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Original articles

New methods for quasi-interpolation approximations: Resolution of odd-degree singularities

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ABSTRACT

In this paper, we study functional approximations where we choose the so-called radial basis function method and more specifically, quasi-interpolation. From the various available approaches to the latter, we form new quasi-Lagrange functions when the orders of the singularities of the radial function's Fourier transforms at zero do not match the parity of the dimension of the space, and therefore new expansions and coefficients are needed to overcome this problem. We develop explicit constructions of infinite Fourier expansions that provide these coefficients and make an extensive comparison of the approximation qualities and – with a particular focus – polynomial reproduction and uniform approximation order of the various formulae. One of the interesting observations concerns the link between algebraic conditions of expansion coefficients and analytic properties of localness and convergence.

1. Introduction

In this paper, we study functional approximations in one or more variables to approximands that are at a minimum continuous. A variety of approximation methods are available, by splines, multivariable polynomials, trigonometric polynomials etc., but here we choose the so-called radial basis function method.

Among the approximants that can be formed from the linear spaces spanned by shifts $\varphi(\|\cdot - x_j\|)$, where the norm is usually Euclidean and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is the so-called radial basis function, are mostly interpolants (see [4]) and quasi-interpolants (see [6]). Those in turn can be formed for finitely many scattered data or gridded data, often finitely many.

We will study quasi-interpolants in this article because they are relatively simple to form and have excellent convergence properties. Those originate from the spaces spanned by shifts $\psi(\cdot - x_j)$ containing polynomials of some low total degree *and* the ψ themselves coming from the span of shifts $\varphi(\|\cdot - x_j\|)$ decaying quickly. Also, collocation is often explicitly not desired. Those two properties together avail us to establish convergence theorems if the approximands f are smooth enough, the basis ingredients coming from local Taylor expansions.

All this rests on the "quasi-Lagrange functions" satisfying the famous Strang and Fix conditions, which we cite in an appropriate form for the convenience of the reader at the end of the introduction. A basic tool is that of Fourier transforms, in one or more dimensions, of a function f,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^n.$$

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Since the ψ are to be linear combinations of the $\varphi(\|\cdot - x_j\|)$ satisfying the Strang and Fix conditions depends on orders of singularities of the generalised Fourier transforms of the radial basis functions φ. In order to resolve those singularities at the origin, the coefficients forming the quasi-Lagrange functions from the φs are designed such that there is a high-order contact between the functions at zero.

This works very well if the parities of the singularities and the orders of the zero of the trigonometric polynomials in Fourier space that are formed from the aforementioned coefficients match and are even, but if the orders of the singularities are odd, extra work has to be done. Roughly speaking, it is no longer possible to have trigonometric *polynomials* coming from the said coefficients of ψ as a linear combination of shifts $\varphi(\|\cdot - x_j\|)$, but they end up in infinite series in order to have the required high order contact absorbing the singularities of different parities.

We aim to undertake an analysis of this phenomenon, our prime examples being multiquadrics, which ought to be used in odd dimensions so that the order of the singularity at the origin is even, and thin-plate splines, which ought to be used in even dimensions so that the order of the singularity at the origin is again even. And we intend to employ these radial functions just in the spaces of "wrong" dimensions, i.e., even and odd respectively and develop and compare different methods to resolve the mentioned problems (cardinal functions, infinite expansions for quasi-Lagrange functions and others). The specific examples we study are the thin-plate spline in dimension one in Section 2 and then compare to the generalised multiquadric in one dimension, which is in the "right" dimension (see Section 3). For the latter we give an explicit expression of the quasi-Lagrange function. We put particular emphasis not only on comparisons but on the order of decay (localness) of the ψs and of course on the polynomial precision of the generated vector spaces. In Section 4 we introduce a new approach of constructing quasi-interpolants, starting from the Fourier domain. Throughout, we shall need the following famous result of Strang and Fix (short SF-conditions). (See, among many other possible references, [6, Theorem 2.2]).

Theorem 1 (*Strang and Fix conditions*).
Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

1. *there exists a positive ℓ such that for some nonnegative integer m, when $\|x\| \to \infty$, $|\psi(x)| = O(\|x\|^{-n-m-\ell})$, which immediately implies m-fold differentiability of the Fourier transform,*
2. *$D^\alpha \hat{\psi}(0) = 0$, $\forall \alpha \in \mathbb{N}_0^n$, $1 \le |\alpha| \le m$, and $\hat{\psi}(0) = 1$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$,*
3. *$D^\alpha \hat{\psi}(2\pi j) = 0$, $\forall j \in \mathbb{Z}^n \setminus \{0\}$ and $\forall \alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$.*

Then the quasi-interpolant

$$Q_h f(x) = \sum_{j\in\mathbb{Z}^n} f(jh)\psi(x/h - j), \qquad x \in \mathbb{R}^n, \tag{1}$$

is well-defined and exact on the linear space $\mathbb{P}_m$ of polynomial functions of degree less than or equal to m. The approximation error can be estimated by

$$\|Q_h f - f\|_\infty = \begin{cases} O(h^{m+\ell}), & \text{when } 0 < \ell < 1, \\ O(h^{m+1}\log(1/h)), & \text{when } \ell = 1, \\ O(h^{m+1}), & \text{when } \ell > 1, \end{cases}$$

for $h \to 0$ and a bounded function $f \in C^{m+1}(\mathbb{R}^n)$ with bounded derivatives.

2. The thin-plate spline in one-dimension using infinitely many coefficients

Along this section and as outlined in the introduction, we consider the radial basis function: $\varphi(r) = r^2 \log(r)$. Therefore the relevant generalised Fourier transform is (see [9, Chapter 4.6]):

$$\hat{\varphi}(r) = -2\pi \left(\left(1 + \frac{1}{2} - \gamma \right) \delta''(r) - r^{-3} \right),$$

where, as usual, δ denotes the Delta-distribution and γ is Euler's constant ($\gamma \equiv 0.577216\ldots$).

At this step, we define μ as the real number (if it exists) satisfying $\hat{\varphi}(r) \doteq r^{-\mu} + O(r^{-\mu+1})$, $r \to 0^+$. Here, the $\doteq$ means equality up to a nonzero constant multiple.

We have $\mu = 3$ in this particular case. The quasi-Lagrange functions will take the form

$$\psi(x) = \sum_{j\in\mathbb{Z}} \lambda_j \varphi(|x - j|), \qquad x \in \mathbb{R}. \tag{2}$$

We study the resulting schemes for different choices of the λ coefficients in this "wrong" (odd) dimension, where we cannot achieve the SF-conditions using only finitely many coefficients. This is because the trigonometric expansions with odd-order zeros at the origin, resolving odd order singularities, will always be infinite unlike the even power singularities. We present several approaches to compute the λ coefficients.

2.1. Cardinal interpolation, infinite expansions from shifts of the radial function

An Ansatz that always works in forming of Lagrange functions, no longer quasi-Lagrange functions, from equally spaced shifts, is

$$\chi(x) = \psi(x) = \sum_{j\in\mathbb{Z}} \lambda_j \varphi(|x-j|), \qquad x \in \mathbb{R},$$

which satisfy $\chi(k) = \psi(k) = \delta_{0k}$ for all integers k, where we use the standard notation χ for the cardinal functions.

The coefficients λ_j come from the Fourier expansion within the Wiener algebra of the reciprocal of the so-called symbol, $\sigma(\vartheta) = \sum_{\ell\in\mathbb{Z}} \hat{\varphi}(\vartheta + 2\pi\ell)$, so that

$$\lambda_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\vartheta j}}{\sigma(\vartheta)} \mathrm{d}\vartheta, \qquad j \in \mathbb{Z}.$$

The localness or asymptotic decay of $\psi(x)$ is identified as $|\psi(x)| = O((1+|x|)^{-4})$ for all (in particular large in modulus) x. For a suitable result see [4, Theorem 4.3], and here $\mu = 3$.

As mentioned previously, the polynomial reproduction of the quasi-interpolant Q_h, that is $Q_h p = p$ for certain low-degree polynomials p, is crucial. In this case we get the polynomial reproduction: $\mathbb{P}_2$-reproduction because $\mu = 3$, (see [4, Theorem 4.4]).

A by now standard theorem then delivers a uniform approximation error for suitably smooth approximands $f \in C^4(\mathbb{R})$ with bounded derivatives

$$\left\| f - \sum_{k\in\mathbb{Z}} f(kh)\psi(h^{-1}\cdot - k) \right\|_\infty = O(h^3|\log h|)$$

for $h \to 0$ (see [4, Theorem 4.6] or the convergence results in [5]). We have to apply the remark in the proof, and not Theorem 4.6 itself, because μ is an integer or we can apply Theorem 1 with $m = 2, \ell = 1, n = 1$.

Cardinal interpolation always works without demands for certain parities of dimensions and orders of singularities; this is because no polynomials of trigonometric type but infinite expansions are used. We now turn to "genuine" quasi-interpolations where no cardinal conditions are demanded.

2.2. Quasi-interpolation without cardinality conditions

We now wish to go away from the well-known cardinal function approach and use straight quasi-interpolation instead. That brings us to the problem, when the parity of the radial function's generalised Fourier transform's singularity at zero is odd, we can no longer form a trigonometric *polynomial* q, say, that matches the degree of the said singularity. This comes from the fact that only even powers of trigonometric expansions have finitely many coefficients when written as Fourier series expansions. So we need to use expansions of periodic series with infinitely many coefficients λ_j coming from the Fourier coefficients of $(2-2\cos x)^{3/2}$ or for instance $|\sin x|^3$. These are of course by no means unique, but mere examples. Therefore we end up in expressions like (2).

We are interested in applying the SF-conditions to check the degree of polynomial reproduction of the quasi-interpolation with thin-plate splines in one dimension. They depend on the properties of the (classical) Fourier transforms of ψ. By straightforward computations and dividing by 2π in order to normalise it can be shown that

$$\hat{\psi}(0) = 1, \qquad \frac{d\,\hat{\psi}}{d\xi}(0) = 0, \qquad \frac{d^k\hat{\psi}}{d\xi^k}(2\pi j) = 0,\ \forall j \in \mathbb{Z}\setminus\{0\},\ k = 0,1, \qquad \frac{d^2\,\hat{\psi}(\xi)}{d\xi^2}(0) \neq 0.$$

As the SF-condition for first degree derivative is satisfied but the second derivative is not satisfied at $\xi = 0$, we have $\mathbb{P}_1$-reproduction at a maximum (but not any higher). We arrive at

Theorem 2. *Let φ be the thin-plate spline radial basis function. With $n = 1$ and the Fourier coefficients of $(2-2\cos x)^{3/2}$ being the λ_j, the quasi-interpolation*

$$Qf = \sum_{k\in\mathbb{Z}} f(k)\psi(\cdot - k) \tag{3}$$

is exact for linear polynomials p.

The next question is the decay of the quasi-Lagrange functions ψ: We expect $|\psi(x)| = O((1+|x|)^{-4})$ (see Theorem 11 for $M = 3$). This is one order better than the routinely required third order decay which would suffice for absolute convergence of the quasi-interpolant when at most linearly growing approximands (in particular linear polynomials) are inserted.

This leads us to the question of approximation error; a routine result gives us from the identified polynomial reproduction and the order of decay of the quasi-Lagrange functions $O(h^2)$ for $h \to 0$. (See Theorem 1 with $m = 1, \ell = 2, n = 1$.) Notice the absence of the logarithmic term due to the one order faster decay of the quasi-Lagrange function.

2.3. An intermediate formulation of ψ and its Fourier transform

Another scheme is the separation of the Fourier transform's singularity into two factors: one that resolves the odd singularities degree separately, and leaving an even and negative power, and then using the classical approach for the high even order contact at the origin.

So we begin in trying to improve the polynomial reproduction using the above scheme and the situation at zero. We set therefore

$$\hat{\psi}(\xi) = P(\xi)|\sin(\xi)|\hat{\varphi}(|\xi|), \qquad \xi \in \mathbb{R},$$

P being $P(\xi) = \sum_{k=-N}^{N} \mu_k e^{ik\xi}$ a suitable trigonometric polynomial. In summary, we set this new scheme in this way:

We have to fix the coefficients of the quasi-Lagrange functions. This always begins with setting the coefficients λ_k to be the (infinitely many) Fourier coefficients of the expansion of

$$P(\xi)|\sin(\xi)|, \qquad -\pi \le \xi \le \pi,$$

– this is by no means unique, we could use for instance

$$P(\xi)(1-\cos(\xi))^{1/2}, \qquad -\pi \le \xi \le \pi.$$

Many other choices of (roots of) trigonometric functions are possible. Depending on those coefficients, especially which trigonometric expansions they form and which orders their zeros have, we will arrive at a reproduction of polynomials, for example $\mathbb{P}_2$-reproduction.

In order to compute the terms and derivatives that will serve to verify up to which order the SF-conditions hold at zero, we will use

$$\hat{\psi}(\xi) = 2\pi \frac{P(\xi)|\sin(\xi)|}{|\xi|^3} \equiv 2\pi \frac{P(\xi)\sin(\xi)}{\xi^3}$$

as ξ goes to 0 because the singular term in $\hat{\varphi}(r)$ is a constant multiple of r^{-3} at the origin. In order to satisfy the conditions at the zero for $\mathbb{P}_2$-reproduction we demand according to the Strang and Fix approach

$$\hat{\psi}(\xi) = 1 + O(\xi^3). \tag{4}$$

Therefore, close to the origin, we can expand

$$2\pi \frac{P(\xi)\sin(\xi)}{\xi^3} = 2\pi \sum_{k=-N}^{N} \mu_k e^{ik\xi} \left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} \pm \cdots\right) \times \xi^{-3}$$

which is

$$2\pi \sum_{k=-N}^{N} \sum_{j=0}^{\infty} \mu_k \frac{(ik\xi)^j}{j!} \left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} \pm \cdots\right) \times \xi^{-3}.$$

Now, by imposing the condition (4) we have

$$1 + O(\xi^3) = 2\pi \sum_{k=-N}^{N} \sum_{j=0}^{\infty} \mu_k \frac{(ik\xi)^j}{j!} \left(\xi^{-2} - \frac{1}{3!} + \frac{\xi^2}{5!} - \frac{\xi^4}{7!} \pm \cdots\right).$$

The above equation will give conditions on the μ_ks which are not unique. Specifically, for $N = 2$ we obtain

$$\sum_{k=-2}^{2} \mu_k = 0, \ \sum_{k=-2}^{2} k\mu_k = 0, \ -\pi \sum_{k=-2}^{2} \left(k^2 + \frac{1}{3}\right) \mu_k = 1,$$

and further

$$-\frac{1}{6} \sum_{k=-2}^{2} \left(k + k^3\right) \mu_k = 0, \ \frac{1}{12} \sum_{k=-2}^{2} \left(\frac{1}{10} + k^2 + \frac{k^4}{2}\right) \mu_k = 0.$$

This can be formulated equivalently as

$$\sum_{k=-2}^{2} \mu_k = \sum_{k=-2}^{2} k\mu_k = 0, \quad \sum_{k=-2}^{2} k^2\mu_k = -\frac{1}{\pi}, \quad \sum_{k=-2}^{2} k^3\mu_k = 0, \quad \sum_{k=-2}^{2} k^4\mu_k = \frac{2}{\pi}. \tag{5}$$

We will then choose

$$\mu_{-2} = \frac{1}{8\pi}, \ \mu_{-1} = -\frac{1}{\pi}, \ \mu_0 = \frac{7}{4\pi}, \ \mu_1 = -\frac{1}{\pi}, \ \mu_2 = \frac{1}{8\pi}$$

or

$$P(\xi) = \frac{1}{8\pi} e^{-2i\xi} - \frac{1}{\pi} e^{-i\xi} + \frac{7}{4\pi} - \frac{1}{\pi} e^{i\xi} + \frac{1}{8\pi} e^{2i\xi},$$

i.e.

$$P(\xi) = \frac{7}{4\pi} - \frac{2}{\pi}\cos(\xi) + \frac{1}{4\pi}\cos(2\xi).$$

We can easily verify that the upper bound on the right-hand side of (4) holds for $\hat{\psi}(\xi) = P(\xi)|\sin(\xi)|\hat{\varphi}(|\xi|)$.

Now, let us study the behaviour of the derivatives $\hat{\psi}^{(\ell)}(2\pi j)$, $j \in \mathbb{Z} \setminus \{0\}$. The purpose of this is checking the SF-conditions:

1. For $\ell = 0$ we have

$$\hat{\psi}(2\pi j) = 0, \quad j \in \mathbb{Z} \setminus \{0\},$$

because of the $|\sin(\xi)|$-term and the continuity of $\hat{\varphi}(\xi)$ away from the origin.

2. For $\ell = 1$

$$\frac{d\hat{\psi}}{d\xi}(\xi) = P'(\xi)|\sin(\xi)|\hat{\varphi}(\xi) + P(\xi)\frac{\sin(\xi)\cos(\xi)}{|\sin(\xi)|}\hat{\varphi}(\xi) + P(\xi)|\sin(\xi)|\hat{\varphi}'(\xi) \tag{6}$$

which vanishes for all $2\pi j$, $j \in \mathbb{Z} \setminus \{0\}$: the first and the third term clearly vanish; the second term of the right-hand part of (6) vanishes due to $\sum_{k=-2}^{2} \mu_k = 0$.

3. And for $\ell = 2$ the derivatives of the first and third terms of (6) vanish at $2\pi j$ with $j \in \mathbb{Z} \setminus \{0\}$ (we now have to use that $\sum_{k=-2}^{2} k\mu_k = 0$). In fact, problems could come from the derivative of the second term of (6) which we will therefore have to compute explicitly. It is

$$\begin{aligned} &P'(\xi)\frac{\sin(\xi)\cos(\xi)}{|\sin(\xi)|}\hat{\varphi}(\xi) + P(\xi)\frac{\cos^2(\xi)}{|\sin(\xi)|}\hat{\varphi}(\xi) - P(\xi)|\sin(\xi)|\hat{\varphi}(\xi) \\ &+ P(\xi)\frac{\sin(\xi)\cos(\xi)}{|\sin(\xi)|}\hat{\varphi}'(\xi) - P(\xi)\frac{\cos^2(\xi)}{|\sin(\xi)|}\hat{\varphi}(\xi) \\ &= P'(\xi)\frac{\sin(\xi)\cos(\xi)}{|\sin(\xi)|}\hat{\varphi}(\xi) - P(\xi)|\sin(\xi)|\hat{\varphi}(\xi) + P(\xi)\frac{\sin(\xi)\cos(\xi)}{|\sin(\xi)|}\hat{\varphi}'(\xi). \end{aligned}$$

In summary, the second order derivatives of $\hat{\psi}$ vanish at the points $2\pi j$, $j \in \mathbb{Z} \setminus \{0\}$ and we have all the requirements for getting $\mathbb{P}_2$−reproduction.

We sum our findings up in the following result.

Theorem 3. *Let φ be the thin-plate spline radial basis function, $n = 1$ and the Fourier coefficients of $P(\cdot) \times |\sin(\cdot)|$, P as above, being the quasi-Lagrange function's coefficients λ_j in (2). Then the quasi-interpolation (3) satisfies*

$$Qp \equiv p$$

for quadratic polynomials p.

Remark 4. At the points $x = 2\pi j, j \in \mathbb{Z} \setminus \{0\}$ the third derivative of $\hat{\psi}$ has a jump discontinuity, so we will not be able to satisfy high enough degree SF-conditions that we could successfully impose in order to arrive at $\mathbb{P}_3$-reproduction.

We note that a convenient way to show that the second order SF-conditions, and only those degrees, are satisfied at the $\xi = 2\pi j, j \in \mathbb{Z} \setminus \{0\}$ (while they would hold at zero up to order three, although that does not help) is to notice the following: if we define $g(\xi) = \frac{1}{2}\sin|\xi| + \frac{1}{2}\sin|\xi - \pi|$, then

$$|\sin\xi| = \sum_{k=-\infty}^{\infty} g(\xi - k\pi), \qquad \xi \in \mathbb{R}.$$

Therefore, using P as above and expanding about the origin we get near zero

$$\begin{aligned} P(\xi)|\sin\xi|\hat{\varphi}(|\xi|) &= 2\pi P(\xi)|\sin\xi| \times |\xi|^{-3} = (\sin\xi)2\pi\xi^{-3}\left(\frac{\xi^2}{2\pi} + \frac{\xi^4}{12\pi} - \frac{7\xi^6}{360\pi} + \cdots\right) \\ &= \left(\xi - \frac{\xi^3}{6} + \cdots\right) \times \left(\frac{1}{\xi} + \frac{\xi}{6} - \frac{7\xi^3}{180} + \cdots\right) \end{aligned}$$

which is

$$1 - \frac{7}{120}\xi^4 + O(\xi^6)$$

near zero, so third order SF-conditions are possible, but since

$$P(\xi)|\sin\xi|\hat{\varphi}(|\xi|) = |\xi|^{-3}\left(|\xi - 2\pi j|^3 + O(|\xi - 2\pi j|^7)\right)$$

near $\xi = 2\pi j, j \in \mathbb{Z} \setminus \{0\}$, only second order SF-conditions are satisfied. The mentioned third degree derivative discontinuity can be seen as well.

In fact, using the above form of $|\sin|$ we can compute its generalised Fourier transform which could then be used to compute some ψ explicitly.

The details of the computation are given in Appendix A.1 Lemma 14, where we show that the generalised inverse Fourier transform $\mathcal{F}^{-1}$ of $|\sin|$ is

$$\mathcal{F}^{-1}|\sin|(x) = \frac{1}{\pi} \times \frac{1 + \exp(ix\pi)}{1 - x^2} \times \mathcal{D}_2(x).$$

Here $\mathcal{D}_2$ is the Dirac comb

$$\mathcal{D}_2 = \sum_{k=-\infty}^{\infty} \delta(\cdot - 2k).$$

We always have to study the decay of ψ, because the resolution of the singularities at the origin will not deliver the desired polynomial precision of

$$\sum_{k\in\mathbb{Z}} p(k)\psi(\cdot - k) \equiv p \tag{7}$$

unless the series above converge absolutely. For this we will need at least an asymptotic decay of $O((1+|x|)^{-3-\epsilon})$ for the quasi-Lagrange function in order to get the summability of the series for the aforementioned quadratic polynomial reproduction. The asymptotic decay is fairly easily established (see, for instance, [4] or [6]) by exploiting the differentiability properties of the quasi-Lagrange function's Fourier transform. At even order multiples of π this $\hat{\psi}$ is infinitely smooth, but at odd multiples of π, we observe the following behaviours. We get about $\xi = (2j+1)\pi$ (but not about $\xi = 2j\pi$)

$$\lim_{\xi\to-\pi^-} \hat{\psi}'(\xi) = -\frac{8}{\pi^3}, \quad \lim_{\xi\to-\pi^+} \hat{\psi}'(\xi) = \frac{8}{\pi^3}.$$

As $\hat{\psi}(\xi) \notin C^1(\mathbb{R})$, then the maximum decay we can obtain for $\psi(x)$ will be $O((1+|x|)^{-2})$ according to our previous remark.

Finally we note that the approximation order (with the established decay and [6, Theorem 2.2] with $m=0$) will be $O(h\,|\log h|)$.

Remark 5. So far, we have only considered pointwise function evaluation as a mean to put the approximand's information into the quasi-interpolant. It is entirely possible to formulate the latter with different linear functionals applied to the approximation:

$$Qf(x) = \sum_{j\in\mathbb{Z}} \lambda_j(f)\psi(x-j)$$

where $\lambda_j(f)$ are suitable functionals. For example, they could be related to the polynomial $P(x)$, i.e.

$$\lambda_j(f) = \sum_{k=-N}^{N} \mu_k f(j-k) \quad \text{or} \quad \lambda_j(f) = \sum_{k=0}^{N} \mu_k f^{(k)}(j)$$

being μ_k the coefficients of $P(x)$ in order to get polynomial reproduction; we may also take local integrals.

That Ansatz would amount to a preconditioning of the approximand before it is fed into the quasi-interpolation procedure.

As soon as the decay of ψ in absolute terms is not sufficient for the absolute convergence (summability) of the series in (7) the functionals must help in order to improve the decay of the function ψ because we need the summability of the series so that the polynomial precision is formed in a well-defined way. However, we do not provide any further detailed analysis of this aspect in this work (but see [6]).

3. Finitely many coefficients for the quasi-Lagrange functions

In odd dimension the "natural degree" i.e., even order singularity at the origin, radial basis function using multiquadrics and their ilk will be $\varphi(r) = (r^2+c^2)^{3/2}$ instead of $\varphi(r) = r^2\log(r)$. In this case using only finitely many coefficients is possible. We present two schemes for the use of generalised multiquadrics, a classical one to resolve the singularity and another which improves the polynomial reproduction order. We finish this section with a short subsection on the possible use of B-splines.

3.1. The classical scheme with the "natural degree" radial basis function in $\mathbb{R}$

The radial basis function is in the first place $\varphi(r) = (r^2+c^2)^{3/2}$ which we shall call the generalised multiquadric function. Its Fourier transform has the property that near zero we have that $\hat{\varphi}(r) \doteq r^{-4}$, as required (even order), and therefore in our notation above $\mu = 4$.

The quasi-Lagrange function's now available *finite number* of coefficients λ_j are the Fourier coefficients of $(1-\cos(\xi))^2$. The latter is a trigonometric polynomial with an even order zero at the origin. Thus, the λ_j will have finite support with respect to their indices. One way of choosing the Fourier transform is, as a consequence, $\hat{\psi}(\xi) = (1-\cos(\xi))^2\hat{\varphi}(|\xi|)$.

It is interesting to verify the polynomial precision results for the cases in the classical way of analysis. Taking into account that the (distributional) Fourier transform, $\hat{\Phi}(\xi)$, of the generalised multiquadric $\Phi(x) = (c^2+\|x\|^2)^\beta$, $x\in\mathbb{R}^n$, $c>0$, $\beta\in\mathbb{R}\setminus\mathbb{N}_0$ is

$$\hat{\Phi}(\xi) = (2\pi)^{n/2}\frac{2^{1+\beta}}{\Gamma(-\beta)}\left(\frac{c}{\|\xi\|}\right)^{\beta+\frac{n}{2}} \times K_{\beta+\frac{n}{2}}(c\|\xi\|), \quad \xi\neq 0,$$

according to [9] we have, in our case ($n=1, \beta=3/2$),

$$\hat{\varphi}(\xi) = (2\pi)^{1/2}\frac{2^{5/2}}{\Gamma(-3/2)}\frac{c^2}{\xi^2}K_2(c|\xi|).$$

Furthermore, from [11] we read

$$\begin{aligned}\frac{c^s}{\|\xi\|^s}K_s(c\|\xi\|) = {} & 2^{s-1}\frac{1}{\|\xi\|^{2s}}\sum_{k=0}^{s-1}\frac{(s-k-1)!}{k!(-4)^k}(c\|\xi\|)^{2k}+ \\ & -\left(\frac{-c^2}{2}\right)^s \log\|\xi\| \sum_{k=0}^{\infty}\frac{(c\|\xi\|)^{2k}}{4^k k!\Gamma(s+k+1)}+ \\ & -\left(\frac{-c^2}{2}\right)^s \log c \sum_{k=0}^{\infty}\frac{(c\|\xi\|)^{2k}}{4^k k!\Gamma(s+k+1)}+ \\ & +\left(\frac{-c^2}{2}\right)^s \sum_{k=0}^{\infty}\left(\frac{\log 2}{4^k k!\Gamma(s+k+1)}+\frac{1}{2}\frac{\Psi(k+1)+\Psi(s+k+1)}{4^k(s+k)!k!}\right) \\ & \times (c\|\xi\|)^{2k},\end{aligned}$$

where $\Psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the Digamma function, and Γ is the Gamma function. We summarise it in the following result.

Theorem 6. *The quasi-Lagrange function satisfies the bound* $|\psi(x)| = O((1+|x|)^{-5})$.

This localness, i.e., decay of ψ, is identified by differentiating its Fourier transform and because $\hat{\psi}(\xi)$ can be expanded, about $\xi=0$, as

$$\tilde{a}+a\xi^2+b\xi^4\log|\xi|+\cdots, \quad \tilde{a},a,b\in\mathbb{R},$$

where the dots mean higher order terms (higher powers of ξ and/or logarithms) in the expansion. Indeed, this comes from the previous expansion of $\frac{c^2}{\|\xi\|^2}K_2(c\|\xi\|)$ and the expansion of $(1-\cos(\xi))^2$ given by $\frac{\xi^4}{4}$, around $\xi=0$. Therefore, by applying the inverse Fourier transform we would obtain the above-mentioned decay (see [9], for instance).

In order to prove that the polynomial reproduction is $\mathbb{P}_1$, we make some explicit computations. First we derive the suitable normalisation constant, by noting that

$$\begin{aligned}\hat{\varphi}(|\xi|) &= (2\pi)^{1/2}\frac{2^{5/2}}{\Gamma(-3/2)}\frac{c^2}{\xi^2}K_2(c|\xi|) \\ &= \frac{12}{\xi^4}-\frac{3c^2}{\xi^2}+6\left(\frac{1}{16}c^4\left(\frac{3}{2}-2\gamma\right)+\frac{1}{8}c^4\log(2)-\frac{1}{8}c^4\log(c)-\frac{1}{8}c^4\log(|\xi|)\right)+\cdots.\end{aligned}$$

Since we multiply by $(1-\cos(\xi))^2=\frac{\xi^4}{4}+\cdots$, $\xi\to 0$, we would have $\hat{\psi}(0)=3$. Therefore, in this case, we should divide by 3. It results,

$$\hat{\psi}(0)=1, \qquad \frac{d\hat{\psi}}{d\xi}(0)=0, \qquad \frac{d^k\hat{\psi}}{d\xi^k}(2\pi j)=0,\ \forall j\in\mathbb{Z}\setminus\{0\},\ k=0,1.$$

As $\frac{d^2\hat{\psi}(\xi)}{d\xi^2}(0)\neq 0$ the SF-condition for the second derivative is not satisfied at $\xi=0$. Therefore we will have $\mathbb{P}_1$ reproduction at most.

Theorem 7. *Let φ be the generalised multiquadric radial basis function, with $n=1$ and the Fourier coefficients of $\frac{1}{3}(2-2\cos x)^2$ being the λ_j in (2). The quasi-interpolation (3) satisfies*

$$Qp \equiv p$$

for linear polynomials p.

The approximation error, determined by Theorem 1 with $m=1, \ell=3$, and $n=1$, exhibits an asymptotic order of $O(h^2)$.

Remark 8. As

$$(1-\cos\xi)^2=\frac{1}{4}e^{-i2\xi}-e^{-i\xi}+\frac{3}{2}-e^{i\xi}+\frac{1}{4}e^{i2\xi}, \qquad -\pi\le\xi\le\pi,$$

the kernel ψ is given by the finite sum

$$\psi(x)=\frac{1}{4}\varphi(|x+2|)-\varphi(|x+1|)+\frac{3}{2}\varphi(|x|)-\varphi(|x-1|)+\frac{1}{4}\varphi(|x-2|),\ x\in\mathbb{R}.$$

We can also use Wolfram Mathematica software to check that this finite sum satisfies

$$|\psi(x)|=O((1+|x|)^{-5}), \quad |x|\to\infty.$$

3.2. Improving the reproduction

Starting with the above explicit construction, we can improve the above scheme to $\mathbb{P}_3$-reproduction changing $(1-\cos(\xi))^2$ by a suitable trigonometric polynomial $P(\xi)$. We consider

- as the radial basis function we take the generalised multiquadric function defined by $\varphi(r) = (r^2 + c^2)^{3/2}$,
- as the finite sum of exponentials in order to resolve the generalised multiquadrics' singularity $P(\xi) = \sum_{k=-N}^{N} \mu_k e^{ik\xi}$,
- and therefore, finally, as Fourier transform $\hat{\psi}(\xi) = P(\xi)\hat{\varphi}(|\xi|)$.

We adjust the coefficients μ_k of $P(\xi)$ in such a way that the expansion of $\hat{\psi}(\xi)$ around $\xi = 0$ is $1 + O\left(\xi^4 \log|\xi|\right)$. To do that, we impose that, in that expansion, the coefficient of ξ^0 is 1 and the coefficients of $\xi^{-4}, \xi^{-3}, \xi^{-2}, \xi^{-1}, \xi, \xi^2, \xi^3$ vanish (even the vanishing of the coefficient of ξ^4 may be added, but not the one of $\xi^4 \log(|\xi|)$, which is not compatible with the previous conditions due to the log-term). The system to be solved collecting all of these requirements is

$$\sum_{k=-N}^{N} \mu_k = \sum_{k=-N}^{N} k\mu_k = \sum_{k=-N}^{N} k^2\mu_k = \sum_{k=-N}^{N} k^3\mu_k = 0, \quad \sum_{k=-N}^{N} k^4\mu_k = 2, \quad \sum_{k=-N}^{N} k^5\mu_k = 0,$$
$$\sum_{k=-N}^{N} k^6\mu_k = -15c^2, \quad \sum_{k=-N}^{N} k^7\mu_k = 0, \quad \sum_{k=-N}^{N} k^8\mu_k = \frac{105}{2}c^4(1 + 4\gamma - 4\log 2 + 4\log c).$$

Its solution for $N = 4$ is:

$$\begin{aligned}
\mu_{-4} &= \frac{1}{11520}(28 + 60c^2 + 15c^4 + 60c^4\gamma - 60c^4\log 2 + 60c^4\log c),\\
\mu_{-3} &= \frac{1}{480}(-16 - 30c^2 - 5c^4 - 20c^4\gamma + 20c^4\log 2 - 20c^4\log c),\\
\mu_{-2} &= \frac{1}{2880}(676 + 780c^2 + 105c^4 + 420c^4\gamma - 420c^4\log 2 + 420c^4\log c),\\
\mu_{-1} &= \frac{1}{1440}(-976 - 870c^2 - 105c^4 - 420c^4\gamma + 420c^4\log 2 - 420c^4\log c),\\
\mu_0 &= \frac{1}{384}(364 + 300c^2 + 35c^4 + 140c^4\gamma - 140c^4\log 2 + 140c^4\log c),\\
&\text{with } \mu_i = \mu_{-i} \text{ for } i = 1, 2, 3, 4.
\end{aligned} \tag{8}$$

By inserting these coefficients into $P(\xi)$ we obtain that the expansion of $\hat{\psi}(\xi)$ around $\xi = 0$ is

$$1 - \frac{c^4}{16}\xi^4 \log|\xi| + \cdots.$$

Therefore, by applying the inverse Fourier transform (see, e.g., [9]), we obtain a *decay of order −5 for ψ:*

$$|\psi(x)| = O((1 + |x|)^{-5}), \quad |x| \to \infty.$$

On the other hand, for this choice of $P(\xi)$, the (distributional) Fourier transform of ψ,

$$\hat{\psi}(\xi) = (2\pi)^{1/2}\frac{2^{5/2}}{\Gamma(-3/2)}\frac{c^2}{\xi^2}K_2(c|\xi|)P(\xi), \qquad \xi \in \mathbb{R},$$

satisfies the conditions

$$\hat{\psi}(0) = 1, \quad \frac{d^k\hat{\psi}}{d\xi^k}(2\pi j) = 0, \forall j \in \mathbb{Z}, k = 1, 2, 3.$$

Therefore, according to the SF-conditions, $\mathbb{P}_3$-reproduction is reached by the quasi-interpolant. As $\frac{d^4\hat{\psi}}{d\xi^4}(0) = \infty$ it is not possible to achieve $\mathbb{P}_4$-reproduction in this case.

Theorem 9. *Let φ be the generalised multiquadric function, with $n = 1$ and the explicit λ_j as above for general parameters c in (2). Then the quasi-interpolation (3) is exact for cubic polynomials p.*

Putting all together, the *approximation order of the quasi-interpolant is* $h^4|\log h|$ (Theorem 1 with $m = 3, \ell = 1, n = 1$).

Remark 10. The kernel ψ is given by the finite sum

$$\psi(x) = \sum_{k=-4}^{4} \mu_k \varphi(|x - k|), \qquad x \in \mathbb{R},$$

where μ_k, $k = -4, \ldots, 4$ are defined in (8). Using Wolfram Mathematica software we are also able to check that this finite sum satisfies

$$|\psi(x)| = O((1 + |x|)^{-5}), \quad |x| \to \infty.$$

3.3. Scheme with the cubic B-spline

The basic ideas of forming quasi-interpolants, quasi-Lagrange functions and providing polynomial precision stem from B-spline quasi-interpolation. This one has the same behaviour than the function of the previous section at $r = 0$, i.e., we set

- as a radial basis function $\varphi(r) = r^3$ and therefore we have the Fourier transform $\hat{\varphi}(r)$ to be a constant multiple of r^{-4},
- the quasi-Lagrange functions ψ are in this case the compactly supported normalised B-splines that have well-known analytic Fourier transforms $\hat{N}_k(\xi) = \left(\hat{N}_0(\xi)\right)^{k+1}$ being $\hat{N}_0(\xi) = \dfrac{1-e^{-i\xi}}{i\xi}$,
- and our well-known coefficients λ_j are the (finitely supported) Fourier coefficients of $(1-\cos(\xi))^2$.
- the decay of $\psi(x)$ is clear, we notice that it is faster than $O((1+|x|)^{-k})$ for any $k \in \mathbb{N}$.

Studying this setup using Theorem 1 with $\ell > 1, n = 1$ gives

- as far as polynomial precision is concerned, $\mathbb{P}_1$-reproduction. The same case as in the first part of Section 3.1. By the way, we could improve the polynomial exactness to $\mathbb{P}_3$ by adding a suitable polynomial $P(x)$.
- Finally, the approximation error is $O(h^{m+1})$, the exponent being $m = 1$ or $m = 3$ depending on polynomial reproduction order in our particular cases.

4. A further construction in the Fourier domain

Looking closely again at the SF-conditions we notice that the second and third set of conditions can be easily satisfied starting the construction in the Fourier domain. When we start in the Fourier domain the more important part is to ensure that the first condition is satisfied. In order to deduce the decay conditions of the function from the Fourier transform we establish the following result.

Theorem 11. *Let f be a symmetric real valued function satisfying*

1. $f \in C^{M-1}(\mathbb{R})$ *with all derivatives absolutely integrable*
2. $f^{(M)}$ *is a locally absolutely continuous function and it has bounded variation on* $\mathbb{R}$,

then $|\hat{f}(\omega)| = o(|\omega|^{-M-1})$, $\omega \to \pm\infty$.

Proof. It follows immediately from the Riemann–Lebesgue Lemma and by integration by parts that $|\hat{f}(\omega)| = o(\omega^{-M})$, $\omega \to \pm\infty$.

Also, $f^{(M)}$ has a well-defined Fourier transform with bounded L^1-norm (see the first display in [10, Theorem 1]) and it even has an integrable derivative, so we have once more by the Riemann–Lebesgue Lemma that even $|\hat{f}(\omega)| = o(|\omega^{-M-1}|)$. Notice for the argument of partial differentiation where we integrate the exponential and differentiate the other factor in the integrand, integrability is sufficient. This concludes the proof. □

4.1. Foundation of the construction

By Theorem 11, it will be useful to have a function ψ with Fourier transform of type $\hat{\psi}(\xi) = \dfrac{Q(\xi)}{|\xi|^3}$, with $Q(\xi)$ such that $\hat{\psi}(\xi)$ satisfies the following conditions:

1. It belongs to $C^2(\mathbb{R})$ and $\hat{\psi}'''(\xi)$ is a piecewise continuous function, and it has bounded variation in $\mathbb{R}$. Moreover $\hat{\psi}(\xi)$ and its derivatives up to the third order have to be integrable.
2. The SF-conditions must be satisfied in order to get $\mathbb{P}_2$−reproduction which requires that $Q(\xi)$ must have a zero of order 3 at the origin.

We think that the point is that we must avoid jumps in the Fourier transform. More in detail, if $\hat{f}^{(M)}(\omega)$ has a jump then it will have a behaviour like a multiple and shift of the Heaviside function. So, we will obtain for $f(x)$ an asymptotic decay of $O(|x|^{-M-1})$ at maximum, although we also need some other conditions as absolute integrability and bounded variation of the function and its derivatives. The reason is that the derivative picks up a Dirac delta, which has a Fourier transform of constant modulus.

With the above conditions, we have found the example

$$\hat{\psi}(\xi) = e^{-a\xi^4 + \frac{\xi^2}{2}} \left|\frac{\sin\xi}{\xi}\right|^3, \quad a > 0.$$

The function in the Fourier domain is plotted in Fig. 1. This function satisfies all the requirements. Moreover, as the third derivative of $\hat{\psi}(\xi)$ has a jump at $\xi = \pi$, visualised in Fig. 2, we would obtain an asymptotic decay of $(1+|x|)^{-4}$. In fact, computations with Wolfram Mathematica software indicate that this is so. We have the following properties:

1. Decay of $\psi(x)$ turns out to be $\psi(x) = o((1+|x|)^{-4})$: to find that, we apply Theorem 11 with $M = 3$.
2. Polynomial precision: $\mathbb{P}_2$−reproduction.
3. Approximation error: $O(h^3|\log h|)$, (use Theorem 1 with $m = 2, \ell = 1, n = 1$).

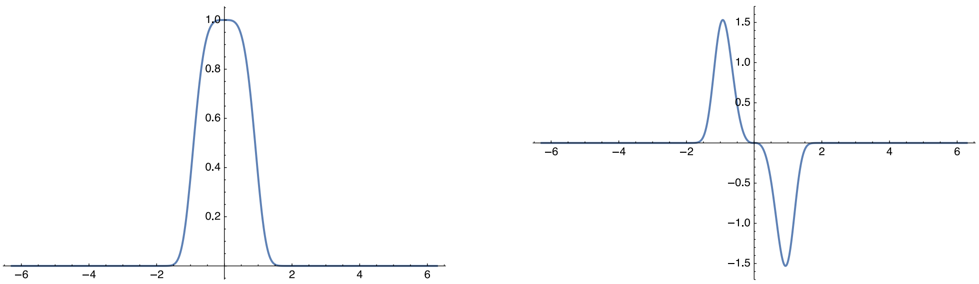

Fig. 1. Graphs of $\hat{\psi}$ (left) and $\hat{\psi}'$ (right).

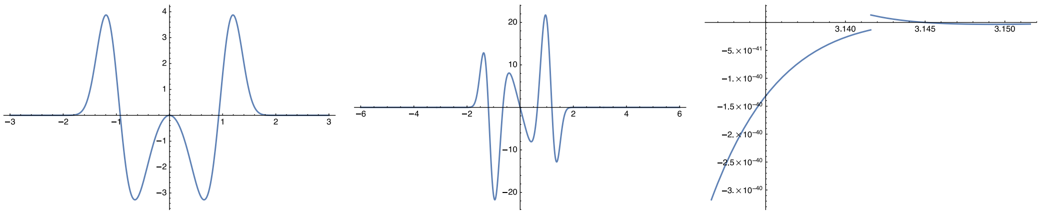

Fig. 2. Graphs of $\hat{\psi}''$ (left), $\hat{\psi}^{(3)}$ (centre), and $\hat{\psi}^{(3)}$ around π (right).

4.2. Generalisation of the construction

We now give a general description of a method to derive similar quasi-Lagrange functions with higher order polynomial reproduction. We look for a ψ with Fourier transform of type

$$\hat{\psi}(\xi) = \frac{e^{p(\xi)}|\sin(\xi)|^{m+1}}{|\xi|^{m+1}}, \tag{9}$$

for a polynomial p. Further $\hat{\psi}(\xi)$ should satisfy the following conditions:

1. It belongs to $C^m(\mathbb{R})$. Moreover, $\hat{\psi}^{m+1}(\xi)$ is a piecewise continuous function and has bounded variation on $\mathbb{R}$. Furthermore, $\hat{\psi}(\xi)$ and its derivatives up to order m have to be integrable.
2. The SF-conditions cited in Theorem 1 must be satisfied in order to get $\mathbb{P}_m$-reproduction.

The properties can be translated into conditions on $p(\xi)$, in (9). Since for m even

$$g(\xi) = |\sin(\xi)|^{m+1} = \sin^{m+1}(\xi)\,\mathrm{sign}(\sin(\xi))$$

is in $C^\infty(\mathbb{R}\setminus\pi\mathbb{Z})$, it is further in $C^m(\mathbb{R})$ since $g^{(j)}(k\pi)=0$ for all $k\in\mathbb{Z}$ and $j\le m$.

Combining this with the higher order product rules and continuity of $|\xi|^{-m-1}$ outside zero, $\hat{\psi}(\xi)$ satisfies the third condition of Theorem 1 since $e^{p(\xi)}$ is $C^\infty(\mathbb{R})$.

Part 2 of the second condition is satisfied if $e^{p(0)}=1$, which is equivalent to assuming $p(0)=0$. For the first part of the second condition of Theorem 1, we need to compute the derivative of $\hat{\psi}(\xi)$ near zero. We find that using for $|\xi|<\pi$:

$$\begin{aligned}\hat{\psi}'(\xi) =&\ e^{p(\xi)}\frac{p'(\xi)|\sin(\xi)|^{m+1}}{|\xi|^{m+1}} + e^{p(\xi)}\frac{(m+1)\sin^m(\xi)\cos(\xi)\,\mathrm{sign}(\sin(\xi))}{|\xi|^{m+1}}\\ &+ e^{p(\xi)}\frac{|\sin(\xi)|^{m+1}(-m-1)\,\mathrm{sign}(\xi)}{\xi^{m+2}}\\ =&\ e^{p(\xi)}\frac{p'(\xi)\sin^{m+1}(\xi)\xi + (m+1)\sin^m(\xi)\cos(\xi)\xi + \sin^{m+1}(\xi)(-m-1)}{\xi^{m+2}},\end{aligned}$$

where we used that m is even, and that for $|\xi|<\pi$, $\mathrm{sign}(\sin(\xi)) = \mathrm{sign}(\xi)$. The condition $\hat{\psi}'(0)=0$ is therefore satisfied if

$$p'(\xi)\sin^{m+1}(\xi)\xi + (m+1)\sin^m(\xi)\left(\cos(\xi)\xi - \sin(\xi)\right) = O(\xi^{m+3}),$$

taking into account that $\lim \frac{\sin(x)}{x} = 1$ for $x\to 0$. It remains to show that

$$p'(\xi)\sin(\xi)\xi + (m+1)\left(\cos(\xi)\xi - \sin(\xi)\right) = O(\xi^3).$$

Using the Taylor series of the trigonometric functions we can show

$$p'(\xi)\left(\xi^2 - \frac{\xi^4}{6}\right) + (m+1)\left(-\frac{1}{3}\xi^3 + \frac{1}{30}\xi^5\right) + \cdots = O(\xi^3),$$

which is true if

$$p'(\xi) = O(\xi).$$

This condition can be satisfied for $\ell = 1$ if $p(\xi) = \sum_{k=2}^{j} a_k \xi^k$ for any choice of coefficients a_k with $a_j < 0$, which ensures integrability. For higher order derivatives one needs to ensure that the Taylor expansion near zero of

$$e^{p(\xi)} \sin^{m+1}(\xi) = \xi^{m+1} + O(\xi^{2m+2}). \tag{10}$$

Theorem 12. *Let $\hat{\psi}$ be of the form (9) and $p(\xi)$ satisfies (10) for an even m. Then the quasi-interpolation (3) is exact for polynomials of degree m.*

Remark 13.

1. For the case of $m = 2$ this gives

$$\begin{aligned} Q(\xi)\sin^{m+1}(\xi) &= e^{p(\xi)} \left(\frac{\xi}{1} - \frac{\xi^3}{6} + \cdots\right)^3 \\ &= \left(1 + p'(0)\xi + \left(p''(0) + p'(0)^2\right)\frac{\xi^2}{2} + \cdots\right)\left(\frac{\xi}{1} - \frac{\xi^3}{6} + \cdots\right)^3 \\ &= \left(1 + p''(0)\frac{\xi^2}{2}\right)\left(\xi^3 - \frac{\xi^5}{2} + \cdots\right) \end{aligned}$$

   which is equal to $\xi^3 + O(\xi^6)$ if we choose $p(\xi) = -\xi^4 + \frac{\xi^2}{2}$.
2. In order to compute the function ψ it is helpful to note that the inverse Fourier transform of e^{-x^4} has been computed in [2] and a series representation of the Fourier transform of $e^{-\|x\|^\beta}$ is given in Appendix A.2 to also compute higher order polynomial reproduction properties.

5. Summary of the results

In Section 2 we compared two options of constructing quasi-interpolants using thin-plate splines and an infinite number of linear combinations to cardinal interpolation. The results are summarised in Table 1.

In Section 3 we studied multiquadric and polyharmonic-spline based quasi-interpolants. The result for the use of these basis functions with and without forming adequate finite linear combinations are displayed in Table 2. Two alternative constructions can be found in Section 4 - where a new radial basis function characterised in the Fourier domain which gives good approximation properties is introduced - (see Table 3). For the general construction described in Section 4.2 the approximation order can even be increased to order $h^{m+1}|\log h|$.

Table 1
$\varphi(r) = r^2 \log r$.

Method	λs	Reproduction	Decay	Approximation order
Cardinal	Infinite	$\mathbb{P}_2$	$O((1+\lvert x\rvert)^{-4})$	$O(h^3\lvert\log h\rvert)$
Section 2.2	Infinite	$\mathbb{P}_1$	$O((1+\lvert x\rvert)^{-4})$	$O(h^2)$
Section 2.3	Infinite	$\mathbb{P}_2$	$O((1+\lvert x\rvert)^{-2})$	$O(h\lvert\log h\rvert)$

Table 2
The asterisk means that we included a suitable polynomial.

RBF $\varphi(r)$	λ's	Reproduction	Decay	Approximation order
$(r^2+c^2)^{3/2}$	Finite	$\mathbb{P}_1$	$O((1+\lvert x\rvert)^{-5})$	$O(h^2)$
$(r^2+c^2)^{3/2}$	Finite*	$\mathbb{P}_3$	$O((1+\lvert x\rvert)^{-5})$	$O(h^4\lvert\log h\rvert)$
Cubic B-spline	Finite	$\mathbb{P}_1$	$O((1+\lvert x\rvert)^{-k}),\ \forall k\in\mathbb{N}$	$O(h^2)$
Cubic B-spline	Finite*	$\mathbb{P}_3$	$O((1+\lvert x\rvert)^{-k}),\ \forall k\in\mathbb{N}$	$O(h^4)$

Table 3
The function $\hat{\psi}(\xi) = e^{-ax^4+\frac{x^2}{2}}\left|\frac{\sin\xi}{\xi}\right|^3$, $\quad a > 0$.

Function	λ's	Reproduction	Decay	Approximation order
$\hat{\psi}(\xi)$	Infinite	$\mathbb{P}_2$	$O((1+\lvert x\rvert)^{-4})$	$O(h^3\lvert\log h\rvert)$

Appendix

A.1. Computation of the inverse Fourier transform of $|\sin(\xi)|$

Lemma 14. *The inverse Fourier transform of* $|\sin(x)|$ *is*

$$\mathcal{F}^{-1}|\sin|(x) = \frac{1}{\pi} \times \frac{1+\exp(ix\pi)}{1-x^2} \times \mathcal{D}_2(x).$$

Here $\mathcal{D}_2$ *is the Dirac comb*

$$\mathcal{D}_2 = \sum_{k=-\infty}^{\infty} \delta(\cdot - 2k).$$

Proof. In Section 2 we used that if $g(\xi) = \frac{1}{2}\sin|\xi| + \frac{1}{2}\sin|\xi - \pi|$, then

$$|\sin\xi| = \sum_{k=-\infty}^{\infty} g(\xi - k\pi), \qquad \xi \in \mathbb{R}.$$

In order to give the inverse Fourier transform of $|\sin\xi|$ we note first that the generalised inverse Fourier transform $\mathcal{F}^{-1}$ of $\sin|\xi|$ is

$$\begin{aligned}
\mathcal{F}^{-1}\sin|\cdot|(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin|\xi| \exp(i\xi x)\, d\xi = \frac{1}{\pi}\int_0^{\infty} \sin\xi \cos(\xi)\, d\xi \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0_+}\int_0^{\infty} \exp(-\epsilon\xi)\sin\xi\cos\xi x\, d\xi \\
&= \frac{1}{\pi}\sqrt{\frac{\pi}{2}}\lim_{\epsilon\to 0_+}\sqrt{\frac{2}{\pi}}\int_0^{\infty} \exp(-\epsilon\xi)\sin\xi\cos\xi x\, d\xi \\
&= \frac{1}{\pi}\sqrt{\frac{\pi}{2}}\lim_{\epsilon\to 0_+}\mathcal{F}_{\cos}\left(\exp(-\epsilon\xi)\sin\xi\right)(x) \\
&= \frac{1}{\sqrt{2\pi}}\lim_{\epsilon\to 0_+}\frac{1}{\sqrt{2\pi}}\left(\frac{1+x}{\epsilon^2+(1+x)^2} + \frac{1-x}{\epsilon^2+(1-x)^2}\right) \\
&= \frac{1}{2\pi}\frac{2}{1-x^2} = \frac{1}{\pi}\frac{1}{1-x^2},
\end{aligned}$$

where we used the cosine transform, $\mathcal{F}_{\cos}$, given in [7], (17.34.22[7]). This gives

$$\mathcal{F}^{-1}\sin|\cdot|(x) = \frac{1}{\pi}\frac{1}{1-x^2}.$$

Therefore, the generalised inverse Fourier transform of g is

$$\mathcal{F}^{-1}g(x) = \frac{1}{2\pi}\left(\frac{1}{1-x^2} + \frac{\exp(ix\pi)}{1-x^2}\right).$$

This gives according to [12]

$$\mathcal{F}^{-1}|\sin|(x) = \frac{1}{\pi} \times \frac{1+\exp(ix\pi)}{1-x^2} \times \mathcal{D}_2(x).$$

First, we have

$$|\sin(x)| = \sum_{k=-\infty}^{\infty} g(x-k\pi) = \left(\sum_{k=-\infty}^{\infty} \delta(\cdot - k\pi) * g\right)(x)$$

Now, from $\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}(\mathcal{F}f)(-x)$ and $\mathcal{F}^{-1}(f*g)(x) = (2\pi)^n\,\mathcal{F}^{-1}f(x)\cdot\mathcal{F}^{-1}g(x)$ it follows:

$$\begin{aligned}
\mathcal{F}^{-1}|\sin(\cdot)|(x) =&\ 2\pi\mathcal{F}^{-1}\left(\sum_{k=-\infty}^{\infty}\delta(\cdot - k\pi)\right)(x) \times \mathcal{F}^{-1}g(x) \\
=&\ 2\pi\left(\frac{1}{\pi}\left(\sum_{k=-\infty}^{\infty}\delta(\cdot - 2k)\right)(x)\right) \times \left(\frac{1}{2\pi}\left(\frac{1}{1-x^2} + \frac{\exp(ix\pi)}{1-x^2}\right)\right) \\
=&\ \mathcal{D}_2(x) \times \frac{1}{\pi}\left(\frac{1+\exp(ix\pi)}{1-x^2}\right). \quad \square
\end{aligned}$$

A.2. Inverse Fourier transform of $\exp(-\|x\|^\beta)$

We now want to investigate the class of inverse n-dimensional Fourier transforms of the functions

$$E(\xi) = e^{-\|\xi\|^\beta},$$

which are integrable for $\beta > 0$. The presented results are based on the results given in the thesis of one of the authors [8, Chapter 3.3]. We start by gathering informations about the special choices of β which have already been considered.

- $\beta = 1$: In this case the function is

  $$E(\xi) = e^{-\|\xi\|},$$

  which is the Poisson kernel. Its inverse Fourier transform is

  $$\mathcal{F}^{-1}E(\xi) = \frac{1}{2\pi}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\frac{1}{(1+\|\xi\|^2)^{\frac{n}{2}+\frac{1}{2}}},$$

  which is a special case of the generalised inverse multiquadric, $\varphi(r) = (1+r^2)^{\alpha/2}$, with $\alpha = -n-1$,
- $\beta = 2$: The function is the Gaussian basis function $E(\xi) = e^{-\|\xi\|^2}$, which has the inverse Fourier transform $\mathcal{F}^{-1}E(\xi) = (1/4\pi)^{n/2}e^{-\|\xi\|^2/4}$ which is also a Gaussian basis function,
- $\beta = 2N$: The function is $E(\xi) = e^{-\|\xi\|^{2N}}$; its Fourier transform was considered, for the case $n = 1$, in [2]. The Fourier transforms of $E(\xi) = e^{-A|\xi|^{2N}}$ have therein been approximated without giving a representation different from the obvious integral description. For the special case $\beta = 4$ the resulting radial basis function is called the inverse quartic Gaussian ($\beta = 4$). A series representation has been computed using Matlab by Boyd in [3] and takes the form

  $$\begin{aligned}\mathcal{F}^{-1}E(\xi) = {} & \frac{1}{2^{3/2}}\sum_{k=0}^{\infty}\frac{\Gamma(1/2)}{\Gamma(1/2+N)\Gamma(3/4+k)}\frac{\left(\frac{|\xi|}{4}\right)^{4k}}{k!}\\ & - \frac{1}{8\pi}\Gamma(3/4)|\xi|^2\sum_{k=0}^{\infty}\frac{\Gamma(5/4)\Gamma(3/2)}{\Gamma(3/2+k)\Gamma(5/4+k)}\frac{\left(\frac{|\xi|}{4}\right)^{4k}}{k!}.\end{aligned}$$

We now give a representation of the inverse Fourier transform of $E(\xi) = e^{-\|\xi\|^{\beta}}$.

We focus on the case $\beta > 1$ using the series representation of the Bessel function. However, to be able to compute the Fourier transform we need to prove this additional lemma first.

Lemma 15. *The series*

$$\sum_{k\geq 0}(-1)^k a^{2k}\frac{\Gamma\left(\frac{n+2k}{\beta}\right)}{\Gamma(k+1)\Gamma(k+\frac{n}{2})},\quad a\in\mathbb{R},$$

is absolutely convergent for every $\beta > 1$.

Proof. We are going to prove that by applying the root test to the series the resulting limit is 0. First, we have

$$0 \leq \lim_{k\to\infty}\left|(-1)^k a^{2k}\frac{\Gamma\left(\frac{n+2k}{\beta}\right)}{\Gamma(k+1)\Gamma(k+\frac{n}{2})}\right|^{\frac{1}{k}} = \lim_{k\to\infty}\left(a^{2k}\frac{\Gamma\left(\frac{n+2k}{\beta}\right)}{\Gamma(k+1)\Gamma(k+\frac{n}{2})}\right)^{\frac{1}{k}}.$$

Applying formula 8.327, 1* of [7] we have that for large k

$$\begin{aligned}\left(\frac{a^{2k}\Gamma\left(\frac{n+2k}{\beta}\right)}{\Gamma(k+1)\Gamma(k+\frac{n}{2})}\right)^{\frac{1}{k}} &\leq \left(\frac{a^{2k}\sqrt{2\pi}\sqrt{\frac{n+2k}{\beta}-1}\left(\frac{n+2k}{\beta}-1\right)^{\frac{n+2k}{\beta}-1}e^k e^{k+\frac{n}{2}-1}c}{\sqrt{2\pi}\sqrt{k}k^k\sqrt{2\pi}\sqrt{k+\frac{n}{2}-1}\left(k+\frac{n}{2}-1\right)^{k+\frac{n}{2}-1}e^{\frac{n+2k}{\beta}-1}}\right)^{\frac{1}{k}}\\ &= \left(\frac{ce^{n(\frac{1}{2}-\frac{1}{\beta})}\left(\frac{n+2k}{\beta}-1\right)^{\frac{n}{\beta}-\frac{1}{2}}}{\sqrt{k}\sqrt{2\pi}\left(k+\frac{n}{2}-1\right)^{\frac{n}{2}-\frac{1}{2}}}\times\frac{a^{2k}e^{2k(1-\frac{1}{\beta})}\left(\frac{n+2k}{\beta}-1\right)^{\frac{2k}{\beta}}}{k^k(k+\frac{n}{2}-1)^k}\right)^{\frac{1}{k}}\\ &= \left(\frac{ce^{n(\frac{1}{2}-\frac{1}{\beta})}\left(\frac{n+2k}{\beta}-1\right)^{\frac{n}{\beta}-\frac{1}{2}}}{\sqrt{k}\sqrt{2\pi}\left(k+\frac{n}{2}-1\right)^{\frac{n}{2}-\frac{1}{2}}}\right)^{\frac{1}{k}}\times\left(\frac{a^2e^{2(1-\frac{1}{\beta})}\left(\frac{n+2k}{\beta}-1\right)^{\frac{2}{\beta}}}{k(k+\frac{n}{2}-1)}\right)\end{aligned}$$

where c is a constant that comes from the remainder of the series in the numerator. The last expression, for large k involves a (first) fraction that tends to one and a second fraction that is of order $O(k^{\frac{2}{\beta}-2})$. As $\beta > 1$, the absolute convergence of the series for any $a \in \mathbb{R}$ follows. □

Lemma 16. *The inverse Fourier transform of* $E(x) = e^{-\|x\|^\beta}$, $x \in \mathbb{R}^n$, $\beta > 1$ *is*

$$\mathcal{F}^{-1}E(\xi) = \frac{2^{1-n}}{\pi^{n/2}}\frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\Gamma\left(\frac{n+2k}{\beta}\right).$$

Proof. We use the formula for Fourier transforms of radial functions to compute the inverse Fourier transform; this is applicable because $E \in L^1(\mathbb{R}^n)$, for all $\beta > 1$, and $n \in \mathbb{N}$. We then use the series representation of the Bessel function ([1] (9.1.10))

$$\begin{aligned}\mathcal{F}^{-1}E(\xi) &= \frac{1}{\sqrt{2\pi}^n}\|\xi\|^{-(\frac{n-2}{2})}\int_0^\infty e^{-t^\beta}t^{n/2}J_{\frac{n-2}{2}}(\|\xi\|t)\,dt\\ &= \frac{1}{\sqrt{2\pi}^n}\|\xi\|^{-(\frac{n-2}{2})}\int_0^\infty e^{-t^\beta}t^{n/2}\sum_{k=0}^{\infty}\frac{(-1)^k(\|\xi\|t/2)^{2k+\frac{n}{2}-1}}{k!\Gamma(k+\frac{n}{2})}\,dt\\ &= \frac{1}{\sqrt{2\pi}^n}2^{-\frac{n}{2}+1}\int_0^\infty e^{-t^\beta}t^{n-1}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|t}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\,dt\\ &= \frac{1}{\sqrt{2\pi}^n}2^{-\frac{n}{2}+1}\lim_{u\to\infty}\int_0^u e^{-t^\beta}t^{n-1}\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{\|\xi\|t}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\,dt.\end{aligned}$$

The sum inside the integrand can be bounded as follows:

$$\left|\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{\|\xi\|t}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\right| \le \sum_{k=0}^{\infty}\left|\frac{\left(\frac{\|\xi\|t}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\right| \le e^{\frac{1}{4}(\|\xi\|t)^2}+c, \quad c>0,$$

with $\frac{1}{\Gamma(k+n/2)} < 1$, for $(k+n/2) > 2$, which gives an integrable majorant on $[0,u]$. Thereby we get

$$\begin{aligned}\mathcal{F}^{-1}E(\xi) &= \frac{1}{\sqrt{2\pi}^n}2^{-\frac{n}{2}+1}\lim_{u\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|}{2}\right)^{2k}}{k!\Gamma\left(k+\frac{n}{2}\right)}\int_0^u e^{-t^\beta}t^{n-1+2k}\,dt\\ &= \frac{1}{\sqrt{2\pi}^n}2^{-\frac{n}{2}+1}\lim_{u\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|}{2}\right)^{2k}}{k!\Gamma(k+\frac{n}{2})}\int_0^{u^\beta}e^{-z}z^{\frac{n+2k}{\beta}-1}\frac{1}{\beta}\,dz\\ &= \frac{1}{\sqrt{2\pi}^n}2^{-\frac{n}{2}+1}\frac{1}{\beta}\lim_{u\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|}{2}\right)^{2k}}{k!\Gamma\left(k+\frac{n}{2}\right)}\gamma\left(\frac{n+2k}{\beta},u^\beta\right),\end{aligned}$$

where we have used the expression 8.350.1 of [7] in the last equality. Here $\gamma(\cdot,\cdot)$ is the incomplete Γ-function. We know that $\gamma\left(\frac{n+2k}{\beta},u^\beta\right) \le \Gamma\left(\frac{n+2k}{\beta}\right)$ for all $\beta > 1$ and applying Lemma 15 we get a convergent majorant. So, we have

$$\mathcal{F}^{-1}E(\xi) = \frac{2^{1-n}}{\pi^{n/2}}\frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\|\xi\|}{2}\right)^{2k}}{k!\Gamma\left(k+\frac{n}{2}\right)}\Gamma\left(\frac{n+2k}{\beta}\right). \quad \square$$

The last series is absolutely convergent for $\beta > 1$ and can be further simplified for many values of β by applying the doubling or tripling formulas for the Gamma function. For the application in Section 4 we are specifically interested in the case $n = 1$ and $Q(x) = \exp(-|x|^\beta)$. In this case, and if $\beta = 2N$, $N \in \mathbb{N}$ our formula simplifies to:

$$\mathcal{F}^{-1}E(\xi) = \frac{1}{2N\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{|\xi|}{2}\right)^{2k}}{k!\Gamma(k+\frac{1}{2})}\Gamma\left(\frac{1+2k}{2N}\right). \tag{11}$$

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Ninth Dover printing, tenth GPO printing ed., Dover, New York, 1964.

[2] J.P. Boyd, The Fourier transform of the quartic Gaussian $\exp(-Ax^4)$: Hypergeometric functions, power series, steepest descent asymptotics and hyperasymptotics and extensions to $\exp(-Ax^{2N})$, Appl. Math. Comput. 241 (2014) 75–87.

[3] J.P. Boyd, P.W. McCauley, Quartic Gaussian and inverse-quartic Gaussian radial basis functions: The importance of a nonnegative Fourier transform, Comput. Math. Appl. 65 (2013) 75–88.

[4] M.D. Buhmann, Radial Basis Functions: theory and implementations, Cambridge University Press, 2003.

[5] M.D. Buhmann, Feng Dai, Pointwise approximation with quasi-interpolation by radial basis functions, J. Approx. Theory (2014) 156–192.

[6] M.D. Buhmann, J. Jäger, Quasi-Interpolation, Cambridge University Press, 2022.

[7] I.S. Gradsteyn, I.M. Ryzhik, Tables, Series, Integrals, Products, Academic Press, New York, 1980.
[8] J. Jäger, Advances in Radial and Spherical Basis Function Interpolation (Ph.D. thesis), Justus-Liebig-Universität Gießen, 2018.
[9] D.S. Jones, The Theory of Generalised Functions, Cambridge University Press, 1982.
[10] E. Liflyand, The Fourier transform of a function of bounded variation: symmetry and asymmetry, J. Fourier Anal. Appl. (2018) 525–544.
[11] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, The NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.
[12] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, 1972.