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Iterative schemes for linear equations of the second kind and related inverse problems

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ABSTRACT

This paper consists of two parts. The first one deals with the generation of an iterative algorithm to obtain an approximate solution of a linear equation of the second kind in a Banach space. This generation is based on a perturbed version of the geometric series theorem which, in particular, allows us to find a family of unisolvent linear Fredholm integral equations of the second kind, even when the associated linear operator has norm greater than or equal to 1. When we consider Fredholm equations of this type and linear Volterra integral equations of second kind, the numerical schemes obtained when appropriate Schauder bases are also introduced in the spaces where the equations operate, enable us to approximate their respective solutions iteratively. The second part of this work focuses on the design of a numerical method for solving an inverse problem associated with a linear equation of the second kind in a Banach space, a method which we apply to problems of parameter estimation related to the two classes of integral equations mentioned above.

1. Introduction

In this paper we deal with numerical problems associated with the well-known geometric series theorem. Recall that this result establishes, in its most popular version, that given a Banach space X, for a continuous, linear and contractive (norm less than one) operator $L : X \longrightarrow X$, $I - L$ is bijective, continuous and linear, and moreover, its inverse operator coincides with the sum of the Neumann series of L:

$$(I-L)^{-1} = \sum_{j=0}^{\infty} L^j.$$

The fact that it is sufficient that the Neumann series be convergent for the same conclusion to be obtained, leads to consider various generalisations of the previous version, in particular the one that requires absolute convergence of the Neumann series. See also [3], which uses the contractivity of a power of the operator, or [17] for a condition in terms of the spectral radius of the operator.

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In the context of the numerical study of linear equations of the second kind, and more specifically in that of integrals, the geometric series theorem has turned out to be an extremely useful technique, since it allows us to approximate the solution $x^{\bullet}$ of such an equation

$$x - Lx = y$$

by means of a partial sum of the series $\sum_{j=0}^{\infty} L^j y$. The Volterra integral equation, when working in continuous or p-integrable function spaces, is a clear example of absolute convergence of the Neumann series, while for the Fredholm integral equation in the same function spaces, the original version (contractive operator) is typically used. In this respect, it is worth noting that all linear integral Volterra equations of the second kind admit a unique solution and a numerical treatment from the associated Neumann series, but for the Fredholm equation this is guaranteed when the operator is contractive. In this paper, we extend the class of Fredholm integral equations to which the geometric series theorem –and hence its numerical analysis– can be applied when absolute convergence of the Neumann series is required, since we prove that for certain linear Fredholm integral equations of the second kind whose integral operators are not contractive, the corresponding Neumann series are convergent, because they are absolutely convergent.

The need to introduce numerical algorithms to solve second kind linear integral equations that fall within the scope of the geometric series theorem is due to the fact that the explicit calculation of the sum $\sum_{j=0}^{\infty} L^j y$, even of one of the terms $L^j y$ (which can be the second one, Ly), is not always possible. The way in which such a sum is approximated is very diverse (see, for instance, [4,5,15,16]). We adopt here an approach derived from the use of suitable Schauder bases for the spaces involved in the integral equation in question, along the lines of other works on integral, integro-differential and differential equations ([6–8,13]). Thanks to this and to a generalised geometric series theorem, we develop algorithms for the numerical approximation of the solution of the linear Volterra equation of second kind, as well as that of certain linear Fredholm equations of second kind. In this way, we extend the algorithms developed in [10], which are confined to the Fredholm equations in the contractive case.

In addition to the numerical solution of a linear equation of the second kind, from an applied point of view, the study of its inverse problem consisting of identifying some parameters of such an equation, assuming that we know a solution –typically an approximate one, since it usually comes from an experimental observation– is of interest. In this context of inverse problems, the original problem is often referred to as the direct problem. Many techniques are commonly used for this purpose (see [2,11,25]). We opt here for the type of algorithm that is designed when the collage theorem or one of its generalisations is applied as has been done for other problems in [1,9,18–20].

The structure of the paper is as follows. Section 2 deals with the problem of approximating the solution of a linear equation of second kind, $(I - L)(\cdot) = y$, an equation which will be our direct problem throughout the paper. To this end, we give an extension of the geometric series theorem, a perturbed version of it which, with the aid of certain Schauder bases in suitable Banach spaces, allows us to establish algorithms for approximating the solution of a wide range of linear integral equations of the second kind: that of Volterra and a family of Fredholm, which are not reduced to those of contractive integral operators. In Section 3 we design collage-based algorithms for solving linear inverse problems of the second kind, and more specifically, for the estimation of parameters in a linear equation of the second kind. Both the algorithms for the direct and the inverse problem are illustrated with several integral examples. Finally, some conclusions are drawn in Section 4.

2. Numerical treatment of the direct problem

In this section, we focus our efforts on the design of an iterative algorithm for numerically solving a linear equation of the second kind in a Banach space. That algorithm relies on a generalised perturbed version of the geometric series theorem, Theorem 2.1, the error control that this provides and the use of some properties of certain Schauder bases associated with the Banach space where the equation is defined. In addition, we show that we can apply the algorithm to some linear Fredholm integral equations of the second kind, not only to the typical contractive case but also for other ones, and to any linear Volterra integral equation of the second kind. We also include some numerical tests.

2.1. The perturbed geometric series theorem and an associated algorithm

In what follows, for a Banach space X, $\mathcal{L}(X)$ denotes the Banach space of the continuous and linear operators from X to X, endowed with its usual operator norm. In the following extension of the geometric series theorem, some perturbations are allowed. The idea behind them is to approximate the calculus of the powers involved in the Neumann series.

Theorem 2.1. *Suppose that X is a Banach space and that $L \in \mathcal{L}(X)$ in such a way that its Neumann series $\sum_{j=0}^{\infty} L^j$ converges. Let $y,\ y_0 \in X$, $n \in \mathbb{N}$ and $L_0, L_1, \ldots, L_n \in \mathcal{L}(X)$. Then, the linear equation of the second kind*

$$(I - L)(\cdot) = y$$

has one and only one solution $x^\bullet \in X$ and

$$\left\| \sum_{j=0}^{n} L_j y_0 - x^\bullet \right\| \leq \sum_{j=0}^{n} \|L_j y_0 - L^j y_0\| + \left\| \sum_{j=0}^{n} L^j \right\| \|y_0 - y\| + \left\| \sum_{j\geq n+1} L^j \right\| \|y\|.$$

Proof. We begin with an argument which is similar to that for the classical geometric series theorem. Let us first observe that the operator $I - L$ is bijective. Indeed, since

$$n \in \mathbb{N} \ \Rightarrow \ (I - L) \sum_{j=0}^{n} L^j = I - L^{n+1} = \left(\sum_{j=0}^{n} L^j \right) (I - L)$$

and the series $\sum_{j=0}^{\infty} L^j$ is convergent, we arrive at

$$(I - L) \sum_{j=0}^{\infty} L^j = I = \left(\sum_{j=0}^{\infty} L^j \right) (I - L),$$

that is, the operator $I - L$ is (linear and) bijective (and continuous, according to the classical Banach isomorphism theorem), with

$$(I - L)^{-1} = \sum_{j=0}^{\infty} L^j,$$

and, in particular, the linear equation of the second kind $(I - L)(\cdot) = y$ is uniquely solvable and its solution is

$$x^\bullet = (I - L)^{-1} y = \sum_{j=0}^{\infty} L^j y.$$

To conclude the proof, it suffices to consider the inequalities

$$\begin{aligned} \left\| \sum_{j=0}^{n} L_j y_0 - x^\bullet \right\| &\leq \left\| \sum_{j=0}^{n} L_j y_0 - \sum_{j=0}^{\infty} L^j y_0 \right\| + \left\| \sum_{j=0}^{n} L^j y_0 - \sum_{j=0}^{\infty} L^j y \right\| + \left\| \sum_{j=0}^{\infty} L^j y - x^\bullet \right\| \\ &\leq \sum_{j=0}^{n} \|L_j y_0 - L^j y_0\| + \left\| \sum_{j=0}^{n} L^j \right\| \|y_0 - y\| + \left\| \sum_{j\geq n+1} L^j \right\| \|y\|. \quad \square \end{aligned}$$

Remark 2.2. The fact that, in the previous theorem, the linear equation of the second kind $(I - L)(\cdot) = y$ admits a unique solution, given any $y \in X$, can be equivalently reformulated by saying that the operator $I - L$ is bijective, provided that the series $\sum_{j=0}^{\infty} L^j$ is convergent, and then

$$(I - L)^{-1} = \sum_{j=0}^{\infty} L^j.$$

Let us also observe that the inequality

$$\|x^\bullet\| \leq \left\| \sum_{j=0}^{\infty} L^j \right\| \|y\|,$$

is valid for the solution $x^\bullet$ of the equation $(I - L)(\cdot) = y$.

One of the hypotheses in Theorem 2.1, the convergence of the Neumann series, is satisfied when some more restrictive conditions are assumed. For instance, we can take into account that for a series in a Banach space, its absolute convergence guarantees its convergence. Moreover, we have the following well-known chain of implications for a Banach space X and an operator $L \in \mathcal{L}(X)$:

$$\|L\| < 1 \Rightarrow \sum_{j=0}^{\infty} \|L^j\| < \infty \Rightarrow \sum_{j=0}^{\infty} L^j \text{ converges.}$$

Some applications of both implications constitute the object of study of the next two subsections.

The following corollary, a straightforward consequence of the previous result, will be the key piece to establish our iterative scheme for solving integral equations.

Corollary 2.3. *Let X be a Banach space and $L \in \mathcal{L}(X)$ such that its Neumann series $\sum_{j=0}^{\infty} L^j$ converges. Let $y,\ y_0 \in X$, $n \in \mathbb{N}$, $L_0, L_1, \dots, L_n \in \mathcal{L}(X)$ and $\varepsilon, \varepsilon_0, \dots, \varepsilon_n > 0$, such that*

$$\sum_{j=0}^{n} \varepsilon_j < \frac{\varepsilon}{2},$$

$$j = 0, 1, \dots, n \Rightarrow \|L_j y_0 - L^j y_0\| < \varepsilon_j$$

and

$$\left\| \sum_{j=0}^{n} L^j \right\| \|y_0 - y\| + \left\| \sum_{j \geq n+1} L^j \right\| \|y\| < \frac{\varepsilon}{2}.$$

If $x^\bullet \in X$ is the unique solution of the linear equation of the second kind $(I - L)(\cdot) = y$, then

$$\left\| \sum_{j=0}^{n} L_j y_0 - x^\bullet \right\| < \varepsilon.$$

Since for large enough n and sufficiently small $\|y_0 - y\|$, we can get $\left\| \sum_{j=0}^{n} L^j \right\| \|y_0 - y\| + \left\| \sum_{j \geq n+1} L^j \right\| \|y\| < \frac{\varepsilon}{2}$, then the question that immediately arises is how to construct the approximating operators $L_0, L_1, \dots, L_n$ satisfying the conditions $\|L_j y_0 - L^j y_0\| < \varepsilon_j$, $j = 0, 1, \dots, n$. The answer will obviously depend on the nature of L. In Subsection 2.4 we will explicitly propose simple and easily computable constructions for the two types of integral equations that will be discussed next: the Volterra and Fredholm integral equations of the second kind. However, we can consider the following general **iterative scheme for a linear equation of the second kind** $(I - L)(\cdot) = y$:

1. First, we introduce the analytical data: the Banach space X, the operator $L \in \mathcal{L}(X)$ whose Neumann series converges and $y \in X$.
2. Next, we construct the approximating data: the tolerance $\varepsilon > 0$, $y_0 \in X$, $n \in \mathbb{N}$, $\varepsilon_0, \dots, \varepsilon_n > 0$ and $L_0, L_1, \dots, L_n \in \mathcal{L}(X)$ fulfilling the assumptions in Corollary 2.3.
3. We obtain the approximate solution

$$\sum_{j=0}^{n} L_j(y_0).$$

4. The absolute error committed is less than the tolerance ε.

2.2. The Fredholm integral equation

We focus on the study of linear Fredholm integral equations of the second kind in the space of continuous functions $C[a, b]$, endowed with its usual max-norm, that is, equations of the form

$$x(t) = y(t) + \int_a^b k(t, s) x(s)\, ds, \qquad (a \leq t \leq b), \tag{1}$$

where $x \in C[a, b]$ is the unknown function and $y \in C[0, 1]$ and $k \in C([a, b]^2)$ are given functions. For these Fredholm integral equations, typically is assumed that its associated integral operator $L : C[a, b] \longrightarrow C[a, b]$ defined as

$$Lx(t) := \int_a^b k(t, s) x(s)\, ds, \qquad (a \leq t \leq b), \tag{2}$$

is contractive, *i.e.*, $\|L\| < 1$, equivalently

$$\max_{a\le t\le b} \int_a^b |k(t,s)|\, ds < 1.$$

Indeed, a stronger condition than $\|L\| < 1$ is usually assumed: $\|k\|_\infty (b-a) < 1$.

Obviously, when L is contractive, our iterative numerical scheme applies. However, now we observe that we can find linear Fredholm integral equations of the second kind whose linear integral operator determines a convergent Neumann series, in fact an absolutely convergent one, despite the fact that the operator is not contractive. In this regard, it is worth mentioning that a simple calculation yields, for any $j \ge 1$, $x \in C[a,b]$ and $t \in [a,b]$,

$$L^j x(t) = \int_a^b \left(\cdots \int_a^b k(t,t_1)k(t_2,t_3)\cdots k(t_{j-1},t_j)x(t_j)\, dt_j \right) \cdots dt_1. \tag{3}$$

More specifically, we consider the family of linear Fredholm integral equations of the second kind in $C[0,1]$ with kernel functions of the form

$$k(t,s) := \alpha t + \beta s, \qquad (0 \le t, s \le 1). \tag{4}$$

In particular, we will prove that, under certain conditions on the reals α and β, the linear and continuous operator $L : C[0,1] \longrightarrow C[0,1]$ associated to such an equation,

$$Lx(t) := \int_0^1 (\alpha t + \beta s) x(s)\, ds, \tag{5}$$

satisfies

$$\|L\| \ge 1$$

but

$$\sum_{j=0}^{\infty} \|L^j\| < \infty.$$

The calculation of $\|L\|$ is easy and well-known. Assuming $\alpha, \beta \ge 0$, one has that k is non-negative, so

$$\begin{aligned}
\|L\| &= \max_{0\le t\le 1} \int_0^1 k(t,s)\, ds \\
&= \max_{0\le t\le 1} \int_0^1 (\alpha t + \beta s)\, ds \\
&= \max_{0\le t\le 1} \left(\alpha t + \frac{\beta}{2} \right) \\
&= \alpha + \frac{\beta}{2}.
\end{aligned}$$

However, the absolute convergence of the power series of L is proved in a more laborious way and requires a couple of prior results. The first of these generalises the calculation of L above.

Lemma 2.4. *Let $\alpha, \beta \ge 0$ and let us define*

$$f_1(t) := \int_0^1 (\alpha t + \beta s)\, ds, \qquad (0 \le t \le 1)$$

and for all $j \ge 2$.

$$f_j(t) := \int_0^1 (\alpha t + \beta s) f_{j-1}(s)\, ds, \qquad (0 \le t \le 1).$$

Then, these functions are non-negative and non-decreasing, and moreover, for each $j \geq 1$ we have

$$\|L^j\| = f_j(1).$$

Proof. Let

$$k(t,s) := \alpha t + \beta s,$$

for $0 \leq t, s \leq 1$. It is clear that if $j \geq 1$ and $t \in [0,1]$ then

$$f_j(t) = \int_0^1 \left(\cdots \int_0^1 k(t,t_1)k(t_1,t_2)\cdots k(t_{j-1},t_j)\,dt_j \right) \cdots dt_1,$$

which, together with the non-negativity of k and of $\dfrac{\partial k}{\partial t}$, since $\alpha, \beta \geq 0$, implies the non-negativity and non-decreasing of all the functions f_j.

Finally, given $j \geq 1$, $x \in C[0,1]$ and $t \in [0,1]$, we deduce from the expression of $L^j x(t)$ given in (3) and the fact that k is non-negative that

$$\|L^j\| = \max_{0 \leq t \leq 1} f_j(t),$$

which in view of the non-decreasing nature of f_j is equivalent to

$$\|L^j\| = f_j(1). \quad \square$$

Lemma 2.5. *Suppose that $\alpha, \beta \geq 0$ and that $\alpha + \beta > 0$. Then, with the notations of Lemma 2.4,*

$$j \geq 2 \;\Rightarrow\; f_j(1) \leq \left(\alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} \right) f_{j-1}(1).$$

Proof. Let $j \geq 2$. Let us first note an elementary fact, but necessary for what follows:

$$f_j(1) \leq (\alpha+\beta) \int_0^1 f_{j-1}(s)\,ds. \tag{6}$$

In fact, it is only necessary to take into account, being $\alpha, \beta \geq 0$, that

$$\begin{aligned} f_j(1) &= \int_0^1 (\alpha + \beta s) f_{j-1}(s)\,ds \\ &\leq (\alpha+\beta) \int_0^1 f_{j-1}(s)\,ds. \end{aligned}$$

Then,

$$\begin{aligned} f_j(1) &= \int_0^1 (\alpha + \beta s) f_{j-1}\,ds \\ &= \alpha \int_0^1 f_{j-1}(s)\,ds + \beta \int_0^1 s f_{j-1}(s)\,ds \end{aligned}$$

If we apply the integration by parts formula to the second integral ($sds = dv$, $f_{j-1}(s) = u$), and taking into account that

$$f'_{j-1}(s) = \alpha \int_0^1 f_{j-2}(\xi)\,d\xi,$$

we obtain

$$\begin{aligned}
\alpha \int_0^1 f_{j-1}(s)\,ds + \beta \int_0^1 s f_{j-1}(s)\,ds &= \alpha \int_0^1 f_{j-1}(s)\,ds + \frac{\beta}{2} f_{j-1}(1) - \beta \int_0^1 \frac{s^2}{2} \left(\int_0^1 \alpha f_{j-2}(\xi)\,d\xi \right) ds \\
&\leq \alpha f_{j-1}(1) + \frac{\beta}{2} f_{j-1}(1) - \frac{\alpha\beta}{2}\frac{1}{3} \int_0^1 f_{j-2}(\xi)\,d\xi \qquad (\text{Lemma } 2.4,\ \alpha \geq 0) \\
&\leq \left(\alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} \right) f_{j-1}(1), \qquad (\text{inequality } (6))
\end{aligned}$$

as we wanted to prove. □

We are already in a position to state the announced absolute convergence for the Neumann series of the linear operator of the linear Fredholm integral equation of the second kind (1) with kernel of the form (4), which implies, as a particular case of Theorem 2.1, the existence of a unique solution for such an integral equation that can be approximated by means of the iterative scheme proposed in the previous section. We will return to this issue in Subsection 2.4.

Theorem 2.6. *For the continuous and linear operator $L : C[0,1] \longrightarrow C[0,1]$ defined for each $x \in C[0,1]$ and $t \in [0,1]$ as*

$$Lx(t) := \int_0^1 (\alpha t + \beta s) x(s)\,ds,$$

where $\alpha, \beta \geq 0$,

$$\alpha + \frac{\beta}{2} \geq 1 \tag{7}$$

and

$$0 < \alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} < 1, \tag{8}$$

we have that

$$\|L\| \geq 1$$

and

$$\sum_{j=0}^{\infty} \|L^j\| < \infty.$$

Proof. Since Lemma 2.4 states, in particular, that

$$\|L\| = \alpha + \frac{\beta}{2},$$

in view of condition (7) on α and β,

$$\|L\| \geq 1.$$

On the other hand, $\alpha + \beta > 0$, according to inequality (7) and the fact that $\beta \geq 0$. Then, Lemmas 2.4 and 2.5 guarantee that

$$j \geq 2 \ \Rightarrow\ \|L^j\| \leq \delta \|L^{j-1}\|,$$

where

$$\delta := \alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)},$$

and so

$$j \geq 1 \ \Rightarrow\ \|L^j\| \leq \delta^{j-1} \|L\|,$$

which implies the convergence of the series $\sum_{j=0}^{\infty} \|L^j\|$ because condition (8) is nothing more than the fact that $0 < \delta < 1$. □

Remark 2.7. It is worth mentioning that the set

$$C := \left\{(\alpha, \beta) \in \mathbb{R}^2 :\ \alpha, \beta \geq 0,\ \alpha + \frac{\beta}{2} \geq 1 \text{ and } 0 < \alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} < 1\right\}$$

is nonempty. For instance,

$$\left(\frac{3}{4}, \frac{1}{2}\right) \in C,$$

and the corresponding operator L satisfies

$$\|L\| = 1,$$

while

$$\sum_{j=0}^{\infty} \|L^j\| < \infty,$$

with

$$\alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} = 0.95.$$

We also consider

$$(0.6, 0.81) \in C,$$

whose associated operator satisfies

$$\|L\| = 1.005,$$
$$\sum_{j=0}^{\infty} \|L^j\| < \infty,$$

and

$$\alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)} = \frac{8907}{9400} < 0.9476.$$

Remark 2.8. If we write

$$\delta := \alpha + \frac{\beta}{2} - \frac{\alpha\beta}{6(\alpha+\beta)},$$

in the proof of Theorem 2.6 we have established the inequality

$$\|L^j\| \leq \delta^{j-1}\|L\|,$$

whenever $j \geq 1$. Therefore, a measure of the speed of convergence of the Neumann series of L is given by that number $\delta \in (0, 1)$: the smaller it is, the faster the convergence. In addition,

$$\sum_{j=0}^{\infty} \|L^j\| = 1 + \sum_{j=0}^{\infty} \|L^j\| \leq 1 + \sum_{j=0}^{\infty} \delta^j = 1 + \frac{\|L\|}{1-\delta} = 1 + \frac{\alpha + \frac{\beta}{2}}{1-\delta}.$$

2.3. The Volterra integral equation

Now we return to the analysis of the convergence of the Neumann series from its absolute convergence. A paradigmatic example of this is given by the linear Volterra integral equations of the second kind. To be more precise, let $X = C[a, b]$ or $L^p[a, b]$ $(1 \leq p < \infty)$, and consider the linear Volterra integral equations of the second kind

$$x(t) = y(t) + \int_a^t k(t, s)x(s)\, ds, \qquad (a \leq t \leq b), \tag{9}$$

where $x \in X$ is the function to be found. The given functions are $y \in X$ and k such that $k \in C[a,b]^2$, if $X = C[a,b]$, or $k \in L^\infty[a,b]^2$, when $X = L^p[a,b]$. The corresponding, clearly well-defined, linear and continuous operator $L \in \mathcal{L}(X)$ associated with this equation is given for each $x \in E$ and each $a \le t \le b$ by

$$Lx(t) := \int_a^t k(t,s)x(s)\,ds, \tag{10}$$

and then,

$$j \in \mathbb{N} \;\Rightarrow\; L^j x(t) = \int_a^t \left(\cdots \int_a^{t_{j-1}} k(t,t_1)k(t_1,t_2)\cdots k(t_{j-1},t_j)\,dt_j \right) \cdots)\,dt_1. \tag{11}$$

The following result establishes the convergence of the Neumann series of the L operator associated with a Volterra equation in two frameworks, in fact, its absolute convergence, and thus, according to Theorem 2.1, the unique solvability of the linear Volterra integral equation of the second kind (9), whose unique solution will be approximated in Subsection 2.4 in an easily computable way.

Theorem 2.9. *Let $X = C[a,b]$ or $L^p[a,b]$ $(1 < p < \infty)$ and for each $x \in X$ and $t \in [a,b]$ let us consider the linear Volterra operator $L : X \longrightarrow X$ given by*

$$Lx(t) := \int_a^t k(t,s)x(s)\,ds, \tag{12}$$

where $k \in C[a,b]^2$, if $X = C[a,b]$, or $k \in L^\infty[a,b]^2$, when $X = L^p[a,b]$. Then, for $X = C[a,b]$,

$$j \ge 1 \;\Rightarrow\; \|L^j\| \le \frac{\|k\|_\infty^j}{j!}(b-a)^j,$$

while for $X = L^p[a,b]$,

$$j \ge 1 \;\Rightarrow\; \|L^j\| \le \|k\|_\infty (b-a)\frac{\|k\|_\infty^{j-1}(b-a)^{j-1}}{(j-1)!}.$$

Proof. Let us first assume that $X = C[a,b]$. We include the proof for the sake of completeness, since it is well-known (see, for instance, [3] for a similar reasoning, or [12] for a proof based on the use of other techniques). In view of the equality

$$\int_a^t \left(\int_a^{t_1} \left(\cdots \int_a^{t_{j-1}} dt_j \right) dt_{j-1} \cdots \right) dt_1 = \frac{(t-a)^j}{j!}$$

and (11), for all $j \ge 1$, $x \in X$ and $a \le t \le b$ we have that

$$\begin{aligned} |L^j x(t)| &\le \int_a^t \left(\int_a^{t_1} \left(\cdots \int_a^{t_{j-1}} |k(t,t_1)||k(t_1,t_2)|\cdots|k(t_{j-1},t_j)||x(t)|\,dt_j \right) dt_{j-1} \cdots \right) dt_1 \\ &\le \|k\|_\infty^j \frac{(t-a)^j}{j!}\|x\|_\infty, \end{aligned}$$

which clearly implies the announced bound.

And finally, if $X = L^p[a,b]$, $(1 < p < \infty)$, $j \ge 1$, $x \in X$ and $a \le t \le b$, then, the Hölder inequality leads to

$$\int_a^b |x(t)|\,dt \le (b-a)^{1/q}\|x\|_p,$$

where q is the conjugate exponent of p, that is, $1/p + 1/q = 1$. Hence, making use once again of (11),

$$\begin{aligned}
|L^j x(t)| &\leq \|k\|_\infty^j \int_a^t \left(\int_a^{t_1} \left(\cdots \int_a^{t_{j-1}} |x(t)|\, dt_j \right) dt_{j-1} \cdots \right) dt_1 \\
&\leq \|k\|_\infty^j \int_a^t \left(\int_a^{t_1} \left(\cdots \int_a^{t_{j-1}} (t_{j-1}-a)^{1/q} \|x\|_p\, dt_j \right) dt_{j-1} \cdots \right) dt_1 \\
&= \|k\|_\infty^j \frac{(t-a)^{j-1+1/q}}{\left(1+\frac{1}{q}\right)\cdots\left((j-1)+\frac{1}{q}\right)} \|x\|_p .
\end{aligned}$$

Therefore,

$$\|L^j x\|_p \leq \|k\|_\infty^j (b-a)^{1/p} \frac{(b-a)^{j-1+1/q}}{\left(1+\frac{1}{q}\right)\cdots\left((j-1)+\frac{1}{q}\right)} \|x\|_p ,$$

and from this inequality, we can easily establish the bound in $L^p[a,b]$. $\square$

2.4. Numerical examples

In the following we will make a concrete proposal for the construction of the approximating operators L_j for the above equations, where for simplicity in the exposition we consider $[a,b]=[0,1]$, in the following three cases: Fredholm linear integral equation (1) with kernel of the form (5) in the Banach space $X=C[0,1]$, and Volterra linear integral equation (9) in the two scenarios considered in Theorem 2.9, *i.e.*, $X=C[0,1]$ or $X=L^p[0,1]$.

First of all, we recall that a sequence $\{e_r\}_{r\in\mathbb{N}}$ in a Banach space X is a *Schauder basis* if every element $x \in X$ can be uniquely represented as $x=\sum_{r=1}^\infty \alpha_r e_r$ for a sequence of reals $\{\alpha_r\}_{r\in\mathbb{N}}$. If we define for each $r\in\mathbb{N}$ the linear operator $P_r : X \longrightarrow X$, known as the *r-th projection* associated with the basis, as $P_r x := \sum_{k=0}^r \alpha_k e_k$, for each fixed x, it is easy to prove that it is a continuous operator and

$$\lim_{r\to\infty} \|P_r x - x\| = 0.$$

The Faber-Schauder system, which is constructed from a dense set of points $\{t_n\}_{n\in\mathbb{N}}$ with $t_1=0$ and $t_2=1$, is the usual Schauder basis in $C[0,1]$, and the Haar system is also in $L^p[0,1]$ (see [21,24]). We also remember that it is possible to tensorially construct bases $\{e_r^{(d)}\}_{r\in\mathbb{N}}$ in $X=C[0,1]^d$ or $X=L^p[0,1]^d$, respectively, from a basis $\{e_r^{(1)}\}_{r\in\mathbb{N}}$ in $X=C[0,1]$ or $X=L^p[0,1]$ (see [14,22]). These tensor bases $\{e_r^{(d)}\}_{r\in\mathbb{N}}$ are of separate variables because they are defined as:

$$e_r^{(d)}(t_1,\dots,t_d) := e_{\alpha_1}(t_1)\cdots e_{\alpha_d}(t_d), \qquad ((t_1,\dots,t_d)\in[0,1]^d,\ \tau(r)=(\alpha_1,\dots,\alpha_d))$$

where $\tau:\mathbb{N}\longrightarrow\mathbb{N}^d$ is a bijection that establishes the square ordering introduced in [14]. We can reorder the bases of $C[0,1]^d$ and $L^p[0,1]^d$ so that for each $m\in\mathbb{N}$, the m^d first elements correspond to $(\alpha_1,\alpha_2,\dots,\alpha_d)$ being $1\leq\alpha_i\leq m$. For these reordered bases, we maintain the same previous notation, $\{e_r^{(d)}\}_{r\in\mathbb{N}^d}$ for the basis and $\{P_r^{(d)}\}_{r\in\mathbb{N}}$ for the sequence of projections. Note that we do this rearrangement to reduce computational cost and achieve higher accuracy with fewer iterations.

Secondly, taking into account (3) and (11) and considering the function

$$\Phi_j(y)(t,t_1,\dots,t_j) := k(t,t_1)k(t_1,t_2)\cdots k(t_{j-1},t_j)y(t_j), \qquad (0\leq t,t_1,\dots,t_j\leq 1)$$

we observe that if $y\in C[0,1]$ then $\Phi_j(y)\in C[0,1]^{j+1}$ and $\Phi_j(y)\in L^p[0,1]^{j+1}$ when $y\in L^p[0,1]$.

We fix a Schauder basis $\{e_r^{(1)}\}_{r\in\mathbb{N}}$ in $C[0,1]$ or $L^p[0,1]$, and for each $d\in\mathbb{N}$, the tensor bases $\{e_r^{(d)}\}_{r\in\mathbb{N}}$ in $C[0,1]^d$ or $L^p[0,1]^d$ rearranged as above together with the projections on these bases $\{P_r^{(d)}\}_{r\in\mathbb{N}}$.

Given $m\in\mathbb{N}$, in each of the three cases studied in this subsection, we consider the following well defined operators L_j: The first one, $L_j : C[0,1]\longrightarrow C[0,1]$,

$$L_j(y)(t) := \int_0^1 \left(\cdots \int_0^1 P_{m^{j+1}}^{(j+1)}(\Phi_j(y)(t,t_1,\dots,t_j)dt_j \right) \cdots dt_1, \tag{13}$$

the second one, $L_j : C[0,1]\longrightarrow C[0,1]$,

Table 1
$\|x^* - x^{(2,9)}\|$ for Example 2.10 considering Faber-Schauder basis of decimal hat functions and Haar basis in $C[0,1]$.

t	HWM [23]	Proposed method (Faber-Schauder basis)	Proposed method (Haar basis)
0	6.1×10^{-5}	0	0
0.2	1.8×10^{-3}	4.3×10^{-7}	1.3×10^{-2}
0.4	1.2×10^{-3}	8.1×10^{-6}	3.7×10^{-2}
0.6	1.6×10^{-3}	7.1×10^{-5}	3.1×10^{-2}
0.8	5.3×10^{-3}	3.5×10^{-4}	1.3×10^{-3}
1	2.3×10^{-1}	1.2×10^{-3}	7.7×10^{-2}

$$L_j(y)(t) := \int_0^t \left(\int_0^{t_1} \left(\cdots \int_0^{t_{j-1}} P^{(j+1)}_{m^{j+1}}(\Phi_j(y)(t, t_1, \ldots, t_j) dt_j \right) \cdots \right) dt_1, \tag{14}$$

and the last one, $L_j : L^p[0,1] \longrightarrow L^p[0,1]$,

$$L_j(y)(t) := \int_0^t \left(\int_0^{t_1} \left(\cdots \int_0^{t_{j-1}} P^{(j+1)}_{m^{j+1}}(\Phi_j(y)(t, t_1, \ldots, t_j) dt_j \right) \cdots \right) dt_1. \tag{15}$$

In each of these scenarios, we can find sufficiently large $m \in \mathbb{N}$ so that $\|L^j(y) - L_j(y)\|$ is small enough in such a way that we can apply Corollary 2.3 and, consequently, we can consider as an approximation of the unique solution

$$x^{(n,m)} := \sum_{j=0}^{n} L_j y.$$

We now show the numerical results obtained by the proposed method to several specific examples. All numerical examples have been performed using Mathematica 12 on an 11th Gen Intel(R) Core(TM) i7-1165G7 system under Windows 11 Home operating system.

Example 2.10. We consider the equation of the Example 6.1 in [23]:

$$x(t) = \cos(t) + \int_0^t (s-t)\cos(s-t)x(s)\, ds,$$

whose solution is $x^\bullet(t) = \frac{1}{3}(2\cos(\sqrt{3}t) + 1)$.

We compare the exact and approximate solutions in Table 1 and the absolute error of the proposed method with the Haar wavelets method (HWM, [23]).

The errors obtained with our method using the Faber-Schauder basis are comparable to the order of those obtained computing the involved sixth projection in [23], which requires the resolution of a linear system of 64 equations, something that is not necessary in our case, with the advantage that this entails.

Example 2.11. We consider the Fredholm integral equation whose associated linear operator has norm greater than 1 but its Neumann series is absolutely convergent:

$$x(t) = -0.09 - 0.075t + t^7 + \int_0^1 (0.6t + 0.81s)x(s)\, ds,$$

with solution $x^\bullet(t) = t^7$. Table 2 shows the numerical results obtained for $n = 2, 3$ and 4.

3. The inverse problem: estimation of parameters for linear equations of the second kind

In this section we address an inverse problem of parameter estimation in a linear equation of the second kind. As mentioned in the introduction, there is a variety of techniques that allow us to tackle this type of problem. In particular, we are going to focus on the use of collage-type results that avoid the use of regularisation techniques and whose idea is to approximate an element in a complete metric space by fixed points of a family of operators in the same space. Specifically, the technique we propose is based on the following generalized version of the collage theorem in a linear framework.

Table 2
$\|x^{\bullet} - x^{(n,9)}\|$ for Example 2.11 using the Faber-Schauder basis of decimal hat functions in $C[0,1]$.

t	$n=2$	$n=3$	$n=4$
0	3.1×10^{-2}	1.6×10^{-2}	5.4×10^{-3}
0.2	3.9×10^{-2}	2.1×10^{-2}	7.9×10^{-3}
0.4	4.8×10^{-2}	2.6×10^{-2}	1.1×10^{-2}
0.6	5.7×10^{-2}	3.2×10^{-2}	1.5×10^{-2}
0.8	6.6×10^{-2}	3.7×10^{-2}	1.5×10^{-2}
1	7.5×10^{-2}	4.2×10^{-2}	1.7×10^{-2}

Theorem 3.1. *Let X be a Banach space, Λ a nonempty set, and for each $\lambda \in \Lambda$, let $L_\lambda \in \mathcal{L}(X)$ be such that the series $\sum_{j=0}^{\infty} L_\lambda^j$ is convergent. If $y \in X$ and for each $\lambda \in \Lambda$, $x_\lambda \in X$ is the unique solution of the linear equation of the second kind $(I - L_\lambda)(\cdot) = y$, then*

$$\left.\begin{array}{l} x \in X \\ \lambda \in \Lambda \end{array}\right\} \Rightarrow \|x - x_\lambda\| \le \left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| \|(I - L_\lambda)x - y\| .$$

Proof. It suffices to observe that, given $\lambda \in \Lambda$ and $x \in X$,

$$(I - L_\lambda)(x - x_\lambda) = (I - L_\lambda)(x) - y,$$

so, Theorem 2.1 (see also Remark 2.2) guarantees that

$$x - x_\lambda = \left(\sum_{j=0}^{\infty} L_\lambda^j \right) ((I - L_\lambda)(x) - y))$$

and therefore

$$\|x - x_\lambda\| \le \left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| \|(I - L_\lambda)(x) - y)\|. \quad \square$$

The mentioned above parameter estimation problem for a linear equation of the second kind reads as follows. Let X be a Banach space, $y \in X$ and Λ be a nonempty set. For each $\lambda \in \Lambda$, assume that $L_\lambda \in \mathcal{L}(X)$ is such that its Neumann series $\sum_{j=0}^{\infty} L_\lambda^j$ converges, and that $x_\lambda \in X$ is the unique solution of the linear equation of the second kind

$$(I - L_\lambda)(\cdot) = y.$$

Let $x^{\bullet} \in X$ be a target element, experimentally obtained when the problem is applied to a real world model, that is, $x^{\bullet}$ is the solution, or an approximation, of one of the problems $(I - L_\lambda)(\cdot) = y$. We want to estimate the parameter $\lambda^{\bullet} \in \Lambda$ such that $(I - L_{\lambda^{\bullet}})(x^{\bullet}) = y$. To this end, we search among all the parameters $\lambda \in \Lambda$ one $\lambda^{\bullet} \in \Lambda$, that makes $\|x^{\bullet} - x_\lambda\|$ minimum, or if not possible, that $\|x^{\bullet} - x_{\lambda^{\bullet}}\|$ be close enough to $\inf_{\lambda \in \Lambda} \|x^{\bullet} - x_\lambda\|$.

According to the extended version of the collage result given in Theorem 3.1,

$$\inf_{\lambda \in \Lambda} \|x^{\bullet} - x_\lambda\| \le \inf_{\lambda \in \Lambda} \left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| \left\|(I - L_\lambda)x^{\bullet} - y\right\| .$$

If we also suppose that

$$\rho := \sup_{\lambda \in \Lambda} \left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| < \infty, \tag{16}$$

then the previous inequality gives

$$\inf_{\lambda \in \Lambda} \|x^{\bullet} - x_\lambda\| \le \rho \inf_{\lambda \in \Lambda} \|(I - L_\lambda)x^{\bullet} - y\|,$$

and so, we want to minimize $\|(I - L_\lambda)x^{\bullet} - y\|$ with $\lambda \in \Lambda$.

It is worth mentioning that condition (16) is not very restrictive. For instance, typically in applications, Λ is a nonempty and compact subset of $\mathbb{R}^N$, for some $N \ge 1$, and the function $f : \Lambda \longrightarrow \mathbb{R}$ defined at each $\lambda \in \Lambda$ as

$$f(\lambda) := \left\| \sum_{j=0}^{\infty} L_\lambda^j \right\|$$

is continuous. Then, condition (16) is fulfilled.

Thus for instance, for such a compact subset Λ of $\mathbb{R}^N$ and for $X = C[a, b]$ or $L^p[a, b]$, $(1 < p < \infty)$, we consider the linear Volterra integral operator $L_\lambda \in \mathcal{L}(X)$ defined as

$$L_\lambda x := \int_a^t k_\lambda(\cdot, s)x(s)\, ds, \qquad (x \in X),$$

where $k_\lambda \in C[a, b]^2$ if $X = C[a, b]$ or $k_\lambda \in L^\infty[a, b]^2$ when $X = L^p[a, b]$. Then, in view of Theorem 2.9 we have that for $X = C[a, b]$

$$\left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| \le e^{(b-a)\|k_\lambda\|_\infty},$$

while for $X = L^p[a, b]$,

$$\left\| \sum_{j=0}^{\infty} L_\lambda^j \right\| \le 1 + (b-a)\|k_\lambda\|_\infty e^{(b-a)\|k_\lambda\|_\infty}.$$

Hence, if the function

$$g(\lambda) := \|k_\lambda\|_\infty, \qquad (\lambda \in \Lambda)$$

is continuous, then condition (16) is valid.

The linear Fredholm integral equations admit a similar reasoning.

Now we illustrate this parameter estimation technique with two numerical examples to which the classical collage theorem cannot be applied. The results have been obtained under the same conditions as in the previous section.

Example 3.2. We consider the family of Volterra integral equations

$$x(t) = \cos(t) + \int_0^t k_\lambda(t, s)x(s)\, ds,$$

with $k_\lambda(t, s) = (\lambda_1 s - \lambda_2 t)\cos(s - t)$, where $\lambda = (\lambda_1, \lambda_2) \in [0, 2] \times [0, 2]$. We consider as target element x the piecewise linear function constructed from the data considering the Faber-Schauder basis of Example 2.10, which corresponds to the value $\lambda = (1, 1)$. Then we solve the associated minimization problem and we obtain $\lambda^\bullet = (0.993, 1.003)$ as the estimated parameters.

Example 3.3. We consider the family of Fredholm integral equations

$$x(t) = \lambda_1 + \lambda_2 t + t^7 + \int_0^1 (0.6t + 0.81s)x(s)\, ds,$$

with $\lambda = (\lambda_1, \lambda_2) \in [-0.3, 0.3] \times [-0.3, 0.3]$. We consider as target element x the piecewise linear function constructed from the data of Example 2.11 for $n = 3$ which corresponds to $\lambda = (-0.09, -0.075)$. We solve our minimization problem and we obtain $\lambda^\bullet = (-0.1, -0.1)$ as the estimated parameters.

4. Conclusions

In this paper we have designed an iterative algorithm to solve linear equations of the second kind in a Banach space. This algorithm combines the generalisation of the geometric series theorem and the use of Schauder bases in suitable spaces. It has been applied to solve both certain linear Fredholm integral equations whose associated linear operator is not contractive and linear Volterra integral equations. The simplicity of the algorithm, in which no system of linear equations needs to be solved, makes it a powerful tool to be applied. Moreover, we have stated a generalised version of collage theorem that can be applied to parameter estimation for, among others, the two types of linear integral equations indicated above. The ideas that have allowed us to establish the previous results can be a source of inspiration to approach the study of certain types of singular integral equations.

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