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Article

A Shape-Preserving Variational Spline Approximation Problem for Hole Filling in Generalized Offset Surfaces

Abdelouahed Kouibia [1,*], Miguel Pasadas [1] and Loubna Omri [2]

[1] Department of Applied Mathematics, University of Granada, 18071 Granada, Spain; mpasadas@ugr.es
[2] FSJES of Tetuan, University Abdelmalek Essaidi, Tetuan 93030, Morocco; loubna.omri@etu.uae.ac.ma
* Correspondence: kouibia@ugr.es

Abstract: In the study of some real cases, it is possible to encounter well-defined geometric conditions, of an industrial or design type—for example, the case of a specific volume within each of several holes. In most of these cases, it is recommended to fulfil a function defined in a domain in which information is missing in one or more sub-domains (holes) of the global set, where the function data are not known. The problem of filling holes or completing a surface in three dimensions appears in many fields of computing, such as computer-aided geometric design (CAGD). A method to solve the shape-preserving variational spline approximation problem for hole filling in generalized offset surfaces is presented. The existence and uniqueness of the solution of the studied method are established, as well as the computation, and certain convergence results are analyzed. A graphic and numerical example complete this study to demonstrate the effectiveness of the presented method. This manuscript presents the resolution of a complicated problem due to the study of some criteria that can be traduced via an approximation problem related to generalized offset surfaces with holes and also the preservation of the shape of such surfaces.

Keywords: shape preservation; generalized offset surfaces; hole filling; spline approximation; variational splines

MSC: 65D05; 65D07; 65D10

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1. Introduction

For some real applications, the study of the generalized offset surface problem has been examined, such as in computer-aided geometric design and generally in the natural sciences domain.

In [1], the authors study the problem of defining fair filling patches under some conditions while in [2] the authors present how to fill polygonal holes with minimal energy surfaces using Powell–Sabin type triangulations. Other studies have aimed to solve the problem of the flatness of filler patches, such as transfinite interpolation as studied in [3] and biharmonic optimization in [4]. Meanwhile [5], using B-spline surfaces, some processes manipulate weakly defined control points, as in [6]; for the filler patch, they study several functions that have some geometric properties imposed.

The method of Skinning or lofting in surface generation is frequently found in the design domain in industries such as automobile manufacturing and aircraft and ship construction. The authors in [7] propose the combination of T-spline technology and surface skinning modeling by using a new process for local shape preservation in T-spline surface Skinning with a smaller computational cost. In [8], the authors propose to numerically solve Caputo's time fractional diffusion equation using modified cubic exponential B-spline placement. The work in [9] presents an improvement in some numerical methods to find the non-polynomial fractional spline, aiming to solve the fractional Korteweg–de Vries (KdV) equation with respect to time, as well as similar problems in various scientific fields, such as plasma physics and mechanics. Meanwhile, in [10], the quintic B-spline differential quadrature method is modified. This then implies that the algebraic system obtained does not have abstract points. In this case, the system of equations is solved without the need for an additional equation. This analysis develops numerical solutions of the KdV equation. For design, in [11], an algorithm is presented for the effective study of displaced surfaces for polygonal meshes. The process is good with respect to degenerate configurations and computes displacements free of (self-)intersection. Furthermore, in [12], the authors study a method to find a displaced surface with a variable thickness so that the solid volume between the surface and its displacement minimizes a certain functional and at the same time verifies a set of conditions. These conditions and the functional can be approximated in a simple way. In [13], the authors introduce a set of partial shape-preserving spline basis functions to smoothly combine a collection of shape primitives with flexible blending range control. These types of spline basis functions can be considered a generalization of traditional B-spline basis functions, where the shape primitives used are control polygons or control points. In [14], the author presents a study using a single rectangular B-spline surface trimmed to fill an n-sided hole, based on an energy minimization or variational B-Spline technique. The method is justified by taking a fraction of a second to fill n-sided holes with high-quality waterproof B-spline surfaces under complicated conditions. In [15], the authors develop a local approach to shape preservation. The objective of their construction is to follow local changes in data and shape preservation that does not involve overshooting. In [16], the authors discuss how to repair a polygonal mesh in which some holes have been detected. The objective of this article is twofold. First, the authors offer a review of the most relevant gap filling methods and highlight their importance, the context in which each method is applied and the results obtained. Second, the authors present a comparative study to evaluate the parameters of all mentioned methods. The work in [17] presents a detail-preserving variational method for curves and surfaces, which integrates several approaches; the authors introduce a method that enables users to modify a free-form curve or surface while preserving its details. The final point that interpolates the B-splines and the corresponding wavelets of the curve or surface are used as the underlying representation. In [18], the authors present a fast variational approach for the interactive design of multi-resolution curves and surfaces. By converting the imposed conditions to different resolution levels, which are lower than the finest resolution level, an optimization system with fewer unknowns is needed. The system is solved in real time even if the number of checkpoints is large. Furthermore, during deformation, the method can preserve the multi-resolution details. All of this is obtained by optimizing the energy of the deformation of the curve or surface at the corresponding level, instead of the total energy.

The authors in [19] seek to approximate three-dimensional models of watertight surfaces that contain holes, in which the holes are too geometrically complicated to fill, through the study of triangulation estimation.

Using biquadratic spline functions that are of C^1, an approximation study is presented in [20] to fill holes in some generalized surfaces, while, in [21], by using variational splines, some generalized offset surfaces are approximated.

The resolution of a complicated problem is presented due to the combination of many conditions. Besides being an approximation problem regarding generalized offset surfaces with holes, it also involves the preservation of the shape of this type of surface. To this end, we present an approximation method to preserve the shape of generalized offset surfaces with holes.

We highlight the advantages of this work with respect to those existing in the literature (see, for example, [7]). On one hand, we compute the resulting function; on the other hand, the analysis of some convergence results is performed.

To demonstrate the usefulness of the method, some examples are studied to approximate the shape of the function inside the hole by using some given data of the function outside the hole only. Please note that the Abstract of this manuscript has been published in the ICRAMCS 2023 Proceedings, ISSN 2605-7700; for more details, please consult [22].

The organization of the rest of the manuscript is as follows. The formulation of the global problem and some preliminary results are presented in Section 2. In Section 3, a construction function outside the hole is presented. In Section 4, a construction function inside the hole is studied. Section 5 is devoted to constructing the approximating solution over the entire domain. In Section 6, some numerical and graphical examples are illustrated to show the effectiveness of the studied method. This manuscript ends with the conclusions and future perspectives.

2. Notations and Formulation of the Global Problem

Let $n, m \in \mathbb{N}$, with $n > 0,\ m > 0$, and we denote by $\langle \cdot \rangle_n$ and $\langle \cdot, \cdot \rangle_n$, respectively, the square norm and inner product in $\mathbb{R}^n$. Let Ω be a non-empty polygonal bounded subset of $\mathbb{R}^2$, let $\Pi_m(\Omega)$ be the restriction to Ω of the polynomials of degree less than m defined on $\mathbb{R}^2$, and let $H^m(\Omega;\mathbb{R}^3)$ be the usual Sobolev space of order m. We shall use in this space $H^m(\Omega;\mathbb{R}^3)$ the norm

$$\|u\|_{m,\Omega} = \Big(\sum_{|\alpha| \le m} \int_\Omega \langle \partial^\alpha u(x) \rangle_3^2 \, dx \Big)^{\frac{1}{2}},$$

the semi-norms

$$|u|_{l,\Omega} = \Big(\sum_{|\alpha| = l} \int_\Omega \langle \partial^\alpha u(x) \rangle_3^2 \, dx \Big)^{\frac{1}{2}},$$

and the corresponding inner semi-products

$$(u,v)_{l,\Omega} = \sum_{|\alpha|=l} \int_\Omega \langle \partial^\alpha u(x), \partial^\alpha v(x) \rangle_3 \, dx, \text{ for } l = 0, \dots, m.$$

Let f and g be two real functions. We write

$$f(d) = O(g(d)), \text{ as } d \to d_0$$

if there exists some real constant $C > 0$ and $\eta > 0$ such that

$$\forall d \in \mathbb{R},\ |d - d_0| < \eta,\ |f(d)| \le C\, g(d).$$

Likewise, we write

$$f(d) = o(g(d)), \text{ as } d \to d_0$$

if $\lim\limits_{d \to d_0} \dfrac{f(d)}{g(d)} = 0$.

Now, let us consider a subset $\mathcal{H}$ of real positive numbers of which 0 is an accumulation point. In this case, for any $h \in \mathcal{H}$, let us also consider a partition $\mathcal{T}_h$ of $\overline{\Omega}$ using rectangles K with diameters $h_K \le h$.

Now, we designate by $\mathcal{V}_h$ a finite element space constructed on the partition $\mathcal{T}_h$ from a generic finite element of class C^k, with $k \ge 1$, such that

$$\mathcal{V}_h \subset H^m(\Omega) \cap C^k(\overline{\Omega}). \tag{1}$$

For any $h \in \mathcal{H}$, let us denote $V_h = (\mathcal{V}_h)^3$. Then, $V_h \subset H^m(\Omega;\mathbb{R}^3) \cap C^k(\overline{\Omega};\mathbb{R}^3)$.

For a regular parametric surface defined for a differentiable function $\phi : \Omega \to \mathbb{R}$, we consider its associated parametric representation $r(u,v) = (u, v, \phi(u,v))$, its two unit tangent vectors in the directions of u and v, and its normal vector given by

$$\overrightarrow{e}_1 = \frac{r_u(u,v)}{\langle r_u(u,v) \rangle_3}, \quad \overrightarrow{e}_2 = \frac{r_v(u,v)}{\langle r_v(u,v) \rangle_3}, \quad \overrightarrow{n}(u,v) = \frac{r_u(u,v) \times r_v(u,v)}{\langle r_u(u,v) \times r_v(u,v) \rangle_3}, \tag{2}$$

respectively, where $\times$ stands for the cross-product in $\mathbb{R}^3$.

Let us describe the direction determined by the vectors $d_1(u,v)e_1(u,v)$, $d_2(u,v)e_2(u,v)$ and $d_3(u,v)n(u,v)$, where d_1, d_2, d_3 are functions of $H^m(\Omega)$. In this case, the generalized surface $r^0(u,v)$ is defined by

$$r^0(u,v) = r(u,v) + d_1(u,v)\overrightarrow{e}_1(u,v) + d_2(u,v)\overrightarrow{e}_2(u,v) + d_3(u,v)\overrightarrow{n}(u,v). \tag{3}$$

Let us consider as a hole an open set $H \subset \Omega$ and let $H_h = \bigcup\limits_{\substack{K\subset\mathcal{T}_h \\ K\cap H\neq\emptyset}} K$; moreover, a surface with some hole $\mathcal{S}$ is defined by the function $f \in H^{m+1}(\Omega - H; \mathbb{R}^3)$, and its generalized offset surface is defined by the function

$$s_f(u,v) = f(u,v) + d_1(u,v)\overrightarrow{e}_1(u,v) + d_2(u,v)\overrightarrow{e}_2(u,v) + d_3(u,v)\overrightarrow{n}(u,v),$$

for all $(u,v) \in \Omega - H$. Obviously, it is verified that $s_f \in H^m(\Omega - H; \mathbb{R}^3)$.

Finally, let V_1^h and V_2^h be the functional spaces of the restrictions to $\Omega - H_h$ and H_h, respectively, of the space V_h.

We wish to approximate s_f from the finite point set $s_f(A)$ and fill the holes through the construction of a function of the type

$$\sigma^h(u,v) = \begin{cases} \sigma_1^h(u,v), & (u,v) \in \Omega - H_h, \\ \sigma_2^h(u,v), & (u,v) \in H_h, \end{cases}$$

such that $\sigma_1^h \in V_1^h$ approximates s_f over $\Omega - H_h$, $\sigma_2^h \in V_2^h$ preserves the shape of f from $\Omega - H_h$ to H_h by the data proceeding from σ_1^h, and $\sigma^h \in C^k(\Omega; \mathbb{R}^3)$.

To achieve this, we proceed to study the following stages:

- firstly, using a smoothness approach method, we seek to find the generalized offset function $\sigma_1^h \in V_1^h$ that approximates s_f over $\bar{\Omega} - H_h$ and compute it;
- secondly, we seek to construct the function $\sigma_2^h \in V_2^h$ in order to fill the holes and also compute it using an interpolation method;
- finally, we achieve the construction of the function $\sigma^h \in V_h$ approaching s_f over $\Omega - H$ and fill and preserve the "shape" of f on the hole H.

3. Constructing the Function σ_1^h Outside the Hole

For each $M \in \mathbb{N}$, with $M > 0$, let $A^M = \{a_1, \ldots, a_M\}$ be a finite $\Pi_{m-1}(\Omega - H_h)$-unisolvent set of M points of $\Omega - H$, i.e., for any $p \in \Pi_{m-1}(\Omega - H_h)$, verifying $p(a_i) = 0$ for $i = 0, \ldots, M$ one has $p = 0$.

Suppose that

$$\sup_{x\in\Omega-H_h} \min_{1\leq i\leq M} \langle x - a_i \rangle_2 = O\left(\frac{1}{M}\right), \quad M \to +\infty. \tag{4}$$

For any $r = 0, \ldots, k$ and $j = 0, \ldots, r$, let L_j^r be the functional operator given by

$$L_j^r(v)(x) = \partial^{(r,r-j)} v(x), \quad \forall\, x \in \Omega,\ \forall\, v \in C^k(\Omega; \mathbb{R}^3),$$

and let $\rho_j^r : C^k(\Omega - H_h; \mathbb{R}^3) \to \mathbb{R}^{M,3}$ be the functional operator given by

$$\rho_j^r(v) = \left(L_j^r(v)(a_i)\right)_{1\leq i\leq M}.$$

Now, let $J_1 :\ H^m(\Omega - H_h; \mathbb{R}^3) \to \mathbb{R}$ be the functional given by

$$J_1^h(v) = \sum_{r=0}^{k} \sum_{j=0}^{r} \langle \rho_j^r(v) - \rho_j^r(s_f) \rangle_{M,3}^2 + \varepsilon |v|_{m,\Omega-H_h}^2,$$

where $\langle A\rangle_{M,3} = \langle A,A\rangle_{M,3}^{\frac{1}{2}}$, $\langle A,B\rangle_{M,3} = \sum_{i=1}^{M}\langle a_i,b_i\rangle_3$, for any $A = (a_i)_{1\le i\le M}$, $B = (b_j)_{1\le j\le M} \in \mathbb{R}^{M,3}$ and $\varepsilon > 0$.

We consider the following minimization problem: find $\sigma_1^h \in V_1^h$ such that

$$\forall v \in V_1^h, \quad J_1^h(\sigma_1^h) \le J_1^h(v). \tag{5}$$

Proposition 1. *The solution of problem (5) is unique. It is called the generalized offset variational spline in V_1^h relative to M, A^M, $(L_j^r(s_f))_{\substack{0\le j\le r\\ 0\le r\le k}}$ and ε, which also is the unique solution of the following variational problem: find $\sigma_1^h \in V_1^h$ such that, for any $v \in V_1^h$, one has*

$$\sum_{r=0}^{k}\sum_{j=0}^{r}\langle \rho_j^r(v),\rho_j^r(\sigma_1^h)\rangle_{M,3} + \varepsilon(v,\sigma_1^h)_{m,\Omega-H_h} = \sum_{r=0}^{k}\sum_{j=0}^{r}\langle \rho_j^r(v),\rho_j^r(s_f)\rangle_{M,3}. \tag{6}$$

Proof. By adapting the notations and using the Lax–Milgram lemma of [23], the proof is obtained, which is similar to that of Proposition 1 of [20]. □

3.1. Computing the Function σ_1^h

The generalized offset variational spline $\sigma_1^h \in V_1^h$ relative to the data M, A^M, $(L_j^r(s_f))_{\substack{0\le j\le r\\ 0\le r\le k}}$ and ε is computed in the following way: let $\{v_1,\ldots,v_{N_1}\}$ be the basis functions of the restrictions over $\Omega - H_h$ of the space V_1^h.

Thus, $\sigma_1^h = \sum_{i=1}^{N_1}\alpha_i v_i$, where $\alpha_1,\ldots.\alpha_{N_1} \in \mathbb{R}$ are the solutions of the linear system

$$\left(\sum_{r=0}^{k}\sum_{j=0}^{r}\mathcal{A}_j^r + \varepsilon\,\mathcal{R}\right)\alpha = \sum_{r=0}^{k}\sum_{j=0}^{r}\mathcal{B}_j^r, \tag{7}$$

where

$$\mathcal{A}_j^r = \Big(\langle \rho_j^r(v_i),\rho_j^r(v_j)\rangle_{M,3}\Big)_{1\le i,j\le N_1},$$
$$\mathcal{R} = \big((v_i,v_j)_{m,\Omega}\big)_{1\le i,j\le N_1},$$
$$\alpha = (\alpha_1,\ldots,\alpha_{N_1})^T,$$
$$\mathcal{B}_j^r = \Big(\langle \rho_j^r(v_i),\rho_j^r(s_f)\rangle_{M,3}\Big)_{1\le i\le N_1}.$$

We do not need all of the data of s_f on $\Omega - H_h$ but only some of them to resolve the linear system (7).

3.2. Convergence

From the definition of H_h, it is verified that, for any $x \in \Omega - H$, there exists an open set $\omega_x \subset \Omega - H$ and $h_0 \in \mathcal{H}$ such that $x \in \omega_x$, and, for any $h \le h_0$, one has $\omega_x \subset H_h$. Following the same proof as in [20] [Theorem 1], the local convergence result is obtained.

Theorem 1. *If hypothesis (4) is satisfied, one has*

$$\varepsilon = O(M^2), \quad M \to +\infty, \tag{8}$$

and

$$\frac{M^2h^{2m}}{\varepsilon} = o(1), \quad M \to +\infty. \tag{9}$$

Let $x \in \Omega - H$; then,

$$\lim_{M \to +\infty} \| s_f - \sigma_1^h \|_{m,\omega_x} = 0.$$

4. Constructing the Function σ_2^h over H_h

Formulation of the Problem over H_h

Let $\overline{\sigma}_1^h$ be the function given by

$$\overline{\sigma}_1^h(u,v) = \sigma_1^h(u,v) - d_1(u,v)\overrightarrow{e}_1(u,v) - d_2(u,v)\overrightarrow{e}_2(u,v) - d_3(u,v)\overrightarrow{n}(u,v),$$

for any $(u,v) \in \Omega - H$.

Then, $\overline{\sigma}_1^h \in H^m(\Omega - H; \mathbb{R}^3)$ and, from Theorem 1, under adequate hypotheses, it is verified that, for any $x \in \Omega - H$, there exists an open set $\omega_x \subset \Omega - H$ and $h_0 \in \mathcal{H}$ such that, for any $h \leq h_0$, one has $\omega_x \subset \Omega - H_h$ and

$$\lim_{M \to +\infty} \| f - \overline{\sigma}_1^h \|_{m,\omega_x} = 0.$$

Now, suppose that f verifies that

$$\tau_i^r L_i^r(f)(x) \geq 0, \quad \forall\, r = 0, \dots, k,\ \forall\, i = 0, \dots, r, \quad \forall\, x \in H,$$

where $\tau_i^r \in \{0,1\}$, for any $r = 0, \dots, k$ and $i = 0, \dots, r$.

Remark 1. *Observe that if $\tau_0^0 = 1$, then f is a positive function on H.*

In addition, if $k = 1$ and $\tau_0^1 = \tau_1^1 = 1$, then f is monotonic with respect to both coordinate axes.

Now, for any $h \in \mathcal{H}$, let us consider that Γ_h stands for the set of the knots of H_h, i.e., for all $x \in \Gamma_h$, x is associated with a degree of freedom of V_2^h. Let Δ_h stand for the set of the knots of Γ_h belonging to the boundary of H_h, i.e., $\Delta_h = \Gamma_h \cap \partial H_h$, and consider

$$\mathcal{F} = \{\mathcal{L}_1, \dots, \mathcal{L}_N\}$$

the set of the degrees of freedom of V_2^h associated with the knots of Δ_h. Let

$$\begin{aligned} M_h = \{ v \in V_2^h \mid\ & \mathcal{L}_i(v) = \mathcal{L}_i(\overline{\sigma}_1^h),\ i = 1, \dots, N, \\ & \tau_i^r L_i^r(v)(x) \geq 0,\ r = 0, \dots, k,\ i = 0, \dots, r,\ x \in H \} \end{aligned}$$

and

$$\begin{aligned} M_h^0 = \{ v \in V_2^h \mid\ & \mathcal{L}_i(v) = 0,\ i = 1, \dots, N, \\ & \tau_i^r L_i^r(v)(x) = 0,\ r = 0, \dots, k,\ i = 0, \dots, r,\ x \in H \}. \end{aligned}$$

Finally, consider the following minimization problem: find $\overline{\sigma}_2^h \in M_h$ such that

$$|\overline{\sigma}_2^h|_{m,H} \leq |v|_{m,H},\ \forall\, v \in M_h. \tag{10}$$

Theorem 2. *The unique solution of problem (10) is named the shape-preserving interpolation variational spline relative to f and $\mathcal{F}$. It is also characterized as the unique solution of the following variational problem: find $\overline{\sigma}_2^h \in M_h$ such that*

$$\forall\, v \in M_h^0, \quad (\overline{\sigma}_2^h, v)_{m,H} = 0.$$

Proof. It is clear that M_h is a non-empty closed convex set of $H^m(H; \mathbb{R}^3)$. Moreover, the application given by

$$((u,v)) = \langle \rho^h(u), \rho^h(v) \rangle_{N,3} + (u,v)_{m,H}, \quad u, v \in H^m(H; \mathbb{R}^3),$$

where $\rho^h(w) = \sum_{i=1}^{N} \mathcal{L}_i(w)$, for any $w \in H^m(H;\mathbb{R}^3)$, is an inner product in $H^m(H;\mathbb{R}^3)$, with the associated norm $[[w]] = ((w,w))^{\frac{1}{2}}$. This norm is equivalent to the Sobolev norm $\|w\|_{m,H}$ introduced in Section 2.

Since, for any $v \in M_h$, it is verified that $\rho^h v = \rho^h(\overline{\sigma}_1^h)$, problem (10) can be traduced to find $\overline{\sigma}_2 \in M_h$, verifying $[[\overline{\sigma}_2]] \leq [[v]]$, for any $v \in M_h$.

By using the theorem of projection on a closed convex set, it is deduced that there exists a unique $\overline{\sigma}_2^h \in M_h$, which is the projection of function 0 over M_h such that $[[\overline{\sigma}_2^h]] \leq [[v]]$, for any $v \in M_h$, and verifying

$$\overline{\sigma}_2^h \in M_h, \quad ((-\overline{\sigma}_2^h, w - \overline{\sigma}_2)) \leq 0.$$

Let $v \in M_h^0$; then, $\overline{\sigma}_2^h - v \in M_h$ and thus $((-\overline{\sigma}_2^h, v)) \leq 0$. Taking into account that M_h^0 is a linear subspace and $\rho^h(v) = 0$, we obtain the result. □

Now, let us consider the two sets $\overline{M}_h$ and $\overline{M}_h^0$ defined by

$$\overline{M}_h = \{v \in V_2^h \mid \mathcal{L}_i(v) = \mathcal{L}_i(\overline{\sigma}_1^h),\ i = 1,\dots,N,$$
$$\tau_i^r L_i^r(v)(x) \geq 0,\ r = 0,\dots,k,\ i = 0,\dots,r,\ x \in \Gamma_h\}$$

and

$$\overline{M}_h^0 = \{v \in V_2^h \mid \mathcal{L}_i(v) = 0,\ i = 1,\dots,N,$$
$$\tau_i^r L_i^r(v)(x) = 0,\ r = 0,\dots,k,\ i = 0,\dots,r,\ x \in \Gamma_h\},$$

In this case, the minimization problem considered is to find $\tilde{\sigma}_2^h \in \overline{M}_h$ such that

$$|\tilde{\sigma}_2^h|_{m,H} \leq |v|_{m,H}, \quad \forall\, v \in \overline{M}_h. \tag{11}$$

The proof of the following theorem is similar to that of Theorem 2.

Theorem 3. *The solution of problem (11) is unique, named the shape pseudo-preserving interpolation variational spline relative to f and $\mathcal{F}$. It is also characterized as the unique solution of the following variational problem: find $\tilde{\sigma}_2^h \in \overline{M}_h$ such that*

$$(\tilde{\sigma}_2^h, v)_{m,H} = 0, \quad \forall\, v \in \overline{M}_h^0.$$

Now, by reasoning as in [24] [Section 4], we can construct an algorithm to calculate a finite sequence $(\tilde{\sigma}_{2,\ell}^h)_{1\leq\ell\leq\eta_h}$ such that $\tilde{\sigma}_{2,\eta_h}^h = \tilde{\sigma}_2^h$. Moreover, by adapting the notations and results in [24] [Theorem 12], one can have

$$\lim_{h\to 0} \|\tilde{\sigma}_2^h - \overline{\sigma}_2^h\|_{m,H} = 0. \tag{12}$$

Then, $\tilde{\sigma}_2^h$ tends toward the shape preservation of f on H as h tends to 0.

Remark 2. *For the construction of the finite sequence, the formulation of the algorithm can be consulted in great detail in [25].*

5. Construction of the Solution σ^h over Ω

Let $\sigma_2^h \in V_2^h$ be the function defined by

$$\sigma_2^h = \tilde{\sigma}_2^h + d_1 \overrightarrow{e}_1(u,v) + d_2 \overrightarrow{e}_2(u,v) + d_3 \overrightarrow{n}(u,v).$$

Hence, the solution of the initial problem is given by

$$\sigma^h = \begin{cases} \sigma_1^h & \text{on } \Omega - H_h, \\ \sigma_2^h & \text{on } H_h. \end{cases}$$

In this case, the following properties are verified.

- For construction, one has that $\sigma^h \in V_h$.
- Moreover, from Theorem 1, under adequate conditions, σ_h locally converges to s_f in $H^m(\Omega - H)$.
- Finally, from the interpolation conditions in H_h, it results from (12) that σ_2^h tends to preserve the shape of the function s_f in H_h; then, we can conclude that σ_h tends to preserve the "shape" of s_f in H_h .

6. Graphic and Numerical Example

Consider the regular surface $\mathcal{S}$ parameterized by

$$f(u,v) = \left\{u, v, \frac{1}{1 + e^{12.73 - 13.5\sqrt{0.1+u^2+v^2}}}\right\}, \quad (u,v) \in \Omega = (0,1) \times (0,1),$$

and $s_f(u,v)$ parameterizes the generalized offset surface with variable offset distances and directions determined by the vectors $d_1(u,v)\overrightarrow{e}_1(u,v)$, $d_2(u,v)\overrightarrow{e}_2(u,v)$ and $d_3(u,v)\overrightarrow{n}(u,v)$, being

$$d_1(u,v) = 0.02u, \; d_2(u,v) = 0.02v, \; d_3(u,v) = 0.25(u^2 + v^2 - u - v) + 0.175.$$

Let $H = H_1 \cup H_2$ be the open subset of Ω given by the points sets $H_1 : 3(x - 0.4)^2 + 12(y - 0.22)^2 < 0.1$ and $H_2 : 4(x - 0.6)^2 + 9(y - 0.73)^2 < 0.2$.

In this case, $\mathcal{V}_h$ is the finite element space constructed from the generic Bogner–Fox–Schmit element of class one over a partition $\mathcal{T}_h$ of 6×9 equal rectangles of $\overline{\Omega}$ (Figure 1).

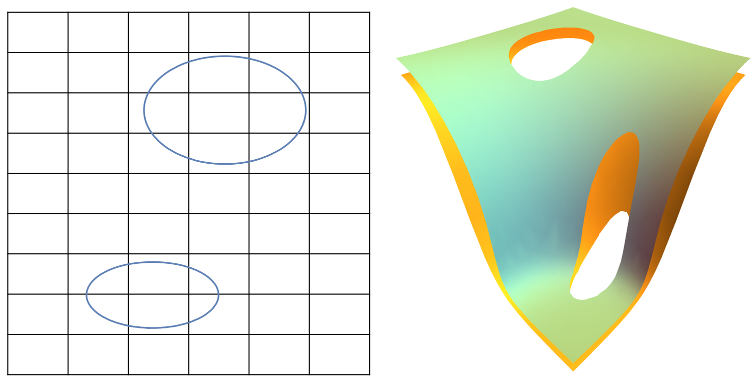

Figure 1. The hole H and the partition $\mathcal{T}_h$, on the **left**, and the graphs of f and s_f over $\Omega - H$ from bottom to top, on the **right**.

Now, we consider the set $H_h = \bigcup_{\substack{K \subset \mathcal{T}_h \\ K \cap H \neq \emptyset}} K$ and a set $A^M \subset \Omega - H_h$ of $M = 4000$ approximation random points (Figure 2).

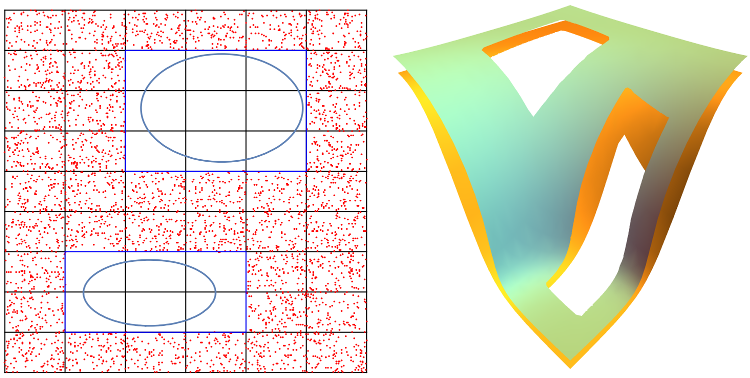

Figure 2. The hole H_h and the approximation point set A^M, on the **left**, and the graphs of f and s_f over $\Omega - H_h$ from bottom to top, on the **right**.

Applying the above approximation method, we construct the smoothing generalized offset variational spline $\sigma_1^h \in V_h^1$ relative to A^M, $(s_f(a))_{a\in A^M}$ and $\varepsilon = 10^{-12}$. We have computed an estimation of the relative error via the expression $E_r = \sqrt{\frac{\sum_{i=1}^{5000} \langle s_f(\xi_i) - \sigma_1^h(\xi_i)\rangle_3^2}{\sum_{i=1}^{5000} \langle s_f(\xi_i)\rangle_3^2}}$, with $\{\xi_1, \dots \xi_{5000}\} \subset \Omega - H_h$ being a random set of points. In this case, it is obtained that $E_r = 2.36006 \times 10^{-4}$ (Figure 3).

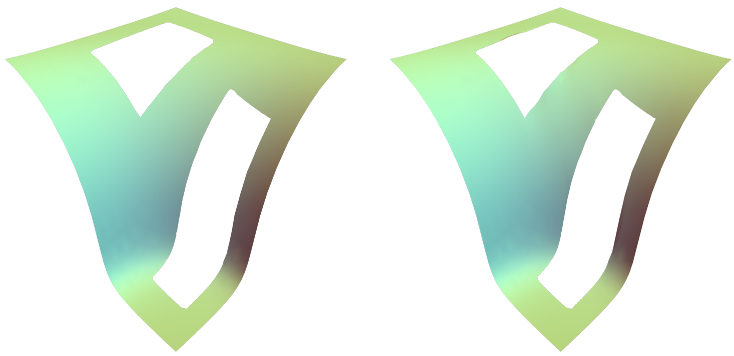

Figure 3. The graphs of s_f and σ_1^h over $\Omega - H_h$, from **left** to **right**. $E_r = 2.36006 \times 10^{-4}$.

Next, we construct the interpolating variational spline $\sigma_2^h \in V_h^2$ relative to σ_1^h and $\mathcal{F}$ and the freedom degree set of V_2^h associated with the knot set belonging to ∂H_h, and we obtain the approximation σ^h of s_f over Ω (Figure 4).

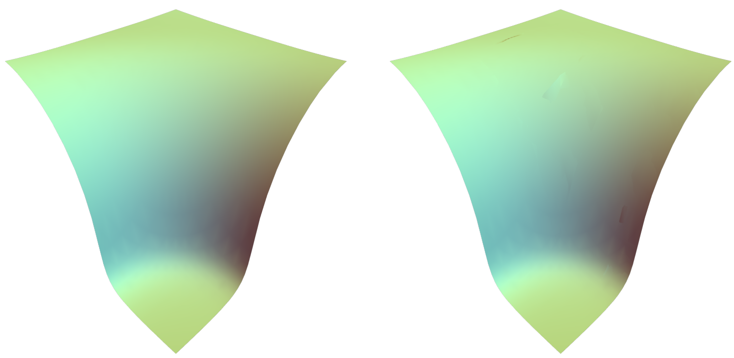

Figure 4. The graphs of s_f and σ^h over Ω, from **left** to **right**.

Table 1 is given to present the relative error estimations E_r from different values approximating points M and the smoothness parameter ε.

Table 1. Computation of the relative error estimations E_r for different values of the approximation M and the smoothness parameter ε.

M	ε	E_r
500	10^{-7}	4.8358×10^{-3}
	10^{-9}	2.7416×10^{-3}
	10^{-12}	2.7314×10^{-3}
1000	10^{-7}	2.4427×10^{-3}
	10^{-9}	2.3178×10^{-3}
	10^{-12}	2.3157×10^{-3}
1500	10^{-7}	1.2414×10^{-3}
	10^{-9}	1.0115×10^{-3}
	10^{-12}	1.0096×10^{-3}
1000	10^{-9}	6.5812×10^{-3}
	10^{-12}	5.1828×10^{-4}
	10^{-14}	8.5806×10^{-4}
1500	10^{-9}	5.5815×10^{-4}
	10^{-12}	4.0570×10^{-4}
	10^{-14}	3.6152×10^{-4}
2500	10^{-9}	3.9301×10^{-4}
	10^{-12}	2.0592×10^{-4}
	10^{-14}	1.3303×10^{-4}

7. Conclusions and Perspectives

From the data presented in Table 1, one can verify the justification of the convergence results and the effectiveness of the studied method. In fact, using small values of the approximating points M, a good order of approximation represented by the relative errors E_r considered is reached. One can observe that the estimation of the error E_r decreases as the point M tends to increase, and the same occurs for the estimation of the error E_r as the parameter ε. This indicates the agreement between the theory of the convergence result and the numerical theory.

In short, the computation of the order of the estimation of the relative errors is similar if we compare it with the known offset surfaces with holes studied in [20], although, in this work, the shape preservation criterion is added, whose study is supposed to be more complex. Hence, the analysis of the results given in Table 1 and Figures 1–4 shows the validity of the presented approximation method. Moreover, we highlight the advantages of this work with respect to those existing in the literature (see, for example, [7]). First, the computation of the solution of the problem is studied; second, some convergence results of the errors are analyzed.

We can conclude that the presented investigation enables the resolution of a complicated problem due to the study of various conditions. Indeed, the problem of the approximation and/or interpolation of generalized offset surfaces with holes, on the one hand, and the preservation of the shape of this type of surface, on the other hand, is studied at the same time.

As future research, it can be proposed to conduct a similar study with other functions, such as radial functions. There is also the possibility of imposing more shape conservation conditions, such as convexity conditions.

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