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ORIGINAL PAPER

Understanding the onto-semiotic approach in mathematics education through the lens of the cultural historical activity theory

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Abstract
Research in mathematics education can be understood as a system of activities addressing the basic and applied problems related to teaching and learning of mathematics. Such a system includes the activities of foundation, planning, implementation, evaluation of mathematics instruction, and teacher professional development, which are supported by different theories. This diversity of theories raises interest in their comparison, coordination, and possible integration. The paper aims to present a case of application of the Cultural Historical Activity Theory (CHAT), in its 3rd and 4th generation versions, to analyze the emergence of the Onto-semiotic Approach to mathematical knowledge and instruction as a theoretical framework that addresses the study of the five partial activities mentioned above. This use of the CHAT can be useful in studies on theory articulation by focusing not only on the subjects, the object, and the instruments but also on the community context, the ecological-normative environment in which these activities take place, and the dilemmas or contradictions between theories.

Keywords Mathematics education · Cultural historical activity theory · Onto-semiotic approach · Networking theories

1 Introduction

The existence of different theoretical frameworks in mathematics education has given place to a research field interested in their comparison and possible articulation (Bikner-Ahsbahs & Prediger, 2014; Prediger et al., 2008). In these studies, a theory is understood as a system of principles, paradigmatic issues, and methods (Radford, 2008), or of research praxeologies (Artigue & Bosch, 2014) made up of tasks, techniques, technologies (justification of techniques) and theories (justification of technologies). Both interpretations of a theory do not emphasize explicitly in their conceptualization aspects such as the historical and socio-cultural context in which the research activity takes place. Consequently, it is useful to adopt a broader perspective by applying the activity system view[1], which deals with the analysis of complex activities in which several partial activities are interrelated. The Cultural Historical Activity Theory (CHAT) is today one of the most influential and progressive schools of thought in child development and elementary education and is also present in a wide range of other disciplines. “This is not a project for a science of everything. But it does point to a potential for a progressive, critical new approach across a range of disciplines, and an improved possibility for interdisciplinary work” (Blunden, 2010, p. 3). A central aim of the CHAT is to develop the notion of activity as a scientific concept that makes sense in multiple fields, including psychology, sociology, political science, linguistics, etc. Interpreting theories as activity systems introduces the community, historical and cultural components, the ecological-normative setting in which they develop, their object or motive (answering some questions) and the instruments (principles and methods) built and

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1 In the third and fourth generations of the Cultural Historical Activity Theory (CHAT) (Engeström, 1987; Roth & Lee, 2007; Engeström & Sannino, 2021).

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applied in the analysis. In this paper, we consider that the variety of theories can be assumed as a plurality of activity systems within communities of practice. Recognizing and solving contradictions, dilemmas, gaps, or duplicities between these theories can advance the global enterprise of the field.

Mathematics education is viewed from different perspectives, sometimes including the mathematics teaching and learning system as a whole, with its various sub-systems: curricular development, teacher education, the mathematics classroom, research in mathematics education, and so on. Steiner (1990) identifies various disciplines related to mathematics education, such as mathematics, epistemology and philosophy of mathematics, history of mathematics, psychology, sociology, and pedagogy. The activity of theorizing or grounding is seen by Steiner as carried out by a community of people in mathematics education, which is regarded as an academic field and a domain of interaction between research, development, and practice. Other communities are interested in planning/design or management of teaching, assessment of learning, or teacher education.

In this article, we consider mathematics education research as a system of activities carried out by individual people or teams in communities interested in the problems of grounding research, dissemination of knowledge, and mathematics education practice[2]. The application of the CHAT notion of activity system could serves to describe and analyze mathematics education research as a system composed by five sub-activities: Foundation, Planning, Implementation, Evaluation, and Teacher Professional Development. We understand these five activities in a general sense; however, it is possible to further unpacked them into partial activities. For example, in this work, we identify six partial activities in teacher professional development. Moreover, since we interpret theories as activities carried out by groups or communities of researchers, mathematics education is constituted by a network of diverse activity systems.

Identifying the different elements of each partial activity and their relationships can reveal contradictions or tensions and progress in the elaboration of a modular and inclusive theoretical system (Ruthven, 2014) that addresses the complexity of the mathematics education activity. In addition, the CHAT can help in the analysis and articulation of mathematics education theories by emphasizing the communities involved and the ecological context in which they take place, as well as the subjects, objects/motives, and instruments.

Specifically, the aim of this paper is to apply the CHAT tools, as a case study, to describe and analyze the construction of the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction as a modular and inclusive theoretical system (Godino & Batanero, 1994; Godino et al., 2007). A characteristic of CHAT is the identification of contradictions or dilemmas between different activity systems. This feature helps understand the construction of OSA because it leads us to focus on the tensions or dilemmas between different theories used in mathematics education research. In this way, we can relate the OSA theoretical instruments to taking a position before various dilemmas between theories on the foundations, design, implementation, and evaluation of educational-instructional processes.

This paper is organized into the following sections. First, we describe the structure of the activity according to the third and fourth generation CHAT model proposed by Engeström (1987). We follow with a brief synthesis of the emergence and development of the OSA as an open and modular theoretical system for mathematics education. The next sections describe the tools developed within OSA to address the Foundations of mathematics education, Planning and design, Implementation, Evaluation and Teacher Professional Development. We end with a synthesis of the dilemmas posed by various mathematics education theories, some of which are addressed by the OSA, while others prefigure a research agenda on theory articulation supported by the CHAT as a basic meta-theory.

To clarify the different theoretical tools developed in OSA, we describe their application to analyze the design and implementation of an instructional problem on elementary probability directed to prospective primary school teachers (Godino et al., 2019).

2 Activity structure in the CHAT

The activity theory has evolved in a succession of four generations of theorizing and research, each of which developed its own analysis unit (Engeström & Sannino, 2021). The first generation was embodied in the work of Vygotsky (1997), who considered *culturally mediated action* as the main object of research. Leont'ev (1978) elaborated the *activity system* as the unit of analysis of the second generation, understanding activity as a relatively enduring, communal system in which the division of labor separates

[2] From Steiner's (1990) perspective, research in mathematics education is a part of mathematics education. The OSA has developed a system of tools to support research on mathematics education problems, taking a broad view of the nature and diversity of these problems. In some cases, research is a matter of understanding, which leads to describing, explaining, and predicting phenomena (basic research, or philosophical reflection). In other cases, it is a matter of intervening in an informed manner in educational-instructional processes to optimize their design, implementation, and evaluation (applied research). We do not even discard the relevance and usefulness of considering as research the work done by teachers themselves (action research and reflective practice) when it is done in a systematic and informed way to improve the learning of their students.

different goal-oriented actions and combines them to serve a collective object.

> The activity concept therefore differs from the kind of events educators usually denote by activity, which are structures that allow children to become engaged, involved, and busy and that one might better refer to as tasks (Roth & Lee, 2007, p. 201).

In mathematics education, the notion of activity system has been applied at both micro (classroom learning) and macro (school and society) levels, in which the classroom context is nested.

> Nesting the micro activity system within broader contexts may provide educational researchers with further understanding of how micro contexts are influenced and dependent upon larger and powerful entities such as the institutional and cultural-historical contexts levels (Núñez, 2009, p. 11).

The fundamental concept of the CHAT is human activity, which is understood as an intentional, mediated, and transformative interaction between humans and the world. In a broad sense, any interaction of a subject with the world can be qualified as activity, although in the CHAT, the term activity has a narrower meaning. It refers to a specific level of subject-object interaction, where the object has the status of a motive, i.e., an object satisfies a certain need of the subject. Leont'ev distinguishes between action and activity. While an action is carried out by an individual or a group to fulfil some goal, an activity is carried out by a community (by unfolding a division of labor and various production tools). Both action and activity are opposed to operations which are habitual behaviors triggered by particular conditions (Bakhurst, 2009, p. 199–200).

The triangular structure (Fig. 1) of any activity system includes six elements and their respective interactions: subject, object, instruments, rules, community, and division of labor. Another central concept is that of *contradiction*, which is the source of change and development in such systems. Expansive learning takes place, i.e., the generation of new ways and instruments for performing actions and operations, by addressing contradictions between the elements of a system or between two or more activity systems.

The *subject* is the individual(s) involved in the activity, in the case of education, the students, in-service or prospective teachers, teacher educators, etc. The *object* (final purpose for the subject behavior) is the matter or problem at which the subject activity is directed and is transformed into results with the help of physical and symbolic tools.

Object/motives reflect collective interest, the interests of the collective, and therefore are general. They reflect generalized needs satisfied in and through the network of collective activities (Roth &Radford, 2011, p. 14).

If the student is the subject of the activity, the object is usually mathematical problem solving, practice of algorithms, preparation of assessments; in the teacher education practice, the object may be improving teaching, learning mathematical practices, or developing skills to motivate students.

Instruments are whatever is used in producing changes in thinking, believing, or belonging (psychological tools, such as mathematical concepts, procedures, and language), as well as material tools (such as computers, mathematical software, etc.). The *community* is composed of the subject and other individuals who intend to achieve a shared object; they are usually organized to congregate at a common place and time (teacher and students, family, friends, educational leaders). In some cases of applying the CHAT, community members do not gather in a common space and time, e.g., the community of people developing and applying a theory. Likewise, individuals can be grouped together to

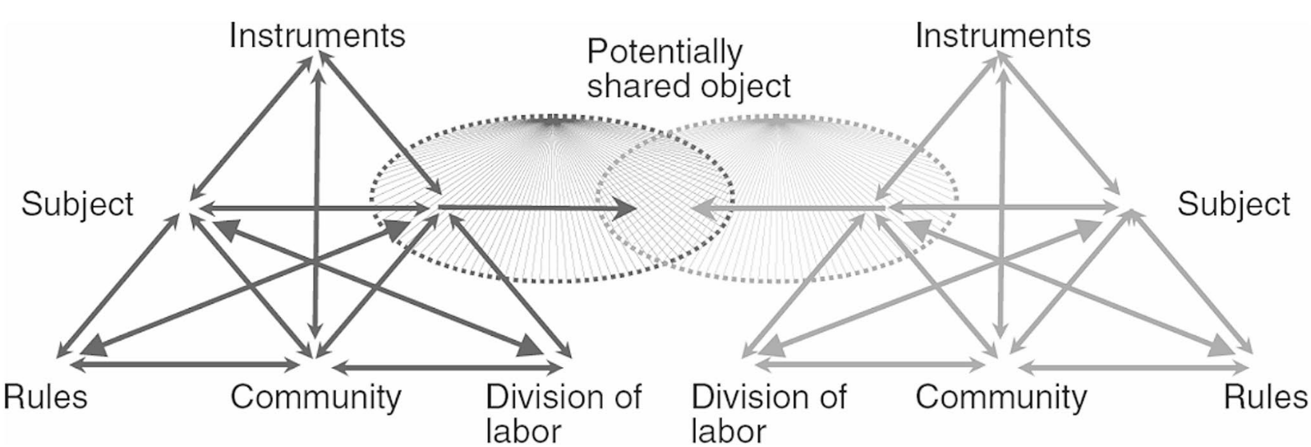

Fig. 1 Two activity systems and a potentially shared object (CHAT third generation model) (from Engeström, 2009, p. 305)

form a unique entity, a collective subject, with shared goals, interests, or characteristics. Community brings individuals together through social norms and division of labor. *Rules* are social norms, conventions or traditions established by the community to govern its members and can be implicit (didactic contract) or explicit (curriculum).

Since activity systems are increasingly interconnected and interdependent, many recent studies take a constellation of two or more activity systems that have a partially shared object as their unit of analysis. These interconnected activity systems may lead to a producer-customer relationship, a partnership, a network, an alliance, or some other collaboration pattern. The set of at least two activity systems connected by a partially shared object is the main analysis unit of third-generation activity theory (Fig. 1).

Increasingly complex problems with broad societal ramifications, such as climate change or pandemics, connect many activity systems across national borders (Engeström, 2009). They tend to transcend the boundaries of a specific activity, or a single society and their study is the focus of the activity theory fourth generation, which aims to solve critical societal problems (Engeström & Sannino, 2021). Despite their differences, the four generation authors share some foundational ideas, as they all consider that work should be analyzed as an object-oriented practice, mediated by instruments, and changing through its inherent contradictions.

3 OSA emergence and development

A need to clarify fundamental notions to describe cognitive phenomena, which were characterized through different constructs, such as knowledge, conception, concept, schema, operative invariant, meaning, or praxeology arose at the beginning of the 1990s, in the context of a Mathematics Education theoretical course in a doctoral program at the University of Granada, Spain. Recognition of the disparity and dilemmas associated to these cognitive and epistemic notions in theoretical frameworks such as the Theory of Didactic Situations in Mathematics (Brousseau, 2002), Conceptual Fields Theory (Vergnaud, 1990), Registers of Semiotic Representation Theory (Duval, 1995), and the Anthropological Theory of Didactic (Chevallard, 1992), motivated research resulting in the OSA early works.

The initial problem originating the OSA first development stage was clarifying the meaning of a mathematical object, its relationship with other constructs, such as concept, conception, and understanding (Godino & Batanero, 1994). The distinction between personal and institutional features of meaning was essential to articulate the epistemological and cognitive approaches in mathematics education. In the second stage (from 1998 onwards), the theoretical framework was extended (Godino et al., 2007) to describe the mathematical activity and the communication of its productions. Progress was made in the development of a specific ontology and semiotics to study the interpretation of mathematical sign systems playing a role in didactic interactions. The development of a mathematical knowledge theory (see, Font et al., 2013), with anthropological (Wittgenstein, 1953; Bloor, 1983), pragmatic (Peirce, 1931-58), and semiotic (Hjelmslev, 1943) bases provided the grounds to articulate some theories of learning and teaching mathematics and to address the following aspects related to the design of instructional processes and teachers' education:

- Analyzing the implementation of mathematical instructional processes (Godino et al., 2007).
- Studying the normative dimension of teaching and learning processes (Molina et al., 2021), and identifying connections and complementarities between didactic contract and socio-mathematical norms.
- Expanding and systematizing the didactic suitability criteria (Breda et al., 2018) that articulate the scientific and technological facets of mathematics education.
- Developing an integrative model of mathematics teacher knowledge and competencies based on the OSA assumptions and tools (Godino et al., 2017).

In the OSA, mathematics education knowledge has scientific and technological character since it is aimed at understanding (describing and explaining) the mathematics education activities and developing applied resources. This point of view makes it possible to address the existing dilemma between different conceptions or paradigms of mathematics education research, those that emphasize its character as a science (Gascón & Nicolás, 2017), whose objective is the understanding of educational phenomena and those that consider education as a socio-technology (Bunge, 1999) and emphasize the component of intervention on the practice for its improvement.

The method used to analyze the construction of the OSA through CHAT consists of identifying the six elements — subject, object, instruments, rules, community, and division of labor —that characterize each of the five partial activities mentioned in Sect. 1 and some dilemmas between different theories, paradigms, or research approaches to which the OSA tries to respond. The strategy followed by the OSA to address these dilemmas or controversies between theories can be either blending or complementarity (Scheiner, 2020).

In the following section, we discuss theoretical problems of ontological, epistemological, and semiotic clarification of mathematical knowledge. We also describe in latter sections those problems related to teaching and learning processes to make them as suitable as possible. There are, therefore, activities in mathematics education aimed at understanding mathematics and learning processes, and activities oriented towards educational practice, which are discussed in the sections on planning, implementation, evaluation, and teacher professional development.

4 Foundations of mathematics education research

In this article, the object/motive of the founding activity of mathematics education research is the elaboration of models of mathematical activity, emergent objects, meaning and characterization of mathematical cognition, understood from the individual and cultural points of view. It is a basic research that serves to understand the nature of school mathematics and as foundation for activities related to the design of educational-instructional processes.

The subject of the founding activity is the individual researcher or the research team, which are members of one or more communities belonging to various disciplines interested in the teaching and learning of mathematics (mathematics, epistemology, ontology, semiotics, psychology, pedagogy, etc.). The problem of articulating paradigms, methodologies and knowledge arises because each community usually is interested in partial aspects. Even though it may be justified to divide a complex problem into subproblems, it seems necessary to address the coherent articulation of various approaches and solutions (Prediger et al., 2008). It is like assembling a puzzle where each piece is a partial solution, and the goal is for all of them to fit together harmoniously. Sometimes, partial solutions may appear contradictory or challenging to integrate. The key is to find the theoretical or conceptual framework that unites them coherently.

Each discipline involved in the founding activity (mainly epistemology, psychology, and semiotics) involves different research paradigms that constitute the rules, habits, or implicit traditions assumed by their respective communities. In OSA, it is considered necessary to select and articulate these paradigms in a coherent manner through a strategy based on blending and the search for complementarities between theories. The normative component (rules) of the founding activity can be interpreted more broadly in terms of the ecological niche[3] (Alley, 1985) in which the activity takes place, i.e., the social environment that supports and conditions its development. Thus, the Theory of Didactical Situation in Mathematics (Brousseau) and the Anthropological Theory of Didactic (Chevallard) were initially linked to the IREMs (Research centers in mathematics education for training teachers), associated to mathematics departments, which could explain the interest of these researchers in developing epistemological models of mathematics as an entry point to the didactics of mathematics. The training and affiliation to psychology or education departments of researchers from the Conceptual Field Theory (Vergnaud) or Registers of Semiotic Representation Theory (Duval) might be a reason for justifying their cognitive perspective. In the same way, the academic environment of a Mathematics Education Theory course explains the interest in the clarification and articulation of theories that prompted the OSA emergence.

The vertex of instruments (Fig. 1) includes the conceptual and methodological tools developed by the subjects to address the tasks required by the activity object/motive. Specifically, the OSA develops instruments to address the founding activity, in relation to the nature of professional and school mathematical knowledge. The connections, concordances, and complementarities of the OSA tools to carry out the epistemic, cognitive, and semiotic analysis with respect to those developed by the French mathematics didactics are analyzed by Godino et al. (2006).

The OSA researchers consider necessary to begin this founding activity by problematizing the nature of the mathematics to be taught. This principle is shared by the epistemological approach (fundamental didactics) in mathematics education research (Gascón, 1998). From an educational point of view, mathematics is conceived with a dual nature, firstly, as a system of objects and secondly, as a system of practices. Mathematical practices _ actions carried out by people when faced with specific types of problem situations _ are the origin and *raison d'être* of mathematical abstractions, ideas, or objects (Font et al., 2013).

Some examples of dilemmas in the foundations of mathematics education (nature of mathematical objects and their emergence, knowledge, and meaning) are:

- Platonism (mathematical objects as pre-existing entities) *versus* nominalism (mathematical objects reduced to names or symbols).
- Mentalism (mathematical objects as mental entities) *versus* culturalism (mathematical objects as cultural-discursive entities).
- Meaning as use *versus* meaning as mental referents of terms or symbols.

[3] This is a metaphorical use in the field of epistemology of science of the biological concept of econiche, the specific role that a species occupies within an ecosystem. Alley (1985) standardized the use of econiche by limiting the concept to functional relationships between organisms and their environment, including epistemic relationships, i.e., the exchange of information through perception and cognition. Several authors have used this metaphor to describe the ecology of knowledge (Chevallard, 1991; Toulmin, 1977).

The onto-semiotic configuration of practices, objects, and processes tool (Fig. 2) synthesizes the OSA position in facing these dilemmas.

At the center of the diagram are the problems, the operative and discursive practices to solve them, and the ecological context in which problem-solving occurs. Various types of objects (languages, concepts-definitions, propositions, procedures, and arguments) emerge from this activity, understood as functional entities (playing a specific role in mathematical activity) (Font et al., 2013). The centrality of problems indicates the assumption of mathematics as a human activity and as a system of objects, as opposed to Platonist, empiric-realist, and nominalist positions.

The institutional-personal duality, through which practices and objects are viewed, indicates the proposal to resolve the dilemma between epistemological and cognitive approaches by assuming their complementarity. From the point of view of mathematics education, it is necessary assuming that mathematics has a double dimension: cultural (institutional, epistemic) and individual (personal, cognitive), describable through the epistemic and cognitive configurations with a similar structure. For the dilemma or tension between semiotic theories that propose the use as meaning of words and symbols (Wittgenstein) versus theories for which meaning is the concept referred to (Vygotsky), OSA assumes that both positions are complementary. Using the duality expression-content, applicable to practices (uses) and objects (referents and references), meaning is defined as the content of any semiotic function (relation between expression and content); this content can be a system of practices (uses) or any object.

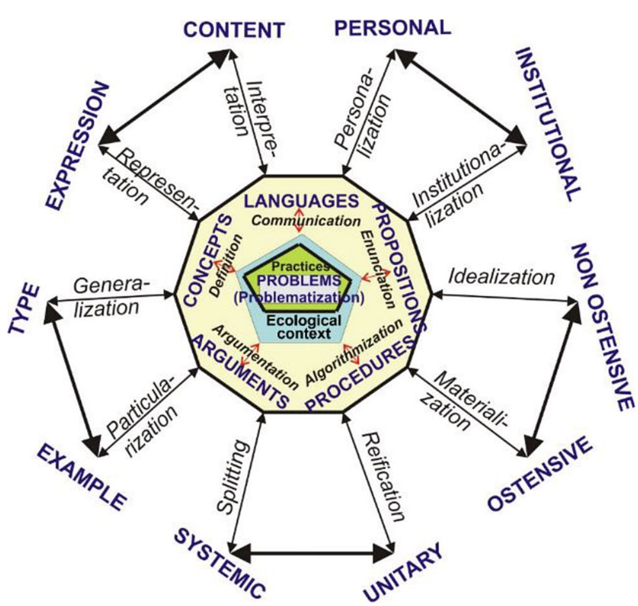

Fig. 2 Onto-semiotic configuration of practices, objects, and processes (Godino, 2023, p. 17)

To clarify the meaning of the OSA constructs, we include below a summarized description of the training experience analyzed by Godino et al. (2019). In this experience, the authors proposed the following task to a sample of 58 prospective primary school as a part of a mathematics education course:

> We will play with two dice. We throw the dice and add the points obtained. If the result is 6, 7, 8, or 9, player A wins a counter; if the result is different, then player B wins a counter. (a) Do you prefer to be player A or B? (b) Is this game fair? Justify your answers.

The aim of the task is that participants elaborate the sample space of the experiment (throwing two dice), as well as the distribution of the random variable "sum of two dice". Then, they have to discover that the game is unfair, as Player A has a much higher probability of winning the game (intended institutional knowledge, represented in Fig. 3. This institutional knowledge includes *concepts* (for example, random experiment, events, simple and compound probability, random variable, distribution), *languages* (verbal, graphical tabular), *properties* (symmetry and the mode of the variable distribution), *procedures* (computations, representing graphs) and *arguments* to explain the solution. Thus, mathematical practice is meaningful if students articulate the mathematical objects that constitute a contextualized and coherent network.

However, some participants did not achieve all this institutional knowledge; for example, some students assumed that all the results of the sum of both dice were equiprobable, while others failed to represent all the possible sums or compared distributions using absolute frequencies instead of relative frequencies. Therefore, we observe that the personal meanings that students attribute to the problem and to the mathematical objects involved do not match the institutional meanings intended by the trainer.

5 Instructional planning and design

The object/motive of planning and design is to select mathematical content for teaching and learning, which also implies its transformation or preparation (Scheiner et al., 2022), producing the curriculum, specific lessons and other resources for the different educational levels and contexts. This work is carried out by various individuals, teams (teachers, authors of books and other study aids) or curricular agents. They are, therefore, part of communities where there is a division of labor among its members: general curricular guidelines are provided by agents appointed by the educational authorities; teachers design

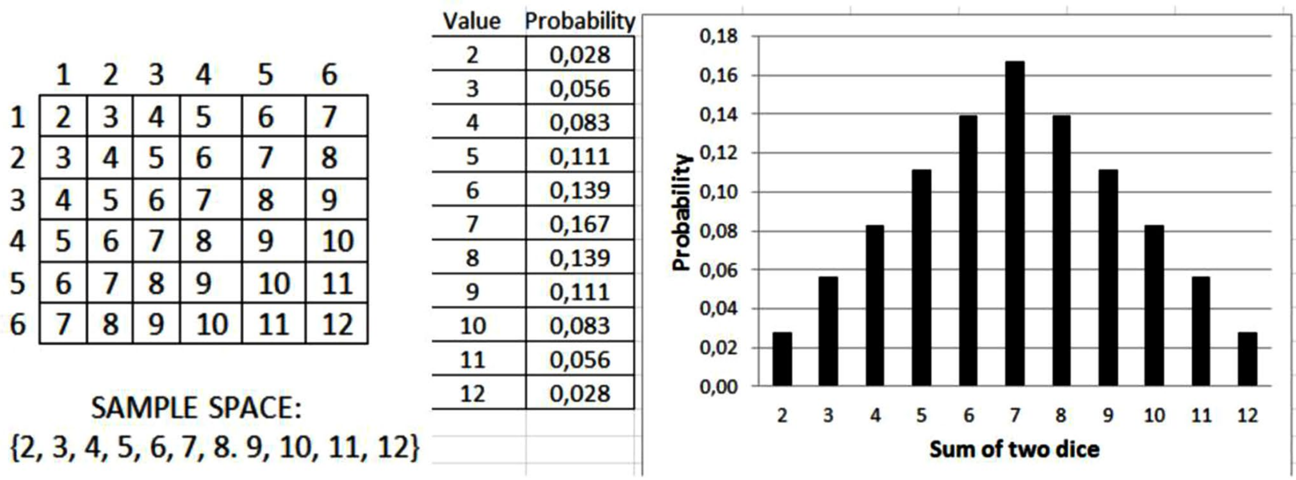

Fig. 3 Intended institutional knowledge in the task proposed to participants (Godino et al., 2019, p. 154)

the lessons, supported by teaching resources developed by authors and publishers. Planning takes place in various settings or ecological niches that support and condition its realization; time, financial means, educational policies, etc. are conditioning factors of the curriculum planning and lesson design. Instructional-design theories (Reigeluth, 1999) and learning sciences (Sawyer, 2014) address these issues.

The planning instruments vary depending on the educational theories used and specifically on the social and educational context norms (Molina et al., 2021). In the OSA framework, we introduced the institutional and personal meanings tool in pragmatic terms as systems of operative and discursive practices (Godino et al., 2021). The types of institutional and personal meanings considered (Godino et al., 2007) provide criteria for curriculum and lesson design, and evaluation of, both the educational process and the students level of acquisition of meanings. The general criterion to assess the design, implementation, and results of the instructional process follows from the coherence of the different personal (respectively, institutional) meanings and from the coupling between personal and institutional meanings at the different moments of the instructional process.

The ecology of meanings metaphor (Godino, 2023) reflects in the OSA the correspondences between the different types of knowledge involved in educational settings. Interpreting the meanings of a mathematical object as systems of practices helps consider these systems as new objects, without neglecting the view of mathematics as activity. The systems of practices involved in solving problem-situations are relative to individuals and communities of practice (institutions); consequently, meanings, and knowledge, are also relative to them. For example, the meaning of random variable presented to participants in Godino et al. (2019) (with only discrete variables and with little use of algebra) is far different from the meaning of random variable in formal probability.

The planning of teaching must consider the specificity of knowledge concerning the community, which leads to exploring partial meanings and articulating them progressively to form a *global or holistic meaning* (Wilhelmi et al., 2007) that serves as a reference model in instructional design.

The types of mathematical objects and the ecology of meanings metaphor help addressing contradictory or partial views on understanding the process of preparing the mathematics content to be taught (Scheiner et al., 2022).The *unpacking metaphor*, which finds its origin in the Anglo-American school of thought of pedagogical reduction of mathematics (Ma, 1999; Ball & Bass, 2000); the elementarization metaphor, developed in the German school of thought of didactic reconstruction of mathematics (Kirsch, 1987), and the recontextualization metaphor, which originates in the French school of thought of didactic transposition (Chevallard, 1991). These views suggest that preparing mathematics for teaching is largely a one-sided process in the sense of an adaptation of the knowledge in question. The OSA proposes a more holistic understanding: preparing mathematics for teaching as ecological engineering. By using the *ecological engineering metaphor*, the preparation of mathematics for teaching is presented as a two-sided process that involves both the adaptation of knowledge and the modification of the teaching and learning environment (Scheiner et al., 2022).

6 Implementing instruction

Instruction is implemented jointly by a teacher and a group of students, and the object/motive is that students apprehend a mathematical knowledge previously transposed in the planning activity. Within the study community (classroom, school) there is a division of labor; the teacher and the students have different roles that are articulated following a system of rules (didactic contract), using specific physical, conceptual, and procedural artifacts. Thus, in the context of developing elementary algebra reasoning, examples of these artefacts are physical, calculator or GeoGebra; conceptual, a function as a model of a physical phenomenon; procedural, a stereotyped technique or algorithm. In the experience by Godino et al. (2019) the table and bar graph displayed in Fig. 3 are artifacts used to visualize the distribution of the sum of two dice.

In all cases, these artefacts are used in a specific way, i.e., they are instrumentalized. In fact, the implementation of instruction takes place in specific environments and circumstances that condition this development (students' abilities and willingness, time available, means, etc.). The diversity of aspects to be considered implies that the optimization has local character and requires the teacher's specific knowledge and skills, as well as the students' interest and perseverance. The complexity of implementation has originated various theories that suggest what tools should be used in each circumstance, what types of interactions are needed, or what rules should be followed to better articulate the teacher and students' roles. Tensions, dilemmas, and complementarities exist between the theoretical frameworks on learning and teaching mathematics that Sfard and Cobb (2014) call *acquisitionism* and *participationism*.

Although the constructivist models of instruction predominate in mathematics education, at least in some countries, some authors discuss the dominance of these models (Godino et al., 2019). Between the extremes that center on either the student or the teacher (Stephan, 2014), in other mixed models, both agents of the educational process play a leading role, which depends on the content to be learned and the students' prior knowledge. Godino et al. (2019) describe complementary constructivist (*student-centered*) and objectivist (*teacher-centered*) didactic models and use them to analyze the probability teaching experience described in the previous sections. Thus, the educator directly explained the basic probability concepts and definitions (teacher-centered), while participants explored the distribution of the sum of two dice by drawing graphs manually and using a spreadsheet (student-centered).

In the OSA, several theoretical artifacts or tools have been proposed to analyze the implementations of instructional processes, as well as suitability criteria to evaluate the use of media and interactions (Hummes et al., 2019). An essential assumption of the OSA didactic model is that the local optimization of mathematics teaching and learning requires considering the triple dialectic between the teacher's work, the students, and the mathematical content. Given the widely adopted principles of socio-constructivist learning, the presence of moments when students take responsibility for learning is positively valued. However, aware of the onto-semiotic complexity of mathematical knowledge, this constructivist learning principle is constrained by the following specific interactional criterion (Godino et al., 2020):

The way of interaction between teachers and learners should be adapted to the moments of the learning process, using a dialogic-collaborative format in the first encounter with the content and granting autonomy to the learner in the moments of practice and application.

The idea of contradiction of the CHAT applied to the implementation activity leads to focus attention on the tension between *constructivist* and *objectivist* positions. The postulate of the onto-semiotic complexity of mathematical knowledge, as well as the essentially regulative nature of mathematical definitions, propositions, and procedures, leads the OSA to elaborate a didactic model (conceptual artifact) of mixed type that proposes collaborative interaction formats in the moments of students' first encounter with the new tasks and of greater autonomy in the exercising and application of knowledge (Godino et al., 2020).

7 Assessment and evaluation of educational processes

The evaluation of the planning and implementation of educational-instructional processes involves the teacher and other agents interested in the overall evaluation of educational systems and learning outcomes (Niss, 1993). Thus, national, and international agencies are interested in the students' learning and the factors that determine this learning and apply standardized tests that often condition the curricula implemented. The object/motive of such evaluative activity is the whole instructional process, involving various facets and their interactions. The assessment also takes place at the local level, i.e., within the classroom, to gather information and make instructional decisions. The desired outcome is information on the learning achieved by students (summative assessment), or on the development of the instructional process at the local level (formative assessment).

Various professional communities are involved at the macro-level (i.e., in external summative evaluation) in the required tasks (design of instruments, implementation, analysis, and interpretation of results, etc.). Evaluation is a community activity at the local level as well, as it involves not only the teacher and the students but also the school and the family. The ecological setting in which the activity takes place is conditioned and supported by rules that regulate its periodicity, forms, procedures, and available means.

There are tensions and dilemmas between *formative* and *summative* evaluation in the assessment of learning at the local (internal to the classroom) and global (external) levels (Stufflebeam et al., 2002). Summative assessment requires developing objective measuring instruments that allow comparisons between groups, schools, and countries to make decisions at the macro level. This evaluation leads to a reduction of complexity, disregarding contextual details that may be essential from an educational point of view. Faced with this dilemma, OSA has taken a stand in favor of formative assessment and evaluation, developing an instrument that allows, more than objective measurement, the analysis of the complexity of the educational-instructional process and that support the systematic reflection of the teacher on teaching practice.

The assessment and evaluation activity within the OSA framework are based by the Theory of Didactical Suitability (Godino et al., 2023), which explicates and structures the axiological principles and optimization criteria of teaching and learning processes. For example, Godino et al. (2019) used the observation to assess different components of didactic suitability in their instructional process: a) *epistemic* (quality of the knowledge effectively implemented in the experience), *cognitive* (participants' difficulties and semiotic conflicts), *affective* (participants' involvement in the activity); *mediational* (use of tools, such as representation, spreadsheet and computer simulation to facilitate learning), *interactional* (interactions in the classroom) and *ecological* (relations of the experience with the curriculum and society).

The suitability notion provides an expanded view of the quality of mathematics instruction (Charalambous & Praetorius, 2018; Hill et al., 2011), by emphasizing an interpretive approach to the network of values at stake in teaching and learning mathematics. This theory highlights the complexity involved in optimizing these processes, where a balance in the implementation of principles related to the different facets and components involved, —which have a strong local component— is needed. The teacher needs to manage this axiological balance by weighing the relative importance of each aspect according to the circumstances of the people involved and the contextual conditioning factors (Breda et al., 2017).

8 Teacher professional development

The object/motive of teacher professional development is to develop teachers' knowledge, competencies, and experience throughout their professional practice; it includes initial and in-service education. Mathematics teacher education can be seen as a system involving several activities with internal connections, which should be considered in the design of training programs for the teachers (Fig. 4):

1) In *Mathematics teacher education*, the subject is the teacher educator, and the object/motive is the design, implementation, and evaluation of programs for teacher professional development. Usually, the teacher educator is part of a university department or center, within which the responsibility for the program design is shared, following curricular regulations.
2) The subject of *Teacher's learning* is the mathematics teacher. The object/motive is to learn to teach mathematics, acquiring didactic-mathematical knowledge and competencies selected by the educator (or by the trainee teacher in the case of self-training processes).
3) In *Mathematics teaching*, the subject is the mathematics teacher, and the object/motive is to plan, implement and evaluate mathematics instruction processes. The teacher expects to optimize the students' mathematical learning, through the optimal selection of appropriate instructional resources and content, and the management of interactions, by following criteria of didactical suitability.
4) In *Mathematical learning*, the mathematics student (subject) tries to achieve mathematical knowledge understanding and mathematical competence (object) through the resources provided by the teacher.
5) Mathematical thinking and competence are developed through the subject's involvement (whether student, teacher, educator, or researcher) in the solution of progressively more complex problems. The object/motive of *Mathematical activity* is to acquire mathematical problem-solving knowledge and competence. The instruments are the material resources that support the activity performance (means of calculation and representation) and the mathematical models (based on concepts, properties, and procedures) used.
6) The training, teaching, and learning of mathematical and didactic content should consider the results of *research in mathematics education* which is another more global activity. The individual researcher or research team is the subject of this activity, and its outcome is knowledge and resources to understand and improve the

mathematics teaching and learning and teacher education processes. In some circumstances, the same person can also act as both researcher and educator (in action research and self-training processes).

In mathematics teacher education, several theoretical models exist that propose categories of knowledge that teachers should have to favor the students' learning (Chapman, 2020; Wood, 2008). There are also other models with principles that efficient training programs should comply with (AMTE, 2017). However, these models are often partial, not explicitly grounded, or do not have the required level of detail. The conceptual and methodological tools developed in OSA (onto-semiotic configuration, ecology of meanings, didactic configuration and trajectory, didactic suitability, normative dimension) converge towards a mathematics teacher education theory (Godino et al., 2017; Pino-Fan et al., 2023) that attempts to address the shortcomings of other models. The knowledge and competent use of OSA instruments could serve educators and teachers to understand the complexity of educational-instructional processes, enabling them to reflect, inquire, and make decisions considering the various interconnected facets that condition mathematics teaching and learning activities.

9 Synthesis of dilemmas and conflicts in mathematics education addressed by the OSA

In the previous sections, we described mathematics education research as a complex social system that involves at least the activities of Foundation, Planning, Implementation, Evaluation, and Teacher Professional Development. We also argued that mathematics education theories can be conceived as systems of activities that attempt to answer the object/motive questions of all or part of these activities. This analysis has been applied as a case study to the OSA.

The structure of the activity systems proposed by the CHAT leads to the study of the historical-cultural and community dimensions of theories, as well as the ecological-normative context in which they attempt to provide

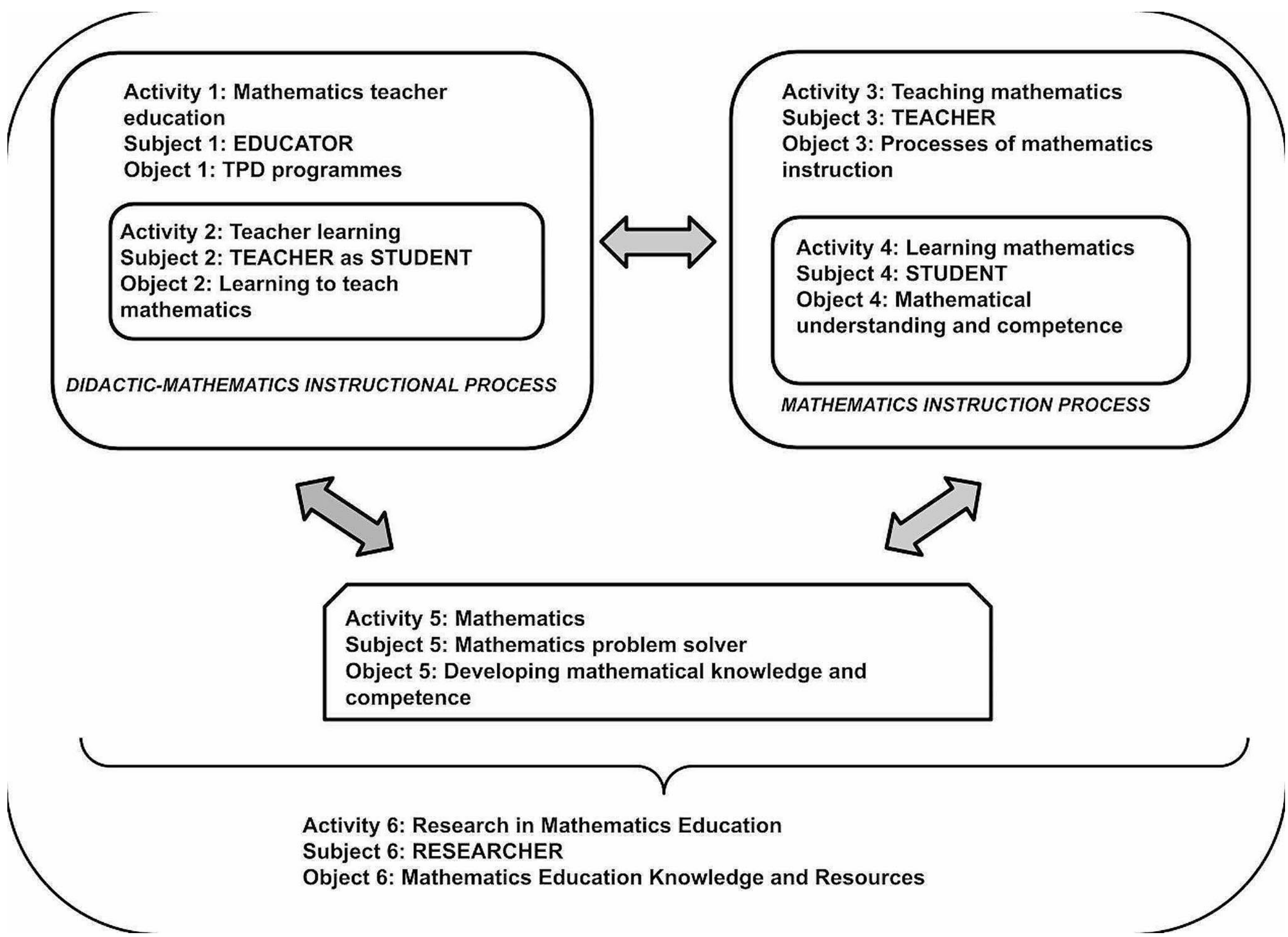

Fig. 4 Research, teaching and learning activities involved in mathematics teacher education

instrument-mediated answers to the questions that constitute their *raison d'être*. The CHAT notion of contradiction, which includes dilemmas, tensions, and conflicts between elements of activity (Nuñez, 2009), or between related activities, clarifies the reasons for changing systems and identifying unresolved contradictions that need to be addressed with new developments.

In Fig. 5, we present the OSA as a mathematics education theoretical system attempting to tackle the complexity of issues that characterize its different activity systems, and the related conflicts. The analysis performed in this paper help to recognize the interdependent relationships between the partial activity systems that make up mathematics education research. We recall that the assumptions about the nature of the mathematical activity and the analytical tools developed determine essential aspects of the planning, implementation, and evaluation of mathematical instructional processes.

The extent to which the instruments developed in OSA solve the contradictions between different theories needs further study. Specifically, the tensions between theories emphasizing the epistemic or the cognitive side, mathematics as a problem-solving activity or as a system of cultural objects, or between didactic models centered on the learner (constructivism) or the teacher (objectivism). These dilemmas in the foundations of mathematics education were revealed by comparing theories of French mathematics didactics that emphasize epistemological versus cognitive approaches and motivated the introduction in the OSA of the dialectic between institutional and personal dimensions of mathematical practices, meanings, and objects.

The dialogic-collaborative didactic model in the student's first encounter with new content is an instrument in the implementation activity, which solves the dilemma described in the foundational activity. Some didactic theories, such as

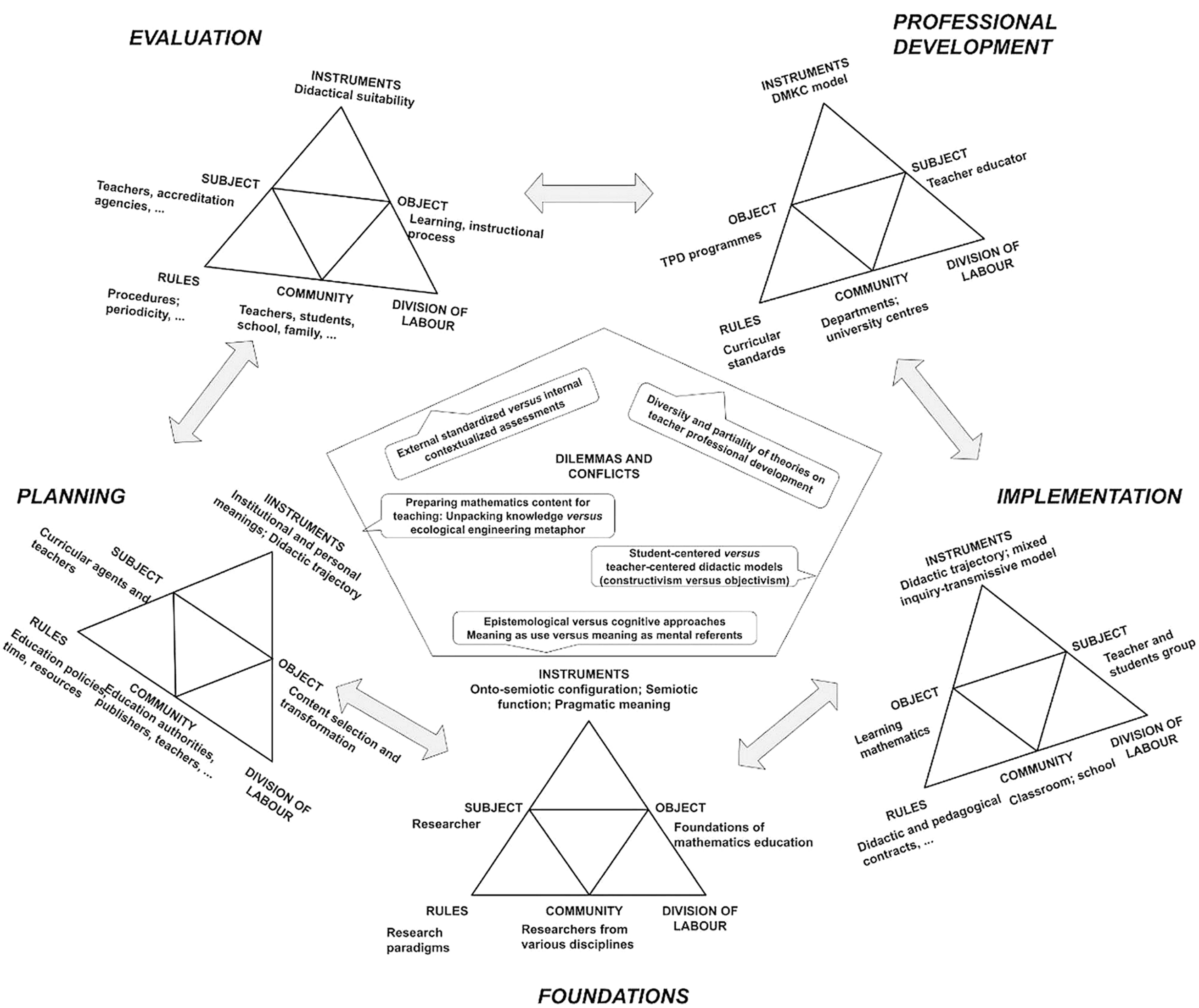

Fig. 5 Dilemmas, conflicts, and interdependencies between activity systems in OSA

the Theory of Objectification (Radford, 2021) advocate the application of a collaborative model, joint work of teacher and students, as preferable to constructivist, or traditional teacher-centered alternatives. In contrast, the educational-instructional model proposed by the OSA is more open, by assuming that the optimization of learning can be achieved with the suitable articulation of different types of didactic configurations. Didactical suitability helps to clarify and weigh the role of standardized external evaluation by showing the complexity of facets and components to be considered and the difficult balance of principles and values to be reconciled to optimize educational-instructional processes. Both summative and formative evaluation carried out by the teacher is essential to appreciate the relative importance of each aspect according to the context and circumstances of the people involved.

10 Conclusions

As a result of the analysis carried out in this paper, we make two original uses of the CHAT: (1) to view mathematics education research as an activity system formed by five partial activities (foundation, planning, implementation, evaluation, and teacher education) and (2) to consider a theory as an activity system. Using the triangular model for activity systems (Fig. 1) leads to broadening the view on theories toward the historical-cultural (community) context and the ecological (normative) niche in which they develop. These uses of the CHAT have made it possible to understand OSA as the study of the five partial activities for which specific conceptual and methodological tools have been developed. In addition, the activity of theoretical elaboration of OSA is presented as a proposal to address dilemmas or controversies (Fig. 4) existing among various theories used in mathematics education, also understood as activity systems.

The development of the OSA can be seen as a version of expansive learning. Engeström (1987) proposes that learning, within the framework of the CHAT, not only involves the assimilation of existing knowledge but also the creation of new knowledge and practices. The collective subject formed by individuals interested in the development of the OSA draws on tools developed within other systems of activity (theories). Nevertheless, through collaborative and socially mediated actions, it seeks to expand the original activities, generating new concepts, tools, and ways of approaching research in mathematics education. Instead of uncritically adopting other existing theories, the OSA collective subject actively seeks to contribute to the transformation of its research learning environments through the elaboration of new theoretical tools.

This work of reflection and analysis cannot be considered finished, but it is necessary to go deeper into each of the dilemmas and to show to what extent OSA proposes a strategy of blending, complementarity, or interplay (Scheiner, 2020) among the theories mentioned. In the 30 years since the first publication of OSA (Godino & Batanero, 1994), its theoretical development has progressed significantly (Font et al., 2013; Godino et al., 2007, 2023). However, it remains necessary to further develop the identification of concordances and complementarities with other theories and their effective application in the design, implementation, and evaluation of instructional educational processes, as well as in teacher training.

The analysis of some of the elements that characterize an activity system according to CHAT, as well as the description of the dilemmas in each of the five partial activities, should be further developed in future studies. The description of the instruments elaborated by the OSA (instrument vertex of Fig. 1), barely sketched in this article, has been made in several publications cited in the references, as well as the reason or motive for its construction. However, the identification of the research community, the division of labor among its members, and, particularly, the ecological niche in which the activities take place require new developments. This will allow us to offer a more complete historical-cultural perspective of OSA and its relationship with the remaining theories and research paradigms. The interpretation we propose of the vertex rules (Fig. 1) as the ecological niche in which the activity (in our case, a theory) develops allows us to think not only about the structures of support and conditioning but also about the relationships of symbiosis and competition with other theories, issues that, due to space limitations, is not addressed in this paper.

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Declarations

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