Each elementary practice performed to solve a problem, which may be intra-mathematical, such as the demonstration that 2+3=5 or involving an extra-mathematical context, plays a role in the solution process. Table 3.1 summarizes the operational/pragmatist use or meaning of the practices required to demonstrate proposition 2+3=5 (Figure 3.3).

Table 3.1. Use and intentionality of practices to demonstrate 2+3=5

Sequence of elementary practices	Use / intentionality
1) The symbols, 2, 3, and 5 represent natural numbers.	To attribute meaning to symbols 2, 3, and 5 as natural numbers.
2) The natural numbers are a set of symbols satisfying Peano's axioms. In particular, there is a first element, 1, and a following (successor) function, s:N→N, injective, is defined. In such a set, the sum + is defined recursively as follows: n+1=s(n); n+s(m)=s(n+m)	To evoke the rules that define natural numbers and their sum in the framework of a specific axiomatic theory.
3) In the sequence, 2 is the successor of 1, 2=s(1)=1+1, 3 is the successor of 2, 3=s(2)=2+1, and 5 is the successor of 4, which is the next to 3, 5=s(4)=s(s(s(3)).	To interpret the meanings of symbols 2, 3, and 5 in Peano's axiomatic theory of natural numbers.
4) Sign = indicates the equivalence of two expressions.	To evoke the meaning of equality of natural numbers as the equivalence of two expressions.
5) The expression 2+3 represents the sum of the natural numbers 2 and 3.	To interpret the meaning of + as the sum of natural numbers.
6) Considering the definition of the sum of natural numbers and successors, we obtain the following: 2+3=2+s(2)=s(2+2)=s(2+s(1))=s(s(2+1)) =s(s(3))=s(4)=5.	Apply the rules that define the following function (successor) and sum of natural numbers.
7) Therefore, the expressions 2+3 and 5 are equivalent.	Fix the new rule of use of numerical symbols (to state the truth of the proposition).

3.3.3. Meaning and semiotic function

Between symbol 2 and the concept of number 2, as well as between the concept of natural number and the system of operative and discursive practices from which this mathematical object emerges, a relationship is established that OSA calls a semiotic function (Section 3.1.3). The semiotic function is understood as the correspondence between an antecedent object (expression/signifier) and a consequent object (content/meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. This notion is intended to include any use given to meaning, meaning is the content of a semiotic function.

Each elementary practice that constitutes the text of the demonstration of proposition 2+3=5 (Figure 3.3) has a function or role in the argumentative process, and this role can be assigned as the operational meaning of the practices (Table 3.1). However, in the realization of each practice and in the conjunction of all or part of them, a network of objects is involved (Table 3.2), whose identification is necessary to understand and manage teaching and learning processes.

Table 3.2. Objects involved in the practices to demonstrate 2+3=5

Sequence of elementary practices	Intervening objects
1) The symbols, 2, 3, and 5 represent natural numbers.	Languages: symbolic; natural. Concepts: natural numbers.
2) The natural numbers are a set of symbols satisfying Peano's axioms. In particular, there is a first element, 1, and a following (successor) function, s:N→N, injective, is defined. In such a set, the sum + is defined recursively as follows: n+1=s(n); n+s(m)=s(n+m)	Language: natural, symbolic. Concepts: natural number; set of symbols; injective following function, first element; successor; sum. Propositions: Peano's axioms.
3) In the sequence, 2 is the successor of 1, 2=s(1)=1+1, 3 is the successor of 2, 3=s(2)=2+1, and 5 is the successor of 4, which is the next to 3, 5=s(4)=s(s(s(3)).	Languages: natural; symbolic. Concepts: sequence; successor, sum. Proposition: 2 is the successor of 1; 3, the successor of 2 and 5 is the successor of the successor of 3. Arguments: convention based on the properties of the following function.
4) Sign = indicates the equivalence of two expressions.	Languages: symbolic; natural. Concepts: equivalence of expressions; equality.
5) The expression 2+3 represents the sum of the natural numbers 2 and 3.	Languages: natural and symbolic. Concepts: addition of natural numbers.
6) Considering the definition of the sum of natural numbers and successors, we obtain the following: 2+3=2+s(2)=s(2+2)=s(2+s(1))=s(s(2+1)) =s(s(3))=s(4)=5.	Languages: natural and symbolic. Proposition: 2+3= 5. Procedure: addition and successor operations. Argument: deductive, which is based on the definition of the sum of natural numbers and the following function.
7) Therefore, the expressions 2+3 and 5 are equivalent.	Languages: natural and symbolic Proposition: statement of practice 7). Rationale: deductive sequence of practices 1) to 6).

The semiotic function can be viewed as an interpretation of the Peircean sign.

> A representation is that character of a thing by virtue of which, for the production of a certain mental effect, it may stand in place of another thing. The thing having this character I term a representamen, the mental effect, or thought, its interpretant, the thing for which it stands, its object. (Peirce, 1931-58, CP 1.564)

In OSA, the Peircean interpreter is conceived as the rule (habit, norm) of correspondence between the representamen and the object, established by a person or within an institution, in the corresponding interpretative act (personal or institutional meanings). When, for example, in practice 1) it is stated that 2 refers to the “concept of natural number two” (Figure 3.3), we follow a convention (habit, rule) that is learned in the community of school mathematical practices. Between sign 2 and concept two, there is an interpreter that is nothing more than a cultural convention followed by the subject performing the interpretation.

Furthermore, in OSA, it is assumed that every entity that participates in a process of semiosis, interpretation, or a language game is an object that can play the role of an expression (signifier), content (signified), or interpreter (rule that relates expression and content). The systems of operative and discursive practices themselves are objects that can be components of the semiotic function. Thus, any use that can be made of the word meaning is modeled.

The pragmatist/anthropological semiotics assumed by OSA presupposes that the objects that are put into correspondence in semiotic functions (functives) are not only ostensive linguistic objects (words, symbols, expressions, diagrams etc.), but that concepts, propositions, procedures, arguments, and even problem situations can also be antecedents of semiotic functions. It makes sense and is necessary to ask about the meaning of the concept of number, as well as the meaning of propositions, procedures, arguments, situations, and representations involved in numerical practices. The functives in the semiotic function can also be unitary or systemic entities, particular or general, material or immaterial, personal or institutional. Thus, a variety of meanings are generated that orient and support the realization of ontosemiotic analyses of mathematical activity at the macro and micro levels, both from epistemic (institutional) and cognitive (personal) perspectives (Font et al., 2013).

3.3.4. Relativity of practices, objects, and meanings

In OSA, mathematical practices are assumed to occur in an ecological background (material, biological, and social), which determines the institutional, personal, and contextual relativity of practices, objects, and meanings regarding *language games* and *form of life* (Wittgenstein, 1953). Therefore, a sociocultural perspective on semiosis is assumed in which the social, cultural, and historical dimensions of signs are emphasized. “In these perspectives, signs are understood not as artifacts to which an individual resorts to represent or present knowledge, but as artifacts of communication and signification” (Presmeg et al., 2018, p. 4).

In the example described above (Figure 3.3), the context of modular arithmetic changes the meaning of 2+3, just as the meaning of the concept of natural numbers is changed by changing the axiomatics, or the set construction of numbers is adopted. The meaning of numbers is different in different communities of practice formed by different cultural groups or at different historical moments.

An aim of didactic-mathematical analysis should be to characterize the various meanings of objects and their interrelationships, constructing a global meaning that serves as a reference for the analysis of mathematical instructional processes. This is the first level of ontosemiotic analysis of mathematical activity through which one becomes aware of the plurality and relativity of the meanings of mathematical objects. The first level involves identifying, classifying, and describing the problem situations in which the object in question intervenes, as well as the mathematical practices through which answers are given to these problems.

The social, material, and biological context (ecological background), which sustains and conditions mathematical activity, implies the relativity of practices, objects, and meanings. Both practices and objects can be viewed from different polarities. Such practices are relative to different institutional (historical-cultural) frameworks and contexts of object use.

For education in general and mathematics education in particular, a holistic theory of meaning that includes the personal-institutional duality for meanings is needed. Both cognitive semiotics and epistemic/cultural semiotics are required: meanings are established between individual persons in discursive and operative practices; but also, between a person and the cultural knowledge whose learning is intended. In mathematical culture, terms, symbols, concepts, etc., have a crystallized, socially shared meaning, formed through historical cultural processes. This meaning results from multiple discursive and operative practices between individual subjects, which are mediated by different languages and artifacts. This approach is consistent with the cultural semiotics proposed by Radford (2006) for the meaning of mathematical concepts: "mathematical objects are conceptual forms of historically, socially, and culturally embodied, reflective, mediated activity". (Radford, 2006, p. 59)

From the perspective of education, meanings should not be reduced to mental or cultural objects; it is necessary to attribute them a dual personal and institutional nature to account for the dialectical relationship established between them in teaching and learning processes.

3.4. An ontosemiotic approach to mathematical cognition

The starting point of OSA was the need to clarify the meaning of mathematical objects from the personal (individual cognition) and institutional (objective, cultural cognition) point of view (Godino & Batanero, 1994; 1998). In other words, it has been a concern to understand the origin and nature of mathematical knowledge from a semiotic perspective, the role of signs and languages in mathematical activity, emerging objects, and the relationships between them. This approach belongs to cognitive semiotics (Zlatev, 2012; Paulucci, 2021) and cultural semiotics (Eco, 1976; Lotman, 1990).

3.4.1. Knowledge and understanding

As seen in Godino and Batanero (1994), the anthropological and pragmatist theory of mathematical practices, objects, and meanings is used to propose a way of understanding what it means to know/understand an object in terms of the coupling of meanings.

> Definition 10: Meaning of an institutional object O_I for a subject p from the perspective of institution I: It is the subsystem of personal practices associated with a field of problems that is considered appropriate for solving those problems in institution I.
>
> Consequently, from the same field of problems C that in an institution I have given rise to an object O_I with meaning $S(O_I)$, a person can give rise to an object O_p with personal meaning $S(O_p)$. The intersection of these two systems of practices is what, from the perspective of the institution, are considered correct manifestations, that is, what the person "knows" or "understands" about the object O_I from the perspective of I. The remaining personal practices will be considered "erroneous" from the institutional perspective. (Godino & Batanero, 1994, p. 342)

In an ideal situation and within an institution, we say that subjects "understand" the meaning of the object O_I—or that they have "grasped the meaning" of a concept—if they can recognize its properties, justify them with valid arguments, use the characteristic representations, relate it to other mathematical objects, and use this object in all the variety of prototypical problem situations within the corresponding institution. The understanding achieved by a subject is unlikely to be complete or null; rather, it encompasses partial aspects of various components and levels of abstraction.

In summary, the mathematical cognition model proposed by OSA includes the following dimensions:

1) *Personal and institutional dimension*: If we accept the pragmatic and relativistic conception of mathematics underlying OSA, a theory of mathematical understanding that is useful and effective in explaining teaching and learning phenomena must recognize the dialectical duality between the personal and institutional facets of cognition. Because each person is born into a family and develops into a member of different institutions and cultural contexts, the psychological processes involved in cognition, including linguistic and conceptual objects, are mediated by institutional meanings, that is, by problem situations, semiotic instruments, habits, and shared conventions. The notion of personal cognition of a concept derived from OSA is the construction or appropriation of the institutional meaning of that object. Therefore, understanding is no longer a mental process; it is now a social and interactive process. It cannot be reduced merely to a mental experience; rather, it involves the whole sphere of the individual. Understanding "is the way we are meaningfully situated in our world through our bodily interactions, our cultural institutions, our linguistic tradition and our cultural context". (Johnson, 1987, p. 102)

2) *Human action and intentionality*: The OSA cognition model is based on the notion of meaningful prototypical practice, which is defined as individuals' actions in their attempts to solve a class of problem situations to which they recognize or attribute a purpose (a "why"). These practices are situated expressive forms that involve a problem situation, an institutional context, a person, and the semiotic instruments that mediate the action. Since mathematical objects are conceived as emerging from systems of meaningful prototypical practices, understanding the object (in an integral or systemic sense) also requires that the subject identifies a purpose in the object—an intentionality (Maier, 1992) as the basis of understanding.

3) *Systemic and dynamic character*: Since we conceive the "systemic meaning of an object" as an entity composed of elements and relative to institutional contexts, the knowledge and understanding of a concept by a subject, at a time and under given circumstances, will imply the appropriation of the different elements that make up the corresponding institutional meanings. This includes justification through valid reasoning of propositions. Recognizing the systemic complexity of an object's meaning also implies that its appropriation by the subject is a dynamic, progressive, and non-linear process (Pirie & Kieren, 1994), because of the different domains of experience and institutional contexts in which they participate.

4) *Practical and discursive dimensions*: The practical component (praxis) is linked to mathematical competence and, therefore, mastery of problem-solving techniques. The discursive/relational component is connected to the idea of understanding and is formed by a system of rules and justifications, including arguments, definitions of concepts, and properties on which they are based. Both components rely on the use of linguistic resources and material artifacts; thus, mathematical

> language (in its various registers) constitutes a third component, without which the previous ones cannot develop. Knowledge, understanding, and competence are closely linked and consist of different elements. They depend on the institution (historical-cultural context) from which they are developed and evaluated.

The semiotic function construct considers the referential dimension characteristic of realist theories of meaning and can be related to the processes of understanding knowledge, which are understood primarily as the connection between objects. It considers the pragmatic dimension, meaning as use, as a system of practices, which implies incorporating the competency component of knowledge, i.e., knowing how to act efficiently in certain situations. The variety of meanings accounts for the variety of ways of understanding and acting of the subject involved in discursive and operative activity. Thus, the ontosemiotic theory of meaning fits within the perspectives of cognitive semiotics (Zlatev, 2012; Paolucci, 2021). The theory of cognition based on OSA is not limited to the interpretation of signs (semiotics), as it also proposes an ontology for mathematics, a furnishing of the world (Bunge, 2011) intertwined with ostensive and non-ostensive systems of signs. Personal meanings account for individual cognition (including beliefs and affects), while institutional meanings account for institutional cognition (historical-cultural knowledge).

In OSA, cognition is essentially related to the ability to act and solve problems. However, this does not imply rejecting the representational dimension in mathematical cognition processes. Discursive practices refer to a world of objects that necessarily accompany operative practices, actions that effectively change the world and simultaneously understand it. The modality of personal cognition postulated by OSA is of an enactive nature (Lakoff & Núñez, 2000):

> From an enactive point of view, every cognition, perception, or thought is the result of a living body engaging in its own environment. However, from a cognitive semiotics' point of view, this environment is not a "natural" one, but a semiotic environment crowded with objects, norms, habits, institutions, and artefacts that shape our minds. (Paolucci, 2021, p. 10)

3.4.2. Knowledge and beliefs

We present below our view of the theoretical problem of characterizing the construct belief and its relationship with knowledge, although no consensus exists on this issue, as has been showed by various studies on the subject (Pajares, 1992; Leder et al., 2002).

Since the belief of subject X about an object O is a mental construct, OSA interprets it in terms of cognitive configuration, i.e., as a system of personal practices (what X does and says) to solve the type of problem situations in which O intervenes, together with the objects and processes that accompany the practices. If O is a proposition, a statement that can be qualified as true or false and the subject can elaborate a valid justification in the corresponding institutional framework, then X's belief about O is knowledge. Qualifying a belief as knowledge requires judgments to be true in a frame of reference and to be validly justified.

Beliefs may be based on personal experience, tradition, or authority, whereas knowledge is a judgment whose truth or certainty is established by valid evidence or arguments within the mathematical community. To declare a judgment or assertion as knowledge, it is therefore necessary to provide a valid justification for its truthfulness. Examples:

1) "Two apples plus three apples make five apples". This statement expresses true empirical knowledge. If we take two apples and add three more, we can prove that the set formed by the apples is five apples. This is an empirical argumentation because the proposition involves perceptible

objects. The proposition is true because it corresponds to empirical reality; it is factual truth.

2) "2+3 = 5". This proposition expresses true mathematical formal knowledge. In fact, 2 refers to the second position in the natural numerical series; the symbol +3 means that three more positions must be counted. This is a rational argumentation, a logical consequence, based on the previously agreed meanings of symbols 2, +, 3, and 5. The proposition is true because it is based on the coherence of the argumentation from the previous mathematical postulates and knowledge; it is a truth of reason.

3) “Peter believes that the earth is flat”. This statement contains two empirical propositions. One is that “the earth is flat”, the other is that “Peter believes (that the earth is flat)”. The first proposition is false. By applying observation and scientific reasoning to analyze and evaluate information, it is possible to argue that this statement does not correspond to reality. Peter’s belief can be true if it is proven that he indeed thinks in that way; he can even justify his proposition using personal arguments. Judged from an institutional point of view (scientific community), the personal knowledge of Peter would be false.

OSA assumes the need to use different theories of truth (Habermas, 2002; Nicolás & Frápolis, 1997), considering the diversity of mathematical and didactic knowledge and the different institutional contexts in which educational-instructional activity occurs. The theory of truth as coherence is relevant when dealing with formal mathematical knowledge. However, no single method exists to develop or justify a mathematical proposition. At early educational levels, learning arithmetic, for example, may require children to work with concrete numbers and construct knowledge of numbers through empirical arguments. In this case, the truth of the propositions is established in correspondence with reality. Likewise, the results of didactic experimentation involve empirical objects and processes, so conjectures are validated in terms of truth as correspondence with facts and results. We also consider the discursive and consensual theory of truth (Habermas, 1997) when developing criteria for didactic suitability (Chapter 5). The criteria are derived from OSA ontological, epistemological and semiotic assumptions and usually constitute shared and rationally justified value judgements within mathematics education.

3.5. Educational mathematics as an ecology of meanings

Toulmin (1977) introduced the expression *intellectual ecology* in the epistemology of knowledge to describe the questions of function and the adaptation of concepts and methods of thought to the real needs and demands of problematic situations. Morin (1992) considered the belief in the physical reality of ideas as inadequate as the denial of a kind of reality and objective existence to the habitat, life, customs, and organization of ideas. For Morin, ideas (and therefore mathematical notions), besides constituting instruments of knowledge, have their own characteristic existence. White (1983) stated that within the framework of mathematical culture, actions and reactions occur between different elements. "One concept reacts on others; ideas mix, merge, form new syntheses" (White, 1983, p. 274).

Ecological metaphors about ideas are useful for analyzing the relationships between school and expert mathematics. To describe the processes of selection and elaboration of school mathematics, these relationships are often subordinate, which is why the metaphors of didactic transposition, elementarization, and transformation are used (Scheiner et al., 2022). In OSA, the metaphor of the ecology of meanings is proposed to describe these processes and the relationships between different types of mathematics (Godino & Batanero, 1998). Each mathematical object has different meanings, with different degrees of generality and levels of formalization; therefore, educational agents select and sequence appropriate meanings according to the context, students’ abilities and motivations. The ecological metaphor reflects well the phenomena of competition, symbiosis, collaboration, and, in a certain sense, the trophic chains established between different types of mathematical knowledge (Godino, 1994). Only the knowledge best adapted to the given context survives or thrives.

The ecological metaphor of school knowledge assumes multiple and diverse mathematics as not only a starting point (professional contexts) but also a point of arrival (school contexts). The progressive growth of knowledge throughout the curriculum can be explained more precisely as a process of mutation driven by educational action that transforms from simpler to more complex forms. This contrasts with the notion of transposition or transformation from more abstract to more elementary forms (Scheiner et al., 2022). The ecology of meanings, i.e., understanding the meanings of concepts systemically and pragmatically (Godino et al., 2021), more accurately reflects the correspondence between the different types of knowledge involved in educational settings. Interpreting the meanings of a mathematical object in terms of systems of practices facilitates the consideration of such systems, and consequently pragmatic meanings, as new objects that relate to others to form new structures.

3.6. Pragmatic meanings and ontosemiotic configurations. An example of articulation

One aim of didactic-mathematical analysis should be to characterize the diverse meanings of objects and their interrelationships, thus constructing a global meaning that serves as a reference for the analysis of educational-instructional processes. This is the first level of ontosemiotic analysis of mathematical activity through which one notices the plurality and relativity of the meanings of mathematical objects. This first level involves identifying, classifying, and describing the problem situations in which the object in question intervenes, as well as the mathematical practices (operative, discursive, and normative) through which these problems are answered. In this way, one moves from the mathematical object, which initially becomes a black box, a label referring to a mental, ideal, or abstract entity, to the practices involved in using such an object.

Once a meaning for a mathematical object has been identified, one has a type of problem situation that can be concretized in a prototypical exemplar and the sequence of practices necessary to solve it. The identification of the web of interrelated objects involved in these practices is necessary to manage mathematical study processes and to become aware of the ontosemiotic complexity of mathematical activity as an explanatory factor of learning and teaching difficulties. The notion of ontosemiotic configuration of practices, objects, and processes guides this second level of didactic-mathematical analysis in the OSA framework.

In this section, we exemplify the articulated use of pragmatic meanings and ontosemiotic configuration tools for the case of mathematical object proportionality following Godino et al. (2017). Figure 3.5 illustrates the relationship between these two theoretical tools.

3.6.1. Pragmatic meanings of proportionality

The *universe of meanings* of proportionality can be classified according to different criteria, in particular, the context or field of application and the level of algebraization of the mathematical practices involved. Some application contexts of the notions of ratio and proportion (daily life, scientific-technical, artistic, geometric, probabilistic, statistical, etc.) involve the participation of specific objects and processes from these fields in the practices of solving problems, as revealed by many studies on the nature and development of proportional reasoning (Freudenthal, 1983; Lamon, 2007; Tourniaire & Pulos, 1985). It is possible to delineate variants of meanings specific to some fields of application of proportionality (geometric, probabilistic, etc.) and, as we shall see below, according to the level of algebraization of the mathematical activity performed to solve the problems.

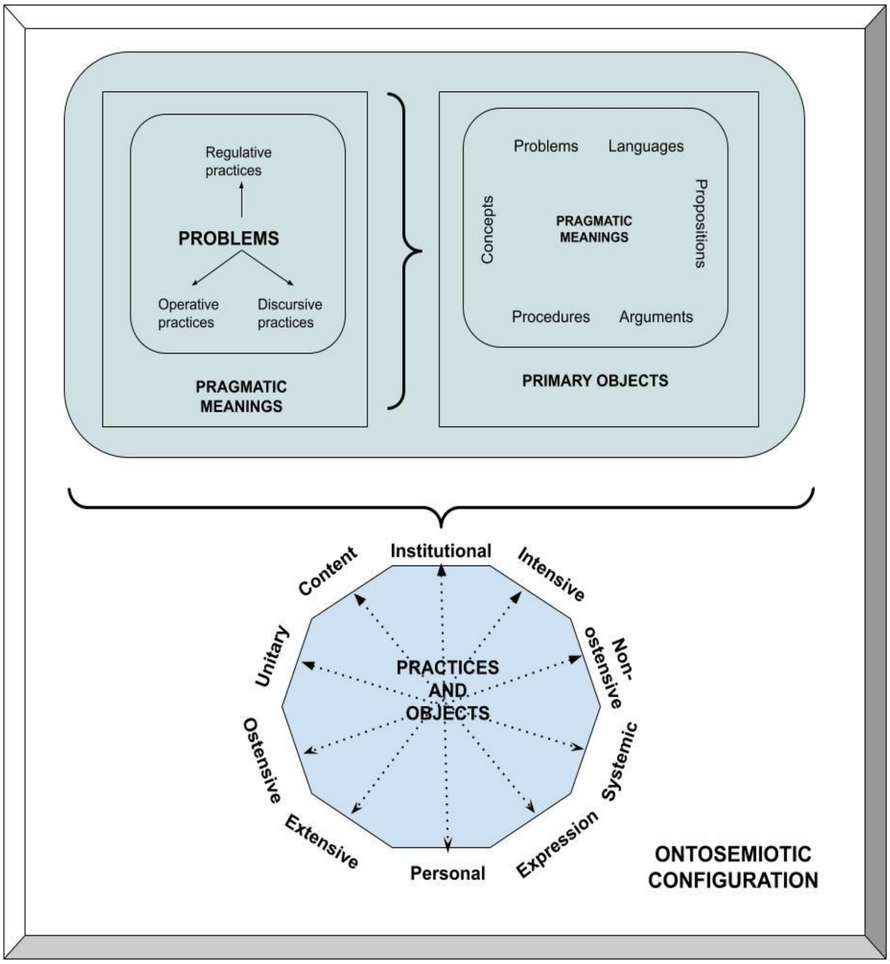


Figure 3.5. Articulation of pragmatic meanings and ontosemiotic configuration

In solving contextualized proportionality problems, magnitudes (lengths, areas, volumes, speeds, densities, etc.) and their respective measures are involved. At a stage of the problem-solving process, the relationships established between the quantities (ratios, proportions) are expressed using the numerical values of the measures, operations are performed with the corresponding real numbers, and finally, the solution is interpreted in terms of the context. In the phase of intra-mathematical modeling, the three meanings of proportionality described in this work are brought into play, together with the pragmatic meanings linked to the application contexts. These three meanings, along with the informal/qualitative ones, are neither exhaustive nor independent, making it possible to identify partial meanings within each category and mathematical practices that involve several of them. It is important to consider diverse meanings in the design of instructional processes, which should occur over an extended period (primary and secondary education) and in different content areas, as described by Wilhelmi (2017) and Burgos and Godino (2020).

Arithmetic meaning

We use the following missing value problem to illustrate the various systems of practices through which its solution can be approached:

A 500-g coffee package is priced at just 5 euros. What is the price of a 450-g package?

Applying arithmetic procedures (multiplication, division) characterizes the arithmetic meaning as follows:

1) In everyday buying and selling situations, it is usually assumed that when buying smaller quantities of coffee, each gram costs the same.
2) Consequently, if double, triple, etc., the amount of product is bought, then double, triple, etc., the price should be paid. Similarly, if half, a third, etc., of the product is bought, then half, a third, etc., the price should be paid.
3) If a 500 g package of coffee is sold for 5 euros, the price of 100 grams of coffee (five times less) should be one-fifth of 5 euros, that is, 1 euro.
4) The price of 50 grams (half of 100 grams) should be half, that is, 50 cents.
5) Thus, 450 grams of coffee should cost $4\times1 + 0.50 = 4.50$; that is, 4 euros and 50 cents.

Practice 1) has a discursive-descriptive character for the problem situation, while the remaining practices have a normative and operational character. The solution involves particular numerical values, and arithmetic operations are applied to these values; therefore, according to Godino et al. (2014), the mathematical activity performed is level 0 of algebraization because no algebraic objects and processes are involved.

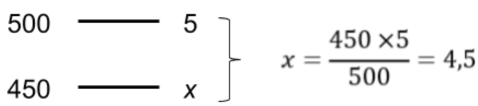
Proto-algebraic meaning

The proto-algebraic meaning is centered on applying the notion of proportion and solving an equation of the form Ax = B, as shown in the following sequence of practices:

1) It is assumed that if double, triple, etc., the amount of product is bought, then double, triple, etc., the price should be paid.
2) Therefore, the relationship established between the amount of the product bought and the price paid is directly proportional.
3) In direct proportionality, the ratios of the corresponding quantities are equal: $5/500 = x/450$; where x is the price at which 450 grams of coffee should be sold.
4) In any proportion, the cross-product equality holds:
5) $5 \times 450 = 500 \times x$,
6) Thus, $x = (5 \times 450)/500 = 4.5$.
7) Therefore, the price of the package should be 4.5 euros.

Although solving a missing value problem based on the use of ratios and proportions involves an unknown and formulating an equation, the algebraization activity performed is of level 2 (proto-algebraic), according to the model by Godino et al. (2014), because the unknown is isolated in one member of the equation established by the proportion.

A diagrammatic variant of this solution technique is known as the “rule of three”, which in a way “hides” involving ratios and proportion, potentially leading to a “degenerate” meaning of arithmetic proportionality.



Algebraic-functional meaning

A proper algebraic meaning is characterized by applying the notion of linear function and using solution techniques based on the properties of these functions:

$f(a+b)=f(a)+f(b)$, $f(ka)=kf(a)$.

One such technique can be applied as follows:

1) It is assumed that if double, triple, etc., the amount of product is bought, then double, triple, etc., the price should be paid. The amount paid for two different coffee packages is equal to the price of one package that weighs the same as the two packages combined.
2) Therefore, the established correspondence between the set of product quantities (Q) and the set of prices paid (P), $f\colon Q \to P$, is linear.
3) In every linear function *f*, the image of the sum of quantities is the sum of the images, $f(a+b)=f(a)+f(b)$, and the image of the product of a quantity by a real number is the product of the image of the quantity by that number, $f(ka)=kf(a)$.
4) The coefficient k of the linear function is the proportionality coefficient in the case of direct proportionality relationships between magnitudes (ratio).
5) Applying these properties to the case, we have:

$f(500g)=5€$; $500f(1g)=5€$; $f(1g)=\frac{5}{500}€$ [A gram of coffee costs 1 cent.]

6) $450f(1g)=450\times\frac{5}{500}€$; $f(450g) = 4{,}5€$.
7) Thus, the price of the 450-g package is 4.5 euros.

The diagrammatic representations of solutions involving the notion of a function are presented in Figure 3.6. In these cases, the mathematical activity can be classified as proto-algebraic level 1.

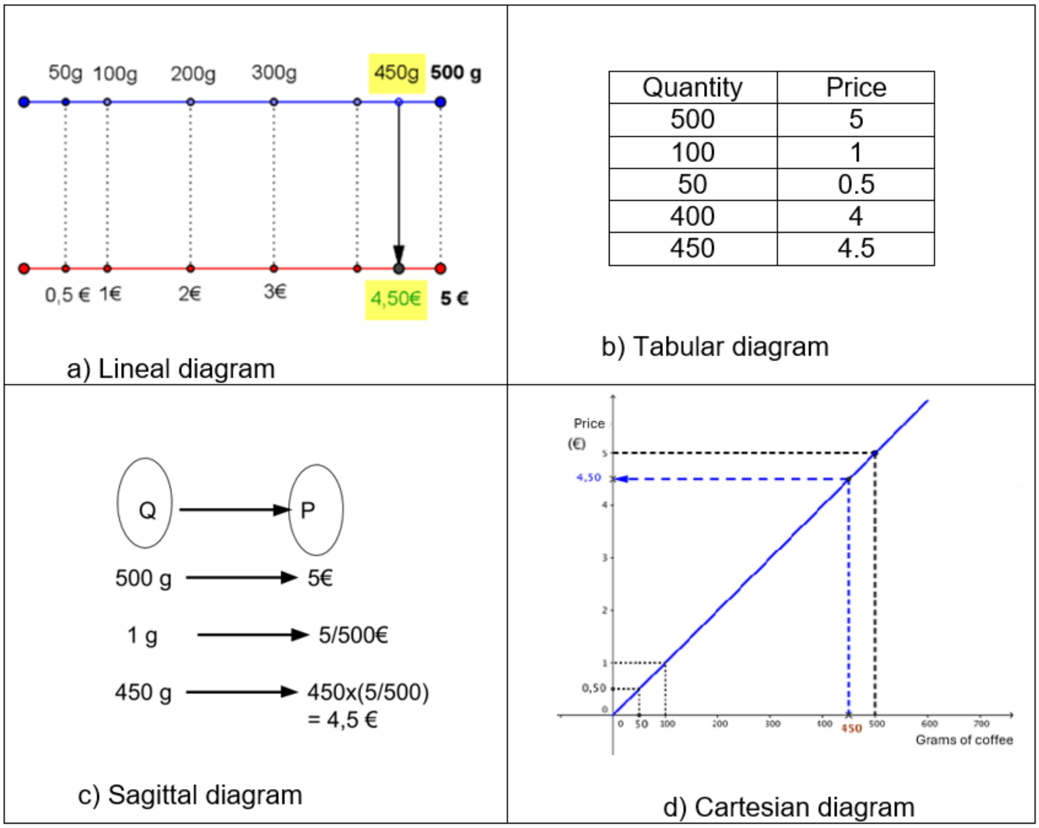


Figure 3.6. Diagrammatic solutions

Some authors (Bolea et al., 2001; Obando et al., 2014) have emphasized proportional reasoning as involving a linear function in a two-variable system. Thus, the mathematical model is a function of the form $y = k \cdot x$, where k is the constant ratio, which is known as the proportionality constant. Although it is referred to as a “linear function” in the singular, knowledge of the structure of a family of functions is involved because k acts as a parameter, representing an initial contact with level four of algebraization, as defined by Godino et al. (2015).

Given the mathematical effectiveness of algebraic reasoning, it seems desirable from the perspective of epistemic suitability (see Chapter 5) that instructional processes tend to achieve the algebraic level of meaning required for proportional reasoning. However, it does not seem ideal, from a cognitive and affective perspective, to dispense with the preceding levels.

However, the resolution of problems involving proportionality in daily and professional life can be ideal through the application of procedures typical of arithmetic meaning.

3.6.2. Ontosemiotic configurations

In this section, we analyze the practices corresponding to the proto-algebraic and algebraic-functional solution of the problem (cost of the coffee package) by applying the notion of ontosemiotic configuration. The aim is to identify the types of mathematical objects and processes put into play and therefore the knowledge involved in each case to form an expected or expert solution.

Proto-algebraic configuration

The first column of Table 3.3 includes the sequence of elementary practices of a possible proto-algebraic solution expected for the problem. The second column shows the role and intentionality of each practice in the sequence of practices included in column 1, and the third column indicates the conceptual, propositional, procedural, and argumentative objects involved in these practices. In this way, semiotic functions (relationship between expression and content), established between the ostensive objects of textualized practices, and the non-ostensive objects referred to by them (processes of signification/interpretation) are made explicit. It is therefore assumed that the elementary practices reported in column 1 are constituted by written expression in natural, numerical, and symbolic language and, therefore, ostensive of the actions that the epistemic subject performs to solve the problem. Non-ostensive elements that necessarily intervene in the subject's actions are referred to in the other columns.

Table 3.3. Ontosemiotic configuration of the proto-algebraic solution

Sequence of elementary practices to solve the task	Use and intentionality of practices	Objects referred to in the practices (concepts, propositions, procedures...)
1) It is assumed that if double, triple, etc., the amount of product is bought, then double, triple, etc., the price should be paid.	Indicate that the conditions for the application of direct proportionality are fulfilled in the context of the problem.	*Concepts*: multiplication; unlimited sequence; functional correspondence, magnitude, quantity, measure. *Proposition P1*: statement of practice 1. *Argument*: pragmatic convention
2) Therefore, the relationship established between the amount of the product bought and the price paid is directly proportional.	State that the relationship established between the heterogeneous quantities is one of direct proportionality.	*Concepts*: ratio, quantity, product direct proportionality *Proposition P2*: the relation between both magnitudes is of direct proportionality. *Argument:* the conditions defining direct proportionality are met.
3) In direct proportionality, the ratios of the corresponding quantities are equal: $5/500 = x/450$; where x is the price at which 450 grams of coffee should be sold.	To represent with a literal symbol the missing value.	*Concepts:* direct proportionality, equality, equation, ratio of quantities, unit price, proportion, unknown *Proposition P3*: the ratios are equal *Argument:* because the unit price is the same in both packages
4) In any proportion, the cross-product equality holds:	Relate the unknown with the data.	*Concepts:* equality, proportion, product *Proposition:* statement of 4) *Argument:* based on a property of proportions
5) $5\times450 = 500\times x$,		*Concepts*: equality, ratio, equation *Procedure:* to isolate the unknown. *Argument*: arithmetical properties, deductive
6) Thus, $x = (5\times450)/500 = 4.5$.	Operate with the unknown	*Concepts*: magnitude, quantity, measure, unit. *Proposition*: the price of the package is 4,5€. *Argument:* sequence of practices 1) to 5)

The analysis of the practices performed in this solution procedure reveals that they involve the use of an unknown, and to find it, a first-degree equation is established. As mentioned above, the activity has a proto-algebraic level 2 character according to Godino et al. (2014).

Algebraic-functional configuration

Table 3.4 shows the sequence of practices put into play in the solution that we have called algebraic-functional to the problem of the price of a package of coffee, the intentionality of these practices, and the objects referred to in the practices.

Table 3.4. ontosemiotic configuration of the algebraic solution

Sequence of elementary practices to solve the task	Use and intentionality of practices	Objects referred to in the practices (concepts, propositions, procedures...)
1) It is assumed that if double, triple, etc., the amount of product is bought, then double, triple, etc., the price should be paid. Additionally, the amount paid for two different coffee packages is equal to the price of one package that weighs the same as the two packages combined.	Explain that the conditions for applying the linear function between sets of magnitude quantities in the context of the problem.	*Concepts*: multiplication, unlimited sequence, proportionality, functional correspondence, magnitude, quantity. *Proposition P1*: the statement of practice 1). *Argument*: pragmatic convention
2) Therefore, the correspondence established between the set of product quantities (Q) and the set of prices paid (P), $f: Q \to P$, is linear.	Declare that the relationship established between heterogeneous magnitudes is linear.	*Concepts*: set, correspondence, magnitude, quantity, measurement, linear relationship. *Proposition P2*: The correspondence between sets of quantities is linear. *Argument:* the conditions that define the linear function according to 1) are met.
3) In every linear function f, the image of the sum of quantities is the sum of the images, $f(a+b) = f(a) +$	Explain the definition conditions of linear functions in two ways: natural	*Concepts*: sum of quantities, product by a scalar, original and image of a function, linear function, product.

	$f(b)$, and the image of the product of a quantity by a real number is the product of the image of the quantity by that number, $f(ka) = kf(a)$.	language; literal symbolic language.	*Procedure*: natural language translation into symbolic.
4)	The coefficient k of the linear function is the proportionality coefficient in the case of direct proportionality relationships between magnitudes (ratio).	Interpret the coefficient k of the line functions in terms of the context of the problem (coefficient of proportionality or both times one).	*Concepts*: direct proportionality, magnitude, coefficient of proportionality.
5)	Applying these properties to the case, we have: $f(500g) = 5€;\ 500f(1g) = 5€;\ f(1g) = \frac{5}{500}€$ [A gram of coffee costs 1 cent]	Calculate the unit cost.	*Concepts*: linear function; equality; proportionality. *Procedures:* translation from natural language (statement) to symbolic; calculation of the proportionality coefficient based on the definition conditions of a linear function.
6)	$450f(1g) = 450 \times \frac{5}{500}€;\ f(450g) = 4{,}5€$.	Calculate the missing value.	*Concepts*: linear function, equality, proportionality. *Procedure*: calculation of the missing value based on the definition conditions of the linear function.
7)	Thus, the price of a 450-g package is 4.5 euros	Interpret the numerical result as a solution to the problem.	*Proposition*: package price is €4.5. *Argument*: sequence of practices 1) to 6).

Since the solutions to the problem, which have been analyzed from an institutional point of view, are expected or expert solutions, such systems of practices and configurations have an epistemic character. The same technique can be applied to the students' answers to obtain the corresponding cognitive configurations. The ostensive-non-ostensive duality is useful here by distinguishing between the textualized practices located in the first column as ostensive objects that evoke and represent the conceptual, propositional, procedural, and argumentative objects identified in the third column. Ostensive objects also play an instrumental role, as shown in the second column.

3.7. Concordances and complementarities between semiotic theories

In this section, we analyze the concordances and complementarities between the OSA theory of meaning and cognition and the theories of meaning described in section 3.2—Frege, Vergnaud, and Steinbring—as well as between the Theory of Registers of Semiotic Representation (Duval, 1995; 2006) and the OSA.

3.7.1. Theories of meaning *versus* OSA

The use of the terms meaning and sense by different authors and disciplines is linked to the notion of object and, in mathematics, to the nature of abstract objects. Therefore, semiotics is essentially linked to ontology, the different types of objects that signs refer to, and the various modalities in which objects can participate in communication and interpretation. The answers to the question of meaning by Frege, Vergnaud, and Steinbring substantially differ regarding the nature of the objects referred to or represented by signs, although all three models are triadic. Frege assumes a Platonic, transcendentalist position on the reference (the referred object). The barycenter, for example, is unique and can be represented in different ways, each providing a distinct meaning. In a way, Vergnaud's and Steinbring's models respond similarly to the question of what, for instance, the word "number" represents: it represents the (ideal, abstract) concept of number; but for the question of what the number means, or what the number is, the answer differs: a heterogeneous system formed by three components (triplet): situations, invariants, representations (Vergnaud); the triplet sign, object, concept (Steinbring).

In the OSA, we find significant differences in the response to the question of the meaning of a mathematical concept, considering that these objects cannot be detached from mathematical practices, assuming an anthropological perspective for mathematics, i.e., conceiving mathematics as a human activity. Additionally, objects and practices can be viewed from institutional and personal perspectives and systemic and unitary perspectives.

When an object intervenes in a unitary manner, the answer to what its meaning is would be one of its possible definitions (rules that intensionally define the concept). When the object intervenes systemically, the answer to this question would be the system of operative and discursive practices in which the object critically intervenes, thus including one of the possible definitions, along with the situations, languages, properties and arguments involved (partial meaning). It is also necessary in the epistemological and didactic analysis of a mathematical object to consider the diversity of partial meanings an object can have and their articulation into a global meaning, as seen in Batanero and Díaz (2007) and Burgos et al. (2022) for probability, Burgos and Godino (2020) for proportionality, and Wilhelmi et al. (2007) for equality of real numbers.

The ontosemiotic position advances in the progressive complexity of mathematical concepts by first connecting it with human activity and is mediated by linguistic and material artifacts involved in solving specific problem situations. Subsequently, it is established that each sense or partial meaning is linked to a specific rule (concept-definition) for using linguistic elements in a class of situations (contexts, phenomena) and to other procedural, propositional, and argumentative objects. Finally, the various partial meanings are organized into a holistic meaning formed by the web of senses and accompanying objects (Figure 3.7).

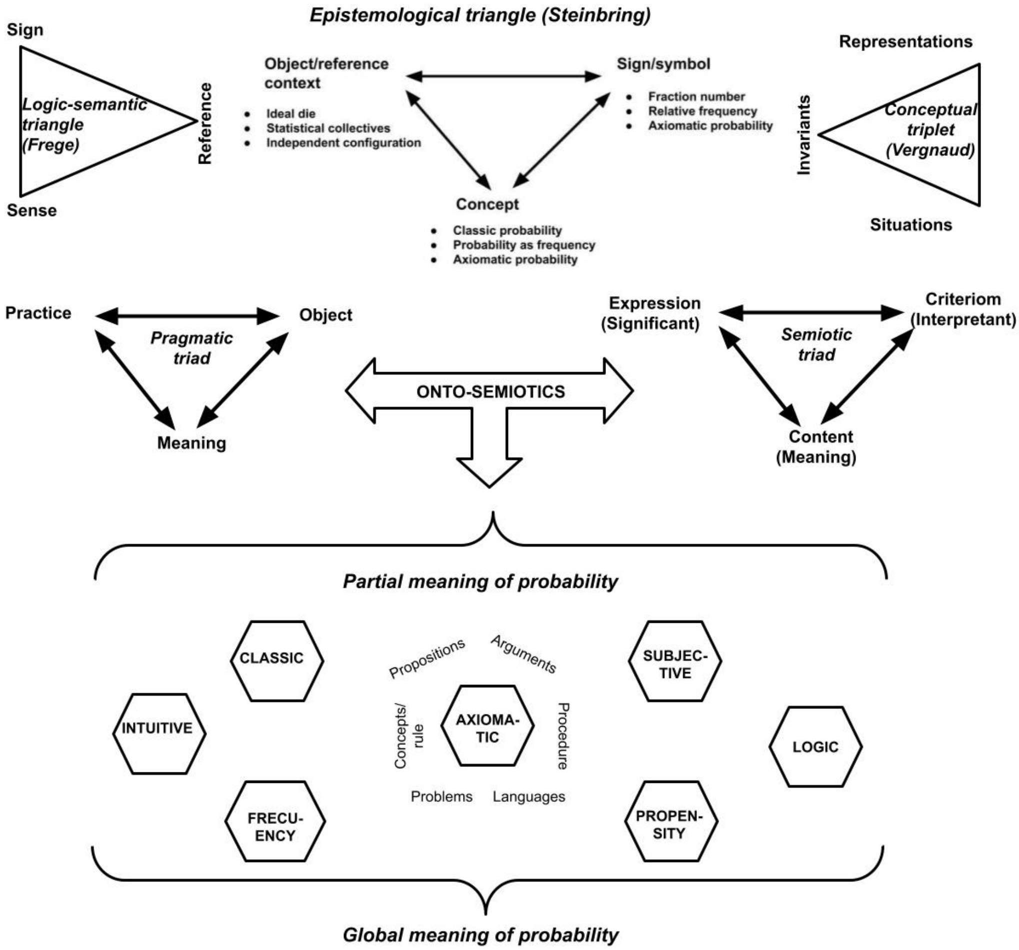


Figure 3.7. From the epistemological triangle to the web of onto-semiotic configurations

The OSA also considers the use of meaning in its functional or operational interpretation, that is, the use of objects in various practices. For example, numerical symbols not only refer to the corresponding concepts but are also instruments for counting, numbering, ordering, and so on. As stated in Figure 3.7, from the ontosemiotic perspective of mathematical knowledge, it is useful to complement the semiotic triad (expression, content, criterion) with the pragmatist triad (practice, object, meaning), to articulate the anthropological analysis of mathematical activity with the analysis of texts reflecting that activity.

Figure 3.7 also suggests that the partial meanings of probability (intuitive, classical, subjective, frequentist, logical, propensity, and axiomatic) reflect different senses (Frege) and the composition of the concept of probability (Steinbring). Each partial meaning can be analyzed in terms of ontosemiotic configurations formed by six components (problems, languages, concepts/rules, propositions, procedures, and arguments), which expand Vergnaud's conceptual triplet.

3.7.2. Registers of semiotic representation versus OSA

The Theory of Registers of Semiotic Representation (TRSR) (Duval, 1995; 2006) enables the analysis of the various types of material representations used in performing mathematical tasks, their transformations, and their role in understanding mathematics. The availability and use of different semiotic representation systems, their transformations, and conversions are essential for understanding, constructing, and communicating mathematics. It is also assumed that the production and apprehension of material representations are not spontaneous and that their mastery must be planned in teaching. Godino et al. (2016) explored the possibilities of articulating this theoretical framework using the OSA ontosemiotic configuration tool. Below, we present a synthesis of this work.

In an initial approach, it can be anticipated that the notion of a register of semiotic representation, its various types and the operations of treatment and conversion between registers, allow for the development of an analysis of linguistic elements, thus enriching OSA. Similarly, the notion of configuration of objects and processes can enrich the TRSR by enabling a detailed analysis of the knowledge involved in transformations between representation registers in mathematical practices (Pino-Fan et al., 2015). Although there are significant differences in the ontological and semiotic assumptions of both theoretical frameworks, it is hypothesized that a certain articulation between them is possible, allowing for more detailed cognitive and epistemic analyses of mathematical activity and consequently contributing to the understanding of teaching and learning processes.

Ontological and semiotic assumptions of both theoretical frameworks

TRSR implicitly assumes an empiricist position regarding the nature of mathematical objects. This is inferred from the postulate that the use of two or more representations is necessary for understanding an object and that such an object is “the invariant of a set of phenomena or the invariant of some multiplicity of possible representations”. The assumed semiotics emphasizes the representational/referential facet, although the instrumental/ operational valence of symbolic inscriptions is also recognized: “The writing of a number represents a number and has an operative meaning linked to the treatments used to perform the operations. Treatments are not the same for decimal and fractional writing”. (Duval, 1995, p. 64)

The OSA ontology is pragmatist-anthropological, and its semiotics is essentially Peircean-Wittgensteinian. Material or external representations have a representational valence when referring to another non-ostensive object, and an operational valence, as ostensive objects are used for mathematical work without necessarily representing another object. Following Peircean semiotics, the antecedent of semiotic functions can be non-ostensive objects.

In TRSR, the mathematical object is seen as a “knowledge object,” as a cognitive entity residing in the mind of the individual subject, and an essential issue is the study of the cognitive operations necessary to perform different mathematical tasks, whether calculations, reasoning, or the use of a figure in a geometric proof. When a subject is unable to perform a particular conversion or treatment, from the TRSR perspective, it can only be said that they lack knowledge of the corresponding mathematical object. Here, OSA complements the representational analysis of TRSR.

Adopting the anthropological postulate for non-ostensive objects (concepts, propositions, procedures) allows answering questions such as: Why semiotic representations are necessary in mathematical activity? What is the relationship between a mathematical object and its various representations? Mathematical objects are the grammatical rules of the languages we use to describe our worlds, making their use (semiotic representations) indispensable. There can be no grammar without language. Furthermore, grammatical rules should not be confused with linguistic statements.

Semiotic function versus representation

The notion of a register of semiotic representation and its types, treatments and conversions between registers provides an analytical resource that develops and complements the primary object language category of the OSA. TRSR expands the language category of OSA by distinguishing different types and revealing the essential role of transformations performed between (and within) different types of languages, which are now considered RSR. Given the intra-discursive nature of mathematical objects, it is necessary to consider the web of objects involved in transformations performed using semiotic representations.

The exemplar-type duality (extensive-intensive) applies to all primary objects, including linguistic elements. This allows us to interpret the relationship between "semiotic representation" and "register of semiotic representation" as extensive-intensive. A semiotic representation is a particular exemplar, and a register is a type or class of representation. Specific transformations must be possible following a set of rules among the constituent elements of the types of representation.

The semiotic function broadens the notion of representation (Godino & Font, 2010). The pragmatist/anthropological semiotics assumed by OSA proposes that the objects brought into correspondence in semiotic functions (functives) are not only ostensive linguistic objects (words, symbols, expressions, diagrams), but also concepts, propositions, procedures, arguments, and even situations that can be antecedents of semiotic functions. Functives can also be unitary or systemic entities, particular or general, material or immaterial, personal, or institutional.

Each RSR used to represent and operate a mathematical object provides a specific meaning for that object. Understanding an object in its entirety requires the articulation of different partial meanings (or senses), which cannot be achieved spontaneously. Using natural language, numerical (decimal, fractional), algebraic, diagrams, geometric figures, Cartesian graphs, and tables are different RSRs, each posing specific learning challenges. It is not sufficient to know the correspondence rules between two different registers, is not sufficient for them to be mobilized and used appropriately.

3.8. Examples of ontosemiotic analysis of mathematical cognition

3.8.1. Natural numbers as cultural and personal objects

The nature of natural numbers and their relationship with sets is an issue of interest to both mathematics and the philosophy of mathematics. However, numbers are also essential tools in our daily and professional lives, making them a crucial topic of study in schools from the earliest levels.

It is necessary to distinguish between the practical and "informal" uses of numbers (answering questions such as "How many elements are there?" or "What position does an object occupy?") and the "formal" uses (what numbers are and how numerical systems are constructed). The latter refers to the foundations of mathematics as an organized body of knowledge. Within these two broad contexts of use (or institutional frameworks), it is possible to identify various historical moments in which these questions are addressed with different resources and approaches, involving specific operative and discursive practices. Retrospectively, certain invariances allow us to speak of the "natural number" in the singular. However, from a local perspective, it seems necessary to distinguish between the various natural numbers "handled" by primitive peoples and ancient cultures (Egyptians, Romans, Chinese, etc.), as well as between the numerical practices currently performed in preschool or primary school and those performed by 19th-century logicist mathematicians or Hilbertian axiomatic formulations.

We refer to Godino et al. (2011) for an analysis of the informal characteristics of numbers and formal semiotic systems from an institutional perspective. The informal semiotic systems in which natural numbers are used are characterized by specific problems (describing the numerosity of collections of things) and the use of linguistic resources, procedures, properties, concepts, and justifications to solve these empirical problems. From a formal perspective, the mathematical entities involved in cardinality and arithmetic calculations are analyzed structurally within the internal framework of mathematics. Numbers are no longer considered a means of expressing quantities (number of people or things) but are interpreted as elements of a structure characterized according to set theory, Peano's axioms, or equivalent systems. The ontosemiotic analysis conducted by Godino et al. (2011) of the natural number, from an epistemic or institutional perspective, justifies recognizing the plurality of numbers and their meanings when interested in teaching and learning processes at various educational levels.

The theoretical tools introduced in OSA, such as the system of practices and configuration of objects and processes, can be used to describe and understand semiotic systems formed by students' responses to specific mathematical tasks. Godino et al. (2011) illustrated this by analyzing the responses of a first-grade child to a counting and writing task involving numbers greater than 10 in the decimal numbering system. This allows understanding the complexity of this process and anticipating informed interventions to promote learning.

3.8.2. Meanings of the concept of function and development of functional reasoning

Godino et al. (2024) studied the diversity of meanings of the function and its progressive articulation, considering the levels of generality and formalization that emerge in the stages of historical evolution. According to previous research, they identified partial meanings of the function (operational-tabular, operational-graphic, algebraic-geometric, analytic, arbitrary correspondence between numerical sets, and set-theoretic) that can be considered part of the global reference meaning in planning and managing teaching and learning processes of functions. This study provides a complementary view to the numerous investigations that have described the phylogenesis of the concept of function in mathematics from a historical and epistemological perspective.

The diagram in Figure 3.8 summarizes the evolution of the meanings of the function concept and the levels of functional reasoning. From the moment explicit definitions of functions appear (J. Bernoulli, Euler), a substantial change occurs in the ontological nature of the concept and the type of activity in which it participates. As occurs at the ontogenetic level, as proposed by theories of cognitive development (Piaget, Dubinski, Sfard), there is a transition from the operational, processual stage to the objectual stage, in which the concept becomes part of cognitive schemas that allow the subject to understand, make decisions, and act in similar situations.

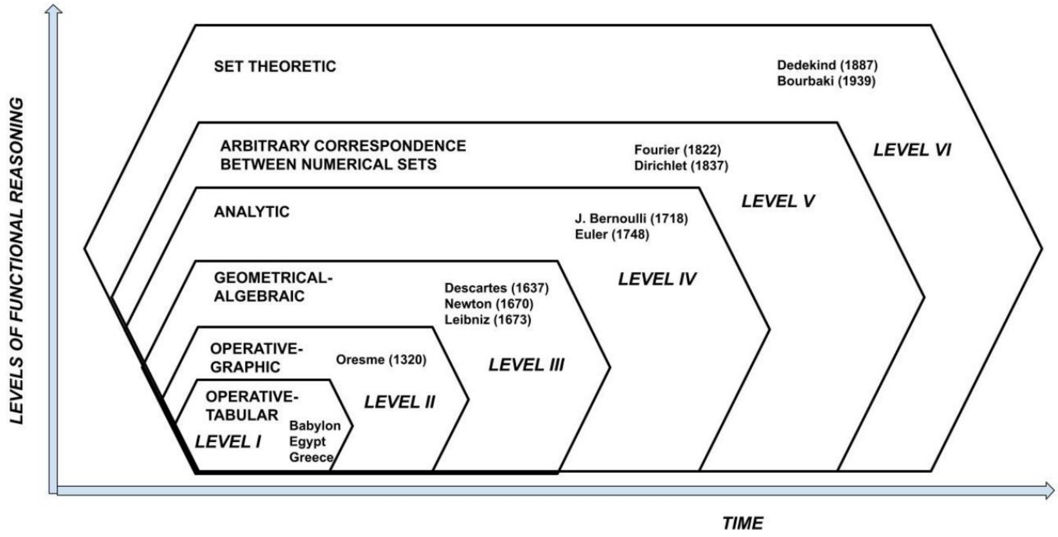


Figure 3.8. Evolution of the meanings of the function concept. Levels of functional reasoning (Godino et al., 2024, p. 29)

At the phylogenetic level, the function becomes part of the repertoire of mathematical objects, like numbers, geometric figures, and equations. Different types of functions have been invented to model various phenomena. Their specific properties (continuity, differentiability, etc.) are studied, allowing the definition of new functions, and they play a role in a new ecological niche characterized by formalization, generalization, and rigor.

The historical evolution of the concept of function reflects the inherent tendency of mathematical work to generalize concepts and procedures to solve increasingly complex and general problems. This tendency arises from "the practical necessity of unifying, through underlying general principles, those aspects of numerous theories that promise to have more than transient interest" (Bell, 1945, p. 470). Thus, the formulation of the function in terms of the correspondence between the elements of sets according to arbitrary criteria, not necessarily through analytical expressions, responds to the need to account for functions that could neither be drawn nor expressed algebraically, such as the Dirichlet function. Another qualitative leap is the use of structural algebraic language in the study of functions, fundamentally addressing the preservation of structures through the application of morphisms (functions that preserve structure).

As Freudenthal (1983) highlighted, there is a great phenomenological variety in which the function object is involved, which, along with various forms of expression, procedures, propositions, and arguments, characterizes functional reasoning. Is it possible to identify a common trait that justifies using the same term "function" to name this variety of meanings? The notions of dependence, covariation, and prediction connect the first three meanings or uses of functions (Figure 3.8). This dependence can be expressed in tabular, graphical, or analytical forms; however, in all cases, variable elements of numerical sets are related to other numbers. The idea of variability and dependence has been lost in a more general and abstract set-theoretic meaning than in previous studies, but the idea of a connection or correspondence between objects based on some type of rule or criterion persists.

3.8.3. Other examples of institutional meaning reconstruction

Batanero and Díaz (2007) applied OSA theoretical notions to analyze the historical emergence of probability and its different current meanings (intuitive, classical, frequentist, propensity, logical, subjective, and axiomatic). They also described mathematical activity as a chain of semiotic functions and used the idea of semiotic conflict to provide an alternative explanation for some widespread probabilistic errors.

Font and Contreras (2008) applied the notion of semiotic function and the OSA mathematical ontology to analyze the processes of generalization and particularization in mathematics teaching and learning. Using the definition of a function derivative in a high school textbook as a context for reflection, these authors address the following issues:

– Delimiting the processes of particularization and generalization with respect to the processes of materialization and idealization.
– Developing a typology of generalization processes.
– Clarifying the role of the generic element in the particular-general relationship.
– Studying the relationship between generalization processes and other mathematical processes.

Montiel et al. (2009) applied OSA to analyze the mathematical notion of different coordinate systems, as well as some situations and actions of university students related to these coordinate systems in the context of multivariable calculus. The authors identify the objects that emerge from mathematical activity and make an initial attempt to describe an epistemic network for this activity. In another paper, Montiel et al. (2012) address different coordinate systems through the process of base change, as developed in the context of linear algebra, as well as the similarity relationship between matrices that represent the same linear transformation concerning different bases.

3.9. Ontosemiotic approach to affective domain in mathematics education[10]

Beltrán-Pellicer and Godino (2020) developed a model for analyzing the affective domain in mathematics education by applying the notions of practices, objects, and dualities from OSA. In their article, the authors address the following questions: Is an ontosemiotic approach relevant for studying the affective domain? Is it possible to provide new insights into affect in mathematics education through the OSA theoretical lens? Which theoretical models of affect can be incorporated and aligned with this

[10] The content of this section 3.9 is based on Beltrán-Pellicer and Godino (2020).

approach? The following sections describe the main features of the proposed model.

3.9.1. Primary affective entities

Following the pragmatic OSA epistemological assumptions, we inquire about the affective meaning of certain signs in any possible register or representation, which can be verbal or written expressions, observable behaviors, etc. This meaning should be sought in the systems of practices that a person engages in to solve a problem or toward a practice, object, mathematical process, or situation involving the study of mathematics.

There is consensus in mathematics education research that the affective domain comprises three components: emotions, attitudes, and beliefs. The origins of this classification are traced to McLeod (1992). In this study, we use this ontology of affective objects, adding values, which is a construct included in DeBellis and Goldin's (2006).

Affective situations

When a student faces a problem situation, an affective situation arises that juxtaposes the cognitive one, incorporating personal meanings in the form of emotions, attitudes, beliefs, and values. For instance, mental blocks toward a type of problem situation, a persevering attitude that facilitates applying problem-solving heuristics, or a specific belief about the nature of mathematical objects can be involved. In fact, all problem situations that require active student participation are strongly affective. Once the situation is presented, each student's personal beliefs come into play, whether toward mathematics as a subject or the context in which the situation is framed.

However, affective situations arise not only in response to problem situations; teaching and learning ecosystems provide constant reference points for the affective domain. Thus, there are situations of production, communication, and individual mathematical study. For example, in the class session itself, beliefs may emerge that influence the student's attitude that day without the need to propose any problem-situation. Therefore, it is feasible to describe an affective configuration for each of these situations, which includes the circumstances of each component of the affective domain: emotions, attitudes, beliefs, and values. Because we are interested in the relationships between affect and mathematical learning, we limit affective situations to circumstances involving mathematical content. The teacher can pose situations that specifically engage students' beliefs about a specific mathematical object.

Affective practices

Affective practices are actions or manifestations that accompany any mathematical practice: expressions about emotions, attitudes, beliefs, or values related to objects. Each of these affective expressions can vary in intensity throughout practice or even disappear, giving rise to new manifestations. Much of the affective trajectory remains hidden from the teacher because not all affective states are expressed. Moreover, it is not possible for one person to observe the entire group and interpret each student's small gestures or signs. However, an observation record, like a class diary (Porlán & Martín, 1991), helps collect data for later reflection. In addition, tools can be incorporated into teaching practice to gather information about the affective domain. For example, the "mood map of problems" (Gómez-Chacón, 2000), which the authors used in previous research (Beltrán-Pellicer, 2015; Beltrán-Pellicer & Godino, 2017). Each student draws pictograms from 14 possible ones (or makes marks on a worksheet) to express their feelings while solving a problem or task. The 14 pictograms represent 14 emotions: curiosity, greatness, boredom, indifference, mental block, desperation, calm, excitement, hurry, confusion, brainstorming, pleasure, fun, and confidence. This map aims at a dual objective. On the one hand, it is a meta-affective practice where students become aware of their own emotional dynamics while solving a mathematical situation. On the other hand, the information can be collected by the teacher, highlighting the affective factors that facilitate progress and reflecting on those that block or hinder progress.

Intervening and emerging objects

Although the categorization of the affective domain into emotions, attitudes, and beliefs is accepted by the research community, with values added, the meaning of these constructs remains controversial. To describe and catalog the affective objects that intervene or emerge in mathematical practices, we use the tetrahedral model proposed by DeBellis and Goldin (2006), where the meanings of affective constructs are described as follows (p. 135):

- Emotions: Rapidly changing feelings experienced consciously, preconsciously, or unconsciously during mathematical (or other) activities. Emotions vary from mild to intense and are embedded locally and contextually.
- Attitudes: Describe orientations or predispositions toward certain sets of emotional sensations (positive or negative) in particular (mathematical) contexts. This differs from the more common view of attitudes as predispositions toward certain behavioral patterns. Attitudes are moderately stable, implying an interactive balance between affect and cognition.
- Beliefs: To attribute truth or external validity to a system of propositions or other cognitive configurations. Beliefs are typically very stable, cognitive, and structured, intertwined with emotions and attitudes that contribute to their stabilization.
- Values: The ethical and moral components refer to deeply appreciated personal truths and commitments. They help motivate long-term decisions or establish short-term priorities. They can be highly structured to form value systems.

Given the interaction with the cognitive domain, it may be convenient to consider the various modes of expressing affects, such as gestures and ordinary language terms (Álvarez, 2012), as a category of affective objects that constitute the ostensible facet of affects. Emotions, attitudes, beliefs, and values are related to mathematical situations, practices, and primary mathematical objects. Therefore, it makes sense to investigate the affective components of proofs, procedures, and representations. Figure 3.9 summarizes the main affective-cognitive categories.

The characteristics of affective languages, which could be considered a fifth category of affective objects, expand the semiotic registers and representations that emerge from practices because a significant portion of the affective load is expressed non-verbally within a system of information transmission, where each element is interpreted by the different agents involved (teacher, students). Therefore, emotions can arise as an instant emotional response to a sensory stimulus, which may be mathematical (a problem field) or not (going to school). Although this distinction seems trivial, the origins of emotions are complex to interpret.

Affective languages deserve special attention, which is reflected in the key position presented in Figure 3.9. Language, in its different registers, constitutes not only a communicative vehicle but also a tool of meaning, composed of signs constantly interpreted. In the affective domain, nonverbal communication plays a fundamental role (Knapp et al., 2013). Just as students' productions, both written (also in their different registers) and verbal, provide indicators of the cognitive domain, a large part of affective information is transmitted through facial expressions, gestures, postures, and movements.

The meta-study by Harris and Rosenthal (2005) shows how students improve in certain facets when the teacher's nonverbal language includes signs of immediacy, such as gesturing while speaking, not sitting behind their desk, looking at the students while speaking, smiling, and using a nonmonotonous tone. Thus, students show interest in the course and the teacher, pay attention, and perceive that they have learned a lot in class (Rocca, 2004). The results of their study also revealed correlations between the teacher's nonverbal language and students' cognitive performance, although this is still under study (Witt et al., 2004). All these affective languages align with interaction patterns that can be grouped into one of the following three dimensions (Rompelman, 2002): the opportunity to respond in a trusting climate, the possibility of feedback, and consideration toward people (respect). Harris and Rosenthal (2005) also mentioned the difficulty of empirical investigation in a classroom environment because of the necessary apparatus to capture all the nonverbal information.

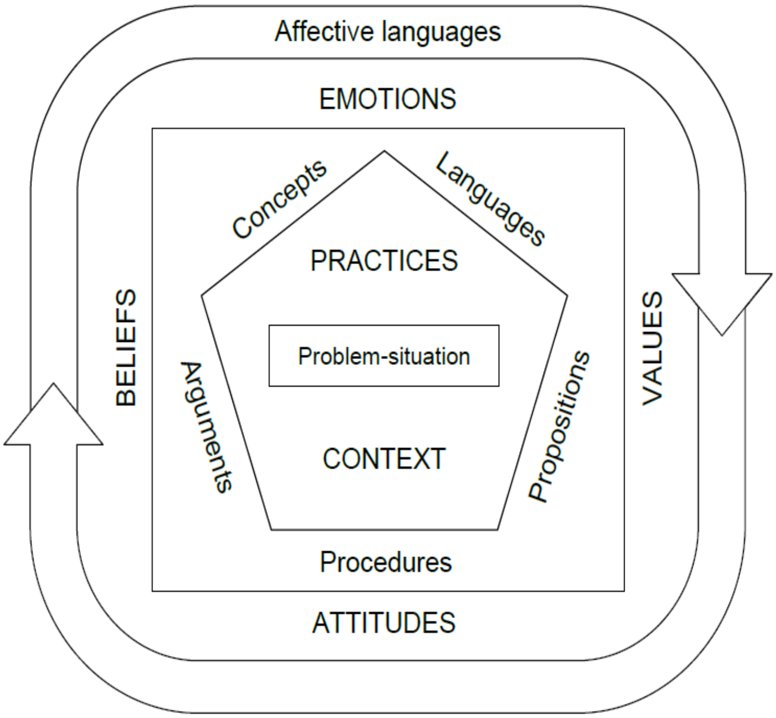


Figure 3.9. Primary affective and cognitive categories (Beltrán-Pellicer & Godino, 2020, p. 7)

Some other authors agree with this comment. Mitchell (2013) noted that given the positive relationship between the teacher's nonverbal language and students' attitudes, it is important that the teacher is not only enthusiastic about content but should also show enthusiasm to have a positive impact on students' learning.

3.9.2. Contextual dualities

Next, we analyze the four types of affective entities in DeBellis and Goldin's (2006) tetrahedral model from the perspective of the five pairs of contextual dualities introduced in the OSA: personal-institutional, expression-content, ostensive-non-ostensive, intensive-extensive, and unitary-systemic. We consider that this analysis allows us to articulate aspects of the affective domain that are treated non-systematically or tangentially in the literature. Figure 3.10 synthesizes these dualities into a single diagram, which we will refer to later.

Personal-Institutional

Affective objects and processes are often considered psychological entities, referring to individuals' almost stable mental states, traits, and dispositions. However, from an educational point of view, achieving affective states that interact positively with the cognitive domain should be of interest to the teacher, that is, to educational institutions. The existence of research on affectivity indicates that it is possible to identify phenomena, regularities, and shared conceptualizations that confer a certain degree of objectivity to affects and their influence on learning. The affective domain, therefore, has an institutional facet and is concretized in affective rules that condition teaching work. The personal-institutional distinction, for both cognitive and affective facets, allows us to focus on the dialectic between these dimensions, thus becoming aware of the various institutional conditions in which affective phenomena occur.

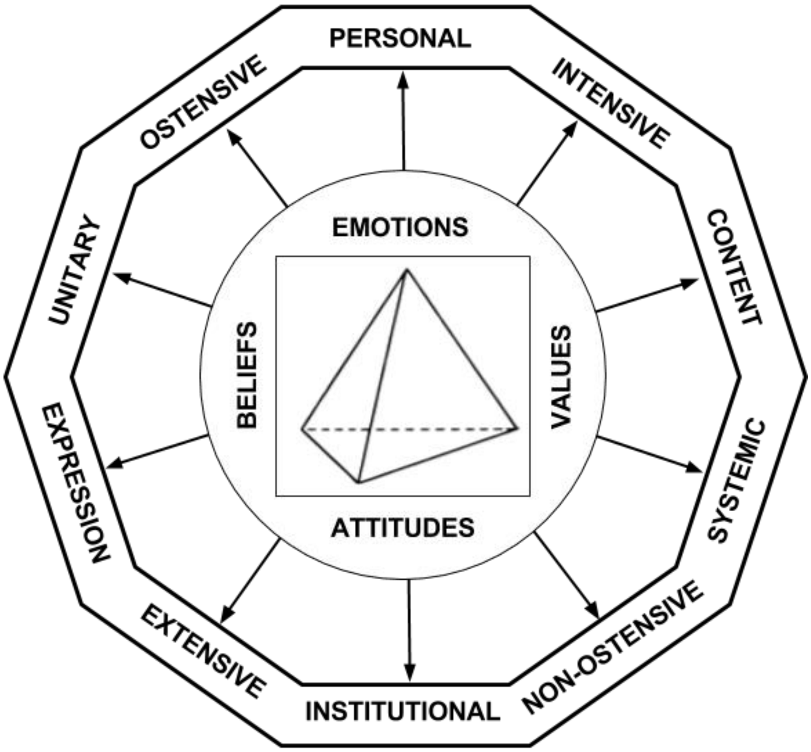


Figure 3.10. Contextual dualities for the affective domain (Beltrán-Pellicer & Godino, 2020, p. 10)

Different curricular regulations, such as the Spanish one (MEFP, 2022), establish guidelines from a socio-affective perspective, primarily concerning attitudes and values.

> The socio-affective sense integrates knowledge, skills, and attitudes to understand emotions, propose and achieve goals, and increase the capacity to make responsible and informed decisions. This aims to improve students' mathematics performance, reduce negative attitudes toward it, promote active learning, and eradicate preconceived ideas related to gender or the myth of indispensable innate talent (MEFP, 2022, p.142).

The institutional dimension is essentially static. However, these norms are first interpreted by the teacher, who, while planning each class, must incorporate the corresponding curricular guidelines and confront their own belief and value systems. At the second level, when the teacher effectively implements the sessions, emotions (instantaneous affective states) toward that group of students and specific content interact with attitudes, beliefs, and values. This forms a system within the personal dimension (of the teacher), which is continually reinforced. That is, emotions arise as representations (ostensible or not) of attitudes, beliefs, and values. These latter categories are reinforced or modified by the persistence of these emotions over time. The same occurs with students when they interact with institutional norms.

Furthermore, on the personal dimension is where other interactions occur between affective entities and other types of entities, such as those in the cognitive domain (Whitson, 1997). These interactions are processes of interpretation, and therefore of signification, of elements from one domain (e.g., epistemic or cognitive) that play the role of signs for the other domain (e.g., affective).

Expression-content (Affective semiotic functions)

Affective objects cannot be conceived as isolated entities; rather, they give rise to interpretative processes by individuals or institutions. In other words, they act as antecedents and consequents of semiotic functions. Goldin (2000, p. 211) attributes representational valence to affect as follows: "Note that the very notion of code suggests that something is being encoded, that affective configurations can signify or represent information". This component of Goldin's theory of representations finds a natural fit in OSA through the notion of semiotic function.

The (pragmatic) meaning of affect can be defined as a system of affective practices in which affect plays a relevant role. That is, the effects or consequences of an affect on the realization of a mathematical practice. Another use of the term affective meaning can be referential when the expression or antecedent of a semiotic function is an affective linguistic expression, and the consequent or meaning is the affect it refers to. Therefore, one can speak of emotional, attitudinal, etc., meaning of an affective expression. Thus, authors like Flavier et al. (2002) identify Peirce's sign object as the student's concern in a given situation, which opens the possibility of subjective judgments depending on prior experience. The representamen or sign, in turn, is an element of the considered situation that, in our case, would be each of the categories of mathematical objects. To complete the triadic conception of the sign, the interpretant represents the mobilization of knowledge during a situation. The notion of semiotic function is also useful as an entity that relates affective entities to each other from both a referential and operational perspective. It also connects affective entities with cognitive and epistemic entities.

The exchange of information between representational systems, as mentioned by Goldin (2000, p. 211) and DeBellis and Goldin (2006, p. 133), can also be interpreted using the notion of a semiotic function. In this way, the meanings encapsulated in each representation of the affective domain relate, through semiotic functions, to representations of other domains such as verbal, visual, formal, planning, and execution systems (Goldin and Kaput, 1996). Conversely, a function that has a visual representation (imagistic) as the starting domain, for example, can transfer that meaning to the arrival domain, evoking an affect related to the meaning encapsulated by the function.

Ostensive - non-ostensive

Affects are mental (or ideal) entities; that is, they are not ostensibly by nature (they are not directly perceptible). However, they manifest through concrete gestures and expressions, that is, through ostensive manifestations. DeBellis and Goldin (2006) studied how to infer internal entities from available observations and the exchange of information (interactions) between affective representation systems and other representational systems involved in problem-solving situations.

Since mathematical objects (concepts, procedures, propositions, and arguments) require linguistic elements for manifestation, language itself (in its various manifestations and registers) is considered within the OSA as a mathematical object, that is, an object that intervenes in mathematical practice. Identifying affective objects is even more difficult. Analogous to mathematical objects, their knowledge is only possible through external manifestations. However, meanings linked to an individual's affective states often remain unconscious or preconscious, are difficult for the individual to verbalize, and are subject to complicated interpretation by external observers (DeBellis & Goldin, 2006, p. 133). Affective signs are small gestures in body language, changes in voice intonation, sighs, facial expressions, etc., whose precise meaning is, at the very least, ambiguous. However, their effectiveness as a communication system is evident because they provoke emotional reactions in other subjects who interpret these signs, often unconsciously or preconsciously.

Extensive-intensive

Goldin (1988) introduced the distinction between local and global affect. Local affect refers to changing and instantaneous affective states that appear during problem-solving situations, constituting an internal representation system at the same level as visual representation (imagistic), formal notations, verbal representations, and the meta-system formed by planning and executive control (DeBellis & Goldin, 1991, p. 29). Furthermore, attitudes that directly depend on belief and value systems constitute the global affect. Like local affect, global affect can be expressed in any type of situation, but its entities are not so changeable or easily modified.

Local affect comprises emotions experienced in different problem situations proposed to students. Therefore, it includes manifestations (when emotions are externalized) or feelings (when instant affective entities remain internalized) of an ephemeral and particular nature at a specific moment. If individuals experience the same affective states in similar situations, these states are reinforced, thus configuring a system (global affect) in which attitudes, beliefs, and even values come into play. In other words, it is a process of generalization, so when the teacher proposes a situation that evokes already lived tasks and activities, the student performs an action of particularization because emotions depend on their own attitudes and beliefs, which in turn are configured as a generalization of emotions.

Unitary-systemic

A characteristic affective trait of a person (e.g., a negative attitude toward mathematics) can be interpreted as the result (unitarization) of a sequence of negative affective experiences related to learning mathematics. Research on the origin of such an affective trait and the design of strategies for its change may require analyzing and decomposing this trait into partial aspects. Contextual dualities apply to each of the affective (and cognitive) entities in the model, as represented in Figure 3.10. The unitary-systemic duality arises when considering the different objects on which emotions, attitudes, beliefs, and values are directed. The interrelationship between them forms a person's affective system.

In the field of mathematics education, McLeod (1992) distinguished different types of beliefs: about mathematics, about the self, about mathematics teaching, and about the social context. The same happens with attitudes, being possible to distinguish between attitudes toward mathematics (interest, satisfaction, curiosity, etc.) as well as mathematical attitudes (flexibility in the choice of techniques and strategies, critical spirit, etc.) (Callejo, 1994; Gil et al., 2005).

The teacher must pay attention to the unitary manifestations of belief systems and the attitudes that emerge from them, taking note of emotions and instant affective responses that favor appropriate mathematical attitudes for the different types of situations that occur in the teaching-learning processes.

3.9.3. Affectivity dynamics

Thus far, we have presented a static view of affect and its relationships with cognitive and epistemic domains. The dynamics of affect must be investigated in instructional processes. The OSA introduced some theoretical notions that can help study the dynamic aspects of affect in mathematics education. This is the case of the notions of configuration and didactic trajectory, which are described in Chapter 4. A didactic configuration is any segment of didactic activity (teaching and learning) performed between the beginning and end of a task (problem situation). Therefore, it includes the actions of students and teachers, and the resources planned or used to conduct the task. The sequence of didactic configurations forms a didactic trajectory. In every didactic configuration, there are: a) an epistemic configuration (system of practices, objects, and institutional mathematical processes required in the task), b) an instructional configuration (system of teacher/student functions and instructional means used besides to the interaction between the different components), and c) a cognitive-affective configuration (system of personal mathematical practices, objects, and processes describing learning and the affective components that accompany it).

From an instructional point of view, affect in mathematics education must be analyzed, planned, implemented, and evaluated, as in the rest of the facets. Research on the relationship between the affective domain and mathematics usually focuses on interactions with the cognitive domain. However, considering the epistemic component, classroom interaction patterns, resource use, and other ecological conditions that determine study processes in educational institutions (curriculum, social and political factors, etc.) also seems necessary.

Identifying affective trajectory and its interaction with epistemic configurations and the cognitive domain allows teachers to use this information to suggest problem-solving strategies for students (Caballero et al., 2017). An emotional state that may initially seem negative, such as mental block or despair, can be the start of an affective trajectory that catalyzes a cognitive sub-trajectory, leading to the use of heuristics to solve the corresponding problem situation. This sub-trajectory interacts again with the affective domain in a continuous feedback loop, favoring the emergence of positive emotions.

3.10. Synthesis of ontosemiotic theory of meaning and mathematical cognition

The aim of this chapter has been to present the theory of meaning elaborated from the ontological and epistemological assumptions of OSA, as well as the theoretical elements of mathematical cognition (personal and institutional) based on ontosemiotics. From this perspective, the study of signs must be conducted together with an analysis of the objects referred to by the signs and their nature and function. We have shown the sources on which ontosemiotics is based and its attempt to combine realist and operational theories about meaning when the problem is approached from the context of the construction and diffusion of mathematical knowledge, that is, the context of education. We have also started the study of the concordances and complementarities of OSA with other models regarding meaning applicable to mathematical knowledge to initiate the articulation of several theories.

Table 3.5 includes a synthesis of the theory by answering the questions in the guide proposed by Michie et al. (2014) for theory analysis.

Table 3.5. Synthesis of the ontosemiotic theory of meaning and mathematical cognition

Elements	Description
Summary. What is the theory about and what are its main propositions?	The ontosemiotic theory of meaning and mathematical cognition develops a global vision of the meaning of mathematical objects as the basis of mathematical cognition from individual (personal) and social (institutional) perspectives. Meaning is the content of any semiotic function, understood as a relationship between two objects (functives), one functioning as an expression (signifier) and the other as content (signified), related according to a criterion or rule of correspondence (interpretant). Functives can be elements of the various languages used in mathematical practice and other types of objects of the OSA ontology (concepts, propositions, procedures, arguments), including the system or practices themselves. In this way, the theory articulates realistic (referential) and pragmatic (operational) assumptions about meaning. The semiotic function construct serves as a basis for defining the knowledge and understanding of mathematics in terms of the webs of semiotic functions that a subject (person or institution) can establish between the objects involved in the practices required for problem solving.
Scope/Objective. What phenomena does the theory explain?	This theory addresses the phenomena related to the processes of representation, interpretation, meaning, and communication in mathematical activity from both institutional (epistemic, cultural) and personal (mental, psychological) perspectives. It also studies from an ontosemiotic perspective (objects and meanings) the nature and emergence of mathematical knowledge, understanding, and competence.
Justification. Why is this theory necessary and how does it improve on previous theories?	It is necessary to develop a theory that articulates realistic and pragmatic positions of meaning in the case of mathematics, which serves as a basis for describing and understanding the processes of mathematical cognition, from the individual (mental) and social (institutional) points of view. An ontosemiotic perspective provides a necessary and effective point of view to address the study of mathematical cognition, as well as the processes of mathematical representation, interpretation, and communication. Existing theories in mathematics education are partial regarding the dilemma between realism and pragmatism and between individual and social cognition. Nor do they consider the diversity of objects that intervene in mathematical activity, reducing semiotics to the study of languages, or ostensive objects detached from the constructs and practices from which they emerge.
Hypotheses. What specific hypotheses does the proposed theory propose, and	Both the signifiers (expression) of semiotic functions and meanings (content) can be ostensive (material) or non-ostensive (construct) objects. The criteria or codes of correspondence between functives can be personal or cultural rules or habits.

how do they differ from other theories?	Meaning has a public, socially shared facet and another private, mental, and idiosyncratic facet for each subject. Mathematical knowledge includes an understanding of the web of objects and relationships involved in mathematical problem-solving practices and competence in conducting them efficiently. Other theories contemplate partial visions of meaning and knowledge (they emphasize the mental or sociocultural component).
Constructs. What elements constitute the theory?	Constructs that make up the theory: – System of operational and discursive mathematical practices. – Typology of primary mathematical objects (languages, problems, concepts-definition, propositions, procedures, arguments). – Semiotic function: A correspondence between an antecedent object (expression, signifier) and another consequent object (content, meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. – Pragmatic meaning of an object: The correspondence between an object and a system of practices in which the object intervenes. – Types of institutional and personal pragmatic meanings. – Individual cognition: network of personal semiotic functions. – Institutional (social) cognition: network of social/shared semiotic functions. – Ecology of meanings
Relations. How are the elements of the theory related to each other?	The types of mathematical objects and systems of practices are the functives of semiotic functions; that is, they function as expression and content. The different types of languages are usually the functive expression, but the remaining types of objects also function as antecedents of semiotic functions. For this reason, the ontosemiotic character of this theory is emphasized, and not simply as semiotics. Mathematical knowledge, from both social and institutional perspectives, is conceived and analyzed in terms of the webs of semiotic functions.

Origin. On which theories is it based, and how?	The notion of sign function from linguistics (Hjemslev) is adopted and interpreted, which is complemented by Peirce's notion of an interpretant for the code or rule of correspondence between functives. It is also based on the ontosemiotic theory of mathematical activity and emergent objects (Chapter 2), adopting the types of mathematical objects and systems of practices as the functives of semiotic functions. It is also based on semiotic theories of cognition and culture (Eco and Lotman). In this way, realistic semiotic positions are articulated (words, signs become significant because an object, a concept or a proposition is assigned as meaning) and pragmatist positions (signs become significant by the fact that they perform a certain function in a linguistic game, because they are used in this game in a certain way and for a specific purpose).
Similarity. Which theories are most similar to this theory?	The notion of semiotic function is related to Peirce's triadic sign. Meaning as a system of operative and discursive practices is an interpretation of Peirce's pragmatic maxim. The pragmatic meaning of a concept is related to Vergnaud's conceptual triplet. The types of languages and their use as representations of other types of objects correspond to Duval's theory of semiotic representation registers. Knowledge as a web of semiotic functions corresponds to relational theories of understanding (Skemp) and mathematical competence (know-how) in problem solving (Mason, Schoenfeld, ...)
Complementarity. With which theories can it be complemented?	Other theories can complement some aspects of the ontosemiotic theory of meaning and cognition. For example, Duval's register theory of semiotic representation explicitly enriches and develops language types. Other theories of cultural semiotics (Lotman, Eco) and cognitive semiotics (Enactivism, Lakoff and Nuñez) can complement the analytical tools of ontosemiotics.
Operationalization. How are the constructs measured or identified?	The constructs of the theory are unmeasurable traits. These are descriptive categories of different types of objects and meanings. A method to delimit the various meanings of mathematical objects and, therefore, to reconstruct epistemological and cognitive reference models is the analysis of the systems of practices (personal and institutional) and the ontosemiotic configurations involved in them.
Uses. What can the theory be used for?	The ontosemiotic theory of meaning is used as a tool to analyze and understand processes of representation and interpretation in mathematical activity (construction and communication of knowledge). The reconstruction of the reference meanings of mathematical objects serves as the basis for the educational-instructional design, allowing different senses or partial meanings to be recognized and a representative sample adapted to the context to be selected. This allows us to recognize the complexity of objects and processes involved in mathematical

	and didactic activity and to develop an educational-instructional model that considers this complexity into account (Chapter 4).

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Chapter 4

Educational design theory in mathematics based on OSA

Introduction

In the educational sciences, various educational design models and theories have been developed for specific areas.

> An instructional-design theory is a theory that offers explicit guidance on how to better help people learn and develop. The kinds of learning and development may include cognitive, emotional, social, physical, and spiritual. (Reigeluth, 1999, p. 5)

These theories are practice-oriented and address different aspects of the educational process, such as the organization of content, the selection of teaching methods, activities sequencing, interactions between teachers and students, learning assessment and the use of educational technologies. Studies of the design and implementation of effective learning environments, together with basic scientific research on these processes, are addressed by the Learning Sciences (Sawyer, 2014), which adopts the approach of "use-inspired basic research" (Stokes, 1997).

In this chapter, we present the assumptions and theoretical tools developed in OSA to address the design, implementation, and evaluation of mathematics teaching and learning processes. The goal of these didactic activities is to ensure that students: 1) Acquire the mathematical knowledge and skills necessary to function in daily and professional life; 2) Develop the ability to reason, analyze, solve problems, and make decisions (logical and critical thinking); 3) Acquire ethical and civic training to become informed, responsible, and committed citizens (Niss, 1996).

The theory of educational design in mathematics presented here provides assumptions and constructs to describe, explain, and design the organized processes of teaching and learning mathematical content, considering contextual factors that condition and make them possible. The model is based on the theory of mathematical activity and emergent objects described in Chapter 2 and the theory of meaning and mathematical cognition presented in Chapter 3. Educational processes involve systems of mathematical practices (institutional meanings), subjects (students) whose commitment is the personal appropriation of these practices (personal meanings), the teacher or manager of the process, and specific instructional resources. In summary, the theory elaborated helps to describe, explain, and predict what teachers and students do when studying mathematical content in each context (scientific component of mathematics education) and what they should do to optimize that activity (technological component) when the postulates of the OSA on mathematical knowledge are assumed.

In sections 4.1 to 4.4, we describe the constructs of the theory. First, we present the structure of the facets and components of an educational-instructional process and the notion of didactic configuration as a unit of analysis (Section 4.1). The normative and meta-normative dimensions, including the construct didactic suitability as a normative aspect, are addressed in Section 4.2. It continues with a description of the types of theoretical configurations and the didactic reference model based on OSA (Section 4.3), followed by the dynamics of an educational process analyzed using the didactic trajectory tool and its types (Section 4.4). These constructs, together with those elaborated in the theories described in Chapters 2 and 3 (configuration of practices, objects and processes, and types of pragmatic meanings), are used to analyze the activities of planning (preliminary study) (Section 4.5), design and a priori analysis of instructional tasks (Section 4.6), implementation (Section 4.7), and retrospective analysis (Section 4.8). These analyses are exemplified through their application to prospective teachers' statistical training experiences (Godino et al., 2014). The concordances and complementarities of the OSA-based instructional design theory with related theoretical perspectives are discussed in Section 4.9. A synthesis of the theory is presented in Section 4.10.

4.1. Structure model of an educational-instructional process

We are interested in developing theoretical tools to understand the complexity of factors that intervene in the teaching and learning processes of mathematics, including the processes of information acquisition, knowledge appropriation, and specific skills, as well as the personal, social, and emotional development of students through mathematics study. This understanding forms the foundation for the planning, implementation, and evaluation of educational-instructional processes. The broad educational perspective that we assume, which includes the acquisition of specific knowledge and skills (instruction), but aspires to the integral development of the individual, leads us to develop a complex model for the structure of educational-instructional processes, as reflected in Figure 4.1.

An educational-instructional process comprises six interconnected facets: epistemic (institutional meanings, processes and relationships), ecological (interdisciplinary connections, curriculum, ...), interactional (teacher, teacher functions; student, student functions), mediational (material resources, study support, ...), cognitive (personal meanings, processes and relationships), and affective (emotions, attitudes, beliefs, ...). In each facet, we can identify a set of components, sub-components, and specific elements of the content (algebra, geometry, etc.), which are sequenced in time, accounting for the dynamics of educational-instructional processes.

Therefore, we have a model with four levels of analysis and interactions between various facets and components.

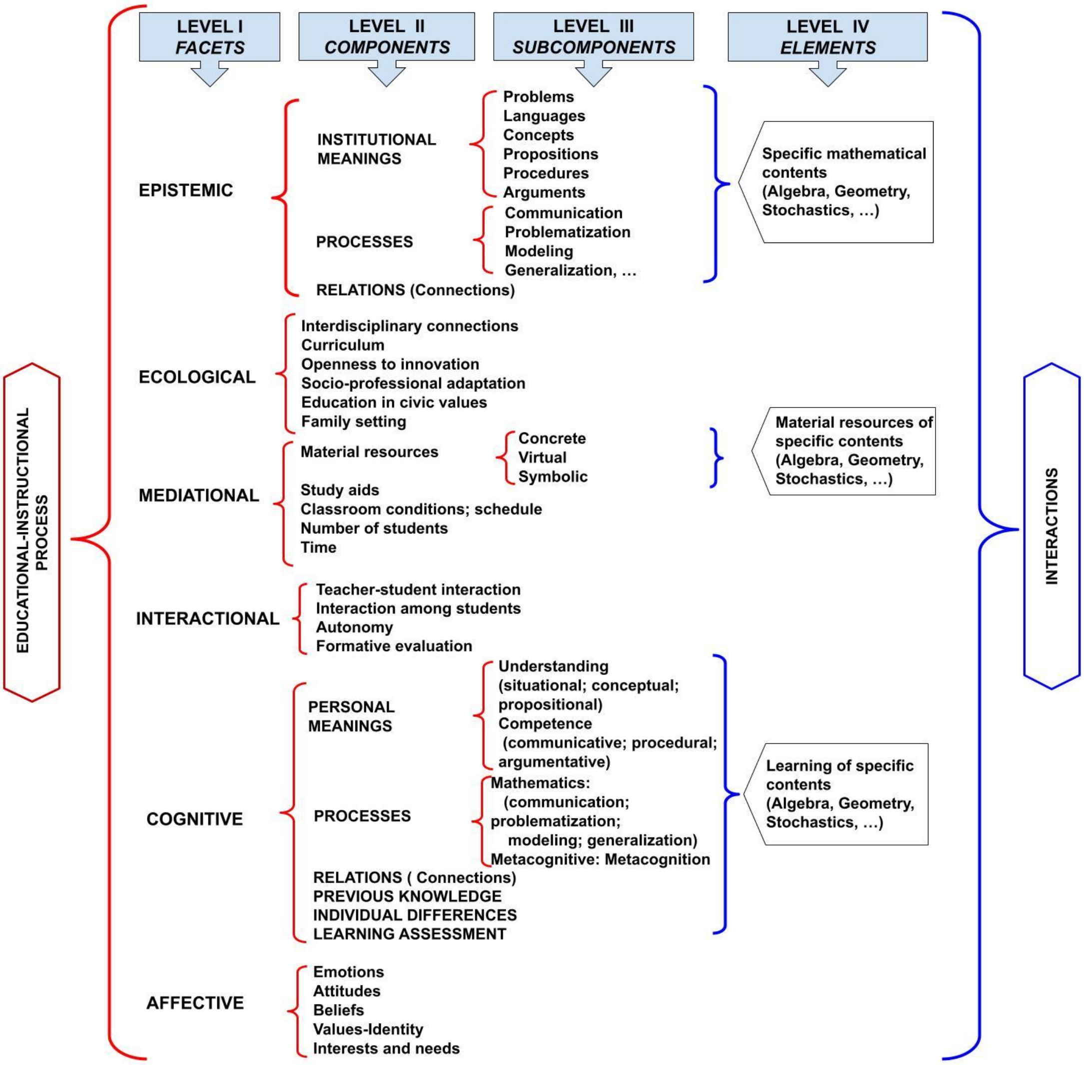

Figure 4.1. Structure model of an educational-instructional process (Godino et al., 2021, p. 10, modified version)

As a unit of analysis that considers and articulates the diversity of aspects that intervene in an educational-instructional process, we have introduced the *didactic configuration* construct, which consists of any segment of mathematical and didactic activity included between the beginning and the end of the resolution of a situation-problem or task. Each segment includes both the actions of the students and teachers as well as the material, epistemic, and cognitive means (prior knowledge and skills) used to address the task. The task can be solved in a short time (a few minutes) or be a longer project that requires a longer period, giving rise to micro, macro, or meso didactic configurations. Figure 4.2 summarizes the structure of a didactic configuration.

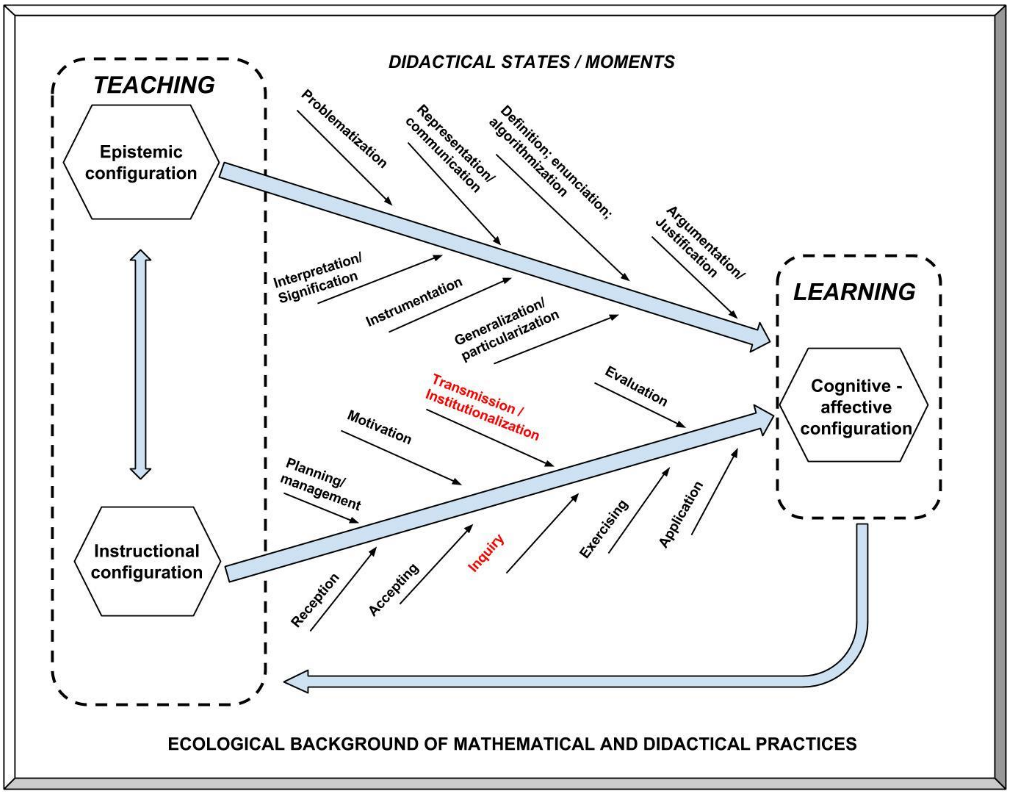

Figure 4.2. Components and dynamics of a didactic configuration (Godino, 2014, p. 31).

In a didactic configuration, we can differentiate three components: a) an epistemic configuration (system of institutional mathematical practices, objects and processes required in the task), b) an instructional configuration (system of teaching functions, learners and educational media used, as well as their interactions), and c) a cognitive-affective configuration (system of personal mathematical practices, objects and processes that describe learning and the affective components that accompany it).

The ontosemiotic theory of mathematical activity described in Chapter 2 serves as a basis for proposing a model for analyzing epistemic configurations that are designed and implemented in an educational-instructional process. These are the types of states or moments related to the management of instructional content: problematization, representation/communication, definition, enunciation, algorithmization, argumentation, interpretation/signification, instrumentation, generalization/particularization.

The instructional configuration includes the teacher’s roles or functions, for which we propose the following categorization (Godino et al., 2006):

- P1: Planning: design of the process and selection of the contents and meanings to be studied (construction of the intended meaning and the intended epistemic trajectory).
- P2: Motivation: creating a climate of affectivity, respect, and encouragement for individual and cooperative work in the instructional process.
- P3: Assignment of tasks: direction and control of the study process, time allocation, adaptation of tasks; orientation and stimulation of the student's functions.
- P4: Regulation: setting rules (definitions, statements, justifications, problem solving, exemplifications), recall and interpretation of prior knowledge necessary for the progression of the study, and readaptation of the planning.
- P5: Evaluation: observation and assessment of the state of learning achieved at critical moments (initial, final, and during the process) and resolution of individual difficulties observed.

- P6: Research: reflection on and analysis of the development of the process to introduce changes in future implementations and articulate moments and parts of the study process.

The following relationship constitutes an inventory of potential student roles in the instructional configuration:

- A1: Acceptance of educational commitment, adoption of a positive attitude toward study, and cooperation with peers.
- A2: Exploration, inquiry, seeking conjecture, and ways of answering the questions posed.
- A3: Recalling, interpreting, and following rules (concepts and propositions) and the meaning of linguistic elements in each situation.
- A4: Formulation of solutions to problems or tasks proposed, either to the teacher, to the whole class, or within a group.
- A5: Argumentation and justification of conjectures (to the teacher or classmates).
- A6: Receive information about how to perform, describe, name, and validate.
- A7: Demanding information: states where learners ask for information from teachers or other peers (e.g., when they do not understand the meaning of the language used or do not remember necessary prior knowledge).
- A8: Drill: performance of routine tasks to master specific techniques.
- A9: Assessment: states where the learner performs assessment tests proposed by the teacher or self-assessment.

The types of teacher and student actions listed above are types of didactic practices, i.e., classes or categories of actions (operative and discursive) that they perform jointly to address the study of the intended content — be it the resolution of a type of problem and the mathematical objects involved, or the development of specific skills. These practices occur in an ecological context (material, biological, social) that supports and conditions them.

4.2. Normative dimension[11]

Like any social activity, education is explicitly and implicitly normative. From the most general level of curricular guidelines, often set by official decrees, to the behaviors of courtesy and mutual respect between teachers and students, teaching and learning processes are governed by norms, conventions, habits, customs, and traditions. All these normative elements constitute what Godino et al. (2009) call the "normative dimension of study processes". This influence "from the shadows" means that norms are rarely questioned, which seriously conditions initiatives aimed at improving the processes of teaching and learning mathematics: without changing the rules, it is not possible to modify the processes governed by them. Consequently, a priority undertaking should be the study of this "normative dimension" to, on the one hand, be able to describe more precisely the functioning of educational-instructional processes and, on the other hand, influence aspects of the normative dimension (modifying them if necessary) to facilitate the improvement of these processes.

The topic of norms has been the subject of research in mathematics education, mainly by authors who base their work on symbolic interactionism (Blumer, 1969), and have introduced notions such as interaction patterns, social and socio-mathematic norms (Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996). Brousseau (1988; 1997) developed the notion of didactic contract as a key element of the Theory of Didactic Situations in Mathematics (TSDM). In all these cases, it is a matter of considering the norms, habits, and conventions, generally implicit, that

[11] The content of this section is based on the articles by Godino et al. (2009) and D'Amore and Godino (2007).

regulate the functioning of the mathematics classroom and condition to a greater or lesser extent the knowledge that students construct. The focus of this study has been mainly on the interactions between teachers and students when studying specific mathematical topics.

The OSA framework, and especially the model of the structure of an educational-instructional process (Figure 4.1), provides constructs that allow us to classify the web of social and mathematical norms on which the teaching and learning of mathematics are supported and conditioned, as shown in Figure 4.3.

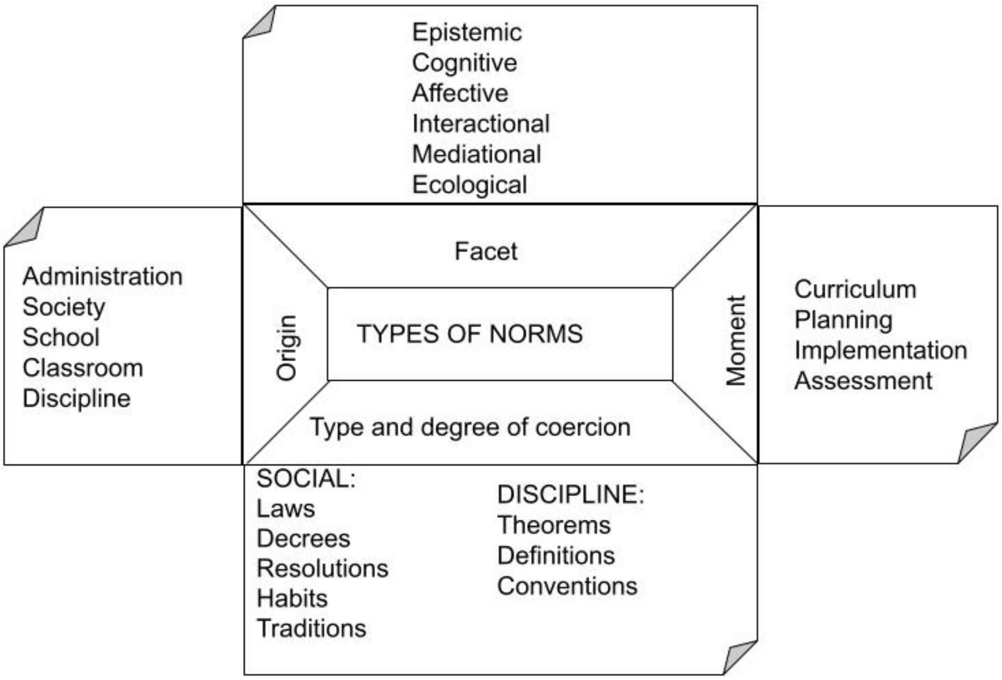

Figure 4.3. Normative dimension. Types of norms (D'Amore et al., 2007, p. 10)

The adopted global perspective classifies norms in two complementary directions:

1) Timing of intervention: Curricular design, planning, implementation, or evaluation. Norms manifest not only during teacher-students

interactions (implementation) but also in planning, evaluation, and the curricular design phase, where reference meanings are configured, guiding and conditioning the intended, implemented, and evaluated meanings.

2) Aspects of the study process addressed by the norm: Epistemic, ecological, interactional, mediational, cognitive, and affective. This allows a focus on regulatory norms:
   - The teacher's work with respect to mathematical knowledge (understood as a system of institutional practices).
   - Relationship with the environment (sociocultural, political, labor, etc.) where the instructional process occurs (ecological aspect).
   - Use of technological and temporal resources (mediational aspect).
   - Teacher-student and student-student interactions.
   - Students' work related to mathematical knowledge (assumed as a system of personal practices).
   - The affectivity of the individuals involved in the study process.

Norms can also be classified by their origin (administration, society, school, classroom, discipline) and type and degree of coercion (social, disciplinary), as indicated in Figure 4.3. Below, we describe some characteristics of norms linked to each aspect of mathematics instruction processes.

4.2.1. Epistemic norms

In mathematics classes, a basic commitment is established: teaching and learning mathematics. Epistemic norms determine the possible mathematical activities in a given educational process. They regulate mathematical content, the types of situations suitable for learning, and the representations used for various contents. In OSA terminology, epistemic norms determine epistemic configurations and the mathematical practices that enable them.

Additionally, each component of the epistemic configuration is related to meta-epistemic norms (usually considered socio-mathematical norms). For instance, students must answer questions such as, “What is a problem?”, “When is a problem solved?” and “What rules should be followed to solve it?” Similarly, regarding the argument component, students must understand what an argument is in mathematics and when it is considered valid.

The system of didactic suitability criteria in the various aspects and components of an educational process, based on OSA assumptions and tools, defines what can be considered “good mathematics,” “good teaching,” and “good learning,” thus characterizing a “meta-reference contract” that can be used to evaluate the norms effectively supporting and conditioning the implemented processes.

4.2.2. Ecological norms

Identifying ecological norms involves gathering information about the social, political, and economic environment of the school, as this influences the types of mathematical practices conducted in the classroom. Society charges the school with educating its citizens, committing them to their community, ensuring the adoption of democratic values, guaranteeing everyone’s rights, and fostering civic duties. The educational institution provides initial training to competent professionals for their future professional practice. Therefore, when making decisions about educational process goals, it is essential to consider the broad social sectors not directly related to the educational situation but affected by it: society as a whole, which will be served by the new professionals.

The ecological aspect of the normative dimension relates to the content to be taught because the intended meanings specified in the curricular guidelines aim to contribute to students’ socioprofessional formation.

Adhering to programs is another requirement for teachers because students' achievement in learning constitutes the starting point for subsequent studies. The obligation to ensure a certain level of knowledge and to inform society about it underlies the mathematics teacher's duty to conduct summative evaluations that inform families and society about students' achievement of mathematical levels.

4.2.3. Interaction norms

The interaction modes between a teacher and students are subject to rules, habits, traditions, commitments, and agreements intended to achieve teaching and learning objective. The interactional aspect of the normative dimension is a system of norms that regulates interactions among people involved in mathematics study processes. Effective realization of a study process may involve changes in interactions from initially planned modalities, depending on the assumed educational paradigm. In a social constructivist model, the teacher's key role is to find good situations and create an environment in which students construct knowledge cooperatively with their peers. In an expository teaching model, the teacher's role is to present the content, and the student's role is to retain it.

4.2.4. Mediational norms

Teaching and learning rely on technical means (books, computers, etc.) and are distributed over time, which is also a resource. Diverse rules that govern the instructional process condition the use of both types of means. This system of rules related to the use of technical and temporal means constitutes the mediational aspect of the normative dimension.

Schools must have classrooms, physical spaces where groups of students meet with a teacher; today, schools must also have blackboards, chalkboards, erasers, computers, projection screens, and interactive whiteboards. At some levels, teachers must have specific manipulative materials and software available; students often need printed or digital study materials. The appropriate use of all these resources is subject to specific technical rules that teachers must follow.

Time is a scarce resource; its management is primarily the teacher's responsibility although some study time is the student's responsibility. Official norms almost rigidly regulate class duration, as is the total time allocated for developing each course's study program development.

4.2.5. Cognitive norms

In OSA, teaching involves students' participation in the community of practices that support institutional meanings, and learning ultimately entails students' appropriation of these meanings. Cognitive norms regulate the personal domain (as opposed to institutional) within mathematics instructional processes. Among other aspects, this normative facet establishes that students must learn and that the institution must ensure:

- Students must possess the necessary prior knowledge.
- They will be taught within their zone of proximal development.
- The institution will adapt to students' diversity.

4.2.6. Affective norms

Another dimension to consider in mathematics instructional processes is affectivity, motivation, emotions, and beliefs. Students must be motivated, have a positive attitude, and not have math phobia. The focus often shifts to the teacher, who "must" motivate students; choose "attractive" content and creating an affective classroom climate conducive to learning. These are general clauses in the affective aspect of the normative dimension that do not indicate the types of teaching actions that may be available to mathematics teachers. An affective rule would be that the teacher must seek or invent rich mathematical situations that belong to the students' short- and medium-term interest fields.

4.2.7. Meta-normative dimension

In developing the didactic contract, Chevallard (1988, p. 58) identified a structure that remains unchanged through changes and breaks in the didactic progression, “it is the set of clauses that manage, in a given field, any adherence to a contract, ensuring its effectiveness, regardless of particular contents”. Chevallard called this set of defining clauses a meta-contract. D’Amore et al. (2007) extended the idea of the meta-contract by introducing a meta-normative dimension that refers to any reflection, expectation, or evaluation of the norms involved in educational processes. For example, “students' learning must be assessed” is a general pedagogical norm. “Students must not cheat on exams” is a metanorm. Given the existence of epistemic, cognitive, and instructional norms, meta-epistemic, metacognitive, and meta-instructional norms can also be identified.

As previously mentioned, epistemic norms regulate mathematical content, learning situations, representations, definitions, propositions, procedures, and arguments. Thus, epistemic configurations regulating mathematical practices within specific institutional frameworks must be considered. Each component of the epistemic configuration is related to meta-epistemic norms (usually considered socio-mathematical norms). Consequently, epistemic configurations are associated with a system of metanorms that can be socially shared (meta-epistemic configuration) or personal to the students involved in the corresponding learning processes (meta-cognitive configuration). Figure 4.4 illustrates these three blocks of the meta-normative dimension with examples of metanorms.

The teacher expects students to rely on a prior epistemic configuration to perform mathematical practices that yield an emerging epistemic configuration; this realization will be regulated by the meta-epistemic configuration (which, as mentioned, coexists with others over time). To this end, the teacher implements an instructional configuration regulated by a meta-instructional configuration. Moreover, the teacher intends for students to personalize epistemic configurations into personal configurations, meta-epistemic configurations into mathematical metacognition, and instructional configurations into didactic metacognition.

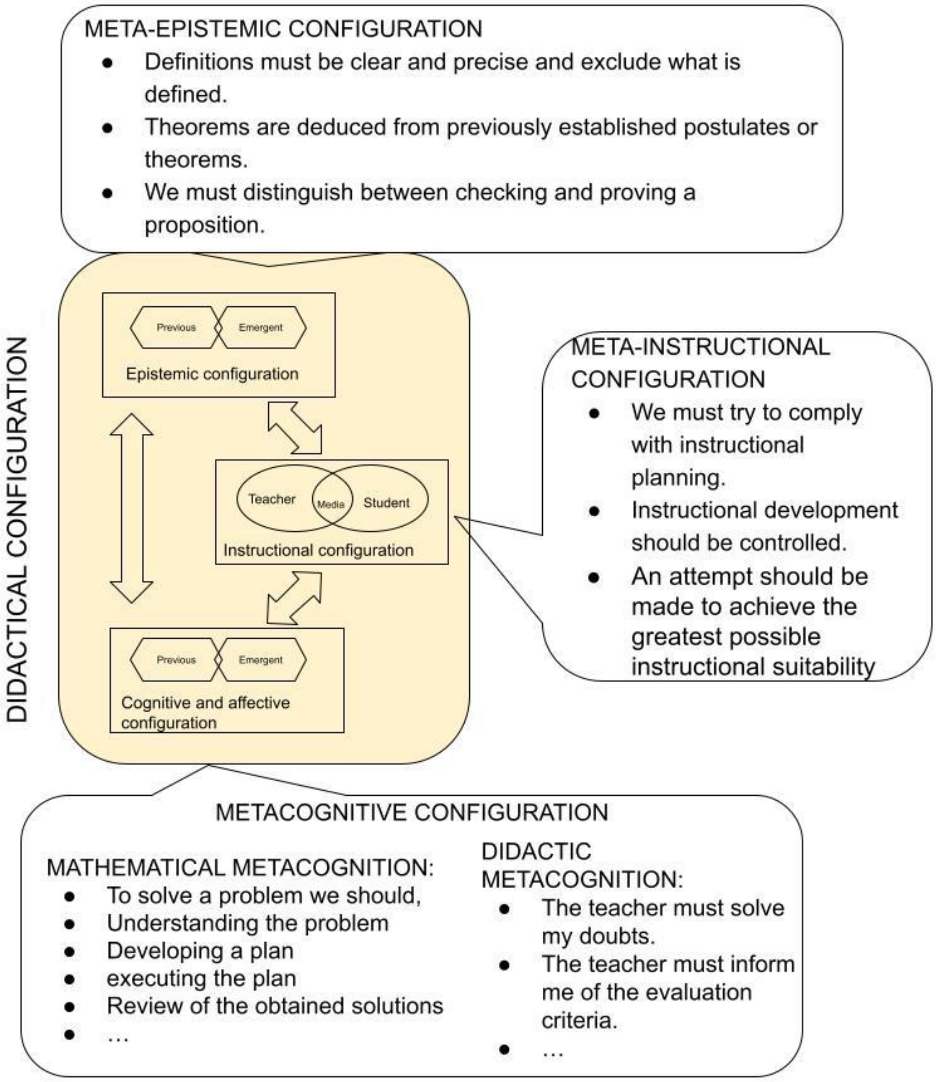

Figure 4.4. Meta-normative dimension (D'Amore et al., 2007, p. 13)

The constitution of these "meta" configurations (meta-epistemic, meta-instructional, and meta-cognitive) often emerges from implicit processes rather than explicit ones and is based on certain habits or ways of acting. For example, a teacher's habit of including keywords such as "add" or "sum" in the statement of problems solved by addition can elicit knowledge that forms part of the metacognitive configuration of the members of the institution: problems involving addition incorporate the keyword "add" in their statement.

4.2.8. Didactic suitability criteria

The notion of didactic suitability has been introduced in the OSA as a tool for transitioning from descriptive-explanatory didactics to normative didactics, that is, one oriented toward effective classroom interventions.

> The didactical suitability of an instructional process is the degree to which such a process (or a part of it) meets certain characteristics that qualify it as optimal or adequate for achieving the adaptation between the students' personal meanings (learning) and the intended or implemented institutional meanings (teaching), considering the circumstances and available resources (environment). These institutional meanings are also representative of the global reference meaning. (Godino et al., 2023, p. 4)

It is assumed that in social and educational sciences, it is possible to formulate suitability criteria in the form of value judgments, "one should do this and not that," in circumstances where such value judgments have a social character, and it is possible to provide a foundation for their formulation. They entail rationality and thus can be subject to scientific scrutiny (Bunge, 1999; Rugina, 1998).

The theory of didactic suitability—which will be developed in Chapter 5—contributes to the study of the normative dimension by: 1) Structuring categories (facets, components) of the system of norms or criteria for designing, implementing, and evaluating educational experiences; 2) Explicitly stating suitability criteria with different levels of generality for the various facets and components. These criteria are based on explicit theoretical assumptions about mathematical knowledge, its teaching, and learning (those assumed by the OSA), which generally align with those proposed by other theories.

The universe of suitability criteria can be hierarchically categorized, distinguishing between general criteria (linked to each of the six facets of an educational-instructional process), partial criteria (associated with the different components of each facet), and specific criteria (related to particular aspects of mathematical content). For example, specific criteria for teaching proportionality include providing students with opportunities to distinguish between multiplicative and additive situations and avoiding algorithmic memorizing of the rule of three. Suitability norms, therefore, can, in some cases, take the form of principles, such as those formulated in general terms for the facets or, in others, be interpreted as rules, for example, those related to the learning of specific contents. A suitability norm with a clear rule-like character could be, “the instructional process should avoid the transmission of erroneous knowledge”. In Chapter 5, the system of suitability criteria for the different facets and components of an instructional process (Figure 4.1) is described to guide the design, implementation, and evaluation of such processes.

An educational process at its various levels—micro (a lesson), meso (a topic), macro (program)—can be characterized by identifying the system of norms that regulate it. The community that designs, implements, and evaluates an educational process follows a system of norms, both explicit and implicit, through which it strives to optimize its development, that is, to achieve the best possible learning and teaching. Therefore, such a system of norms constitutes a system of didactic suitability criteria, which depends on the assumptions and values assumed about the learning and teaching of the intended content. Each teacher, community, and theoretical approach has its own system of didactic suitability criteria. Thus, the curricular guidelines of a country regulate what mathematics to teach, what means to use, and how to evaluate at each educational level. There is no doubt that these norms are promulgated with the positive intention of optimizing teaching and learning processes. Each country or region has its curricular norms, which may largely coincide, although there can also be differences, considering some preferences, specific needs, or contextual restrictions (economic resources, etc.). Each teacher or group of teachers designs educational processes, concretizes curricular designs and adapts them to the group of students and available means. They also organize and manage the class in a particular way based on their beliefs, knowledge, and value system. All this is done with the "best intention" of enhancing student learning.

The system of didactic suitability criteria based on OSA can be used to design educational processes and assess effectively implemented processes. If the design is based on other principles, the didactic suitability criteria are used as metanorms (reflection, evaluation) of the norms followed by other educational designs. For this evaluative purpose, it is necessary to convert the system of criteria (norms) into another system of rubrics (indicators) to measure the degree of compliance with the norms.

4.3. Reference didactic configurations. OSA didactic model

The analysis of didactic configurations implemented in an educational-instructional process and of those that can potentially be designed for their implementation can be facilitated if some theoretical models are used as references. In this section, we describe four theoretical configurations that can play this role and are designated as magisterial, adidactic, personal, and dialogic configurations (Godino et al., 2006).

The Theory of Didactic Situations in Mathematics (TSDM) (Brousseau, 1997) proposes an optimal way of organizing the work of the teacher and the students regarding intended mathematical knowledge in relation to students' learning. The sequence of adidactic situations of action, formulation, validation, and the didactic situation of institutionalization specify the student's roles in interaction with the environment (which includes the teacher, intended knowledge, and specific material and cognitive resources) and can be interpreted as a theoretical didactic configuration. However, we know that this is not the only type of didactic configuration that can be and is in fact implemented. In the proposed TSDM framework, it is not claimed that all mathematical knowledge can and should be studied in this way.

We all have in mind a traditional or classical way of teaching mathematics based on magisterial presentation, followed by exercises of applying the presented knowledge. The discursive component of the meaning of mathematical objects (definitions, statements, demonstrations) is presented first, and it is left to the students themselves to make sense of the discourse through examples, exercises, and applications. This is a *topogenetic* decision: "first, I, the teacher, give you the general rules, then you apply them". In fact, in this type of didactic configuration, moments of exploration, formulation, and validation are not necessarily suppressed but remain under the student's responsibility or are brought into play in isolated moments of evaluation.

An intermediate variant, which we call dialogic, can be defined between the types of configurations described (which we will designate as adidactic and magisterial, respectively), respecting the moment of exploration but with the teacher basically assuming formulation and validation. Institutionalization (regulation) occurs through a contextualized dialog between the teacher and the students, who have had the opportunity to take on the task, become familiar with it, and possibly outline some solution technique.

Another basic type of didactic configuration occurs when the student resolves the problem-situation (or the performance of a task) without direct intervention from the teacher. In practice, this occurs when students solve exercises proposed by the teacher or are included in textbooks and can solve them. This is a type of didactic configuration in which self-study is essentially predominant. In Figure 4.4, the four vertices of a square represent the four types of theoretical didactic configurations described. The empirical didactic configurations that occur in the sample trajectories can be represented by a point inside the square and are close to these theoretical configurations. Throughout a mathematical instructional process, empirical configurations oscillate around these theoretical types.

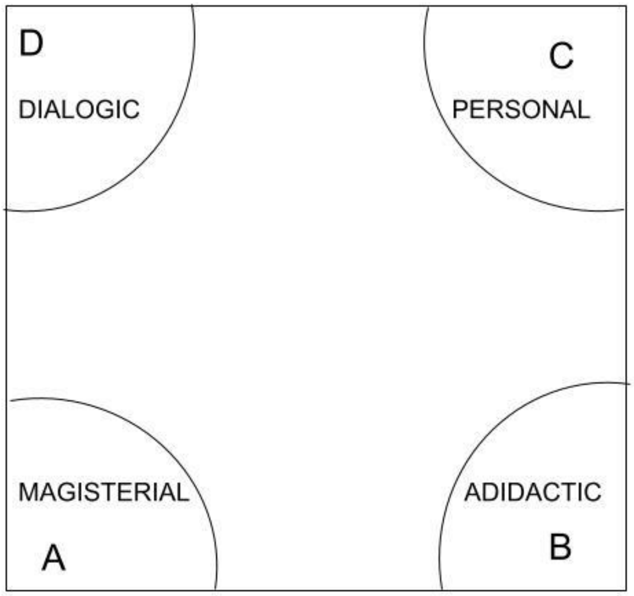

Figure 4.4. Theoretical didactic configurations (Godino et al., 2006, p. 69)

These theoretical types of didactic configurations can be considered patterns of didactic interaction, i.e., regularities in the modes of interaction in the development of educational-instructional processes.

OSA didactic model

In OSA, various types of didactic configurations that promote learning are assumed, depending on the types of knowledge sought, the initial state of knowledge of the subjects, context, and circumstances of the educational-instructional process. Inquiry-based (adidactic-constructivist), collaborative, and transmissive (magisterial) didactic configurations can occur sequentially, although without a rigid pre-established order (Figure 4.5).

When learning new and complex content, the transmission of information at specific times by a teacher or student leader within work teams can be crucial. Such transmission can be meaningful when students participate in activities and work collaboratively. The didactic configuration tool helps us understand the dynamics and complexity of the interactions between the content, teacher, learners, and environment. Learning can be optimized through a mixed model that articulates the transmission of information, inquiry and collaboration and is managed through didactic suitability criteria interpreted and adapted to the context by the teacher.

In the moments or phases of a student's first encounter with a specific meaning of a mathematical content or object, it is considered that a dialogic-collaborative configuration can optimize learning. In this type of configuration, the teacher and students work together to solve problems that bring the object into play in a critical manner; the first encounter should therefore be supported by expert intervention from the teacher.

The teaching-learning process could thus achieve greater epistemic and ecological suitability. When the rules and circumstances of application that characterize the object of learning are understood, one can tend toward higher levels of cognitive and affective suitability by proposing to deepen the study of the object (exercise and application situations) through didactic configurations that progressively and in a controlled manner attribute greater autonomy to the student (Figure 4.5).

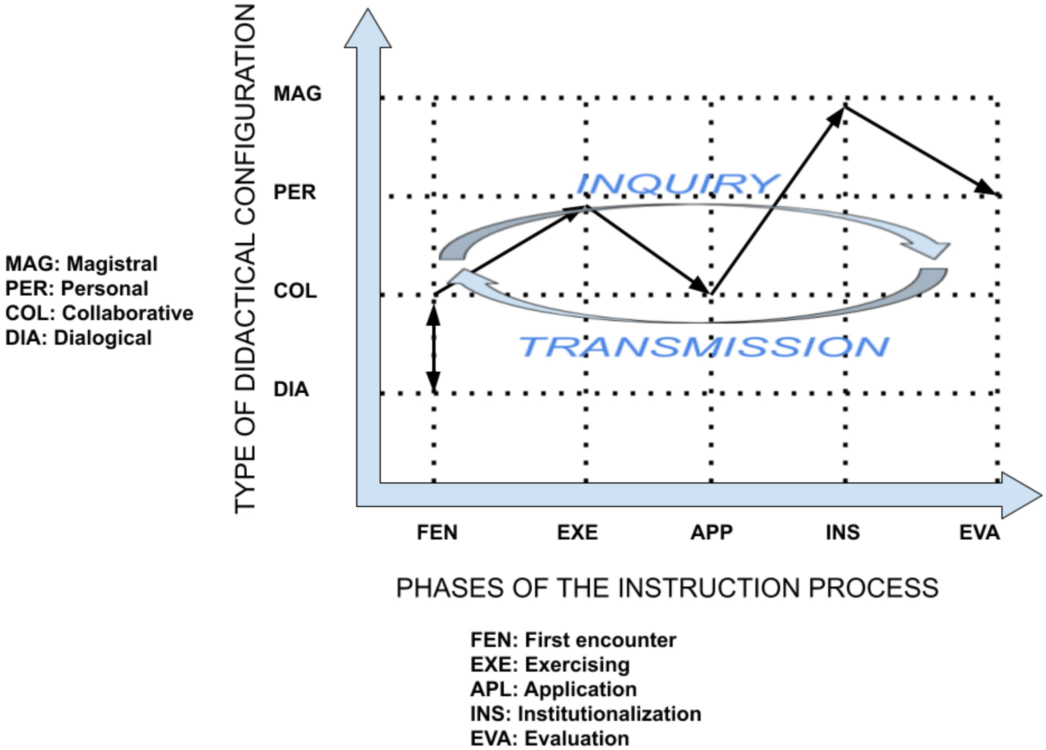

Figure 4.5. Didactic model based on OSA (Godino & Burgos, 2020, p. 97)

Some didactic theories, such as the Theory of Objectification (Radford, 2006; 2014), argue that a collaborative model—joint work of teachers and students—is preferable to a constructivist or traditional teacher-centered alternative. The didactic model proposed by OSA is more open by assuming that learning optimization can be achieved through the ideal articulation of different types of didactic configurations. This mixed didactic model articulates the theoretical frameworks of learning and teaching mathematics that Sfard and Cobb (2014) call acquisitionism and participationism. The first framework comprises approaches that present mathematics as pre-established structures and procedures and consider learning to be acquired by the student. The acquired entities may be called knowledge, schemas, or conceptions, and the acquisition process may be passive, by mere transmission, or active, through the learner's constructive efforts. The second framework describes mathematics as a form of human activity that has evolved historically, rather than as something acquired; thus, learning mathematics is the process of becoming a participant in this type of activity. Sfard (1998) presented acquisition and participation as metaphors for learning and teaching that should be seen as complementary. This complementarity is consubstantial in the mixed didactic model based on OSA because, on the one hand, it is based on an ontological, semiotic, and epistemological model of mathematics in which mathematics is conceived both as a human activity and as a system of cultural objects. The appropriation (acquisition) of these cultural artifacts is an essential objective of educational-instructional activities. On the other hand, it is assumed that mathematical objects emerge from the activity of people in their interactions with the environment and through communication with other people. Therefore, it is justified that dialog, collaboration, and participation in communities of practice are key aspects of learning and teaching.

4.4. Dynamics of an educational-instructional process

An educational-instructional process takes place over time and consists of successive tasks, which are solved interactively by the students and teachers and supported by the available material, epistemic and cognitive resources. That is, it occurs through the sequencing of different didactic configurations (Figure 4.6).

It is natural to model the temporal distribution of functions and components using stochastic processes, considering such functions and components as their possible states. The stochastic character derives from the diversity and complexity of the nondeterministic factors involved in educational processes. Even if planning has been careful, there are always random elements that produce changes in each trajectory, given the need to adapt the planned teaching to the characteristics and requirements of the students and the context.

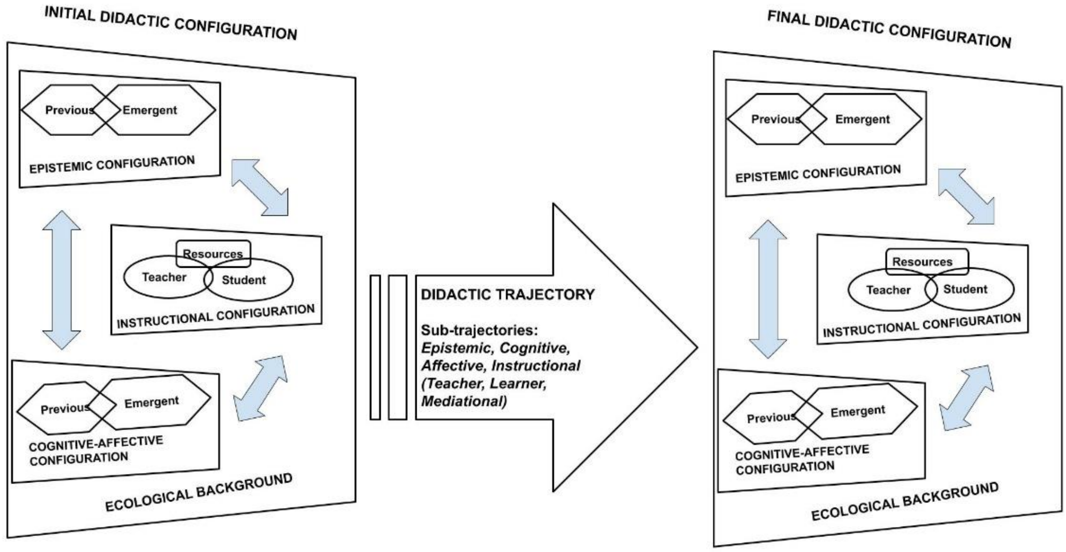

Figure 4.6. Didactic trajectory (Godino et al., 2008, p. 13)

In each implementation of the instructional process (each experience of teaching a mathematical content), a series of possible states occurs. This means that a sample trajectory of the process is produced, describing a particular sequence of functions or components that occur over time. We distinguish five types of sample subtrajectories:

1) Epistemic subtrajectory: The distribution over time of the components of the implemented institutional meaning (problems, language, definitions, procedures, properties, arguments) in a certain order.
2) Ecological subtrajectory: The distribution over time of elements related to the ecological context in which the process develops (interdisciplinary connections, curriculum, etc.).
3) Instructional subtrajectory: Composed of three partial subtrajectories:

– *Teacher*: The tasks and actions carried out by the teacher throughout the instructional process.
– *Learner*: Actions performed by students (one for each student).
– *Mediational*: Technological resources such as books, notes, manipulatives, software, etc.

4) Cognitive subtrajectories: Chronogenesis of students' personal meanings (learning).
5) Affective subtrajectories: The temporal distribution of affective states (attitudes, values, beliefs) of each student in relation to the mathematical objects and study process followed.

The construct of didactic trajectory refers to the articulation of the five described partial subtrajectories. When observing an educational process, the sequences in time of possible states constitute empirical sample trajectories.

4.4.1. Epistemic subtrajectory

The epistemic analysis of an instructional process involves its decomposition into units of analysis to characterize the type of mathematical activity that is effectively implemented. This requires identifying mathematical objects, their relationships and groupings, and the ecological relationships established among them. To support this, the notions of epistemic (or mathematical) configuration, epistemic trajectory, and potential states of these trajectories are introduced. The epistemic trajectory represents the time distribution of the mathematical objects in the instructional process. We distinguish six possible states within it according to the type of entity—and mathematical process—being studied at each moment:

- E1: Situational: A certain type of problem is stated.
- E2: Procedural: The development or study of a method to solve problems is addressed.
- E3: Linguistics: notations and graphical representations, etc., are introduced.
- E4: Conceptual: Definitions of the objects are developed or interpreted.

- E5: Propositional: The properties are stated and interpreted.
- E6: Argumentative: Adopted actions or stated properties are justified.

These states occur throughout the instructional process related to a mathematical topic or content. The classification of mathematical entities into the categories we have defined is not absolute because, as functional entities, it depends on the chosen level of analysis and the language games in which they are generated. Thus, the identification of trajectory states can be considered subjective. However, if two people participate in the same language game and adopt the same perspective, they will progressively agree on the categorization of a certain unit of analysis. Analysis of an epistemic trajectory allows for the effective characterization of its institutional meaning and ontosemiotic complexity. To perform the analysis, its development is divided into units of analysis according to the different problem situations (or tasks) proposed. We call epistemic configuration the system of objects and semiotic functions established among them when solving a problem. Therefore, it is a segment of the epistemic trajectory.

Epistemic analysis will characterize epistemic configurations, their sequencing, and articulation. This study examines the chronogenesis of mathematical knowledge in schools and its ontosemiotic complexity. Within each configuration, additional elementary units for analysis are defined according to trajectory states and are called epistemic units. Over time, a collection of problem situations is posed and solved around which epistemic configurations are constructed. The sequence of these configurations ultimately constitutes the "system of mathematical practices" that establishes the implemented institutional meaning of the object under study.

4.4.2. Instructional subtrajectory

The instructional subtrajectory comprises the sequencing over time of the teacher and learner roles listed in Section 4.2 and the temporal distribution of the resources used. Various means or resources can be used as study aids in the instructional process. This includes means of presenting information in class (interactive whiteboard, etc.), calculation and graphing devices (calculators, computers), manipulative materials, etc. The use of these resources (type, modality, sequencing, articulation with other elements of the process, etc.) should be the focus of attention in both teaching practice and didactic research. The notion of instructional subtrajectory serves as a tool for analyzing the potential and effectively implemented uses of instructional resources and their consequences for learning.

4.4.3. Cognitive subtrajectory

In OSA, the notion of personal meaning is introduced to designate students' knowledge. These meanings are conceived, like institutional meanings, as "systems of operative and discursive practices" that students can employ for certain types of problems. Personal meanings are progressively constructed throughout the instructional process, starting with initial meanings and progressing to final (achieved or learned) meanings. These personal meanings, evaluated at a given moment, constitute the students' cognitive configurations, i.e., the state of their mathematical knowledge and skills. The notion of cognitive trajectory refers to the chronogenesis of a personal system of practices, which can be modeled as a stochastic process. The chronogenesis of personal meanings is a dimension of the study process that cannot be characterized by simply recording the class development audio visually because it is relative to each learner and occurs both in and outside the classroom. It will be necessary to examine "class notes", complete initial and final questionnaires and assessment tests, conduct interviews, and so forth. The teacher's interactions with students while they solve tasks in class and during the segments when this activity occurs allow partial access to the progressive construction of students' knowledge and skills and to make decisions about the epistemic and instructional subtrajectories.

4.4.4. Affective subtrajectory

Other conditioning factors of the educational-instructional process are affective states (interest, personal commitment, feelings of self-esteem, aversion, etc.). The "devolution" process introduced by the Theory of Didactic Situations in Mathematics (Brousseau, 1997) responds to the need for students to assume as their own the problem-situations proposed by the teacher as a means of constructing mathematical knowledge. Feedback is a component of emotional trajectories. Although it is important to consider students' affective trajectory in any instructional process, it can be crucial in processes involving groups of students with special educational needs (students with disabilities, immigrant students with difficulties, etc.).

4.4.5. Complexity of didactic interactions

The relationships between teaching and learning are not linear but cyclical and complex (Figure 4.5). In moments of inquiry, students interact with the epistemic configuration without the teacher's intervention (or with minimal influence). This interaction conditions the teacher's interventions, which should be anticipated in the instructional configuration, perhaps not entirely in content but in their nature, necessity, and usefulness. The cognitive trajectory produces examples, meanings, arguments, etc., that condition the instructional process and, consequently, influence the epistemic and instructional configurations, enabling or restricting learning.

To conduct a comprehensive didactic analysis and explain the dynamics of the development of educational-instructional processes, it is necessary to add the normative dimension (see Section 4.2) to the six-facet structural model (Figure 4.1). This dimension refers to the web of norms and metanorms that condition and support process development. It goes beyond the descriptive aspect of understanding what happens and provides explanations of why things happen. To analyze the effects of norms as support and constraints of the development of an instructional process, it is useful to introduce the construct of normative trajectory, which is formed by the sequence of temporal moments in which these norms are established or modified.

4.5. Preliminary analysis: Reconstruction of reference meanings

In research oriented to educational design or didactic engineering (Artigue, 2011; Godino et al., 2014), four phases or stages are considered (Figure 4.7):

- Preliminary study (foundations) of the process in the epistemic-ecological, cognitive-affective and instructional facets.
- Planning or designing the didactic trajectory. Selecting problems, sequencing, and a priori analysis of them with indication of the students' expected behaviors and teacher's planning-controlled interventions.
- Implementation of the didactic trajectory. Observation of the interactions between people and resources and evaluation of the learning achieved.
- Evaluation or retrospective analysis, which is followed by a comparison between what was foreseen in the design and what was observed in the implementation. It also reflects on the norms that condition the instructional process and on didactic suitability.

The distinction between the six facets in each phase helps systematically analyze the factors involved in each phase. The arrows in the diagram indicate the cyclical nature of the educational process and the interaction between its different stages.

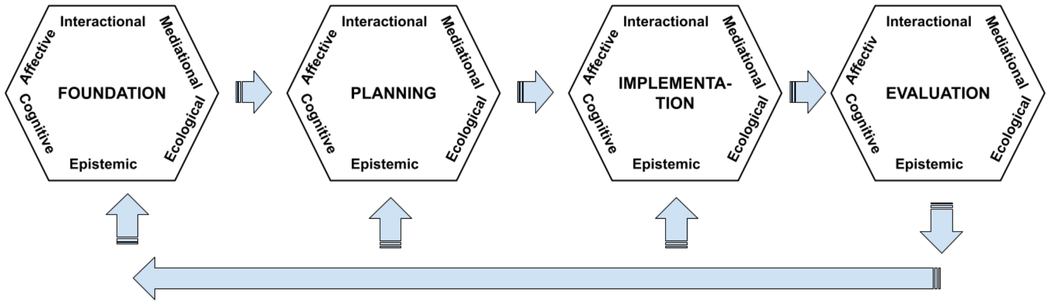

Figure 4.7. Phases and facets of educational-instructional processes.

In this section, we describe theoretical tools for the preliminary analysis (or grounding) of designs based on OSA. Selection of mathematical content for teaching and learning occurs. This selection also implies its transformation or preparation (Scheiner et al., 2022) to the corresponding educational level and context, resulting in the curriculum as well as specific lessons and resources for different educational levels. Individual subjects, teams (teachers, authors of books, and other study aids), or curricular agents performed this work. They are therefore part of communities among whose members there is a division of labor: general curricular guidelines are provided by agents appointed by the educational authorities; teachers design lessons, supported using didactic resources developed by authors and publishers, taking into account the school's planning and departmental agreements. Planning work occurs in various environments or ecological niches that support and condition its realization. Time, economic means, educational policies, etc., are conditioning factors for curricular planning and lesson design.

Planning instruments vary according to the different educational theories on which they are based. The types of personal and institutional meanings proposed by the OSA (Figure 4.8) provide criteria for curriculum, lesson, and assessment processes.

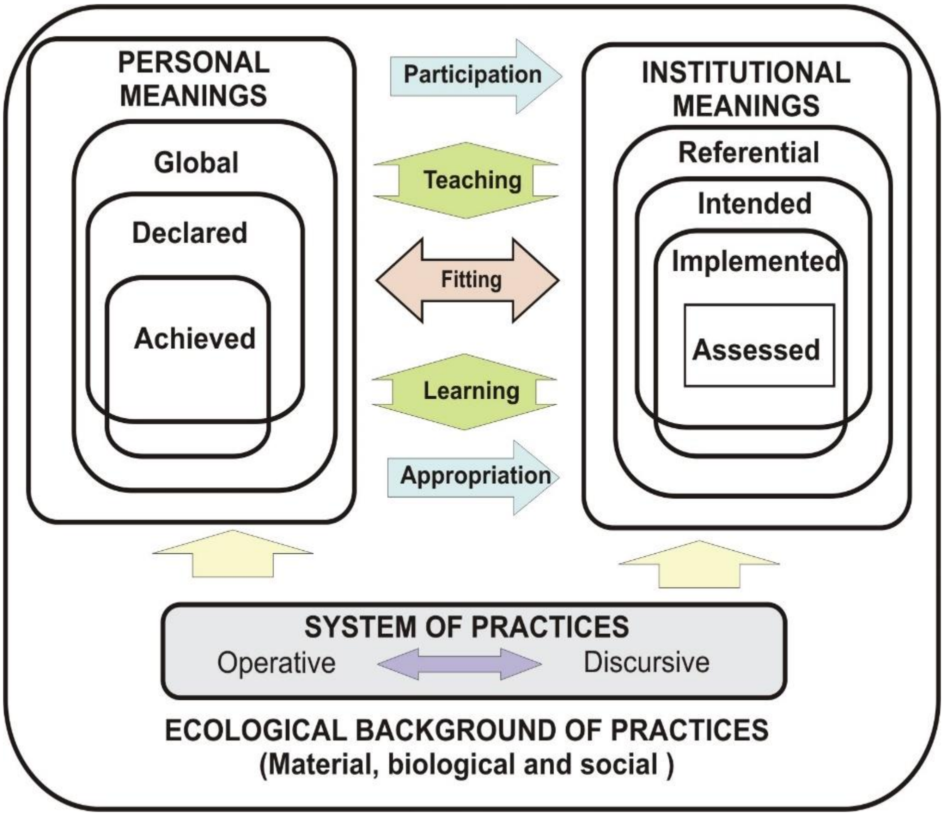

Figure 4.8. Institutional and personal meanings of mathematical objects (Godino et al., 2008, p. 6).

Learning is conceived in terms of students' appropriation of institutional meaning through their participation in the mathematical practices required for problem solving. Teaching considers students' initial meanings so that progressive coupling of meanings occurs. The construct *semiotic conflict* helps identify aspects and moments in which mismatches occur during meaning coupling. This describes any disparity or discordance between the meanings attributed to an expression by two individuals (persons or institutions). If the disparity occurs between institutional meanings, we speak of epistemic semiotic conflicts, whereas, if the disparity occurs between practices that form the personal meaning of the same subject, we designate them as cognitive semiotic conflicts. When a disparity occurs between the practices (discursive and operative) of two different subjects in communicative interaction (for example, student-student or student-teacher), we will speak of interactional (semiotic) conflicts.

The study of the processes of didactic transposition, transformation, and adaptation of mathematical contents to be studied at an educational level or a specific educational process is carried out in OSA by means of the metaphor of the ecology of meanings (Chapter 3). Interpreting the meanings of a mathematical object in terms of systems of practices facilitates the consideration of these systems, and consequently, of pragmatic meanings, as new objects, without dismissing the view of mathematics as an activity. It is the mathematical activity that must be transformed; the mathematical practices that a student performs from the first level of education are based on types of problems and resources that are very different from the level of abstraction that characterizes professional mathematical work. Recognizing different levels or degrees of generality and formalization in mathematical activity (Chapter 2) is fundamental to address the problem of its reproduction in learning environments.

Teaching planning must consider the specificity of knowledge, which leads to exploring partial meanings and their progressive articulation in a global or holistic meaning that serves as a reference model in instructional design. The curricular design of mathematical content at different educational levels requires considering the diversity of meanings, progressive articulation, and degrees of generality and formalization.

4.5.1. Application example of preliminary analysis tools[12]

This section exemplifies the use of preliminary analysis tools based on OSA in a case study of mathematics training of prospective elementary school teachers (Rivas, 2014; Godino et al., 2014), specifically in the design,

[12] Examples of the application of instructional design tools based on OSA were more extensively developed in the Rivas (2014)' s doctoral thesis and in Godino et al. (2014).

implementation, and evaluation of the topic of introduction to statistics and probability. Formative experiences were conducted under usual conditions in a subject focused on laying the mathematical foundation for teaching with limited time.

A preliminary analysis of meaning is a step in the design and implementation of a formative experience (Godino et al., 2014). In statistics education, a particularly relevant type of problem is data analysis projects, in which students are involved in the resolution of a case study intended to make sense of the operative and discursive practices of statistics and the objects involved. Instead of introducing decontextualized concepts and techniques or applying them only to typical problems difficult to find in real life, the different phases of a statistical investigation are presented: statement of a problem, decision on the data to be collected, data collection and analysis, and drawing conclusions on the problem posed (Batanero et al. 2011, p. 15).

In the training experience described (Godino et al., 2014), to select data analysis projects on which to base the study of statistics and to contemplate the results of previous research, the text by Batanero (2001) and the statistical reasoning model proposed by Wild and Pfannkuch (1999) were considered. Likewise, the recommendations of various authors on the teaching of statistics based on the use of projects (Batanero et al., 2011; Batanero & Díaz, 2011; Nolan & Speed, 1999), some curricular proposals (Franklin et al., 2005), and the systematization of previous research on cognitive and instructional aspects of the subject (Batanero, 2001; Díaz et al., 2008) were considered.

4.6. Design and a priori task analysis

Rivas (2014) and Godino et al. (2014) described the components of the general design of the thematic unit developed in the instructional process. These components include the following items:

- Statistics and its uses.
- Population, sample, and statistical variables.
- Tables and graphs.
- Measures of central tendency and dispersion.
- Random phenomena.
- Concept of probability and different approaches to it.
- Statistics as cultural knowledge.

To develop these contents, the study process is structured around the following three projects:

1) *Typical student* (4.5 hours, 3 sessions): collection, representation, and interpretation of data on the characteristics of the students in the class to describe a representative student profile.
2) *Throwing two dice* (3 hours, 2 sessions): focused on the study of basic probabilistic notions.
3) *Effectiveness of sports training* (1.5 hours, 1 session): focused on the study of basic statistical notions for comparing frequency distributions.

The following items complement these three projects:

a) Using a textbook (Batanero & Godino 2003)
b) List of the solved exercises as supplementary material.
c) A virtual teaching board used as an information repository and asynchronous communication space between students and between students and the teacher.

The students' personal meaning achieved was evaluated through the resolution of a problem situation that required the application of the statistical and probabilistic notions and procedures studied in the class sessions. The evaluation considered attendance and participation in the practical class sessions and the quality of reports submitted to the corresponding team workbooks.

As an example, we present the design of the project "Typical student" (adapted from Batanero, 2001). Although the project might seem "artificial" (there is no real need to choose a typical subject from a group), it has several advantages from the perspective of the training process of prospective school teachers: ease of data collection in class, the possibility of including other statistical variables (height, arm span, etc.). Moreover, it is a project directly applicable to primary education, allowing the study of different types of statistical variables (nominal, ordinal, discrete, and continuous quantitative) to be contextualized and to motivate the emergence of basic statistical notions and techniques.

4.6.1. A priori analysis of a data analysis project

This project involves collecting data from the class on the variables: gender, intensity of sports practice, number of siblings, weight, and amount of money in one's pocket at a given time. The following initial questions are proposed to motivate the use of central tendency and dispersion statistics, as well as the comparison of frequency distributions: What are the characteristics of a typical or representative student in the class? How representative is this student of the class? Are there differences between boys and girls in terms of each of these characteristics?

An analysis of the statistical practices to be implemented to answer the questions and the configuration of the objects and processes involved is included, distinguishing between those that can be assumed to be known by the students and those that constitute new learning objectives. Additionally, some conjectures about potential conflicts in project development, based on previous research and teaching experience, are made.

Type of problem and statistical practices

The project is open-ended because it raises questions that can be interpreted in various ways. As in most statistical projects, it does not suggest the direct application of a statistical technique, especially regarding the representativeness of a typical student. The aim is to motivate the process of reducing statistical data, identifying variables, their values, and frequencies to construct the corresponding frequency distribution. Subsequently, the distribution must be described using measures of central tendency, dispersion, and shape to select an ideal value that "represents" the dataset. Determining the differences in statistical summaries between the two subsamples (boys and girls) motivates the comparison of frequency distributions; in a more advanced course, the significance of the differences between averages and dispersions could be analyzed through inference. It also motivates a graphical comparison (e.g., using side-by-side diagrams) of pairs of distributions.

The statement of this problem situation can be generalized in various ways, as shown in Batanero and Díaz (2011, pp. 73-95). In this case, it is expected that students will follow the following statistical practices:

– Construct frequency distributions of the five variables, identifying the variables and their respective values, recounting the absolute frequencies of each value, and representing these results in a properly labeled tabular layout.
– Calculate averages (mode, median, and mean, discriminating their use according to the type of variable and the shape of the distribution). Assess the representativeness of the averages depending on the existence of skewness or outliers.
– Calculate extreme values (maximum, minimum) and measures of dispersion (range, quartiles, interquartile range, standard deviation), discriminating their use according to the type of variable and shape of distribution.
– Numerically (averages and dispersions) and graphically (side-by-side diagrams) compare the frequency distributions of the two subsamples (boys and girls).
– Assess the relative importance of differences between summary statistics of frequency distributions in subsamples.

It is expected that the tabular, numerical, and graphical reduction of statistical data has been previously studied by most students, so questions 1) and 2) would be considered the application of prior knowledge. Emergent objects and processes that would be included in the expanded content knowledge for trainee teachers should be highlighted:

- The ideal nature of averages (they do not have to correspond to a data point) and their use as representatives of a dataset (sample or population).
- Comparison of frequency distributions; qualitative assessment of differences between averages and dispersions.

The implementation of these statistical practices involves a complex configuration of mathematical objects and processes, the essential elements of which are indicated below.

Linguistic elements

It is very likely that the teacher will need to share with the class the intended institutional meaning of linguistic expressions such as: “characteristics of a typical or representative student of the class”, “how representative is this student of the class”, “differences between boys and girls”, etc. It can be assumed that students are familiar with most linguistic terms and expressions specific to descriptive statistics (absolute frequency, frequency table, mode, mean, median, maximum, minimum, range, bar chart, histogram). However, considering previous research (Díaz et al., 2008), difficulties in labeling frequency tables, side-by-side bar charts, and box plots can be anticipated. Additionally, students unfamiliar with spreadsheets will have difficulties representing a collection of data arranged in columns, statistical variables, and the specific functional language of spreadsheets (data set, calculation rule, result).

Conceptual elements

The following concepts of descriptive statistics are often poorly studied and recognized by students but are essential for understanding the system of operative and discursive practices of statistics:

- Statistical data (trait or contextualized information), statistical individual; data collection (sample, population).
- Statistical variable (trait of statistical individuals that can take different values in a data collection).
- Variability of the trait among individuals: values.
- Qualitative statistical variable: categories.
- Quantitative statistical variables: minimum, maximum, range.

The resolution of the requested tasks requires the application of concepts that students may be familiar with:

- Absolute and relative frequencies; frequency distribution; averages (mode, median, mean); extreme values (minimum, maximum); and dispersion (range; variance, standard deviation).
- Bar and pie charts.

Students may be less familiar with —and thus, they can emerge as emergent objects from the practices required— with the following items:

- Frequency histogram (intervals and class marks, criteria for their selection).
- Side-by-side bar charts and histograms; interpretation and use.
- Skewness of a frequency distribution, positive and negative skewness; its relation to the choice of the average used to represent the data.
- Percentiles, percentile ranges, interquartile ranges, box plot.
- Qualitative assessment of differences in means and dispersions.

Properties

These properties are necessary for project progress, and some of them are implicitly used:

- Averages represent data collection because they indicate trends or central positions of the corresponding frequency distributions.
- The mode is the only average that can be used if the statistical variable is a qualitative attribute; it may not be a unique value.
- The median is more representative than the mean when the distribution is skewed; both statistics coincide when the distribution is symmetrical.

Procedures

The elaboration of frequency tables, calculation of mode, mean, maximum, minimum, and range, and construction of charts are procedures that students either recall or find easy to master. The calculation of median, development of frequency tables grouped into intervals, and construction of box plots and histograms require special attention. Similarly, calculation of percentiles, interquartile range, and standard deviation.

Arguments

Students are expected to justify their answers to the posed questions by developing deductive arguments such as: "Considering the definitions and properties of the averages, the typical subject is a girl who does little sport, has 2.5 siblings, weighs 60 kg, and has 6 € in her pocket" (these are the median values as the distributions are skewed). "Choosing a girl for the variable, gender, is representative because 68% are girls (mode), while boys are infrequent".

Processes

A process of idealization that requires special attention leads to the concept of a typical or representative subject that does not correspond to a variable value. Thus, the median number of siblings was 2.5, which obviously does not correspond to any possible value for the variable. The procedures and properties applied to answer the questions posed in the given specific situation have a general character that the teacher must emphasize. The calculation of the median, determination of percentiles, and representation of histograms should conclude with a statement of general rules applicable to other data analysis situations.

4.7. Instructional implementation

The implementation of instruction is an activity performed jointly by a teacher and a group of students so that the students learn mathematical knowledge that has been previously adapted to the context during the planning activity. Within the study community (classroom, school), a division of labor occurs; teachers and students have different roles that must be articulated following a system of rules (didactic contract), using specific physical, conceptual, and procedural artifacts. The implementation of instruction occurs in specific environments under specific conditions (students' abilities and disposition, time available, means, etc.). The diversity of aspects to be considered makes the optimization of the process have a local character and requires the teacher's specific knowledge and skills as well as students' interest and perseverance.

The complexity of implementation has led to the development of various theories on what tools to use in each circumstance, what types of interactions to conduct, or what rules to follow to articulate the roles of the teacher and students in the most suitable way possible. The notions of didactic configuration and trajectory (Section 4.4) allow for detailed analyses of: a) the progressive deployment of the institutional meanings implemented; b) learning and its dependence on the interaction formats that actually take place; and c) the use of resources and time allocated. In this analysis, the focus is on the following descriptions:

– Content that was effectively dealt with.
– Patterns of teacher- students interaction.
– Recognition of the cognitive and interactional conflicts and how they are addressed by the teacher and students.

We refer to Godino et al. (2014), who describe the implementation of the "Typical student" project, whose a priori analysis we conducted in Section 4.6. The description includes transcript segments corresponding to a sample of significant didactic facts (SDFs). The notion of SDH is the criterion for delimiting didactic configurations, which can be linked to problems, subproblems, or other epistemic, cognitive, or instructional components that characterize the study process. SDHs provide local indicators of didactic suitability in some aspects of the study process. An SDH can contribute to the development of a study process, block its evolution, or limit the functioning of the didactic system. The "meaningfulness" of a fact does not refer then exclusively to its suitability for the development of meanings for students but to its importance for understanding the study process itself.

4.8. Retrospective analysis

The evaluation or retrospective analysis of the educational process consists of contrasting what was foreseen in the design and what was observed in the implementation to identify possible improvements. Godino et al. (2014) described this phase of the design in the case of the "Typical student" project, comparing the significant didactic facts observed with a priori analysis, followed by an assessment of the didactic suitability and the identification of possible improvements. The detailed analysis of the development of the other two data analysis projects used in formative experience (throwing two dice and effective sports training) can be found in the doctoral thesis of Hernán Rivas (2014).

In summary, a comparison of the design with SDHs shows that the objects and processes foreseen in the design have been put into play. From a cognitive perspective, most of the foreseen conflicts and others have manifested. From an instructional point of view, it has been found that the students have been "excessively guided" by the teacher; they have had "little autonomy" to provide answers by themselves to the questions posed. This was the teacher's decision due to frequent blocking of students and limited time to teach the content. It has been revealed that it is quite illusory to pretend that students, in their first encounter with the subject, mobilize the concepts of frequency tables, measures of central tendency, and dispersion to compare two frequency distributions from the instructions given in the project.

The reference elements for assessing the epistemic suitability of the implemented process should correspond to the institutional meaning intended by the teacher. On the other hand, to assess the epistemic suitability of the planning, it is necessary to investigate the elements of the meaning of elementary data analysis in texts and research related to its study at similar educational levels. The elements of reference for the remaining dimensions or facets (ecological, cognitive, affective, interactional, mediational) should be investigated in the texts and publications of didactic research on these aspects.

4.9. Theoretical perspectives related to OSA educational design theory

In this section, we describe other theoretical perspectives related to the problems addressed by the OSA-based design theory described in this chapter. We consider it of interest to analyze the dilemma between constructivist and objectivist positions in the field of education, which allows us to situate and understand the mixed didactic model that we propose. We then present the general characteristics of design-based research, within which the theory is included. Specific theories that include a design component elaborated in mathematics education, such as the Theory of Didactic Situations in Mathematics (Brousseau, 1997), the Theory of Didactic Moments (Chevallard, 1999), and Realistic Mathematics Education (Freudenthal, 1991), will be described and compared with OSA in Chapter 7.

4.9.1. The dilemma of constructivism versus objectivism[13]

Various forms of constructivism share, among other assumptions, the idea that learning is an active process, that knowledge is constructed rather than innate or passively absorbed, and that for effective learning to occur, students need to be presented with significant, open-ended, and challenging problems (Ernest, 1994; Fox, 2001).

> The arguments that human beings are active agents constructing knowledge by themselves have made people believe that instructional activities should encourage learners to construct knowledge through their own participations. This constructivist view plays an important role in science teaching and learning and has become a dominant teaching paradigm. (Zhang, 2016, p. 897)

Ideas for inquiry-based teaching and learning in mathematics and sciences have played a significant role in the curricular orientations of various countries, in projects, research centers, and reform initiatives. Linn et al. (2003) defined inquiry-based science learning as follows:

> We define inquiry as engaging students in the intentional process of diagnosing problems, criticizing experiments, distinguishing alternatives, planning investigations, revising views, researching conjectures, searching for information, constructing models, debating with peers, communicating to diverse audiences, and forming coherent arguments. (Linn et al., 2003, p. 518)

In pedagogical models that assume constructivist principles, the role of the teacher is to create a learning environment in which students can interact autonomously. This means that the teacher must carefully select learning tasks and ensure that the student has the cognitive and material resources required to engage in problem-solving. Additionally, the teacher must create

[13] The content of this section is based on the articles by Godino et al. (2019) and Godino and Burgos (2020).

cognitive scaffolding, an "architecture of choices", that supports and promotes the construction of knowledge by the students themselves. In a sense, it involves implementing a "libertarian paternalistic" pedagogy in the sense of Thaler and Sunstein (2008), based on "nudge" type interventions.

> A nudge, as we will use this term, is any aspect of the architecture of choice that modifies the people's behaviour in a predictable manner without prohibiting any option or significantly changing their economic incentives. (Thaler & Sunstein, 2008, p. 6)

In mathematical learning, the use of problem situations (applications to daily life, other fields of knowledge, or problems internal to the discipline itself) is considered essential for students to make sense of the conceptual structures that shape mathematics as a cultural reality. These problems constitute the starting point of mathematical practice; therefore, problem-solving activity, its formulation, communication, and justification are considered key to developing the ability to tackle non-routine problems. This is the main objective of the tradition known as problem-solving (Schoenfeld, 1992), which focuses on identifying heuristics and metacognitive strategies. It is also the goal of other theoretical models, such as the Theory of Didactical Situations in Mathematics (Brousseau, 1997) and Realistic Mathematics Education (Freudenthal, 1973; 1991).

However, there are positions opposed to constructivism, such as those of Mayer (2004) and Kirschner et al. (2006), among others, who justify through various studies the greater effectiveness of instructional models in which the teacher and the transmission of knowledge play a predominant role. These positions are related either to objectivist philosophical theories (Jonassen, 1991) or to empirical research over the last half-century on direct instruction or lesson-based pedagogy (Boghossian, 2006). Sweller et al. (2007) found that minimal guidance during instruction was significantly less effective and efficient than specifically designed guidance to support the cognitive processing necessary for learning. These authors claim that, in general, the effects of unguided discovery tasks are limited compared to enhanced discovery tasks. Opportunities for constructive learning may not arise when students are left without guidance.

> Perhaps the findings of these meta-analyses can help steer the debate away from issues of unguided discovery toward a fruitful discussion and empirical research on how to best implement cognitive scaffolding, how to provide classroom feedback, how to create worked examples for various content areas, and when to provide direct forms of instruction during learning (Alfieri et al., 2011, p.13).

Cognitive reasons can be provided to favor a didactic model based on the transmission of knowledge (objectivism) over models based on autonomous construction (constructivism). Kirschner et al. (2006) noted that constructivist positions, through minimally guided instruction, contradict the architecture of human cognition and impose a heavy cognitive load that hinders learning. Other reasons for rejecting constructivist positions come from cultural psychology.

> Accounts of cognitive development have often portrayed children as independent scientists who gather first-hand data and form theories about the natural world. I argue that this metaphor is inappropriate for children's cultural learning. In that domain, children are better seen as anthropologists who attend to, engage with, and learn from members of their culture. (Harris, 2012, p. 259)

The metaphor of the child as a natural scientist is useful for describing how children make sense of the universal regularities of the natural world, which they can observe for themselves, regardless of their cultural environment. However, the metaphor is misleading when used to explain cognitive development comprehensively and globally. Children are born into cultural worlds that mediate their encounters with physical and biological worlds. To access this cultural world, children require a socially oriented mode of learning (learning through participant observation). "The mastery of normative regularities requires cultural learning". (Harris, 2012, p. 269)

The debate between direct teaching, linked to objectivist positions on mathematical and scientific knowledge that advocate a central role for teachers in guiding learning and minimally guided teaching, usually referred to as constructivist teaching models, is not clearly resolved in the research literature. Advocates of problem-based and inquiry-based learning focus on the amount of guidance and the situations in which such guidance is provided. They believe that guidance given includes extensive support and immersion in real-life situations helps students make sense of scientific content.

Zhang (2016) asserted that the tension between these two positions lies not in whether one favors presenting more or less guidance or support to students but between explicitly presenting solutions to learners or letting them discover them. “For proponents of direct instruction, the explicit presentation of solutions and the demonstration of the process to achieve the solutions are essential guidance” (Zhang, 2016, p. 908). Expecting students to discover, explore, and find the solutions according to “inquiry-based education” eliminates the need to present solutions. In constructivist positions, even if some degree of information transmission from teacher to student is admitted, it is still essential to *hide* part of the content. For direct instruction proponents who adopt cognitive load theory with an emphasis on worked examples, providing solutions is essential. For example, in the field of proportional reasoning development, Bentley and Yates (2017) used the “worked examples” didactic model to present the unit reduction strategy, i.e., helping students adopt a step-by-step analysis of missing-value problems by first easily recognizing a unit and then using it to solve the problem. In their research, they reported positive results when applying this didactic strategy to students with high and low socioeconomic status.

In OSA, a new variable is introduced into the debate surrounding these models. This involves recognizing the ontosemiotic complexity of mathematical and scientific knowledge, which must be considered in instructional processes designed to optimize student learning. By assuming anthropological, semiotic, and pragmatist assumptions about mathematical (or scientific) knowledge, it can be concluded that an essential part of the knowledge that students must learn are conceptual, propositional, and procedural rules agreed upon within the community of mathematical (or scientific) practices. To solve the problems constituting the educational objective and develop mathematical reasoning skills, students start from prior knowledge, which centrally includes rules that must be available to understand and tackle the task. Expecting students to discover these rules can be an excessive challenge for most students. Considering the ontosemiotic complexity of mathematical knowledge and recognizing the central role of problem-solving as the *raison d'être* of content, this leads to the assumptions of a mixed educational-instructional model that is presented as a solution to the dilemma between inquiry and transmission (Figure 4.5).

4.9.2. Design-based research

To bridge the gap between theoretical research and teaching practice, "design-based research" (DBR) (Brown, 1992; Kelly et al., 2008) was developed. DBR comprises a family of methodological approaches aimed at studying context-specific learning. These approaches utilize instructional design and systematic research on instructional strategies and tools to achieve interdependence. It is understood that the research includes not only the design phase but also classroom experimentation and the evaluation of results. In mathematics education, DBR is conducted by applying different foundational theories to the design and interpretation of results. Artigue (2015) describes the methodology of *didactic engineering* as a type of DBR, based on the Theory of Didactic Situations, the Anthropological Theory of Didactics, and other theories. Didactic engineering (DE) (Artigue, 1989; 2011) is usually presented with a dual facet: as a "research methodology" and as a set of means or resources for teaching specific topics, developed considering research results. “Didactical engineering thus emerged as a research and development methodology based on classroom realizations in form of sequences of lessons, informed by theory and putting to the test theoretical ideas”. (Artigue, 2015, p. 469)

Godino et al. (2013) analyzed the characteristics of DBR and DE and concluded that since DE can be based on different theoretical frameworks, DE and instructional design describe the same type of didactic research. Instructional design or didactic engineering research, regardless of the underlying theory, addresses questions such as: What learning outcomes are obtained if a specific educational intervention is conducted in a given context? This corresponds to the predictive scheme: If X, then Y. Because these studies were conducted in real educational contexts, they considered the richness and complexity of factors that condition teaching practices. Consequently, the resources developed and the knowledge obtained from these studies can solve practical problems that are usually related to what mathematics is taught and how. However, since the knowledge provided by this type of research is predictive, it cannot derive the evaluations and norms of action required for efficient practice intervention. Bridging the gap between scientific knowledge and teaching practice requires developing theories that explicitly state the system of axiological principles and evaluative and normative criteria for efficient educational action derived from theoretical and applied research. The Theory of Didactical Suitability, described in Chapter 5, provides this interface in the OSA.

The instructional model assumed in OSA is based on the principles of cultural/discursive psychology (Lerman, 2001), which attributes a key role to the “zone of proximal development” (Vygotsky, 1934). Contrary to constructivist models, student autonomy in learning is the result of this process and not a prerequisite. However, given the central role that the anthropological perspective of knowledge plays in solving problems and the activity involved in solving them, the search, selection, and adaptation of good problem situations and the involvement of students in their resolution are also principles of meaningful mathematical instruction. This assumption leads to a mixed instructional model in which the construction and transmission of knowledge are articulated dialectically (Godino & Burgos, 2020; Godino et al., 2020) and is summarized in the following principles:

- Learning enables students to apply appropriate institutional meanings and objects to solve specific problems and develop as individuals.
- The study of students' personal meanings is an essential component of educational problems because the appropriation of intended institutional meanings is conditioned by students' initial personal meanings.

The institutional meanings ultimately implemented in an instructional process may differ from the intended and reference meanings because of constraints imposed by students' cognitive abilities, available resources, and the social and educational context. However, the meanings of intended and implemented institutional objects in educational contexts are expected to represent a representative sample of the overall reference meaning.

4.10. Synthesis of the theory of instructional design in mathematics based on OSA

In Table 1, we present a synthesis of the characteristics of the Ontosemiotic Theory of Educational Design in Mathematics, following Michie et al. (2014) on elements for the description of theories in the field of social and behavioral sciences.

Table 4.1. Synthesis of the theory of educational design based on OSA

Elements	**Description**
Summary. What is the theory about and what are its main propositions?	The Onto-Semiotic Theory of Educational Design in Mathematics provides assumptions and theoretical tools for designing teaching and learning processes in mathematics based on the onto-semiotic theory of mathematical activity and emergent objects, as well as the onto-semiotic theory of meaning and mathematical cognition proposed by the OSA (Chapters 2 and 3). This theory includes a model of the structure and dynamics of educational processes that considers the various facets and components that characterize these processes. It proposes a model of categories of norms and metanorms, including criteria of didactical suitability, that allows for explaining didactical phenomena and provides guidelines for optimizing educational processes. The constructs of didactical configuration and trajectory enable detailed analyses (descriptive and explanatory) of the design and implementation of educational processes. Complemented by the postulate of the onto-semiotic complexity of content and the construct of didactical suitability, these analyses allow for the development of a mixed didactical model that resolves the dilemma between constructivist (inquiry-based) and objectivist (transmissive) models to optimize mathematical learning.
Scope/Objective. What phenomena does the theory explain?	The goal of the theory of educational design is to understand the processes of teaching and learning mathematics, the facets, components, and interactions involved in design, implementation, and evaluation, and to construct a system of criteria based on OSA and didactical research results to optimize the development of these processes. Therefore, it includes assumptions and tools for the scientific (descriptive, explanatory, and predictive) component as well as for the technological (prescriptive) component of mathematics education.
Justification. Why is this theory necessary and how does it improve on previous theories?	This theory emerges from the consideration that existing educational design theories—even those specific to mathematics—are not based on explicit models regarding the specific nature of mathematical knowledge. The onto-semiotic theory of mathematical activity allows for awareness of the onto-semiotic complexity of teaching contents and provides tools for preliminary analysis of the meanings of mathematical objects. This includes an a priori analysis of tasks, identification of significant didactical facts, and criteria for optimizing the epistemic and cognitive trajectories of educational processes in mathematics.

Hypotheses. What specific hypotheses does the proposed theory propose, and how do they differ from other theories?	This theory postulates that the design of educational processes should be grounded in specific theories about the nature of the content being designed. In the case of mathematics, the theory of educational design in mathematics assumes the ontological, semiotic, and epistemological postulates of the OSA regarding mathematical activity, as well as the processes and emergent objects thereof. It posits that during students' initial encounters with new content, collaborative didactical configurations optimize didactical suitability by considering the onto-semiotic complexity of the content under study. Student autonomy emerges from didactic activities and is not a prerequisite, as constructivist theories postulate. .
Constructs. What elements constitute the theory?	The theory includes the following constructs: – Facets, components, subcomponents, and elements of an educational process. – Normative dimension, meta-normative dimension, and suitability criteria. – Didactical configuration and epistemic, ecological, instructional (interactional, mediational), normative, and cognitive-affective subconfigurations. – Didactical trajectory and epistemic-ecological, instructional, normative, and cognitive-affective subtrajectories. – Institutional and personal meanings. Ecology and complexity of meanings. – Semiotic conflicts (epistemic, cognitive, instructional). – Reference didactical configurations. Didactical model based on OSA. – Didactical suitability. Criteria and indicators.
Relations. How are the elements of the theory related to each other?	The model of structure of an educational process forms the basis for the systematic analysis of the design, implementation, and evaluation, allowing for the categorization of types of configurations, didactical trajectories, norms, and metanorms. Institutional and personal meanings, pragmatically understood according to OSA, enable the study of the reconstruction and ecological adaptation of meanings in educational contexts during the planning phase of teaching. The identification of semiotic conflicts provides explanations and criteria for studying classroom interactions. The construct of didactical suitability, in its various facets and components, offers criteria for decision-making during planning and implementation phases and suitability indicators for evaluation. The construct of the onto-semiotic complexity of meanings and reference didactical configurations forms the basis of the mixed inquiry-transmissive didactical model based on OSA.

Origin. On which theories is it based, and how?	The theory of educational design is based on the onto-semiotic theory of mathematical activity and emergent objects by adopting the types of objects and mathematical processes proposed by this theory as components and elements of the epistemic and cognitive facets of educational processes in mathematics. The postulate of institutional and personal relativity of the meaning of mathematical objects in relation to institutional frameworks and contexts of use connects the theory of educational design with historical-cultural theories in mathematics education, as well as situated cognition theories.
Similarity. Which theories are most similar to this theory?	The OSA-based theory of educational design has connections with didactical engineering (Artigue), based on the Theory of Didactic Situations in Mathematics (Brousseau), as well as the Theory of Didactical Moments and its development in the Theory of Study and Research Paths (Chevallard). There are also concordances with Realistic Mathematics Education (Freudenthal). The onto-semiotic assumptions about mathematical knowledge and the specific constructs underlying the theory extend the types of analyses that can be conducted, explanations of didactical phenomena, and especially the mixed inquiry-transmissive didactical model based on OSA. The system of didactical suitability criteria explicates and expands those derived from the abovementioned theories.
Complementarity. With which theories can it be complemented?	OSA aspires to develop a comprehensive theoretical system to address the ontological, semiotic, epistemological, educational-instructional problems involved in the teaching and learning processes of mathematics by applying a theoretical hybridization strategy. It can be complemented by curricular theories that explicitly address the development of affective and ecological facets of educational-instructional processes, such as critical and inclusive mathematics education (the role of mathematics in society, social justice, equity).
Operationalization. How are the constructs measured or identified?	The constructs of the theory are unmeasurable or gradable traits, except for didactical suitability. These are descriptive categories of the various aspects of an educational process that are understood as multidimensional or multifaceted constructs. Sections 4.1–4.8 describe the constructs that configure this theory.

Uses. What can the theory be used for?	The OSA-based theory of educational design can be used to plan, implement, and evaluate educational processes in mathematics at the micro (lessons), meso (topics), and macro (programs) levels. It can also serve as an instrument for describing, explaining, and evaluating educational processes designed from other theoretical perspectives, helping to identify aspects that can be improved. Therefore, it is a resource for design research and teachers' reflection on their own practice.

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Chapter 5

Theory of didactic suitability based on OSA

Introduction

The evaluation of educational and instructional planning and implementation processes involves teachers and other stakeholders interested in the overall assessment and learning of students. Thus, various national and international bodies are interested in the results of student learning and the factors that determine them and apply standardized tests that often condition the curricula implemented. The object/subject of this evaluative activity is the educational-instructional process, thus involving its various facets and interactions. At the local level, i.e., within the classroom, evaluative activities are conducted to gather information and make instructional decisions. The desired outcome is information about the learning achieved by the students (summative evaluation) or the development of the instructional process at the local level (formative evaluation).

At the macro level, i.e., in external summative assessment, various professional communities are involved in the required tasks (design of instruments, implementation, analysis and interpretation of results, etc.). At the local level, assessment is also a community activity that involves not only teachers and students but also schools and families. The ecological environment in which the activity occurs is conditioned and supported by rules that regulate periodicity, forms, procedures, and tools. Macro-level learning assessment processes require creating objective measurement instruments that allow comparisons between groups, schools, and countries to make macro-level decisions. This assessment reduces complexity by eliminating contextual details that may be essential from an educational perspective. From this, the following general dilemma arises regarding assessment in mathematics education: "How can we assess the essential components of mathematical knowledge, understanding, thinking, creativity, problem solving, and general ability without seriously distorting them" (Niss, 1993, p. 27).

In this chapter, we develop the notion of didactic suitability as mentioned in Chapter 4 as an element of the normative dimension, i.e. as the system of criteria that implicitly or explicitly guides the aim of optimizing educational-instructional processes. This theoretical tool can be used in the planning and implementation phases, but particularly in the phase of retrospective analysis or evaluation of learning, identifying conditioning factors in the development of the educational process, and improvements. Suitability is closely related to the quality of instruction and the instruments used for measuring learning, although it focuses on the local optimization of mathematics education processes. It focuses on recognizing the complexity of the facets and components that condition these processes and on developing criteria or guidelines to help teachers, who must make final decisions about the relative weight of the criteria in each circumstance, to act.

In Section 5.1, we describe the notion of instructional quality by quoting various studies that developed the construct and the instruments for its measurement. We justify the usefulness of providing an extended view of quality using a qualitative approach, as proposed by didactic suitability. In Section 5.2, we present how we conceptualize didactic suitability, its definition, and the structure of its facets and components. We speak of a theory of didactic suitability based on OSA (TDS-OSA) by considering that both the definition of suitability and the system of criteria developed in Section 5.3 are based on the assumptions and constructs of the theories of mathematical activity, emergent objects, and meaning that constitute OSA (Chapters 2 and 3), as well as the theory of instructional design described in Chapter 4. In principle, every theory used in mathematics education, including every teacher, has its own theory of didactic suitability. The suitability criteria listed in the various tables of Section 5.3 apply to any mathematical content. However, from didactic research on specific contents, suitability criteria are derived at a more detailed level that must be observed to optimize the corresponding educational processes. In Section 5.4, we include a system of suitability criteria for the study of proportionality, while in Section 5.5, we present an example of applying TDS-OSA in an experience of teaching proportionality to secondary school students. Analyzing the consistencies and complementarities between OSA-based suitability criteria and other theories used in mathematics education is the aim of Section 5.6. We conclude the chapter by answering the questions proposed by Michie et al. (2014) as a synthesis of a theory in the social and behavioral sciences.

5.1. Quality of instruction and its measurement[14]

Various educational studies have been interested in developing instruments for observing and measuring the quality of instruction, either through generic or content-specific characteristics, or a combination of both. Charalambous and Praetorius (2018) quote, among others, the Elementary Mathematics Classroom Observation Form (Thompson & Davis, 2014), Instructional Quality Assessment (IQA, Matsumura et al. 2008), Mathematical Quality of Instruction (MQI, Hill et al., 2011), and Mathematics-Scan (M-Scan, Walkowiak et al., 2014) projects. Most of these papers seek to provide valid and reliable information for educational

[14] The content of sections 5.1 to 5.3 is based on the paper by Godino, J. D., Batanero, C. & Burgos, M. (2023). Theory of didactical suitability: An enlarged view of the quality of mathematics instruction. *EURASIA Journal of Mathematics, Science and Technology Education, 19*(6), em2270.

authorities to make decisions on reform plans, accreditation, and teacher selection processes.

A distinctive feature of instructional quality studies is the observation of samples of classes, schools, teachers, and students' productions to statistically relate certain teaching variables to learning. Protocols for observing classes and student work were constructed, and explicit criteria for assigning scores were established by external evaluators (Boston, 2012; Hill et al., 2011). Recommendations for improving instruction at the school or district level are provided.

Instruments for measuring the quality of instruction usually assess aspects of educational practices empirically associated with students' learning. Instrument feasibility and technical quality are sought to ensure reliable use when assigning scores to classroom observations and student productions. These measurement requirements may reduce the generalizability of the results because important aspects of instruction (e.g. misconceptions about mathematics or the role assigned to mathematical processes) are not captured.

Although assessment of a few well-chosen aspects of instruction may provide useful information for improving instruction, a comprehensive instrument can help realize the complexity of educational processes and identify significant variables. Optimizing teaching and learning processes often requires prioritizing some principles and neglecting others, depending on the context and circumstances of learners. It is necessary to develop instruments to systematically analyze the different facets and components of the educational-instructional process that teachers can use to reflect on their practice and make informed decisions to progressively improve it.

In the OSA-based Theory of Didactic Suitability (TDS-OSA), as described in this chapter, we attempt to complement the efforts of quantitative quality measurement with a qualitative approach, which focuses on teachers' initiative and responsibility to decide on their own teaching practices. This reflective activity must be supported by specific instruments to reveal the complexity of the processes and the difficulty of balancing sometimes conflicting teaching principles. The results of this work have implications for research on teacher education, particularly for teachers' reflection and decision-making about practice (Karsenty & Arcavi, 2017; Tzur, 2001).

5.2. Conceptualizing didactic suitability

In OSA, we developed two tools to support the evaluation of educational-instructional processes:

1) The structure model described in Section 4.1 (Chapter 4) for which we distinguish six facets and their respective interactions: epistemic, ecological, mediational, interactional, cognitive, and affective. For each facet we differentiate various components and sub-components (Figure 4.1).
2) The theory of didactic suitability in which we suggest, in addition to the construct of didactic suitability, a system of criteria—principles or norms—to optimize educational instructional processes in their different facets and components. The criteria formulation described in Section 5.3 is based on the OSA assumptions and constructs, although, as explained in that section, the criteria are usually shared by other theories, curricular orientations, and schools of thought.

5.2.1. Definition and structure of didactic suitability

In Godino et al. (2006), we considered the didactic suitability of an instructional process as we faced the challenge of moving from the analysis and description of processes to didactic engineering, understood as a discipline that guides the design, implementation, and evaluation of mathematics teaching and learning. Specifically, we asked ourselves, "What criteria can we derive for the suitability of didactic configurations and trajectories from the ontological-semiotic approach to mathematical cognition?" In this study, we consider that the overall suitability of a didactic configuration and trajectory must be assessed by considering various facets or dimensions. For teaching and learning configurations, we assessed suitability by considering the possibilities of identifying conflicts and negotiating meanings.

This theoretical tool has been widely used in various research works, as described by Malet et al. (2021), and we can see its progressive refinement and changes in the formulation of suitability criteria and indicators (Breda et al., 2018; Godino, 2013). In Godino et al. (2023) we propose the following conceptualization of didactic suitability:

> The didactic suitability of an instructional process is defined as the degree to which that process (or a part) meets certain characteristics that allow it to be qualified as optimal or adequate to achieve the fit between the personal meanings achieved by learners (learning) and the intended or implemented institutional meanings (teaching), taking into account the circumstances and resources available (environment). These institutional meanings are also representative of the global meaning of reference. (Godino et al., 2023, p. 4)

This statement describes the conditions for an instructional process to be suitable, which is initially linked to the optimization or adequacy of the coupling between teaching and learning and the implementation of rich mathematics, considering the multiple factors involved in these processes. From this, it is possible to state an overall criterion (principle) for didactical suitability:

> One should ensure that students learn the mathematics intended to be taught, with such mathematics representing the overall meaning of the same and considering personal, contextual and temporal circumstances.

This formulation incorporates the social values of mathematics education, such as avoiding school failure and efficient use of resources. Instructional suitability is a gradable trait of educational processes that involves the coherent articulation of six partial aspects of suitability related to its facets and components (Figure 5.1). Section 5.3 provides a detailed description of the sub-criteria for facets and components.

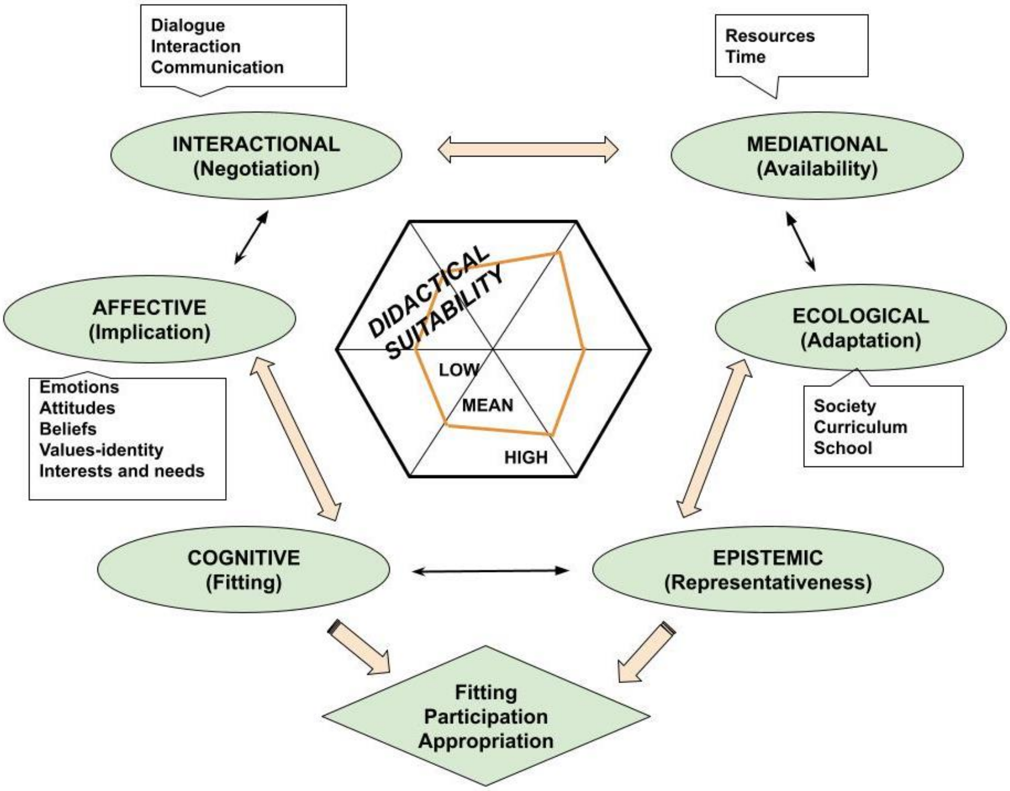

Figure 5.1. Didactic suitability. General facets and criteria (Godino, 2013, p. 116)

We assume that in the social and educational sciences, it is possible to develop suitability criteria as value judgements if such judgements have a social character and it is possible to substantiate their formulation. Such judgments involve rationality and can therefore be subject to scientific scrutiny (Bunge, 1999; Lacey, 1999; Rugina, 1998).

The partial-suitability criteria for each facet can be refined using the components provided by various OSA tools. Thus, for example, epistemic suitability refers to the degree to which the institutional meanings of the content and the configurations of objects and processes implemented represent the overall meaning of reference, considering the contextual and personal circumstances of the subjects involved. Cognitive suitability refers to the degree to which the learning objectives are cognitively challenging and achievable for the learners, and the personal meanings achieved are consistent with the planned institutional meanings, considering their personal and contextual circumstances. Joint optimization of partial suitability may be conflicting in specific contexts and circumstances:

> this leads, first, to treating suitability criteria jointly (and not as independent criteria as is often done in quality) and, second, to questioning or relativizing the validity of a given criterion in a specific context, which leads to giving different relative weights to each criterion depending on the context. In this way, the relative weight of each partial suitability criterion no longer depends only on external factors (the existence of prior community consensus) but, to a greater extent, on internal factors (the conflict that the suitability criterion generates with the context and other criteria) (Breda et al., 2018, p. 265).

The second level for suitability analysis (Figure 4.1, Chapter 4) is determined by the components of each facet (in varying quantities), some of which apply to any discipline, while others are specific to mathematics. For the epistemic and cognitive facets, it is possible and convenient to propose a Level III analysis that distinguishes sub-components determined by the elements that characterize mathematical knowledge according to OSA. When the educational-instructional process under analysis refers to specific content, for example, probability, we can define a Level IV of analysis in the epistemic and cognitive facets, considering specific features of teaching and learning of this content (Beltrán-Pellicer et al., 2018).

The actions and resources used in the epistemic, ecological, interactional, and mediational facets are intended for students' learning, in which both cognitive and affective aspects are contemplated. There are complex interactions between the different facets as educational-instructional processes happen within recursive and open social systems in which interactions between elements rely on interpretation and negotiation of meanings, as well as on a complex web of interwoven values.

> Social systems are often semiotic systems in the sense that interactions between elements are not based on physical strength but on meaning and interpretation. In these terms, education can be characterized as an open recursive semiotic system. It is a semiotic system because the exchanges between teachers and learners do not involve physical force but meaning. The system functions recursively because teachers and learners act on the basis of their interpretations and understandings. Educational systems are generally open because they interact with their environment (albeit under conditions of reduced complexity). (Biesta, 2010, p. 497)

In TDS-OSA, the axiological principles and criteria for optimizing teaching and learning assumed by the research community are explicit and structured, incorporating some of their own derived from ontosemiotic assumptions about mathematical knowledge. This theory provides an expanded vision of the quality of mathematics instruction studies (Charalambous & Praetorius, 2018), emphasizing an interpretative approach to the web of values at stake in mathematics teaching and learning processes. The complexity associated with optimizing such processes becomes obvious because it is necessary to achieve a balance in implementing principles related to different facets and components, which have a strong local component. The teacher should manage this axiological balance by weighing the relative importance of each aspect according to the circumstances and context.

Fulfillment of the suitability criteria is associated with several empirical indicators. For example, it is assumed that students' mathematical understanding and competence stem from the use of different representations and their transformation and conversion. An instructional process should ensure that students have opportunities to use different representation, transformation, and processing systems. The suitability of a specific instructional process is based on the observation of empirical indicators of the use of representational systems. The criterion is a norm (sometimes used as a principle, in other cases, as a rule) that should be followed to achieve high suitability for an instructional process; the indicator is the observable manifestation of the application of the criterion; criteria are grouped or categorized according to facets and components.

5.2.2. A broader view of suitability

Didactic suitability criteria are heuristic principles that synthesize the results of mathematics education research. Various theories share most of them, as outlined in Godino (2013) and Godino (2021). They are offered as tools to support the educational professional's individual or collaborative enquiry, which must interpret and adapt the global and specific criteria for each component of the instructional process to their particular circumstances and weigh their relative relevance.

The notion of suitability can be applied not only to didactic activities but also to any human activity, thus establishing a link between scientific-technological research and reflective practice. This helps in decision-making regarding addressing the triple dialectic between the Aims, Values and Means, which lies on the reflective practitioner. Value judgments can be rationally analyzed, compared, and articulated. All human activities, including mathematics, incorporate principles of efficiency; therefore, didactic suitability can be extended, e.g., to mathematical or economic suitability.

With this overview, a Theory of Suitable Activity will be a system of value judgements—*one should do this and not that*—on how to perform an activity in the best possible way, considering the specific context and circumstances. These theories may be implicit or explicit, personal or social, spontaneous, or based on basic or applied research, as well as on the reflective practice of the actors involved. For didactic activities (teaching and learning), each teacher will have his/her own “theory of what can be best” depending on their personal knowledge, beliefs, and values.

5.3. System of didactic suitability criteria

In this section, we describe the general suitability criteria for the different facets, justifying their rationality in the OSA assumptions and their consistency with other general or specific educational theories of mathematics education. The complete system of didactic suitability criteria for the facets and components configures the instrument Guide for the Analysis of the Didactic Suitability of Mathematical Instruction Processes (GADS-MIP), which is formed by the set of tables included in this section.

The suitability criteria should be understood as principles to be followed to ensure that the instructional process is suitable for each facet, considering its components. In previous studies (Godino, 2013; Breda et al., 2018), suitability indicators were developed for these components and understood as features that should be observed in a suitable instructional process. To assign a greater or lesser degree of suitability, it is necessary to develop rubrics with rules for assigning numerical values to the degree of compliance with each indicator. This quantitative orientation in the assessment of suitability has not been developed in the TDS-OSA because the main prior use of the GADS-MIP instrument has been for professional teacher development and not for comparing and ranking lessons or teachers’ quality.

5.3.1. Epistemic facet

For TDS-OSA, it is essential to assess the quality of the content being taught and learned so that the epistemic facet has a prominent place in analysis levels I, II, and III (Figure 4.1, Chapter 4), demonstrating the complexity of measuring the quality of institutional mathematical knowledge. In Table 5.1A, we describe the general criteria for epistemic suitability and its components.

Table 5.1A. Suitability criteria for the epistemic facet and its components

General criterion of the epistemic facet	*Component-specific criteria according to the components*
The partial institutional meanings of the content and the configurations of objects and processes linked to each meaning, implemented throughout the instructional process, should be articulated, representative of the reference global meaning. and consider the contextual and personal circumstances of the subjects involved.	*Meaning*
	– Select the partial meanings whose study is adapted to the contextual and students' personal circumstances, contextualizing them with understandable problem-situations. – Consider a representative sample of the primary objects involved in the mathematical activity (situations, languages, concepts, properties, procedures, and arguments) that are involved in the partial meanings of the content.
	Relations (Connections)
	– Relate the partial meanings to each other and to the objects involved in the corresponding practices, as well as to the content of other topics that the student already knows.
	Processes
	– Consider the diversity of processes from which the objects involved in mathematical practices emerge (problematization, representation, definition, generalization, modeling, ...).

The implemented mathematical content must meet specific requirements to ensure epistemic suitability for the instructional process. It must include rich, optimal, or adequate mathematics, according to the students' contextual and personal circumstances (ecological and cognitive facet). The ontosemiotic model of mathematical knowledge provides elements for characterizing such mathematics for different components and subcomponents of the epistemic facet (Chapter 2). A specific instructional process occurs in a particular environment and is performed over a limited time interval; therefore, it is inevitable that some partial meanings of the object in question and the configurations of objects and processes associated with the meanings are selected. Nevertheless, globally (throughout education), the set of meanings must be representative[15].

With this formulation of the epistemic suitability general criterion, there is no single but diverse “good mathematics”, since for each content different “correct” meanings can be identified, which vary in generality, formalization, and the objects and processes involved[16]. The optimization of learning is to be local, i.e., adapted to the context, subjects, and circumstances.

For high epistemic suitability of the instructional process, the task design should have the characteristics stated in Table 5.1B. Considering the anthropological view of mathematics assumed in the OSA, i.e., mathematics as an activity of people and as a system of cultural objects emerging from it, problem solving is fundamental to the instructional processes. This is reflected in the general criterion and in the criteria linked to the components: meanings (contextualization through situations-problems understandable to students), relationships or connections between meanings, objects (situations-problems), and processes (problematization).

Table 5.1B. Suitability criteria for Level III subcomponents of the epistemic facet

Subcomponents	*Specific criteria*
Problem-situations	– Present a representative and articulated sample of contextualization, exercise, application situations, and problem generation (problematization).
Languages	– Use a representative sample of different modes of mathematical expression (verbal, graphic, symbolic…), translations, and conversions between them.
Rules (concepts, propositions, procedures)	– Propose definitions and procedures that are clear, correct, and adapted to the educational level to which they are addressed.

[15] The requirement that the meanings, objects, and processes implemented should represent the intended institutional meaning implies that there should be no mathematical errors in the teacher’s planning and presentations. For this reason, “absence of errors” in the epistemic facet is not included as a Level II component, as some models do, e.g., MQI (Hill et al., 2011) and Breda et al., (2018). The absence of epistemic conflicts is considered a criterion related to the definitions, propositions, and procedures (Level III) sub-components.

[16] Examples of the reconstruction of the global meaning of some mathematical objects are described in Batanero and Díaz (2007), Burgos and Godino (2020), Wilhelmi et al. (2007), among other works.

	– Correct presentation of the fundamental statements and procedures of the topic for the given educational level. – Propose situations in which students must generate or negotiate definitions, propositions, or procedures.
Arguments	– Propose definitions and procedures that are clear, correct and adapted to the educational level to which they are addressed. – Correctly present the fundamental statements and procedures of the topic for the given educational level. – Propose situations where students have to generate or negotiate definitions, propositions, and procedures.

The epistemic suitability criteria (Tables 5.1A and 5.1B) are consistent with the principles assumed by the Theory of Didactic Situations in Mathematics (TDSM) (Brousseau, 2002) and the Realistic Mathematics Education (RME) (Van den Heuvel-Panhuizen & Wijers, 2005), based on Freudenthal's (1983; 1991) didactic phenomenology. These theories and curricular proposals (such as NCTM, 2000) propose the use of problem situations to contextualize mathematical ideas and generate them from the resolution, communication, and generalization of solutions. The activity and reality principles of RME support the consideration of epistemic suitability criteria. For Freudenthal, there is no mathematics without mathematization: it is "all organizing activity of the mathematician, whether it involves mathematical contents and expressions, or more naïve, intuitive, or lived experiences, expressed in everyday language" (Freudenthal, 1991, p. 31). This activity applies to solving environmental problems or reorganizing mathematical knowledge.

Thus, the selection and adaptation of problem situations are central to achieving high epistemic suitability. However, although problem situations are a central element, high epistemic suitability also requires attention to the various representations or means of expression (which is consistent with the work of Duval, 1995), definitions, procedures, propositions, and their associated arguments. Such tasks should provide students with a variety of ways to approach them, involve a variety of representations and require students to conjecture, interpret, and justify solutions (Hanna and Villiers, 2012). All these processes characterizing rich mathematics must be relativized to context, subject, local, and temporal circumstances.

Attention should also be paid to the connections between different parts of the mathematical content and the various types of objects and processes. Mathematics is an integrated field of study; "in a coherent curriculum, mathematical ideas are related and build on each other" (NCTM, 2000, p.14). This position is consistent with the "Principle of interconnectedness" of the RME: Mathematical content blocks (numeracy and calculus, algebra, geometry, ...) cannot be treated as separate entities. Problem situations should include interrelated mathematical content. Problem solving in rich contexts often requires the application of various mathematical tools and knowledge.

5.3.2. Ecological facet

Ecological suitability is the degree to which an educational action to learn mathematics is suitable for the environment in which it is used. By environment, we mean everything that conditions teaching and learning outside the classroom: society, schools, pedagogy, mathematics education. The instructional process occurs in an educational context that fixes goals and values for educating citizens and professionals. These aims and values are interpreted and specified within the educational project of the school or department that coordinates the actions of teachers. The teacher does not work in isolation in the classroom but is part of a community of study and inquiry that provides useful knowledge about good mathematical and didactic practices that should be observed. In Table 5.2., we describe the general ecological suitability criteria and the criteria for the respective components.

Table 5.2. Suitability criteria for the ecological facet and its components

General criterion for the ecological facet	*Specific criteria according to the components*
The educational-instructional process should agree with the educational project of the center and society, considering the conditioning factors of the setting in which it is developed and innovations based on educational research.	*Interdisciplinary and intra-disciplinary connections* – Relate the contents with other intra- and interdisciplinary contents. .
	Curriculum – Propose a progressive and articulated study of the various partial meanings of mathematical contents at different educational levels by graduating the generality and formalization with which these meanings are approached.
	Openness to innovation – Implement innovations that are based on research and best practices recognized. – Integrating the use of new technologies (calculators, computers, ICT, etc.) in the educational project.
	Socioprofessional and cultural adaptation – Ensure that the educational-instructional process as a whole contributes to the students' socio-professional growth.
	Education in civic values – Include in the design and implementation of the educational-instructional process the education of students on democratic values and critical thinking.
	Family setting - Stimulate and support, to the extent possible, the student's learning outside school and his/her development as a person.

Critical mathematics education (Skovsmose, 2012) provides ideas for making citizens an active part of a democratic society. Beyond individual mathematical learning, it is necessary to reflect on the collective consequences of such learning in contemporary society. At school, mathematical practices can exert an enormous influence in two completely opposite ways: mathematics, reduced to mere routine calculations, can reinforce passive and complacent attitudes, and mathematics in its broadest sense can develop critical and alternative thinking.

Components of the ecological facet include the connections between various content blocks and disciplinary areas, which influence mathematical content richness and are also related to epistemic suitability. Other aspects of a cross-cutting nature are also considered, whose implementation is the responsibility not only of the teacher but also of other actors. Such is the case of the curriculum, which should consider the results of mathematics education research, consider the social and professional training of students, and value education. As research evidence shows, the family environment is also mentioned as a learning determinant, although “in most cases, however, it is not desirable to remove children from their families simply to improve their chances of educational success at some point in time” (Biesta, 2010, p. 501). This observation reveals the complexity of achieving an axiological balance in educational-instructional processes.

5.3.3. Mediational facet

This facet includes different resources that condition and support mathematics teaching and learning. In addition to concrete materials and technological tools, such as calculators and computers, study aids (textbooks, activity books, educational videos, ...), the number of students assigned to the teacher, the timetable in which lessons occur, classroom material conditions, and the total time allocated to study and its distribution are also included. Table 5.3 sets out the general criterion for mediational suitability and specific criteria for its components.

Table 5.3. Criteria of suitability for the mediational facet and its components

General criterion of the mediational facet	*Component-specific criteria*
	Material resources (Concrete, virtual, and symbolic) – Distinguish mathematical objects (regulative, non-ostensive) from their respective concrete, visual or symbolic representations in mathematical and didactic practices.

Adequate resources should be available for the optimal development of teaching and learning processes.	– Articulate the use of configurations of objects and processes based on alphanumeric representations with those based on concrete representations to progressively enhance the processes of generalization, calculation, and mathematical proof.
	Study aides (textbooks, exercise books, educational videos, …) – Make critical and reflective use of curricular materials (textbooks or activity worksheets in physical or virtual format, etc.) or educational videos, deciding when and how to use them to support the study process.
	Number of students, schedule, and classroom conditions – Optimize the number of students to provide personalized attention. – Adapt the classroom and the distribution of students to facilitate interactions. – Provide a schedule of class sessions that promotes student attention and commitment.
	Time (collective teaching/tutoring; learning time) – Appropriate time (face-to-face and non-face-to-face) for the intended teaching. – Adequate time should be assigned to the most important contents and those that are difficult to understand.

In recent decades, there has been a broad consensus in mathematics education on the use of manipulative materials and virtual resources to support teaching and learning, as they "concretize and visualize" mathematical concepts. "Technology is essential in the learning and teaching of mathematics. It can positively influence what is taught and, in turn, increase student learning" (NCTM, 2000, p. 24). This professional organization also considers calculators and other technological tools, such as algebraic calculus systems, dynamic geometry software, applets, spreadsheets, and interactive presentation devices, to be vital components of high-quality mathematics education.

Other studies (e.g. McNeil and Jarvin, 2007; Uttal et al., 1997) take a more critical approach to manipulative use. Uttal et al. (1997) asserted that a sharp distinction between concrete and symbolic forms of mathematical expression is not useful. There is no guarantee that students will make the connections between manipulative and more traditional mathematical expressions because the manipulative should represent something different, i.e. it is also a symbol.

> A concrete manipulative may be interesting to children, but it alone is not enough to advance their knowledge of mathematics or concepts. To learn mathematics from manipulatives, children must perceive and understand the relationships between manipulatives and other forms of mathematical expression. (Uttal et al., 1997, p. 38)

In OSA, as we have already mentioned, the relationships between material representations and visualizations (ostensive objects) of mathematical concepts, propositions, and procedures are complex because these have a regulative nature (non-ostensive objects) and should not be confused with their representations (Godino et al., 2007; Font et al., 2013). For example, the rational number “one-third” can be referred to in mathematical practice by the symbolic expression 1/3. It can also be represented by a pie chart in which the unit disk is divided into three equal parts, and one portion, which is one-third of the unit, is set aside. However, any fraction equal to 1/3 also represents the rational one-third. Progress in mathematical understanding therefore requires distinguishing the mathematical object from its ostensive representations (whether visual or manipulative), which materialize the mathematical object in iconic or indexical ways.

We must also recognize the different efficiency of symbolic representations compared to iconic and indexical representations for calculation, generalization, and demonstration processes. Mathematical activity is usually performed with the support of means of expression and calculation whose nature may be tangible or manipulative (abacus, geo-plane, ...), visual-diagrammatic (Cartesian graphs, probabilistic simulators, ...), or alphanumeric symbolic. Any of these means of expression is dialectically related to non-ostensive mathematical objects that regulate the development of operative and discursive mathematical practices and provide answers to problem situations.

From these ontosemiotic considerations, a specific criterion for the suitability of using material resources in mathematical instruction is postulated:

> Mathematical objects (regulative, non-ostensive) should be distinguished from their respective concrete, visual, or symbolic representations in mathematical and didactic practice.

We must remember the dialectic between the configurations of objects and processes based on the use of manipulative-visual resources and the analytical configurations grounded on symbolic representations. Synergistic relations are established between these two types of configurations, which are intertwined in mathematical practices. Configurations based on concrete and visual representations play a key role not only in school mathematics but also in the generation of conjectures, induction, and explanation, while analytical configurations are essential in generalization, calculation, and justification processes. This leads to another specific criterion of mediational suitability:

> Using configurations of objects and processes based on alphanumeric representations should be articulated with those based on concrete representations to progressively enhance the processes of generalization, calculation, and mathematical proof.

Bartolini and Martignone (2020) distinguished concrete from virtual manipulation. The former are physical artifacts that students can manipulate to offer tangible experiences in school mathematics activities, while the latter are digitally manipulated to offer visual experiences. However, alphanumeric symbols, which are part of the language category of the epistemic facet, are also “manipulated”, they are objects of processing and conversion between different registers (Duval, 2006). The articulation of the use of these means of symbolic expression with material resources, as pointed out by Uttal et al. (1997), leads us to consider the usefulness of distinguishing three sub-components in the category of material resources: concrete, virtual, and symbolic manipulative. For the three types of resources, there is a wide variety of devices or artifacts depending on the mathematical content to be addressed: arithmetic (abacus, rulers, fraction wall, ...), geometry (geoboard, GeoGebra, ...), statistics (simulators, graphics, ...), and algebra (algebraic balance, ...). These devices are used for Level IV analysis of the material resources component of the mediational facet.

5.3.4. Interactional facet

Although there is a debate between knowledge-transmitting and knowledge-constructing models (as we show in Chapter 4), the outcome is currently weighted in favor of the latter.

> The constructivist learning framework provides a foundation for mathematics reforms in K-12. Many prospective teachers across the United States are being trained to believe that this is how students learn best. (Andrew, 2007, p. 157)

This preference for learner-centered learning models is evident in the curricular orientations of various countries, which adopt constructivist or socio-constructivist theoretical frameworks, as observed in the NCTM:

> Students learn more and better when they take control of their learning by defining goals and monitoring their progress. When challenged with appropriately chosen tasks, students gain confidence in their ability to tackle difficult problems, desire to work things out for themselves, show flexibility in exploring mathematical ideas and trying alternative solution paths, and a willingness to persevere. (NCTM, 2000, p. 20)

Likewise, educational research attributes much importance to discourse, dialog and conversation in the classroom:

> The nature of mathematical discourse is central to classroom practice. If we seriously accept that teachers need opportunities to learn from their practice, the development of mathematical conversations enables teachers to continuously learn from their students. Mathematical conversations that focus on students' ideas can provide teachers with a window on students'

> thinking in ways that individual student work does not. (Franke et al., 2007, p. 237)

These trends justify the TDS-OSA following general criterion of interactional suitability:

> Interaction patterns should allow identification of potential semiotic conflicts, putting in place adequate means for their resolution, favoring progressive autonomy in learning and the development of students' communicative competences.

Table 5.4 includes the suitability criteria linked to the interactions between the teacher and the students and among the students themselves. Considering socioconstructivist learning principles, the presence of moments in which students take responsibility for learning is positively valued. However, when becoming aware of the ontosemiotic complexity of mathematical knowledge, in TDS-OSA, this constructivist principle is qualified in the sense marked by the following specific interactional criterion (Godino et al., 2020):

> The modes of teacher-student interaction should be adapted by considering the moments of the study process, applying a dialogic-collaborative format in the moments of the first encounter with the content, and attributing autonomy to the student in the moments of exercise and application.

Table 5.4. Suitability criteria for the interactional facet and its components

General criterion for the interactional facet	*Component-specific criteria*
Interaction patterns should help identify potential semiotic conflicts, provide adequate means for their resolution, favor progressive autonomy in learning, and develop students'	*Teacher-students' interactions* – Adapt the interaction modes considering the moments of the study process, applying a dialogic-collaborative format in the first encounter with the content and attributing autonomy to the student in exercise and application. – Make an adequate presentation of the topic (clear and well-organized presentation, not speaking too fast, emphasizing the key concepts of the topic, etc.). – Recognize and resolving student conflicts (appropriate questions and answers are asked, etc.). – Seek consensus based on the best argument.

communicative competences.	– Use rhetorical and argumentative devices to engage and capture the students' attention. – Facilitate the inclusion of students in the dynamics of the class. – Encourage participation and active engagement of all students.
	Interactions among students – Encourage dialog and communication among students. – Enhance group inclusion and avoiding exclusion.
	Autonomy – Provide times when students take responsibility for the study (pose questions and present solutions; explore examples and counterexamples to investigate and conjecture; use a variety of tools to reason, make connections, solve problems, and communicate).
	Formative assessment – Systematically observe students' cognitive progress and use the information obtained to make decisions about instruction development.

Accepting the autonomy principle in learning is an essential feature of Brousseau's (2002) Theory of Didactic Situations, in which situations of action, communication, and validation are conceived as *adidactic* moments of study processes, i.e. situations in which learners are protagonists in the construction of the intended knowledge. Likewise, Realistic Mathematics Education (RME) assumes a principle of interaction, according to which mathematics teaching is considered a social activity. Interactions between students and between students and teachers can lead each student to reflect on the input of others and thus reach higher levels of understanding. Rather than recipients of ready-made mathematics, students are active participants in the teaching-learning process, where they develop tools and understanding and share their experiences with others. Explicit negotiation, intervention, discussion, cooperation and evaluation are essential in constructive learning in which the learner's informal approaches are a platform for achieving formal methods. In this interactive instruction, learners are encouraged to explain, justify, agree, and disagree, question alternatives, and reflect (Van den Heuvel-Panhuizen & Wijers, 2005, p. 290).

A fundamental principle of Freudenthal (1991) is that students should be given a "guided opportunity" to "reinvent" mathematics. In RME, this implies that both teachers and educational programs play a fundamental role in how students acquire knowledge. They direct the learning process, but not in a fixed way, by demonstrating what students must learn. This contradicts the activity principle (Van den Heuvel-Panhuizen & Wijers, 2005) and leads to false perceptions. Students require space and tools to construct their mathematical knowledge. To achieve this goal, teachers must provide students with a learning environment in which the construction process can emerge.

Deciding on the progression of the study, both by teachers and students, requires the implementation of observation and survey procedures for a formative evaluation of learning. Deciding on the progression of the study, both by teachers and students, requires the implementation of observation and survey procedures for a formative evaluation of learning.

5.3.5. Cognitive facet

OSA assumes that learning involves students' appropriation of the planned institutional meanings, which presupposes their recognition of and interrelation with the objects involved in the mathematical practices that determine the meanings. Progressive coupling between students' initial personal meanings and planned or effectively implemented institutional meanings is achieved through their participation in the community of practice generated in the classroom. In Table 5.5A, we include the general criterion of cognitive suitability and specific criteria for its components.

Table 5.5A. Suitability criteria for the cognitive facet and its components

General criterion of the cognitive facet	*Component-specific criteria*
Learning objectives should	*Personal meanings* – Promote understanding of problem situations, representations, concepts, and properties.

present students with both personal and contextual challenges. Students' personal meanings should be consistent with the planned institutional meanings. The assessment of learning should serve to improve the instructional process.	– Develop communicative, procedural, and argumentative competence.
	Processes – Promote the development of students' competence to implement content-specific mathematical processes (modeling, generalization, problem posing and solving, proof, representation, ...) and metacognitive processes (reflection on one's own mathematical thought processes).
	Relations (Connections) – Promote relational learning so that students can understand and relate to different meanings in the teaching process and the objects involved.
	Previous knowledge – Consider the previous knowledge that students must address when studying the intended content.
	Individual differences – Support students' learning by considering individual differences in prior knowledge, learning styles, and levels of understanding and competence.
	Learning assessment – Regular checking of learning progress to facilitate improvement (formative assessment).

Cognitive suitability is attributed to the instructional process as a gradable trait linked to achievement of learning objectives that demand attainable effort. This trait is consistent with rich mathematics and can be adapted to personal and contextual circumstances. The general criterion of cognitive suitability is inspired by the concept of the zone of proximal development (Vygotsky, 1934), in which learning objectives should involve the development of valuable mathematical knowledge and skills requiring attainable effort with the support of the teacher and peers, considering individual prior knowledge and abilities, and the principle of equity (NCTM, 2000). This model assumes relational learning and an understanding of institutional meanings. The assessment of the learning achieved should account for the students' personal characteristics and the different levels of understanding and competence they can attain. Table 5.5B describes specific criteria for the Level III sub-components of the cognitive facet.

Table 5.5B. Suitability criteria for the Level III subcomponents of the cognitive facet

Subcomponents	*Specific criteria*
Situational understanding	– Promote and evaluate the correct resolution of problem situations and learning tasks that pose an achievable challenge for students.
Communicative competence	– Promote and assess communicative competence in different modes of correct mathematical expression.
Conceptual and propositional understanding; Procedural competence	– Promote and assess conceptual and propositional understanding. – Promote and assess correct procedural competence.
Argumentative competence	– Promote and evaluate argumentative competence.

Three of the six mathematics education principles described by NCTM (2000) are related to cognitive suitability. The principle of equity states, "Excellence in mathematics education requires equity, high expectations and strong support for all students" (p. 16). It requires reasonable and appropriate accommodations and inclusion of motivating content to promote access and achievement for all students. The learning principle assumes that "Students should learn mathematics by understanding it, actively constructing new knowledge from their prior knowledge and experiences" (p. 16). Likewise, the assessment principle states that, "Assessment should support relevant mathematics learning and provide useful information to both teachers and students" (p. 16).

5.3.6. Affective facet

The resolution of any mathematical problem involves an affective situation for the subject that brings into play not only his/her knowledge to solve it but also mobilizes emotions, attitudes, beliefs, and values that condition his/her response. Affective processes are psychological entities that describe varying degrees of mental states, traits, or dispositions toward a subject's actions. However, from the didactic point of view, the achievement of affective states that interact positively with the cognitive domain must be considered by educational authorities and teachers (Gómez-Chacón, 2000), whose work is conditioned by institutional norms of affective nature.

The affective suitability of the process is based on students' degree of involvement, interest, motivation, self-esteem, and willingness. Beliefs in mathematics and its study also influence learning and therefore need to be acknowledged. In Table 5.6, we describe the general criterion of affective suitability and specific criteria for the different components of this facet, which are not unique to mathematics instruction (i.e., they have a general character). These criteria are consistent with principles assumed by various studies on the interactions between cognitive and affective domains in mathematical learning (Beltrán-Pellicer & Godino, 2020; Gómez-Chacón, 2000; McLeod, 1992).

Table 5.6. Suitability criteria for the affective facet and its components

General criterion for the affective facet	*Component-specific criteria*
The instructional process should achieve the highest possible degree of student involvement (interest, motivation, self-esteem) and should consider students' beliefs about mathematics and its learning.	*Emotions* – Design situations to identify and discuss emotions to avoid rejection, phobia, or fear of mathematics. – Highligh the esthetic and precision qualities of mathematics.
	Attitudes – Promote that students assume responsibility for learning and attempt to complete tasks with perseverance, both of which require personal inquiry as well as the reception and retention of knowledge. – Favor argumentation in situations of equality; the argument is valued in itself and not by the person who voices it.
	Beliefs – Identify students' beliefs about mathematics and its teaching that may condition learning and consider them in the instructional process.
	Values-identity – Promote self-esteem so that students feel capable of contributing conjectures and solutions to the problems they face, relying on mathematical arguments to

	convince others of the validity of their assertions, thus building a positive mathematical identity.
	Interests and needs – Propose tasks that are of interest to the students and are within their reach. – Propose situations that allow the assessment of the usefulness of mathematics in daily and professional life.

5.3.7. Interactions between facets

In the preceding sections, we have described the suitability criteria for the six facets involved in an instructional process. As shown in Figure 4.1 (Chapter 4), these facets are not independent; in fact, there are interactions between them. For instance, the use of a technological resource can help to tackle certain types of problems and the corresponding configurations of objects and processes, leading to new forms of representation, argumentation, and generalization. The forms of interaction between teachers and students, interest, motivation, and ultimately learning can also be affected.

Godino (2013, p. 127) includes some suitability criteria related to interactions between facets, which are described in terms of indicators. For example, an indicator of suitability in the interaction between epistemic and ecological facets states that “The curriculum proposes the study of problems in various fields such as schools, everyday life and work”. This indicator can be described using the following criteria: “The curriculum should propose the study of problems ....”, which implies a value attributable to a varying extent to the instructional process: that the curriculum proposes the study of problems in a variety of fields is valued as a positive trait. The same approach can be adopted with the remaining indicators of interactions between facets.

Table 5.7. Components and indicators of the interaction between suitability facets

Components	*Indicators*
Epistemic-ecological	- The curriculum proposes the study of problems in a varied domains such as schools, everyday life, and work.

Epistemic-cognitive-affective	- The content of the study (phenomena explored in the different areas, developing and justifying conjectures) makes sense for students with different levels and grades. - Students are confident in their abilities to tackle difficult problems and maintain perseverance when tasks are complex. - Students are encouraged to reflect on their reasoning during problem-solving processes to apply and adapt strategies developed in other problems and contexts. - Tasks selected by teachers for assessment are representative of the learning intended.
Epistemic-cognitive mediational	- The use of technological resources induces positive changes in teaching content, modes of interaction, motivation, and student learning.
Cognitive-affective-interactional	- Explanations given by students include mathematical and rational arguments, not just descriptions of procedures. - Include motivating content with reasonable and appropriate adaptations that promote access and achievement for all learners.
Ecological-instructional (teachers' role and education)	- The teacher is caring and dedicated to his or her students. - The teacher has a deep knowledge and understanding of the mathematics that he/she teaches and uses this knowledge flexibly in his/her teaching tasks. - The teacher has ample opportunities and support to increase and frequently update his/her didactical-mathematical knowledge.

Inclusive mathematics education (Gervasoni & Peter-Koop, 2020; Ross, 2019) requires accounting for the interactions between various facets and components. Interactions among cognitive and affective facets (individual differences in students' knowledge, skills, attitudes, beliefs), ecological facet (values education, socio-professional development), and mediational and interactional facets (use of diverse resources, collaborative work). It also requires bearing in mind the epistemic facet to select diverse situations and representations that are mathematically relevant and allow students of different motivations and abilities to engage in meaningful learning of mathematics.

The NCTM (2000) calls for attention to the connections between cognitive-affective and instructional issues: "Effective mathematics teaching requires knowing and understanding what students know and need to learn about mathematics; and then motivating and supporting them to learn it well" (p. 17). The adoption of the interaction principle in RME implies that teaching the whole class plays an important role. This does not mean that the whole class is carried together and that every student follows the same path and achieves the same developmental level at the same time. In contrast, in RME, children are considered individuals, each following an individual learning path. This view of learning often leads to advocating the division of classes into small groups of students who each follow their own learning. However, in RME, there is a strong preference for keeping the class as an organizational unit and adapting education to students' different abilities. This can be achieved by providing students with problems that can be solved according to different levels of understanding.

The use of models in RME relates to mediational, epistemic (representational, phenomenological), cognitive, and instructional issues. It is argued that models serve as a key tool to bridge the gap between informal, context-related, and formal mathematics. First, students develop strategies that are closely related to their context. Later, some aspects of the context situation can be generalized, meaning that the context becomes a model that can support solutions to other related problems. Finally, the models allow students to gain more formal mathematical knowledge. To fulfill the bridging function between formal and informal levels, models must move from a "model of" a particular situation to a "model for" all equivalent situations (Van den Heuvel-Panhuizen & Wijers, 2005, p. 289).

The RME reality principle links epistemic and cognitive aspects. The overall aim is to ensure that students can use their mathematical knowledge and tools to solve real problems. This principle is not only recognizable at the end of the learning process; in mathematics applications, reality is also considered a source of learning mathematics. A real context refers to both problematic situations in everyday life and real situations for learners. Just as mathematics emerged from the mathematization of reality, learning should also have originated from this reality. Instead of starting with certain abstractions or definitions to be applied later, one should start with rich contexts that require mathematical organization or contexts that can be mathematized (Freudenthal, 1968).

The time devoted to teaching and learning and its management by the teacher and students determine the components of the didactic suitability of a study process. This factor has been included as another resource in the mediational facet, together with technological resources. However, time also interacts with various other aspects. Table 5.8 includes indicators of time suitability in relation to epistemic, cognitive, instructional, and ecological facets.

Table 5.8: Temporal suitability components and indicators

Components	*Indicators*
Temporal-epistemic	- The content and its various meanings are distributed rationally over the allotted study time.
Temporal-cognitive	- Learning objectives consider the developmental stages of learners' evolutionary development.
Temporal-instructional	- Instructional time management considers the various moments required to develop different types of learning (exploration, formulation, communication, validation, institutionalization, exercise, evaluation).
Temporal-ecological	- The time assigned to the study process in the curriculum design is adequate to allow learning of the programed content.

The NCTM (2000) curriculum principle relates the epistemic facet (inclusion of relevant mathematics and set of activities), connection, and articulation between the different levels: "A curriculum is more than a set of activities. It must be coherent, focused on relevant mathematics, and well-articulated across the different levels" (p. 15). Also, the RME includes a principle related to learning levels. Learning mathematics means students move through different understanding levels: from the ability to invent informal solutions related to the context, to creating different levels of shortcuts and schematizations, to acquiring a knowledge of underlying principles and discerning broader relationships. The condition for reaching the next level is the ability to reflect on the activities performed. An interaction can prompt this reflection.

5.4. Suitability criteria for specific contents

The set of tables 5.1 to 5.7 (Section 5.3) constitute the GADS-MIP guide for analyzing and assessing the suitability of educational-instructional processes for any mathematics content. Its application in particular experiences involves specific mathematical topics for which multiple research results exist. These findings provide knowledge that can be interpreted as specific suitability criteria for teaching and learning the content under investigation. To a certain extent, having a system of suitability criteria for the different facets and components (Tables 5.1 to 5.7) does not avoid the effort of carefully reviewing the mathematics education literature, to complete generic suitability criteria with others specific to each content. Castillo et al. (2022) developed an example of suitability criteria for proportionality. Although their study was focused on producing a guide for the analysis of textbook lessons, which can be interpreted as study processes planned by the authors, the criteria systems can serve to analyze other processes. Beltrán-Pellicer et al. (2018) developed specific indicators of didactic suitability for probability. They also describe their use by a teacher as a tool for reflecting on the experience of teaching probability in secondary education.

5.5. Application example of didactic suitability theory. Reflecting on an experience of teaching proportionality[17]

In this section, we present an example of the use of didactic suitability, as described by Aroza et al. (2016). This is a teaching experience conducted during the practical phase of a master's degree in initial secondary mathematics teacher training.

5.5.1. Describing the teaching experience

The didactic unit was taught to a group of first-year secondary school pupils (12-13-year-olds). It was a group of 30 pupils, comprising 8 girls and 22 boys, with a high proportion of immigrant students (40%, 9 different nationalities of origin), which implied a certain heterogeneity and cultural diversity. It was difficult to maintain a suitable atmosphere, and it was necessary to draw their attention often. More than half of the students had negative attitudes toward study and mathematics, partly because of the curricular mismatch that 17 of them had carried over from primary school, with many difficulties in understanding basic mathematical and procedural concepts. The remaining students exhibited motivation and interest in the subject.

The study process followed the orientation and content proposed in the textbook (Colera & Gaztelu, 2010) used at the school. Following the textbook, the didactic unit was taught in 11 sessions, where, besides explanations (theoretical notions), tasks were solved. The last session was reserved for an evaluation test to assess the students' level of understanding and learning in solving the different types of proportionality and percentage tasks. The textbook authors emphasized a view of mathematics as rules or algorithms

[17] The content of this section is based on the paper, Aroza, C. J., Godino, J. D. y Beltrán-Pellicer, P. (2016). Iniciación a la innovación e investigación educativa mediante el análisis de la idoneidad didáctica de una experiencia de enseñanza sobre proporcionalidad. *Aires*, *6*(1), 1-29.

to be followed, illustrated with examples of how to interpret such rules, followed by procedural exercises for mastering their application.

Sessions usually began by correcting the homework that the students were given at home and providing a reminder of what had been studied in the previous session. In addition, during the explanation, questions were asked about the content that the students had already acquired so that they could maintain their level of attention and follow the explanation. To develop the theoretical explanations, some task examples contextualized in real-life situations were then performed, and special attention was paid to the errors and difficulties that could arise. It was necessary to emphasize the key procedures and notations used, trying to justify them as much as possible. In this task-solving phase, students were encouraged to participate in class through frequent questions and trips to the blackboard to maintain their attention levels. The order and contents of the textbook were always followed, using it as a script so that students could easily access the subject matter, although sometimes examples and tasks that were not in the textbook were provided.

Regarding their way of working in class, students individually performed some tasks related to the explanation given earlier. However, the students could discuss their doubts with their classmates, while the teacher tried to resolve other doubts for the remaining students in a personalized way. Later on, or in subsequent sessions, the teacher usually corrected the tasks on the blackboard. Sometimes, the students themselves indicated the necessary steps, and the teacher wrote them on the blackboard; on sporadic occasions, the students went to the blackboard to solve the tasks. During the task corrections, the aim was to place great emphasis on the errors made so that students would not commit them in future situations.

At the end of the didactic unit, the students were examined using a written assessment test to determine whether they had learned the content and achieved the proposed objectives. The test, which was passed by only 57% of the students, comprised 10 tasks. Aroza et al. (2016) described the types of tasks and explanations in textbooks, the applied learning assessment test, and the main errors made by students.

5.5.2. Didactic-mathematical knowledge of proportionality and percentages

The aim of the training process described by Aroza et al. (2016) was to assist prospective teachers in carrying out an analysis of the didactic suitability of the teaching process implemented and to identify proposals for informed change. The didactic suitability criteria proposed in Godino (2013) were used, complemented with criteria derived from a compilation and synthesis of the main research and innovations related to the teaching and learning of proportionality and percentages. With the help of the teacher educator, the didactic-mathematical knowledge of proportionality was identified, synthesized, and classified according to epistemic, ecological, mediational, interactional, cognitive, and affective facets.

5.5.3. Assessment of didactic suitability and proposals for change

The application of didactic suitability leads to the following questions:

1) What is the degree of didactic suitability of the teaching-learning process in relation to proportionality and percentages, as experienced by the participants during the teaching practice period in the first year of secondary education?
2) What changes could be made in the design and implementation of the study process to increase its didactic suitability?

Below, we summarize the analysis and assessment of the report described by Aroza et al. (2016).

1) *Epistemic and ecological facets*

The study process followed the contents and orientations proposed in the textbook (Colera & Gaztelu, 2010), which was used for the 1st year of secondary school at the institute where teaching practices were conducted. In this textbook, throughout the teaching unit, only an "arithmetic" approach to proportionality is proposed, with no geometric or algebraic development, even preceded by some activities of an intuitive and qualitative nature. The proportionality concept, from this arithmetical approach, is basically reduced to the transmission of an algorithm (the rule of three), which needs to be known how to apply and operate in each case. To develop this method, the textbook reduced the concept of "proportion" to a new name for two equivalent fractions and that of "ratio" to a new name for the fraction, with no other treatment or task to help the learner develop proportional reasoning through reflection. From the perspective of structuring content, this algorithm should not have been introduced until the learner had mastery of other, more intuitive testing and solving methods. Its content is basically procedural, making the "rule of three" the only method of solving proportionality problems, focusing the pupil on a purely mechanical approach, devoid of concepts, reasoning, and reflection on whether the problems are proportionality problems.

However, due to the contexts in which the concepts are dealt with and particularly the tasks proposed throughout the development of the didactic unit, the study of some other intra-disciplinary content such as rational numbers, the equivalence of fractions, decimal numbers, and the decimal metric system was enriched; other interdisciplinary content such as physics, chemistry, and economics served to recognize and apply properties of proportionality and percentages. All this contributed to the students' socio-cultural and vocational training, with the part of the book dealing with percentages standing out in this respect.

Regarding the series of tasks in the textbook and proposed to the students, they provide a representative sample for exercising and applying the intended content, although activities in which the students have to formulate their own proportionality and percentage problems as recommended by the indicators of epistemic suitability are omitted.

The mathematical language used was appropriate for the first level of secondary education, although, both in the tasks and in the conceptual developments, there was a poor typology of mathematical expressions and representations, with only symbolic and numerical language used as tables of values. The reason for this is the lack of geometrical and algebraic approaches to proportionality, which mainly use graphic language (linear function) and manipulative language (construction of similar figures) in their development.

From an ecological perspective, proportionality is introduced in the book through examples, which lead to defining when there is direct or inverse proportionality between two magnitudes, although fundamental conceptual definitions such as “ratio”, “proportion” and “constant of proportionality”, which are not mentioned in some sections of the book and are included as key concepts in the curricular orientations for this level of education, are missing.

Because of the above reasons, the epistemic and ecological suitability of the teaching process was low.

2) Cognitive and affective facets

One major error and difficulty detected in some pupils in the two assessment instruments applied was their inability to distinguish between direct and inverse proportionality or between proportional and non-proportional situations. The textbook devoted very little content to this issue, and it would have been desirable to devote more time and even more emphasis to it, giving greater importance to proportionality from a qualitative approach and then moving on to the quantitative aspect.

Regarding the temporal sequencing of the curricular content, it was sufficiently justified (supported by what was previously stated in the section on the epistemic facet) to delay the study of proportionality and percentages in the school teaching program, placing teaching units related to the similarity of figures and linear functions before it to support the different treatments required by the subject.

From the arithmetic approach to proportionality proposed in the textbook, the prior knowledge required to study proportionality and percentages are fractions and their equivalence, solving basic arithmetic problems, operations with decimal numbers, and relations between fractions and decimal numbers. All the content was taught in previous units according to the school's didactic program. However, the results obtained in the two evaluations performed (correction of class tasks and formative assessment) were clearly unsatisfactory, although proportionality and percentages were taught at an accessible level of difficulty in accordance with that of the first year of secondary school. Many students experienced serious difficulties and errors when operating with decimal numbers and fractions, and an initial session should have been devoted to reviewing this prior knowledge.

One positive aspect of the textbook is that, in the homework section, all the tasks were marked with a triangle code, according to their difficulty level, which simplified the curricular adaptation when proposing reinforcement tasks for some pupils and extension tasks for others. As a result, it was easier for all students in the class to achieve the intended learning of the didactic unit, starting from their own personal level of knowledge.

To assess the students' rate and level of learning of the content taught, two assessment instruments were used: the collection and correction of a representative set of tasks halfway through the unit and a formative assessment test at the end of the unit. These two instruments considered different levels of acquisition of the intended learning and, once corrected, were distributed among the students so that they could check and revise where they had made mistakes. In addition, with the evaluation instrument applied halfway through the didactic unit, the aim was to detect where the most common difficulties and errors had occurred to adapt and redirect the teaching, placing more emphasis on the key concepts and procedures involved.

The content and series of tasks proposed in the textbook were familiar with the context. This fact greatly enriched the teaching proposal, not only in the attentional and motivational aspects, since the students valued the usefulness of this part of mathematics in their lives, but also because it facilitated their understanding when receiving instructions on how to deal with problems.

The dynamic developed throughout the sessions aimed to systematize and encourage students to work consistently: pay attention to the teacher's explanations, start working on assigned tasks in class, and finish their homework at home. Performing the tasks in pairs and socializing their corrections throughout the class sessions, with frequent questions and sporadic trips to the blackboard, contributed to boosting pupils' self-esteem when facing proportionality problems. All the students always showed a very positive attitude toward this type of work strategy, encouraging their participation in the tasks.

Because of the above reasons, the cognitive-affective suitability of the implemented process was medium to low.

3) Interactional and mediational facets

The modes of classroom interaction in the teaching experience responded to a traditional model: the teacher first explained concepts and procedures, exemplifying them in everyday contexts to make them clearer and emphasizing the key contents, and then the students performed various tasks related to what had been taught. It would have been desirable to introduce some changes in the teaching process, aimed at students raising questions and presenting solutions; exploring examples and counter-examples to investigate and conjecture; and using a wider variety of tools to reason, argue, make connections, solve difficulties and communicate them.

There were a few moments when students were granted autonomy, except for individual work on homework assignments. When working on homework assignments in class, students could consult in pairs with their desk partners, which encouraged dialog, argumentation, and communication between them. The same was true of some students' sporadic trips to the blackboard during the task-solving phase or of questions addressed to the students during the teacher's explanation phase. These teaching practices not only helped to engage and capture students' attention and motivation but also facilitated their inclusion in the classroom dynamics.

In this teaching experience, only the resources available to the students in the classroom were used: a blackboard, projector, textbook, and calculator. No manipulative or other technological resources were used because they were not considered necessary to support the teaching and learning of the planned content.

Regarding the time component of instructional suitability, there was little use of the sessions devoted to developing the most important conceptual and procedural content of proportionality and percentages in the four approaches (intuitive, geometric, arithmetic, and algebraic).

The number of students (30) and their distribution were ideal, but the schedule for mathematics classes was not adequate. Of the five hours of lessons per week, three were held in the morning before break, but the remaining two hours were held at the end of the morning and on the last day of the week. This practice did not encourage adequate levels of student attention and motivation in class, making it difficult for students to manage their behaviors during their working hours. Improving the interactional and mediational suitability of the process would lead us to include activities and tasks that use manipulative materials and computer resources (Godino & Batanero, 2003), which can constitute novel and useful tools to achieve the intended learning. We conclude that applying the didactic suitability criteria helps systematize didactic knowledge and its application to the reflection and progressive improvement of teaching practice.

5.6. Concordances and complementarities with other theories[18]

In Godino (2021), we began comparing the didactic principles of the Theory of Didactic Situations in Mathematics (TDSM, Brousseau), the Anthropological Theory of Didactic (ATD, Chevallard), the Realistic Mathematics Education (RME, Freudenthal), and the suitability criteria based on OSA. The following is a synthesis of the results of this comparison. We consider it relevant to interpret some didactic principles of TSDM, ATD, and RME as suitability criteria by applying the facets and components of didactic suitability proposed by OSA. This makes it possible to identify some consistencies and complementarities between these theoretical frameworks. However, we recognize that the analysis conducted here is limited, given its complexity; its extension and deepening should be the subject of further work.

5.6.1. Epistemic facet

The four theories are consistent in attributing a central role to problem situations (questions, tasks) to achieve high epistemic suitability in instructional processes. Characterizing the fundamental situations of the different subjects included in the school mathematics curriculum is a priority aim of the TDSM. The notion of the ATD study and research pathway fixes the attention on searching for generative questions in mathematical praxeologies that constitute the purpose of an educational project.

[18] The content of this section is based on the paper, Godino, J. D. (2021). De la ingeniería a la idoneidad didáctica en educación matemática. *Revemop*, e202129, 1-26, 2021.

Likewise, the RME principles of activity and reality can be interpreted as indicators of epistemic suitability. The RME proposes as a heuristic for the design of situations that give meaning to mathematical objects (concepts, procedures, etc.) the *didactic phenomenology*, comprising the search in the history and epistemology of mathematics for types of phenomena of real life or internal to mathematics itself that are organized by such objects and considered by Freudhental as mental objects. “On the assumption that mathematics has arisen as a result of solving practical problems, we can assume that current applications encompass the phenomena that originally had to be organized”. (Gravemeijer, 2020, p. 226)

The main distinction between OSA and ATD is the proposed level of disaggregation for mathematical praxeologies. The notion of a system of practices (operative and discursive) linked to resolving a certain type of problem-situation in which the intervention of a certain mathematical object is decisive for its realization is central to OSA and can be assimilated to the notion of praxeology in ATD (Godino et al., 2006). However, while the structure of a praxeology is analyzed by distinguishing the quatern <task, technique, technology, theory> OSA considers a more explicit detail of the various objects and processes involved in mathematical activity. The notion of configuration of primary objects (problems, languages, concepts-definitions, procedures, propositions and arguments) and the processes of representation, definition, enunciation, argumentation, generalization, among others, allow for a level of analysis complementary to that of praxeology. Consequently, the application of the notion of configuration of objects and processes introduces explicit criteria of epistemic suitability, referring to linguistic elements (representations, their conversions and treatments) and to the respective processes of representation and communication (duly contemplated in the TDSM with the *adidactic* situations of formulation/communication).

The ATD notions of technology and theory in OSA are replaced and broken down by the notion of "configuration of objects and processes", which leads to the formulation of suitability criteria for the management of different types of objects (concepts, propositions, procedure). In ATD, the procedural component (work of the technique) is explicitly recognized as the key to the construction of knowledge, which remains diffuse in TDSM. TDSM, ATD, and TDS-OSA are consistent in attributing a central role to argumentative/validating objects and corresponding validation/justification processes (validation situations, technological-theoretical moment). Attention should also be paid to the connections between the different parts of the mathematical content and the articulation of the various partial meanings of the objects under study (Wilhelmi et al., 2007; Godino et al., 2011). Mathematics is an integrated field of study. This position is consistent with the "Principle of interconnectedness" of MRE. Blocks of mathematical content (numeracy and calculus, algebra, geometry, ...) cannot be treated as separate entities. Problem situations should include interrelated mathematical content. In addition, solving rich context problems often means that several mathematical tools and understandings are required to be applied (duly covered in TDSM with adidactic formulation/communication situations).

5.6.2. Cognitive facet

The cognitive dimension is accounted for in ATD through the notion of "personal relation to the object" and in TDSM with the distinction between knowledge and knowing. The emphasis on the institutional dimension of knowledge (ATD) and didactic situations (TDSM) has meant that the focus of didactic analysis is on mathematical knowledge (its organization and ecology) and the mathematics classroom as an institution or community. However, in Chevallard (2009), we found a reference to what can be described as the need to consider the subject's prior knowledge for the development of an activity or project:

> Given a project of activity in which a given institution or person intends to be involved, what is the praxeological equipment that is considered indispensable or simply useful in the conception and realization of that project? (Chevallard, 2009, p. 29)

In OSA, we postulate a dialectical relationship between institutional and personal so that alongside the configurations of objects and processes in the epistemic (institutional) sense, the corresponding cognitive configurations are introduced, with elements similar to those of epistemic configurations. Accordingly, cognitive suitability criteria related to learning are formulated. The various modes of assessment should indicate that learners achieve appropriation of the intended knowledge (including different levels of understanding and competence): conceptual and propositional understanding; communicative and argumentative competence; procedural understanding or competence; and metacognitive competence.

The principle of levels in RME is related to the cognitive facet proposed by TDS-OSA. It underlines that learning mathematics involves learners going through several levels of understanding: from informal solutions related to the context, through the creation of several levels of shortcuts and schemata, to the acquisition of knowledge about how concepts and strategies are related.

5.6.3. Affective facet

The TDSM notion of *devolution* can be interpreted as a component of the affective facet. The RME principles of activity and reality incorporate aspects related to the affective dimension of learning. It is recommended that learners actively engage in mathematics learning by practicing it themselves.

The students explicitly valued the use of realistic situations, and the informal solutions they developed in their efforts to find solutions to these situations were considered.

The four theoretical models considered should adopt or develop explicit models on components and suitability indicators related to the conglomerate of affective notions (interests, attitudes, emotions, beliefs), since they interact with the cognitive facet and condition learning. Beltrán-Pellicer and Godino (2020) developed a model of analysis of the affective domain in mathematics education from OSA.

5.6.4. Interactional facet

Both the TDSM (with the types of situations it proposes) and the ATD (six moments of the study process) provide criteria for the suitability of the modes of interaction between teacher and students. In the case of RME, the principle of interactivity recognizes that learning mathematics is not only an individual activity but also a social activity. Therefore, RME favors whole-class discussions and group work, which gives students the opportunity to share their strategies and inventions with one another. Likewise, the principle of guidance implies that teachers should play a proactive role in students' learning (Freudenthal's guided reinvention).

All four theories are consistent with the socio-constructivist assumptions of learning: an instructional process with high interactional suitability contemplates moments in which students take responsibility for the study (raise questions and present solutions; explore examples and counterexamples to investigate and conjecture; use a variety of tools to reason, make connections, solve problems and communicate them). In the case of TDS-OSA, it is considered that in the moments of institutionalization, the teacher should make an adequate presentation of the topic, recognize and resolve students' conflicts, favor consensus based on the best argument and use various rhetorical and argumentative resources to involve and capture students' attention. However, these moments of institutionalization can occur at any point in the instructional process: the moment of students' first encounter with new type of problem or content, or the moment students remember forgotten content (Godino & Burgos, 2020).

5.6.5. Mediational facet

The notion of *milieu*, which is understood as the "antagonistic" context or environment that a subject experiences to win the "game" of learning, is central to TDSM. It is a complex and rich notion that includes elements of diverse nature, prior knowledge, the teacher's actions, and the material means used to pose the problem and to explore possible solutions. In ATD, the milieu is not assumed to be given at the beginning with the didactic system (teacher, students, question), as in TDSM; the didactic system produces and organizes the milieu with which, dialectically, the answer to the question is generated. The mediational facet introduced in TDS-OSA considers only material or technological resources (artifacts) that can intervene in the intended mathematical practice and are therefore a component of the milieu of TDSM.

Using technological resources is not explicitly mentioned in the six RME principles; it is implicit in the reality principle and in the use of models in the level principle. From the RME perspective, Drijvers (2020) considered that the correspondence with the use of digital technology is not obvious. Guided reinvention may be challenged by the rigid nature of tools, and the phenomena that form the starting point of mathematics learning may change in a technology-rich classroom. In terms of didactic phenomenology, he concluded that phenomena could change in a technology-rich classroom. The digital environment itself can be a significant phenomenon for learners.

5.6.6. Ecological facet

ATD grants a central role to the identification of restrictions and conditioning (levels of co-determination) when implementing didactic organizations, as well as in the articulation of different mathematical praxeologies. It is proposed to avoid the study of specific and isolated praxeologies. These are the suitability components described by the OSA as part of the ecological facet, which is implicit in the TDSM. In the case of RME, the principles of reality and interweaving include aspects of the ecological facet, connection with real-life situations, and integration between different content blocks. The TDS-OSA proposes considering, besides the connections between different mathematical contents/topics/praxeologies, and the interdisciplinary connections, the following ecological components:

- Adaptation to the curriculum.
- Openness toward didactic innovation.
- Socioprofessional and cultural adaptation; and
- Education in values.

5.7. Synthesis of didactic suitability theory based on OSA

In Table 5.11, we include a synthesis of the elements that characterize the theory of didactic suitability based on OSA, responding to the questions proposed by Michie et al. (2014) as a description of a theory in the field of social and behavioral sciences.

Table 5.11. Synthesis of TDS-OSA

Elements	Description
Summary. What is the theory about and what are its main propositions?	Develops a system of criteria for the local optimization of the design, implementation, and evaluation of educational-instructional processes in mathematics, based on the assumptions and constructs of OSA. The suitability criteria are value judgements about the preferred didactic actions that should be performed in the different facets and components that structure educational processes (epistemic, ecological, mediational, interactional, cognitive and affective) to optimize mathematical learning, considering personal circumstances and educational context.

Scope/Objective. What phenomena does the theory explain?	The aim is to optimize educational-instructional processes in mathematics by helping to design and implement good mathematics, select resources, and devise didactic strategies to optimize learning. It is not an explanatory theory; rather, it develops a system of criteria (rules or principles) on preferred actions to optimize mathematics teaching and learning processes. The criteria were based on the assumptions of OSA mathematics. Agreements with similar criteria based on other theories of mathematics education were also identified.
Justification. Why is this theory necessary and how does it improve on previous theories?	When, in addition to a scientific component (descriptive, explanatory, predictive), mathematics education has a technological component (prescriptive), guidelines need to be developed to indicate the types of actions that should be implemented to improve educational processes. These criteria must be based on research results and should be the rational consequences of the assumed theoretical assumptions. The TDS-OSA provides an expanded vision of theoretical models for the quality of mathematics instruction by considering contextual circumstances when developing suitability criteria and considering them as weightable principles and not as general rules of action.
Hypotheses. What specific hypotheses does the proposed theory propose, and how do they differ from other theories?	Optimizing suitability requires weighting the criteria for the different facets and components according to the circumstances of the context (subjects, resources, educational purposes) while considering the interactions between the facets. The optimization is local, where the teacher weights the relative importance of the suitability criteria according to context and subject. Other theories on the quality of instruction attribute a more essentialist character to the standards and attend to or give priority to certain aspects of the educational process.
Constructs. What elements constitute the theory?	Didactic suitability is a gradable feature of educational-instructional processes. It uses the notions of personal and institutional meaning that are understood within the OSA framework, as well as the structural model of the facets and components of an educational-instructional process. The suitability criteria can be general, referring to each facet or partial and relating to components and sub-components. They are understood as principles that can be weighed according to context and not as general rules of action.
Relations. How are the elements of the theory related to each other?	The suitability criteria are developed considering the OSA constructs and postulates of mathematical activity and meaning. As a whole, they are structured using the facets, components, and subcomponents of an educational-instructional process.
Origin. On which theories is it based, and how?	This is based on the assumption of the scientific and technological nature of mathematics education, which leads to the need to seek criteria to optimize didactic activities. Although any educational theory, including each teacher, has its own

	criteria for suitability, the TDS-OSA is based on the postulates and constructs of the theory of mathematical activity and emergent objects (Chapter 2), the theory of meaning and mathematical cognition (Chapter 3), and the theory of educational design based on the OSA (Chapter 4).
Similarity. Which theories are most similar to this theory?	This theory is related to different theoretical models of mathematics instruction quality.
Complementarity. With which theories can it be complemented?	The TDS-OSA is open to the refinement of criteria and the incorporation of new criteria from theories of instructional quality and other theoretical models of mathematics education consistent with the assumptions of the OSA. The suitability criteria for the study of specific content must be supported by the results of didactic research.
Operationalization. How are the constructs measured or identified?	Didactic suitability is a gradable feature of educational-instructional processes; it can be high or low. The criteria are expressed as value judgments such as "this should be done." The system of suitability criteria for the different aspects and components provides a guide for systematic reflection on teaching practices. However, a system of rubrics with observable indicators that allows objective measurement of the degree of compliance with the criteria.
Uses. What can the theory be used for?	The TDS-OSA provides a guide for designing locally suitable (optimal) instructional processes in mathematics to achieve the planned educational purposes. It helps to become aware of the complexity of achieving a balanced balance between the different facets involved (epistemic, ecological, mediational, interactional, cognitive and affective). The guide can also be used to evaluate the design and implementation of instructional processes and to help identify aspects that can be improved. Therefore, it is a resource for mathematics teachers to reflect on their own practice.

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Chapter 6

Theory of teacher professional development based on the OSA

Introduction

In Chapter 5, the Theory of Didactic Suitability was developed based on the Onto-Semiotic Approach (TDS-OSA), providing an expanded view of theories regarding the quality of mathematics instruction. In this theory, we identify a system of criteria (standards, principles, or norms) that should be met by educational-instructional processes in mathematics to optimize school mathematics, teaching, and learning locally. Since the teacher is the main agent in the design, implementation, and evaluation of these processes, we must address the following questions: a) What knowledge and competencies should teachers possess to perform their teaching duties optimally? b) What characteristics should ideal programs for mathematics teacher training have? c) What knowledge and competencies should mathematics teacher educators possess to perform their training duties optimally?

In this chapter, we address these issues by developing a theory of professional development for mathematics teachers based on the assumptions of the OSA (Professional Development Theory based on the Onto-Semiotic Approach, PDT-OSA). This theory includes the following:

1) A system of knowledge and competencies that teachers should possess for the design, implementation, and evaluation of suitable educational-

instructional processes in mathematics (Teacher DMKC Model; Godino et al., 2017).

2) A system of criteria or standards for the characteristics of an ideal mathematics teacher training program that equips teachers to design, implement, and evaluate educational-instructional processes in mathematics with high didactic suitability.
3) A system of knowledge and competencies that mathematics teacher educators should possess to design, implement, and evaluate suitable training processes (Educator DMKC Model).

The Teacher DMKC Model is based on the TDS-OSA, meaning that the knowledge and competencies teachers should possess are dependent on the characteristics that educational-instructional processes in mathematics should possess. Conversely, the Educator DMKC Model depends on the characteristics of the mathematics didactic training process. It is necessary to start from the system of suitability criteria for educational-instructional processes in mathematics (Chapter 5) and interpret what knowledge is required. This involves clarifying what constitutes an effective mathematics instruction program and identifying what is needed to achieve it.

In the PDT-OSA, we address the characterization of mathematics teacher educators' work, thus extending both the TDS-OSA and the DMKC Model. To this end, we must identify the suitability criteria for training processes in mathematics didactics and infer the knowledge and competencies of teacher educators, which will include those related to mathematics instruction and the didactic-mathematical education of teachers.

The scope or objective of the PDT-OSA aligns, in the case of mathematics education, with the research problem that seeks to "build the foundations of a pedagogy of teacher education in the form of fundamental principles for teacher education programs and practices" (Korthagen et al., 2006, p. 1022). Thus, we articulate two lines of research in the field of mathematics teacher education: one that focuses on developing categories of knowledge and competencies and the other that identifies efficiency principles for teacher training programs.

In Section 6.1, the conceptualization of professional development and a synthesis of the background on systems of categories of mathematics teacher knowledge and efficient professional development programs are included. The structure of the facets and components of an educational-instructional process proposed in Chapter 4 (Figure 4.1) serves as the basis for structuring the system of didactic suitability criteria and the knowledge and competencies of teachers and educators. In Section 6.2, we expand this structuring by identifying the phases of foundation, planning, implementation, and evaluation, as well as various activities in the teacher training processes. In Section 6.3, we interpret the system of didactic suitability criteria for educational-instructional processes developed in Chapter 5 in terms of suitability criteria for teacher education processes, understanding that teachers must acquire training that enables them to design, implement, and evaluate mathematics instruction processes that optimize didactic suitability. This professional task implies that mathematics teachers should acquire the knowledge and competencies described in Section 6.4. The relationship we establish between didactic suitability criteria and didactic-mathematical knowledge, based on the facets, components, subcomponents, and elements of an educational-instructional process (Chapter 4), allows us to refine and expand the previously developed model of teacher knowledge (Godino, 2009; Pino-Fan & Godino, 2015). In Section 6.5, we extend the use of the suitability criteria, which were applied in Chapter 5 to mathematics instruction processes, to teacher education processes, which allows us to develop a system of knowledge and competencies for mathematics teacher educators (Section 6.6) and a guide for analyzing the suitability of training processes (Section 6.7). To demonstrate the use of the PDT-OSA, in Section 6.8, we describe an example of research on teacher education. Concordances and complementarities with other theories and models of professional development are included in Section 6.9. Finally, in Section 6.10, we address the questions posed by Michie et al. (2014) to summarize social and behavioral theories concerning the theory presented in this chapter.

6.1. Teacher professional development: Conceptualization and background

Research on mathematics teacher education has grown substantially over the last 20 years, as evidenced by articles published in the Journal of Mathematics Teacher Education, the Handbooks of Mathematics Teacher Education (Chapman, 2020; Wood, 2008), and the ICMI Study on the topic (Ball & Even, 2008). The knowledge required by teacher educators has received limited but increasing attention in recent years, driven by the interest of many countries in improving teacher quality and enhancing performance in international mathematics assessment tests (Beswick & Goos, 2018). Among the research topics in this field, Goos and Beswick (2021) highlighted the following:

– The nature of the expertise (knowledge, competencies, specialization) of mathematics teacher educators.
– Learning and development as a mathematics teacher educator.
– Methodological challenges in researching expertise, learning, and development of mathematics teacher educators.

Teacher Professional Development (TPD) is studied and presented in the relevant literature in various ways (Bautista & Ortega-Ruiz, 2015). However, central to such efforts is the consensus that professional development involves teachers acquiring relevant content, learning to learn, and applying their knowledge to benefit student learning.

> Teacher professional learning is a complex process that requires the cognitive and emotional involvement of teachers individually and

> collectively, the capacity and willingness to examine where each one stands in terms of convictions and beliefs, and the perusal and enactment of appropriate alternatives for improvement or change. (Avalos, 2011, p. 10)

In the TPD literature, training programs are considered effective when an explicit relationship between the program, the improvement of teaching practice, and student learning exist (Desimone & Pak, 2017). Effective professional development involves structured professional learning that changes teachers' practices and improves student learning (Darling-Hammond et al., 2017).

Documents on standards for mathematics teacher education, such as NCTM (2014) and AMTE (2017), have proposed systems of criteria and indicators regarding the specific knowledge, skills, and dispositions of a good mathematics teacher, as well as the characteristics of an effective teacher education program. Therefore, they reflect models of the mathematical and didactic knowledge that mathematics teachers should possess and the professional knowledge of teacher educators.

6.1.1. Mathematics teachers' and teacher educators' knowledge

Various publications have proposed principles and standards to achieve quality mathematics teaching (NCTM, 2000; 2014), as well as standards for developing effective mathematics teacher education programs (AMTE, 2017; Beisiegel et al., 2018; Desimone & Garet, 2015; Rasch et al., 2020). As a result of these studies, we found various systems of categories of mathematics teacher knowledge (Ball, Thames et al., 2008; Carrillo et al., 2018; Godino et al., 2017; Rowland et al., 2005) and mathematics teacher educator knowledge (Castro-Superfine et al., 2020; Escudero-Ávila et al., 2021; Leikin et al., 2018), as well as lists of principles or quality criteria for mathematics instruction and the effectiveness of teacher education programs (Bostic et al., 2021; Charalambous & Praetorius, 2018).

However, the theoretical foundation and rationale of such categorization systems are usually not explicit, confusing, or diverse, and very generic knowledge categories or domains have been proposed (Godino, 2009). "There is room for increasingly detailed research on the specific knowledge that Mathematics Teacher Educators (MTEs) employ in the various facets of their work" (Beswick & Goos, 2018, p. 425). It would be useful to move from models of teacher knowledge categories to systems of principles grounded in teaching practice. In mathematics education, the categories of mathematical and didactic knowledge and instructional models should be grounded in a prior model that explicates the nature of mathematical activity and the objects and relationships involved in it, i.e., an epistemological, ontological and semiotic reference model. The learning and teaching model, along with its principles and quality standards, should be coherent with and supported by this reference model.

6.1.2. Characteristics of effective professional development programs

Heck et al. (2019) noted a considerable body of literature, with some empirical support, that outline guiding principles for the design and implementation of effective professional development. Their work focuses on six elements commonly cited in the literature: (1) duration, (2) content focus, (3) coherence, (4) active/practice-based learning, (5) collective participation, and (6) expert facilitation.

AMTE (2017) proposed standards and indicators for mathematics teachers' knowledge, skills, and dispositions, as well as the characteristics of an effective professional development program. Park et al. (2018) defined professional development as any activity aimed at (a) developing teachers' knowledge, skills, and experience and, (b) preparing teachers to improve their pedagogical performance in current or future school roles.

The TRU (Teaching for Robust Understanding) framework (Schoenfeld, 2013; 2018) emphasizes experiences that determine student learning. This framework is a tool for designing and implementing professional development activities. It is rooted in principles of student-centered instruction and distinguishes five dimensions of instructional processes: mathematical content, cognitive demand, equitable access to content, agency, ownership and identity, and formative assessment. In this way, the types of instruction that make students knowledgeable, flexible, resourceful, and problem-solvers are characterized.

In their Learning Mathematics for Teaching project, Hill et al. (2008) developed an instrument to more effectively measure the quality of mathematics instruction, believing that a good measurement tool would enable teacher educators to improve teaching and learning. Hill et al. (2011) proposed a conceptual framework to specify and assess the mathematical characteristics of classroom work. This project introduced the MQI (Mathematical Quality of Instruction) construct, accompanied by a detailed coding guide for evaluating various criteria. By "mathematical quality of instruction", Hill et al. defined the mathematical content available to students during instruction (p. 30). The MQI framework includes six constructs and their corresponding scales: richness and development of mathematics, response to students, connection of classroom practice with mathematics, language, equity, and mathematical errors. This line of research derives the MKT (Mathematical Knowledge for Teaching) model (Ball, Thames et al., 2008) of categories of mathematics teacher knowledge.

Another line of research in mathematics teacher education promotes reflective practice (Schön, 1983; Tzur, 2001), whether as future teachers, practicing teachers, or teacher educators. The goal is to train reflective professionals (Llinares & Krainer, 2006) as a strategy to improve mathematics teaching and learning. This reflection on different aspects and moments of practice can be guided (Nolan, 2008) not only by educators in the case of prospective teachers but also through conceptual tools that draw attention to critical aspects of practice. These guides provide a structure for holistic (Klein, 2008), articulated (Ash & Clayton, 2004), guided (Husu et al., 2008), and critical (Harrison et al., 2005) reflection.

Reflection and research on practice have also been proposed for teacher educators, involving learning about certain content or pedagogical aspects, such as discourse and problem-solving (Chapman, 2009). Developing a professional vision of mathematical learning and teaching experience is proposed through "professional teacher noticing" (Dindyal et al., 2021; Mason, 2002), a line of research that has gained considerable attention (König et al., 2022; Schack et al., 2017). Various theoretical lenses and methodological strategies have been developed to promote this skill (Fernández & Choy, 2019), such as framing theory (Scheiner, 2023), hypothetical learning trajectories (Simon, 1995; Simon & Tzur, 2004), and professional discussions based on Lesson Study (Lee & Choy, 2017).

6.2. Structure of teacher education processes

In the OSA, the educational-instructional process is the unit of analysis in mathematics education activity[19], and it is understood as the articulation of two partial activities: teaching and learning of mathematical content, dispositions, and skills. In teacher education, the unit of analysis comprises formative processes related to the content and skills of mathematics didactics, where teaching and learning converge. For a detailed analysis of educational-instructional processes, we distinguish the phases of

[19] We understand the notion of activity in the sense proposed by the Cultural Historical Theory of Activity (CHAT) in its second and third generation version (Engënstrom, 1987). Activity is the unit of analysis whose structure is given by six elements: subject, object, instruments, community, rules, and division of labor. We consider that the structure of an educational-instructional process distinguishing phases, facets, and components allows for more detailed and explanatory analyses than those provided by the CHAT.

Foundation, Planning, Implementation, and Evaluation (Figure 6.1), which are partial activities of the overall mathematics education activity and phases in their temporal development. In each phase, it is necessary to distinguish six facets: epistemic, ecological, mediational, interactional, cognitive, and affective (Chapter 4).

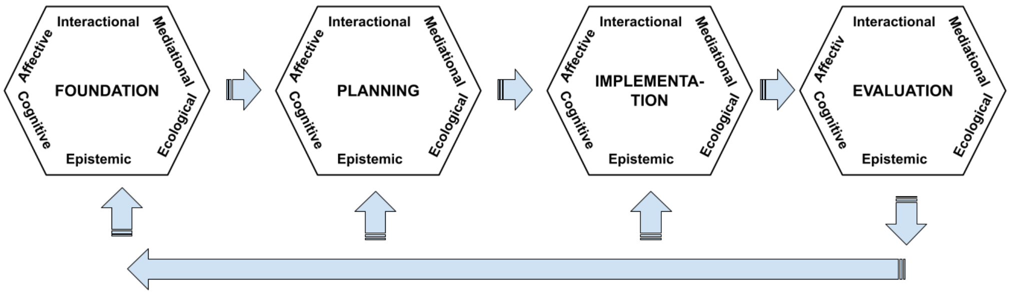

Figure 6.1. Phases and facets of educational-instructional processes

Figure 4.1 (Chapter 4) describes the components, subcomponents, and elements proposed by the OSA for analyzing these six facets. The same structure applies to teacher education processes. In this case, the knowledge, skills, and dispositions involved in mathematical instruction processes should form part of the epistemic facet (institutional knowledge) of the training process. The teacher educator must also consider the knowledge, skills, and dispositions of the other facets of the training process, which aim at teaching and learning of didactic-mathematical content.

The components and subcomponents of the epistemic and cognitive facets included in Figure 4.1 (Chapter 4) are derived from the onto-semiotic configuration in its epistemic version (institutional meanings) and cognitive version (personal meanings), and from the types of mathematical objects and processes involved. Level IV of analysis (Elements) describes the different blocks of mathematical content (arithmetic, geometry, algebra, statistics, etc.) on which research in mathematics education has produced knowledge and resources that should be considered by teachers and educators.

If the instructional process concerns mathematics, the structure presented in Figure 4.1 (Chapter 4) applies to the knowledge the teacher employs about the mathematics to be taught and other involved facets. Therefore, a model that develops other related models, such as the MKT (Ball, Thames et al., 2008) and DMK (Pino-Fan & Godino, 2015; Godino et al., 2017), is proposed.

The focus of this chapter is the teacher education process in mathematics, which involves training activities and teacher learning (Figure 6.2). The optimal development is closely related to mathematics instructional processes. In Figure 6.2, Activity 1 (teacher education) involves the educator designing programs and training actions for TPD. Activity 2 (teacher learning) involves the teacher learning to teach mathematics. Activity 3 (teaching mathematics) involves the teacher conducting instructional processes in mathematical content, thus involving Activity 4 of mathematical learning, aimed at achieving students' mathematical understanding and competence.

As shown in Figure 6.2, Process I on mathematical instruction (teaching and learning activities in mathematics) is nested within Process II on mathematics didactics. Both constitute the focus of teachers' professional learning. The didactic suitability criteria (Chapter 5) for Process I (Figure 6.2) of teaching and learning mathematics will be interpreted in terms of the mathematical-didactic knowledge and competencies of the mathematics teacher (Teacher DMKC model). Teacher educators should consider these professional knowledge and competencies when designing and implementing training programs (Process II). Furthermore, how teachers learn, the affectivity involved, resources, and interaction patterns between educators and teachers should also be considered in the design and evaluation of such programs. Consequently, we will develop a model of knowledge and competencies for teacher educators (Educator DMKC model), which includes the Teacher DMKC model but also incorporates specific knowledge and competencies that promote teacher learning (i.e., the teacher as a student of mathematics didactics).

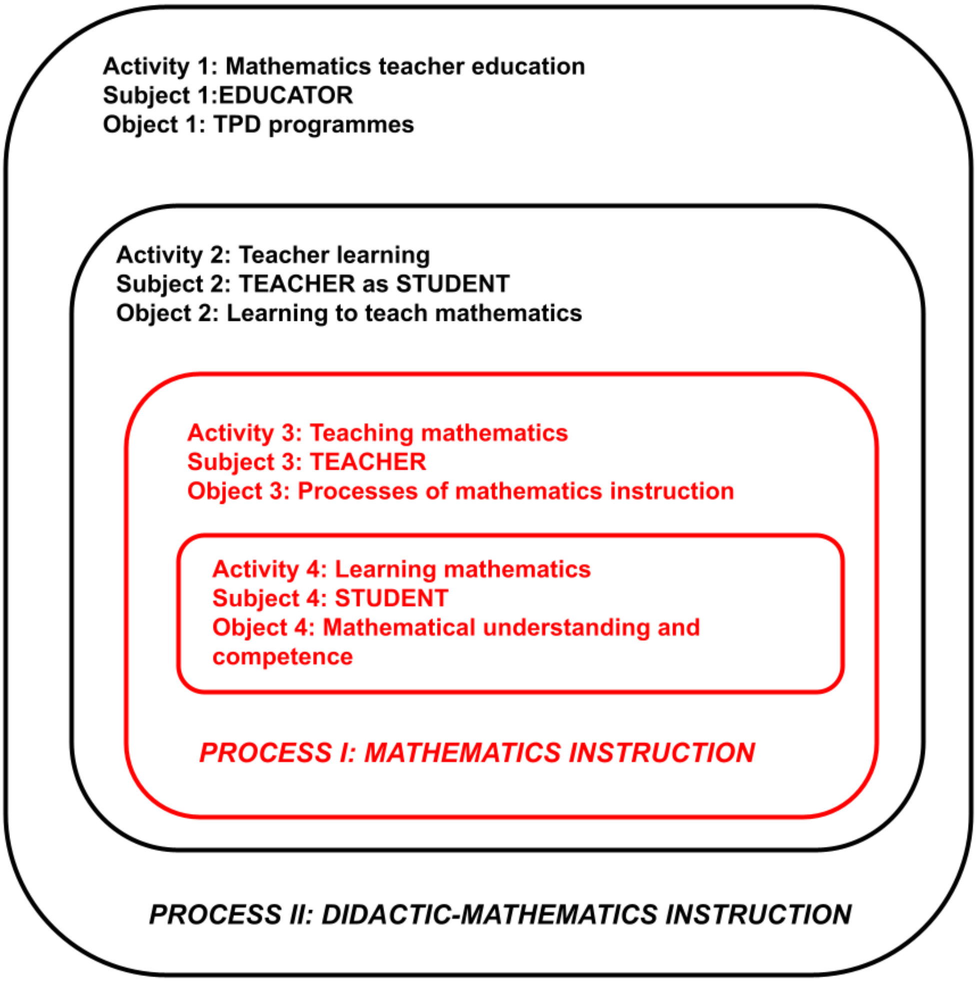

Figure 6.2. Activities involved in the training process

6.3. Didactic-mathematical knowledge and competencies in mathematics instruction

As previously indicated, the system of suitability criteria formulated in Chapter 5 can be used to categorize the teacher's didactic-mathematical knowledge, guiding its identification and formulation according to facets, components, subcomponents, and elements of specific content, under the educational-instructional design theory presented in Chapter 4.

In this section, we describe the knowledge system derived from the TDS-OSA for the process of mathematical instruction. In doing so, we expanded and revised the previous model of didactic-mathematical knowledge (DMK) (Godino, 2009; Pino-Fan & Godino, 2015). The training process should ensure that mathematics teachers acquire the knowledge and competencies that enable them to underpin, design, implement, and evaluate mathematical instruction processes with high didactic suitability. This implies competence to weigh the partial suitability criteria of epistemic (content), ecological (context), mediational (resources), interactional (interactions), cognitive (learning), and affective (students' emotions, beliefs, values) dimensions, considering the circumstances that condition mathematical instruction processes. This didactic dimension must be complemented with knowledge related to the normative and meta-normative dimensions (Chapter 4).

In the following sections, we identify the knowledge related to different facets and components. The justifications, which are based on the coherence with the assumptions of the OSA and the concordances (consensus) with other theories provided in Chapter 5 for the suitability criteria, serve as the foundation for the formulated knowledge.

6.3.1. Knowledge of characteristics of mathematical content (epistemic and ecological facets)

Based on the model of mathematical activity proposed by the OSA, the TPD program should equip teachers to ensure that the mathematics they teach (epistemic and ecological facets) possess the characteristics stated in Table 6.1.

Table 6.1. Knowledge of characteristics of mathematical content (epistemic and ecological facets)

Facet criteria	**Specific criteria by component**
Epistemic facet: The formative process promotes teachers' adoption of an anthropological view of mathematics, i.e., as a human activity focused on solving problems from which mathematical objects emerge and give meaning. Teachers recognize various partial meanings, objects, and processes and develop instruction with varying degrees of generality and formalization.	**Meanings and mathematical objects** – Consider the various partial meanings of the content and primary objects involved in each (situations, languages, concepts and properties, procedures, and arguments) and select those that are better adapted to the contextual and personal circumstances of the subjects involved. *Problem situations:* – Select and adapt mathematical problems/tasks that give meaning to mathematical knowledge and distinguish situations of contextualization, exercise, application, and problem generation. *Languages:* – Recognize the central role of mathematical languages (representations) and their types, transformations, and conversions in building and communicating mathematical knowledge. – Manage (know and use) different modes of mathematical expression and how they are related, recognizing their relevance according to the educational level. *Rules (concepts, propositions, procedures)* – Understand mathematics as an interconnected system of rules (concepts, procedures, and properties). – Select and correctly present definitions, propositions, and procedures adapted to the educational level. *Arguments:* – Recognize the central role of argumentation in building mathematical knowledge and diversity of proof methods. – Elaborate correct explanations, proofs, and demonstrations appropriate to the educational level. **Relations (connections)** – The partial meanings studied should be related to each other. The objects involved in the

	corresponding practices and other subjects that the student is familiar with are also related.
	Processes
	– Considering the diversity of processes from which the objects involved in mathematical practices emerge (problematization, representation, definition, generalization, modeling, etc.).
Ecological facet: The formative process promotes teachers' knowledge, skills, and disposition so that the mathematical instruction they design and implement corresponds to the educational project of the center and society. This process also considers the framework of the environment in which it occurs and innovations based on didactic research.	**Intra- and interdisciplinary connections** – Relate the content with other intra- and interdisciplinary content. **Curriculum** – Follow mathematics curriculum guidelines and their rationale. **Openness to didactic innovation** – Introduce innovations based on research and recognize best practices. – Integrate the use of new technologies (calculators, computers, ICT, etc.) in the educational project. **Socio-professional and cultural adaptation** – Ensure that the educational-instructional process contributes to the socio-professional education of students. **Education in civic values** – Include in the design and implementation of the educational-instructional process the education of students on democratic values and critical thinking. **Family setting** – The process stimulates and supports students' learning outside school and their individual development as a person.

The mathematical content that the teacher implements in the classroom must meet certain characteristics to optimize the development of the instructional process; mathematics must be rich, optimal, or adequate, according to the contextual (ecological facet) and personal circumstances of the students (cognitive facet). A specific instructional process occurs in a particular environment and is typically conducted over a bounded time interval. It is therefore inevitable that the teacher knows how to select some partial meanings of the object in question and the configurations of the associated objects and processes associated with them, but globally (throughout education), the set of meanings must represent the one previously established as a reference.

The teacher must know how to mobilize diverse representations of mathematical objects, solve tasks through different procedures, link mathematical objects of the educational level in which he/she teaches and of previous and subsequent levels, understand and mobilize the diversity of partial meanings for the same mathematical object, provide diverse justifications and arguments and identify the knowledge put into play in resolving problems.

The ecological facet of didactic-mathematical knowledge refers to knowledge about the mathematics curriculum of the educational level in which the study of the mathematical object is envisaged, its relations with other curricula and the relations that such curriculum has with the social, political and economic aspects that support and condition the teaching and learning process. The aspects covered within this facet of knowledge include Shulman's (1987, p. 8) proposals on curricular knowledge, knowledge of educational contexts and of the aims, purposes, and values of education, and Grossman's (1990, p. 9) knowledge of the horizontal and vertical curriculum for a subject and knowledge of the context.

6.3.2. Knowledge of the characteristics of mediational and interactional facets

The TPD program should provide opportunities for learning the knowledge and competencies listed in Table 6.2 regarding the mediational and interactional facets of the instructional processes that teachers design, implement, and evaluate. The mediational facet includes various resources that condition and support the teaching and learning of mathematics. In addition to concrete and technological material resources, such as calculators and computers, study aids (textbooks, activity notebooks, educational videos, ...), the number of students assigned to the teacher, the timetable in which classes occur, the material conditions of the classroom, as well as the total time assigned to study and its distribution are also considered. As can be seen, linking the interactional and mediational facets develops and enriches the notion of "content knowledge and teaching" raised by Ball et al. (2008, p. 401).

Table 6.2. Knowledge of the characteristics of interactional and mediational facets of mathematical instructional processes

Facet criteria	**Specific criteria**
Interactional facet: The formative process fosters patterns of interaction that help identify potential semiotic conflicts, select appropriate means to resolve them, promote progressive autonomy in learning, and develop students' communicative competence.	**Teacher-students' interactions** – Adapt the interaction between teachers and learners to the moments of the learning process using a dialogic collaborative format in the first encounter with the content and grant autonomy to the learner during the moments of practice and application. – Make an adequate presentation of the topic (clear and well-organized presentation, do not speak too fast, emphasize the key concepts of the topic, etc.). – Recognize and resolve student conflicts (ask questions and provide appropriate answers, etc.). – Seek consensus based on the best argument. – Use rhetorical and argumentative devices to engage and capture students' attention. – Facilitate the inclusion of students in the dynamics of the class. – Encourage participation and active engagement of all students. **Interactions among students** – Encourage dialog and communication among students. – Enhance group inclusion and avoid exclusion. **Autonomy** – Provide times when students take responsibility for the study (pose questions and present solutions; explore examples and counterexamples

	– to investigate and conjecture; use a variety of tools to reason, make connections, solve problems, and communicate). **Formative assessment** – Systematically observe students' cognitive progress and use the information obtained to decide on instructional development.
Mediational facet: The formative process fosters teachers' knowledge, skills, and disposition to use appropriate material and time resources to develop mathematics teaching and learning processes.	**Material resources (concrete, virtual and symbolic)** – Distinguish mathematical objects (regulative, non-ostensive) from their respective concrete, visual, or symbolic representations in mathematical and didactic practices. – Articulate alphanumeric and concrete representations of objects and processes to improve generalizability, calculation, and mathematical proof. **Study aids (textbooks, workbooks, educational videos, ...)** – Make critical and reflective use of curricular materials or other educational resources (textbooks or activity workbooks in physical or virtual format, educational videos, etc.), deciding when and how to use them to support the instructional process. **Number of students and classroom conditions** – Optimize as much as possible the number of students to provide personalized attention. – Adapt classroom and student distribution as much as possible to facilitate interactions. – Ensure a class session schedule that favors students' attention and commitment. **Time** (collective teaching/tutoring; learning time) – Provide adequate time (face-to-face and non-face-to-face) for teaching. – Provide sufficient time for important and challenging content.

The interactional facet involves the knowledge necessary to foresee, implement, and evaluate sequences of interactions between agents that participate in the teaching and learning process, oriented to the fixation and negotiation of meaning (learning) of students. These interactions are not only established between teachers and students but also between students and resources and between teachers, resources, and students. The mediational facet includes the teacher's knowledge that he/she must use and evaluate the relevance of the use of technological materials and resources to enhance the learning of a specific mathematical object, as well as the time allocated to the different learning actions and processes.

6.3.3. Knowledge of student learning characteristics (cognitive and affective facets)

The TPD program should provide opportunities for learning the knowledge and competencies listed in Table 6.3 regarding the cognitive and affective facets of the instructional processes they design, implement, and assess.

Table 6.3. Knowledge about the characteristics of students' mathematical learning (cognitive and affective facets).

Facet criteria	**Specific criteria by component**
Cognitive facet: The formative process advances teachers' knowledge, skills, and dispositions in such a way that learning goals are achievable cognitive challenges for students given their personal and contextual circumstances, that the personal meanings achieved by students are consistent with planned institutional meanings, and that assessment of learning serves to improve the instructional process.	**Personal meanings (learning)** – Promote understanding of situations-problems, representations, concepts, and properties. – Develop communication, procedural, and argumentative skills.
	Relations (connections) – Promote relational learning to help students understand and relate to different meanings and objects in the teaching process.
	Processes – Promote the development of students' competence to implement content-specific mathematical (modeling, generalization, problem posing and solving, proof, representation, ...) and metacognitive processes (reflection on own mathematical thinking).
	Previous knowledge

	– Consider students' previous knowledge when studying the intended content.
	Individual differences – Support students' learning by recognizing differences in prior knowledge, learning styles, and levels of understanding and proficiency.
	Learning assessment – Regularly checking learning progress to make instructional decisions regarding improvement (formative assessment).
Affective facet: The formative process promotes teachers' knowledge, skills, and disposition for implementing mathematics instruction to achieve the highest possible level of student engagement (interest, motivation, self-esteem), considering their beliefs about mathematics and their learning.	**Emotions** – Plan situations to identify and discuss emotions to avoid rejection, phobia, or fear of mathematics. – Highlight the aesthetic and precision qualities of mathematics. **Attitudes** – Promote students' responsibility for learning by attempting to complete tasks with perseverance, both of which require personal inquiry and the reception and retention of knowledge. – To favor argumentation in situations of equality; the argument is valued in itself and not by who says it. **Beliefs** – Identify and deal with students' beliefs about mathematics and its teaching that may condition learning and the instructional process. **Values-identity** – Promote self-esteem so that students feel capable of contributing to conjectures and solutions to problems by relying on mathematical arguments to convince others of the validity of their assertions, thus building a positive mathematical identity. **Interests and needs** – Propose tasks that interest students and are within their reach. – Propose situations that highlight the usefulness of mathematics in daily and professional life.

Progressive coupling between students' initial personal and institutional meanings is achieved through their participation in the community of practices generated in the classroom. The cognitive and affective facets, as defined in the OSA, together provide a better approach and understanding of the knowledge that mathematics teachers should have about the characteristics and aspects related to students' ways of thinking, knowing, acting, and feeling regarding their mathematical activities. The cognitive facet, on the one hand, provides teachers with the necessary knowledge to "reflect and evaluate" the proximity or degree of alignment between personal meanings (students' knowledge) with institutional meanings (knowledge from a historical-cultural point of view). For this purpose, the teacher must be able to foresee (during planning or design) and address (during implementation), based on the students' productions or expected productions, possible responses to a problem, misconceptions, conflicts, or errors that arise in the solution, and links (mathematically correct or not) between the mathematical object studied and other mathematical objects required to solve the problem.

The affective facet attends to the knowledge necessary to understand and manage students' moods, aspects that motivate them or not to solve problems, and so forth. It is knowledge that helps teachers describe students' experiences and sensations within a specific class or with a specific mathematical problem at a specific educational level, considering aspects linked to the ecological facet.

These two facets (cognitive and affective) integrate and expand the ideas of Shulman (1987, p. 8) —knowledge about students and their characteristics—, Schoenfeld and Kilpatrick (2008) —about knowing students as thinking and learning individuals—, Grossman (1990, p. 8) —about understanding students, their conceptions, and misconceptions of particular topics—, and Hill et al. (2008, p. 375) —about content knowledge and students.

6.3.4. Extended DMK Model

Figure 6.3 summarizes the categories of didactic-mathematical knowledge required for the design, implementation, and evaluation of educational instruction processes in mathematics that teachers should optimize and are classified according to their facets and components. At the bottom of Figure 6.3, we also include information on two categories of teacher knowledge related to the mathematical dimension: the mathematical knowledge *per se* that the teacher must possess. As explained in Pino-Fan and Godino (2015), *common content knowledge* is the knowledge about a specific mathematical object (for example, the derivative) that suffices to solve the problems or tasks proposed in the mathematics curriculum (or study plans) and in textbooks at a certain educational level (for example, high school). This type of knowledge is shared between the teacher and students. *Extended content knowledge* is the knowledge the teacher must have about the mathematical notions being studied at a moment (for example, the derivative) and those that are further ahead in the curriculum (for example, the integral in high school or the fundamental theorem of calculus and differential equations in university). Extended content knowledge provides the teacher with a mathematical foundation to present new mathematical challenges in the classroom, link the mathematical object being studied with other mathematical notions, and guide students toward studying subsequent mathematical notions related to the object of study.

The version of the DMK presented in this section develops the version included in Godino (2009) and Pino-Fan and Godino (2015) in several aspects. We retain the six facets, but we reorganize their components, mainly those corresponding to the epistemic and cognitive facets, according to the model of the structure of an educational-instructional process elaborated in Chapter 4. In addition, we include descriptions of general knowledge for each facet, which is specific to different components. We consider that the mathematical dimension of the teacher's knowledge (Pino-Fan & Godino, 2015) is included within the epistemic facet by considering that the teacher's mathematical education should have a specialized orientation for teaching.

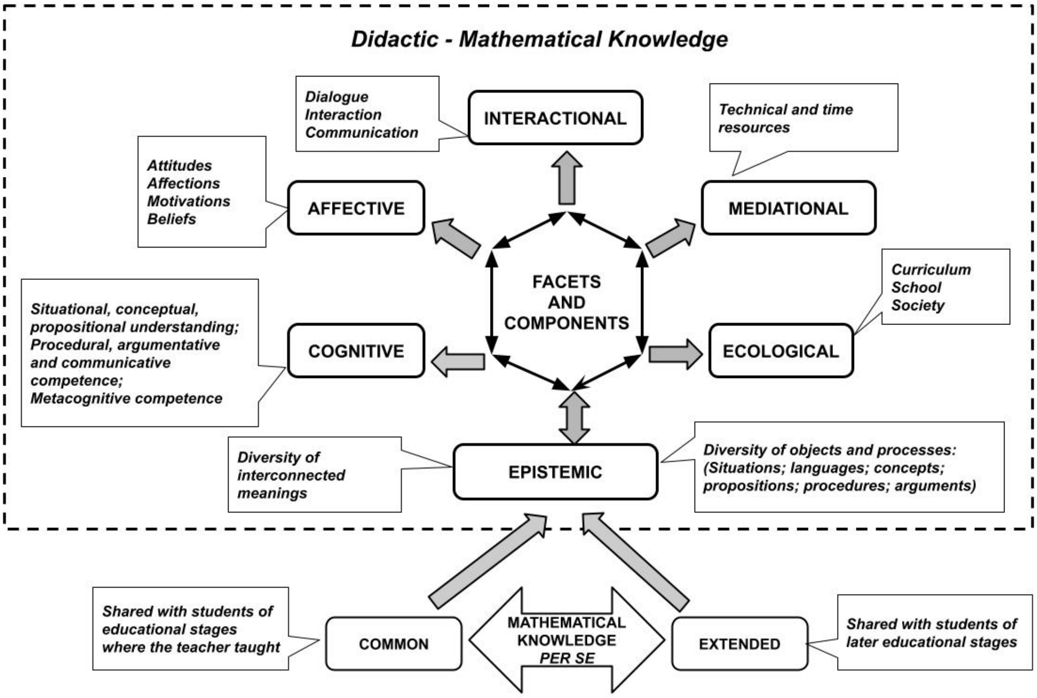

Figure 6.3. Facets and components of teacher knowledge (Godino et al., 2016, p. 292).

6.3.5. Model of didactic-mathematical competencies

Teachers must develop not only the knowledge necessary to understand teaching and learning processes but also competencies, i.e., skills and disposition to perform the required actions. In previous sections, we have used the structure of educational processes and the concept of didactic suitability to reformulate the DMK model of didactic-mathematical knowledge. The OSA provides various tools that help not only analyze and understand educational-instructional processes but also intervene in the design, planning, and evaluation activities. One aim of training processes should be for teachers to develop competencies when using these tools. This involves developing the general competency of didactic design and intervention, composed of the five sub-competencies described below, following the work of Godino et al. (2017).

Competency in analyzing global meanings

This competency is required when the teacher addresses the following questions:

- What are the meanings of the mathematical objects involved in studying the intended content?
- How are they interconnected?

In the preliminary phase of the instructional design process, meanings are pragmatically understood as systems of practices aimed at constructing a reference model that delineates the types of problems addressed and the operative and discursive practices required for their resolution. Suppose that fractions are being studied. The teacher must be able to characterize both institutional practices (different institutional meanings of fractions, such as ratio, part-whole, etc.), considering the various contexts where such problems are presented and the student's expected personal practices (personal meanings that students may acquire about fractions).

Knowledge of the notion "systems of operative and discursive mathematical practices, and their various types" (Godino et al., 2007, p. 129) corresponds to a competency in analyzing global meanings. The focus is on identifying problem situations that provide partial meanings or senses for the objects or mathematical topics under study and the operative and discursive practices that must be employed in their resolution. For instance, find situations that give sense to the different meanings of fractions.

Competency in the ontosemiotic analysis of mathematical practices

Resolving mathematical tasks involves and emerges from a network of objects that make the corresponding practices possible. Students must explicitly recognize these objects to progress in knowledge construction. The teacher's identification of the objects and processes involved in mathematical practices is a competency that will enable them to understand the progression of learning, manage necessary institutionalization processes, and evaluate students' mathematical competencies. This involves answering the following questions:

- What are the configurations of objects and mathematical processes involved in the practices that constitute the various meanings of the intended content? (epistemic configurations).
- What are the configurations of objects and processes employed by students when solving problems? (cognitive configurations).

Mathematics teachers must know and understand the concept of the configuration of objects and processes and be able to competently use it in didactic design processes. This relates to competency in the ontosemiotic analysis of mathematical practices used in instructional tasks.

Competency in analyzing and managing didactic configurations

The mathematics teacher must understand the notion of didactic configuration (Chapter 4), which is introduced as a tool for analyzing personal and material interactions in mathematical study processes. This means knowing didactic configurations that can be implemented and their effects on student learning. They must have the competency to use these configurations appropriately when implementing instructional designs (i.e., the skills to manage didactic configurations). They should be able to address the teaching problem of how to teach specific content, concretized in the OSA:

- What types of interactions between people and resources are implemented in instructional processes and what consequences do they have for learning?

- How can interactions be managed to optimize learning?

Competency in normative analysis

The various phases of the didactic design process are supported and dependent on a complex network of norms of different origins and natures (Godino et al., 2009) and meta-norms (D'Amore et al., 2007), whose explicit recognition is necessary to understand the development of mathematical study processes and guide them toward optimal levels of suitability. For example, when studying fractions, norms regarding their notation or graphical representation appear. There are also non-mathematical norms, such as the time dedicated to the fraction topic, the textbook students use, and the dates when evaluations are conducted. The mathematics teacher must know, understand, and value the normative dimension and use it competently, making it necessary to design training actions for its instrumental use. This involves developing competence in the normative analysis of mathematical study processes to answer the following questions:

- What norms condition the development of instructional processes?
- Who, how, and when are the norms established?
- Which norms can be changed to optimize mathematical learning?

Competency in analyzing and assessing didactic suitability

The professional questions mentioned above involve a microscopic view of teaching practice, i.e., performing detailed analyses of problem-solving or specific teaching and learning activities. In OSA, the notion of didactic suitability was introduced, guiding the macroscopic analysis of mathematical study processes. Given a specific topic in a certain educational context, for example, the study of quadratic equations in secondary education, the notion of didactic suitability leads to questions such as the following:

- What is the degree of didactic suitability of the implemented teaching-learning process on quadratic equations?

– What changes should be made in the design and implementation of the study process to increase its didactic suitability in the next experimentation cycle?

To elaborate a well-founded judgment on the didactic suitability of a mathematical study process, it is essential to reconstruct the didactic reference meanings of the corresponding topic. This requires a systematic review of research and innovations in mathematics education concerning epistemic, ecological, cognitive, affective, interactional, and mediational aspects. This leads to the following preliminary question:

– What are the didactic-mathematical knowledge results from previous research and innovations in teaching and learning quadratic equations?

Didactic suitability has been introduced as a support tool for global reflection on didactic practice, its assessment, and progressive improvement. The mathematics teacher must know, understand, and value this tool and acquire the competency to use it appropriately. This is the competency in analyzing the didactic suitability of mathematical study processes.

General competency in didactic analysis and intervention

The competencies described above are sub-competencies of a broader one, specific to the mathematics teacher, which is didactic analysis and intervention, as represented in Figure 6.5. The articulation of didactic competencies and knowledge is naturally accomplished in the OSA. Mathematical and didactic practices are actions oriented toward solving a problem or performing a task (they are not mere behaviors or actions). These practices can be discursive-declarative, indicating the possession of knowledge, or operative-procedural, indicating capacity or competency. Both types are interwoven; thus, efficient execution of operative practices involves activating declarative knowledge, which refers to the description of the instruments used or previously obtained results that need to be activated. Likewise, understanding declarative knowledge requires engaging in situations that provide the rationale for such knowledge and being involved (disposition for action) in its efficient resolution (Figure 6.4).

Including the competencies described in this section in the model of didactic-mathematical knowledge gives rise to the model of didactic-mathematical knowledge and competencies of mathematics teachers (Godino et al., 2017) (Teacher DMKC Model).

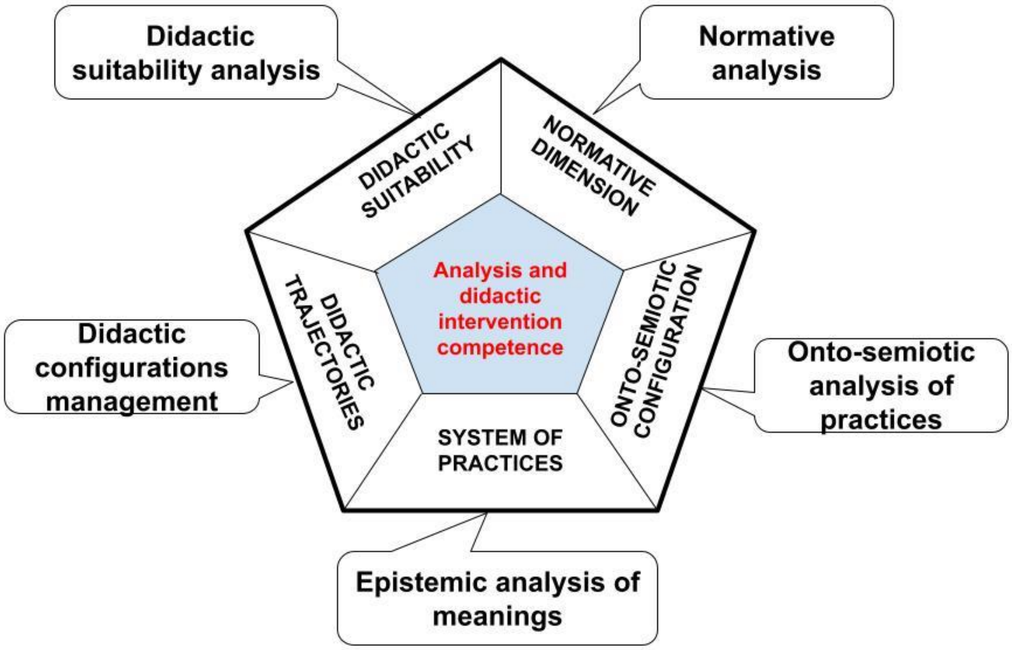

Figure 6.4. Components of didactic analysis and intervention competence (Godino et al., 2016, p. 295)

6.3.6. Edumat-Teacher books and the DMKC-Teacher model

The systems of knowledge categories that mathematics teachers should possess to promote learning serve as *containers* for classifying knowledge according to various criteria, but they do not specify what this knowledge entails for different content areas (arithmetic, geometry, etc.). The Edumat-Teachers books (Godino et al., 2004a; 2004b) complement these theoretical models by effectively developing mathematical and didactic knowledge for designing programs and training activities in mathematics (for primary school mathematics teachers) and in mathematics education (for teacher educators). We analyze the characteristics of these books from the perspective of the DMCK model.

Mathematics for teachers: Common and expanded content knowledge

The book *Mathematics for Teachers* (Godino et al., 2004a) is a resource that includes the knowledge that teachers should possess to design mathematical instruction processes at various levels of primary education. It defines what can be considered "appropriate mathematics" for both primary school students (common knowledge) and teachers responsible for their education (expanded knowledge). Let us examine the characteristics of the instructional mathematics processes proposed in the text for different facets of the DMKC model.

Epistemic and ecological facets

The text includes the various blocks of curricular content appropriate for primary education: number systems; proportionality; geometry; magnitudes; stochastics; algebraic reasoning. The number systems block is the most extensive, comprising six chapters (Natural Numbers; Numeration Systems; Addition and Subtraction; Multiplication and Division; Fractions and Rational Numbers; Decimal Numbers and Expressions; Positive and Negative Numbers). Geometry is covered in three chapters (Geometric Figures; Geometric Transformations. Symmetry and Similarity; Spatial Orientation). Magnitudes include one chapter on the concept of magnitude and its measurement and another on geometric magnitudes. The stochastics block is grouped into two chapters (Statistics; Probability), while proportionality and algebraic reasoning are each developed in one chapter. Each chapter includes two sections:

A: Professional contextualization. This section presents a collection of problems and exercises extracted from primary education textbooks. The teacher trainees are asked to solve these problems, analyze the concepts and procedures and develop other related problems. These tasks share a problem-solving-focused vision of mathematics with teacher trainees to develop problem formulation and analytical competence.

B: Mathematical knowledge. Mathematics is understood as both a problem-solving activity and a system of related objects (knowledge). Therefore, each chapter describes the relevant knowledge. Each lesson includes introductory examples that motivate the content and a final section called the Mathematics Workshop, where complementary problems are proposed for resolution.

For the network of conceptual objects that characterize each content, various meanings (intuitive and formal), definitions, properties, procedures with their justifications, and the use of various representation systems are studied. For example, the study of numbers begins in early childhood education and progresses through successive levels of complexity in primary and secondary education. This progression gives teachers a broad view of the various meanings and their increasing generality and formalization, enabling them to design well-founded learning trajectories for different levels of primary education.

Interactional and mediational facets

The model of teaching and learning mathematics proposed implicitly in the book Mathematics for Teachers (i.e., the modes of teacher-student-content interaction), both for children and for teachers, is made explicit in the monograph *Fundamentals of Teaching and Learning Mathematics* (Godino, Batanero, & Font, 2003). While not disregarding constructivist approaches to mathematics education, it is necessary to recognize the teachers' crucial roles in organizing, directing, and promoting student learning. Meaningful mathematical instruction should play a key role in social interaction, cooperation, teacher discourse, communication, and subject interaction in problem situations. To achieve meaningful mathematical learning, a trainee teacher must understand the complexity of the teaching task. It is necessary to design and manage various types of didactic situations, implement diverse interaction patterns, and consider the often-implicit norms that regulate and condition teaching and learning.

Regarding the use of resources or media for teaching and learning (mediational facet), teachers must have a favorable attitude toward using manipulative materials as elements of didactic situations. It is necessary to adopt a critical stance toward the indiscriminate use of such resources. We argue that manipulative materials (whether tangible or graphic-textual) can serve as bridges between reality and mathematical objects, but precautions must be taken to avoid blind empiricism and sterile formalism.

Cognitive and affective facets

The mathematical learning processes proposed in the textbook Mathematics for Teachers aim to build students' prior knowledge and develop the new knowledge and competencies required for teaching primary education. The first section of each chapter, A) Professional Contextualization, aims to evoke knowledge specific to primary education (common content knowledge) and simultaneously motivate (affective facet) the study by relating it to the exercise of the profession. The mathematics studied is closely related to the teacher's professional needs. Content included in each chapter ensures an understanding of the types of mathematical situations specific to primary education, as well as comprehension of concepts and propositions and development of procedural, argumentative, and communicative competencies involved in solving problem situations.

The process of studying mathematics proposed for trainee teachers, supported using the books *Mathematics for Teachers* and *Fundamentals of Teaching and Learning Mathematics*, exhibits suitable characteristics

(Godino et al., 2023) in various facets, so that the didactic model they experience in their training process can be transferred to the primary education levels they must design and implement.

6.4. Criteria for the suitability of teacher education and learning activities

We have elaborated on the model of the structure of an educational-instructional process included in Figure 4.1 (Chapter 4) considering the teaching and learning of mathematical content. However, it also applies to other contents, in particular to didactic-mathematical competencies and knowledge, the learning of which is the object/motive of teacher professional development activities.

We then develop criteria for the suitability of formative processes. In Section 6.6, these criteria are interpreted in terms of knowledge and competencies for educators, thus giving rise to the DMKC-Educator model. The general criterion for the suitability of educational processes is expressed in the following terms:

> The formative process should ensure that mathematics teachers acquire the knowledge and competencies to substantiate, design, implement, and evaluate educational-instructional processes of mathematics with high didactic suitability (epistemic and cognitive facets). In addition, the teacher must have and use appropriate training resources, implement interaction patterns that optimize teacher learning, and consider the affective and ecological factors involved.

The epistemic facet of the formative process (Process II, Figure 6.2) is constituted by the system of knowledge and competencies of the DMKC-Teacher model, referring to mathematical instruction processes, which are the object of the teaching activity. The didactic suitability criteria of Chapter 5 are interpreted as the epistemic suitability criteria for the formative process. We then elaborate on the suitability criteria of the remaining facets.

6.4.1. Criteria for the interactional, mediational, and ecological facets of formative processes

The partial suitability criteria for the interactional, mediational, and ecological facets of the didactic-mathematical teacher education process are presented in Table 6.4. We also express specific criteria for the components of the respective facets.

Table 6.4. Criteria for interactional, mediational, and ecological facets of the formative process.

Facet criteria	Specific criteria
Interactional facet The didactic configurations and trajectories of the formative process help identify potential semiotic conflicts in learning content and provide appropriate resolution means.	The formative process should include the following aspects: a) Consider the role of different interaction patterns in mathematical learning (dialectic between student autonomy and institutionalization). b) Apply strategies for the formative evaluation of teachers' learning. c) Identify and solve conflicts of meaning and learning difficulties related to classroom interaction. d) Develop communicative competencies and autonomous work of teachers.
Mediational facet The formative process manages adequate material and temporal resources to implement the formative tasks.	The formative process should include the following aspects: a) Acknowledge the role of manipulative and computer resources in mathematical and didactic-mathematical learning, including their possibilities and limitations. b) Provide adequate teaching time for different training tasks. c) Integrate the use of information and communication technologies and material resources in the formative tasks.
Ecological facet The formative process should be in line with the educational project of the center and society, considering the conditioning factors of the environment in which it is developed and innovations	The formative process should include the following aspects: a) Curricular guidelines for teacher professional development and their rationale. b) Results of research on teacher education. c) The search for, selection, and adaptation of good practices that involve the real context and interdisciplinarity of teacher education and the use of technology.

based on educational research.	d) Conditioning factors and restrictions of the social environment in the teaching and learning of mathematics and its didactics (economic, political, cultural factors). e) Teachers' democratic values and critical thinking.

Using manipulative and computer resources in a relevant and timely manner for the learning of specific mathematical topics is a component of specialized content knowledge and is therefore part of the teacher's learning expectations. Computers and audio-visual resources are appropriate for the presentation of cases related to teaching practices and their retrospective analyses. Likewise, the resources available for virtual communication (forums and virtual platforms) should also be used.

Given the extent of the didactic-mathematical knowledge related to the different blocks of content and specific topics, it is likely that there will not be enough time for a systematic study of them during the teaching time assigned to the subject. This will lead to the selection of some thematic units whose planning and didactic analysis will be performed in the time available; such units must have prototypical characteristics, i.e., they must represent the set of topics to be studied. The content and formative activities revolve around the professional development of mathematics teachers by considering and integrating the contributions of the remaining subjects in the curriculum and disciplinary areas.

Ecological suitability refers to the degree to which the professional development program is adequate within the environment in which it is developed, i.e., the socio-professional context and the curricula or programs established by the educational authority. In the formative process, the planning of training activities should be adapted to these guidelines. In addition, the results of research on teacher education should be considered in the different dimensions involved (characterization of the didactic-mathematical knowledge and competencies required by teachers for the exercise of their profession, difficulties and limitations in their acquisition, proposals for intervention with teachers, etc.).

6.4.2. Criteria for the cognitive and affective facets of the training process

The partial suitability criteria for the cognitive and affective facets of the didactic-mathematical teacher education process are presented in Table 6.5.

Table 6.5. Criteria for characteristics of the didactic-mathematical learning of teachers

Facet criteria	**Specific criteria by component**
Cognitive facet: The learning objectives of mathematical content *per se* and specialized didactic-mathematical knowledge ensure that teachers acquire the necessary competencies for planning, implementing, and evaluating mathematical instructional processes with high didactic suitability.	*Personal meanings* The formative process should include the following aspects: a) Promote an anthropological vision of mathematics that acknowledges the diversity of meanings and configurations of objects and processes. b) Recognize the implications of this vision for the management of interaction patterns and the use of didactic resources.
	Relations (connections) The learning of mathematical and didactic-mathematical knowledge should be relational so that teachers can understand and relate to the different meanings and objects included in the teaching process.
	Processes The formative process should include the following aspects: a) Promote the teacher's competence in implementing mathematical processes specific to the mathematical content (modeling, generalization, problem-solving or problem posing, proof, representation ...) and metacognitive (reflection on one's own mathematical thinking processes). b) Promote the teacher's competency in planning, implementing, and evaluating mathematical instruction processes.
	Previous knowledge

	The formative process considers teachers' prior mathematical content and specialised (didactic-mathematical) knowledge.
	Individual differences The formative process bears in mind teachers' differences and learning styles in relation to mathematical content and specialized (didactic-mathematical) knowledge.
	Assessment of teacher learning Apply evaluation instruments (observation scripts, interviews, questionnaires, essay tests, portfolios) to evaluate teachers' levels of conceptual and propositional comprehension, communicative, argumentative, and metacognitive competence, and procedural fluency in mathematical and specialized knowledge.
Affective facet: The educator needs to achieve the teacher's highest possible involvement (interest, motivation, self-esteem, willingness) in the formative process and consider the teacher's beliefs and values regarding mathematics and its teaching.	The formative process should include the following aspects: a) Search for, select, and adapt tasks/situations that pertain to the teachers' fields of interest and are applicable in daily and professional life. b) Organize and manage classroom interactions that promote self-esteem, participation, perseverance, and responsibility in the study of all participants. c) To evaluate teachers' beliefs and values about mathematics and its teaching by reflecting on them and their possible evolution.

The main indicator of cognitive suitability of the formative process is the effective achievement of learning expectations on the mathematical and didactic-mathematical content of the teachers, for whose formative and summative evaluation the educator should apply the system of methods and techniques usual in educational research (written tests, questionnaires, observation and interview scripts, portfolios).

An adequate theory-practice connection and the selection of real situations that teachers may encounter in their future professional practice to analyze and reflect upon will be indicators of affective suitability, as they will help to foster the interest, motivation and commitment of teachers in training. Special consideration will be given to the component of teachers' beliefs and values about mathematics and its teaching, a component that several authors have included within the affective dimension (DeBellis & Goldin, 2006; Goldin, 2000; Philipp, 2007).

6.5. System of knowledge and competencies of mathematics teacher educators

In Section 6.4., we have identified a system of criteria for the suitability of mathematics teacher education processes, i.e., of the teaching and learning processes of mathematics didactics (Process II of Figure 6.2). Since these processes are designed, implemented, and evaluated by the teacher educator, it is necessary to investigate the system of knowledge and competencies of the teacher educator required to perform the activities that constitute process II (Figure 6.2). The criteria system can be interpreted in terms of knowledge and competencies.

6.5.1. The DMKC-Educator model

The DMKC teacher model constitutes the epistemic facet of the DMKC educator model, i.e., the educator should possess the mathematical and didactical-mathematical knowledge and competences of the mathematics teacher; otherwise, we would be in a situation of "teaching what one does not know", which does not seem pertinent. The educator should possess the knowledge and competences to base, design, implement, and evaluate suitable formative processes in mathematics didactics. This involves weighing the partial criteria of suitability (epistemic, ecological, mediational, interactional, cognitive, and affective) considering the circumstances that condition professional teacher development in mathematics.

Figure 6.5 presents the elements of the DMKC educator model and its relationships with the DMKC teacher model.

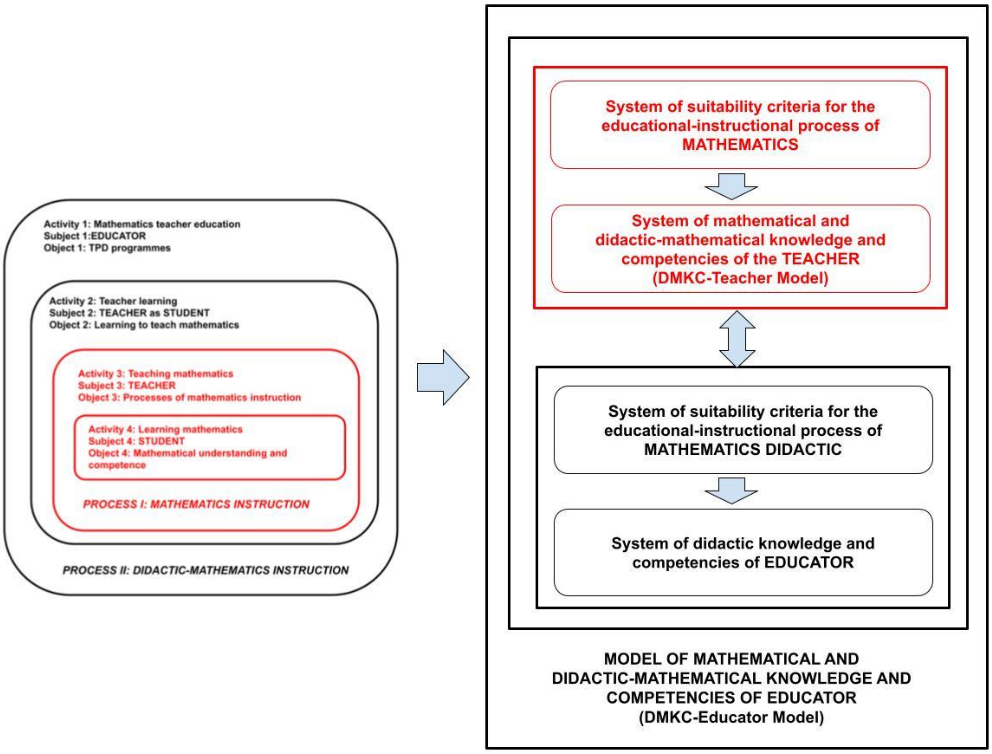

Figure 6.5. Teacher educator knowledge and competency model

6.5.2. Edumat-Teacher books and the DMKC-Educator model

The *Didactic of Mathematics for Teachers* (Godino et al., 2004b) is a resource that develops the knowledge of mathematics didactics that educators should consider when designing teacher education processes. This book includes seven blocks of didactic content. The first monograph, previously published as *Foundations of mathematics teaching and learning for teachers* (Godino, Batanero, & Font, 2003), is constituted by four chapters: Educational perspective of mathematics; Teaching and learning of mathematics; Mathematical curriculum for primary education; and Resources for the study of mathematics. Each of these chapters includes three sections: Professional contextualization; Didactic knowledge; Didactic seminar. The book also includes a list of complementary bibliographical references.

In this monograph, we offer an overview of mathematics education. We attempt to create a space for reflection and study on mathematics as an object of teaching and learning and on the conceptual and methodological tools that the didactics of mathematics generate as a field of research. The six principles of the NCTM (2000) —equity, curriculum, teaching, learning, assessment, and technology— describe crucial issues that, although not specific to school mathematics, are deeply interconnected with mathematics programs. They must be considered in the development of curriculum proposals, the selection of materials, the planning of instructional units, the design of assessments, instructional decisions in classrooms, and the establishment of support programs for teachers' professional development.

Each chapter of the monograph is structured into three sections. In the first section, which we call Contextualization, we propose an initial situation of collective reflection and discussion of an aspect of the topic. In the second section of Knowledge Development, we present the main positions and information, as well as a collection of activities and tasks inserted in the text that can serve as introductory situations for the different sections or as a complement to and evaluation of the study. The third section of the Didactic Seminar includes a collection of "problems of didactics of mathematics" that extend the reflection and analysis of the knowledge proposed on each topic.

Didactics of mathematical content blocks. Specialized knowledge of content

The *Didactics of mathematics for teachers* (Godino et al., 2004) includes, besides the monograph *Fundamentals of teaching and learning mathematics*, six other blocks of didactic content that refer to specific didactic-mathematical knowledge of the mathematical content blocks: number systems; proportionality; geometry; magnitudes; stochastics; algebraic reasoning. Each chapter includes the following sections:

Curricular orientations; Cognitive development and progression in learning; Conflicts in learning; Assessment instruments; Situations and resources; Didactic workshop (Analysis of school texts, Design of didactic units; Analysis of answers to evaluation tasks).

These sections contemplate didactic-mathematical knowledge, which includes aspects of mathematical cognition of specific contents (cognitive development, learning conflicts, assessment instruments), ecological (curricular orientations), mediational (situations and resources). The *Didactics Workshop* addresses aspects of the mediational and interactional facet of mathematics teacher–educator knowledge by indicating how to contextualize didactic-mathematical knowledge.

The *Edumat-Teachers*' books are valuable resources for mathematics teachers and teacher educators, considering the different facets involved in the educational-instructional processes of mathematics and didactics of mathematics. A new development of this project is yet to be addressed. The monograph *Foundations of mathematics teaching and learning* will be extended by presenting the didactic analysis tools provided by the theoretical framework of OSA. Likewise, the remaining monographs will update the results of the didactic research and include specific workshops so that teachers can use these tools in their reflection on their teaching practices.

6.6. Guidelines for analyzing the didactic suitability of TPD processes

In this chapter, we have addressed the problem of identifying and structuring a system of principles or criteria for the design of education processes suitable for the professional development of mathematics teachers. The ideal development of Process I (mathematics instruction) (Figure 6.2) requires the teacher to apply the suitability criteria presented in Chapter 5 (Tables 5.1A-B, 5.2, 5.3, 5.4, 5.5A-B and 5.6, 5.7 and 5.8). This set of tables constitutes a Didactic Suitability Analysis Guide for mathematics teachers

(DSAG-Teacher). Supplementing Tables 6.4 and 6.5 of this chapter provide a support instrument for the analysis and reflection on the didactic suitability of teacher educators' education processes. We refer to this instrument as the DSAG-Educator.

Teacher educators should have and deploy a system of specific knowledge and competencies to manage (design, implement and evaluate) mathematics teacher education processes. The category system of suitability criteria that forms the DSAG-Educator instrument distinguishes facets, components, subcomponents, and content elements for the two related processes (mathematics instruction and didactic-mathematics education). Hence, it can be interpreted as a system of categories of knowledge and competencies for teacher educators. Each criterion, general or partial, is associated with or can be considered a category of specific teacher–educator knowledge and competencies.

The DSAG-Educator instrument is a resource for reflection on the design, implementation, and evaluation phases of educational and training experiences. In the design phase, the aim is to anticipate and plan a suitable instructional process that can be adapted to the different facets and components of the given context. During implementation, the guide helps identify critical points in interaction patterns and to recognize semiotic conflicts related to interpreting tasks and discourse in the classroom, as well as trainee teachers' prior knowledge and attitudes. In the evaluation phase, it helps to identify weaknesses observed in the design and implementation of facets, components, and interactions, as well as possible improvements in future interventions.

With the DSAG-Educator instrument, we have interpreted, extended, and applied the concept of didactic suitability, which was first created for mathematical instructional processes (Godino, 2013; Godino et al., 2023). We apply the suitability construct to an activity whose subject is the educator of teachers and whose aim is to develop professional knowledge (mathematics, its teaching and learning) among teachers.

The rationale underlying the general and specific criteria of the different facets and components are the anthropological and pragmatist assumptions of OSA and their implications in educational-instructional processes. In principle, each theory, school of thought, or even teacher educator has its own system of suitability criteria for improving teacher education, although these criteria are often not specified. This opens a field of inquiry for identifying commonalities and complementarities among different models and moving toward a unified model.

6.7. Research example performed using theoretical tools

We include in this section a synthesis of the article by Godino et al. (2018) in which we applied tools from the DMKC-Teacher model to the analysis of professional knowledge in the design and management of a class on the similarity of triangles. We performed a retrospective analysis of formative action with prospective secondary mathematics teachers who were presented with a videotaped class episode on triangle similarity and asked to conduct a didactic analysis. The instructions given to prospective teachers were to describe, explain, and evaluate the mathematical content put into play; the roles of the teacher and students; the use of instructional resources; and the recognition of norms as explanatory factors of behaviors. This type of training action indicates the need for and usefulness of specific theoretical tools that can help teachers systematically reflect on teaching practices and make justified decisions about future teaching tasks.

6.7.1. Description of formative actions

This activity has its origins in a set of activities of initiation to research in mathematics education, which were proposed by Godino and Neto (2013). This was implemented as part of formative actions in the framework of a master's course for educating secondary mathematics teachers and consists of three phases:

Phase 1: Text commentary

Reading and discussion of a document on the characteristics of an ideal mathematics classroom, taken from the NCTM curriculum guidelines (2000, p. 3): A vision of school mathematics. The aim was to first elaborate on the ideal characteristics of a mathematics class. Work is done in small groups using a reflection guide with a motivating focus to discuss previous ideas, beliefs, and conceptions that prospective teachers have about mathematics and the complex processes of its teaching and learning. The phase closes with a reflection on the need to know and be competent in using specific tools that allow the teacher to systematically evaluate his/her practice; it is not only about describing and explaining what is happening in that ideal class, but also about reflecting on what aspects could be improved.

Phase 2: Implementation

It is proposed to watch a fragment of a videotaped secondary school mathematics class in which it is possible to observe 9 minutes of a class taught in Mexico. The video shows a first stage inside the classroom, where students work in groups solving problems related to the calculation of inaccessible heights, followed by the sharing of tasks; in the second stage, students perform fieldwork in the schoolyard, solving problems related to the calculation of heights of real objects (trees, poles, ...) from the measurement of their shadows. In Table 1, we included the transcript of the video to facilitate the analysis of the participants' answers.

Table 1. Transcription of the dialogs produced in the episode (videotape available at http://www.youtube.com/watch?v=60s_0Ya2-d8).

1T	Good afternoon, everyone.
2Ss	Good afternoon.
3T	Look, today we will work on a new task. We are in the module: shape, space, and measurement, focusing on geometric shapes and the subtopic (emphasizes) of

	similarity.
4T	We will work in the usual way, as always, as we have been doing.
5T	Professor Martín Eduardo is here to document the classes we conduct and how we work. So, please work as you normally would.
6T	We have completed this task today.
DISTRIBUTION OF TASKS [00:52]	
7T	Now, you can turn over your paper and read the task.
READING OF TASKS [01:07]	
8T	Alright, young people. Have you read the task?
9T	Can anyone tell me what we should be doing with this task?
10T	Mr. Legarre.
11S	Based on the drawing provided, calculate the height.
12T	Very good. What do the rest of you think? All clear?
13Ss	Yes
VERBALIZATION [01:49]	
14T	Calculate tree height in a drawing.
15T	Are we all clear?
16Ss	Yes.
17T	Go ahead. The tree height is calculated using the information provided.
18T	Now. Now. Look.
USE OF ICT [02:18]	
19T	On the board, on the projector, we already see the problem we are solving.
20T	Use the knowledge acquired in the previous instructions, because there, you calculated the values of the measures of some triangles with their homologous sides.
21T	You also determined the proportionality value.
DIDACTIC SITUATIONS [02:52]	
STUDENTS SPEAKING SPANISH [03:18]	
22T	Is it clear?
STUDENTS SPEAKING NAHUATL DIALECT [ ]	
23T	You have two approaches. When solving a problem, a single method. You can also use another method to verify that you are correct. The most accurate method should be "this" (the teacher points to the student's paper).
24T	Lo más correcto es que sea "esto" (el profesor señala el folio del alumno).
SHARING RESULTS [03:44]	
25S	The answer to the question is 5.23 (She explains the procedure they followed by writing the calculation on the board).
26S	Thus, we employ a rule of three, where X = 5.23.
27T	You obtained the same result for both methods. Excellent.
28T	The height of the tree was 5.23 m.
29	(The class goes out to work in the schoolyard).
30T	"This," times "this," divided by "this" gives the height of the post. (The teacher explains some students and writes in their notebook).
31S	Ah!
32T	You will now do the same. With the meter in hand, you can find a small tree and measure its shadow.

SUPPLEMENTARY ACTIVITIES [06:41]	
33	We should present evidence to school supervision of the current work on secondary education reform. We would appreciate it if you could briefly comment on what you are doing and tell us what grade this group is in, what task they are working on, and what part of mathematics is being covered at this time.
34T	This group here is 3ºA.
35T	We are working on the similarity of triangles. Some exercises outlined in the reform focus on similarity. Therefore, we address some related problems.
36T	We went out to the field to make it more practical, so the students have concrete evidence of what it means to calculate the heights of some trees/posts, which are very difficult to measure from below.
37T	With the similarity of triangles, this problem can be solved.
38T	They measure the shadow of some objects and, based on this, determine the height.
39E	Thank you very much, teacher. These are the tasks currently being developed under reform. Are we looking at any specific task?
39T	Certainly, similarity of triangle.

After viewing the class episode, the prospective teachers are given the reflection task presented in Chart 2, and they work in groups.

Chart 2. Didactic reflection task. (Giacomone et al., 2018, p. 9)

In the following link, we can find a video of a mathematics class: http://www.youtube.com/watch?v=60s_0Ya2-d8. After watching the video and working in teams, prepare a report addressing the following questions:

1. Description: *What happens?*
  a. What mathematical content is being studied?
  b. What meanings characterize the studied content?
  c. What is the context and educational level of the class?
  d. What does the teacher do?
  e. What do students do in this class?
  f. What resources are used?
  g. What prior knowledge must the students possess to tackle this task?
  h. What learning difficulties/conflicts do learners manifest?
  i. What norms (regulations, habits, customs) enable and condition class development?

2. Explanation: *Why did it happen?*
  a. Why is that content being studied?
  b. Why was a realistic problem used to study the content?
  c. Why did the teacher behave in the way he did?
  d. Why do students behave the way they do?

3. Evaluation: *What could be improved?*
Provide a reasoned judgment on the observed teaching in the following aspects and indicate some changes that could be introduced to improve it:
  a. Epistemic (mathematical content studied)

b. Ecological (relations with other topics, curriculum)
c. Cognitive (prior knowledge, learning, etc.)
d. Affective (interest, motivation, etc.)
e. Interactional (modes of interaction between teacher and students)
f. Mediational (resources used)

4. *Limitations of the available information*:
What additional information would be necessary to make the analysis more precise and well-founded?

Phase 3: Introduction of the reflection tool

Reading and discussion of the article: *Indicators of Didactic Suitability in Teaching and Learning Processes of Mathematics* (Godino, 2013). In this phase, the article previously read by the students is discussed. This document presents the notion of didactic suitability and its indicators and highlights the concordances between the selected criteria and those proposed by various authors and theoretical frameworks.

6.7.2. Analysis of the knowledge and competencies of the teacher managing the video-recorded class

Although the video segment only provides a glimpse of a small part of the class session, the experience with prospective teachers provoked an initial reflection on the dimensions of a mathematical study process and enabled the identification of some didactic-mathematical knowledge. In the following sections, we include possible interventions that instructors can make during the discussion phase of the responses given by prospective teachers to the questions posed in the task instructions. We also indicate the OSA theoretical tools to facilitate a more systematic analysis of the corresponding facets. The mastery of these tools should be the subject of the design of other formative interventions.

We begin with a section that describes the need to conduct a preliminary study of the problem situation to reconstruct the global meaning of proportionality, serving as a reference for the remaining analyses. For this purpose, the results of research on the meanings of proportionality (epistemic facet), learning processes (cognitive facet), and instructional resources (interactional and mediational facets) are considered. The position of the topic in the curriculum and its connections with other topics and disciplinary areas (ecological facet) should also be considered. In this example, we include only partial information about the epistemic facet (institutional meanings of proportionality).

Preliminary Study: Reconstruction of the reference meaning of proportionality

In the described episode, the students solve the following task: "If the length of the shadow of a tree is 12 m and that of a post of 1.5 m is 2.25 m, what is the height of the tree?" The solution involves an arithmetic meaning of proportionality based on establishing the equality of ratios:

$$\frac{12}{2{,}25} = \frac{x}{1{,}5}$$

Alternatively, the constant of proportionality can be found using the following unit reduction procedure (algebraic-functional meaning):

2'25 m → 1'5 m

1 m → 1'5/2'25 m

12 m → (1'5/2'25)×12 m = 8 m.

$$y = \frac{1}{15}x$$

In both cases, it is necessary to ensure that the conditions for applying a version of Thales' Theorem (Font et al., 2017) are satisfied. Therefore, a geometric meaning of proportionality is involved (Figure 6.6).

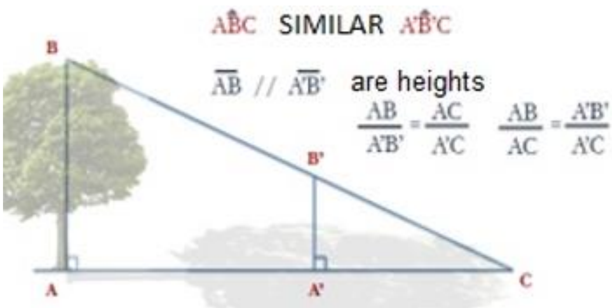

Figure 6.6. Graphical and symbolic representations

If the solution is justified by applying the "similarity of triangles", it will be necessary to justify that the triangles formed by the objects and their respective shadows are indeed similar. This requires showing that both triangles can be put into "Thales' position", justifying the proportionality of the corresponding segments.

Due to the mechanical use of algorithms and rules, it is possible to solve a proportionality problem without guaranteeing that proportional reasoning will occur. The widespread use of algorithms, such as the rule of three, often leads students to apply them to problems that are not proportional. This may produce in students the "illusion of linearity" (assuming that relationships between variables are linear when they are not).

Conducting a preliminary study of the content is a way to reflect on the different meanings and connections among them. Thus, the mathematical problem studied in the episode is a potential situation that could contribute to discussions with prospective teachers about the necessity of recognizing that mathematical objects have diverse meanings (see Chapter 3).

Description

Items a and b of the Guide (Chart 2) draw students' attention to the content being studied in the episode. A detailed content analysis is required to understand the learning difficulties (item h) and the prior knowledge required (item g). It is not sufficient to mention that the episode studies "the similarity of triangles" or "proportionality."

Analysis of mathematical objects and processes

In the transcript, we find this fragment of dialog:

- 9T: Can anyone tell me what we are supposed to do in this task?

- 11S: Based on the drawing provided, calculate the height.

- 17T: Go ahead. The tree height is calculated using the information provided.

The task involves calculating the inaccessible height of a tree by applying the proportionality of the homologous sides in similar triangles, as previously studied. Note that this is an application exercise.

- 20T: Use the knowledge gained from previous tasks, where the values of the measures of some triangles with their homologous sides.

- 21T: Have you also determined the proportionality value.

In solving the task, prior concepts are used: height of an object; triangles; homologous sides; proportionality; procedure: rule of three; proposition: the answer to the problem is 5.23; arithmetic calculations with/without a calculator; concepts of decimal numbers; units of measurement; and measuring with a graduated ruler or tape measure.

It can be observed that the application of the similarity of triangles is not problematized, nor are there moments requiring justification of solutions and procedures (imprecise measurement of shadows); that is: why is it possible to solve the task using the rule of three (for example)? Why is it possible to apply Thales' theorem?

Because of the sun's distance, the rays are parallel; therefore, Thales' theorem can be applied; the triangles formed by the tree and its shadow, and the post and its shadow can be placed under Thales' position.

Analysis of didactic processes

Items d (What does the teacher do?), e (What does the student do?), and f (What resources are used?) aim to initiate reflection on interaction processes in the classroom. Students are expected to make observations such as: In the observed class, the teacher gives instructions; distributes material; asks what should be done according to the task; authorizes the use of calculators and indicates the use of knowledge worked on in previous classes; asks, monitors, and provides feedback on students' work; and directs the sharing of solutions. In the second part of the video, the teacher helps students apply classroom-learned procedures to calculate the heights of trees and other real objects. The students in the classroom: read the task; recall solutions to previous tasks related to the similarity of triangles; apply that knowledge to the task (calculate the height of a tree represented on paper); and practice applying the rule of three. In fieldwork: measure shadows; work in teams.

Instructional resources used in the teaching/learning process include a learning guide; notebooks; paper, pencil, calculator; elements of the environment (trees, shadows); a graduated ruler, meter, and foot for measuring shadows; and a blackboard and projector.

It is necessary to discuss with students the delicate issue of articulating different modes of interaction in the classroom: individual, team, and teacher roles as manager and transmitter of knowledge. Ultimately, adopting a critical attitude toward traditional didactic models centered on the teacher, as well as toward naïve constructivist models centered on the student (see Chapter 4). Systematic reflection on the processes of interaction and mediation in the classroom requires the application of specific analytical tools, such as the notion of didactic configuration (Chapter 4).

6.7.3. Explanation. Analysis of norms and metanorms

Questions a, b, c, and d in section 2) of the Guide (Explanation) are proposed to provoke reflection on the network of norms that condition and support the development of teaching and learning processes. The development of the episode was guided by the *Reform* (curricular guidelines of the SEP of Mexico): working in teams to solve problems should be encouraged; this form of work has become a class habit that establishes the way of working. Regarding teacher-student interaction modes, a situation (written task) is proposed for each student; students are grouped around tables; first, they work individually but with the freedom to consult and exchange ideas and solutions; solutions are shared. Students consult the teacher; the teacher explains the task development.

Systematic reflection on norms that condition and support the teaching and learning processes of mathematics can be undertaken within the DMKC model using the normative dimension tool (see Chapter 4).

6.7.4. Assessment. Analyzing didactic suitability

Question 3) posed in the Guide (Chart 2), "what could be improved?" is broken down considering the six facets proposed in the Theory of Didactic Suitability (see Chapter 5). The system of criteria and empirical indicators identified in each facet constitutes a guide for systematic analysis and reflection that provides knowledge for the progressive improvement of teaching and learning processes. The didactic suitability tool applied to the episode case assists in the formulation of the following evaluative judgments.

a) Epistemic (mathematical content studied)

- Apply Thales' theorem to justify the similarity of triangles and to accept the proportional relationship between the lengths of homologous sides.
- Encourage students to develop conjectures rather than applying previously practiced procedure.
- Justifying the validity and equivalence of the procedures.
- Lack of precision in the language and the referred concepts: "value of measures of some triangles with their homologous sides" (20T).
- Avoid solving tasks by applying the three rules mechanically.
- Employing a functional approach to solving proportionality problems.
- Discuss the problem of measurement precision and acquire the necessary skills for accurate length measurement.

The analysis of the episode reveals the need for teachers to recognize the key role that argumentation, validation, institutionalization, and generalization (modeling approach using the linear function of the studied phenomenon) play in achieving high epistemic suitability in the teaching and learning process. Additionally, it recognizes mathematical connections: proportionality and linear function; Thales' theorem and the similarity of triangles.

b) Ecological (relations with other topics, curriculum)

- The content corresponds to the topics required in the curriculum and contributes to students' mathematical education.
- Emphasizing the connections between topics (similarity of triangles, Thales' theorem, proportionality, and linear function).
- From a mathematical perspective, the task allows the implementation of significant and relevant mathematical practices (knowledge and competencies): geometric proportionality; linear function; similarity of triangles; calculation of inaccessible heights and distances.
- It is a practical topic that can be used in everyday life (realistic context).
- There is no evidence of stimulating critical thinking.

c) Cognitive (prior knowledge, learning, etc.)

- The aim is to apply previously learned calculation rules; calculation of a term of a proportion knowing the other three. The intended content is within students' reach and poses an accessible challenge.
- No information is available on whether students know Thales' theorem.
- No curricular adaptations are required.
- Students can answer the task using two methods (it is not seen in the video fragment what those two methods might be).
- The degree of achieved learning cannot be showed even though learning is primarily procedural.

- The team and dialogical work format indicates moments of formative evaluation.

d) Affective (emotions, attitudes, beliefs, etc.)

- This task shows the utility of mathematics in everyday life. Students appear interested in the task.
- Teaching could be accompanied by a historical contextualization of the content in Ancient Greece and Egypt.
- A problem with the legend told by Plutarch, in which Thales applied his theorem to calculate Giza's height pyramids could be proposed.
- No argument is observed although teamwork is clear.
- The quality of precision in mathematical work is not highlighted (imprecise measurements).

e) Interactional (modes of interaction between teacher and students)

- Although students share solutions, moments of solution justification and institutionalization by the teacher are lacking.
- Students have a certain degree of autonomy in solving calculation and measurement tasks but not in communicating and discussing results.
- Moments of formative evaluation by the teacher were observed.

f) Mediational (resources used)

- Calculators are used for the rule of three calculations.
- Given that the teacher has access to a computer and projector, they can present illustrative situations and other methods of estimating inaccessible distances. Tape measures were not applied. Students measure distances with a graduated ruler and with steps, which can also be used to discuss different instruments and units of measurement.

Examples of applying the didactic suitability tool, complementary to the one presented in this section, include Aroza et al. (2016), Beltrán-Pellicer and

Godino (2017), Breda et al. (2017), Castro et al. (2014), and Posadas and Godino (2017).

6.7.5. Limitations of the available information and final reflections

To make the analysis of the knowledge involved in the classroom episode more precise and substantiated, additional information is necessary. Specifically:

- Worksheets from sessions in which the notion of the similarity of triangles and its relationship with Thales' theorem were introduced.
- Complete recording/transcription of the class to verify whether there were indeed moments of validation and institutionalization.
- Observation of the teacher's role in monitoring the work of different teams (identification of conflicts and ways of resolving them; formative evaluation).
- Moments of individualized summative evaluation to access the learning achieved.

The activity described in this formative experience should be considered as a first encounter for trainee teachers, allowing them to develop prior ideas about the facets and components involved in the complex reality of a mathematics class. It also serves as a contribution to the teacher educator, as it highlights the need for specific theoretical tools to support systematic reflection on these facets and components. These tools should be the subjects of study in new situations that focus on each of the mentioned tools.

Various studies have been conducted in initial and ongoing training contexts, designing and implementing training cycles for teachers or prospective teachers to develop the competencies of this model and learn the corresponding knowledge (e.g., Pochulu et al., 2016; Rubio, 2012; Seckel, 2016). In these cases, training cycles are often conducted in workshop format and designed as learning environments, so that: 1) participants engage actively through the analysis of classroom episodes; 2) the types of analysis proposed by the model emerge from the collective discussion within the larger group.

6.8. Concordances and complementarities with other theories

In Pino-Fan and Godino (2015), we analyzed the concordances and complementarities between the Didactic-Mathematical Knowledge (DMK) model and other knowledge models proposed by various authors: PCK (Shulman, 1987), MKT (Ball et al., 2008), KQ (Knowledge Quartet, Rowland et al., 2005), among others. In this section, we present other theories and models regarding the characteristics that mathematics teacher education programs should possess.

AMTE (2017) proposed a system of standards and indicators on specific knowledge, skills, and dispositions that a good mathematics teacher should possess and the characteristics that an education program for these teachers should meet. Regarding knowledge, they proposed four standards:

C1. Knowledge of mathematics for teaching.

C2. Pedagogical knowledge and practices for teaching mathematics.

C3. Students as learners of mathematics; and

C4. Social contexts of teaching and learning mathematics.

Regarding training program characteristics, AMTE (2017) proposed five standards:

P1. Partnerships (cooperation between all program stakeholders).

P2. Opportunities to learn mathematics (with emphasis on understanding the essential ideas of mathematics for teaching).

P3. Opportunities to learn how to teach mathematics (integration of mathematics, instructional practices, knowledge of students as learners, and social contexts).

P4. Opportunities for learning in the clinical setting (own and others' teaching).

P5. Recruitment and retention of teacher candidates (representative of diverse communities).

Park et al. (2018) defined TPD as any activity intended to (1) develop teachers' knowledge, skills, and expertise and (2) prepare teachers to improve their educational performance in current or future school roles. These authors, drawing from various publications (Beisiegel et al., 2018; Desimone and Garet, 2015), propose the following nine characteristics that efficient mathematics education programs should possess:

1) Focus on content: Develop well-organized content and pedagogical knowledge of the discipline and how students learn that content.
2) Active learning: Mathematics teachers should actively engage in meaningful discussions about instructional goals, student tasks, instructional strategies, student thinking, and practice.
3) Promote consistency: Align with teachers' professional development goals and district, state, and national standards and assessments.
4) Duration: The duration must be sufficient, including the duration of the activity and contact hours.
5) Collective participants: Involve groups of teachers from the same school, grade level, or subject area to build an interactive learning community.
6) Teacher outcomes: Assessment tools that can be used to measure the extent of teacher knowledge, skills, and changes in instructional practice.
7) Research-based models: The rationale for understanding the relationships among research-based models that involve student

thinking, new strategies, theories of teaching and learning, and instructional practices.

8) Student-provided data: This considers students' prior knowledge of mathematics. Understanding how students think helps teachers obtain information about effective instructional approaches for them; and
9) Promote changes in teachers' beliefs and attitudes about mathematics teaching to improve classroom practices.

In relation to these models, we question whether it is possible and advisable to structure, substantiate, and expand the list of these principles to produce a more comprehensive and detailed tool that provides efficient support in the development of TPD programs and specific actions. The application of OSA categories and methodological tools allows us to provide an affirmative answer to this question, which is concretized in the development of the theory presented in this chapter.

6.9. Synthesis of the teacher professional development theory based on OSA

In Table 6.6, we include a summary of the theory of teacher professional development based on OSA, responding to the questions proposed by Michie et al. (2014) as descriptors of a theory in the field of social and behavioral sciences.

Table 6.6. Synthesis of the theory of teacher professional development based on OSA

Elements	Description
Summary. What is the theory about, and what are its main propositions?	Develop a model of didactic-mathematical knowledge and competencies for teachers to optimize mathematics education and instruction processes. Additionally, a system of principles or criteria for evaluating the suitability of mathematics teacher education programs and actions related to mathematics didactics. This should consider the structure of processes and activities, such as foundation, design, planning, and evaluation. The system of suitability criteria is formulated in terms of value judgements, i.e., actions that should be taken to optimize the teaching and learning process of mathematics (teacher) and the training process in the didactics of mathematics (educator). The systems of suitability criteria underpin the respective knowledge and competency models for teachers and educators.
Scope/Objective. What phenomena does the theory explain?	The objective is to optimize mathematics teacher education by developing a guide for analyzing the suitability of education programs for teacher educators. The developed system of suitability criteria and categories of knowledge and competencies, both for teachers and educators, aids in designing, implementing, and evaluating educational instructional processes in mathematics and the didactics of mathematics.
Justification. Why is this theory necessary and how does it improve on previous theories?	In mathematics teacher education, various theoretical models have proposed systems of knowledge categories that teachers should possess to facilitate student learning exist. In addition, there are other models with principles that efficient training programs should fulfill. However, these models are often partial, are not explicitly grounded, or lack the required details. This theory addresses these deficiencies.
Hypotheses. What specific hypotheses does the proposed theory propose, and how do they differ from other theories?	It is assumed that the foundation, design, implementation, and evaluation of mathematics teacher education processes are complex and require consideration of various facets, components, and their interactions. It is possible and necessary to identify criteria (principles) that aid in the optimal development of education processes, based on explicit theories of didactic-mathematical knowledge and research results on these processes.
Constructs. What elements constitute the theory?	– Model of phases and structure of an educational-instructional process in mathematics, distinguishing epistemic, ecological, mediational, interactional, cognitive, and affective facets in each of the foundation, design, implementation, and evaluation phases. – Model of phases and facets of an educational-instructional process in the didactics of mathematics.

	– System of suitability criteria for mathematical instruction activity. – System of categories of mathematical and didactic-mathematical knowledge and competencies for mathematics teachers. – System of suitability criteria for teacher education and learning activities. – System of categories of didactic-mathematical knowledge and competencies for teacher educators.
Relations. How are the elements of the theory related to each other?	The structural and phase model of an educational-instructional process is used to elaborate and organize a system of suitability criteria for mathematical instruction activities developed by teachers and training activities of teacher educators. The suitability criteria systems determine the categories of knowledge and competencies for mathematics teachers and teacher educators.
Origin. On which theories is it based, and how?	This theory is based on ontosemiotic theories of mathematical activity, meaning, mathematical cognition, and educational design theory in mathematics based on OSA. The components and sub-components of the epistemic and cognitive facets correspond to the categories of objects, processes, and meanings proposed in the cited theories. The suitability criteria for educational programs are based on the theory of didactic suitability.
Similarity. Which theories are most similar to this theory?	This theory relates to theories of mathematical instruction quality and proposes categories of teacher knowledge and efficiency principles for educational programs. Identifying the concordances and complementarities between this theory and other theories is a topic that requires further research.
Complementarity. With which theories can it be complemented?	The identification of concordances and complementarities between this theory and other theories is a research topic.
Operationalization. How are the constructs measured or identified?	The suitability criteria for mathematical instruction processes and teacher education activities are value judgments that can be graded to ensure compliance with specific processes. Systems of rubrics with observable indicators of compliance with the suitability criteria and efficiency principles of the programs are pending development.
Uses. What can the theory be used for?	This theory is used for the design, implementation, and evaluation of mathematics teacher education programs and actions. The developed system of categories of didactic-mathematical knowledge and competencies and the suitability criteria for education programs can be used to describe and understand the activities of educators and mathematics teachers and identify potential improvements.

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Chapter 7
The OSA theoretical system

Introduction

In this chapter, we demonstrate the relevance of considering OSA as a theoretical system by analyzing the interrelationships between the partial theories described in the previous chapters and the need and usefulness of elaborating this system to address the complexity of mathematics education problems. Although it is possible to investigate partial problems related to epistemic, cognitive, etc. aspects, we need to consider the interrelationships between the different facets. Therefore, the development of an integrative theoretical framework that underpins the design of mathematics educational-instructional processes and the education of teachers in specific theories of mathematical meaning and cognition is relevant for research and practice in mathematics education.

In Section 7.1, we present an overview of the theoretical tools that compose OSA and their interconnections. Synthesis of the dilemmas between various paradigms or approaches to mathematics education research that motivate the construction of OSA is included in Section 7.2. Section 7.3 outlines our approach to research problems in OSA and describes the features of the corresponding methodological approach. In Section 7.4, we summarize a research example in which most theoretical OSA tools are used. In Section 7.5, we explore the OSA's concordances and complementarities with six other theoretical frameworks: didactic situation theory, anthropological theory, realist mathematics education, APOS theory, objectification theory, and the ethnomathematical program. Section 7.6

describes the scope of OSA applications and dissemination included in the web repository: http://enfoqueontosemiotico.ugr.es. Finally, in Section 7.7, we present a synthesis of OSA's philosophical postulates and, in Section 7.8, an overall summary of OSA in response to questions by Michie et al. (2014).

7.1. Connecting the OSA theoretical tools

The OSA aims to build a system of conceptual and methodological tools that enable for macro- and micro-level analyses of the epistemic, ecological, instructional, cognitive, and affective dimensions and interactions involved in teaching and learning mathematics processes. The general notion of a mathematical object, its types and relationships with mathematical practices, the different polarities from which it can be considered, and the concept of semiotic function configure an ontosemiotic approach—an ontological and semiotic model—to mathematical knowledge that enriches, complements, and articulates the partial ontologies that characterize other theoretical models in mathematics education.

The theoretical models described in chapters 2 to 6, together with their interconnections shown in Figure 7.1, allow us to consider OSA as an inclusive, open, and dynamic theoretical system. This system stems from reflection on different theoretical frameworks in mathematics education and from conducting multiple empirical investigations in research projects and doctoral programs (Godino, 2022). It considers the different dimensions and analysis levels required by research on educational-instructional processes in different contexts and provides tools for comprehensive didactic analysis that substantiates the teaching and learning processes of mathematics, according to its various dimensions and phases.

The pair <system of practices, configuration of objects and processes> is original to OSA and is key to conducting an a priori analysis of the mathematical knowledge involved in problem solving at both the macro-level (emergence and articulation of partial meanings of the object) and micro-level (identification of the network of objects and processes involved in mathematical practices). These a priori analyses are essential to design, implement, and evaluate educational-instructional processes because they allow the informed selection of meanings whose appropriation by students is proposed as an educational aim. In addition, they allow the elaboration of the epistemic trajectories involved in resolving learning tasks and the anticipation of expected cognitive trajectories.

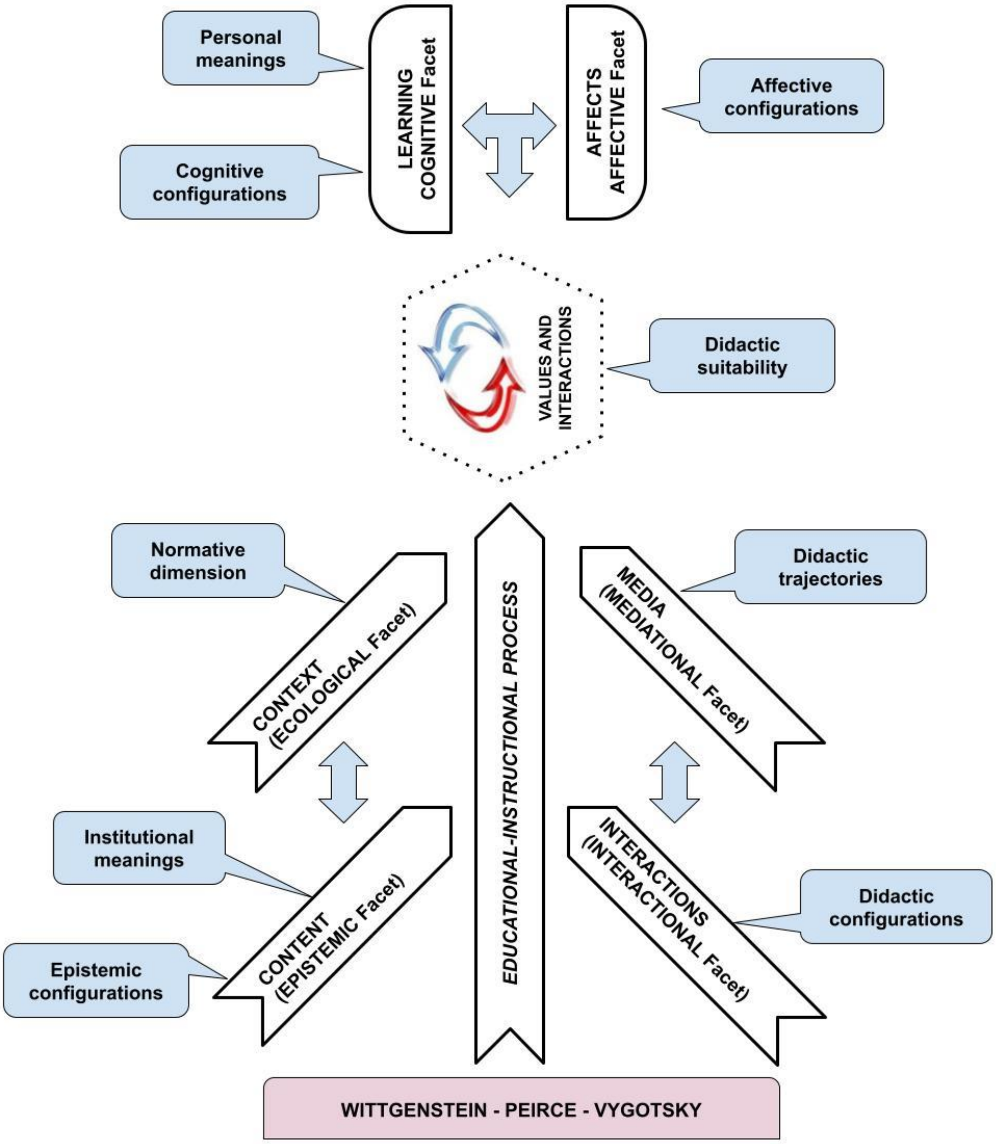

Figure 7.1. OSA tools and connections

OSA assumes a complex research paradigm based on a holistic and systemic approach (Cohen et al., 2007) because it considers that it is necessary to address epistemological and ontological problems specific to basic research, oriented toward understanding phenomena, and instructional design problems (focused on the solution of practical problems of teaching and learning). It also engages with teacher education, recognizing its fundamental role in implementing effective changes in mathematics education through action research and reflective practice.

To bridge the gap between scientific-technological research and reflective practice, we developed the Theory of Didactic Suitability (OSA module), which addresses the axiological problem of identifying and structuring action values and norms to optimize educational processes. This theory lays the foundation for a research program aimed at identifying the value judgements involved in each facet and component of an educational-instructional process and at comparing and articulating the criteria proposed by different theoretical frameworks.

7.2. Mathematics education dilemmas and conflicts addressed by OSA

In OSA, we conceive mathematics education as a complex social system that involves activities related to the educational-instructional processes of grounding, design/planning, implementation, evaluation, and teacher professional development. Likewise, we consider mathematics education theories as activity systems (Godino et al., 2024) in the context of the Cultural Historical Activity Theory (CHAT) (Engeström, 1987; Roth and Lee, 2007), which provides answers to questions that are the object/motive of partial activities. The proposed CHAT structure of activity systems allows us to consider the historical-cultural and community dimensions of theories and the ecological regulatory context in which they seek to provide tool-mediated answers to the questions that constitute their raison d'être. The notion of contradiction, which includes dilemmas, tensions, or conflicts between different elements of an activity (Núñez, 2009) or related activities, helps explain the reasons for changing systems and identify unresolved contradictions that need to be addressed in new developments.

The application of CHAT to the entire analysis of mathematics education and the theories developed within it can help understand new aspects of its organization and development. This modeling of mathematics education enables us to structure the OSA theoretical system based on the five partial activities and highlight the relationships between its ontological and semiotic assumptions about mathematical activity and the educational-instructional model that is consistent with these assumptions. Using the triangular model for activity systems extends the perspective of theories to the historical-cultural (community) context and the ecological (regulatory) niche in which they develop. Likewise, the idea of contradiction or dilemma between the components of a system or between two or more activity systems allows us to reinterpret the reasons behind the emergence of some OSA tools and assumptions (Godino et al., 2024).

In Figure 7.2, we indicate some dilemmas raised in the analysis of mathematics education theories that are addressed by the OSA partial theories. Specifically, there are tensions between theories that emphasize the epistemic or cognitive facet—mathematics seen as a problem-solving activity or a system of cultural objects, or between student-centered (constructivism) and teacher-centered (objectivism) didactic models. These dilemmas in the foundations of mathematics education - revealed by comparing theories such as TDSM (Brousseau) and ATD (Chevallard) with TCC (Vergnaud) and TRRS (Duval) - encouraged the introduction into OSA of the dialectic between institutional and personal dimensions, attributed to mathematical practices, meanings, and objects.

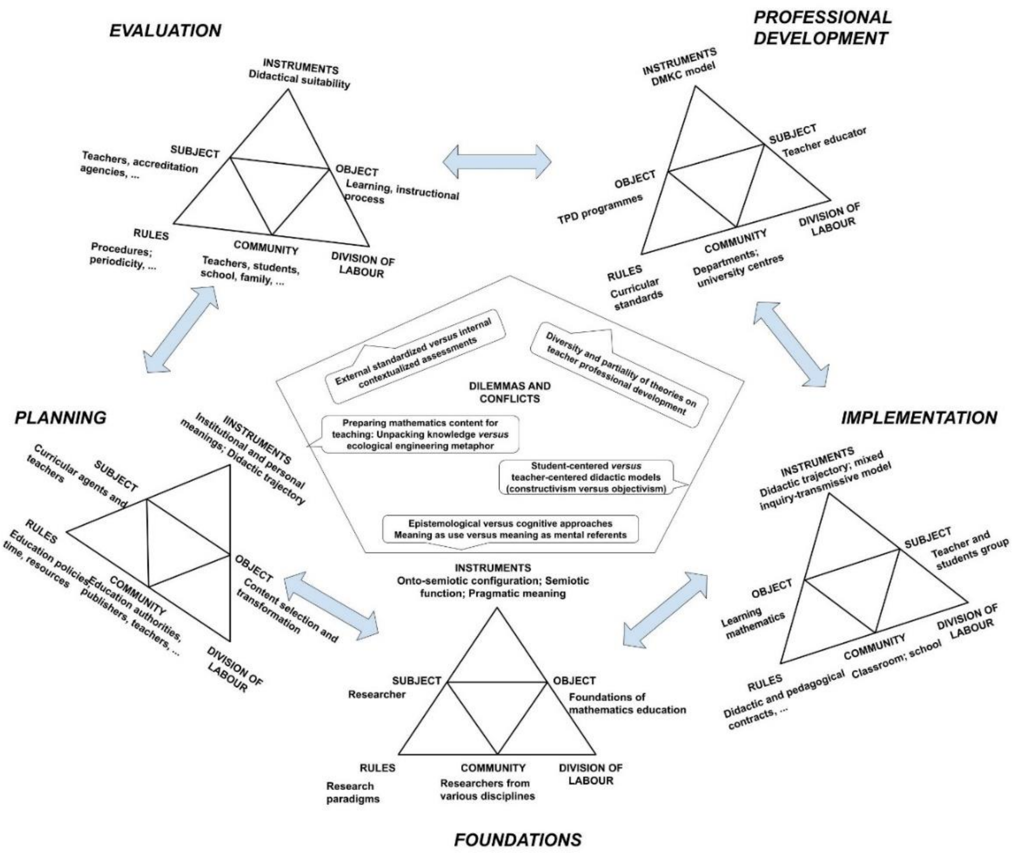

Figure 7.2. Dilemmas, conflicts, and interdependencies among activity systems (Godino et al., 2024)

Besides to the above dilemmas, there is a tension between referential (realist) and operational (pragmatic) theories of meaning. In realist theories, linguistic expressions are related to certain entities (objects, attributes, facts). In pragmatic theories, the meaning of linguistic expressions depends on the context in which they are used; similarly, the meaning of abstract objects must be inferred from their use. This dilemma is addressed in OSA by considering systems of practices as objects to which conceptual terms and expressions refer, together with the notion of semiotic function (Godino et al., 2021): the system of practices is the consequent object or meaning of conceptual terms or symbolizations, which participate as antecedents or signifiers of a semiotic function. The dual pragmatist-referential vision of meanings enables the combined conception of mathematics as a problem-solving activity with the vision of mathematics as a system of cultural objects, considering the abstract object as the unitary regulative entity of this activity (Chapter 2).

OSA addresses the dilemma between pedagogical theories that propose student-centered (constructivism) or teacher-centered (objectivism) didactic models. By acknowledging the complexity of objects and processes involved in mathematical activity and postulating the regulative nature of mathematical concepts, propositions, and procedures (Font et al., 2013), the optimization of didactic appropriateness requires the application of a mixed model that dialectically articulates the interactions between the teacher, learner, and content. The dialogic-collaborative didactic model in the student's first encounter with new content (Chapter 4) is an instrument in the implementation activity, derived from the solution of the dilemma discussed in the foundational activity.

Some didactic theories, such as the Theory of Objectification (Radford, 2014), advocate the application of a collaborative model, in which teachers and students work together, as preferable to constructivist or traditional teacher-centered alternatives. The educational-instructional model proposed by OSA is more inclusive, assuming that learning optimization can be achieved through appropriate articulation of different types of didactic configurations.

Didactic suitability helps clarify and weigh the role of standardized external evaluations by describing the complexity of facets and components to be considered and the difficult balance of principles and values to be reconciled to optimize educational-instructional processes. Both summative and formative evaluations performed by the teacher are essential to determine the relative importance of each aspect in relation to the context and circumstances of the participants.

Regarding the dilemmas facing teacher education, Figure 7.2 indicates the diversity and partiality of teacher professional development theories; we can add the tension between the understanding and mastery of knowledge about teaching and learning and the development of professional competences, i.e. effective action on practice. The model of mathematical didactical knowledge and competences (CCDM, Chapter 6) coherently articulates the development of teachers' knowledge of the various facets and components involved in mathematical instruction and professional competence.

7.3. Research problems and methods from OSA perspective[20]

Research in mathematics education has evolved in response to changes in predominant theoretical frameworks, including experimental psychology, constructivism, and sociocultural approaches. During the 1990s, the focus of mathematics education research shifted largely from cognitive to social: from theories that focus on individual thinking processes to "theories that see meaning, thinking, and reasoning as products of social activity" (Lerman, 2000, p. 23). These trends have produced biases and partialities in mathematics education research topics. Inglis and Foster (2018) conclude that "mathematics education would benefit from greater interaction between research in experimental psychology and sociocultural research" (p. 494).

We approach research problems by assuming particular principles and methods specific to the theoretical frameworks from the problems formulated and the results interpreted. Therefore, it is necessary to reflect on the choice and implications of the conceptual model employed (even if tacitly). As Schoenfeld put: "For example, which phenomena are not considered in this perspective, which are given significant importance, and how can these theoretical biases shape the interpretation of the situation?"

[20] The content of this section is mainly based on Godino et al. (2021).

In this section, we illustrate how the OSA toolkit enables researchers to develop research questions that address cognitive and sociocultural dimensions as well as the scientific, theoretical, and technological practical components of educational research. The basic unit of didactic analysis for OSA is the educational-instructional process in mathematics (student learning) or mathematics education (teacher training). In both cases, six facets and their interactions are relevant: epistemic, ecological, mediational, interactional, cognitive, and affective (Figure 7.1). These facets are used as main categories to classify the focus of didactic analysis and intervention while still considering the overall scope of the educational phenomenon.

7.3.1. Classification of research problems

A first criterion to classify the problems is the mathematical content that is the focus of the research, i.e. Arithmetic, Geometry, Measurement, Algebra, Statistics, Calculus, etc. The educational level (early childhood education, primary, secondary, university, teacher training, etc.) to which the instructional process concerns is another criterion for organizing the research questions. In addition to content and educational level, the focus of the research may be one or more of the following categories:

- *Epistemic*: investigates the mathematical content itself and the different forms in which this content can be introduced (more or less formal, almost complete) in mathematical activity.
- *Ecological:* This approach focuses on the relationships of mathematical content with other disciplines and the curricular, socio-professional, political, and economic factors that condition mathematics instruction processes.
- *Mediational:* Analysis of resources (technological, material and temporal) to enhance student learning.

- *Interactional:* Study of teacher and student roles in task management, identification and resolution of learning conflicts, and types of interactions that can be established in the classroom.
- *Cognitive:* This study investigates how students learn, reason, and understand mathematics, their problem-solving strategies, the difficulties or semiotic conflicts they exhibit in the instructional process, and how their learning progresses.
- *Affective:* This approach focuses on students' affective, emotional, and attitudinal aspects in relation to mathematical objects and the instructional process being implemented.

The didactic analysis of teacher training processes must remember the six aforementioned facets, which in this case refer to didactic-mathematical knowledge, the development of professional competences, and the study of their conditioning factors (Chapter 6).

The classification of research problems according to their focus can be complemented with the following typology, which is characterized by its intentionality or purpose:

- Descriptive of meanings, processes, and factors (What is ...? How is ...?).
- Explanatory of teaching and learning processes and the effects of intervening factors (Why ...? What changes?).
- Predictive or of implementing actions to achieve an aim (e.g., how to design or motivate ...? What will happen if I change ...?).
- Evaluative of the appropriateness of an instructional process or any of its components (To what extent is it suitable...?).

In addition, research is distinguishable in terms of scope, depending on whether it is a case study, a sample (which can be probabilistic or not), or a census (Cohen et al., 2007). Figure 7.3 summarizes the proposed criteria for classifying research problems.

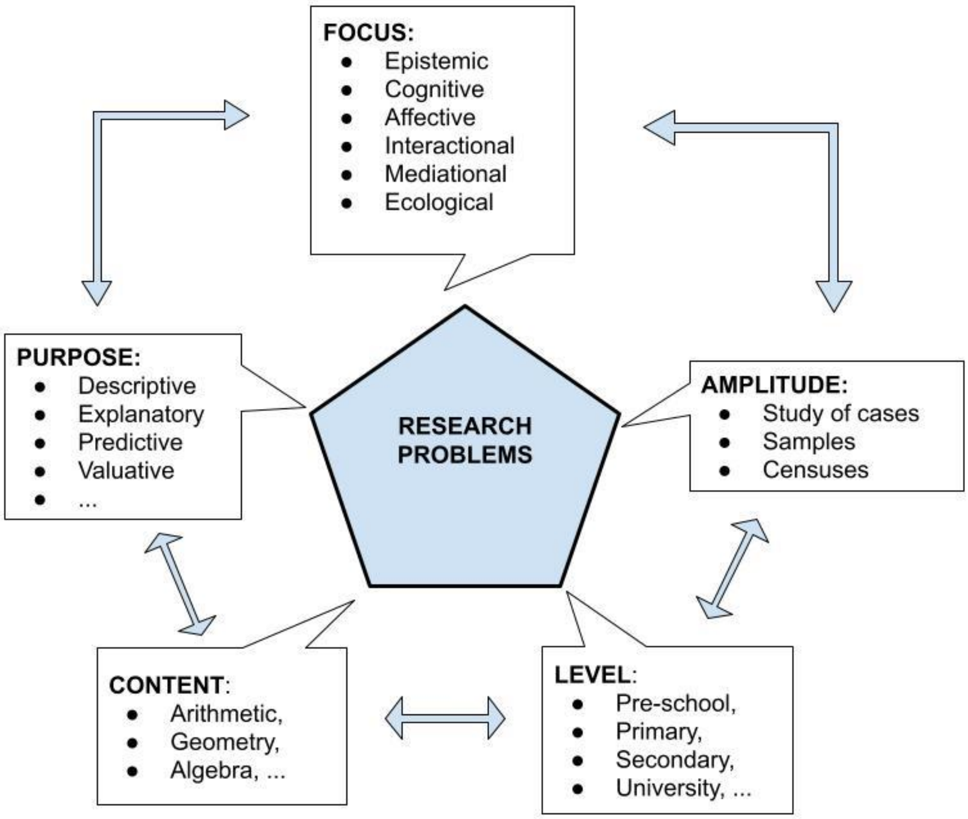

Figure 7.3. Classification of research problems

We can define specific problems by fixing mathematical content, a facet, a purpose, and a generality or breadth; however, problems can also be defined by encompassing several of these categories or different contents. The analytical process of developing research problems should not ignore the global nature of the facets, components, and variables involved in mathematical instruction to ensure its importance and relevance. Thus, we highlight instructional design research (Kelly et al., 2014) as a type of research that considers the interactions between different facets and components. This research has a predictive purpose; that is, it responds to the scheme "If we apply treatment X in a set of circumstances, then we get results Y". In principle, this research is closer to teaching practice issues and needs because it considers the different facets and components involved in instructional processes. The assumption underlying the conduct of experiments is that similar results are obtained under conditions like those of the experiment. This potential generality of results can occur not only in quantitative studies but also in rich and dense anthropological descriptions, typical of the human sciences, where implementing all the requirements of experimental or quasi-experimental research is difficult (Schoenfeld, 2007).

7.3.2. Methodological approach

There is a strong interest in using qualitative research methods in mathematics education and other social sciences, although this does not imply that quantitative methods should be neglected when seeking more broadly generalizable results. There is a growing recognition of the complexity of the problems addressed in the social sciences and the need to adopt a pragmatist perspective regarding the use of mixed methodologies. This perspective allows us to understand educational activities in the context in which they occur while providing generalizable recommendations to support educational decision-making (Hart et al., 2009). This pragmatist perspective accepts a wide variety of methods to address complex research questions. Qualitative and quantitative methods can be applied in the same research project with careful planning and recognition of the potential contribution of each approach (Johnson & Onwuegbuzie, 2004). It is important to be aware of the interconnectedness of the question statement and methods with the theoretical framework principles and conceptual tools within which the research project is framed. "Methodologies are part of the theoretical frameworks used in the research, and therefore deeply connected to the theory’s principles and paradigmatic issues" (Bikner-Ahsbahs et al., 2015, p. 533).

Ontosemiotic analysis as a technique for determining meanings

The epistemological question, i.e., the description of how mathematical knowledge emerges and develops from an institutional point of view, is investigated in the OSA according to the following methodological principle:

> The institutional genesis of mathematical knowledge is investigated through the following: 1) the identification and categorization of the situations-problems that require a response in which the object intervenes; this sometimes also requires a historical study; 2) the description of the sequences of practices that are put into play in the resolution (Godino et al., 2019, p. 39).

To study the nature of mathematical objects and their knowledge, we apply the ontosemiotic configuration tool of practices, objects, and processes in its dual versions, epistemic (institutional meanings) and cognitive (personal meanings). The ontosemiotic analysis — i.e., the characterization of the systems of practices, the objects involved in these practices, and the semiotic functions they establish — provides an answer to the ontological and semiotic-cognitive problems of mathematics education, permitting the description of institutional and personal knowledge (Chapters 2 and 3). At a microscopic level, it allows us to identify the meanings put into play in a specific mathematical activity, for example, the use of terms and expressions. At a more general or macroscopic level, it helps to describe the semiotic structure of a complex mathematical organization implemented in a particular study process or by a student when working with a task or throughout an instructional process (Burgos et al., 2021; Font & Contreras, 2008).

At both levels, ontosemiotic analysis helps identify discordances or disparities between the meanings attributed to the expressions of two subjects (persons or institutions) in didactic interactions. These semiotic conflicts can explain, at least partially, learners' potential or actual difficulties in the instructional process and identify the limitations of mathematical competences and understandings involved. The information obtained from this analysis is necessary to approach the design and implementation of the instructional process using rigorous criteria and to determine the materials and time resources.

Methodological tools for analyzing educational-instructional problems

To address the educational-instructional problem, i.e., the inquiry into teaching and learning, how they relate together, and the identification of the conditioning factors determining their optimization, OSA has developed several specific methodological tools, in particular the notion of didactic configuration (Chapter 4). There are three components in every didactic configuration: a) an epistemic configuration (system of institutional mathematical practices, objects and processes required in the task), b) an instructional configuration (system of teaching roles, learners and instructional media used and their interactions) and c) a cognitive-affective configuration (system of personal mathematical practices, objects and processes describing learning and the associated affective components). The sequence of didactic configurations (didactic trajectory) accounts for the articulation of different configurations and their evolution over time.

For identifying conditioning factors in instructional processes, OSA offers the normative dimension tool. This approach considers the norms, habits, and conventions, generally implicit, that regulate the operation of mathematics classrooms and condition, to a varying degree, the knowledge that students construct. Regarding the optimization of instructional processes, a didactical suitability methodological tool was developed (Chapter 5).

In Godino et al. (2014), we find the application of the OSA's various theoretical notions to address the overall educational-instructional problem through an interpretation of the didactic engineering methodology, which is understood in a generalized sense, as described in Godino et al. (2013).

Methodology to assess mathematical knowledge

Wheeler (1993) posed the evaluation problem of mathematical knowledge from an epistemological dimension:

> If we need to assess students' mathematical knowledge for multiple purposes, the first question to be addressed concerns the nature of the knowledge itself. The reason given by this author is as follows: "How can we assess what we do not know? one purpose?" (Wheeler, 1993, p. 87).

In OSA, this problem corresponds to the characterization of meanings. Specifically, one purpose of the epistemology of mathematical knowledge proposed by OSA is to provide criteria for the elaboration of a theory of its evaluation, which previously needs to adopt or elaborate a theory about its nature, variety, and structure.

The determination of personal knowledge necessarily requires inference processes based on the sets of practices observed in the evaluation situation, whose validity and reliability must be guaranteed. The complexity of this process is deduced from the fact that not only are there interrelationships between knowledge referring to different mathematical objects but also that, even for a given mathematical object, a subject's knowledge about it cannot be reduced to a dichotomous state (knows or does not know) or a unidimensional degree or percentage. This makes it difficult to apply classical psychometric theories of domain mastery or latent trait (Snow & Lohman, 1991; Webb, 1992).

The observable character of social practices allows appropriate phenomenological and epistemological studies to determine the associated problem field and institutional meanings for an object embodied in the corresponding epistemic configurations. Analysis of the didactic variables of the problem field provides a criterion for structuring the population of possible tasks from which a representative sample can be drawn if the content validity of the assessment instrument is to be guaranteed. These two elements provide initial reference points for designing relevant situations to assess personal knowledge and to design appropriate didactic engineering.

7.4. OSA as a theoretical framework for research

Each theory in the OSA makes it possible to address specific research questions on partial aspects relevant to mathematics education. However, the approach and solution to substantive problems in teaching and learning mathematical content when involving various factors requires a systematic articulation of the different theories within the OSA.

As an example of the use of the OSA theoretical system to design, implement, and evaluate teaching and learning processes of specific mathematical content, in this section, we describe the questions of a research project on teacher education. This example was a doctoral thesis by Verón (2023) entitled "Ontosemiotic model of the concept of differential. Implications for mathematics teacher education".

The author is a mathematics teacher at the *Instituto Superior de Formación Docente* in Argentina, teaching a Seminar on Didactics of Mathematics in the fourth year of the Secondary Education Teacher Training Course in Mathematics. Prospective mathematics teachers (PMT) take courses in Mathematical Analysis in their curriculum and must be trained to teach the fundamentals of infinitesimal calculus to high school and university students. From his teaching experience, the author knows that the concept of differential calculus is difficult for students. At the beginning of his training as a researcher, he decided to investigate how he should train prospective teachers so that they can teach the concept of differential well. Before addressing an actual educational-instructional problem, it is necessary to problematize the nature of mathematical activity, emerging objects, and their meanings (Verón & Giacomone, 2021):

–What is the concept of differential? (ontological problem)

–What different meanings are referred to as differential? (semiotic problem)

To answer these questions, historical and epistemological works on infinitesimal calculus have been studied, with a focus on the concept of differential. In this way, the author attempts to identify the types of intra-mathematical and extra-mathematical problems in whose solution the differential object intervenes, that is, to answer the following questions:

- Which types of problems and systems of operational and discursive practices solve these problems in which the concept of differential occurs?
- What are the various pragmatic meanings of the differential, what elements allow them to be distinguished in terms of generality and formalization, and how are they articulated?

It is also necessary to analyze mathematical cognition to characterize the types of personal meanings of students regarding the differential through a study of the mathematics education bibliography, answering the following question:

- What types of personal meanings (knowledge, incorrect conceptions) do Calculus students possess regarding the differential?

With the information provided by answering these questions, the educational-instructional problem of developing students' mathematical and didactic-mathematical knowledge of the differential object is approached in a grounded manner. Having fixed the subjects and educational context, in this case, prospective secondary school mathematics teachers who have already studied differential in mathematics courses, the problem of developing mathematical knowledge, specialized didactic-mathematical knowledge and competences concerning analysis and didactic intervention is faced. It is therefore necessary to address issues related to didactic design for learning both mathematical and didactic-mathematical content.

- Which partial meanings of the differential should be selected so that the PMTs can deepen their common and extended knowledge of this object and its relationships to other content?
- Which problems can be selected to generate partial meanings and determine their relationships?
- What kind of didactical configurations are suitable for studying selected partial meanings and the specialized didactical-mathematical knowledge of differential concepts?

Next, it is necessary to raise questions related to the development of PMTs' analysis and didactic intervention competences. It is necessary for PMTs to become familiar with the OSA's tools for analyzing mathematical and didactic activities so that they can select problems, identify meanings, and reconstruct the configurations of mathematical practices, objects, and processes.

- What training actions are necessary and possible to implement as part of a training program to develop prospective teachers' knowledge and competence in the ontosemiotic analysis of mathematical activities involved in using differentials?
- What aspects and criteria should teachers consider when optimizing the teaching and learning processes of the differential concept?
- What kind of training actions are necessary and possible to implement in a training program to develop the prospective teachers' knowledge and competent use of didactical suitability and to systematically reflect on the process of studying the differential?

In the different chapters of his thesis, Verón describes the a priori analysis, design, and implementation of educational tasks related to the concept of differential, adapted to the context and the temporal and technological resources available, and raises questions related to the evaluation of that experience (Verón et al., 2024):

– What is the degree of didactic suitability of the training process regarding the global meaning of the concept of differential, which is implemented in the initial training of mathematics teachers?
– What changes should be made in the design and implementation of the training process to increase its didactic suitability for future application in mathematics teachers' initial training?

7.5. Concordances and complementarities with other theories

In this section, we identify the concordances and complementarities of OSA with other theoretical frameworks that address the educational-instructional design issues. These are the Theory of Didactic Situations in Mathematics (TDSM, Brousseau), the Anthropological Theory of Didactics (ATD, Chevallard), Realistic Mathematics Theory (RME, Freudenthal), APOS Theory (Dubinsky), Cultural Theory of Objectification (TO, Radford), and the Ethnomathematics Program (D'Ambrosio). For each of these theories, we identify the assumptions and constructs that characterize them in the facets indicated in Figure 7.1 and the concordances and complementarities with those proposed by the OSA.

7.5.1. Theory of didactic situations

Main theoretical elements

Epistemic and ecological facets

In the TDSM framework, the knowledge to be taught has a cultural existence pre-existing and, to some extent, independent of the people and institutions interested in its construction and communication. Mathematical knowledge is a special form of institutionalized knowledge that is usually recorded in an axiomatic form that depersonalizes and decontextualizes it.

"This knowledge, whose text already exists, is not a direct production of the teacher, it is a cultural object, quoted or recited" (Brousseau, 1986, p. 73). Brousseau uses the term "knowledge" in connection with the qualifier "formal knowledge", "erudite knowledge", "theoretical knowledge", "practical knowledge", which indicates that it is interpreted as something external or institutional, as an element of reference for teaching and learning. Didactic transposition accounts for the adaptations of this knowledge to be studied in the school context.

In TDSM, meaningful mathematics learning is a fundamental objective.

> The meaning of mathematical knowledge is defined - not only by the set of situations in which this knowledge is achieved as mathematical theory (semantics in Carnap's sense), not only by the set of situations in which the subject has found this knowledge as a solution, but also by the set of conceptions, previous choices he rejects, errors he avoids, economies he provides, formulations he takes up, etc. (Brousseau, 1983, p. 170).

Affective and cognitive facets

Among the notions used in TDSM to refer to the " subject's knowledge", we find the use of "representation" in the sense of internal representation; sometimes Brousseau uses the expression "implicit models" for such knowledge and representations. He interprets implicit models as "forms of knowledge" that neither function completely independently nor in a fully integrated manner to control the subject's interactions. The notion of model is central to describing computational procedures, formulation results, and learners' knowledge when confronted with a situation. Thus, he defined:

- Action model: The calculation procedure produces either a strategy (valid for all cases) or a tactic (specific for some concrete cases).
- Explicit model: Result of a formulation situation that can be developed using known or new signs and rules.
- Implicit model: Simplified representation of sufficient knowledge to characterize observed behavior in a situation.

The TDSM respects the contributions of psychology when studying the construction of knowledge by a subject. Knowledge evolves according to complex processes. Explaining these evolutions solely through effective interactions with the environment is certainly a mistake because very early children can internalize the situations that interest them and operate with their "internal representations", which are important mental experiences. In this way, they solve problems of assimilation (increasing already acquired schemas by adding new facts) or accommodation (reorganization of schemas to learn new questions or to resolve contradictions). For the learner to "construct" knowledge, it is necessary for them to take a personal interest in solving problems posed in a didactic situation. In this case, it is said that the situation has been returned to the learner. "Devolution is the act by which the teacher makes the learner accept responsibility for an (adidactic) learning situation or problem and accepts the consequences of this transfer of responsibility" (Brousseau, 2002, p.230). The expectation is that through interaction with an appropriate medium, learners will progressively construct knowledge collectively, rejecting or adapting their initial strategies if necessary.

Instructional facet

In this facet, we find the Piagetian constructivist postulate and constructs such as the didactic contract, types of didactic situations, didactic obstacles, and didactic phenomena. The central aim of the TDSM is to investigate the conditions that teaching should meet to provide meaning to the mathematical knowledge that is the learning objective. Its basic hypothesis is that the knowledge constructed or used in each situation is defined by its constraints. Thus, by creating certain artificial restrictions, the teacher can encourage students to construct a particular type of knowledge. This hypothesis is certainly closer to constructivism than to approaches derived from the Vygostskian notion of the zone of proximal development (Sierpinska & Lerman, 1996).

A basic construct of the TDSM is the didactic situation defined as the set of explicit and implicit relationships established between a learner or a group of learners, some environment (including instruments or materials), and the teacher for enabling learners to learn—that is, to reconstruct—some knowledge. The situations are knowledge-specific. The learning theory assumed by the TDSM is constructivist because it is interested in determining how subjects construct and communicate mathematical knowledge in problem solving. Problems should be selected in such a way that they optimize the learning adaptive dimension and students' autonomy.

> The pupil's intellectual work must, at some points, be comparable to a mathematician's scientific work. To know mathematics is not only to learn definitions and theorems, to recognize when to use and apply them; we know that doing mathematics involves dealing with problems (...) A good reproduction by the pupil of a scientific activity requires him to intervene, to formulate, to test, to construct models, languages, concepts, theories, to exchange them with others, to recognize those which conform to the culture, to take those which are useful to him, etc. (Brousseau, 2002, p. 22).

In the TDSM, the artificial genesis of a mathematical concept results from a sequence of the following types of situations or states of a didactic contract:

- Action-focused situations in which students first attempt to solve a problem proposed by the teacher.
- Communication-focused situations in which students communicate the results of their work with other students and the teacher.
- Validation-focused situations in which theoretical rather than empirical arguments are used.
- Institutionalization situations, in which the results of the negotiations and conventions of the previous phases are summarized, with a focus on "important" facts, procedures, ideas, and official terminology.

From the institutionalization phase onwards, the meaning of terms is no longer an object of negotiation but of correction, regarding to definitions, notations, theorems, and accepted procedures. Within each of these situations, there is a didactic component, i.e. a space and time where the management of the situation falls entirely on the students. This is considered the most important part because, in fact, the aim of teaching is what Brousseau calls the return of the problem to students.

OSA interpretation

We believe that the progress made by the theory of situations in genetically connecting mathematical knowledge to problem-situations is fundamental; however, we believe that the analysis of the constituents of knowledge is insufficient because situations are not the only constituents of the same. In the TDSM, even if implicit, proposals are found to progress in the controlled decomposition of knowledge. Although there are situations of action that provide the occasion for developing and applying mathematical techniques to solve problems, situations of formulation-communication in which linguistic instruments intervene essentially, and situations of validation in which validating objects (argumentations or demonstrations) intervene, concepts and theorems are essential constituents of the discursive component of knowledge, both in their personal (conceptions, concepts and theorems in action) and institutional (mathematical concepts and theorems) versions.

The TDSM is an experimental epistemology of mathematics, a theory about the characteristics that teaching-learning situations must allow students to reconstruct and reinvent mathematical knowledge autonomously by solving problems, especially those chosen by the teacher. The fundamental epistemological assumption is that knowledge emerges from problem-solving activities from both professional and educational perspectives. This postulate is fully shared with OSA. However, in the TDSM, the ingredients of this activity—the diversity of objects involved—, are not modeled, except for the problem component, the resolution of which gives meaning to knowledge. The mathematical world furniture in the TDSM is excessively austere from the OSA perspective. The ontology and semiotics of mathematics, from both professional mathematical culture and educational mathematics perspectives, are limited, which have consequences for modeling the subject's cognition and for the design, implementation, and evaluation of educational-instructional processes.

We interpret the characteristics that TDSM attributes to the meaning or sense of knowledge in terms of the pragmatic meaning of an object proposed by OSA, as a system of operative and discursive practices in which that knowledge (object, knowledge) participates relevantly to respond to a class of problems. Absent in the TDSM is the explicit recognition of the plurality of meanings of an object (knowledge), its relativity to the institutional framework, the subject, and the contexts of use. From our perspective, the TDSM and the research methodology described as didactic engineering are not conceived as an "instructional theory", but constitute an experimental epistemology for the didactics of mathematics. Likewise, they incorporate or assume a constructivist-Piagetian theory for mathematical learning and a positivist-experimental approach to the didactics of mathematics, whose aim must be to discover didactic phenomena and construct teaching situations that necessarily produce the intended learning.

7.5.2. Anthropological theory in didactics of mathematics

Main theoretical elements

Epistemic and ecological facets

The Anthropological Theory in Didactics of Mathematics (ATD) that Chevallard and collaborators have developed (Chevallard, 1992; 1997; 1999) provides basic elements of an epistemology of mathematics, which broaden and deepen the theory of knowledge that serves as the basis for the TDSM. The notions of mathematical praxeology and institutional and personal relation to the object are useful extensions of the TDSM concepts of knowledge and knowing. This is Chevallard's (1999, p. 229) definition of praxeology:

> Around a type of task T, in principle, a triplet is found formed by at least one technique, τ, by a technology of τ, θ, and by a theory of θ, Θ. The total, indicated by [T/τ/θ/Θ], constitutes a punctual praxeology, where this last qualifier means that it is a praxeology relative to a single task, T. Such a praxeology—or praxeological organization—is thus constituted by a practical-technical block, [T/τ], and by a technological-theoretical block [θ/Θ].

Techniques are described as ways to perform tasks. A technique is not necessarily algorithmic or quasi-algorithmic; it is used only in rare cases.

Both TDSM and ATD share the view of mathematics as a human activity, oriented toward resolving certain types of tasks or problem questions, as with OSA. In ATD, doing mathematics comprises activating a mathematical organization, i.e., solving certain types of problems with certain types of techniques (the know-how) in an intelligible, justified, and reasoned way (through the corresponding knowledge). The ATD highlight questions often co-disciplinary in which various praxeological systems are involved. The tool scale of co-determination levels helps to focus attention on the different types of constraints to which didactic action is subject, from the level of civilization to the level of the specific mathematical subject addressed (Chevallard, 2019).

Cognitive and affective facets

ATD describes the cognitive dimension in terms of the personal relationship to the object, which includes all the other concepts proposed by psychology (conception, intuition, schema, internal representation, etc.).

> An object exists as soon as person X or an institution I recognizes this object as an existent (for him/her). More precisely, object O will be said to exist for X (resp., for I) if there is an object, which I represent by R (X, O) [resp., R(O)], which I call personal relation from X to O (resp., institutional relation from I to O). (Chevallard, 1992, p. 9)

This notion has not been developed by postulating the prior and determinants of the characterization of mathematical praxeologies and the study of institutional relations. In fact, local praxeology constitutes the minimum unit of analysis of didactic processes (Bosch & Gascón, 2005). There is no question of structures or mental models in this notion, but an attitude (rapport), a "relation to" and a "functioning with" regarding what an institution defines as knowledge; it is known or not, only in relation to the opinion of an institution, not in an absolute sense (Arsac, 1992). Therefore, the psychology of learning or knowledge studies is not of interest, but rather the anthropological analyses of institutions.

Instructional facet

The Theory of Didactic Moments, complemented by the Research and Study Path (RSP), extends and qualifies the types of adidactic situations proposed in the TDSM and provides criteria for designing and managing instructional processes. The aim of a teaching-learning process can be formulated in terms of the components of the mathematical organizations (mathematical praxeologies) that are to be reconstructed: what types of problems should one be able to solve, with what types of techniques, based on what descriptive and justifying elements, in what theoretical framework, etc.

ATD proposes a model for studying mathematics in terms of didactic moments (Chevallard, 1997). The essential types of didactic moments in studying a mathematical organization are: the first encounter moment, exploratory; technique work, technological-theoretical; institutionalization; and evaluation. The instructional design component of ATD was reinforced by introducing the notion of RSP and the changes in the educational paradigm that it entails (Chevallard, 2009).

This involves placing the "questioning of the world" as the starting point for didactic action, i.e. starting from questions (situations-problems) central to mathematics or multidisciplinary, instead of starting with knowledge, considered as "works or monuments" that are "visited". Instead of students encountering the mathematical works of the program through a multiplicity of study and enquiry activities, each of which starts from a different question and mobilizes different "auxiliary" works, an investigation is made into how to achieve a strong degree of integration, deriving a whole set of questions Qi from a "parent" question Q*. "In this way, the Q* question requires an enquiry, which takes the form of a certain path of study and research" (Chevallard, 2009, p. 26).

A key issue here is the generativity of the starting Q* question, which should allow the generation of derived questions that broaden the range of praxeologies involved and, thus, can be studied. The design of RSPs must be such that, (1) they have a broad mathematical orientation and are not focused on an isolated and specific concept or topic; (2) the program of a course can be studied through a finite number of "big questions": an RSP appears as a true "discovery tour", as a "program of study and research".

OSA interpretation

From the OSA perspective, ATD theoretical approaches have some limitations in supporting research in mathematics education. We highlight the following points:

- The epistemological and anti-psychological emphasis, by which the psychological explanation of some didactic phenomena is not granted space, limits the use of the anthropological perspective in the study of educational-instructional processes.
- The desire to redirect everything toward the institution without valuing and studying the individual is limited. In our opinion, the complex phenomenon of mathematics learning is not entirely explicable in terms of adherence to a particular institution.
- ATD offers powerful theoretical tools to study mathematical organizations, their ecological relationships, and the institutional constraints that condition their evolution and development. However,

the subject-institution identification prevents it from accounting for the conditions under which learning occurs.

The level of analysis of mathematical organizations allowed by the ATD in terms of quartet tasks, techniques, technologies, and theories does not deepen the ontosemiotic complexity of these organizations. Making explicit the system of conceptual, propositional, and argumentative rules in the technological-theoretical block proposed by OSA enables us to recognize the complexity of representation and interpretation processes and the capacities necessary for students to understand and follow these rules. Didactic research should focus not only on the ecology of mathematical organizations but also on their accompanying cognitive-affective phenomena, which may explain learning difficulties and enable identification of the didactic resources necessary for their achievement.

7.5.3. Realistic mathematics education

Realistic Mathematics Education (RME) is largely based on Freudenthal's (1973; 1983; 1991) reflections on mathematics and its learning. Although it initially emerged in the Netherlands at the IOWO and developed at the Freudenthal Institute (Utrecht University), RME has expanded to other countries around the world (Phan et al., 2022). It is presented as an innovative theory of instruction specific to mathematics, with one distinctive feature being that it gives a prominent position in the learning process to the use of "realistic" situations. These situations serve as a source to start the development of mathematical concepts, tools, and procedures and as a context in which learners can, at a later stage, apply their mathematical knowledge, which then gradually becomes more formal and general and less context-specific (Van den Heuvel-Panhuizen & Drijvers, 2014, p. 521).

Main theoretical elements

Epistemic and ecological facets

Freudenthal considered mathematics as a human activity that should not be learned as a closed system but as a mathematization of reality. Mathematicizing is not only axiomatizing, formalizing, and schematizing but also every organizing activity of the mathematician, which can refer to mathematical content or expression, even realized in the most naïve, intuitive, or lived experience and expressed in everyday language.

Freudenthal (1983) proposed mathematical concepts, structures, and ideas to organize phenomena in both the real world and mathematics. By using geometrical figures, such as triangles, parallelograms, rhombuses, or squares, the world of the phenomena of shapes is organized; numbers organize the phenomena of quantity. At a higher level, the geometrical figure phenomenon is organized by geometrical constructions and demonstrations; the decimal system organizes the number phenomenon.

The phenomenology of a mathematical concept, structure, or idea means, in Freudenthal's terminology, to describe this construct (noumenon) in relation to the phenomena that it permits organizing. If, in this relation between noumenon (construct) and phenomenon (phenomenon), the didactic element is emphasized, i.e., if attention is paid to how such a relation is acquired in a teaching-learning process, one speaks of the didactic phenomenology of this noumenon.

A related element of the ecological facet of mathematical knowledge is the *interweaving principle* (Van den Heuvel-Panhuizen, 1996), where mathematical content domains, such as number, geometry, measurement, and data processing, are assumed not to be isolated chapters of the curriculum but are strongly integrated. Students are offered problems in which they can use various mathematical tools and knowledge. This principle also applies to domains. For example, in the domain of number sense, mental arithmetic, estimation, and algorithms are taught in close connection.

Cognitive and affective facets

For Freudenthal, a mathematical concept (number, group, etc.) is a cultural object fixed by decontextualized and depersonalized definitions and properties. He proposes that learning should occur through resolving problems belonging to the subject's sphere of "reality", rather than through a more or less formal "acquisition of the concept", understood in mathematical culture Through phenomenology, the subject does not acquire the concept but forms a mental object through which he interprets and understands the phenomena for which the mathematical object (number, function) is a means of organization.

> Leibniz and John Bernoulli used the word "function" for something that was no more than a mental object, and only with the first appearance of a symbol letter for a function in the works of D'Alembert and Euler the way was paved for the concept of function. The distance between a mental object and concept depends on the subject matter, but even more on the individual and his/her particular situation (Freudenthal, 1991, p. 19).

He rejects concept concretizations as a learning tool, seeing them as usually false or too rough to reflect the essential features of concepts, even if, by a variety of "concrete materials", one wishes to capture more than one facet. Didactically, this means putting the cart before the horse: teaching abstractions by making them concrete. What a didactic phenomenology can do is to take the opposite approach: beginning with the phenomena that need to be organized and teaching the student to manipulate those organizing tools from that starting point. In the didactic phenomenology of length, numbers, etc., phenomena organized by length, number, etc., are shown as widely as possible. To teach groups, instead of starting with the group concept and seeking materials that make that concept concrete, one should first seek phenomena that could compel the student to constitute the mental object that is mathematically formulated by the group concept. If, at a given age, such phenomena are not available to students, one abandons the futile attempt to inculcate the group concept. For this opposite approach, Freudenthal avoids talking of concept acquisition. Instead, he speaks of the constitution of mental objects, which, in his view, precedes concept acquisition and can be highly effective even if concept acquisition does not follow.

The level principle, which underlines that learning mathematics involves learners going through several levels of understanding, is assumed: from informal solutions related to the context through the creation of several levels of shortcuts and schematizations to the acquisition of knowledge about how concepts and strategies are related.

Instructional facet

The instructional component of MRE is reflected in the principles of activity, reality, interactivity, and orientation (Van den Heuvel-Panhuizen, 1996):

– Activity principle: Students should be active participants in the learning process. It also emphasizes that mathematics can be best learned by doing mathematics, which is strongly reflected in Freudenthal's interpretation of mathematics as a human activity and his idea of mathematization.

– Reality principle: This principle expresses the importance of developing students' ability to apply mathematics in solving "real life" problems. This means that mathematics education should start from problem situations that are meaningful to students, which gives them the opportunity to attribute meaning to the mathematical constructions they develop while solving the problems. Instead of starting with the teaching of abstractions or definitions that will be applied later, in RME teaching starts with problems in rich contexts that require mathematical organization or, in other words, that can be mathematised and put learners on track of informal context-related solution strategies as a first step in the learning process.

– Principle of interactivity: Learning mathematics is not only an individual activity but also a social activity. Whole-class discussions and group work are encouraged, giving students the opportunity to share their strategies and inventions with others. In this way, students can acquire ideas to improve their strategies. The interaction also provokes reflection, which enables students to reach higher levels of understanding.

– Guiding principle: This refers to the "guided reinvention" of mathematics. Teachers should play a proactive role in students' learning, and educational programs should contain scenarios that can function as levers to improve students' understanding. Teaching and learning programs must be based on long-term, coherent teaching-learning trajectories.

OSA interpretation

The consideration of mathematics as a human activity and its central role in in resolving internal or external problems, including those of everyday life, is consistent with anthropological approaches to the philosophy of mathematics and the OSA. The MRE assumes that mathematical concepts and structures serve to organize phenomena, both of the real world and mathematics itself; thus, we can infer that mathematics, in addition to being an activity, is a system of objects with a reality external to the subject. This is another concordance with the ontological presuppositions of OSA. However, in MRE, we do not find a clear and explicit position on the nature and diversity of mathematical objects and the attribution of multiple meanings. We understand institutional mathematics is conceived in the MRE in a formal, abstract, axiomatic, decontextualized, and depersonalized manner, stripped of any sensory connotation. For OSA, there also exist applied mathematics and school mathematics that do not have these characteristics and can serve as a reference to guide educational-instructional processes.

The relations of mathematics to the real world and its emergence from problem solving, mediated by material and linguistic artifacts, form a basic postulate of OSA's holistic, pluralistic, and ecological vision, which is consistent with the RME's principle of entanglement. Freudenthal's phenomenological analyses of mathematical concepts and structures are undoubtedly rich, but the ontosemiotic configuration tool of practices, objects, and processes can complement them and help to understand learning conflicts.

According to Freudenthal, learning should be oriented toward the constitution of mental objects and not toward the acquisition of concepts. We can infer that he assumed a notion of mathematical concepts as cultural, abstract, formal, and decontextualized objects, while the mental object reflects the cognitive state of the subject when approaching the resolution of realistic problems. He believed that the constitution of mental objects must be established before the acquisition of concepts. These distinctions can be related to the personal and institutional duality of OSA practices, objects, and processes. The constructs of cognitive and epistemic configuration can help describe the processes of the constitution of (personal) mental objects and their relation to institutional ones, which have diverse meanings, not only the formal ones emphasized by Freudenthal.

The MRE level principle in mathematical learning recognizes that students go through different levels of understanding mathematical objects. This is consistent with the recognition of a diversity of institutional meanings with varying degrees of formalization. These factors are considered in the design of educational-instructional processes and thus in students' learning.

The RME principles of activity, reality, and interactivity in the design of instructional processes are compatible with the OSA ontosemiotic model. The principle of guidance or guided reinvention, in which a proactive role for the teacher in learning is recognized, is also acceptable in the first approximation. However, configuration and didactic trajectory tools, supported by the ontosemiotic model of mathematical knowledge, provide elements of detailed analysis of teaching and learning activities. The recognition of the ontosemiotic complexity of learning objects leads to the proposal that the students' first encounter with a new object may require the implementation of a different type of didactic configuration than in the case of exercise or application moments (Chapter 4). Likewise, the OSA's anthropological and conventionalist assumptions about regulative mathematical objects (definitions, propositions, procedures) qualify the constructivist assumptions of learning, such as the idea of students' reinvention of knowledge.

7.5.4. APOS theory

APOS (acronym for Action, Process, Object, and Schema) is a theory with a cognitive orientation toward mathematics education problems that proposes models to investigate the types of mental constructions a student may engage in while learning mathematical concepts (Arnon et al., 2014). This serves as an evaluative framework because individuals are observed in problem situations in which the researcher attempts to describe their understanding level and the mental structures involved in their learning of the concept. It also provides tools for designing pedagogical activities and environments that promote learning development through a social approach, considering that learning is fostered by cooperative patterns of interaction. Dubinsky (1984) introduced the primary ideas, although the acronym APOS was introduced in Cottrill et al. (1996).

Main theoretical elements

Cognitive and affective facets

The basic principle of APOS theory is that an individual's understanding of a mathematical topic develops through reflection on problems and their solutions in a social context and through the construction of specific mental structures organized in schemas to be used for solving new situations. Starting from the concept of reflective abstraction, "attempts to elaborate a theoretical framework that can be used to describe any mathematical concept together with its acquisition" (Dubinsky, 1991, p. 97). Reflective abstraction is the construction of mental objects and actions on such objects. In the development of logical-mathematical thinking, five types of actions are distinguished: internalization, coordination, encapsulation, generalization, and reversal.

The notion of schema was adopted and interpreted as a coherent collection of objects and processes. "A subject's tendency to invoke a schema in order to understand, deal with, organize, or make sense of a given problem situation is his or her knowledge of a particular mathematical concept" (Dubinsky, 1991, p. 103). Schemes exist for situations involving numbers, arithmetic, functions, propositions, quantifiers, and proofs by induction. These schemas must be interrelated in a large, complex organization. One aim of APOS is to isolate small portions of this complex structure and provide explicit descriptions of possible relationships among schemas. This description of the relations between schemas concerning a concept is the genetic decomposition of the concept. This is a description of the specific mental constructs that learners bring into play when developing their understanding of a mathematical concept.

Instructional facet

APOS has developed a mathematics teaching model based on previously developed cognition theory, which it has termed the ACE (Activities, Classroom discussions, Exercises) cycle, comprising of three components: (A) Activities; (C) Classroom discussions; and (E) Exercises (Arnon et al., 2014). APOS has also developed a curriculum/instructional design research model that can be related to didactic engineering or didactic design research. The model distinguishes three interrelated components: theoretical analysis, instructional design and implementation, and data collection and analysis. The research begins with a theoretical analysis of the mathematical concept cognition under consideration, leading to a preliminary genetic decomposition of the concept. This provides a basis for designing activities that encourage the mental constructs required for analysis. Various pedagogical strategies, such as cooperative learning, small group problem solving, and lectures, can be very effective in helping students learn mathematics. Finally, data collection and analysis are conducted using APOS's theoretical lens on mathematical cognition. The analysis focuses on whether students made the intended mental constructions and on revising their initial genetic decomposition and activities in the case of a negative response, starting a new cycle of research (Arnon et al., 2014).

OSA interpretation

Although a mentalistic language is used, the information used to elaborate a genetic decomposition of the concept (GCD) is constituted by the subjects' operational and discursive practices (manifestations, behaviors), either an ideal epistemic subject (concretized in the researcher elaborating some expected solutions to the tasks) or concrete subjects presented with the tasks. Through the analysis of the subjects' responses, the conclusion can be that the CDF was inadequate, leading to its re-elaboration and re-experimentation. APOS assumes a conceptualist view of mathematics and a mentalist/cognitivist view of mathematical learning. However, its starting point is the Piagetian theory in which the subject's problem-solving activity is key to the genesis of knowledge. Reflective abstraction and the mechanisms of assimilation and accommodation are the basis of the model.

APOS's cognitivist view of mathematical concepts can be enriched with the historical-cultural, institutional approach. This perspective leads us to recognize that each concept has different partial meanings articulated to varying degrees of formalization and generality, and that each partial meaning entails a variety of the object, the learning of which must be the focus of attention. The DGC of, for example, the concept of function, derivative, fraction, etc. should be performed for each variety of such objects. In addition, each meaning entails a configuration of specific practices, processes, objects, and schemas, which must be constructed and articulated. In other words, OSA provides a more complex view of the nature of mathematical knowledge, understanding, and competence, requiring the identification of networks of referential and operational semiotic functions, where APOS observes actions, processes, objects, and schemas associated with a concept.

The ontological and semiotic problematization of mathematical concepts from an institutional, i.e. cultural-historical perspective, is at the foundation of OSA. Mathematical cognition, from a personal point of view, corresponds to institutional cognition. From the perspective of educational-instructional design, the ontosemiotic model, as a first step, leads to the reconstruction of a reference meaning in which various partial meanings or senses of the intended content and associated ontosemiotic configurations are articulated. These tools broaden the perspective of the genetic decomposition of concepts.

7.5.5. Objectivation theory

Radford (2008; 2014) developed the Theory of Objectification (TO), which was inspired by anthropological and cultural-historical schools of knowledge. It is supported by non-rationalist epistemology and ontology that gives rise to an anthropological conception of thinking and to an essentially social conception of learning. This model assumes the following two principles:

1. Psychological dimension must be an object of study in mathematics education.

2. The meanings that circulate in the classroom cannot be confined to the interactive dimensions that occur in the classroom, but must be conceptualized within their cultural-historical dimensions. Learning is viewed as a social activity rooted in a preceding cultural tradition.

Main theoretical elements

Epistemic and ecological facets

The mathematical knowledge and learning epistemological principles that characterize TO are consistent with those assumed by sociocultural approaches. Radford (2018, p. 4066-7) formulates them as follows:

> p1: knowledge is historically generated during the individuals' mathematical activity.
>
> p2: the production of knowledge does not respond to adaptive piloting but is embedded in cultural ways of thinking that are imbricated with a symbolic and material reality that provides the basis for interpreting, understanding and transforming the individual's world and the concepts and ideas that they form

He introduces mathematical objects as constructs of ontological order, defined as fixed patterns of reflexive activity embedded in the constantly changing world of social practice mediated by artifacts. This implies a departure from Platonist and realist ontologies and their corresponding conceptions of mathematical objects as eternal objects that precede the individual's activity. It also introduces a semiotic-cognitive construct of objectification or subjective awareness of cultural objects. Learning is defined as the social process of objectification of external patterns of action fixed in culture that constitute mathematical objects.

Affective and cognitive facets

It accepts a non-mentalistic concept of thought. Thinking is, above all, an active reflection on the world mediated by artifacts, the body (through perception, gestures, movements, etc.), language, signs, etc. Knowing as a process ('knowing') is awareness in the course of a social, emotional, and sensitive process; it is a process mediated by material culture (signs, artifacts, language, etc.), the senses, and the body (through gestures, kinesthetic actions, etc.). The subject participating in objectification is a concrete subject and not the abstract epistemic subject of other theories (such as Piaget's and the Theory of Didactic Situations). It is a subject that feels, enjoys, and suffers. Radford defined the objectification process as follows:

> Subjectification consists of processes through which subjects take positions in cultural practices and become unique historical cultural subjects. Subjectification is the historical process through which the self is created (Radford, 2014, p. 142).

The subject is constituted as such through his or her actions, reflections, joys, and sufferings. However, the actions through which the subject is constituted are immersed in cultural and historical forms of action and relation to others.

Instructional facet

Learning is an activity through which individuals relate not only to the world of cultural objects (subject-object plane) but also to other individuals (subject-subject plane or plane of interaction) and acquire human experience in the common pursuit of the goal and the social use of signs and artifacts (Leontiev, 1993). Teaching and learning not only produce knowledge; they also produce subjectivities. As a result, we should try to understand the production of knowledge and subjectivities in the classroom and promote forms of pedagogical action that can lead to meaningful teaching and learning. Meaningful learning and teaching refer to pedagogical actions that involve:

(1) A deep understanding of mathematical concepts.

(2) The creation of a political and social space within which reflective, caring, and responsible subjectivities can develop.

The essential principle of objectification theory in the educational-instructional dimension is the idea of labor in the sense of Hegel, Marx, Leont'ev, and dialectical materialism. It is through labor or work that individuals continually develop and transform themselves, encountering systems of ideas of culture: scientific, legal, artistic, etc. systems of ideas. It is also through work that we discover cultural ways of being. In this framework, teaching and learning are not two separate processes, but rather a joint labor in the Hegelian sense. They are not two separate activities, one performed by a teacher who guides the learner and the other by a learner who does things by himself and for himself; they are a single, inseparable activity.

This theory adopts the Hegelian sense of objectification: something that is there and appears before the subject and is consequently presented as phenomenological theory. Objectification is the social, corporeal, and symbolically mediated process of becoming aware of and critically discerning historically and culturally constituted forms of expression, action, and reflection (Radford, 2014, p. 141).

OSA interpretation

The epistemological model proposed by OSA is broadly consistent with that of TO. Both theories share similar anthropological assumptions about mathematical activity and emerging sociocultural processes and products. OSA, however, explicitly incorporates the basic elements of the linguistic turn introduced by Wittgenstein in the philosophy of mathematics and the contributions of Peircean semiotics to describe and explain mathematical communication and interpretation.

Both TO and OSA assume the epistemological and ontological principles of mathematical knowledge and learning characteristics of sociocultural approaches. OSA shares a similar anthropological position on the nature of mathematics and emergent objects, but it adopts a broader perspective on mathematical objects, their types, nature, and functions. When one speaks of a mathematical object in TO, one apparently thinks of conceptual objects for which OSA has a double conceptualization:

- from a unitary perspective, as grammatical rules in Wittgenstein's sense (concept-definitions), and
- In a systemic sense, as a configuration of operational, discursive, and normative practices together with a network of other related objects and processes (ontosemiotic configuration).

The ontosemiotic configuration tool (in its double version, epistemic and cognitive), permits a detailed analysis of mathematical activity and the objects involved, which are not reduced to conceptual or abstract objects. Recognition of the complex network of objects and processes involved in problem solving is an explanatory factor of learning and teaching difficulties and a necessary step for the appropriate management of educational-instructional processes.

The objectification process is equivalent in cognitive and educational terms to that of the personalization of institutional/cultural meanings by learners proposed by OSA. Moreover, the view of conceptual objects in their unitary version as socially agreed rules for how languages and artifacts are used helps understand the two sources of learning proposed by TO: contact with the material world, the world of cultural artifacts around us (objects, instruments, etc.), and social interaction. What is to be learned are socially agreed rules for the use of artifacts.

OSA learning can be understood as a progressive coupling of personal and institutional meanings. Teaching involves the participation of the learner in the community of practice fixed by the institution where learning occurs and involves the acquisition by the learner of these institutional meanings. The principle of learning described by Radford can be assumed naturally in OSA:

> p3: learning is the acquisition of a piece of culturally-objectified knowledge that learners achieve through a social process of objectification mediated by signs, language, artefacts and social interaction as learners engage in cultural forms of reflection and action. (Radford, 2018, p. 4067).

It is accepted that learning, as a social process of objectification, entails of endowing meaning to the conceptual objects encountered by the learner in their culture.

The TO educational-instructional model, based on activity theory and the notion of the zone of proximal development, with the principle of "working together" (Radford, 2014), is assumed by OSA, although not exclusively. OSA assumes different types of didactic configurations that promote learning, depending on the types of knowledge sought, the subjects' initial state of knowledge, the instructional process context, and circumstances. Constructivist (autonomist), collaborative, personal, or masterful instructional models may have their place (Godino et al., 2006). When learning new and complex content, the transmission of knowledge at specific times by teachers or learner leaders within teams is crucial. Such transmission can be meaningful when students participate in activities and work collaboratively (Chapter 4).

A substantial difference between TO and OSA is the aims of didactic research. TO essentially aims to describe the subjects' processes of objectification and to relate/explain such processes in terms of teaching. "Objectification research focuses on how culturally and historically encoded forms of thought and action become objects of recognition or objects of consciousness" (D'Amore & Radford, 2017, p. 123). OSA, moreover, assumes the aim of studying the conditions of realization of mathematical and didactic activity in the most suitable way, considering the subjects and circumstances (Godino et al, 2019).

The emphasis of TO on the mathematical education's ethical and political dimension is included in the OSA through the affective (Beltrán-Pellicer and Godino, 2020) and ecological dimensions of didactic suitability, where training in democratic values and critical thinking is a criterion of suitability. The development of these humanist and ethical values should not, however, relegate the development of rationality and mathematical thinking.

We refer readers to Godino et al. (2020), who investigated the concordances and complementarities between TO and OSA based on empirical research on interpreting a Cartesian graph proposed within the TO framework.

7.5.6. Ethnomathematics program

Ethnomathematics is a research program with a consolidated international presence that proposes an expanded vision of mathematics and mathematics education (D'Ambrosio, 1985; D'Ambrosio & Knijnik, 2020; Oliveras & Godino, 2015). Vithal and Skovsmose (1997) describe four facets or fields of study in ethnomathematics:

1) History of mathematics. The traditional view of the history of mathematics has been criticized for ignoring, devaluing, distorting, or marginalizing the contributions of other non-European cultures to the body of knowledge referred to as Western mathematics.
2) Analysis of the mathematics of traditional cultures and indigenous peoples who were colonized while maintaining their original mathematical practices. These practices have been explored in relation to topics such as number systems, symbolism and gestural language, games and puzzles, geometry, space, shapes, forms, patterns, symmetry, art and architecture, time, money, networks, graphs, sand drawings, kinship relations, and artifacts.
3) Mathematics in everyday life. Analysis of mathematics used by different groups in everyday life settings to demonstrate the mathematical knowledge generated in a wide variety of contexts, both by adults and children.
4) The relationship between ethnomathematics and mathematics education. The connections (or lack of them) between the mathematics found in everyday contexts and those studied in the formal school system.

The ethnomathematical research program is interested in the sociocultural origins of mathematical knowledge by considering meaning, thinking, and reasoning as products of diverse social activities. It is part of the perspectives that characterize the social turn in mathematics education research and practice (Lerman, 2000).

Main elements of the program

Epistemic and ecological facets

One major aim of ethnomathematics research is to broaden understandings of the diverse nature of mathematics. This study claims the mathematical character of the practices of diverse cultural groups when dealing with certain professional and everyday activities. Mathematics is not only the product of a professional mathematician's activity, which is characterized by the use of formal languages, deductive argumentation, and the generality of theorems, but also the practices of diverse cultural groups.

Some authors have assumed key notions of Wittgenstein's philosophy, such as language play, forms of life, family resemblances, grammar, and rules, as the philosophical foundation of ethnomathematics (Vilela, 2010; Knijnik, 2012). These notions support and justify the socio-anthropological view of mathematics, characteristic of ethnomathematics, according to which the social practices of other cultures or ethnic groups in certain situations and activities are also mathematical practices.

Also highlighted as a field of enquiry for ethnomathematics is the study of political issues (power relations, dependence, subordination) involved in the development and study of mathematics as an academic discipline. The "naturalized" power relations between epistemological formations linked to social, ethnic, and cultural groups must be recognized (Knijnik, 2012). European mathematics is the only existing form of mathematics that has been introduced into school systems around the world that offers an alternative.

Cognitive and affective facets

A basic thesis of ethnomathematics is that mathematics education can be improved by considering students' cultural backgrounds and by providing an understanding of their achievements, attitudes, and motivations. It is interested in investigating the thought processes that characterize the mathematics of each culture, the cultural conceptions that permeate personal mathematical thinking, and in determining how students' self-esteem is affected by the school marginalization that can result from imposing academic/formal mathematical culture.

The importance of equity in mathematics education must be at the forefront. Therefore, the main goal of educators should be to achieve equity among learners and incorporate ethnomathematics into the classroom.

"Students learn in ways characterized by social and affective approaches, harmony with community, holistic perspectives, dependence on the environment, expressive creativity, and non-verbal communication." (Rosa & Shirley, 2016, p. 39)

It aims to relate the mathematical concepts of the school curriculum to the students' cultural background (D'Ambrosio 2001), thus enhancing their ability to make meaningful connections and deepen their understanding of mathematics.

Instructional facet (interactional and mediational)

From its beginnings, the ethnomathematics program involved two dimensions that have always remained closely related: field research and pedagogical work developed in school based on this research. Among the potential changes in education (curricula, resources and classroom practices) to consider the multicultural background of mathematics classes, Gerder (1996) stated:

- Incorporation of material from diverse cultures into the curriculum, thus valuing all students' cultural backgrounds.
- Incorporation of mathematical ideas from various linguistic and cultural groups within a country or region and/or developed by various social groups, such as basket weavers, potters, and house builders, into teacher education programs.
- Introduction of cultural elements that facilitate learning that most students recognize and value as belonging to their culture in textbooks.
- Development of materials that explore the possibilities of mathematical activities based on artistically appealing designs that belong to the students' culture (possibly in a broad sense) or that of their ancestors/parents.

OSA interpretation

The plural vision of mathematics advocated by ethnomathematics is consistent with OSA when considering how the notion of mathematical practice and the postulate of institutional and personal relativity of practices, objects, and meanings are conceived. Likewise, the consistency between both theoretical frameworks results from how the notion of institution is interpreted in the OSA, which encompasses any cultural or ethnic group, context, or community of practices.

The ontosemiotic configuration tool can characterize, in a detailed manner, the mathematical practices of cultural groups and thus describe and explain the differences and similarities between different "epistemic varieties" of mathematics. OSA postulates a relativism for mathematical practices, objects, and meanings but acknowledges ecological relations between the different epistemological formations that constitute diverse mathematics, whether linked to cultural or professional groups.

The analysis of the ethnomathematics program, in its educational component, reveals that a substantial part of is "instructional design-oriented research" (Oliveras & Godino, 2015). However, it lacks an explicit instructional theory to support the design, implementation, and retrospective analysis of its intended educational interventions. We often find ethnomathematical works that use tools from other frameworks (realistic mathematics education, didactic engineering, etc.), which results in a certain theoretical framework that is not always coherent and productive.

OSA provides analytical tools for analyzing the objects and processes involved in mathematical practices (system of practices, ontosemiotic configuration), tools for analyzing teaching and learning processes in the classroom (didactic configuration and trajectory), and meta-didactic reflection (normative dimension and didactic suitability). Therefore, OSA tools can aid in performing detailed descriptions of the mathematical and didactic practices claimed by Vithal and Skovsmose (1997) for educational experiences based on ethnomathematics. In turn, the ethnomathematical perspective can enrich the ecological facet of the educational-instructional processes proposed by OSA by incorporating analytical categories of the social and political components involved in mathematics education. Moreover, by considering the multicultural factor of educational contexts, this can guide the reconstruction of the reference meanings for the intended contents, which is required in the design of educational-instructional processes based on OSA.

7.5.7. Comparing theories regarding the understanding-use duality

In this section, we compare the five theories and OSA from the point of view of the understanding-use duality, according to the model proposed by Stokes (1997), which applies to both natural and social sciences, to classify types of research. To achieve this, Stokes uses a matrix with four cells, whose rows distinguish whether the research is inspired by the seeking of a fundamental understanding of phenomena and whose columns distinguish whether the research is inspired by practical application or use. Thus, quadrant I is considered basic applied research (e.g., that developed by Louis Pasteur); quadrant II, fundamental or basic-pure research (such as that of Niels Bohr); quadrant III, the identification of singular phenomena; and quadrant IV, pure applied research (e.g., that of Thomas Alva Edison). In Figure 7.4, we interpret these quadrants and indicate the position of OSA, along with the theories discussed in this section.

Scientific and technological research (quadrants I and II) aims to describe, explain, and predict phenomena; it is characterized by generality, control of variables, experimental design, and quantitative methods. Therefore, they are characterized by the paradigm of the natural sciences, basically positivist (Cohen et al., 2007). In the case of education, quadrant III can be represented by naturalistic enquiry in its different versions (ethnographies, case studies, biographies, etc.). Quadrant IV can be characterized by reflective practice (action research in its different versions); the focus is on the improvement of practice, either collaboratively or individually. Quadrants III and IV include investigations of singular phenomena, interpretation, ethnography, participant observation, and qualitative methods.

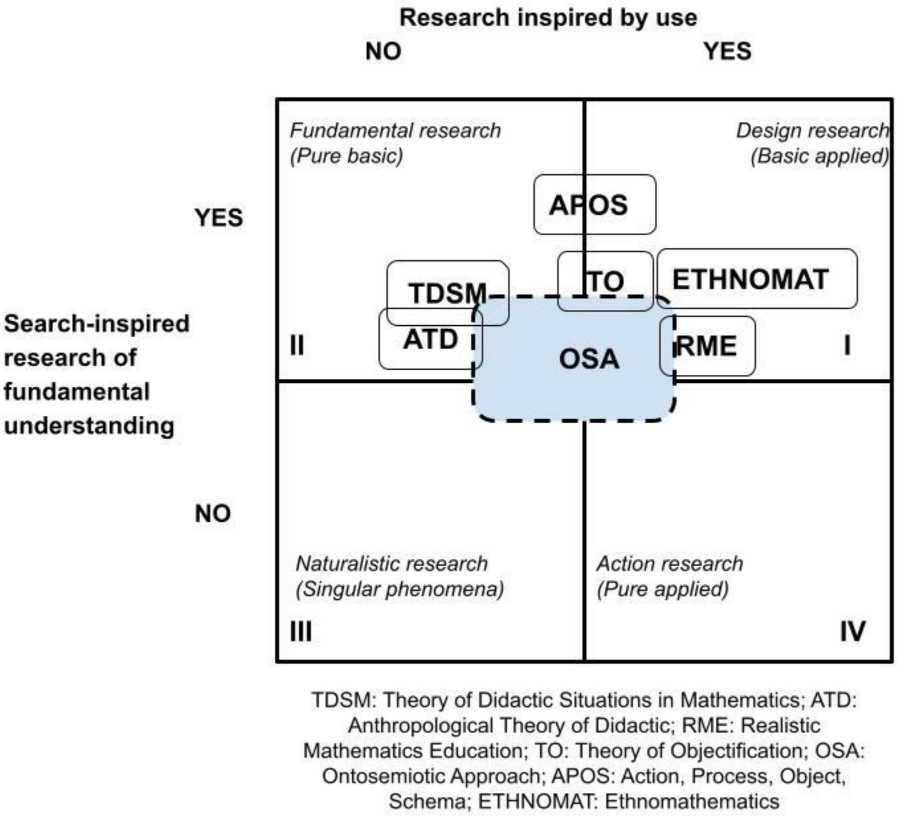

Figure 7.4. Research types according to the understanding-use duality

In Figure 7. 4 we have included, in quadrant I, Realistic Mathematics Education (RME) (Freudenthal, 1991; Van den Heuvel-Panhuizen & Drijvers, 2014) and the Ethnomathematics program (D'Ambrosio, 1985), in quadrant II, the Theory of Didactic Situations in Mathematics (TSDM) (Brousseau, 2002) and the Anthropological Theory of Didactics (ATD), although when didactic developmental engineering (ID-D) is applied, they also intersect with quadrant I. In both APOS and TO theory, we identify features of quadrants I and II.OSA includes principles and tools for conducting research focused on both understanding and use. The OSA Didactic

Suitability module is intended as a tool to support professional enquiry, and therefore, we represent it in the four quadrants.

7.6. OSA applications and diffusion

The main publications reflecting the development of the OSA theoretical tools, their applications to different mathematical contents, in teacher training, as well as on comparison and articulation with other theoretical frameworks are available in the various entries of the web repository: http://enfoqueontosemiotico.ugr.es. This activity has been conducted in the framework of various research projects and postgraduate programs at different universities. In the web repository, there are 106 PhD theses available to date that have been produced using OSA as a theoretical framework.

A specific entry also includes publications in English in the main mathematics education journals, grouped in the same categories as on the main page: synthesis papers, ontosemiotic meanings and configurations, didactic design and analysis, didactic appropriateness, articulation with other theories, teacher training, algebra, arithmetic, calculus, statistics, probability, and combinatorics.

Kaiber et al. (2017) analyzed for the 10-year period prior to 2017 the papers based on OSA presented at Latin American conferences, such as the Reunión Latinoamericana de Educación Matemática (RELME), the Conferencia Interamericana de Educación Matemática (CIAEM), and the Congreso Iberoamericano de Educación Matemática (CIBEM). The analysis of the proceedings of these congresses allowed us to identify 188 articles in which OSA was used as the main theoretical reference for research or as a theoretical guide for the production of analyses. This set of 188 publications comprises 121 articles published in ALME, 26 in the Annals of CIAEM and 41 in the Annals of CIBEM, covering different dimensions or research areas in

Didactics of Mathematics. In addition, papers supported by OSA are common at the SEIEM, CERME, PME, and ICME conferences.

The impact of OSA on postgraduate mathematics education in Brazil was analyzed in an article by Breda et al. (2021). They conducted a meta-analysis of 16 doctoral theses presented from 2005 to 2019 at different Brazilian universities and used OSA tools as a theoretical framework to state the research problem and analyze and interpret the results.

7.7. Synthesis of OSA philosophical postulates

The plurality of paradigms and theories converging in mathematics education and the need to clarify and articulate them are sources of inspiration for the emergence of OSA as a field of scientific and technological enquiry. The construct ontosemiotic configuration that incorporates transdisciplinary elements has been developed to overcome the boundaries between philosophical, psychological, and sociological disciplines to the extent that they are interested in mathematics, its learning and dissemination, as reasoned in Chapter 2. An essential postulate of OSA is the emergence of mathematical constructs (concepts, propositions, etc.) from people's operational and discursive practices when solving problems (Font et al., 2013).

Mathematical constructs or ideas are not independent of people but are simultaneously creation and discovery (Cañón, 1993), thus assuming an anti-Platonist perspective. Mathematical axioms and postulates are inventions that occur in people's brains. Although the propositions derived from them are unknown a priori and apparently have been discovered, this does not justify Platonism.

Ontological dimension

The OSA philosophy of educational mathematics, implicitly embodied in the ontosemiotic configuration construct (Chapter 2), is summarized in the following postulates, using the adapted scheme proposed by Bunge (1983) to characterize his philosophical system.

- Naturalism. Assumes the existence of material objects and rejects the independent existence of ideas, whether physical or formal. At the same time, it rejects physicalism because it denies the fact that all objects are physical entities. Mathematical practices are actions of people and therefore are cerebral and bodily processes (manipulative and gestural); when these practices are shared within a community, they are institutional practices that depend on the cerebral activity of its members and the interpersonal interactions established between them.
- Systemism. It assumes the systems of practices, objects, processes, and contexts in which mathematical activity takes place, articulated in the ontosemiotic configuration construct as its objects of study.
- Emergentism. It assumes that abstract mathematical objects come from other previous entities (the operational and discursive practices) and are not reducible to them.
- Pluralism. The diversity of practices, objects, and processes required for the description and understanding of mathematical activity in its various varieties.
- Dynamism. It assumes that meaning changes with time and personal and contextual circumstances.

Epistemological dimension

- Realism. Both formal and applied mathematical knowledge emerge from operational and discursive practices when solving problems. A kind of virtual or fictional reality refers to objects that emerge from mathematical activities when they interact with perceivable objects and artifacts in the environment.

- Evolutionism. It postulates that personal and institutional meanings evolve over time as subjects progressively address more complex problems. The construction of new knowledge starts from existing knowledge, expanding and correcting previously produced by individuals within historical communities.
- Social constructivism. Cognitive ontosemiotic configurations are created by subjects, and socioepistemic configurations result from interpersonal communication. Knowledge construction occurs for subjects in a community whose norms promote or inhibit investigative activities.
- Moderate rationalism and empiricism. Both reason and experience are necessary to construct mathematical knowledge; mathematical practices can be both operational (involving the use of empirical artifacts) and discursive (involving objects of reason).
- Conventionalism. Concept definitions, propositions, and mathematical procedures are not arbitrary conventional rules but are motivated by the activity of describing and explaining the real world and virtual constructs facts and objects. This conventional character explains the necessity and universality of mathematical constructs.
- Justificationism. It includes arguments as a primary object type. These arguments can be descriptive, explanatory, or justificatory, and different types of reasoning can be used based on both reason and experience. These arguments result from the use of different types of reasoning, based on both reason and experience.

Semiotic dimension

- Realism. In realist theories of meaning (Kutchera, 1975), linguistic expressions have an attributional relation to certain entities (objects, attributes, facts). Words and signs are meaningful when they are assigned an object, concept, or proposition as meaning. In this way, there are entities that are not necessarily concrete but are always objectively given before words, which are their meanings. Ontosemiotics postulates a type of referential semiotic function, designating certain entities by convention. In this way, the representational value of languages is accounted for.

– Pragmatism. In pragmatic (operational) theories, meaning depends on the context in which the words are used. Signs become meaningful because they are playing a certain function in a linguistic game when they are used in this game in a certain way and for a certain purpose. The meanings of mathematical objects as systems of operational and discursive practices imply the acceptance of pragmatic theories and the recognition of the instrumental value of languages.

Ontosemiotics assigns an essential role in the creation and manipulation of sign systems representing different types of objects and instruments of mathematical activity. Thus, representationist and instrumentalist postulates in semiotic cognitive theories are compatible and complementary. The OSA provides a transdisciplinary vision of mathematical activity by considering, in an articulated manner, different views of disciplines interested in mathematical knowledge, its learning, and dissemination. These are the following points:

– Epistemological: mathematics as a particular mode of human activity and its product as a special type of knowledge.
– Ontological: mathematics as a finished product and a system of objects and theories.
– Psychological: a particular type of mental (or cerebral) activity.
– Sociological: mathematics as a social activity and its product as a special cultural artifact.
– Historical: mathematics as a historical process of discovery, invention, and diffusion in a given society.

– Instrumental: mathematics as a tool for science, technology and the humanities.

These different views of mathematics are mutually compatible and even complementary. It would be wrong to adopt one of them to exclude all others because mathematics consists simultaneously of everything that these different perspectives provide.

7.8. Synthesis of OSA theoretical system and open questions

In Table 7.1. we include a synthesis of the 11 features that characterize OSA as a theoretical system, in line with the syntheses of the five theories described in chapters 2 to 6: Brief summary; scope/objective; rationale; hypotheses; constructs; relationships; provenance; similarities; complementarities; operationalization; and uses.

Table 7.1. Synthesis of OSA

Elements	**Description**
Summary. What is the theory about, and what are its main propositions?	The ontosemiotic approach to mathematics education provides a system of constructs, principles and methodological tools for studying and understanding the nature of mathematical activity, mathematical knowledge and teaching and learning processes. This scientific component (descriptive, explanatory and predictive) on mathematics education is complemented by another technological (prescriptive) component formed by a system of criteria or standards to optimize the design, implementation and evaluation of educational-instructional processes and a professional development model. The system comprises five theories: 1) The ontosemiotic theory of mathematical activity. It develops an anthropological and pragmatist vision of mathematics, that is, as a human activity focused on problem solving. This anthropological conception of mathematics as an activity is complemented and articulated with two other conceptions: mathematics as a system of objects and processes, and mathematics as a system of signs. 2) The ontosemiotic theory of meaning and mathematical cognition. Develops a global vision of the meaning of mathematical objects, articulating realistic and pragmatic assumptions as the basis of mathematical cognition from individual (personal) and social (institutional) perspectives.

	3) The theory of educational design in mathematics. It develops assumptions and theoretical tools to describe and design mathematics teaching and learning processes based on the OSA's specific theory of mathematical activity and object meaning. 4) The theory of didactic suitability. It develops a system of criteria for the local optimization of the design, implementation, and evaluation of educational-instructional processes in mathematics based on the OSA assumptions and constructs. Criteria are developed for the epistemic, ecological, mediational, interactional, cognitive, and affective facets of teaching and learning processes. 5) The theory of teacher professional development. It develops a model of mathematics teachers' knowledge and competences that considers the facets, components and sub-components of the educational processes involved in the activities of grounding, design, planning and evaluation. It also includes a system of principles or criteria for the efficiency of teacher education programs.
Scope/Objective. What phenomena does the theory explain?	The OSA theoretical system aims to describe, explain, and predict phenomena related to the design, implementation, and evaluation of mathematical teaching and learning processes at different educational contexts and levels. It also seeks to identify criteria for the local optimization of such processes and, thus, prescribe preferable actions to achieve the intended educational purposes based on the ontosemiotic assumptions of mathematical knowledge.
Justification. Why is this theory necessary and how does it improve on previous theories?	OSA addresses the problem of the diversity and disparity of existing theories in mathematics education by developing a modular and inclusive theoretical system that considers dilemmas regarding ontological, semiotic, cognitive, and epistemological issues in teaching and learning. Other existing theories are enhanced by grounding mathematics instruction and teacher education models into explicit and articulated theories of mathematical activity, meaning, and mathematical cognition.
Hypotheses. What specific hypotheses does the proposed theory propose, and how do they differ from other theories?	To describe and understand educational-instructional processes in mathematics, OSA assumes it is necessary to question the nature of mathematics and, therefore, develop explicit theories about the types and emergence of mathematical objects, the relationship of mathematics with languages, and material reality. It also assumes that mathematics education has a scientific (descriptive, explanatory and predictive) and technological (prescriptive) component and thus requires the development of tools to address the study of scientific and technological issues. The promotion of learning (growth of mathematical knowledge, understanding and competence) requires an appropriate selection of partial meanings of the content and an appropriate

	sequence of didactic configurations that consider the ontosemiotic complexity of the content.
Constructs. What elements constitute the theory?	The constructs comprising each theory in the OSA theoretical system are: – Theory of mathematical activity: mathematical practices; types of objects and process; ontosemiotic configuration. – Theory of meaning and cognition: Semiotic function; types of meaning; knowledge, understanding, mathematical competence. – Theory of educational design: facets and components of an educational-instructional process; didactic configuration; didactic trajectory; normative dimension. – Theory of didactic suitability: didactic suitability; criteria of suitability. – Theory of teacher professional development: Didactic-mathematical knowledge; didactic-mathematical competences; criteria of suitability of training programs.
Relations. How are the elements of the theory related to each other?	The constructs of meaning theory are based on the typology of practices and objects in mathematical activity theory. The theory of educational design draws on mathematical activity, meaning, and cognition theories. The components and sub-components of the theory of suitability are based on the model of meaning and knowledge from the theory of mathematical cognition. The theory of teacher professional development builds on the other theories.
Origin. On which theories is it based, and how?	(See explanation for each theory) The philosophical component is based on the anthropological and conventionalist assumptions of Wittgenstein's philosophy of mathematics and Peirce's pragmatism. It also assumes Vygotsky's cultural-historical view of cognition.
Similarity. Which theories are most similar to this theory?	OSA is related to anthropological theory in didactics (Chevallard, 1992; 1999), theory of didactic situations (Brousseau, 2002), sociocultural theories developed in mathematics education, such as the theory of objectification (Radford, 2006), cognitive theories, such as the theory of conceptual fields (Vergnaud, 1990), APOS (Dubinsky & McDonald, 2001), and theory of semiotic representation registers (Duval, 1995).
Complementarity. With which theories can it be complemented?	Theories related to OSA study partial aspects of mathematics education and provide results that complement analyses of ontosemiotic theories. For example, the Theory of Semiotic Representation Registers (Duval) delves into the types of languages used in mathematical activity and the treatments and conversions that they entail.
Operationalization. How are the constructs	(See explanation for each theory)

measured or identified?	
Uses. What can the theory be used for?	Each partial theory in the OSA system is used to investigate issues relevant to mathematics education. The ontosemiotic theory of mathematical activity enables detailed analyses of mathematical activity and helps us understand the complexity of the objects and processes involved in problem solving, providing a foundation for educational-instructional processes in mathematics. The ontosemiotic theory of meaning and mathematical cognition helps analyze and understand processes of representation and signification in the construction and communication of mathematical knowledge. It helps recognize different or partial meanings of mathematical objects and select a representative sample adapted to the given context. It makes it possible to recognize the web of knowledge involved in mathematical activity and, consequently, to elaborate an educational-instructional model that considers its complexity. Educational design theory is used to plan and implement educational processes in mathematics at the micro (lessons), meso (topics), and macro (programs) levels. It also serves as a tool to describe, explain, and assess educational processes designed from other theoretical perspectives, helping to identify aspects that can be improved. The theory of didactic suitability is a guide for designing locally suitable (optimal) instructional processes in mathematics to achieve the planned educational goals. It helps to become aware of the complexity of finding a weighted balance between the different facets involved (epistemic, ecological, mediational, interactional, cognitive, and affective). It is also used as a guide for evaluating the design and implementation of instructional processes and identifying aspects that can be improved. Therefore, it is a resource for teachers to reflect on their own practice. The theory of teacher professional development is used to design, implement, and evaluate specific mathematics teacher education programs and actions. The developed system of categories of didactical-mathematical knowledge and competences and the suitability criteria of the training programs can be used to describe and understand the activity of mathematics teacher educators and teachers and to identify possible improvements. Overall, the OSA has been developing a system of articulated theoretical instruments for the realization of research activities and teaching practice in mathematics education, considering the complexity of the aspects involved.

The complexity of mathematics education, as a field of research and practice, led us to develop five articulated theories that address issues related to theoretical foundations, design of educational processes, their implementation and evaluation, and teacher professional development. In the different chapters of the book, we justified the need to elaborate on the conceptual and methodological tools that characterize each partial theory. We have also addressed the study of the concordances and complementarities of OSA with various theories, in particular, the theory of didactic situations, the anthropological theory of didactics, realistic mathematics education, APOS theory, objectification theory, and the ethnomathematics program. These studies of theory articulation should be deepened and extended to other theories used in mathematics education, such as those mentioned by Asenova et al. (2024). The extent to which OSA is sufficient as a theoretical system for mathematics education should also be analyzed. Can we reduce to five theories that are necessary and sufficient for studying the scientific and technological research problems posed by mathematics teaching and learning, including teacher training? What changes and developments are necessary in OSA to effectively resolve the dilemmas and contradictions between different theories, avoid redundancies, and provide a shared language?

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https://produccioncientifica.ugr.es/investigadores/355065/detalle

Synopsis

Ontosemiotic Approach in Mathematics Education
Foundations, Tools, and Applications

This book addresses the dilemmas and contradictions posed by the diversity of theories elaborated to understand the complexity of teaching and learning in mathematics education research. The ontosemiotic approach to mathematical knowledge and education is a modular and inclusive theoretical system that addresses this problem. It comprises four articulated partial theories that address ontological, epistemological, and semiotic questions regarding mathematical knowledge and those related to the design, implementation, and evaluation of educational-instructional processes. These theories serve as the basis for developing a fifth theory on educating mathematics teachers and teacher educators. The book presents the initial assumptions and specific tools of each theory, along with examples of their application to different mathematical contents. It also includes a study of concordances and complementarities with other theoretical frameworks, particularly the theory of didactic situations, the anthropological theory of didactics, realistic mathematics education, APOS theory, objectification theory, and the ethnomathematics program. This book is useful for researchers in mathematics education, teacher educators, and mathematics teachers interested in understanding their professional activities and who want to learn tools to reflect on their practice.