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Subvariedades espaciales a través de hipersuperficies luz en variedades de Lorentz

Autor
Rodrigo Morón Sanz

Director: Francisco J. Palomo Ruiz
Dep. de Matemática Aplicada - Universidad de Málaga

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Autor: Rodrigo Morón Sanz
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« In 1905 “L’ Enseignement Mathématique” started an inquiry into the methods of working of mathematicians. The results of this inquiry augmented and developed later by several authors, for instance Carmichael and Hadamard, can be expressed shortly as follows. The faculty of deduction belongs to the conscious mind, the subconscious being in general only able to perform very simple and trivial deductions. On the contrary the faculty of rearranging is typical of the work of the subconscious and is described by Carmichael as consisting of an extremely rapid passing over of innumerable useless combinations till a vital one or some vital ones rise to consciousness, to bring, after a severe control of the conscious mind, new truth to light. »

J. A. Schouten

A mis padres y a mi hermano.

Contents

Chapter 0

Resumen

Desde que Einstein formalizó la extensión del espacio-tiempo de Minkowski a una variedad curva de Lorentz para modelar campos gravitatorios no nulos, la geometría lorentziana ha servido como el marco matemático fundamental para expresar la Relatividad General. Inicialmente, su estudio era predominantemente local, ya que investigaciones globales en geometría lorentziana parecían innecesarias para describir nuestro Universo inmediato. Sin embargo, a partir de la década de 1970, los progresos en las Teorías de Causalidad y Singularidad, impulsados principalmente por las contribuciones de Hawking, Penrose, Geroch, entre otros, han conducido a la reconocida necesidad de desarrollar nuevas metodologías para realizar análisis exhaustivos de los modelos cosmológicos. Estos avances impulsaron significativamente la progresión contemporánea de la geometría lorentziana global, produciendo resultados con interpretaciones físicas inesperadas.

Además, es importante destacar que la exploración puramente geométrica de las variedades lorentzianas ha generado interés en numerosos estudios recientes, estableciéndose como una rama de la Geometría Diferencial caracterizada por problemas no exclusivamente originados en la Física y carentes de aplicaciones directas en ella.

Esencialmente, un tensor métrico lorentziano en una variedad (diferenciable) define una estructura de conos tangentes, que nos permiten clasificar los vectores tangentes en tres tipos: temporales, luminosos y espaciales. Esto se conoce como su carácter causal. En consecuencia, una curva en la variedad se clasifica como temporal, luminosa o espacial si sus vectores tangentes exhiben consistentemente el carácter causal correspondiente. La idea clave de la Relatividad General consiste en conceptualizar la gravedad como un efecto producido por la curvatura del espacio-tiempo. De esta manera, las partículas materiales (luminosas, respectivamente) en “caída libre” serían geodésicas temporales (luminosas, respectivamente) del tensor métrico lorentziano.

Recordemos la noción de estructura conforme en signatura semi-riemanniana. Una estructura conforme semi-riemanniana (M, c) es el par formado por una variedad M y una clase de equivalencia c de métricas semi-riemannianas en M, donde dos métricas están en c si difieren por un factor que es una función diferenciable positiva en la variedad M. A menos que se indique lo contrario, M es una variedad de dimensión $n \geq 2$.

La estructura de conos es un invariante conforme en la geometría lorentziana. Weyl introdujo las estructuras conformes (en signatura lorentziana) para formular una teoría unificada de campos. Weyl escribió “Para derivar los valores de las cantidades g_{ik} a partir de fenómenos observados directamente, usamos señales de luz... Observando la llegada de la luz en los puntos vecinos a O podemos determinar las razones de los valores de los g_{ik}... Sin embargo, es imposible derivar más resultados a partir del fenómeno de la propagación de la luz...” [67, Cap. 4]. Se pueden encontrar más detalles sobre la teoría de Weyl y las ideas físicas detrás de ella en los textos clásicos [29], [10] y [57], mientras que una exposición elemental de la misma está disponible en [1].

En esta tesis, nos centraremos en dos temas aparentemente no relacionados que, tras un examen más detenido, revelan conexiones profundas en las que creemos contribuir a su clarificación y comprensión mejorada. Estos son la geometría lorentziana y la geometría conforme riemanniana. Aunque a primera vista estos dos temas pueden parecer no relacionados, se sabe que han estado conectados desde los tiempos de Cartan, quien introdujo la noción de “espacio generalizado” para construir un puente entre la geometría en el sentido del programa de Erlangen de Felix Klein y la Geometría Diferencial. En el programa de Erlangen, una geometría se da mediante una variedad dotada de una acción transitiva de un grupo de Lie, y así por un espacio homogéneo G/H de un grupo de Lie G. Klein consideraba G/H dotado de la geometría cuyo grupo de automorfismos era G. La idea de Cartan fue asociar a dicho espacio homogéneo una estructura geométrica diferencial, cuyos objetos pueden pensarse como análogos curvos del espacio homogéneo G/H, al igual que las variedades riemannianas pueden pensarse como análogos curvos del espacio euclidiano. En terminología moderna, tales estructuras se llaman geometrías de Cartan, y se definen como fibrados principales dotados de conexiones de Cartan, véase la Definición 3.1. El espacio homogéneo G/H se denomina el modelo homogéneo de la geometría de Cartan. Un estudio exhaustivo de varios ejemplos básicos de geometrías de

Cartan se puede encontrar en el libro [62].

Para Cartan, las estructuras conformes riemannianas n-dimensionales pueden considerarse como análogos curvos del espacio de rayos en el cono de luz futuro del espacio-tiempo de Minkowski $\mathbb{L}^{n+2}$ visto como un subconjunto en el espacio proyectivo $\mathbb{R}P^{n+1}$. Notemos que este espacio de rayos es topológicamente la esfera $\mathbb{S}^n$. Aquí, el grupo ortocrono $O^+(1, n+1)$ actúa como el grupo de transformaciones conformes globales de $\mathbb{S}^n$ con respecto a la clase conforme que contiene la métrica redonda canónica. Denotaremos esta clase conforme por c_0. En este contexto, el par $(\mathbb{S}^n, c_0)$ es denominado el espacio de Möbius. Las geometrías de Cartan que surgen al deformar el espacio de Möbius se conocen como geometrías de Möbius, véase la Definición 3.5. Las geometrías de Cartan se han utilizado también para investigar otros tipos de geometrías. Por ejemplo, [53] está dedicado al estudio de las variedades luminosas vistas como análogos curvos del cono de luz futuro de $\mathbb{L}^{n+2}$. Recordemos que una variedad luminosa no es más que una variedad dotada con un tensor métrico degenerado, consulte la Sección 2.2.

Queremos enfatizar que las ideas de Cartan jugaron un papel significativo en el trabajo de Einstein desarrollando la Relatividad General. Adjuntamos un fragmento de una carta escrita por Cartan y dirigida a Einstein donde se puede ver cómo compartían correspondencia e ideas:

> *«En tus artículos recientes en los Sitzungsberichte dedicados a una nueva teoría de la relatividad generalizada, introdujiste la noción de "Fernparallelismus" en un espacio riemanniano. Ahora bien, la noción de espacio riemanniano dotado de un Fernparallelismus es un caso especial de una noción más general, la de espacio con una conexión euclidiana, que esbozé brevemente en 1922 en un artículo en los Comptes Rendus (vol. 174, pp. 593-595), publicado cuando impartías tus conferencias en el Collège de France; incluso recuerdo que intenté, en casa del Sr. Hadamard, darte el ejemplo más simple de un espacio riemanniano con Fernparallelismus al considerar dos vectores dentro de una esfera que forman el mismo ángulo con las líneas meridianas que pasan por sus orígenes como paralelos: las geodésicas correspondientes son las líneas de rumbo. Este ejemplo se cita en un artículo: "Sur les récentes généralisations de la notion d'espace" (Bull. Sciences math. 48, 1924, pp. 294-320). »*

En este pasaje, se puede leer cómo Cartan le dijo a Einstein que la noción de "Fernparallelismus" era un caso particular de una teoría mucho más general de conexiones que él mismo había desarrollado. La carta completa se puede leer en [26].

Uno de los objetivos principales y el hilo conductor de esta tesis será proporcionar un enfoque novedoso para estudiar y relacionar la geometría lorentziana y la geometría conforme riemanniana. Para ser precisos, vamos a utilizar inmersiones espaciales que factorizan a través de ciertas hipersuperficies luminosas embebidas en variedades lorentzianas. Además de este propósito, estudiaremos estos tipos de inmersiones porque son de interés desde la perspectiva de la teoría de subvariedades. Como es bien sabido, las hipersuperficies luminosas heredan una métrica degenerada de la métrica lorentziana ambiente y desempeñan un papel importante en la Relatividad General como horizontes de sucesos de agujeros negros [33]. La teoría clásica de subvariedades falla para estas hipersuperficies ya que el fibrado normal de tales hipersuperficies está contenido en su fibrado tangente. Creemos que el estudio de inmersiones espaciales de codimensión dos que factorizan a través de una hipersuperficie luminosa puede proporcionar una herramienta para comprender la geometría de tales hipersuperficies y también servir para profundizar en nuestra comprensión de las propias inmersiones. El estudio de inmersiones espaciales de codimensión dos en hipersuperficies luminosas se ha desarrollado previamente en [52], [55] y [56] para el caso de inmersiones compactas en el cono de luz del espacio-tiempo de Minkowski. El caso no compacto se considera en [4] y el estudio de inmersiones atrapadas en hipersuperficies luminosas del espacio-tiempo de de-Sitter aparece en [3]. Este enfoque también se ha aplicado a los espacio-tiempos de Brinkmann. Recordemos que los espacio-tiempos de Brinkmann admiten un campo de vectores luminosos paralelo y entonces, admiten una foliación por hipersuperficies luminosas. Las inmersiones espaciales que se encuentran en tales hipersuperficies se han estudiado en [16] para el caso compacto y en [54] para casos más generales.

También será crucial para nosotros abordar la geometría conforme riemanniana a través de las geometrías de Cartan, las cuales proporcionan herramientas poderosas para reinterpretar la geometría conforme. Uno de los hitos principales en esta tesis será reconstruir tales geometrías de Cartan a partir de inmersiones espaciales. Esta metodología se examinará con detalle en el Capítulo 5. En los Capítulos 2 y 3, introduciremos las nociones necesarias sobre geometría lorentziana, conforme y de Cartan para entender esta nueva perspectiva. El Capítulo 4 está dedicado enteramente al estudio de inmersiones espaciales en una cierta familia de espacio-tiempos. Este capítulo tiene un interés intrínseco desde la perspectiva de la teoría de subvariedades, pero además también tiene aplicaciones en el estudio de las relaciones entre la geometría lorentziana y la geometría conforme. Finalmente, en el Capítulo 6, continuaremos estudiando las relaciones entre ambas geometrías utilizando para ello una versión debilitada de la construcción de variedades ambiente para estructuras conformes riemannianas dada por Fefferman y Graham, ver [31]. Para llevar a cabo tal construcción, es necesario enfatizar que el espacio total $\mathcal{Q}$ del fibrado de escalas de una estructura conforme riemanniana admite naturalmente una métrica degenerada, llamada tensor tautológico, que denotaremos por $\bar{h}$, ver Sección 6.1. Con esto en mente, la construcción de una variedad ambiente esencialmente implica extender localmente la variedad luminosa $(\mathcal{Q}, \bar{h})$ en una variedad lorentziana que la admita como una hipersuperficie luminosa.

Chapter 1

Introduction

Since Einstein's extension of Minkowski spacetime to a curved Lorentzian manifold to model nonzero gravitational fields, Lorentzian geometry has served as the fundamental mathematical framework for expressing General Relativity. Initially, its study was predominantly local, as global investigations of Lorentzian geometry appeared unnecessary for describing our immediate Universe. However, beginning in the 1970s, advancements in Causality and Singularity Theories, primarily through the contributions of Hawking, Penrose, Geroch, and others, prompted consideration of the necessity to develop new methodologies for comprehensive examinations of cosmological models. These advancements significantly spurred the contemporary progression of global Lorentzian geometry, yielding results with unexpected physical interpretations.

Furthermore, it is noteworthy that the purely geometric exploration of Lorentzian manifolds has generated interest in numerous recent studies, establishing itself as a branch of Differential Geometry characterized by problems not exclusively originating from Physics and lacking direct applications therein.

Essentially, a Lorentzian metric tensor on a (smooth) manifold defines a structure of tangent cones, which allow us to classify tangent vectors into three types: timelike, lightlike, and spacelike. This is known as their causal character. Consequently, a curve on the manifold is classified as timelike, lightlike, or spacelike if its tangent vectors consistently exhibit the corresponding causal character. The key idea of General Relativity consists of conceptualizing gravity as an effect produced by the curvature of the spacetime. In this way, material particles (resp. lightlike) in "free fall" would be timelike (resp. lightlike) geodesics of the Lorentzian metric tensor.

Let us recall the notion of a conformal structure in semi-Riemannian signature. A semi-Riemannian conformal structure (M, c) is the pair formed by a manifold M and an equivalence class c of semi-Riemannian metrics on M, where two metrics are in c if they differ by a factor that is a smooth positive function on the manifold M. Unless stated otherwise, M is an $(n \geq 2)$-dimensional manifold.

The structure of cones is a conformal invariant in Lorentzian geometry. Weyl introduced conformal structures (in Lorentzian signature) in order to formulate a unified fields theory. Weyl wrote " To derive the values of the quantities g_{ik} from directly observed phenomena, we use light-signals... By observing the arrival of light at the points neighbouring to O we can thus determine the ratios of the values of the g_{ik}'s... It is impossible, however, to derive any further results from the phenomenon of the propagation of light..." [67, Chap. 4]. Further details of Weyl's theory and the physical ideas behind it can be found in the classical texts [29],[10] and [57], while an elementary exposition of it is available in [1].

In this thesis, we will focus on two seemingly unrelated topics that, upon closer inspection, they reveal deep connections in which we believe we contribute to their clarification and enhanced understanding. These are Lorentzian geometry and Riemannian conformal geometry. Although at first glance these two topics may appear unrelated, it is known that they have been connected since the times of Cartan, who introduced the notion of "espace généralisé" in order to build a bridge between geometry in the sense of Felix Klein's Erlangen program and Differential Geometry. In the Erlangen program, a geometry is given by a manifold endowed with a transitive action of a Lie group, and thus by a homogeneous space G/H of a Lie group G. Klein considered G/H endowed with the geometry whose automorphism group was G. Cartan's idea was to associate to such a homogeneous space a differential geometric structure, whose objects may be thought of as curved analogs of the homogeneous space G/H, just like Riemannian manifolds may be thought of as curved analogs of Euclidean space. In modern terminology, such structures are named Cartan geometries, and they are defined as principal bundles endowed with Cartan connections, see Definition 3.1. The homogeneous space G/H is referred to as the homogeneous model of the Cartan geometry. A comprehensive study of several basic examples of Cartan geometries can be found in the book [62].

For Cartan, n-dimensional Riemannian conformal structures can be considered as curved analogues of the space of rays in the future lightlike cone of the Minkowski spacetime $\mathbb{L}^{n+2}$ seen as a subset in the projective space $\mathbb{R}P^{n+1}$. Note that this space of rays is topologically the sphere $\mathbb{S}^n$. Here, the orthochronous group $O^+(1, n+1)$ acts as the group of global conformal transformations of $\mathbb{S}^n$ with respect to the conformal class containing the canonical round metric. We denote this conformal class by c_0. In this setting, $(\mathbb{S}^n, c_0)$ is called the Möbius space. Cartan geometries that arise from deforming the Möbius space are known as Möbius geometries, Definition 3.5. Cartan geometries have been employed to investigate other types of geometries. For instance, [53] is devoted to study lightlike manifolds as curved analogs of the future lightlike cone of $\mathbb{L}^{n+2}$. Recall that a lightlike manifold is nothing more than a manifold endowed with a degenerate metric tensor, see Section 2.2.

We want to emphasize that Cartan's ideas played a significant role in Einstein's work developing General Relativity. We attach a fragment of a letter written by Cartan and addressed to Einstein where it can be seen how they shared correspondence and ideas:

> *« In your recent articles in the Sitzungsberichte devoted to a new theory of generalized relativity, you introduced, the notion of "Fernparallelismus" in a Riemannian space. Now, the notion of Riemannian space endowed with a Fernparallelismus is a special case of a more general notion, that of space with a Euclidean connection, which I outlined briefly in 1922 in an article in the Comptes Rendus (vol. 174, pp.593-595), published when you gave your lectures at the Collège de France; I even remember trying, at Mr Hadamard's home, to give you the simplest example of a Riemannian space with Fernparallelismus by regarding two vectors within a sphere making the same angle with the meridian lines passing through their origins as parallel: the corresponding geodesics are the rhumb lines. This example is quoted in an article: "Sur les récentes généralisations de la notion d'espace" (Bull. Sciences math. 48, 1924, pp.294-320). »*

In this passage, one can read how Cartan told Einstein that the notion of "Fernparallelismus" was a particular case of a much more general theory of connections that he himself had developed. The entire letter can be read at [26].

One of the main goals and guiding thread of this thesis will be to provide a novel approach to studying and relating Lorentzian geometry and Riemannian conformal geometry. To be precise, we will use spacelike immersions that factor through certain lightlike hypersurfaces embedded in Lorentzian manifolds. In addition to this purpose, we study these types of immersions because they are of interest from the perspective of the theory of submanifolds. As it is well-known, lightlike hypersurfaces inherits a degenerate metric from the Lorentzian ambient metric and play an important role in General Relativity as event horizons of black holes [33]. The classical theory of submanifolds fails for these hypersurfaces since the normal bundle of such hypersurfaces is contained in their tangent bundle. We think that the study of codimension two spacelike immersions which factor through a lightlike hypersurface can provide a tool to understand the geometry of such hypersurfaces and also to serve for deepening our understanding of the immersions themselves. The study of codimension two spacelike immersions in lightlike hypersurfaces has been previously developed in [52], [55] and [56] for the case of compact immersions into the lightlike cone in the Minkowski spacetime. The non-compact case is considered in [4] and the study of trapped immersions into lightlike hypersurfaces of the de-Sitter spacetime appears in [3]. This approach has been also applied to Brinkmann spacetimes. Recall that Brinkmann spacetimes admit a parallel lightlike vector field and then, they have a foliation by lightlike hypersurfaces. Spacelike immersions which lie in such hypersurfaces have been studied in [16] for the compact case and in [54] for more general settings.

It will also be crucial for us to approach Riemannian conformal geometry through Cartan geometries, which provide powerful tools for reinterpreting conformal geometry. One of the main milestones in this thesis will be to reconstruct such Cartan geometries from spacelike immersions. This methodology will be thoroughly examined in Chapter 5. In Chapters 2 and 3, we will introduce the necessary notions of Lorentzian, conformal, and Cartan geometries to understand this new perspective. Chapter 4 is entirely devoted to the study of spacelike immersions in a certain family of spacetimes. This Chapter has intrinsic interest from the perspective of the theory of submanifolds, but it also has applications in the study of the relationships between Lorentzian geometry and conformal geometry. Finally, in Chapter 6, we will continue studying relationships between both geometries focusing on a weakened version of the ambient manifold construction for Riemannian conformal structures given by Fefferman and Graham, see [31]. To carry out such construction, it is necessary to emphasize that the total space $\mathcal{Q}$ of the scale bundle of a Riemannian conformal structure naturally admits a degenerate metric, called the tautological tensor, which we denote by $\bar{h}$, see Section 6.1. With this in mind, the construction of an ambient manifold essentially involves locally extending the lightlike manifold $(\mathcal{Q}, \bar{h})$ into a Lorentzian manifold that admits it as a lightlike hypersurface.

Now that we have introduced the setting of this thesis, we are going to outline the contents of each chapter. We would like to highlight that the material covered in Chapters 4, 5, and 6 is available in [47], [45] and [46], respectively.

From a mathematical perspective, a classical problem in Riemannian conformal geometry is the construction of invariants. The invariants of any Riemannian metric in a conformal class transform according to intricate formulas and they do not provide invariants for the conformal structure. The problem of the construction of conformal invariants remains very difficult. Anyway, there are several equivalent ways of describing conformal structures which immediately lead to a construction of conformal invariants. In Chapter 3, we focus on introducing and studying two of these equivalent ways of describing conformal structures: Möbius geometries and tractor bundles.

For this purpose, we need to introduce a family of tensors which we will refer to as "Schouten type-tensors", Definition 3.12. A tensor D is said to be a "Schouten type-tensor" for a Riemannian conformal structure (M, c) if it is a map such that:

1. $D : c \to \mathcal{T}_{(0,2)}M$ such that for every $g \in c$, the tensor $D(g)$ is symmetric and
2. D satisfies the following conformal transformation law

$$D(e^{2u}g) = D(g) - \frac{\|\nabla u\|^2}{2}g - \mathrm{Hess}(u) + du \otimes du,$$

   where ∇u and $\mathrm{Hess}(u)$ are the gradient and the Hessian of the function $u \in \mathcal{C}^\infty(M)$ for the metric g, respectively. Here, $\|\cdot\|$ denotes the norm with respect to g.

Every Riemannian conformal structure (M, c) endowed with a "Schouten type-tensor" D can be seen equivalently as an admissible Cartan geometry $(p : \mathcal{P} \to M, \omega)$ with homogeneous model the Möbius space, see details in Chapter 3. For dimension $n \geq 3$, every Riemannian conformal structure admits a unique normal Möbius geometry, [20, Theor. 1.6.7]. This is the one constructed by taking the Schouten tensor itself as the "Schouten type-tensor". The Cartan connection ω associated to the normal Möbius geometry is called the normal Cartan connection.

On the other hand, we are interested in the equivalent approach taken by Thomas in the 1920s, [63]. He describes Riemannian conformal structures in terms of vector bundles endowed with certain linear connections, see Definitions 3.7 and 3.8. He named these special bundles and connections as tractor conformal bundles $\mathcal{T} \to M$ and tractor connections $\nabla^{\mathcal{T}}$. The relationship between tractor connections on tractor conformal bundles and Cartan connections on Möbius geometries was given by Čap and Gover in [18] in a more general setting. We want to emphasize that Chapter 3 has been added in an attempt to clarify this relationship. As far as we know, this aspect has not been addressed elsewhere with sufficient detail. There is a one-to-one correspondence between tractor conformal bundles $\mathcal{T} \to M$ endowed with tractor connections $\nabla^{\mathcal{T}}$ and Möbius geometries $(p : \mathcal{P} \to M, \omega)$. The following diagram represents the equivalent constructions that we have just presented:

$$\begin{tikzcd}
 & \big(p : \mathcal{P} \to M, \omega\big) \arrow[dl, <->] \arrow[dr, <->] & \\
\big(M, c, D\big) \arrow[rr, <->] & & \big(\mathcal{T} \to M, \nabla^{\mathcal{T}}\big)
\end{tikzcd}$$

For dimension $n \geq 3$, the pair consisting of the tractor conformal bundle and the tractor connection that corresponds to the normal Möbius geometry is called the normal standard tractor conformal bundle, and the tractor connection is referred to as the normal tractor connection, Definition 3.20. In [19], the normal tractor connection is characterized in terms of its curvature properties. We have included here this result in Theorem 3.22.

It is worth highlighting that “Schouten type-tensors” are truly interesting to study. These tensors are of greater importance in dimension two where there is no canonical choice since the Schouten tensor is not defined in such dimension. In the last Chapter, we will see that all these tensors can be reconstructed through certain Weingarten endomorphisms of spacelike submanifolds. We will also focus on a “normalized” subset within the family of “Schouten type-tensors”, whose elements are known as Möbius structures. A Möbius structure is a triple (M, c, D) where D is a “Schouten type-tensor” that additionally satisfies the condition

$$\mathrm{trace}_g D(g) = \frac{\mathrm{S}^g}{2(n-1)}$$

for every $g \in c$, Definition 3.24. These Möbius structures have been used to study various problems. For instance, the equivalence problem between two dimensional Möbius structures (M, c, D) and a subset of Möbius geometries was addressed in [15] and [14].

In Chapter 4 we begin by introducing a family of spacetimes that we are going to study, to do this we first have to recall some basic facts of the Schwarzschild spacetime. Karl Schwarzschild discovered in 1916 the point-mass solution to Einstein equations that bears his name. Historically, this solution was the first and more important nontrivial solution of the vacuum Einstein equations. The $(m + 2)$-dimensional exterior Schwarzschild spacetime with mass $\mathbf{M} \geq 0$ is equipped with the Lorentzian metric

$$\widetilde{g} = -\Big(1 - \frac{2\mathbf{M}}{r^{m-1}}\Big)dt^2 + \frac{1}{1 - \frac{2\mathbf{M}}{r^{m-1}}}dr^2 + r^2 g_{\mathbb{S}^m},$$

where $(t, r) \in \mathbb{R} \times \mathbb{R}_{>0}$ with $r^{m-1} > 2\mathbf{M}$ and $g_{\mathbb{S}^m}$ denotes the usual round metric of constant sectional curvature 1 on the m-dimensional sphere $\mathbb{S}^m$. For $\mathbf{M} = 0$, the Schwarzschild metric reduces to the Minkowski metric in spherical terms, Example 4.10. The exterior Schwarzschild spacetime has several remarkable properties which we are interested in.

1. For every lightlike vector field ξ in the tr-half plane, the Levi-Civita connection $\widetilde{\nabla}$ of the Schwarzschild metric satisfies $\widetilde{\nabla}\xi = \alpha \otimes \xi$ for some 1-form α.

2. The $(m + 2)$-dimensional exterior Schwarzschild spacetime admits two foliations by lightlike hypersurfaces. In fact, for every lightlike vector field ξ in the tr-half plane, the distribution given by the vector fields $\widetilde{g}$-orthogonal to ξ is involutive and every leaf inherits a degenerate metric from $\widetilde{g}$, Lemma 4.1.

3. The vector field ∂_t is a timelike Killing vector field. That is, the exterior Schwarzschild spacetime is static.

These properties rely on the following facts. The metric $\widetilde{g}$ is a warped product metric given by a Lorentzian metric on an open subset of the tr-plane and a Riemannian metric, [51, Chap. 7]. Moreover, the metric on the tr-plane admits a globally defined lightlike vector field and the function $f^2(r) := 1 - \frac{2\mathbf{M}}{r^{m-1}}$ does depend only on r.

These facts lead us to consider the following family of Lorentzian warped product manifolds.

Definition 1.1. A Lorentzian warped product manifold $(\widetilde{M}, \widetilde{g}) = B \times_\lambda F$ is said to be an $(m+2)$-dimensional generalized (exterior) Schwarzschild spacetime when B is an open subset of $\mathbb{R}^2$ with canonical coordinates (t, r) and metric

$$g_B = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2, \tag{1.1}$$

where $f(r) > 0$, (F, g_F) is a m-dimensional connected Riemannian manifold and $\lambda \in C^\infty(B)$ with $\lambda > 0$ is the warping function. That is, $\widetilde{M} = B \times F$ and

$$\widetilde{g} = \pi_B^*(g_B) + (\lambda \circ \pi_B)^2 \pi_F^*(g_F),$$

where π_B and π_F are the natural projections on B and F, respectively (see Section 2.3).

Note that the Minkowski spacetime can be described in two ways from this setting. Namely, as was mentioned above and for $f(r)^2 = 1$, $\lambda(t,r) = 1$ and $(F, g_F) = \mathbb{E}^m$.

For $\lambda(t,r) = r$ and $(F, g_F) = \mathbb{S}^m$, this class includes relevant spacetimes with spherical symmetry. Namely, we set

$$f^2(r) = 1 - \frac{2\mathrm{M}}{r^{m-1}} + \frac{q^2}{r^{2m-2}} - \frac{2\Lambda r^2}{m(m+1)}, \tag{1.2}$$

where M is called the mass parameter, q is the charge and Λ is a constant function. For $q = \Lambda = 0$, we get the Schwarzschild metric and for $q \neq 0$, $\Lambda = 0$, the Reissner-Nordström metric with total charge q. The de-Sitter and anti-de-Sitter versions correspond to $\Lambda > 0$ and $\Lambda < 0$, respectively, [40].

The generalized Schwarzschild spacetimes admit two lightlike vector fields $\xi, \eta \in \mathfrak{X}(B)$, see (4.2). The distributions D_ξ and D_η defined by the $\widetilde{g}$-orthogonal vector fields to ξ and η, respectively, are involutive, Lemma 4.1. Therefore, we have two transverse foliations by lightlike hypersurfaces of $\widetilde{M}$. It is worth pointing out that the vector field $\partial_t \in \mathfrak{X}(\widetilde{M})$ is Killing if and only if the warping function λ depends only on the (radial) coordinate r. The integral curves of ξ and η are called lightlike geodesic generators of the corresponding lightlike hypersurfaces. Recall that we can scale ξ and η so that they are geodesic vector fields, [32]. As was mentioned, the Minkowski spacetime $\mathbb{L}^{m+2}$ admits two descriptions as generalized Schwarzschild spacetime. Every description provides different foliations by lightlike hypersurfaces, see details in Example 4.10.

The main aim of Chapter 4 is to study spacelike immersions of an n-dimensional manifold M in generalized Schwarzschild spacetimes. Most of our results are focused on the particular situation in which the spacelike immersion of M is contained in a leaf of the above mentioned foliations by lightlike hypersurfaces.

The research on spacelike immersions (definition in Section 2.2) has been developed both from physical and geometric interest. For instance, the Cauchy problem for the Einstein equations is formulated as an initial data problem on a Riemannian manifold which becomes a Cauchy hypersurface in the solution spacetime, see [40, Chap. 7]. Recall also the Penrose incompleteness theorem which relates the existence of a trapped codimension two spacelike immersion with the singularities of certain spacetimes [40, Chap. 7]. We also want to highlight the seminal work [58]. The notion of trapped immersion is usually given in terms of the mean curvature vector field of the immersion, Definition 4.45.

We wish to highlight that the study of spacelike immersions in generalized Friedmann-

Lemaître-Robertson-Walker spacetimes goes back to the seminal work [7]. Since then multiple researchers have developed this topic. These spacetimes are written as $I \times_\lambda F$ with metric $-dt^2 + \lambda^2(t)g_F$. From this point of view, the spacelike immersions in generalized Schwarzschild spacetimes can be seen as the next natural step to shed light in the theory of spacelike submanifolds. As far as we know, there is no many works devoted to this problem. For instance, in the setting of stationary spacetimes, the study of prescribed mean curvature problem in Schwarzschild and Reissner-Nordström spacetimes appears in [25]. On the other hand, the results in [65] have been enriching and have given us a better approach for the development of this Chapter.

The plan of Chapter 4 is as follows. Section 4.1 presents the distributions D_ξ and D_η and also includes several technical results to be used later. Section 4.2 focuses on the family of Lorentzian manifolds we are interested in, Definition 1.1. Since this family of Lorentzian manifolds are warped product manifolds, we particularize the formulas for theirs Levi-Civita connections from [51, Chap. 7]. For a spacelike immersion in a generalized Schwarzschild spacetime $\Psi : M \to B \times_\lambda F$, we have written $\Psi = (\Psi_B, \Psi_F)$, $u := t \circ \Psi_B$ and $v := r \circ \Psi_B$. Lemma 4.8 states that a spacelike immersion factors through an integral hypersurface of D_ξ if and only if

$$\nabla v = (f \circ \Psi_B)^2 \nabla u,$$

where ∇ denotes the gradient operator corresponding to the induced metric g on M. In order to make the presentation of the results more fluid, in the Introduction we specialize our results and discussions to the distribution D_ξ and its integral lightlike hypersurfaces. Almost all the results admit a similar version for the other distribution D_η.

Section 4.3 exhibits several fundamental equations for spacelike immersions in generalized Schwarzschild spacetimes. As a consequence, we obtain an integral characterization of compact spacelike immersions through leaves of D_ξ, Theorem 4.14.

> Assume $\Psi : M \to B \times_\lambda F$ is a compact spacelike immersion in a generalized Schwarzschild spacetime with $f' > 0$ (resp. $f' < 0$). Then
>
> $$\int_M \left[n\, \widetilde{g}(\mathbf{H}, \xi^\perp) + \left(\frac{\xi\lambda}{\lambda} \circ \Psi_B\right) \left[n + 2g(\xi^\top, \eta^\top) \right] \right] d\mu_g \geq 0. \quad \text{(resp. } \leq 0\text{)},$$
>
> where the superscripts $\top$ and $\perp$ denote the tangent and normal parts of the indicated vector fields, respectively. The equality holds if and only if M factors through an integral hypersurface of D_ξ.

Our main results are in Sections 4.4, 4.5 and 4.6, where we will focus on the case of spacelike immersions factoring through a lightlike integral hypersurface of D_ξ or D_η. First, in Section 4.4 we deal with the case of arbitrary codimension. Our maim aim here is to find several conditions which assure that the immersion factors through a slice of the generalized Schwarzschild spacetime. That means the above mentioned functions u and v are constants. Therefore, assuming that we have a spacelike immersion $\Psi : M \to B \times_\lambda F$ through an integral hypersurface of D_ξ, we want to highlight some results here:

- Assume $\eta\lambda \geq 0$ and M compact with $\mathbf{H} = 0$. Then M factors through a slice and the immersion of M in such slice is minimal, Corollary 4.23.
- Assume M compact with $\mathrm{Ric}^g(\nabla v, \nabla v) \leq 0$. The normal vector field $\eta^\perp$ is an umbilic direction if and only if M factors through a slice, Theorem 4.24.
- Assume $\xi\lambda \neq 0$. Then M factors through a slice if and only if $\nabla^\perp \xi = 0$, Theorem 4.30.

The assumption $\xi\lambda \neq 0$ in Theorem 4.30 hold for a wide family of generalized Schwarzschild spacetimes. Indeed, when $\lambda(t, r) = r$, this hypothesis is satisfied. This warping function includes all physically relevant spacetimes of the family, such as the Schwarzschild or Reissner-Nordström spacetimes. For Theorem 4.24, it is a key fact that if $\eta^\perp$ is an umbilic direction then ∇v is a conformal vector field (4.15).

In our notion of generalized Schwarzschild spacetime, the geometry of the Riemannian part F is arbitrary. Nevertheless, the case of spherical symmetry is the most relevant from the physical point of view. Theorem 4.25 and Proposition 4.32 show conditions to ensure that F is a topological sphere. In fact, if we assume M compact and $\eta^\perp$ an umbilic direction, Theorem 4.25 states a condition on the Ricci tensor which shows that, when ∇v is a nonzero vector field, M is isometric to a sphere $\mathbb{S}^n(c)$ of constant sectional curvature c. This result is a consequence of [27, Theor. 1]. Therefore, in case that the codimension of M is two and F is simply-connected, by means of Proposition 4.32, the manifold F must be a topological sphere.

Section 4.5 is devoted to study codimension two ($m = n$) immersions through these lightlike integral hypersurfaces. In the terminology of black holes, a such immersion M is called a cross-section when every lightlike geodesic generator intersects M at most one, [33]. At topological level, every codimension two immersion through a lightlike integral hypersurface is a covering space of the fiber F but not necessarily a Riemannian covering, Proposition 4.32 and

Remark 4.34. The mean curvature vector field of these immersions is obtained in Proposition 4.37 and Corollary 4.38 as follows

$$\mathbf{H} = \left[\frac{\eta\lambda}{\lambda} \circ \Psi_B - \left(\frac{\xi\lambda}{2\lambda} \circ \Psi_B\right) \|\nabla v\|^2 + \frac{1}{n}\Delta v\right] \xi + \left(\frac{\xi\lambda}{\lambda} \circ \Psi_B\right) \ell^\xi,$$

where ℓ^ξ is the normal lightlike vector field to M with $\widetilde{g}(\xi, \ell^\xi) = -1$. Furthermore, we compute that

$$\|\mathbf{H}\|^2 = \frac{1}{v^2} \left((f \circ \Psi_B)^2 - \frac{S^{\Psi_F^*(g_F)} - v^2 S^g}{n(n-1)} \right),$$

where S^g and $S^{\Psi_F^*(g_F)}$ are the scalar curvatures of the induced metric g and $\Psi_F^*(g_F)$ on M, respectively. Since there is no possibility of confusion, we denote the norms of g and $\widetilde{g}$ the same as $\| \cdot \|$.

These results extend previous ones in [3], [4], [52] and [56], see details in Remark 4.39. The second formula shows a relation between the intrinsic and extrinsic geometry of the codimension two immersions through lightlike integral hypersurfaces. A such kind of relation has been previously pointed out for the case of the lightlike cone in the Minkowski spacetime in [52] and [56]. Section 4.5 also contains a characterization of marginally trapped immersions when the warping function λ agrees with the radial coordinate, Corollary 4.47. This result extends [4, Cor. 6.3] where the case of the lightlike cone in the Minkowski spacetime was studied.

We finish this Chapter with Section 4.6. Here we proceed with the study of immersions with parallel mean curvature vector field. That is, we consider the condition $\nabla^\perp \mathbf{H} = 0$. Under a technical condition, Theorem 4.52 provides an intrinsic characterization of the slices as the unique codimension two immersions through lightlike integral hypersurfaces with parallel mean curvature vector field.

In Chapter 5 we will relate tractor conformal bundles and tractor connections with codimension two spacelike immersions in a Lorentzian manifold. Although we can find the definitions of these objects in Section 3.1 (Definitions 3.7 and 3.8), we will include the formal definitions here to facilitate reading.

([19]) A (Riemannian) tractor conformal bundle on a manifold M with $\dim M = n \geq 2$ is a rank $n+2$ real vector bundle $\mathcal{T} \to M$ endowed with a bundle metric $\mathbf{h}$ of Lorentzian signature and with a distinguished oriented lightlike line subbundle $\mathcal{T}^1 \subset \mathcal{T}$.

([14], [19]) A tractor connection $\nabla^{\mathcal{T}}$ on a tractor conformal bundle $\mathcal{T} \to M$ is a linear connection such that $\nabla^{\mathcal{T}}\mathbf{h} = 0$ and the following map β is an isomorphism of vector bundles on M

$$\begin{tikzcd} TM \arrow[rr, "\beta"] \arrow[dr] & & \mathrm{Hom}\big(\mathcal{T}^1, (\mathcal{T}^1)^{\perp}/\mathcal{T}^1\big) \arrow[dl] \\ & M & \end{tikzcd}$$

given by

$$\beta(V_x)\big(\xi\big) = \nabla^{\mathcal{T}}_{V_x}\sigma + \mathcal{T}^1_x, \tag{1.3}$$

where $x \in M$, $V_x \in T_xM$, $\xi \in \mathcal{T}^1_x$ and $\sigma \in \Gamma(\mathcal{T}^1)$ is any section with $\sigma(x) = \xi$.

In Section 3.1 we can see how $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ induces a conformal class of Riemannian metrics c on M. If we start with a Riemannian conformal structure (M, c) and the induced conformal structure on M by means of $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ agrees with c, we say that $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ is a standard tractor conformal bundle for the fixed Riemannian conformal structure.

From the point of view of Lorentzian geometry, there is a setting where several of the above mentioned objects arise in a natural way. Namely, from every spacelike inmersion $\Psi\colon M^n \to (\widetilde{M}^{n+2}, \widetilde{g})$ and each lightlike normal vector field ξ, we can construct a tractor conformal bundle as follows. The vector bundle $\mathcal{T}$ on M is the pullback via Ψ of the tangent bundle of the manifold $\widetilde{M}$ with bundle metric $\widetilde{g}$ and distinguished lightlike line subbundle $\mathcal{T}^1 = \mathrm{Span}\{\xi\}$. The natural choice for a tractor connection is the induced connection $\widetilde{\nabla}$.

Of course, the induced connection $\widetilde{\nabla}$ is always a metric connection but the map β defines an isomorphism of vector bundles if and only if the Weingarten endomorphism A_ξ corresponding to the normal vector field ξ is non-singular at every point $x \in M$, Proposition 5.1. Then, the following natural question is about when $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is a standard tractor conformal bundle for the conformal class of the induced metric on M.

> That is, when the induced metric from Ψ belongs to the equivalence class of the conformal structure deduced from $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$?

This happens if and only if there is a nonvanishing smooth function $\mu \in C^\infty(M)$ such that $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$, Proposition 5.1. There are two mutually disjoint possibilities in order to the condition $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$ holds. Namely, $A_\xi = \mu \cdot \mathrm{Id}$ or M is endowed with an almost product structure P (i.e., $P \in \mathcal{T}_{(1,1)}M$ with $P^2 = \mathrm{Id}$ and $P \neq \pm\mathrm{Id}$) compatible with the induced metric and therefore with its conformal class, see Section 5.1.

The following natural question is addressed in Theorem 5.4 and Corollary 5.7, where we characterize

> when the standard tractor conformal bundle corresponding to a spacelike immersion $\Psi \colon M^n \to (\widetilde{M}^{n+2}, \widetilde{g})$ and a lightlike normal vector field ξ as above is normal.

The normality condition on $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is stated in terms of relationships between the extrinsic and intrinsic geometry of the spacelike immersion. Theorem 5.4 deals with the general case $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$ and Corollary 5.7 with the umbilical one. Although, the normality condition for a tractor connection was stated for $\mathrm{dim} M = n \geq 3$, the curvature properties in Theorem 3.22 have sense for $n \geq 2$. The main results of this Chapter can be summarized as follows.

> Let $\Psi \colon M^n \to (\widetilde{M}^{n+2}, \widetilde{g})$ be a spacelike immersion in a Lorentzian manifold with induced metric g, and let $\xi \in \mathfrak{X}^\perp(M)$ be a lightlike vector field. Let us consider $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ as above. Then,
>
> 1. The induced connection $\widetilde{\nabla}$ is a tractor connection if and only if the Weingarten endomorphism A_ξ is not singular at every point.
> 2. $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is standard for the induced metric g if and only if there is a smooth nonvanishing function $\mu \in C^\infty(M)$ such that $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$.
> 3. Assume $A_\xi = \mu \cdot \mathrm{Id}$ and there is a lightlike vector field $\ell \in \mathfrak{X}^\perp(M)$ such that $\widetilde{g}(\xi, \ell) = -1$. Then, $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is normal if and only if the following conditions hold:
>   (a) $\nabla^\perp \xi = \frac{1}{\mu} d\mu \otimes \xi$, where $\nabla^\perp$ denotes the normal connection.
>   (b) For every $V, W \in \mathfrak{X}(M)$, the Ricci tensor of g satisfies
> $$\mathrm{Ric}^g(V, W) = \frac{n}{2} \|\mathbf{H}\|^2 \, g(V, W) - (n-2) \widetilde{g}(\mathbf{H}, \xi) g(V, A_\ell W),$$
>   where $\mathbf{H}$ is the mean curvature vector field of $\Psi \colon M \to (\widetilde{M}, \widetilde{g})$.
>
>   The general case $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$ is analyzed in Theorem 5.4.

As a direct consequence when $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is normal, the scalar curvature S^g of the induced metric g satisfies $S^g = n(n-1)\|\mathbf{H}\|^2$, moreover (M, g) is Einstein if and only if $\Psi : M \to (\widetilde{M}, \widetilde{g})$ is totally umbilical.

Chapter 5 ends with an application to genereralized Schwarzschild spacetimes (Section 5.2). We assume that $(\widetilde{M}, \widetilde{g})$ now belongs to the said family of spacetimes, then Proposition 5.11 states the following result.

> Let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. Then, $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is a standard tractor conformal bundle if and only if $\xi\lambda \neq 0$.

Furthermore, we particularize Corollary 5.7 to give the hypotheses under which $\widetilde{\nabla}$ is normal, Theorem 5.12. Finally, we assume $\lambda(t, r) = r$ and can give the following characterization, Corollary 5.15.

> Let $\Psi : M \to B \times_r F$ be a totally umbilical codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. Then, $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is normal if and only if $\mathrm{Ric}^g = (n-1)\|\mathbf{H}\|^2\, g$. Furthermore, for dimension $n \geq 3$, the immersion Ψ has parallel mean curvature vector field.

We finish the thesis with Chapter 6. The planning of this Chapter is as follows. Starting from a Riemannian conformal structure (M, c), by setting a metric $g \in c$ and an admissible 1-parameter family $\gamma \colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$, see Definition 6.6, we construct a $(n+2)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$, see Proposition 6.8, such that

1. there is a distinguished lightlike hypersurface $\mathcal{Q} \subset \widetilde{M}$ and

2. every metric in the conformal class $e^{2u}g \in c$ is the induced metric of an immersion from M to $\widetilde{M}$ through $\mathcal{Q}$. Such immersions are defined in (6.12) and are denoted by Ψ^u.

This construction is inspired by the Fefferman and Graham ambient metric for conformal structures in the 1980s, [30] (see also [31]). Roughly speaking, the Fefferman and Graham construction proceeds as follows. Starting from a Riemannian conformal structure (M, c), the space of scales $\mathcal{Q}$ consists of the rays of metrics $y := t^2 g_x$ on T_xM where $x \in M$, $t \in \mathbb{R}_{>0}$ and $g \in c$. The ambient metric $\widetilde{g}$ is defined so that $(\widetilde{M}, \widetilde{g})$ is a Lorentzian manifold that admits $\mathcal{Q}$ as an embedded lightlike hypersurface. The original Fefferman-Graham metric requires certain normalisation condition (see Remark 6.4).

In this thesis, we will adopt the weaker notion of pre-ambient space given in [19], Definition 6.3. The aforementioned manifold $(\widetilde{M}, \widetilde{g})$ provided in Proposition 6.8 is a pre-ambient space where $\widetilde{g}$ is a pre-ambient metric as defined in (6.2). Note that $\widetilde{g}$ is not a warped product metric in general, see Remark 6.9.

Now, every spacelike immersion Ψ^u has codimension two in $(\widetilde{M}, \widetilde{g})$ and its normal bundle is spanned by the lightlike vector fields vector fields ξ^u and ℓ^u given in (6.14). The main aim of this Chapter is to show Theorem 6.20 which states that:

> Assume the admissible 1-parameter family γ satisfies $\mathrm{trace}(\gamma(0)) = \frac{\mathrm{S}^g}{n-1}$, where S^g is the scalar curvature of the fixed metric g. Then, the assignment
>
> $$D : c \to \mathcal{T}_{(0,2)}M, \quad e^{2u}g \mapsto e^{2u}g\left(A_{\ell^u}(-), -\right),$$
>
> defines a Möbius structure for the Riemannian conformal structure (M, c), where A_{ℓ^u} denotes the Weingarten endomorphism of ℓ^u. Moreover, every Möbius structure for a Riemannian conformal structure (M, c) arises in this way. Even more, if we remove the trace hypothesis, what we are doing is recovering all the "Schouten type-tensors" from spacelike immersions, see Remark 6.21.

The content of this Chapter is distributed as follows. In Section 6.1, we recall the notion of Möbius structure on Riemannian conformal structures (M, c) given in Definition 3.24. Then, we show some properties from the Lorentzian geometry perspective of the notion of pre-ambient space. Section 6.2 provides an explicit method to construct examples of pre-ambient spaces and includes several curvature properties of these pre-ambient spaces. In particular, we give conditions which permit to assure that the Ricci tensor of these pre-ambient spaces vanishes along $\mathcal{Q}$, Corollary 6.14.

The main results are in Section 6.3, where, as mentioned, it is essentially shown that Möbius structures agree with certain Weingarten endomorphisms of codimension two spacelike immersions in these pre-ambient spaces, Theorem 6.20. This result is remarkable for conformal structures in surfaces. In fact, there is no preferred Möbius structure on a two dimensional Riemannian conformal structure. Thus, Theorem 6.20 provides an explicit method to construct such structures. Section 6.3 also includes several properties on the family of spacelike immersions Ψ^u. In fact, Corollary 6.18 shows that the normal curvature tensor of such immersions always vanishes. Also, as a consequence of Remark 6.19, the mean curvature vector field of an isometric immersion Ψ^u with induced metric $e^{2u}g$ satisfies

$$||\mathbf{H}^u||^2 = \frac{\mathrm{S}^{e^{2u}g}}{n(n-1)},$$

see details in Remark 6.26. We want to highlight that this formula has already appeared several times earlier in the thesis, playing a fundamental role. Note that the causal character of $\mathbf{H}^u$ in the Lorentzian manifold $\widetilde{M}$ is determined by the sign of the scalar curvature of the metric $e^{2u}g$. Remark 6.26 also includes that $\nabla^{\perp}\mathbf{H}^u = 0$ if and only if $\mathrm{S}^{e^{2u}g}$ is constant (compare with [56, Cor. 3.10]). In particular, when M is compact, the positive answer to the Yamabe problem (see [41]) implies that there exists an immersion Ψ^u with parallel mean curvature vector field. Recall that the positive solution to the Yamabe problem states that on every $(n \geq 3)$-dimensional compact Riemannian conformal structure (M, c) there is a metric $g \in c$ with constant scalar curvature.

Section 6.4 focusses in the two dimensional case, we write down the Codazzi equation in terms of the Cotton-York tensor, Lemma 6.27. Then, Proposition 6.29 shows that tangent spaces of M along these immersions are invariant under the curvature tensor of $(\widetilde{M}, \widetilde{g})$ if and only if the Cotton-York tensor of c vanishes. In the terminology of [15], [60], this means that the Möbius structure D on (M, c) is flat.

Chapter 2

Preliminaries

In this preliminary Chapter we review the necessary definitions and background that will be used throughout the thesis.

2.1 Semi-Riemannian geometry

For semi-Riemannian geometry our basic reference is [51]. All geometric objects of interest will be considered smooth unless otherwise specified. The manifolds are assumed to be Hausdorff, satisfying the second axiom of countability and without boundary. For a manifold M, we denote by $\mathcal{C}^\infty(M)$ the algebra of smooth functions on M, by $\mathfrak{X}(M)$ the Lie algebra and $C^\infty(M)-$module of its tangent vector fields and by $\Omega^1(M,\mathbb{R})$ the $C^\infty(M)-$module of its real valued 1-forms. We write T_xM for the tangent vector space of M at $x \in M$ and TM for the total space of the vector tangent bundle. For a smooth function $f : M \to \widetilde{M}$ between two manifolds M and $\widetilde{M}$ we denote by $T_xf : T_xM \to T_{f(x)}\widetilde{M}$ the differential map of f at $x \in M$.

Let (M, g) be an n-dimensional semi-Riemannian manifold. That is, M is a manifold endowed with a metric tensor g of signature (p, q) where $p + q = n$. Here, p and q denote the number of $-$ and $+$ that appear in the matrix representation of the metric g with respect to any orthonormal basis, respectively. Unless stated otherwise, we assume $n \geq 2$ along this thesis. We write ∇ for the Levi-Civita connection on the semi-Riemannian manifold (M, g). Recall that the Levi-Civita connection is the unique linear connection on TM that preserves the metric g and is torsion-free. Finally, we denote the quadratic form corresponding to g by $\|\cdot\|^2$.

We can distinguish the following types of vectors in a semi-Riemannian manifold with

$p \geq 1$. Let $V_x \in T_xM$ be a tangent vector at a point $x \in M$, then we say that V_x is

- spacelike if $\|V_x\|^2 > 0$ or $V_x = 0$,
- timelike if $\|V_x\|^2 < 0$ and
- lightlike if $\|V_x\|^2 = 0$ and $V_x \neq 0$.

If $\|V_x\|^2 \leq 0$ and $V_x \neq 0$ the tangent vector V_x is said to be causal. These definitions can be extended to the case of a tangent vector field $V \in \mathfrak{X}(M)$ considering that V is spacelike (resp. timelike, lightlike, causal) if $V_x := V(x)$ is a spacelike (resp. timelike, lightlike, causal) vector at every point $x \in M$.

In addition, let us also recall the definition of conformal vector field. A vector field $Z \in \mathfrak{X}(M)$ is said to be conformal if

$$\mathcal{L}_Z g = 2hg,$$

where $\mathcal{L}$ denotes the Lie derivative and h is a function defined on M. Equivalently, this condition can be written

$$g(\nabla_V Z, W) + g(V, \nabla_W Z) = hg(V, W)$$

for every $V, W \in \mathfrak{X}(M)$. A conformal vector field Z is called a Killing vector field when $h = 0$. On the other hand, if h is a non-zero constant function, the vector field Z is said to be homothetic.

Below we include the definitions of some differential operators and tensors associated to the metric g. The gradient vector field, ∇h, of a function $h \in \mathcal{C}^\infty(M)$ is the vector field metrically equivalent to the 1-form $dh \in \Omega^1(M, \mathbb{R})$, that is, it is defined by the relation

$$g(\nabla h, V) = V(h) = dh(V),$$

for every $V \in \mathfrak{X}(M)$. Notice that we use the same notation for both the Levi-Civita connection and the gradient operator. The divergence, $\mathrm{div}(V)$, of a vector field $V \in \mathfrak{X}(M)$ is the smooth function defined by

$$\mathrm{div}(V) = \mathrm{trace}(W \mapsto \nabla_W V).$$

The Hessian operator $\mathrm{Hess}(h)$ of a function $h \in \mathcal{C}^\infty(M)$ is defined as

$$\mathrm{Hess}(h) : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M), \quad (V, W) \mapsto g(\nabla_V \nabla h, W).$$

Lastly, we want to introduce the Laplace operator, Δh, of a function $h \in \mathcal{C}^\infty(M)$. This is the smooth function defined as

$$\Delta h = \operatorname{div}(\nabla h).$$

Our assumption on the sign of the Riemann curvature tensor is

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W,$$

for every $U, V, W \in \mathfrak{X}(M)$. Let us also recall the definitions of the Ricci tensor and the scalar curvature. In terms of a local orthonormal basis $(E_1, \ldots, E_n)$, we define the Ricci tensor of g as

$$\operatorname{Ric}^g(V,W) = \sum_{i=1}^{n} \epsilon_i g\left(R(E_i,V)W, E_i\right),$$

where $\epsilon_i = g(E_i, E_i)$, and the scalar curvature

$$S^g = \sum_{i=1}^{n} \epsilon_i \operatorname{Ric}(E_i, E_i).$$

When $p = 0$ and $q = n$, the pair (M, g) is said to be a Riemannian manifold. In this case, we will adopt the same notation shown along this Section.

When $p = 1$ and $q = n-1$, the pair (M, g) is said to be a Lorentzian manifold. For each $x \in M$, it is well-known that the subset of timelike vectors in T_xM (resp. causal, lightlike) has two connected parts and each one of these parts will be called timelike cone (resp. causal cone, lightlike cone). A Lorentzian manifold is said to be time-orientable if there exists a smooth choice of one of the timelike cones. Being time-orientable is equivalent to the existence of a globally defined timelike vector field on (M, g) (see [51, Lemma 5.32]). A connected time-orientable n-dimensional Lorentzian manifold is called a spacetime [51, Chap. 6]. Unless stated otherwise, throughout this work we will use widetilde notation when we are working in Lorentzian signature, that is, we write $\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{R}, \widetilde{\operatorname{Ric}} \ldots$ We also denote the quadratic form corresponding to $\widetilde{g}$ by $\|\cdot\|^2$.

To conclude this Section we would like to recall the notion of involutive distribution and the well-known Frobenius Theorem (see [66, Chap. 1]). Let d be an integer such that $1 \leq d \leq n$. A d-dimensional distribution D on an n-dimensional manifold M is a smooth choice of a d-dimensional subspace $D(x)$ of T_xM for every $x \in M$. A vector field $V \in \mathfrak{X}(M)$ is said to lie in the distribution D if $V_x \in D(x)$ for every $x \in M$. A distribution is called involutive if for any two vector fields $V, W \in \mathfrak{X}(M)$ that lie in the distribution D, the Lie bracket $[V, W]$ lies in the distribution D as well.

Definition 2.1. *A smooth immersion $\Psi : N \to M$ is said to be an integral manifold of a distribution D on M if*

$$T_p\Psi \cdot (T_pN) = D(\Psi(p)),$$

for every $p \in N$.

Now we can state the Frobenius Theorem ([66, Theor. 1.60]).

Theorem 2.2. *Let D be a d-dimensional involutive distribution on M and $x \in M$. Then there exists an integral manifold $\Psi : N \to M$ passing trough x, that is, it exists a point $p \in N$ such that $\Psi(p) = x$.*

Finally, we give the notion of maximal integral manifold.

Definition 2.3. *A maximal integral submanifold $\Psi : \mathcal{L} \to M$ of a distribution D on M is a connected integral manifold of D whose image in M is not a proper subset of any other connected integral manifold of D.*

As a consequence of the Frobenius Theorem we have the following result ([66, Theor. 1.64]).

Theorem 2.4. *Let D be a d-dimensional involutive distribution on M and $x \in M$. Then through x there passes a unique maximal connected integral manifold $\mathcal{L}$ of D, and every connected integral manifold of D through x is contained in $\mathcal{L}$.*

2.2 Immersions in Lorentzian manifolds

In this Section our basic references are [51] and [24]. A smooth immersion $\Psi : M \to (\widetilde{M}, \widetilde{g})$ of a connected n-dimensional manifold M in an $(m+2)$-dimensional Lorentzian manifold $\widetilde{M}$ is said to be spacelike if the induced metric $g := \Psi^*(\widetilde{g})$ is Riemannian. Here, $\Psi^*(\widetilde{g})$ denotes the pullback of $\widetilde{g}$ by the immersion Ψ. The spacelike immersions have been studied for a long time, both from the physical and mathematical points of view (see for instance [61] and references therein).

Let $\overline{\mathfrak{X}}(M)$ be the $C^\infty(M)-$module of vector fields along the spacelike immersion Ψ, that is, $V \in \overline{\mathfrak{X}}(M)$ when for every $x \in M$ we have $V_x \in T_{\Psi(x)}\widetilde{M}$. Every vector field $X \in \mathfrak{X}(\widetilde{M})$ provides, in a natural way, the vector field $X|_\Psi := X \circ \Psi \in \overline{\mathfrak{X}}(M)$. The set of vector fields $\mathfrak{X}(M)$ may be seen as a $C^\infty(M)-$submodule of $\overline{\mathfrak{X}}(M)$ by meaning of

$$\mathfrak{X}(M) \to \overline{\mathfrak{X}}(M), \quad V \mapsto T\Psi \cdot V,$$

where $(T\Psi \cdot V)(x) := T_x\Psi \cdot V_x$ for all $x \in M$.

As usual, for $V \in \overline{\mathfrak{X}}(M)$, we have the decomposition

$$V = T\Psi \cdot V^\top + V^\perp,$$

where $V_x^\top \in T_xM$ and $V_x^\perp \in (T_x\Psi \cdot T_xM)^\perp$ for all $x \in M$. We call $V^\top$ the tangent part of V and $V^\perp$ the normal part of V. The $C^\infty(M)-$submodule of $\overline{\mathfrak{X}}(M)$ of all normal vector fields along Ψ is denoted by $\mathfrak{X}^\perp(M)$, that is,

$$\mathfrak{X}^\perp(M) = \{V \in \overline{\mathfrak{X}}(M) : V^\top = 0\}.$$

In order to avoid ambiguities, we explicitly write the immersions and differential maps when necessary.

As we have stated in the previous Section, we write $\widetilde{\nabla}$ and ∇ for the Levi-Civita connections of $\widetilde{g}$ and g, respectively. As usual, we also denote by $\widetilde{\nabla}$ the induced connection on M. The decomposition of the induced connection $\widetilde{\nabla}$, into tangent and normal parts, leads to the Gauss and Weingarten formulas of Ψ as follows (see [51, Chap. 4])

$$\widetilde{\nabla}_V W = \nabla_V W + \mathrm{II}(V, W) \qquad \text{and} \qquad \widetilde{\nabla}_V \zeta = -A_\zeta V + \nabla^\perp_V \zeta, \tag{2.1}$$

for every tangent vector fields $V, W \in \mathfrak{X}(M)$ and $\zeta \in \mathfrak{X}^\perp(M)$. Here $\nabla^\perp$ denotes the normal connection on M, II the second fundamental form and A_ζ the Weingarten endomorphism (or shape operator) associated to ζ. Every Weingarten endomorphism A_ζ is self-adjoint and the second fundamental form is symmetric. They are also related by the following formula

$$g\left(A_\zeta V, W\right) = \widetilde{g}\left(\mathrm{II}(V, W), \zeta\right). \tag{2.2}$$

A normal vector field or normal direction ζ is said to be umbilic if $A_\zeta = \mu \cdot \mathrm{Id}$ for a smooth function $\mu \in \mathcal{C}^\infty(M)$. We say that Ψ is totally umbilical when every normal direction is umbilic. On the other hand, when the second fundamental form II is identically zero at every point on M, the immersion Ψ is said to be totally geodesic.

The mean curvature vector field is defined by $\mathbf{H} = \frac{1}{n}\,\mathrm{trace}_g\mathrm{II}$ where trace_g denotes the trace with respect to the metric g. From (2.2) we have

$$\mathrm{trace}(A_\zeta) = n\widetilde{g}(\mathbf{H}, \zeta).$$

We say that the immersion Ψ has parallel mean curvature vector field when $\nabla^\perp_V \mathbf{H} = 0$ for every $V \in \mathfrak{X}(M)$.

Taking into account that our convention on the sign of the Riemann curvature tensor is the opposite to [51], the Gauss equation is given by (e.g., [51, Chap. 4])

$$g(R(U,V)W,X) = \widetilde{g}(\widetilde{R}(U,V)W,X) - \widetilde{g}(\mathrm{II}(U,W),\mathrm{II}(V,X)) + \widetilde{g}(\mathrm{II}(U,X),\mathrm{II}(V,W)), \quad (2.3)$$

for any $U,V,W,X \in \mathfrak{X}(M)$. Also, we let

$$(\nabla_U \mathrm{II})(V,W) := \nabla^{\perp}_U(\mathrm{II}(V,W)) - \mathrm{II}(\nabla_U V, W) - \mathrm{II}(V, \nabla_U W). \quad (2.4)$$

The Codazzi equation reads as follows (see for instance [51, Prop. 4.33])

$$(\nabla_U \mathrm{II})(V,W) - (\nabla_V \mathrm{II})(U,W) = \left(\widetilde{R}(U,V)W\right)^{\perp}. \quad (2.5)$$

The normal curvature tensor $R^{\perp}$ is given by

$$R^{\perp}(V,W)\zeta = \nabla^{\perp}_V \nabla^{\perp}_W \zeta - \nabla^{\perp}_W \nabla^{\perp}_V \zeta - \nabla^{\perp}_{[V,W]}\zeta.$$

A particular case occurs when, working with a codimension two spacelike immersion Ψ, that is $m = n$, we are able to find a global lightlike normal frame $\{\xi, \ell\}$ along Ψ. That is, ξ and ℓ are two globally defined normal vector fields along Ψ which are lightlike with the normalization condition $\widetilde{g}(\xi,\ell) = -1$. Let A_ξ and A_ℓ be the associated Weingarten endomorphisms. Then, for every $V, W \in \mathfrak{X}(M)$, the second fundamental form can be written as

$$\mathrm{II}(V,W) = -g(A_\ell V, W)\xi - g(A_\xi V, W)\ell. \quad (2.6)$$

Taking traces in this expression, we obtain for the mean curvature vector field that

$$\mathbf{H} = -\frac{1}{n}\Big(\mathrm{trace}\,(A_\ell)\xi + \mathrm{trace}\,(A_\xi)\ell\Big). \quad (2.7)$$

On the other hand, we would also like to recall the notions of lightlike manifold and lightlike hypersurface. A lightlike manifold is a pair $(N, \bar{h})$ where N is an $(m+1)$-dimensional manifold and is equipped with a lightlike metric $\bar{h}$. That is, $\bar{h}$ is a symmetric $(0,2)$-tensor field on N such that

1. $\bar{h}(\xi,\xi) \geq 0$ for all $\xi \in \mathfrak{X}(N)$ and
2. for every $y \in N$, the radical $\mathrm{Rad}(\bar{h})(y) = \{\xi_y \in T_yN : \bar{h}(\xi_y, -) = 0\}$ defines a 1-dimensional distribution on N.

A smooth immersion $\Psi\colon N \to (\widetilde{M}, \widetilde{g})$ in an arbitrary $(m+2)$-dimensional Lorentzian manifold is said to be a lightlike hypersurface when the induced tensor $\Psi^*(\widetilde{g})$ is a lightlike metric. The terminology lightlike manifolds stems from General Relativity where lightlike hypersurfaces are models of various types of horizons. Roughly speaking, the horizon of a set A marks the limit of the region of such spacetime controlled by a set A [51, Chap. 14]. Also in General Relativity, the existence of smooth closed achronal totally geodesic lightlike hypersurfaces (the Null Splitting Theorem) [32, Theor. IV.1] has important consequences in order to obtain rigidity results (see for instance [32, Theor. IV.3]). Being achronal means that the timelike curves of the ambient cut at most once to the lightlike hypersurface (see [51, p. 413]). For the notion of totally geodesic lightlike hypersurface see Remark 4.5.

A classical example of a lightlike hypersurface is the lightlike cone with vertex at the origin of the Minkowski spacetime $\mathbb{L}^{m+2}$, defined as

$$\Lambda := \{v \in \mathbb{L}^{m+2} : \langle v, v\rangle = 0, v \neq 0\}.$$

Recall that the Minkowski spacetime $\mathbb{L}^{m+2}$ is the Lorentzian manifold $\left(\mathbb{R}^{m+2}, \langle\cdot,\cdot\rangle\right)$, where $\langle\cdot,\cdot\rangle = -dx_1^2 + dx_2^2 + \ldots + dx_{m+2}^2$ with respect to the canonical coordinate system. At times, we treat $\mathbb{L}^{m+2}$ purely as a Lorentzian vector space. In such instances, we label it as the Minkowski vector space and also denote its scalar product by $\langle\cdot,\cdot\rangle$. Its canonical basis is represented by $(e_1, \ldots, e_{m+2})$, where $e_i = (0, \ldots, 0, \underset{\hat{i}}{1}, 0, \ldots, 0)$.

2.3 Warped product manifolds

Although the definition of warped product is more general, we particularize on the family we are interested in. Let (B, g_B) be a two dimensional oriented Lorentzian manifold and (F, g_F) be an m-dimensional connected Riemannian manifold. Fix $\lambda \in C^\infty(B)$ with $\lambda > 0$, we define the Lorentzian warped product manifold given by the product manifold $\widetilde{M} = B \times F$ endowed with the Lorentzian metric

$$\widetilde{g} = \pi_B^*(g_B) + (\lambda \circ \pi_B)^2 \pi_F^*(g_F),$$

where π_B and π_F are the natural projections on B and F, respectively [51, Chap. 7]. As usual, we denote the Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ as $B \times_\lambda F$ and λ is called the warping function. The sets of vector fields $\mathfrak{X}(B)$ and $\mathfrak{X}(F)$ can be lifted to $\mathfrak{X}(\widetilde{M})$ in a natural way. We denote the sets of all lifts as $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$, respectively. We use the same notation for a vector field and its lift and then, every vector field $E \in \mathfrak{X}(\widetilde{M})$ has a unique expression as $E = X + V$ where $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$.

Let us recall that the Levi-Civita connection of $\widetilde{g}$ is given in [51, Prop. 7.35] as follows. For $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, we have

$$\widetilde{\nabla}_X Y = \nabla^B_X Y, \quad \widetilde{\nabla}_X V = \widetilde{\nabla}_V X = \frac{X\lambda}{\lambda} V, \quad \widetilde{\nabla}_V W = -\frac{\widetilde{g}(V,W)}{\lambda} \nabla^B \lambda + \nabla^F_V W, \tag{2.8}$$

where ∇^B and ∇^F are the Levi-Civita connections of B and F, respectively. For every $h \in \mathcal{C}^\infty(B)$, we write $\nabla^B h$ for the gradient of h with respect to the metric g_B. Besides we have $\widetilde{\nabla}(h \circ \pi_B) = \nabla^B h \circ \pi_B$, [51, Lemma. 7.34]. As was mentioned in the introduction, there are several very relevant examples in this family.

Example 2.5. The $(m+2)$-dimensional Minkowski spacetime $\mathbb{L}^{m+2}$ can be described in two ways as a warped product of this type.

- For the first one we take $(B, g_B) = \mathbb{L}^2$ with canonical coordinates (t, r), $\lambda(t, r) = 1$ and $(F, g_F) = \mathbb{E}^m$, where $\mathbb{E}^m$ denotes the m-dimensional Euclidean space. Hence, we have $\mathbb{L}^{m+2} = \mathbb{L}^2 \times \mathbb{E}^m$.

- The second one is obtained by taking $(B, g_B) = \mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{L}^2$ with the same coordinate system as above, $\lambda(t, r) = r$ and $(F, g_F) = \mathbb{S}^m \subset \mathbb{E}^{m+1}$. Thus, the smooth map

$$\phi : (\mathbb{R} \times \mathbb{R}_{>0}) \times_r \mathbb{S}^m \to \mathbb{L}^{m+2}, \quad (t, r, x) \mapsto (t, rx) \tag{2.9}$$

provides an isometry with the open subset $\{(t, p) \in \mathbb{L}^{m+2} : p \neq 0\}$, see Figure 2.1.

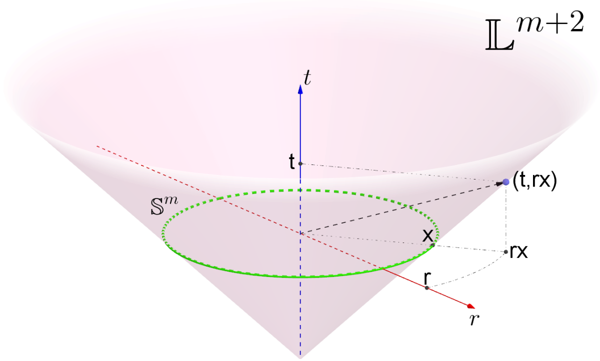

Figure 2.1: Coordinates on $\mathbb{L}^{m+2}$ induced by ϕ.

The family of warped products we are interested in is given in Definition 1.1. That is, the generalized Schwarzschild spacetimes. This family includes the warped products given in Example 2.5 and, as was mentioned, the exterior Schwarzschild spacetime.

Let I be an open interval in $\mathbb{R}$ and (F, g_F) be a connected Riemannian manifold, the spacetimes $(I \times F, -dt^2 + \lambda^2(t) g_F)$ are called generalized Friedmann-Lemaître-Robertson-Walker spacetimes. The study of spacelike hypersurfaces in such spacetimes goes back to the seminal work [7]. Since then, many other researchers have continued this approach. From our point of view, a next natural step could be to study spacelike immersions in warped products where $(I, -dt^2)$ is replaced by a two dimensional oriented Lorentzian manifold (B, g_B). To be precise, in Chapter 4 we will study spacelike immersions in the still quite general ambient of the generalized Schwarzschild spacetimes.

Let us recall that for every $q_0 \in B$, the spacelike immersion $F \hookrightarrow B \times_\lambda F$ given by $x \mapsto (q_0, x)$ is called the slice at level q_0. From [51, Prop. 7.35 (3)], we know that the normal part of $\widetilde{\nabla}_V W$ for $V, W \in \mathfrak{L}(F)$ is

$$\mathrm{II}(V, W) = -\frac{\widetilde{g}(V, W)}{\lambda} \nabla^B \lambda,$$

so the mean curvature vector field of the slice at level q_0 is

$$\mathbf{H} = -\frac{\nabla^B \lambda}{\lambda}(q_0). \tag{2.10}$$

In particular, the slices are totally umbilical spacelike embedded immersions.

2.4 Riemannian conformal geometry

For conformal geometry our basic references are [9] and [11, Chap. 1]. First, we make precise the notion of Riemannian conformal structure.

Definition 2.6. *Two Riemannian metrics g and g' on an n-dimensional manifold M are said to be conformally equivalent when $g' = e^{2u} g$ for a smooth function u on M. The set of all conformally equivalent metrics to a Riemannian metric g is called the conformal class $c = [g]$ of g. A Riemannian conformal structure (M, c) is the pair formed by an n-dimensional manifold M endowed with a conformal class c of Riemannian metrics.*

It is a well-known result that the Levi-Civita connections of two metrics in the same conformal class are related as follows (see [11, Theor. 1.159])

$$\nabla_V^{e^{2u} g} W = \nabla_V^g W + du(V) W + du(W) V - g(V, W) \nabla^g u,$$

for every $V, W \in \mathfrak{X}(M)$. Here $\nabla^{e^{2u}g}$ and ∇^g denote the Levi-Civita connections of $e^{2u}g$ and g, respectively. Thus, we have the following conformal transformation law

$$\mathrm{Ric}^{e^{2u}g} = \mathrm{Ric}^g - (n-2)\mathrm{Hess}(u) - (\Delta u)g - (n-2)\|\nabla u\|^2 g + (n-2)du \otimes du \tag{2.11}$$

where all operators on the right-side of the equality are taken with respect to g. As a consequence we have

$$e^{2u}S^{e^{2u}g} = S^g - 2(n-1)\Delta u - (n-2)(n-1)\|\nabla u\|^2. \tag{2.12}$$

For $(n \geq 3)$-dimensional Riemannian manifolds (M, g), we would like to introduce the Schouten tensor. It is defined by

$$P^g = \frac{1}{n-2}\Big(\mathrm{Ric}^g - \frac{\mathrm{S}^g}{2(n-1)}g\Big). \tag{2.13}$$

Let us note that the Schouten tensor is not defined for dimension $n = 2$ and, furthermore, it satisfies $\mathrm{trace}_g P^g = \frac{\mathrm{S}^g}{2(n-1)}$. Also, as consequence of (2.11) and (2.12), the Schouten tensor satisfies the following conformal transformation law

$$P^{e^{2u}g} = P^g - \frac{\|\nabla u\|^2}{2}g - \mathrm{Hess}(u) + du \otimes du. \tag{2.14}$$

The Schouten tensor plays a fundamental role in conformal geometry, this is clear from the following decomposition formula for the Riemann curvature tensor (see Section 1G in [11]).

Proposition 2.7. *The Riemann curvature tensor of a $(n \geq 3)$-dimensional Riemannian manifolds (M, g) admits a decomposition of the form*

$$R^g = W^g - P^g \owedge g \tag{2.15}$$

with a tensor W so that

$$W^{e^{2u}g} = e^{2u}W^g.$$

W is called the Weyl-tensor. Here $\owedge$ denotes the Kulkarni-Nomizu product of symmetric bilinear forms. That is,

$$\begin{aligned}(\tau_1 \owedge \tau_2)(U, V, W, X) := {} & \tau_1(U, W)\tau_2(V, X) - \tau_1(V, W)\tau_2(U, X) + \tau_1(V, X)\tau_2(U, W) \\ & - \tau_1(U, X)\tau_2(V, W),\end{aligned}$$

where τ_1 and τ_2 are symmetric bilinear forms.

A Riemannian manifold (M, g) is said to be (locally) conformally flat if for any point $x \in M$ there exists an open neighborhood $x \in U \subset M$ with local coordinate system $(x_1, \ldots, x_n)$ such that $g|_U = e^{2u}(dx_1^2 + \ldots + dx_n^2)$ for some smooth function $u \in \mathcal{C}^\infty(M)$. In dimension $(n \geq 4)$, the vanishing of the Weyl-tensor W characterizes local conformal flatness, see for instance [5]. Thus, Proposition 2.7 implies that for conformally flat metrics and dimension $n \geq 4$, the Riemann curvature tensor is governed by the Schouten tensor.

2.5 Principal fiber bundles and associated vector bundles

In this Section our basic references are [9] and [39]. Let $p : \mathcal{P} \to M$ be a principal fiber bundle with structure group G. We denote by ζ_X the fundamental vector field of the G-action on $\mathcal{P}$ defined by the element $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. That is, for every $u \in \mathcal{P}$, the vector field ζ_X is given by

$$\zeta_X(u) := \frac{d}{dt}\bigg|_{t=0} \Big(u \cdot \exp(tX)\Big).$$

Let $u \in \mathcal{P}_x := p^{-1}(x) \subset \mathcal{P}$ be a point in the fiber of $\mathcal{P}$ over $x \in M$. The subspace

$$V_u\mathcal{P} := T_u\mathcal{P}_x = \{\zeta_X(u) : X \in \mathfrak{g}\} \subset T_u\mathcal{P}$$

is called the vertical tangent space in $u \in \mathcal{P}$. A linear subspace $\mathcal{H}_u \subset T_u\mathcal{P}$ is said to be horizontal if $T_u\mathcal{P} = \mathcal{H}_u \oplus V_u\mathcal{P}$. A general connection on $p : \mathcal{P} \to M$ is a horizontal distribution $H \subset T\mathcal{P}$, that is, for every $u \in \mathcal{P}$ we have that $\mathcal{H}_u$ is horizontal. General connections can be defined in more general fiber bundles, not necessarily principal ones.

A principal connection on $\mathcal{P}$ is a general connection whose horizontal distribution is invariant with respect to the principal action of the structure group G, i.e.,

$$\mathcal{H}_{u \cdot g} = T_u r^g \cdot \mathcal{H}_u$$

for every $u \in \mathcal{P}$ and $g \in G$, where r^g is the (principal) right multiplication by g. It is easy to prove that this definition of principal connection can be rewritten in another way. Let us consider a 1-form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$, which satisfies the conditions

(1) $(r^g)^*(\gamma) = \mathrm{Ad}(g^{-1}) \circ \gamma$ for all $g \in G$ and

(2) $\gamma(u)(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$ and $u \in \mathcal{P}$,

where Ad denotes the adjoint representation of G on $\mathfrak{g}$. The 1-form γ defines a right invariant horizontal distribution $\mathcal{H}^\gamma$ on $\mathcal{P}$ by

$$\mathcal{H}^\gamma : u \in \mathcal{P} \to \mathcal{H}^\gamma_u := \operatorname{Ker} \gamma(u) \subset T_u\mathcal{P}.$$

Reciprocally, any right invariant horizontal distribution $\mathcal{H} : u \in \mathcal{P} \to \mathcal{H}_u \subset T_u\mathcal{P}$ gives us a 1-form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$ as above by setting

$$\gamma(u)\,(Y + \zeta_X(u)) := X \in \mathfrak{g},$$

where $Y \in \mathcal{H}_u$. Therefore, we have a one-to-one correspondence between principal connections and 1-forms $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$ satisfying (1) and (2).

Now, let $\rho : G \to GL(V)$ be a representation of the Lie group G over a k-dimensional (real) vector space V. There is a standard way to associate a vector bundle E over M to the principal fiber bundle $\mathcal{P}$ by means of ρ. The total space E is defined to be the orbit space of the right action of G on $\mathcal{P} \times V$ given by

$$(u, v) \cdot g := (u \cdot g,\ \rho(g^{-1})v), \quad (u, v) \in \mathcal{P} \times V,\ g \in G.$$

We denote this orbit space by

$$E := \mathcal{P} \times_G V := (\mathcal{P} \times V)/G$$

and the elements of E by $[u, v] := \{(u \cdot g,\ \rho(g^{-1})v) : g \in G\}$. Thus, E is a vector bundle over M with fiber type V and projection $p_E([u, v]) := p(u)$. Any point u in the fiber $\mathcal{P}_x$ of $\mathcal{P}$ over $x \in M$ gives rise to a linear isomorphism

$$[u] : v \in V \to [u, v] \in E_x := p_E^{-1}(x) \tag{2.16}$$

between the fiber type V and the fiber E_x of E.

Definition 2.8. *Let $p : \mathcal{P} \to M$ be a principal fiber bundle with structure group G and $\rho : G \to GL(V)$ be a representation of the Lie group G over a vector space V. Then, the vector bundle $p_E : E \to M$ is called the associated vector bundle for the representation ρ.*

Conversely, starting from a rank k real vector bundle $\pi_E : E \to M$ with fiber type V. We define the following set

$$\mathcal{P}_x := \{u = (u_1, \ldots, u_k) : u \text{ is a basis of } E_x\},$$

for every $x \in M$. For simplicity, we assume a fixed basis in V. Then, we have the following isomorphism

$$\mathcal{P}_x \simeq \{L_u : V \to E_x \text{ linear isomorphisms}\}.$$

Let us note that we denote by L_u the linear isomorphism associated with the basis u. Thus, $\mathcal{P} := \bigcup_{x\in M} \mathcal{P}_x$ can be endowed, in a natural way, with a smooth manifold structure and a right action by $G = GL(V)$ as follows

$$u \cdot g := L_u \circ g,$$

where $u \in \mathcal{P}$ and $g \in GL(V)$. Note that this action comes from considering

$$V \xrightarrow{g} V \xrightarrow{L_u} E_x.$$

Thus, the natural projection $p : \mathcal{P} \to M$ is a principal fiber bundle with structure group $GL(V)$ and the original vector bundle $\pi_E : E \to M$ can be recovered as the associated vector bundle

$$\mathcal{P} \times_{GL(V)} V \to E, \quad [u, v] \mapsto L_u(v), \tag{2.17}$$

for the (standard) representation $\rho = \mathrm{Id}_{GL(V)}$. Furthermore, the principal fiber bundle $p : \mathcal{P} \to M$ is called the frame bundle of E.

Remark 2.9. Assume $\pi_E : E \to M$ is a rank k real vector bundle endowed with a bundle metric h of Lorentzian signature $(1, k-1)$ on E. Without loss of generality, we consider the fiber type V to be the Minkowski vector space $\mathbb{L}^k$ with the canonical basis $(e_1, \ldots, e_k)$ fixed (see end of the Section 2.2). For each $x \in M$, we define the set

$$\begin{aligned} \mathcal{O}(E_x) : &= \{u = (u_1, \ldots, u_k) : u \text{ is an } h_x\text{-orthonormal basis of } E_x \text{ with } u_1 \text{ timelike}\} \\ &\simeq \left\{L_u : \mathbb{L}^k \to (E_x, h_x) \text{ orthogonal maps}\right\}. \end{aligned}$$

Now, we have that $\mathcal{O}(E) := \bigcup_{x\in M} \mathcal{O}(E_x) \xrightarrow{p} M$ is a principal fiber subbundle of the frame bundle $\mathcal{P}$ of E with structure group $O(1, k-1)$, where p is the natural projection and $O(1, k-1)$ acts on $\mathcal{O}(E)$ in a natural way. Here the action comes from considering

$$\mathbb{L}^k \xrightarrow{g} \mathbb{L}^k \xrightarrow{L_u} (E_x, h_x),$$

where $u \in \mathcal{O}(E_x)$ and $g \in O(1, k-1)$. The vector bundle $\pi_E : E \to M$ can be recovered as the associated vector bundle $\mathcal{O}(E) \times_{O(1,k-1)} \mathbb{L}^k$ for the standard representation. The principal bundle $\mathcal{O}(E)$ is called the orthonormal frame bundle. It should be noted that the orthonormal frame bundle for a bundle metric of signature (p, q) is defined in the same way.

Let us return to the general case $E = \mathcal{P} \times_G V$. The set of smooth sections of the vector bundle E is denoted by $\Gamma(E)$. Any section $\sigma \in \Gamma(E)$ can locally be represented as $\sigma|_U = [s, v]$, where U is an open subset in M, $s : U \to \mathcal{P}$ is a smooth local section of the principal fiber bundle $\mathcal{P}$ and $v \in \mathcal{C}^\infty(U, V)$ is a smooth function on U with values in the vector space V. Any principal connection $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$ induces a linear connection $\nabla^\gamma : \Gamma(E) \to \Gamma(T^*M \otimes E)$ given by

$$(\nabla^\gamma_W \sigma)|_U := [s, W(v) + \rho' (\gamma(Ts \cdot W)) (v)] , \tag{2.18}$$

where $W \in \mathfrak{X}(M)$ and ρ' denotes the differential map at the identity element of the representation ρ, that is, $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$. It is not difficult to show that the linear connection ∇^γ is well-defined. This linear connection is called the induced connection by γ.

Let us focus on the following particular case. Let $(\pi_E : E \to M, h)$ be a rank k real vector bundle endowed with a bundle metric h of Lorentzian signature $(1, k-1)$. Now, we consider it as the associated vector bundle to its orthonormal frame bundle $\mathcal{O}(E)$, that is, $E \simeq \mathcal{O}(E) \times_{O(1,k-1)} \mathbb{L}^k$ (see Remark 2.9). Then, the induced connection ∇^γ by a principal connection $\gamma \in \Omega^1(\mathcal{O}(E), \mathfrak{o}(1, k-1))$ given in formula (2.18) can be reinterpreted in this context as follows. We write

$$\gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1k} \\ \vdots & \ddots & \vdots \\ \gamma_{k1} & \cdots & \gamma_{kk} \end{pmatrix} \qquad \text{with } \gamma_{ij} \in \Omega^1\big(\mathcal{O}(E), \mathbb{R}\big).$$

Since γ takes values in $\mathfrak{o}(1, k-1)$, it follows that $\gamma_{ii} = 0$ for $i \in \{1, \ldots, k\}$. Let $\sigma \in \Gamma(\mathcal{O}(E) \times_{O(1,k-1)} \mathbb{L}^k)$ be a section that is locally given by $\sigma|_U = [s, v]$, where $s = (s_1, \ldots, s_k)$ and $s_i \in \Gamma(E|_U)$ for $i \in \{1, \ldots, k\}$. Note that we can identify s_i with the local section $[s, E_i] \in \Gamma\big((\mathcal{O}(E) \times_{O(1,k-1)} \mathbb{L}^k)|_U\big)$ by means of (2.17), where $E_i(x) = e_i$ for every $x \in U$. Recall that e_i denotes the i-th element of the canonical basis of $\mathbb{L}^k$. Then, taking in mind (2.17) and formula (2.18), it has sense to compute

$$\nabla^\gamma_w s_i = \sum_{j=1}^k \gamma_{ji}(s(x))(T_x s \cdot w) s_j(x) \in E_x, \tag{2.19}$$

for every $x \in U$ and $w \in T_x M$. Furthermore, since γ takes values in $\mathfrak{o}(1, k-1)$, it follows that the linear connection ∇^γ given in (2.19) is metric for the bundle metric h, that is,

$$W\big(h(\sigma_1, \sigma_2)\big) = h\left(\nabla^\gamma_W \sigma_1, \sigma_2\right) + h\left(\sigma_1, \nabla^\gamma_W \sigma_2\right),$$

for all sections $\sigma_1, \sigma_2 \in \Gamma(E)$ and vector fields $W \in \mathfrak{X}(M)$. This equation is equivalent to $\nabla^\gamma h = 0$.

Remark 2.10. Let us outline how a principal connection $\gamma \in \Omega^1(\mathcal{O}(E), \mathfrak{o}(1, k-1))$ is constructed from a metric linear connection. Let ∇ be a metric linear connection on a rank k real vector bundle $(\pi_E : E \to M, h)$, where h is a bundle metric of Lorentzian signature $(1, k-1)$. We consider a local section $s : U \to \mathcal{O}(E)$ with $s = (s_1, \ldots, s_k)$ and we define for each $x \in U$ and $w \in T_x M$,

$$\nabla_w s_i = \sum_{j=1}^{k} \omega^s_{ji}(x)(w) s_j(x).$$

Since ∇ is a metric linear connection, we have

$$\omega^s = \begin{pmatrix} \omega^s_{11} & \cdots & \omega^s_{1k} \\ \vdots & \ddots & \vdots \\ \omega^s_{k1} & \cdots & \omega^s_{kk} \end{pmatrix} \in \Omega^1\big(U, \mathfrak{o}(1, k-1)\big),$$

where $\omega^s_{ji}(x)(w) = \epsilon_j h_x\big(\nabla_w s_i, s_j(x)\big)$ being $\epsilon_j = h_x\big(s_j(x), s_j(x)\big)$. In particular, it follows that $\omega^s_{ii} = 0$ for $i \in \{1, \ldots, k\}$. We have the local trivialization of $p : \mathcal{O}(E) \to M$ associated to s given by

$$\begin{aligned} p^{-1}(U) &\xrightarrow{\psi_s} U \times O(1, k-1) \\ u &\longmapsto (p(u), g(u)), \end{aligned}$$

where $g(u)$ is determinated by the condition $s(p(u)) \cdot g(u) = u$. Now, we define

$$\gamma^s(u)(\xi) := \mathrm{Ad}(g(u)^{-1}) \circ \omega^s(p(u))(T_u p \cdot \xi) + \omega_{O(1,k-1)}(g(u))(T_u g \cdot \xi), \tag{2.20}$$

where $u \in p^{-1}(U)$, $\xi \in T_u\mathcal{O}(E)$ and $\omega_{O(1,k-1)}$ denotes de Maurer-Cartan form of $O(1, k-1)$. It is not difficult to show that γ^s does not depend on the section s taken at the beginning. Therefore, the formula (2.20) defines a principal connection $\gamma \in \Omega^1\big(\mathcal{O}(E), \mathfrak{o}(1, k-1)\big)$. Additionally, the connection ∇ is recovered by means of the induced linear connection ∇^γ given in (2.19).

From Remark 2.10, we can state the following result.

Theorem 2.11. $\big($*See* [20, *Sec. 1.3.5*]$\big)$ *Let $\pi_E : E \to M$ be a rank k real vector bundle endowed with a bundle metric h of Lorentzian signature $(1, k-1)$ on E. Then, there is a bijective correspondence between:*

$$\boxed{\textit{Metric linear connections } \nabla \textit{ on } (\pi_E : E \to M, h)}$$
$$\Updownarrow$$
$$\boxed{\textit{Principal connections } \gamma \in \Omega^1\big(\mathcal{O}(E), \mathfrak{o}(1, k-1)\big)}$$

The above Theorem works for any bundle metric on $\pi_E : E \to M$ of arbitrary signature (p, q). We have stated this Theorem in this way to meet the requirements of this thesis.

Chapter 3

Cartan geometries and tractor conformal bundles

Several problems faced in our work have the roots in Riemannian conformal geometry. In particular, the description of conformal geometry from the point of view of Cartan connections will be a milstone to understand some results here. Therefore, we will give a short introduction to the notion of Cartan geometry and several of its properties. Subsequently, we will introduce the Cartan geometries that interest us. Specifically, we are referring to the so-called Möbius geometries. Finally, we will explore the relation between Möbius geometries and tractor conformal bundles. In this Chapter the basic references are [9] and [20].

3.1 Cartan geometries, tractor conformal bundles and Riemannian conformal structures

Unless stated otherwise, we assume $n \geq 2$ for our purposes.

Definition 3.1. ([20]) *Let $H \subset G$ be a closed Lie subgroup in a Lie group G, and let $\mathfrak{g}$ be the Lie algebra of G. A Cartan geometry of type (G, H) on an n-dimensional manifold M is a principal fiber bundle $p : \mathcal{P} \to M$ with structure group H, which is endowed with a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the Cartan connection. We require that ω is H-equivariant, reproduces the generators of fundamental vector fields and defines an absolute parallelism. More formally, this means that*

1. $(r^h)^*(\omega) = \mathrm{Ad}(h^{-1}) \circ \omega$ *for all $h \in H$, where r^h is the right multiplication by h,*

2. *$\omega(\zeta_X) = X$ for all $X \in \mathfrak{h} \subset \mathfrak{g}$ and*

3. *$\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.*

Note that the third condition implies that the homogeneous space G/H has the same dimension as the base manifold M. Contrary to usual connections, Cartan connections do not allow one to distinguish a right invariant horizontal distribution on $\mathcal{P}$, see Section 2.5. Let us consider the trivial example of Cartan geometry of type (G, H):

Example 3.2. Let G be a Lie group and $\omega_G \in \Omega^1(G, \mathfrak{g})$ its Maurer-Cartan form:

$$\omega_G(g) : X \in T_gG \to T_g\lambda_{g^{-1}} \cdot X \in T_eG = \mathfrak{g},$$

where λ_g denotes the left multiplication by $g \in G$. If $H \subset G$ is a closed subgroup, then the projection $p : G \to G/H$ gives rise to a principal fiber bundle with structure group H over the homogeneous space $M = G/H$. The Maurer-Cartan form ω_G is a Cartan connection of type (G, H) on this homogeneous bundle. The manifold M is the homogeneous model for Cartan geometries of type (G, H). The homogeneous model for Cartan geometries of type (G, H) is also known as the Klein geometry of that type.

Definition 3.3. ([9]) *Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H). Let $\rho : G \to GL(V)$ be a representation of the group G on a k-dimensional vector space V. By restriction, this gives a representation of H on V and an associated vector bundle*

$$E := \mathcal{P} \times_H V.$$

The vector bundle $p_E : E \to M$ is called the tractor bundle for the representation ρ. Let us note that the representation is for the large group G.

Tractor bundles are fundamental in the theory of Cartan geometries since contrary to the case of arbitrary associated vector bundles, on a tractor bundle there is a linear connection associated to the Cartan connection ω. To see this, we extend the principal fiber bundle $\mathcal{P}$ with structure group H to a principal fiber bundle $\bar{\mathcal{P}}$ with structure group G as follows. Since $H \subset G$ is a closed Lie subgroup, we have the action

$$H \times G \to G, \quad (h, g) \mapsto hg.$$

The total space $\bar{\mathcal{P}}$ is defined to be the orbit space of the right action of H on $\mathcal{P} \times G$ given by

$$\big(P \times G\big) \times H \to P \times G, \quad \big((u, g), h\big) \mapsto (u \cdot h, h^{-1}g).$$

That is, the total space is $\bar{\mathcal{P}} := \mathcal{P} \times_H G := (\mathcal{P} \times G)/H$ and we have the canonical embedding $\iota : u \in \mathcal{P} \to [u, e] \in \bar{\mathcal{P}}$. Now, we can extend the Cartan connection ω on $\mathcal{P}$ to a principal connection $\bar{\omega}$ on $\bar{\mathcal{P}}$: First we use the Cartan connection ω on $\mathcal{P}$ and the Maurer-Cartan form ω_G of G (see Example 3.2) to define a 1-form $\hat{\omega} \in \Omega^1(\mathcal{P} \times G, \mathfrak{g})$ by

$$\hat{\omega}(u, g) := \mathrm{Ad}(g^{-1}) \circ (\pi_{\mathcal{P}}^*(\omega))(u, g) + (\pi_G^*(\omega_G))(u, g),$$

where $\pi_{\mathcal{P}}$ and π_G are the projections from $\mathcal{P} \times G$ onto $\mathcal{P}$ and G, respectively. The 1-form $\hat{\omega}$ is invariant under the H-action on $\mathcal{P} \times G$ and hence it projects to a 1-form $\bar{\omega} \in \Omega^1(\bar{\mathcal{P}}, \mathfrak{g})$. A direct calculation shows that $\bar{\omega}$ is indeed a principal connection on the principal fiber bundle $\bar{\mathcal{P}}$ with structure group G. Moreover, for the embedding ι, we have that $\iota^*(\bar{\omega}) = \omega$.

Since $\rho : G \to GL(V)$ is a representation of the group G, we have a natural vector bundle isomorphism

$$E = \mathcal{P} \times_H V \simeq \bar{\mathcal{P}} \times_G V.$$

Therefore, a Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ defines a linear connection ∇^ω on E via its extension to the principal connection $\bar{\omega}$ on $\bar{\mathcal{P}}$ defined by $\nabla^\omega := \nabla^{\bar{\omega}}$. Let σ be a smooth section of the vector bundle E, according to formula (2.18), ∇^ω is given by

$$(\nabla^\omega_W \sigma)|_U := [s,\, W(v) + \rho'\left(\bar{\omega}(Ts \cdot W)\right)(v)]\,, \tag{3.1}$$

where $W \in \mathfrak{X}(M)$ and $\sigma|_U = [s, v]$ for a local section $s : U \subset M \to \mathcal{P}$ and a smooth function $v \in \mathcal{C}^\infty(U, V)$.

Any G-invariant scalar product $\langle \cdot, \cdot \rangle_V$ on V induces a well-defined $\big($by means of (2.16)$\big)$ bundle metric $\mathbf{h}$ on the vector bundle E given by

$$\mathbf{h}([u, v_1], [u, v_2]) := \langle v_1, v_2 \rangle_V, \tag{3.2}$$

where $[u, v_1], [u, v_2] \in E$. By definition, ∇^ω is a metric linear connection with respect to any metric $\mathbf{h}$ as above.

Definition 3.4. *Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H). Let $p_E : E \to M$ be the tractor bundle for the representation $\rho : G \to GL(V)$. The linear connection ∇^ω given in* (3.1) *is called the tractor connection. Furthermore, ∇^ω is metric with respect to any bundle metric on E that comes from a G-invariant inner product defined on V in the sense of* (3.2).

From now on until the end of this Chapter, we will focus on the Cartan geometries we are interested in. We consider V to be the Minkowski vector space $\mathbb{L}^{n+2}$ with the canonical basis fixed. Also, we take $G = O^+(1, n+1)$ the orthochronous group. Letting

$$\Lambda^+ = \{v \in \mathbb{L}^{n+2} : \langle v, v \rangle = 0, v_1 > 0\}$$

for the future lightlike cone with vertex at the origin of $\mathbb{L}^{n+2}$. Then, the action of $O^+(1, n+1)$ descends to a transitive action on the space of rays in Λ^+ seen as a subset in the projective space $\mathbb{R}P^{n+1}$. Note that this space of rays is topologically the sphere $\mathbb{S}^n$. If we fix the lightlike line generated by a vector $\ell \in \Lambda^+$ and consider $H \subset G$ to be its stabiliser subgroup, then we get the homogeneous space description

$$\mathbb{S}^n = O^+(1, n+1)/H.$$

Here, $O^+(1, n+1)$ acts as the group of global conformal transformations of $\mathbb{S}^n$ with respect to the conformal class containing the canonical round metric. We denote this conformal class by c_0. In this setting, $(\mathbb{S}^n, c_0)$ is called the Möbius space.

Now, from Definition 3.4, we state the following particular case of tractor bundle, see [62] and [19].

Definition 3.5. (1) *A Möbius geometry on a manifold M is a Cartan geometry*

$$(p : \mathcal{P} \to M, \omega)$$

of type (G, H), where $G = O^+(1, n+1)$ and $H \subset G$ is the stabiliser subgroup of a fixed lightlike line generated by a vector $\ell \in \Lambda^+$ as above.

(2) *Let $\mathcal{T} := \mathcal{P} \times_H \mathbb{L}^{n+2} \to M$ be the tractor bundle for the restriction of the standard representation $\rho : G \to GL(n+2, \mathbb{R})$. The tractor bundle $\mathcal{T}$ is called the tractor conformal bundle of the Möbius geometry $(p : \mathcal{P} \to M, \omega)$. We write $\nabla^{\mathcal{T}}$ for the corresponding tractor connection ∇^ω on $\mathcal{T}$.*

The name tractor conformal bundle is early justified. By construction, this bundle carries a canonical bundle metric $\mathbf{h}$ of Lorentzian signature given by

$$\mathbf{h}([u, v_1], [u, v_2]) = \langle v_1, v_2 \rangle,$$

where $\langle\, , \rangle$ is the scalar product on $\mathbb{L}^{n+2}$ and $[u, v_1], [u, v_2] \in \mathcal{T}$. As a particular case of (3.1), the following formula holds for the tractor connection

$$\left(\nabla^{\mathcal{T}}_W \sigma\right)\big|_U = [s,\, W(v) + \bar{\omega}(Ts \cdot W)(v)]\,, \tag{3.3}$$

where $W \in \mathfrak{X}(M)$ and $\sigma|_U = [s, v]$ for a local section $s : U \subset M \to \mathcal{P}$ and $v \in \mathcal{C}^\infty(U, \mathbb{L}^{n+2})$.

The lightlike line generated by ℓ in $\mathbb{L}^{n+2}$ used to define H leads to a subbundle $\mathcal{T}^1 \subset \mathcal{T}$ with fiber type the lightlike line as follows

$$\mathcal{T}^1 := \{[u, a\ell] : u \in \mathcal{P}, a \in \mathbb{R}\} \subset \mathcal{T}.$$

We will write $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ to denote the tractor conformal bundle of the Möbius geometry $(p : \mathcal{P} \to M, \omega)$ endowed with the lightlike subbundle $\mathcal{T}^1$, the bundle metric $\mathbf{h}$ and the tractor connection $\nabla^{\mathcal{T}}$.

As was mentioned, let us see that every tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ naturally induces a conformal class of Riemannian metrics on the manifold M. In order to do this, we take $\sigma \in \Gamma(\mathcal{T}^1|_U)$ a nonvanishing local section. Then, we have $\mathbf{h}(\sigma, \sigma) = 0$ and then,

$$0 = W\big(\mathbf{h}(\sigma, \sigma)\big) = 2\mathbf{h}\big(\nabla^{\mathcal{T}}_W \sigma, \sigma\big), \tag{3.4}$$

for every $W \in \mathfrak{X}(M)$. As consequence of (3.4), we conclude that $\nabla^{\mathcal{T}}_W \sigma \in \Gamma\big((\mathcal{T}^1)^\perp\big)$ for every $W \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathcal{T}^1)$. Taking into account that $\mathcal{T}^1 \subset \big(\mathcal{T}^1\big)^\perp$, we can consider the rank n real vector bundle $\mathrm{Hom}\big(\mathcal{T}^1, (\mathcal{T}^1)^\perp/\mathcal{T}^1\big) \to M$. Thus, the fiber over any point $x \in M$ is given by

$$\big\{\lambda : \mathcal{T}^1_x \to (\mathcal{T}^1_x)^\perp/\mathcal{T}^1_x \text{ such that } \lambda \text{ is linear}\big\}.$$

Now, we have the well-defined $\mathcal{C}^\infty(M)$-bilinear bundle map

$$\begin{tikzcd} TM \arrow[rr, "\beta"] \arrow[dr] & & \mathrm{Hom}\big(\mathcal{T}^1, (\mathcal{T}^1)^\perp/\mathcal{T}^1\big) \arrow[dl] \\ & M & \end{tikzcd}$$

given by

$$\beta(W_x)\big([u, a\ell]\big) = \nabla^{\mathcal{T}}_{W_x} \sigma + \mathcal{T}^1_x,$$

where $x \in M$, $W_x \in T_xM$, $[u, a\ell] \in \mathcal{T}^1_x$ and $\sigma \in \Gamma(\mathcal{T}^1)$ is any section with $\sigma(x) = [u, a\ell]$. From formula (3.3), it can be deduced that β is injective and then, it is an isomorphism of vector bundles on M.

Every nonvanishing local section $\sigma \in \Gamma(\mathcal{T}^1|_U)$ provides the vector bundle isomorphism

$$\beta_\sigma \colon TU \to \Big((\mathcal{T}^1)^\perp/\mathcal{T}^1\Big)|_U, \quad W_x \mapsto \nabla^{\mathcal{T}}_{W_x} \sigma + \mathcal{T}^1_x. \tag{3.5}$$

Thus, every nonvanishing local section $\sigma \in \Gamma(\mathcal{T}^1|_U)$ produces a Riemannian metric $\mathbf{h}^\sigma$ on U by means of the formula

$$\mathbf{h}^\sigma(V, W) := \mathbf{h}(\beta_\sigma(V), \beta_\sigma(W)), \tag{3.6}$$

for $V, W \in \mathfrak{X}(U)$. Any other nonvanishing local section $\bar{\sigma} \in \Gamma(\mathcal{T}^1|_U)$ can be written as $\bar{\sigma} = f\sigma$ for some nonvanishing smooth function f on U. From (3.5), it follows that $\beta_{\bar{\sigma}} = \beta_{f\cdot\sigma} = f \cdot \beta_\sigma$. Therefore, different choices of the section σ induce conformally related metrics on U and then, a conformal class c on U. Hence, from a standard argument, we have that $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ induces a conformal class c on M.

Definition 3.6. *Let (M, c) be a given Riemannian conformal structure and $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ be the tractor conformal bundle of a Möbius geometry $(p : \mathcal{P} \to M, \omega)$. When the induced conformal class on M by means of $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ agrees with c, we say that $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ is a standard tractor conformal bundle for (M, c).*

This construction for the tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ starting from a Möbius geometry $(p : \mathcal{P} \to M, \omega)$ can be reversed. This fact motivated the authors of [18] to give an alternative and direct definition for a tractor conformal bundle. We can find it in [18] in a more general context and specialized for ours in [19]. It is important to note that this idea comes from Thomas's work in the 1920s, [63].

Definition 3.7. ([*19*]) *A (Riemannian) tractor conformal bundle on an n-dimensional manifold M is a rank $n+2$ real vector bundle $\mathcal{T} \to M$ endowed with a bundle metric $\mathbf{h}$ of Lorentzian signature and with a distinguished oriented lightlike line subbundle $\mathcal{T}^1 \subset \mathcal{T}$.*

Likewise, the definition of tractor connection can be revisited from this new point of view.

Definition 3.8. ([*19*]) *A tractor connection $\nabla^{\mathcal{T}}$ on a tractor conformal bundle $\mathcal{T} \to M$ is a linear connection such that $\nabla^{\mathcal{T}}\mathbf{h} = 0$ and the following map β is an isomorphism of vector bundles on M*

$$\begin{tikzcd} TM \arrow[rr, "\beta"] \arrow[dr] & & \mathrm{Hom}\big(\mathcal{T}^1, (\mathcal{T}^1)^\perp/\mathcal{T}^1\big) \arrow[dl] \\ & M & \end{tikzcd}$$

given by

$$\beta(W_x)(\xi) = \nabla^{\mathcal{T}}_{W_x}\sigma + \mathcal{T}^1_x, \tag{3.7}$$

where $x \in M$, $W_x \in T_xM$, $\xi \in \mathcal{T}^1_x$ and $\sigma \in \Gamma(\mathcal{T}^1)$ is any section with $\sigma(x) = \xi$.

Let us note that $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$, when viewed from this new perspective, also induces a conformal class c on M in a similar way.

As previously mentioned, by considering $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ from this new vantage point, it is possible to construct a Möbius geometry $(p : \mathcal{P} \to M, \omega)$ such that the tractor conformal bundle constructed from this Möbius geometry $(p : \mathcal{P} \to M, \omega)$ agrees with $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$. For the sake of completeness, we include this construction here.

Consider the Riemannian conformal structure c induced on M from the data $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$. Every choice of a metric $g \in c$ provides us with a decomposition of $\mathcal{T}$ as follows. The metric $g \in c$ is determined by an oriented section $\sigma \in \Gamma(\mathcal{T}^1)$ by the condition $g = \mathbf{h}^\sigma$, where $\mathbf{h}^\sigma$ is given by (3.6). Then, we obtain the decomposition

$$\mathcal{T} \overset{g}{\simeq} \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}},$$

where $\underline{\mathbb{R}}$ denotes the trivial bundle $M \times \mathbb{R} \to M$. The first trivial bundle $M \times \mathbb{R} \to M$ arises from the trivialization of $\mathcal{T}^1$ deduced from σ. The copy of TM is given by means of

$$F \colon TM \to \mathcal{T}, \quad W_x \mapsto \nabla^{\mathcal{T}}_{W_x}\sigma.$$

for every $x \in M$. The second trivial bundle $M \times \mathbb{R} \to M$ comes from the unique lightlike section $\delta \in \Gamma(\mathcal{T})$ such that $\mathbf{h}(\sigma, \delta) = 1$ and $\mathbf{h}(\delta, F(TM)) = 0$. Using this decomposition, any smooth section T of $\mathcal{T} \to M$ can be written as follows

$$T = \begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix}, \text{ with } \alpha, \beta \in \mathcal{C}^\infty(M) \text{ and } W \in \mathfrak{X}(M),$$

and the bundle metric $\mathbf{h}$, which will be denoted by $\mathbf{h}^g$ under this decomposition, is given by

$$\mathbf{h}^g\left(\begin{pmatrix} \alpha_1 \\ W_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ W_2 \\ \beta_2 \end{pmatrix}\right) := \alpha_1\beta_2 + \beta_1\alpha_2 + g(W_1, W_2).$$

This decomposition will be extensively used in this Chapter.

On the other hand, from Remark 2.9 and Theorem 2.11, there exists a principal fiber bundle $\bar{\mathcal{P}} \to M$ with structure group $G = O^+(1, n+1)$ such that $\mathcal{T} \simeq \bar{P} \times_G \mathbb{L}^{n+2}$ and, in addition, it can be endowed with the principal conecction $\gamma \in \Omega^1(\bar{\mathcal{P}}, \mathfrak{g})$ corresponding to the linear connection $\nabla^{\mathcal{T}}$. Also, let us note that $\bar{P} \to M$ is the bundle of $\mathbf{h}$-orthonormal frames which apply the usual time orientation of $\mathbb{L}^{n+2}$ to the time orientation on every $(\mathcal{T}_x, \mathbf{h}_x)$ induced from the oriented lightlike line subbundle $\mathcal{T}^1 \subset \mathcal{T}$. In this context, we denote $\mathcal{O}^+(\mathcal{T})$ rather than $\bar{\mathcal{P}}$. That is,

$$\mathcal{O}^+(\mathcal{T}_x) = \left\{ u \in \mathcal{O}(\mathcal{T}_x) \text{ such that } L_u \colon \mathbb{L}^{n+2} \to (\mathcal{T}_x, \mathbf{h}_x) \text{ preserves the time orientation } \right\}.$$

With the intention of being used later, we are going to describe the Lie algebra $\mathfrak{g} = \mathfrak{o}(1, n+1)$ of G using an appropriate basis. Let $\mathbb{L}^{n+2}$ be the Minkowski vector space and consider the Witt basis given by

$$\left(\ell_- := \frac{1}{\sqrt{2}}(e_{n+2} - e_1), e_2, \dots, e_{n+1}, l_+ := \frac{1}{\sqrt{2}}(e_{n+2} + e_1) \right), \tag{3.8}$$

where $(e_1, \dots, e_{n+2})$ is the canonical (orthonormal) basis of $\mathbb{L}^{n+2}$. Then, the Lie algebra $\mathfrak{g}$ using this Witt basis reads as follows

$$\mathfrak{g} = \left\{ \begin{pmatrix} -a & Z & 0 \\ Y & A & -Z^t \\ 0 & -Y^t & a \end{pmatrix} : a \in \mathbb{R},\ A \in \mathfrak{o}(n),\ Y \in \mathbb{R}^n,\ Z \in (\mathbb{R}^n)^* \right\},$$

see ([9, Sec. 2.2.1]). We are also interested in describing, in the Witt basis given in (3.8), the Lie algebra of the following subgroup of G. Let $H \subset G$ be the stabiliser subgroup of the lightlike line $\mathbb{R} \cdot \ell_-$ in $\mathbb{L}^{n+2}$. Then, we have

$$H = \left\{ \begin{pmatrix} a^{-1} & V & -\frac{a}{2}\|V^t\|^2 \\ 0 & \sigma & -a\sigma V^t \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{R}_{>0},\ \sigma \in O(n),\ V \in (\mathbb{R}^n)^* \right\},$$

where $\|\cdot\|^2$ denotes here the usual Euclidean norm on $\mathbb{R}^n$. Therefore, its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} -a & Z & 0 \\ 0 & A & -Z^t \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{R},\ A \in \mathfrak{o}(n),\ Z \in (\mathbb{R}^n)^* \right\}.$$

We continue with the construction of the principal fiber bundle with structure group H, which supports the desired Cartan connection and is defined in this manner. For each $x \in M$, we define

$$\mathcal{P}_x := \left\{ u \in \mathcal{O}^+(\mathcal{T}_x) : L_u(\ell_-) \in \mathcal{T}^1_x \right\}.$$

We let $\mathcal{P} := \bigcup_{x\in M} \mathcal{P}_x \subset \mathcal{O}^+(\mathcal{T})$. Thus, $p : \mathcal{P} \to M$ is a principal fiber bundle with structure group H as follows. We take $h \in H$ and $u \in \mathcal{P}_x$, then the right action is given by

$$u \cdot h := \mathbb{L}^{n+2} \xrightarrow{h} \mathbb{L}^{n+2} \xrightarrow{L_u} (\mathcal{T}_x, \mathbf{h}_x).$$

Clearly, we have that $(u \cdot h)(\ell_-) \in \mathcal{T}^1_x$ and therefore, the element $u \cdot h \in \mathcal{P}_x$. Note that this action is the restriction of the action that we have in the principal fiber bundle $\mathcal{O}^+(\mathcal{T})$ with structure group G to H.

Theorem 3.9. *Let $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ be a tractor conformal bundle as above (Definitions 3.7 and 3.8) and $\gamma \in \Omega^1\big(\mathcal{O}^+(\mathcal{T}), \mathfrak{o}(1, n+1)\big)$ be the principal connection corresponding to the linear connection $\nabla^{\mathcal{T}}$. Then, $\omega := \gamma|_{\mathcal{P}}$ is a Cartan connection of type (G, H) on $p : \mathcal{P} \subset \mathcal{O}^+(\mathcal{T}) \to M$, where $G = O^+(1, n+1)$ and $H \subset G$ is the stabiliser subgroup of the lightlike line $\mathbb{R} \cdot \ell_-$ in $\mathbb{L}^{n+2}$.*

Proof. Recall that a Cartan connection of type (G, H) satisfies that

(1) $(r^h)^*(\omega) = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$,

(2) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{h} \subset \mathfrak{g}$ and

(3) $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

For (1) and (2), the proofs are a direct consequence of the analogous properties for γ. In fact, for each $h \in H$, $u \in \mathcal{P}$ and $\xi \in T_u\mathcal{P}$ we have by definition

$$\omega(u \cdot h)\left(T_u r^h \cdot \xi\right) = \gamma(u \cdot h)\left(T_u r^h \cdot \xi\right).$$

Since γ is a principal connection it follows that

$$\gamma(u \cdot h)\left(T_u r^h \cdot \xi\right) = \mathrm{Ad}(h^{-1}) \circ \gamma(u)(\xi) = \mathrm{Ad}(h^{-1}) \circ \omega(u)(\xi),$$

and therefore $\omega(u \cdot h)\left(T_u r^h \cdot \xi\right) = \mathrm{Ad}(h^{-1}) \circ \omega(u)(\xi)$. In order to proof (2) we take $X \in \mathfrak{h}$, then

$$\omega(u)(\zeta_X(u)) = \gamma(u)(\zeta_X(u)) = X.$$

We have used again that γ is a principal connection.

Property (3) is the only one that does not have a direct proof, let us see it. First, we take a metric $g \in c$ and for each $x \in M$, we can construct a g-orthonormal local frame $(s_2, \ldots, s_{n+1})$

defined in an open set $U \subset M$ with $x \in U$. The choice of this metric g allows us to write a local section $\tau : U \to \mathcal{O}^+(\mathcal{T})$ as follows

$$\begin{aligned} x \mapsto \tau(x) &:= \big(\tau_1(x), \tau_2(x), \dots, \tau_{n+1}(x), \tau_{n+2}(x)\big) \\ &:= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2(x) \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ s_{n+1}(x) \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Let us note that $\tau(x) \in \mathcal{P}_x$. In fact,

$$\begin{aligned} L_{\tau(x)}(\ell_-) = \frac{1}{\sqrt{2}} L_{\tau(x)}(e_{n+2} - e_1) &= \frac{1}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \\ &= \frac{1}{2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We can consider the "Witt basis" write as follows

$$\hat{\tau} = \left(\tau_- := \frac{1}{\sqrt{2}}(\tau_{n+2} - \tau_1), \tau_2, \dots, \tau_{n+1}, \tau_+ := \frac{1}{\sqrt{2}}(\tau_{n+2} + \tau_1) \right).$$

Taking into account (1), we need to show the property (3) only on $\tau(U)$. Now, for each $\tau(x) \in \mathcal{P}$ with $x \in U$ and each $\xi \in T_{\tau(x)}\mathcal{P}$ we give the following descomposition

$$\xi = \zeta_X(\tau(x)) + T_x\tau \cdot w,$$

where $X \in \mathfrak{h}$ and $w \in T_xM$. As a consequence, we get

$$\omega(\tau(x))(\xi) = X + \omega(\tau(x))\big(T_x\tau \cdot w\big).$$

From Remark 2.10, let us recall how $\omega = \gamma|_{\mathcal{P}}$ is constructed. We consider the local trivialization ψ_τ associated to the local section τ for the principal bundle $p : \mathcal{O}^+(\mathcal{T}) \to M$. That is,

$$\begin{aligned} p^{-1}(U) &\xrightarrow{\psi_\tau} U \times O^+(1, n+1) \\ u &\longmapsto (p(u), g(u)), \end{aligned}$$

where $\tau(p(u)) \cdot g(u) = u$. Since $g(\tau(x)) = e \in O^+(1, n+1)$, it follows that

$$\omega(\tau(x))\big(T_x\tau \cdot w\big) = \omega^\tau(x)(w) + \omega_{O^+(1,n+1)}(e)(T_xg \circ \tau \cdot w) = \omega^\tau(x)(w).$$

Recall that ω^τ is determined by $\nabla^{\mathcal{T}}_w \tau_i = \sum_{j=1}^{n+2} \omega^\tau_{j,i}(x)(w)\tau_j(x)$, and then

$$\omega^\tau = \begin{pmatrix} \omega^\tau_{1,1} & \cdots & \omega^\tau_{1,n+2} \\ \vdots & \ddots & \vdots \\ \omega^\tau_{n+2,1} & \cdots & \omega^\tau_{n+2,n+2} \end{pmatrix},$$

where $\omega^\tau_{j,i}(x)(w) = \epsilon_j \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_i, \tau_j(x)\big)$. As a consequence,

$$\omega(\tau(x))(\xi) = X + \begin{pmatrix} \omega^\tau_{1,1}(x)(w) & \cdots & \omega^\tau_{1,n+2}(x)(w) \\ \vdots & \ddots & \vdots \\ \omega^\tau_{n+2,1}(x)(w) & \cdots & \omega^\tau_{n+2,n+2}(x)(w) \end{pmatrix},$$

where $\xi = \zeta_X(\tau(x)) + T_x\tau \cdot w$. We know that the Lie algebra $\mathfrak{g}$ in the Witt basis (3.8) is represented by

$$\mathfrak{g} = \left\{ \begin{pmatrix} -a & Z & 0 \\ Y & A & -Z^t \\ 0 & -Y^t & a \end{pmatrix} : a \in \mathbb{R},\ A \in \mathfrak{o}(n),\ Y \in \mathbb{R}^n,\ Z \in (\mathbb{R}^n)^* \right\}.$$

Taking $w = 0$, it is clear that $\mathfrak{h} \subset \operatorname{Im} \omega_{\tau(x)}$. On the other hand, taking $X = 0$ we have

$$\omega(\tau(x))(\xi) = \begin{pmatrix} \omega^\tau_{1,1}(x)(w) & \cdots & \omega^\tau_{1,n+2}(x)(w) \\ \vdots & \ddots & \vdots \\ \omega^\tau_{n+2,1}(x)(w) & \cdots & \omega^\tau_{n+2,n+2}(x)(w) \end{pmatrix} \in \mathfrak{g}.$$

Then, the element $\omega(\tau(x))(\xi)$ in this Witt basis is given by

$$\begin{pmatrix} \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_-, \tau_+(x)\big) & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_2, \tau_+(x)\big) & \cdots & 0 \\ \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_-, \tau_2(x)\big) & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_2, \tau_2(x)\big) & \cdots & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_+, \tau_2(x)\big) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_-, \tau_{n+1}(x)\big) & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_2, \tau_{n+1}(x)\big) & \cdots & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_+, \tau_{n+1}(x)\big) \\ 0 & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_2, \tau_-(x)\big) & \cdots & \mathbf{h}_x\big(\nabla^{\mathcal{T}}_w \tau_+, \tau_-(x)\big) \end{pmatrix}.$$

To conclude that ω defines an absolute parallelism we only need to prove that

$$T_xM \to \mathbb{R}^n, \quad w \mapsto \Big(\mathbf{h}_x(\nabla^{\mathcal{T}}_w \tau_-, \tau_2(x)), \ldots, \mathbf{h}_x(\nabla^{\mathcal{T}}_w \tau_-, \tau_{n+1}(x))\Big)^T$$

is a linear isomorphism. In fact, assume $\mathbf{h}_x(\nabla^{\mathcal{T}}_w \tau_-, \tau_i(x)) = 0$ for all $i \in \{2, \ldots, n+1\}$, then $\beta(w)(\tau_-(x)) = 0$ and therefore $\beta(w) = 0$. Taking into account that β is an isomorphism of vector bundles on M, we get $w = 0$.

$\square$

The construction given in Theorem 3.9 is the inverse of the given in Definition 3.5 and conversely. The work done in this Section could be summarized as follows:

$$\begin{array}{c}
\boxed{\text{Möbius geometry } (p:\mathcal{P}\to M,\omega)} \\
\Updownarrow \\
\boxed{\text{Tractor conformal bundle with tractor connection } (\mathcal{T},\mathcal{T}^1,\mathbf{h},\nabla^{\mathcal{T}})} \\
\Downarrow \\
\boxed{\text{Riemannian conformal structure } (M,c)}
\end{array}$$

3.2 From Riemannian conformal structures to standard tractor conformal bundles

Maybe the most difficult point in the study of Cartan geometries is to state the equivalence between a certain Cartan geometry and an underlying geometric structure on a manifold M. In particular, these difficulties arise in the setting of Riemannian conformal structures (M, c). In fact, there are many Möbius geometries $(p : \mathcal{P} \to M, \omega)$ or, equivalently, tractor conformal bundles with tractor connections $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ that induce the same conformal structure on M. To address the equivalence problem between Riemannian conformal structures and Möbius geometries, we introduce the following definition.

Definition 3.10. *A standard tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ for a Riemannian conformal structure (M, c) is said to be admissible when for every metric $g \in c$ and its respective decomposition $\mathcal{T} \overset{g}{\simeq} \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$, the tractor connection satisfies*

(1)

$$\nabla^{\mathcal{T}}_V \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V(\alpha) \\ \alpha V \\ 0 \end{pmatrix},$$

and

(2) *there exists a symmetric tensor $D(g) \in \mathcal{T}_{(0,2)}(M)$ such that*

$$\nabla^{\mathcal{T}}_V \begin{pmatrix} 0 \\ W \\ 0 \end{pmatrix} = \begin{pmatrix} -D(g)(V,W) \\ \nabla_V W \\ -g(V,W) \end{pmatrix},$$

where $\alpha \in \mathcal{C}^\infty(M)$ and $V, W \in \mathfrak{X}(M)$. Let us note that ∇ denotes the Levi-Civita connection of the respective metric $g \in c$.

Remark 3.11. The other positions of the tractor connection $\nabla^{\mathcal{T}}$ of an admissible standard tractor conformal bundle are determined by the metric condition. In fact, for every metric $g \in c$, it is not difficult to show that

$$\nabla^{\mathcal{T}}_V \begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix} = \begin{pmatrix} V(\alpha) - D(g)(V,W) \\ \nabla_V W + \alpha V + \beta \hat{D}(g)(V) \\ V(\beta) - g(V,W) \end{pmatrix},$$

where $\hat{D}(g)(V)$ is the endomorphisms field defined by the equation

$$g\left(\hat{D}(g)(V), W\right) = D(g)(V,W).$$

For a different choice of the metric $e^{2u}g$ in the same conformal class c, the corresponding symmetric tensor $D(e^{2u}g)$ satisfies the following conformal transformation law

$$D(e^{2u}g) = D(g) - \frac{\|\nabla u\|^2}{2} g - \mathrm{Hess}(u) + du \otimes du, \tag{3.9}$$

where ∇u and $\mathrm{Hess}(u)$ are the gradient and the Hessian of the function $u \in \mathcal{C}^\infty(M)$ for the metric g, respectively. Here, $\|\cdot\|$ denotes the norm with respect to g. To see this Remark in detail, see [9, Proposition 2.25].

As was mentioned, the aim of this Section is to construct a Möbius geometry from a Riemannian conformal structure (M, c). This Möbius geometry will correspond to a standard tractor conformal bundle for (M, c). Taking into account Definition 3.10 and Remark 3.11, we need to introduce one more object, that is, a “Schouten type-tensor” D, and give an explicit construction of a standard tractor conformal bundle for (M, c) from the triple (M, c, D). For this construction, we will follow the one briefly stated in [20], made explicit in [8], and described in greater detail in [9]. It is worth noting that other constructions for a standard tractor conformal bundle from a Riemannian conformal structure exist as was described in [17] and [35].

Definition 3.12. *Let* (M, c) *be an* $(n \geq 2)$*-dimensional Riemannian conformal structure. A "Schouten type-tensor"* D *for* c *is a map such that:*

1. $D : c \to \mathcal{T}_{(0,2)}M$ *such that for every* $g \in c$*, the tensor* $D(g)$ *is symmetric and*
2. D *satisfies the conformal transformation law* (3.9).

Also, we denote by $\widehat{D}(g)$ the $(1,1)$-tensor induced by $D(g)$, that is, $\widehat{D}(g)$ is given by the relation $g\left(\widehat{D}(g)(V), W\right) = D(g)(V, W)$, for every $V, W \in \mathfrak{X}(M)$. Let us note that $\widehat{D}(g)$ satisfies a similar conformal transformation law given by

$$\widehat{D}(e^{2u}g)(V) = e^{-2u}\left[\widehat{D}(g)(V) - \frac{\|\nabla u\|^2}{2}V - \nabla_V \nabla u + V(u)\nabla u\right], \tag{3.10}$$

for every $V \in \mathfrak{X}(M)$.

Remark 3.13. *For an* $(n \geq 3)$*-dimensional Riemannian conformal structure* (M, c)*, the conformal transformation law (2.14) for the Schouten tensor implies that* $D(g) = P^g$ *provides a "Schouten type-tensor" for c.*

3.2.1 Construction of the g-tractor vector bundle

Let (M, c) be an n-dimensional Riemannian conformal structure and D be a "Schouten type-tensor" for c. Now, for every $g \in c$, we can construct the following vector bundle from $(M, g, D(g))$ endowed with a linear connection and a bundle metric:

1. Let $\mathcal{T}_g$ be the rank $n + 2$ real vector bundle over M given by

$$\mathcal{T}_g := \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}},$$

where $\underline{\mathbb{R}}$ denotes the trivial bundle $M \times \mathbb{R} \to M$. For any smooth section $\sigma : M \to \mathcal{T}_g$ we write

$$\sigma = \begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix}, \text{ with } \alpha, \beta \in \mathcal{C}^\infty(M) \text{ and } W \in \mathfrak{X}(M).$$

2. On $\mathcal{T}_g$ we consider the linear connection $\nabla^{\mathcal{T}_g}$ defined by

$$\nabla^{\mathcal{T}_g}_V \begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix} := \begin{pmatrix} V(\alpha) - D(g)(V, W) \\ \nabla_V W + \alpha V + \beta\widehat{D}(g)(V) \\ V(\beta) - g(V, W) \end{pmatrix},$$

where ∇ is the Levi-Civita connection of g, and the bundle metric $\mathbf{h}^g$ of Lorentzian signature given by

$$\mathbf{h}^g\left(\begin{pmatrix}\alpha_1\\ W_1\\ \beta_1\end{pmatrix},\begin{pmatrix}\alpha_2\\ W_2\\ \beta_2\end{pmatrix}\right) := \alpha_1\beta_2+\beta_1\alpha_2+g(W_1,W_2).$$

Definition 3.14. *Let (M,c) be an n-dimensional Riemannian conformal structure and D be a "Schouten type-tensor" for c. For every $g\in c$, the triple $(\mathcal{T}_g,\mathbf{h}^g,\nabla^{\mathcal{T}_g})$ is called the g-tractor associated to $(M,g,D(g))$.*

For a Riemannian conformal structure (M,c), the g-tractor associated to $(M,g,D(g))$ depends on g and $D(g)$. However, with the help of the g-tractor, we will be able to define a standard tractor conformal bundle for (M,c) that depends on the choice of D but not on $g\in c$.

Lemma 3.15. *Let (M,g) be a Riemannian manifold. For any section $\sigma\in\mathcal{T}_g$ we define*

$$\sigma' := D_{g,u}\cdot\sigma \ \ \textit{with}\ \ D_{g,u}=\begin{pmatrix} e^{-u} & -e^{-u}du & -e^{-u}\dfrac{\|\nabla u\|^2}{2}\\ 0 & e^{-u}Id & e^{-u}\nabla u\\ 0 & 0 & e^{u}\end{pmatrix},$$

where $u\in\mathcal{C}^\infty(M)$. Then, for every $\sigma,\sigma_1,\sigma_2\in\Gamma(\mathcal{T}_g)$ we have

(1) $\nabla_V^{\mathcal{T}_{g'}}\sigma' = D_{g,u}\nabla_V^{\mathcal{T}_g}\sigma$ *and*

(2) $\mathbf{h}^{g'}(\sigma_1',\sigma_2')=\mathbf{h}^g(\sigma_1,\sigma_2)$,

where $g'=e^{2u}g$ and $V\in\mathfrak{X}(M)$.

Proof. Let us start by proving (1). Note that both members of the equality are $\mathbb{R}$-linear in "σ". Hence, we can check (1) for sections of $\underline{\mathbb{R}}$ and TM.

- For a section $(\alpha,0,0)^t\in\Gamma(\mathcal{T}_g)$, we have

$$\nabla_V^{\mathcal{T}_{g'}}D_{g,u}\begin{pmatrix}\alpha\\0\\0\end{pmatrix}=\nabla_V^{\mathcal{T}_{g'}}\begin{pmatrix}e^{-u}\alpha\\0\\0\end{pmatrix}=\begin{pmatrix}V(\alpha)e^{-u}-\alpha e^{-u}V(u)\\ e^{-u}\alpha V\\ 0\end{pmatrix}$$

and

$$D_{g,u}\nabla_V^{\mathcal{T}_g}\begin{pmatrix}\alpha\\0\\0\end{pmatrix}=D_{g,u}\begin{pmatrix}V(\alpha)\\ \alpha V\\ 0\end{pmatrix}=\begin{pmatrix}V(\alpha)e^{-u}-\alpha e^{-u}V(u)\\ e^{-u}\alpha V\\ 0\end{pmatrix}.$$

- For a section $\big(0,0,\beta\big)^t \in \Gamma(\mathcal{T}_g)$, we compute

$$
\begin{aligned}
\nabla^{\mathcal{T}_{g'}}_V D_{g,u}\begin{pmatrix}0\\0\\\beta\end{pmatrix} &= \nabla^{\mathcal{T}_{g'}}_V\begin{pmatrix}-e^{-u}\dfrac{\|\nabla u\|^2}{2}\beta\\ e^{-u}\beta\nabla u\\ e^u\beta\end{pmatrix}\\
&= V(\beta)\begin{pmatrix}-e^{-u}\dfrac{\|\nabla u\|^2}{2}\\ e^{-u}\nabla u\\ e^u\end{pmatrix}\\
&+\beta\begin{pmatrix}e^{-u}V(u)\dfrac{\|\nabla u\|^2}{2}-e^{-u}g(\nabla_V\nabla u,\nabla u)-e^{-u}D(g')(V,\nabla u)\\ e^{-u}\dfrac{\|\nabla u\|^2}{2}V+e^{-u}\nabla_V\nabla u+V(e^{-u})\nabla u+e^u\widehat{D}(g')(V)\\ 0\end{pmatrix}.
\end{aligned}
$$

As consequence of conformal transformations laws (3.9) and (3.10), we have

$$
\nabla^{\mathcal{T}_{g'}}_V D_{g,u}\begin{pmatrix}0\\0\\\beta\end{pmatrix} = V(\beta)\begin{pmatrix}-e^{-u}\dfrac{\|\nabla u\|^2}{2}\\ e^{-u}\nabla u\\ e^u\end{pmatrix}+\beta\begin{pmatrix}-e^{-u}D(g)(V,\nabla u)\\ e^{-u}\widehat{D}(g)(V)\\ 0\end{pmatrix}.
$$

For the right-hand side, we have

$$
D_{g,u}\nabla^{\mathcal{T}_g}_V\begin{pmatrix}0\\0\\\beta\end{pmatrix} = D_{g,u}\begin{pmatrix}0\\ \beta\widehat{D}(g)(V)\\ V(\beta)\end{pmatrix} = \begin{pmatrix}-e^{-u}V(\beta)\dfrac{\|\nabla u\|^2}{2}-e^{-u}\beta D(g)(V,\nabla u)\\ e^{-u}V(\beta)\nabla u+e^{-u}\beta\widehat{D}(g)(V)\\ e^uV(\beta)\end{pmatrix}.
$$

- Lastly, for a section $\big(0,W,0\big)^t \in \Gamma(\mathcal{T}_g)$, a similar computation, by means of the conformal transformations laws (3.9) and (3.10), ends the proof.

Now, we are going to proof (2). Let us consider $\sigma_1 = \big(\alpha_1, W_1, \beta_1\big)^t$ and $\sigma_2 = \big(\alpha_2, W_2, \beta_2\big)^t$. Then, a direct computation gives

$$
\begin{aligned}
&\mathbf{h}^{g'}(\sigma_1',\sigma_2')\\
&= \mathbf{h}^{g'}\left(\begin{pmatrix}e^{-u}\alpha_1-e^{-u}W_1(u)-e^{-u}\dfrac{\|\nabla u\|^2}{2}\beta_1\\ e^{-u}W_1+e^{-u}\beta_1\nabla u\\ e^u\beta_1\end{pmatrix},\begin{pmatrix}e^{-u}\alpha_2-e^{-u}W_2(u)-e^{-u}\dfrac{\|\nabla u\|^2}{2}\beta_2\\ e^{-u}W_2+e^{-u}\beta_2\nabla u\\ e^u\beta_2\end{pmatrix}\right)\\
&= \alpha_1\beta_2+\beta_1\alpha_2+g(W_1,W_2) = \mathbf{h}^g(\sigma_1,\sigma_2).
\end{aligned}
$$

□

Although the proof has been a direct calculation, it is important to highlight the importance of D satisfying the conformal transformation law. We finish this Subsection by stating the following result.

Lemma 3.16. *Let (M, g) be a Riemannian manifold. Then, the linear connection $\nabla^{\mathcal{T}_g}$ is metric for $\mathbf{h}^g$. That is,*

$$V\big(\mathbf{h}^g(\sigma_1, \sigma_2)\big) = \mathbf{h}^g\left(\nabla^{\mathcal{T}_g}_V \sigma_1, \sigma_2\right) + \mathbf{h}^g\left(\sigma_1, \nabla^{\mathcal{T}_g}_V \sigma_2\right),$$

for every $V \in \mathfrak{X}(M)$ and $\sigma_1, \sigma_2 \in \Gamma(\mathcal{T}_g)$.

The proof of the previous Lemma is a straightforward computation.

3.2.2 Standard tractor conformal bundles from conformal geometry

Let (M, c) be an n-dimensional Riemannian conformal structure and D be a “Schouten type-tensor” for c. For every $g \in c$ and $D(g)$ we are able to build a g-tractor. Now, we are going to contruct a standard tractor conformal bundle $\mathcal{T}$ for (M, c) which will be independent of the choice of $g \in c$. In order to do so we introduce the following notation. Let us take $g \in c$ and we consider $(\mathcal{T}_g, \mathbf{h}^g, \nabla^{\mathcal{T}_g})$ the g-tractor associated to $(M, g, D(g))$. For each $x \in M$, we write the fiber of $\mathcal{T}_g$ over x by

$$(\mathcal{T}_g)_x = \left\{ \left(\begin{pmatrix} a \\ w \\ b \end{pmatrix}, g_x \right) : \, a, b \in \mathbb{R}, \, w \in T_x M \right\}.$$

From Lemma 3.15, we know that there is an isomorphism of vector bundles given by

$$\begin{tikzcd} (\mathcal{T}_g, \mathbf{h}^g) \arrow[rr, "D_{g,u}"] \arrow[dr] & & (\mathcal{T}_{g'}, \mathbf{h}^{g'}) \arrow[dl] \\ & M & \end{tikzcd},$$

which is an isometry on each fiber, where $g' = e^{2u} g$. Now, we consider $T := \bigcup_{g \in c} \mathcal{T}_g$ and define the following equivalence relation. Take $\left((a, w, b)^t, g_x\right) \in (\mathcal{T}_g)_x$ and $\left((a', w', b')^t, g'_x\right) \in (\mathcal{T}_{g'})_x$,

$$\left(\begin{pmatrix} a \\ w \\ b \end{pmatrix}, g_x \right) \sim \left(\begin{pmatrix} a' \\ w' \\ b' \end{pmatrix}, g'_x \right) \Longleftrightarrow \begin{pmatrix} a' \\ w' \\ b' \end{pmatrix} = D_{g,u}(x) \begin{pmatrix} a \\ w \\ b \end{pmatrix}.$$

Definition 3.17. *We denote the quotient set $T/\sim$ by $\mathcal{T}$ and its elements by*

$$\left[\begin{pmatrix} a \\ w \\ b \end{pmatrix}, g_x\right],$$

where $a, b \in \mathbb{R}$, $x \in M$ and $w \in T_xM$.

Lemma 3.18. *Let (M, c) be an n-dimensional Riemannian conformal structure and D be a "Schouten type-tensor" for c. Then, $\mathcal{T}$ has a canonical structure of vector bundle over M.*

Proof. For every metric $g \in c$, we consider the map

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi_g} & \mathcal{T}_g \\ \left[\begin{pmatrix} a \\ w \\ b \end{pmatrix}, g_x\right] & \longmapsto & \left(\begin{pmatrix} a \\ w \\ b \end{pmatrix}, g_x\right). \end{array}$$

This map Φ_g is one-to-one and, since $\mathcal{T}_g$ is a vector bundle, $\mathcal{T}$ can be endowed with a vector bundle structure by means of Φ_g. Furthermore, this vector bundle structure is independent of the choice of $g \in c$. In fact, taking $g, g' \in c$ such that $g' = e^{2u}g$, we have the following commutative diagram

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "\Phi_g"'] \arrow[dr, "\Phi_{g'}"] & \\ \mathcal{T}_g \arrow[rr, "D_{g,u}"] & & \mathcal{T}_{g'}. \end{tikzcd}$$

Given that $D_{g,u}$ is an isomorphism of vector bundles, we conclude the proof. $\square$

Further we can see that $\nabla^{\mathcal{T}_g}$ and $\mathbf{h}^g$ induce analog structures on $\mathcal{T}$:

1. On $\mathcal{T}$ we consider the linear connection $\nabla^{\mathcal{T}}$ defined by

$$\nabla^{\mathcal{T}}_V \left[\begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix}, g\right] := \left[\nabla^{\mathcal{T}_g}_V \begin{pmatrix} \alpha \\ W \\ \beta \end{pmatrix}, g\right],$$

for every $V \in \mathfrak{X}(M)$ and $\left[\left(\alpha, W, \beta\right)^t, g\right] \in \Gamma(\mathcal{T})$. Additionally, we introduce the bundle metric $\mathbf{h}$ of Lorentzian signature given by

$$\mathbf{h}\left(\left[\begin{pmatrix} \alpha_1 \\ W_1 \\ \beta_1 \end{pmatrix}, g\right], \left[\begin{pmatrix} \alpha_2 \\ W_2 \\ \beta_2 \end{pmatrix}, g\right]\right) := \mathbf{h}^g\left(\begin{pmatrix} \alpha_1 \\ W_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ W_2 \\ \beta_2 \end{pmatrix}\right),$$

where $\left[\left(\alpha_1, W_1, \beta_1\right)^t, g\right], \left[\left(\alpha_2, W_2, \beta_2\right)^t, g\right] \in \Gamma(\mathcal{T})$. As consequence of Lemma 3.15, we have that the connection $\nabla^{\mathcal{T}}$ and the bundle metric $\mathbf{h}$ are well-defined.

2. Also, we have the following line subbundle given by

$$\mathcal{T}^1 := \mathbb{R} \cdot \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, g\right] \subset \mathcal{T}.$$

It is not difficult to show that $\mathcal{T}^1$ is well-defined and $\mathbf{h}\big(\mathcal{T}^1, \mathcal{T}^1\big) = 0$.

Lemma 3.19. *The linear connection $\nabla^{\mathcal{T}}$ is metric for $\mathbf{h}$. That is,*

$$V\big(\mathbf{h}(\sigma_1, \sigma_2)\big) = \mathbf{h}\left(\nabla^{\mathcal{T}}_V \sigma_1, \sigma_2\right) + \mathbf{h}\left(\sigma_1, \nabla^{\mathcal{T}}_V \sigma_2\right),$$

where $V \in \mathfrak{X}(M)$ and $\sigma_1, \sigma_2 \in \Gamma(\mathcal{T})$.

Proof. This result is direct from Lemmas 3.15 and 3.16. □

As a consequence, we have that $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ satisfies Definitions 3.7 and 3.8. Furthermore, as we saw in the previous Section, $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ induces a conformal class of Riemannian metrics on M. It is a direct computation, from the definitions of $\mathcal{T}^1$, $\mathbf{h}$ and $\nabla^{\mathcal{T}}$, that we recover the one from which we started, that is, the conformal class c. In other words, we have constructed an admissible standard tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ for (M, c) from (M, c) by means of D. That is, we have the following equivalence:

$$\boxed{\text{Admissible standard tractor conformal bundle } (\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})}$$
$$\Updownarrow$$
$$\boxed{(M, c, D)}$$

3.3 Normal Cartan connections and normal tractor connections

As was wentioned, each "Schouten type-tensor" allows us to construct a Möbius geometry and a standard tractor conformal bundle from a given n-dimensional Riemannian conformal structure (M, c). A natural problem is to find a way to assign a unique Möbius geometry to each Riemannian conformal structure. In other words, finding normalization hypotheses for the Cartan connections such that there is a one-to-one correspondence between Riemannian conformal structures and Möbius geometries. For dimension $n \geq 3$, there is a canonical choice for the "Schouten type-tensor" by means of the Schouten tensor defined in (2.13) for each metric $g \in c$. That is, the "Schouten type-tensor" $P : c \to \mathcal{T}_{(0,2)}M$ given by $P(g) := P^g$.

Definition 3.20. *Let (M, c) be an $(n \geq 3)$-dimensional Riemannian conformal structure and P be the Schouten tensor. Let $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ be the standard tractor conformal bundle constructed from (M, c, P). Then, $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ is called the normal standard tractor conformal bundle and the tractor connection $\nabla^{\mathcal{T}}$ is said to be the normal tractor connection. The Möbius geometry $(p : \mathcal{P} \to M, \omega)$ constructed, by means of Theorem 3.9, from $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ is called the normal Möbius geometry and the connection ω is said to be the normal Cartan connection.*

Remark 3.21. In the normal case, both $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ and $(p : \mathcal{P} \to M, \omega)$ are uniquely determined by the underlying conformal structure up to isomorphism. This is an old result of Elie Cartan that Riemannian conformal structures of dimension ≥ 3 admit a canonical normal Cartan connection. See, for example, [20, Theor. 1.6.7].

Thus, we have the following diagram:

$$\begin{array}{c}
\boxed{\text{Normal Möbius geometry } (p : \mathcal{P} \to M, \omega)} \\
\Updownarrow \\
\boxed{\text{Normal standard tractor conformal bundle } (\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})} \\
\Updownarrow \\
\boxed{(n \geq 3)\text{-dimensional Riemannian conformal structure } (M, c)}
\end{array}$$

If we take an arbitrary standard tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ or equivalently an arbitrary Möbius geometry $(p : \mathcal{P} \to M, \omega)$, a difficult problem arises in determining when the connection $\nabla^{\mathcal{T}}$ or the connection ω is normal. This problem was addressed and resolved in different ways. Here, we are going to present one of them.

Theorem 3.22. ([*19*, *Sec. 2.2*]) *Let $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ be a standard tractor conformal bundle for an $(n \geq 3)$-dimensional Riemannian conformal structure (M, c). Then, $\nabla^{\mathcal{T}}$ is the normal tractor connection if and only if the following conditions hold:*

(1) *Its curvature* $R^{\mathcal{T}}$ *satisfies*

$$R^{\mathcal{T}}(V,W)\sigma := \nabla^{\mathcal{T}}_V \nabla^{\mathcal{T}}_W \sigma - \nabla^{\mathcal{T}}_W \nabla^{\mathcal{T}}_V \sigma - \nabla^{\mathcal{T}}_{[V,W]}\sigma \subset \Gamma(\mathcal{T}^1),$$

for every $V, W \in \mathfrak{X}(M)$ *and* $\sigma \in \Gamma(\mathcal{T}^1)$.

Note that (1) *implies that* $R^{\mathcal{T}}(V,W)$ *induces an endomorphism on* $(\mathcal{T}^1)^{\perp}/\mathcal{T}^1$. *In fact, by means of any section* $\sigma \in \Gamma(\mathcal{T}^1)$, *we can consider* $\mathcal{W} \in \Gamma(\Lambda^2 T^*M \otimes L(TM, TM))$ *as follows*

$$\mathcal{W}(U,V)W = (\beta_\sigma)^{-1}\Big(R^{\mathcal{T}}(U,V)\beta_\sigma(W)\Big), \quad U, V, W \in \mathfrak{X}(M),$$

where β_σ *is given in* (3.5). *Note that* $\mathcal{W}$ *does not depend on the choice of* $\sigma \in \Gamma(\mathcal{T}^1)$.

(2) *The Ricci type contraction of* $\mathcal{W} \in \Gamma(\Lambda^2 T^*M \otimes L(TM, TM))$ *vanishes on* M. *That is, the following equation holds*

$$\sum_{i=1}^{n} \mathbf{h}^{\sigma}\Big(\mathcal{W}(E_i, V)W, E_i\Big) = \sum_{i=1}^{n} \mathbf{h}\Big(R^{\mathcal{T}}(E_i, V)\beta_\sigma(W), \beta_\sigma(E_i)\Big) = 0,$$

for any $V, W \in \mathfrak{X}(M)$ *and* $\sigma \in \Gamma(\mathcal{T}^1)$, *where* $(E_1, \ldots, E_n)$ *is a local orthonormal frame with respect to the Riemannian metric* $\mathbf{h}^{\sigma} \in c$ *as in* (3.6).

In Chapter 5, starting from a Riemannian conformal structure (M, c) we will study when the standard tractor conformal bundle with the normal tractor connection determined by (M, c) can be realized by means of a codimension two spacelike immersion of M in a Lorentzian manifold.

Remark 3.23. There are different ways to construct the normal Cartan connection. One of them is the given in Definition 3.20 by means of the Schouten tensor. Another way to give the normal Cartan connection is defining a "Cartan curvature function" and assuming certain normalization hypotheses over this function, see [20, Def. 1.6.7]. This last point of view is the one originally given. In fact, the proof of Theorem 3.22 consists of translating these normalization hypotheses into terms of the curvature of the tractor connection. We have not included this way because it is technically very complicated and we do not believe it helps to understand our work.

As we have seen, in dimension $n \geq 3$ there is a canonical choice to construct $(\mathcal{T}, \mathcal{T}^1, \mathbf{h}, \nabla^{\mathcal{T}})$ and $(p : \mathcal{P} \to M, \omega)$ from (M, c) by means of the Schouten tensor P. However, this is not the case for two dimensional Riemannian conformal structures where the Schouten tensor is not defined. As we already know, conformal geometry in this dimension is more exotic than in higher dimensions since conformal geometry in dimension two carries no local information. So, a problem that looks interesting from our point of view is to study the family of "Schouten type-tensors" in this case. In chapter 6, we will see how each "Schouten type-tensor" can be recovered by means of certain Weingarten endomorphisms associated with spacelike immersions of M in a family of Lorentzian manifolds, Remark 6.21.

Furthermore, as a consequence of (2.13), we know that

$$\mathrm{trace}_g P^g = \frac{\mathrm{S}^g}{2(n-1)},$$

for every $g \in c$. Therefore, this additional property can be imposed on the "Schouten type-tensors" in order to normalize, in some sense, the family of tensors that we are interested in studying. This additional property leads to the following definition.

Definition 3.24. ([15], [60]) *A Möbius structure on an $(n \geq 2)$-dimensional manifold M is a triple (M, c, D), where (M, c) is a Riemannian conformal structure and D is a "Schouten type-tensor" for c (Definition 3.12) which, in addition, satisfies that*

$$\mathrm{trace}_g D(g) = \frac{\mathrm{S}^g}{2(n-1)},$$

where S^g is the scalar curvature of the metric $g \in c$ and $\mathrm{trace}_g D(g)$ denotes the trace with respect to the metric g of the corresponding tensor $D(g)$.

The conformal transformation law implies that a Möbius structure (M, c, D) is completely determined by the value of D at a single $g \in c$. In fact, the relationship between the scalar curvatures of two conformally related metrics (see formula (2.12)) and the conformal transformation law imply that

$$\mathrm{trace}_{e^{2u}g}\, D(e^{2u}g) = \frac{\mathrm{S}^{e^{2u}g}}{2(n-1)}.$$

Chapter 4

Generalized Schwarzschild spacetimes

The generalized Schwarzschild spacetimes have been introduced as warped manifolds where the base is an open subset of $\mathbb{R}^2$ equipped with a Lorentzian metric and the fiber is a Riemannian manifold, Definition 1.1. This family includes physically relevant spacetimes closely related to models of black holes. The generalized Schwarzschild spacetimes are endowed with involutive distributions which provide foliations by lightlike hypersurfaces. As was mentioned in the Introduction, in this Chapter we study spacelike immersions in the generalized Schwarzschild spacetimes, mainly, under the assumption that such immersions lie in a leaf of the above foliations. In this scenario, we provide an explicit formula for the mean curvature vector field and establish relationships between the extrinsic and intrinsic geometry of these immersions. We have derived several characterizations of the slices, and we delve into the specific case where the warping function is the radial coordinate in detail. This subfamily includes the Schwarzschild and Reissner-Nordström spacetimes.

4.1 Involutive distributions

Although, we are interested here in the class of spacetimes given in Definition 1.1, there are several properties which can be stated in a more general setting. Let (B, g_B) be a two dimensional oriented Lorentzian manifold and (F, g_F) a m-dimensional connected Riemannian manifold. Fix $\lambda \in C^\infty(B)$ with $\lambda > 0$, we are interested in the Lorentzian warped product manifold $(\widetilde{M}, \widetilde{g}) = B \times_\lambda F$ given by the product manifold $\widetilde{M} = B \times F$ endowed with the Lorentzian metric

$$\widetilde{g} = \pi_B^*(g_B) + (\lambda \circ \pi_B)^2 \pi_F^*(g_F),$$

see Section 2.3. Recall that every $E \in \mathfrak{X}(\widetilde{M})$ has a unique expression as $E = X + V$ where $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$.

For our aims here, we assume there exists a global lightlike vector field $\xi \in \mathfrak{X}(B)$. That is, we have $g_B(\xi, \xi) = 0$ and $\xi_q \neq 0$ for all $q \in B$. Taking into account that $\dim B = 2$, it is not difficult to show that there exists a 1-form $\alpha \in \Omega^1(B, \mathbb{R})$ such that

$$\nabla^B \xi = \alpha \otimes \xi. \tag{4.1}$$

The assumption on the existence of the vector field ξ has the following key consequence.

Lemma 4.1. *The distribution* $D_\xi = \{E \in \mathfrak{X}(\widetilde{M}) : \widetilde{g}(E, \xi) = 0\}$ *on* $\widetilde{M}$ *is involutive.*

Proof. For $X + V, Y + W \in D_\xi$ a straightforward computation gives

$$\widetilde{g}([X + V, Y + W], \xi) = g_B([X, Y], \xi) \circ \pi_B.$$

Taking into account that $X + V \in D_\xi$ if and only if $g_B(X, \xi) = 0$, we obtain from (4.1) that

$$g_B([X, Y], \xi) = -g_B(\nabla^B_X \xi, Y) + g_B(\nabla^B_Y \xi, X) = 0.$$

$\square$

Therefore, as consequence of Theorem 2.4 we have that through every point $(q, x) \in \widetilde{M}$ passes a maximal integral submanifold $\mathcal{L}$ of the distribution D_ξ and we have a foliation of the manifold $\widetilde{M}$ by hypersurfaces. If we write $\gamma \colon I \to B$ for the maximal integral curve of the vector field ξ with initial condition $\gamma(0) = q \in B$, then the hypersurface $\mathcal{L}$ is given by

$$\mathcal{L} = \{(\gamma(t), x) \in B \times F : t \in I \ , x \in F\}.$$

$\mathcal{L}$ inherits a degenerate metric tensor from $\widetilde{g}$ whose radical is spanned by the vector field $\xi|_{\mathcal{L}}$ and therefore it is a lightlike hypersurface. Every smooth section $\sigma = (\sigma_B, \mathrm{Id}_F)$ of the natural projection $\mathcal{L} \to F$ provides a spacelike immersion in $\widetilde{M}$ with induced metric $g = \sigma^*(\widetilde{g}) = (\lambda \circ \sigma_B)^2 g_F$. Hence, the induced metric g belongs to the same conformal class of g_F. In particular, $\mathcal{L}$ is a subset of the bundle of scales of F for the conformal class of g_F [20, Chap. 1].

Remark 4.2. The projection $B \times_\lambda F \to B$ is a semi-Riemannian submersion [50] (see also [51, Chap. 7]). In the terminology of semi-Riemannian submersions, every maximal integral submanifold $\mathcal{L}$ of the distribution D_ξ is the horizontal lift of a maximal integral curve of the vector field ξ on B.

Now let $\Psi : M \to B \times_\lambda F$ be an arbitrary spacelike immersion. The immersion Ψ can be written

$$\Psi = (\Psi_B, \Psi_F),$$

where $\Psi_B = \pi_B \circ \Psi$ and $\Psi_F = \pi_F \circ \Psi$. For the vector field $\xi|_\Psi$ we have

$$0 = \widetilde{g}(\xi|_\Psi, \xi|_\Psi) = \widetilde{g}(T\Psi \cdot \xi^\top, T\Psi \cdot \xi^\top) + \widetilde{g}(\xi^\perp, \xi^\perp).$$

Hence, $\xi^\perp$ does not vanish at any point of M and $\widetilde{g}(\xi^\perp, \xi^\perp) \leq 0$, in other words, $\xi^\perp$ is a causal normal vector field. On the other hand, the vector field $\xi^\top$ vanishes identically if and only if $M \subset \mathcal{L}$. In this case, we say that M factors through an integral hypersurface $\mathcal{L}$ of D_ξ.

Remark 4.3. Let recall that the two dimensional manifold B is assumed to be orientable. Thus, the existence of the lightlike vector field ξ implies that there is another lightlike vector field $\eta \in \mathfrak{X}(B)$ which is uniquely determined by the normalization condition $g_B(\xi, \eta) = -1$. As for ξ, we have $\nabla^B \eta = -\alpha \otimes \eta$ and the corresponding distribution D_η is also involutive. Every each maximal integral submanifold $\mathcal{N}$ of D_η inherits a degenerate metric tensor from $\widetilde{g}$ whose radical is now spanned by the restriction of η to $\mathcal{N}$.

In order to be used later, let us recall the notion of parabolic Riemannian manifold.

Definition 4.4. *A (non necessary complete) Riemannian manifold is parabolic if the only subharmonic functions bounded from above that it admits are the constants. That is, a Riemannian manifold (M, g) is parabolic when $\Delta h \geq 0$ and $\sup_M h < +\infty$ for a smooth function $h \in C^\infty(M)$ implies h must be constant (see, for instance, [38] and [36]).*

Note that Δ denotes the Laplace operator of the metric g. From a physical point of view, the parabolicity is equivalent to the recurrence of the Brownian motion on a Riemannian manifold [36]. Let us also recall that every complete Riemannian surface with non-negative Gaussian curvature is parabolic [37]. Even more, every complete Riemannian surface with finite total curvature is parabolic [37]. In arbitrary dimension there is no clear relation between parabolicity and sectional curvature. Nevertheless, there exist sufficient conditions to ensure the parabolicity of a Riemannian manifold of arbitrary dimension based on the volume growth of its geodesic balls [6].

4.2 Generalized Schwarzschild spacetimes

From now on $B \times_\lambda F$ is a generalized Schwarzschild spacetime, Definition 1.1. In order to facilitate reading, let us recall the definition. A Lorentzian warped product manifold $(\widetilde{M}, \widetilde{g}) = B \times_\lambda F$ is said to be an $(m+2)$-dimensional generalized Schwarzschild spacetime when B is an open subset of $\mathbb{R}^2$ with canonical coordinates (t, r) and metric

$$g_B = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2,$$

where $f(r) > 0$, (F, g_F) is a m-dimensional connected Riemannian manifold and $\lambda \in C^\infty(B)$ with $\lambda > 0$ is the warping function.

We can give two lightlike vector fields in B as follows

$$\xi = \frac{1}{f^2}\partial_t + \partial_r \text{ and } \eta = \frac{1}{2}(\partial_t - f^2\partial_r) \tag{4.2}$$

with $\widetilde{g}(\xi, \eta) = -1$. The Levi-Civita connection ∇^B is directly computed from (1.1) as follows

$$\nabla^B_{\partial_t}\partial_t = f^3 f'\partial_r, \qquad \nabla^B_{\partial_t}\partial_r = \frac{f'}{f}\partial_t \qquad \text{and} \qquad \nabla^B_{\partial_r}\partial_r = -\frac{f'}{f}\partial_r$$

where $f' = \partial_r f$. The Gauss curvature of the metric g_B is $K^B = -\left((f')^2 + ff''\right)$. The 1-form defined in (4.1) satisfies

$$\alpha = f'\left(fdt - \frac{1}{f}dr\right). \tag{4.3}$$

In particular, $\alpha(\xi) = \alpha(\eta) = 0$ and then ξ and η are geodesic vector fields. Straightforward computations show for the natural coordinates t and r on B and for the warping function λ that

$$\nabla^B t = -\frac{1}{f^2}\partial_t, \quad \nabla^B r = f^2\partial_r, \quad \nabla^B f = f^2 f'\partial_r \quad \text{and} \quad \nabla^B\lambda = -\frac{\lambda_t}{f^2}\partial_t + f^2\lambda_r\partial_r.$$

Remark 4.5. Let $\mathcal{L}$ be an integral hypersurface of the distribution D_ξ. Recall that the null-Weingarten map b_ξ is defined for every point $p \in \mathcal{L}$ as

$$b_\xi \colon T_x\mathcal{L}/\xi_x \to T_x\mathcal{L}/\xi_x, \quad [v] \mapsto [\widetilde{\nabla}_v\xi],$$

where [ ] denotes the class in the quotient vector space $T_x\mathcal{L}/\xi_x$, see [32]. The lightlike manifold $\mathcal{L}$ is said to be totally geodesic when $b_\xi = 0$. A direct computation from (2.8) gives $b_\xi([v]) = \frac{\xi\lambda}{\lambda}[v]$. This formula implies that $\mathcal{L}$ is a totally geodesic lightlike hypersurface if and only if $\xi\lambda = 0$. In a similar way, the integral hypersurfaces of D_η are totally geodesic lightlike hypersurfaces if and only if $\eta\lambda = 0$.

Remark 4.6. Lorentzian manifolds admitting a global parallel and lightlike vector field ξ were introduced in [13]. Such a Lorentzian manifolds are called Brinkmann spacetimes Hence, a generalized Schwarzschild spacetime is a Brinkmann spacetime if and only if $\alpha = 0$. Recall that a Lorentzian manifold $(\bar{M}, \bar{g})$ is said to be static when admits a Killing vector field K with $\bar{g}(K, K) < 0$. The timelike vector field ∂_t in a generalized Schwarzschild spacetime is Killing if and only if $\lambda_t = 0$. This is the case of the classical Schwarzschild spacetime.

Remark 4.7. In a general setting, given a semi-Riemannian manifold $(\bar{M}, \bar{g})$, a vector field $X \in \mathfrak{X}(\bar{M})$ is said to be recurrent when there is a 1-form α on $\bar{M}$ such that $\bar{\nabla} X = \alpha \otimes X$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$. In particular, equations (4.1) and (2.8) imply that the vector fields ξ and η are recurrent. This property widely generalizes the Brinkmann spacetimes. Lorentzian manifolds with recurrent lightlike vector fields have been studied in [42].

Let M be an $(n \geq 2)$-dimensional manifold such that $m \geq n$. For $\Psi : M \to B \times_\lambda F$ a spacelike immersion in a generalized Schwarzschild spacetime, we have the smooth functions on M given by $u = t \circ \Psi_B$ and $v = r \circ \Psi_B$. We have for the gradients of these functions with respect to the induced metric g that

$$\nabla u = -\frac{1}{(f \circ \Psi_B)^2} \partial_t^\top \quad \text{and} \quad \nabla v = (f \circ \Psi_B)^2 \partial_r^\top \tag{4.4}$$

and therefore

$$\xi^\top = \frac{1}{(f \circ \Psi_B)^2} \nabla v - \nabla u \quad \text{and} \quad \eta^\top = -\frac{1}{2} \left(\nabla v + (f \circ \Psi_B)^2 \nabla u \right). \tag{4.5}$$

These formulas (4.5) lead to the following characterization for spacelike immersions through integral submanifolds of the distributions D_ξ or D_η.

Lemma 4.8. *A spacelike immersion $\Psi : M \to B \times_\lambda F$ in a generalized Schwarzschild spacetime factors through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of the distribution D_ξ (resp. D_η) if and only if*

$$\nabla v = (f \circ \Psi_B)^2 \nabla u \quad (\text{resp.}\, \nabla v = -(f \circ \Psi_B)^2 \nabla u).$$

Remark 4.9. Slices are totally umbilical spacelike embedded immersions (see Section 2.3). Note that every spacelike immersion which factors through an integral hypersurface of D_ξ and, at the same time, through an integral hypersurface of D_η must factors through a slice.

Example 4.10. As was mentioned in the Introduction (see also Example 2.5), the $(m+2)$-dimensional Minkowski spacetime $\mathbb{L}^{m+2}$ can be described in two ways as a generalized Schwarzschild spacetime.

- The first one is $B=\mathbb{R}^2$, $f(r)=1$, $\lambda(t,r)=1$ and $(F,g_F)=\mathbb{E}^m$. The lightlike vector fields in (4.2) are $\xi=\partial_t+\partial_r$ and $\eta=\frac{1}{2}(\partial_t-\partial_r)$. Hence, the leaves of the lightlike foliations are lightlike hyperplanes in $\mathbb{L}^{m+2}$.
- The second one is obtained by taking $B=\mathbb{R}\times\mathbb{R}_{>0}$, $f(r)=1$, $\lambda(t,r)=r$ and $(F,g_F)=\mathbb{S}^m$. The lightlike vector fields in (4.2) are $\xi=\partial_t+\partial_r$ and $\eta=\frac{1}{2}(\partial_t-\partial_r)$. The foliations by lightlike hypersurfaces given in Lemma 4.1 correspond via the isometry (2.9) given in Example 2.5 with the lightlike cones with vertex at the points $(t,0)\in\mathbb{L}^{m+2}$.

4.3 Immersions in generalized Schwarzschild spacetimes

Along this Section

$$\Psi : M \to B\times_\lambda F$$

is a fixed spacelike immersion which does not necessary factors through an integral submanifold of D_ξ or D_η. For every vector field $V\in\mathfrak{X}(M)$ and $x\in M$, we denote

$$V_x^B = T_x\Psi_B\cdot V_x \quad \text{and} \quad V_x^F = T_x\Psi_F\cdot V_x.$$

Since we agree to ignore the differential map of Ψ, this means that $V=V^B+V^F$. We get that

$$V^B = g(V,\nabla u)\partial_t|_\Psi + g(V,\nabla v)\partial_r|_\Psi, \tag{4.6}$$

and from (4.4), we have

$$(V^B)^\top = -(f\circ\Psi_B)^2 V(u)\nabla u + \frac{1}{(f\circ\Psi_B)^2}V(v)\nabla v. \tag{4.7}$$

As a consequence of (2.8), we get

$$\widetilde{\nabla}_V(\xi|_\Psi) = \alpha(V^B)\xi|_\Psi + \Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)V^F.$$

The Gauss and Weingarten formulas (2.1) imply that

$$\nabla_V\xi^\top + \mathrm{II}(V,\xi^\top) - A_{\xi^\perp}V + \nabla_V^\perp\xi^\perp = \alpha(V^B)\xi|_\Psi + \Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)V^F. \tag{4.8}$$

In particular for the tangent parts to M, we get

$$\nabla_V \xi^\top - A_{\xi^\perp} V = \alpha(V^B)\xi^\top + \Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big)(V^F)^\top. \tag{4.9}$$

From (4.7) and taking into account that $(V^F)^T = V - (V^B)^T$, the equation (4.9) reduces to

$$\nabla_V \xi^\top - A_{\xi^\perp} V = \alpha(V^B)\xi^\top + \Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big)\Big(V + (f\circ\Psi_B)^2 V(u)\nabla u - \frac{1}{(f\circ\Psi_B)^2}V(v)\nabla v\Big)$$

and then,

$$\begin{aligned}\operatorname{div}(\xi^\top) - n\,\widetilde{g}(\mathbf{H}, \xi^\perp) = \Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big)\Big[n + (f\circ\Psi_B)^2\|\nabla u\|^2 - \frac{1}{(f\circ\Psi_B)^2}\|\nabla v\|^2\Big] \\ + (\Psi_B^*\alpha)(\xi^\top).\end{aligned} \tag{4.10}$$

Note that $(\Psi_B^*\alpha)(\xi^\top) = \alpha((\xi^\top)^B)$. A straightforward computation from (4.5) and (4.6) gives

$$(\xi^\top)^B = \Big(\frac{1}{(f\circ\Psi_B)^2} g(\nabla u, \nabla v) - \|\nabla u\|^2\Big)\partial_t|_\Psi + \Big(\frac{1}{(f\circ\Psi_B)^2}\|\nabla v\|^2 - g(\nabla u, \nabla v)\Big)\partial_r|_\Psi$$

and then, from (4.3), we have

$$(\Psi_B^*\alpha)(\xi^\top) = -(ff'\circ\Psi_B)\|\xi^\top\|^2.$$

On the other hand, one can compute that

$$\operatorname{trace}_g(\Psi_B^* g_B) = -(f\circ\Psi_B)^2\|\nabla u\|^2 + \frac{1}{(f\circ\Psi_B)^2}\|\nabla v\|^2 = -2g(\xi^\top, \eta^\top),$$

where $\operatorname{trace}_g(\Psi_B^* g_B) = \sum_{i=1}^n (\Psi_B^* g_B)(E_i, E_i)$ for a local orthonormal frame in M.

Remark 4.11. Taking into account that $n = \operatorname{trace}_g(g) = \operatorname{trace}_g(\Psi_B^* g_B) + (\lambda\circ\Psi_B)^2 \operatorname{trace}_g(\Psi_F^* g_F)$, we get $g(\xi^\top, \eta^\top) > -n/2$.

From (4.10), the above computations give the following result.

Lemma 4.12. *Let $\Psi : M \to B \times_\lambda F$ be a spacelike immersion in a generalized Schwarzschild spacetime. Then the following formula holds*

$$\operatorname{div}(\xi^\top) - n\,\widetilde{g}(\mathbf{H}, \xi^\perp) = \Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big)\big[n + 2g(\xi^\top, \eta^\top)\big] - (ff'\circ\Psi_B)\|\xi^\top\|^2.$$

Remark 4.13. We know that $\xi^\top = 0$ for a spacelike immersion which factors through an integral hypersurface $\mathcal{L}$ of D_ξ. In this case, the above Lemma reduces to

$$\widetilde{g}(\mathbf{H}, \xi) = -\frac{\xi\lambda}{\lambda} \circ \Psi_B.$$

The following result provides an integral characterization for compact spacelike immersions through integral hypersurfaces of D_ξ.

Theorem 4.14. *Assume M is compact and $\Psi : M \to B \times_\lambda F$ is a spacelike immersion in a generalized Schwarzschild spacetime with $f' \circ \Psi_B > 0$ (resp. $f' \circ \Psi_B < 0$). Then, we have*

$$\int_M \Big[n\, \widetilde{g}(\mathbf{H}, \xi^\perp) + \Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big) \big[n + 2g(\xi^\top, \eta^\top)\big] \Big]\, d\mu_g \geq 0. \quad (\textit{resp.} \leq 0). \tag{4.11}$$

The equality holds if and only if M factors through an integral hypersurface $\mathcal{L}$ of D_ξ.

Proof. Suppose the case $f' \circ \Psi_B > 0$. The inequality (4.11) is a direct consequence of Lemma 4.12 and the classical divergence theorem. Furthermore, (4.11) becomes an equality if and only if

$$\int_M (f f' \circ \Psi_B) \|\xi^\top\|^2\, d\mu_g = 0.$$

Since $f' \circ \Psi_B > 0$ and the immersion is spacelike, we get $\xi^\top = 0$ and this fact ends the proof. The proof for $f' \circ \Psi_B < 0$ works in a similar way.

$\square$

Remark 4.15. We have for the family of functions given in (1.2) that

$$(ff')(r) = \frac{-2\Lambda r^{2m} + m(m^2-1)(\mathbf{M}r^{m-1} - q^2)}{m(m+1)r^{2m-1}}.$$

Therefore, the assumption $f' \circ \Psi_B > 0$ is satisfied when $\Lambda \leq 0$ and $\mathbf{M}v^{m-1} > q^2$. In particular, it holds for the exterior Schwarzschild spacetime.

We just sketch the proof of Lemma 4.12 for the lightlike vector field η. The condition $\widetilde{g}(\xi, \eta) = -1$ implies $\nabla^B \eta = -\alpha \otimes \eta$ (see Remark 4.3) and then

$$\widetilde{\nabla}_V (\eta|_\Psi) = -\alpha(V^B)\eta|_\Psi + \Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big) V^F.$$

From the Gauss and Weingarten formulas we have

$$\nabla_V \eta^\top + \mathrm{II}(V, \eta^\top) - A_{\eta^\perp} V + \nabla^\perp_V \eta^\perp = -\alpha(V^B)\eta|_\Psi + \Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big) V^F, \tag{4.12}$$

and taking tangent parts, we get

$$\nabla_V \eta^\top - A_{\eta^\perp} V = -\alpha(V^B)\eta^\top + \Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big)(V^F)^\top. \tag{4.13}$$

Hence we have

$$\operatorname{div}(\eta^\top) - n\,\widetilde{g}(\mathbf{H},\eta^\perp) = \Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\left[n + (f\circ\Psi_B)^2\|\nabla u\|^2 - \frac{1}{(f\circ\Psi_B)^2}\|\nabla v\|^2\right]$$
$$- (\Psi_B^*\alpha)(\eta^\top).$$

A straightforward computation from (4.3) shows that

$$(\Psi^*\alpha)(\eta^\top) = \Big(\frac{ff'}{2}\circ\Psi_B\Big)\operatorname{trace}_g(\Psi_B^* g_B).$$

Therefore, the corresponding version of Lemma 4.12 to the vector field η reads as follows.

Lemma 4.16. *Let $\Psi : M \to B\times_\lambda F$ be a spacelike immersion in a generalized Schwarzschild spacetime. Then the following formula holds*

$$\operatorname{div}(\eta^\top) - n\,\widetilde{g}(\mathbf{H},\eta^\perp) = \Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\big[n + 2g(\xi^\top,\eta^\top)\big] + (ff'\circ\Psi_B)g(\xi^\top,\eta^\top). \tag{4.14}$$

When $\Psi(M)$ factors through an integral hypersurface $\mathcal{N}$ of D_η, we have

$$\widetilde{g}(\mathbf{H},\eta) = -\frac{\eta\lambda}{\lambda}\circ\Psi_B.$$

Remark 4.17. From (4.14) and for M compact, we get

$$\int_M \left[n\,\widetilde{g}(\mathbf{H},\eta^\perp) + \Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\big[n + 2g(\xi^\top,\eta^\top)\big]\right]d\mu_g = -\int_M (ff'\circ\Psi_B)g(\xi^\top,\eta^\top)\,d\mu_g.$$

The right-hand side of this integral formula has no prescribed sign although we impose $f'\circ\Psi_B \neq 0$. Hence, a similar result to Theorem 4.14 does not hold for the vector field η.

4.4 Immersions through lightlike integral hypersurfaces

In this Section, we will focus on spacelike immersions $\Psi : M \to B\times_\lambda F$ through lightlike integral hypersurfaces of the distributions D_ξ or D_η. Recall that we write $\mathcal{L}$ (resp. $\mathcal{N}$) for a general integral hypersurface of D_ξ (resp. D_η). The following results specialize several formulas of the above sections for theses cases. For example, the formula (4.5) using Lemma 4.8. From now on until the end of this Chapter we are going to avoid writing the immersion $|_\Psi$.

Lemma 4.18. *Let $\Psi : M \to B\times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. The following formulas hold*

1. $\alpha(V^B) = 0$ *for every $V \in \mathfrak{X}(M)$.*

2. *$\eta^\top = -\nabla v$.*

3. *$V^B = g(V, \nabla v)\xi$. In particular, we get $(V^B)^\top = 0$ and $(V^F)^\top = V$.*

4. *When M has codimension two, the normal bundle $TM^\perp$ is spanned by the normal lightlike vector fields ξ and*
$$\ell^\xi = -\frac{\|\nabla v\|^2}{2}\xi + \eta^\perp.$$
We also have $\widetilde{g}(\xi, \ell^\xi) = -1$.

Lemma 4.19. *Let $\Psi : M \to B\times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{N}$ of D_η. The following formulas hold*

1. *$\alpha(V^B) = -\left(\frac{2f'}{f} \circ \Psi_B\right) g(V, \nabla v)$ for every $V \in \mathfrak{X}(M)$.*

2. *$\xi^\top = \frac{2}{(f \circ \Psi_B)^2}\nabla v$.*

3. *$V^B = -\frac{2}{(f \circ \Psi_B)^2} g(V, \nabla v)\eta$. In particular, we get $(V^B)^\top = 0$ and $(V^F)^\top = V$.*

4. *When M has codimension two, the normal bundle $TM^\perp$ is spanned by the normal null vector fields η and*
$$\ell^\eta = \xi^\perp - \frac{2\|\nabla v\|^2}{(f \circ \Psi_B)^4}\eta.$$
We also have $\widetilde{g}(\eta, \ell^\eta) = -1$.

Remark 4.20. In these cases, the projection $\Psi_F \colon M \to F$ is also an immersion. Indeed, from Lemma 4.18, the equality $T_x\Psi_F \cdot v = 0$ for $x \in M$ with $v \in T_xM$ and $\Psi(M) \subset \mathcal{L}$ give $g(v, v) = 0$ and then $v = 0$. The same argument works for $\mathcal{N}$. For spacelike immersions $\Psi : M \to B \times_\lambda F$ through these lightlike integral hypersurfaces, we have that the induced metric is $g = (\lambda \circ \Psi_B)^2 \Psi_F^*(g_F)$.

Lemma 4.21. *Let $\Psi : M \to B\times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. For every $V \in \mathfrak{X}(M)$, we have*
$$A_\xi V = -\Big(\frac{\xi\lambda}{\lambda} \circ \Psi_B\Big)V \quad \textit{and} \quad A_{\eta^\perp} V = -\Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big)V - \nabla_V \nabla v.$$
In particular, we get $\widetilde{g}(\mathbf{H}, \eta^\perp) = -\frac{\eta\lambda}{\lambda} \circ \Psi_B - \frac{1}{n}\Delta v$.

Proof. Under our assumptions $\xi^\top = 0$ and by means of Lemma 4.18, the equation (4.9) reduces to the announced formula for A_ξ. From Lemma 4.18 we have $\eta^\top = -\nabla v$. Hence, formula (4.13) ends the proof. □

In a similar way we have.

Lemma 4.22. *Let $\Psi : M \to B\times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{N}$ of D_η. For every $V \in \mathfrak{X}(M)$, we have*

$$A_\eta V = -\Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)V \quad \textit{and} \quad A_{\xi^\perp}V = -\Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)V + \frac{2}{(f\circ\Psi_B)^2}\nabla_V\nabla v.$$

In particular, we get $\widetilde{g}(\mathbf{H},\xi^\perp) = -\frac{\xi\lambda}{\lambda}\circ\Psi_B + \frac{2}{n(f\circ\Psi_B)^2}\Delta v$.

In order to avoid ambiguities, we add the following terminology. A function h with values in $\mathbb{R}$ is said to be signed when $h \geq 0$ or $h \leq 0$ on whole domain. The assumptions on the functions $\xi\lambda$ and $\eta\lambda$ in the statements of the following results are only required along Ψ.

Corollary 4.23. *Assume $\eta\lambda$ (resp. $\xi\lambda$) signed and let $\Psi : M \to B\times_\lambda F$ be a compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of the distribution D_ξ (resp. D_η) with $\mathbf{H} = 0$. Then M factors through a slice with $u = t_0$ and $v = r_0$ such that $\nabla^B\lambda(t_0, r_0) = 0$ and M is minimal in F.*

Proof. We give the proof only for the case of D_ξ. From Lemma 4.21, we have

$$\frac{\eta\lambda}{\lambda}\circ\Psi_B + \frac{1}{n}\Delta v = 0.$$

The assumption regarding the sign of $\eta\lambda$ implies that Δv is also signed. As a consequence of Hopf's Theorem (see, for example, [2]), we know that the compactness of M implies that v is a constant function r_0. Furthermore, Lemma 4.8 implies that u is also a constant function t_0. Now, consider the string of smooth maps

$$M \xrightarrow{\Psi_F} F \hookrightarrow B\times_\lambda F,$$

where the second map is the slice at level (t_0, r_0). From [23, Chap. 3], we know that for an orthonormal local frame $(E_1, \cdots, E_n)$ of the induced metric g, we have

$$\mathbf{H} = \mathbf{H}' + \frac{1}{n}\sum_{i=1}^{n}\overline{\mathrm{II}}(T\Psi_F\cdot E_i, T\Psi_F\cdot E_i),$$

where $\mathbf{H}'$ denotes the mean curvature vector field of Ψ_F and $\overline{\mathrm{II}}$ is the second fundamental form of the slice at (t_0, r_0). Therefore, as a consequence of (2.10) and Remark 4.20, we get that

$$\mathbf{H} = \mathbf{H}' - \frac{\nabla^B\lambda}{\lambda}(t_0, r_0).$$

Hence $\mathbf{H} = 0$ provides that $\mathbf{H}' = 0$ and $\nabla^B\lambda(t_0, r_0) = 0$.

□

As a direct consequence of Lemmas 4.21 and 4.22, for a spacelike immersion through an integral hypersurface of D_ξ (resp. D_η), the normal vector field $\eta^\perp$ (resp. $\xi^\perp$) is umbilic if and only if there is $h \in \mathcal{C}^\infty(M)$ such that

$$\nabla_V \nabla v = hV, \tag{4.15}$$

for every $V \in \mathfrak{X}(M)$. Then, in both situations, we have $\mathcal{L}_{\nabla v} g = 2hg$ and therefore ∇v is a conformal vector field in (M, g).

Theorem 4.24. *Let $\Psi : M \to B \times_\lambda F$ be a compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η) with $\mathrm{Ric}^g(\nabla v, \nabla v) \leq 0$, where Ric^g is the Ricci tensor of the induced metric g. Then $\eta^\perp$ (resp. $\xi^\perp$) is an umbilic direction if and only if M factors through a slice.*

Proof. Since ∇v is a conformal vector field and $\mathrm{Ric}^g(\nabla v, \nabla v) \leq 0$ it follows that ∇v is a Killing vector field (see [59, Chap. 5]). This necessarily implies that $h = 0$ in (4.15) and therefore $\Delta v = 0$. Taking into account that M is assumed to be compact, we get that v is a constant function. Furthermore, from Lemma 4.8 the function u must also be a constant. The converse is obvious.

□

Equation (4.15) implies that ∇v is a conformal gradient vector field on the Riemannian manifold (M, g). From a classic result by Obata [48], the existence of such vector fields on compact Riemannian manifolds has been address in [27, Theor. 1]. As a direct consequence of this result, we have.

Theorem 4.25. *Let $\Psi : M \to B \times_\lambda F$ be a compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η) with $\eta^\perp$ (resp. $\xi^\perp$) an umbilic direction. Assume the Ricci tensor of the induced metric g satisfies*

$$0 < \mathrm{Ric}^g \leq (n-1)\left(2 - \frac{nc}{\lambda_1}\right) c,$$

for a constant c where λ_1 is the first non-trivial eigenvalue of the Laplace operator of the metric g. If ∇v is a nonzero vector field, then (M, g) is isometric to the sphere $\mathbb{S}^n(c)$ of constant sectional curvature c.

Remark 4.26. A careful reading of [27, Theor. 1] shows that we only need the above condition on the Ricci tensor for the vector field $\nabla(\frac{\Delta v}{n} + cv)$.

Remark 4.27. From [68, Chap. 1], we know that for every conformal vector field V in an n-dimensional Riemannian manifold, the following formula holds

$$V(S^g) = -\frac{2(n-1)}{n}\Delta(\mathrm{div}(V)) - \frac{2}{n}\mathrm{div}(V) \cdot S^g,$$

where S^g is the scalar curvature of g. Under the assumptions of Theorem 4.25, the vector field ∇v is conformal and the manifold (M, g) is isometric to the sphere $\mathbb{S}^n(c)$. Therefore, the above formula reduces to

$$0 = \Delta(\Delta v) + nc\Delta v.$$

This implies that $\Delta v + nc\, v = k \in \mathbb{R}$. Hence for $w := v - \frac{k}{nc}$, we have that $\Delta w + ncw = 0$ and either $v = \frac{k}{nc}$ or w is an eigenfunction of the Laplace operator of the sphere $\mathbb{S}^n(c)$ corresponding with nc. This is the well-known value of the first non-trivial eigenvalue λ_1 of the Laplace operator of the sphere $\mathbb{S}^n(c)$, [22, Chap. 2]. The space of homogeneous harmonic polynomial of $\mathbb{R}^{n+1}$ of degree 1 restricted to $\mathbb{S}^n(c)$ constitutes the eigenspace corresponding to λ_1. In other words, there is $a \in \mathbb{R}^{n+1}$ such that $v(x) = \langle x, a\rangle + \frac{k}{nc}$, where $\langle\, ,\, \rangle$ is the usual Euclidean inner product.

Proposition 4.28. *Let $\Psi : M \to B \times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. For every $V \in \mathfrak{X}(M)$, we have*

$$\nabla^\perp_V \xi = -\left(\frac{\xi\lambda}{\lambda} \circ \Psi_B\right) g(\nabla v, V)\xi \;\; \textit{and} \;\; \nabla^\perp_V \eta^\perp = -\left(\frac{\eta\lambda}{\lambda} \circ \Psi_B\right) g(\nabla v, V)\xi + \mathrm{II}(\nabla v, V).$$

Proof. The first assertion is a direct computation from equation (4.8) from Lemmas 4.18 and 4.21 taking into account that $V^F - V = -V^B$. On the other hand, Lemmas 4.18 and 4.21 reduce equation (4.12) to

$$-\mathrm{II}(V, \nabla v) + \Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big)V + \nabla^\perp_V \eta^\perp = \Big(\frac{\eta\lambda}{\lambda} \circ \Psi_B\Big)V^F.$$

and again from $V^F - V = -V^B$, the above formula ends the proof.

□

In a similar way we obtain the following lemma.

Proposition 4.29. *Let $\Psi : M \to B \times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{N}$ of D_η. For every $V \in \mathfrak{X}(M)$, we have*

$$\nabla^\perp_V \eta = \frac{2}{(f \circ \Psi_B)^2}\left(ff' \circ \Psi_B + \frac{\eta\lambda}{\lambda} \circ \Psi_B\right) g(\nabla v, V)\eta$$

and

$$\nabla^{\perp}_{V}\xi^{\perp}=\frac{2}{(f\circ\Psi_B)^2}\left[g(\nabla v,V)\left(-\left(ff'\circ\Psi_B\right)\xi^{\perp}+\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)\eta\right)-\mathrm{II}(\nabla v,V)\right].$$

As a direct consequence of Propositions 4.28 and 4.29 we have.

Theorem 4.30. *Let $\Psi : M \to B \times_\lambda F$ be a spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η). Assume $\xi\lambda \neq 0$ (resp. $ff' + \frac{\eta\lambda}{\lambda} \neq 0$), then the following assertions are equivalent*

1. $\nabla^{\perp}_{V}\xi = 0$ (*resp.* $\nabla^{\perp}_{V}\eta = 0$) *for every $V \in \mathfrak{X}(M)$.*
2. *M factors through a slice.*

Remark 4.31. In order to study the applicability of this result to the case $B \times_r \mathbb{S}^m$ where $f^2(r)$ is given in (1.2), recall $\xi\lambda = 1$ and a direct computation shows that $ff' + \frac{\eta\lambda}{\lambda} \neq 0$ if and only if

$$-2\Lambda r^{m+1} + m(m+1)(2m\mathbf{M} - r^{m-1} - (2m-1)q^2 r^{1-m}) \neq 0.$$

For the exterior Schwarzschild spacetime with mass $\mathbf{M}$, we have

$$ff' + \frac{\eta\lambda}{\lambda} = \frac{2m\mathbf{M} - r^{m-1}}{2r^m}.$$

Therefore, the assumption in Theorem 4.30 is always satisfied for the distribution D_ξ but the condition for D_η holds if and only if the value $2m\mathbf{M}$ is not achieved for the function $v^{m-1} \in C^\infty(M)$.

4.5 Codimension two spacelike immersions

From now on, we assume $m = n$, that is, the spacelike immersion $\Psi : M \to B \times_\lambda F$ has codimension two. We begin this Section with a topological result on such spacelike immersions.

Proposition 4.32. *Let $\Psi : M \to B \times_\lambda F$ be a codimension two compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η). Then the map $\Psi_F \colon M \to F$ is a covering map. In particular, F is also compact and when F is simply-connected, Ψ_F is a diffeomorphism.*

Proof. We give the proof only for the case of the distribution D_ξ. We claim that the map $\Psi_F \colon M \to F$ is a local diffeomorphism. Indeed, we know that $\Psi_F \colon M \to F$ is an immersion between manifolds of the same dimension. The compactness of M and the connectedness of F imply that Ψ_F is a covering map (see [28, Proposition 5.6.1] for details). □

Remark 4.33. Under the assumptions of Theorem 4.25, if F is simply-connected necessarily must be a topological sphere.

Remark 4.34. The map $\Psi_F \colon M \to F$ is not a Riemannian covering, in general. In fact, as was mentioned for $\Psi = (\Psi_B, \Psi_F)$, the induced metric on M is given by $g = (\lambda \circ \Psi_B)^2 \Psi_F^*(g_F)$. Also, taking into account the relation between the scalar curvatures of two conformally related metrics (see formula (2.12)), we have that

$$S^{\Psi_F^*(g_F)} = (\lambda \circ \Psi_B)^2 \left(S^g + 2(n-1)\Delta \log(\lambda \circ \Psi_B) - (n-2)(n-1)\|\nabla \log(\lambda \circ \psi_B)\|^2\right),$$

where $S^{\Psi_F^*(g_F)}$ and S^g are the scalar curvatures of $\Psi_F^*(g_F)$ and g, respectively. For $\lambda(t,r) = r$, we have $\lambda \circ \Psi_B = v$ and from a straightforward computation the above formula reads as follows

$$S^{\Psi_F^*(g_F)} = v^2 \left(S^g + \frac{2(n-1)}{v}\Delta v - \frac{n(n-1)}{v^2}\|\nabla v\|^2\right). \tag{4.16}$$

Proposition 4.35. *Assume $\lambda(t,r) = r$ and let $\Psi : M \to B \times_\lambda F$ be a compact codimension two spacelike immersion through an integral hypersurface of D_ξ or D_η. Then*

$$\int_M \left(S^{\Psi_F^*(g_F)} - v^2 S^g\right) d\mu_g \leq 0$$

and the equality holds if and only if M factors through a slice.

Proof. From (4.16), a direct computation gives that

$$\int_M \left(S^{\Psi_F^*(g_F)} - v^2 S^g\right) d\mu_g = (n-1)\int_M (2v\Delta v - n\|\nabla v\|^2) d\mu_g.$$

Taking into account that $\Delta v^2 = 2v\Delta v + 2\|\nabla v\|^2$, the above formula reduces to

$$\int_M \left(S^{\Psi_F^*(g_F)} - v^2 S^g\right) d\mu_g = -(n-1)(n+2)\int_M \|\nabla v\|^2 d\mu_g.$$

Then, Lemma 4.8 ends the proof. □

As a direct consequence of Corollary 4.23 and Proposition 4.32 we have.

Corollary 4.36. *Assume $\eta\lambda$ (resp. $\xi\lambda$) signed. Then every codimension two compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η) with $\mathbf{H} = 0$ factors through a slice and $\Psi_F \colon (M, g) \to (F, \lambda(t_0, r_0)^2 g_F)$ is a Riemannian covering space.*

Proposition 4.37. *Let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η). Then the normal bundle $TM^\perp$ is* *spanned by the vector fields $\xi^\perp$ and $\eta^\perp$. When M factors through an integral hypersurface $\mathcal{L}$ of D_ξ, the mean curvature vector field is*

$$\mathbf{H} = \left[\frac{\eta\lambda}{\lambda}\circ\Psi_B - \Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)\|\nabla v\|^2 + \frac{1}{n}\Delta v\right]\xi + \Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)\eta^\perp \tag{4.17}$$

and when M factors through an integral hypersurface $\mathcal{N}$ of D_η we have

$$\mathbf{H} = \Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\xi^\perp + \left[\frac{\xi\lambda}{\lambda}\circ\Psi_B - \Big(\frac{4\,\eta\lambda}{\lambda f^4}\circ\Psi_B\Big)\|\nabla v\|^2 - \frac{2}{n(f\circ\Psi_B)^2}\Delta v\right]\eta.$$

Proof. The assertion on the normal bundle is a direct consequence of $\xi^\perp = \xi$ (resp. $\eta^\perp = \eta$) and $\widetilde{g}(\xi,\eta) = -1$. Hence, there are smooth functions $a, b \in C^\infty(M)$ such that

$$\mathbf{H} = a\xi + b\eta^\perp$$

where $b = -\widetilde{g}(\mathbf{H},\xi)$ and from Lemma 4.18 we can compute that $a = \widetilde{g}(\mathbf{H}, \|\nabla v\|^2\xi - \eta^\perp)$. Now formula (2.2) and Lemma 4.21 imply that

$$\widetilde{g}(\mathbf{H},\xi) = -\frac{\xi\lambda}{\lambda}\circ\Psi_B \quad \text{and} \quad \widetilde{g}(\mathbf{H},\eta^\perp) = -\frac{\eta\lambda}{\lambda}\circ\Psi_B - \frac{1}{n}\Delta v.$$

This completes the proof for the case of spacelike immersions through integral hypersurfaces of D_ξ. Slight changes in the proof show the formula for the mean curvature vector field in the case D_η.

□

Under the same assumptions of Proposition 4.37 and from Lemmas 4.18 and 4.19, we have.

Corollary 4.38. *For a spacelike immersion $\Psi : M \to B\times_\lambda F$ which factors through an integral hypersurface $\mathcal{L}$ of D_ξ, we have*

$$\mathbf{H} = \left[\frac{\eta\lambda}{\lambda}\circ\Psi_B - \Big(\frac{\xi\lambda}{2\lambda}\circ\Psi_B\Big)\|\nabla v\|^2 + \frac{1}{n}\Delta v\right]\xi + \Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)\ell^\xi.$$

In case that M factors through an integral hypersurface $\mathcal{N}$ of D_η,

$$\mathbf{H} = \Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\ell^\eta + \left[\frac{\xi\lambda}{\lambda}\circ\Psi_B - \Big(\frac{2\,\eta\lambda}{\lambda f^4}\circ\Psi_B\Big)\|\nabla v\|^2 - \frac{2}{n(f\circ\Psi_B)^2}\Delta v\right]\eta.$$

Remark 4.39. This Corollary extends the formulas for the mean curvature vector field of codimension two spacelike immersions through lightlike hyperplanes and cones in the Minkowski spacetime, [4] and [56]. For immersions through lightlike hyperplanes, we particularize Corollary 4.38 for $f(r) = 1$ and $\lambda(t,r) = 1$ (see Example 4.10), then $\mathbf{H} = \frac{1}{n}\Delta v\,\xi$ for M in the lightlike hyperplane

$$\Pi_\xi := \{x \in \mathbb{L}^{n+2} : \langle x, \xi\rangle = 0\}.$$

Similarly, we obtain $\mathbf{H} = -\frac{2}{n}\Delta v\, \eta$ when M factors through the lightlike hyperplane Π_η. It can be easily seen that our formulas for $\mathbf{H}$ coincides with the formula (8.2) given in [4, Sect. 8].

For immersions through lightlike cones we need to take $f(r) = 1$ and $\lambda(t, r) = r$, then we compute the mean curvature as follows

$$\mathbf{H} = \Big(\frac{-1 - \|\nabla v\|^2}{2v} + \frac{\Delta v}{n}\Big)\xi + \frac{1}{v}\ell^\xi, \quad D_\xi \text{ case}$$

and

$$\mathbf{H} = -\frac{1}{2v}\ell^\eta + \Big(\frac{1 + \|\nabla v\|^2}{v} - \frac{2\Delta v}{n}\Big)\eta, \quad D_\eta \text{ case}.$$

Theses formulas agree with [56] and [4, Sect. 6]. In fact, taking $\Lambda^+ \subset \mathbb{L}^{n+2}$ the future lightlike cone with vertex at $0 \in \mathbb{L}^{n+2}$, we know from (2.9) that $\hat{\xi} := r\xi$ satisfies $T\phi \cdot \hat{\xi}$ is the position vector field in Λ^+. Therefore, if we rescale $\hat{l} := \frac{1}{r}l^\xi$, the first formula of $\mathbf{H}$ expressed in terms of $\hat{\xi}$ and $\hat{l}$ coincides with the formula given in [56] and [4, Sect. 6].

Remark 4.40. Assume the warping function λ depends only on the radial coordinate r. According to (4.17), the existence of a codimension two spacelike immersion with $\mathbf{H} = 0$ through an integral hypersurface $\mathcal{L}$ of D_ξ implies $\lambda_r \circ \Psi_B = 0$. In particular, there are no such spacelike immersions in the exterior Schwarzschild spacetime with mass $\mathbf{M}$. The same result remains true for spacelike immersions through an integral hypersurface $\mathcal{N}$ of D_η.

Remark 4.41. Let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ with $\lambda(t, r) = r$. Proposition 4.28 and Corollary 4.38 give that the normal lightlike vector field $\ell := v\,\xi$ satisfies

$$\widetilde{g}(\ell, \mathbf{H}) = -1 \quad \text{and} \quad \nabla^\perp \ell = 0. \tag{4.18}$$

In the case that the spacelike immersion lies in an integral hypersurface $\mathcal{N}$ of D_η, the lightlike normal vector field $\ell := \frac{-2v}{(f\circ\Psi_B)^2}\eta$ satisfies $\widetilde{g}(\ell, \mathbf{H}) = -1$ and $\nabla^\perp \ell = 0$. Assuming that M is compact and F is the round sphere, the existence of a normal lightlike vector field ℓ such that (4.18) holds implies that M lies in an integral lightlike hypersurface of D_ξ or D_η, [65]. The authors of [65] call these hypersurfaces as null hypersurfaces of symmetry. In other words, the null hypersurfaces generated by the round sphere.

Taking into account formulas (2) in Lemmas 4.18 and 4.19, we obtain the following result from Proposition 4.37.

Corollary 4.42. *Under the same assumptions of Proposition 4.37, for a spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ, we have*

$$\|\mathbf{H}\|^2 = -\frac{2\,\eta\lambda\,\xi\lambda}{\lambda^2}\circ\Psi_B - \Big(\frac{2\,\xi\lambda}{n\lambda}\circ\Psi_B\Big)\Delta v + \Big(\frac{\xi\,\lambda}{\lambda}\circ\Psi_B\Big)^2\|\nabla v\|^2$$

and, for a spacelike immersion through an integral hypersurface $\mathcal{N}$ of D_η, we have

$$\|\mathbf{H}\|^2 = -\frac{2\,\eta\lambda\,\xi\lambda}{\lambda^2}\circ\Psi_B + \Big(\frac{4\,\eta\lambda}{n\lambda f^2}\circ\Psi_B\Big)\Delta v + 4\Big(\frac{\eta\,\lambda}{\lambda f^2}\circ\Psi_B\Big)^2\|\nabla v\|^2,$$

where $\|\mathbf{H}\|^2$ denotes $\widetilde{g}(\mathbf{H},\mathbf{H})$.

Corollary 4.43. *Assume $\xi\lambda \neq 0$ (resp. $\eta\lambda \neq 0$). Then, every codimension two compact spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η) with*

$$\|\mathbf{H}\|^2 = -\frac{2\,\eta\lambda\,\xi\lambda}{\lambda^2}\circ\Psi_B$$

factors through a slice.

Proof. We give the proof only for the case of D_ξ. As a consequence of Corollary 4.42, we have

$$\Big(\frac{2\,\xi\lambda}{n\lambda}\circ\Psi_B\Big)\Delta v = \Big(\frac{\xi\,\lambda}{\lambda}\circ\Psi_B\Big)^2\|\nabla v\|^2.$$

Since $\xi\lambda\circ\Psi_B \neq 0$, we deduce that Δv is also signed. We can now proceed analogously to the proof of Theorem 4.24.

□

Remark 4.44. Assume $\lambda(t,r) = r$. Corollary 4.42 provides a formula which relates the mean curvature vector field and the scalar curvatures $S^{\Psi_F^*(g_F)}$ and S^g. In fact, for a codimension two spacelike immersion through an integral hypersurface of D_ξ or D_η, one computes from (4.16) that

$$\|\mathbf{H}\|^2 = \frac{1}{v^2}\left((f\circ\Psi_B)^2 - \frac{S^{\Psi_F^*(g_F)} - v^2 S^g}{n(n-1)}\right). \tag{4.19}$$

If we specialize this formula for the case $n = 2$, we get that

$$\|\mathbf{H}\|^2 = \frac{1}{v^2}\left((f\circ\Psi_B)^2 - K^{\Psi_F^*(g_F)} + v^2 K^g\right).$$

Hence, if we assume M compact, the Gauss-Bonnet formula implies

$$\int_{M^2}\|\mathbf{H}\|^2\,d\mu_g = \int_{M^2}\frac{(f\circ\Psi_B)^2}{v^2}\,d\mu_g,$$

where $d\mu_g$ is the canonical measure associated to the metric g.

Let us recall the following terminology in General Relativity.

Definition 4.45. Let $\Psi : M \to (\widetilde{M}, \widetilde{g})$ be a codimension two spacelike immersion of a manifold M in a spacetime $(\widetilde{M}, \widetilde{g})$. M is said to be trapped when $\mathbf{H}$ is timelike. M is called marginally (resp. weakly) trapped if $\mathbf{H}$ is lightlike (resp. causal) on M.

Remark 4.46. Assume $\lambda(t,r) = r$. Under the hypotheses of Corollary 4.42, we have $\|\mathbf{H}\|^2 \leq 0$ for a spacelike immersion through an integral hypersurface of D_ξ or D_η if and only if

$$\frac{2}{n}\Delta v \geq \frac{(f \circ \Psi_B)^2 + \|\nabla v\|^2}{v}.$$

In particular, there are no compact weakly trapped immersions in this case. Taking into account that for this choice of λ, the manifold $B \times_\lambda F$ is stationary, this result is only a particular case of [43, Theor. 2]. In fact, recall that [43, Theor. 2] states that there is no compact weakly trapped immersions in a stationary spacetime.

From Remark 4.40, there is no point on M where $\mathbf{H} = 0$ and then, from (4.16), we have.

Corollary 4.47. *Assume $\lambda(t,r) = r$ and let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface of D_ξ or D_η. Then, the following assertions are equivalent.*

1. *M is marginally trapped.*

2. *The function v satisfies the equation*

$$2v\Delta v - n\Big[(f \circ \Psi_B)^2 + \|\nabla v\|^2\Big] = 0.$$

3. *The scalar curvature of M satisfies*

$$S^{\Psi_F^*(g_F)} = v^2 S^g + n(n-1)(f \circ \Psi_B)^2.$$

Definition 4.48. *An immersion $\Psi : F \to B \times_\lambda F$ is said to be an (entire) spacelike graph on F when*

$$\Psi(x) = (\Psi_B(x), x)$$

and the induced metric $\Psi^*(\widetilde{g})$ is Riemannian.

Recall that if a spacelike graph on F factors through an integral hypersurface of D_ξ or D_η, the induced metric is $g = (\lambda \circ \Psi_B)^2 g_F$. Assume $\lambda(t,r) = r$. Taking into account that $\Psi_F = \mathrm{Id}_F$ for spacelike graphs, formula (4.19) implies that for every graph factoring through an integral hypersurface of D_ξ or D_η, the mean curvature vector field $\mathbf{H}$ and the scalar curvatures S^{g_F} and S^g are related by

$$\|\mathbf{H}\|^2 = \frac{1}{v^2}\left((f \circ \Psi_B)^2 - \frac{S^{g_F} - v^2 S^g}{n(n-1)}\right). \tag{4.20}$$

Remark 4.49. In the particular case of the exterior Schwarzschild spacetime with mass $\mathbf{M}$, the above formula reduces to

$$\|\mathbf{H}\|^2 = \frac{S^g}{n(n-1)} - \frac{2\mathbf{M}}{v^{n+1}}.$$

Then, a spacelike graph factoring through a lightlike hypersurface of D_ξ or D_η is marginally trapped if and only if $S^g = \frac{2\mathbf{M}n(n-1)}{v^{n+1}}$. Also, taking into account the second description of the Minkowski spacetime in Example 4.10, a spacelike graph factoring through a lightlike cone in the Minkowski spacetime is marginally trapped if and only if $S^g = 0$.

Theorem 4.50. *Assume λ depends only on the radial coordinate r and F is a non-compact parabolic Riemannian manifold (see Definition 4.4). Let $\Psi : F \to B \times_\lambda F$ be a spacelike graph through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_ξ (resp. D_η) with $\mathbf{H} = 0$. Then, $\Psi(F)$ is a totally geodesic slice.*

Proof. We give the proof only for $\mathcal{L}$ since the case of $\mathcal{N}$ is similar. From formula (4.17), we get that $\lambda \circ \Psi_B$ is a constant $k \in \mathbb{R}_{>0}$ and $\Delta v = 0$. Then (4.20) implies that

$$\frac{(k^2 - v^2)S^g}{n(n-1)} = (f \circ \Psi_B)^2.$$

There are two possibilities. The first one is $S^g > 0$ and $v^2 < k^2$, then the parabolicity of F gives that v is constant. The other possibility is $S^g < 0$ and $k^2 < v^2$. Therefore, $-v < -k$ or $v < -k$ with $\Delta v = 0$, again the parabolicity of F shows that v is constant. Now, formula (2.10) ends the proof. □

4.6 Parallel mean curvature

Lemma 4.51. *Let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. For every $V \in \mathfrak{X}(M)$, we have*

$$\widetilde{g}(\nabla^\perp_V \mathbf{H}, \xi) = -g(\nabla v, V)\left(\frac{\xi(\xi\lambda)}{\lambda} \circ \Psi_B\right).$$

For the case of an integral hypersurface $\mathcal{N}$ of D_η, we have

$$\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\eta)=2g(\nabla v,V)\left(\frac{\eta(\eta\lambda)+ff'\,\eta\lambda}{\lambda f^2}\circ\Psi_B\right).$$

When $\lambda(t,r)=r$, the above formulas reduce to $\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\xi)=0$ and $\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\eta)=0$, respectively.

Proof. From Propositions 4.28 and 4.37, we derive

$$\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\xi)=-V\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)-\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)^2 g(\nabla v,V). \tag{4.21}$$

A direct computation from Lemma 4.18 shows that

$$V\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)=V^B\left(\frac{\xi\lambda}{\lambda}\right)=g(\nabla v,V)\Big(\xi\left(\frac{\xi\lambda}{\lambda}\right)\circ\Psi_B\Big).$$

Substituting this formula in (4.21), we get

$$\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\xi)=-V\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)-\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)^2 g(\nabla v,V)=-g(\nabla v,V)\Big(\frac{\xi(\xi\lambda)}{\lambda}\circ\Psi_B\Big).$$

In a similar way, from Propositions 4.29 and 4.37, we have

$$\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\eta)=-V\left(\frac{\eta\lambda}{\lambda}\circ\Psi_B\right)+\frac{2}{(f\circ\Psi_B)^2}\left(\frac{\eta\lambda}{\lambda}\circ\Psi_B\right)\left(ff'\circ\Psi_B+\frac{\eta\lambda}{\lambda}\circ\Psi_B\right)g(\nabla v,V)$$

and Lemma 4.19 ends the proof. □

Theorem 4.52. *Assume the warping function satisfies $\xi(\xi\lambda)\circ\Psi_B\neq 0$ at every point and let $\Psi: M\to B\times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. Then, the following assertions are equivalent*

1. $\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\xi)=0$ *for every $V\in\mathfrak{X}(M)$.*

2. *M factors through a slice.*

3. $\nabla^{\perp}\mathbf{H}=0$.

Proof. Assume that $\widetilde{g}(\nabla^{\perp}_{V}\mathbf{H},\xi)=0$. Then from Lemma 4.51 it is directly follows that v is a constant function. From Lemma 4.8, the function u is also constant and then M factors through a slice. The mean curvature vector field of an immersion Ψ which factors through a slice is computed from (4.17) as follows

$$\mathbf{H}=\Big(\frac{\eta\lambda}{\lambda}\circ\Psi_B\Big)\xi+\Big(\frac{\xi\lambda}{\lambda}\circ\Psi_B\Big)\eta.$$

Hence as a direct consequence of Lemma 4.18 and Proposition 4.28, we get $\nabla^{\perp}\mathbf{H}=0$. The rest of the proof is obvious.

□

In a similar way we have.

Theorem 4.53. *Assume* $\big(\eta(\eta\lambda) + ff'\eta\lambda\big) \circ \Psi_B \neq 0$ *at every point and let* $\Psi : M \to B \times_\lambda F$ *be a codimension two spacelike immersion through an integral hypersurface* $\mathcal{N}$ *of* D_η*. Then, the following assertions are equivalent*

1. $\widetilde{g}(\nabla^\perp_V \mathbf{H}, \eta) = 0$ *for every* $V \in \mathfrak{X}(M)$.
2. M *factors through a slice.*
3. $\nabla^\perp \mathbf{H} = 0$.

Remark 4.54. The proof of Theorems 4.52 and 4.53 does not work for $\lambda(t,r) = r$. In this case, Lemma 4.51 gives $\widetilde{g}(\nabla^\perp_V \mathbf{H}, \xi) = 0$ for a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. Hence, the mean curvature vector field is parallel if and only if $\widetilde{g}(\nabla^\perp_V \mathbf{H}, \eta^\perp) = 0$. Let us note that this last equation is equivalent to $\|\mathbf{H}\|^2$ being a constant function. A similar result is achieved for codimension two spacelike immersions through an integral hypersurface $\mathcal{N}$ of D_η.

Chapter 5

Normal tractor conformal bundles

For every codimension two spacelike immersion in a Lorentzian manifold and each choice of a normal lightlike vector field, we introduce a canonical way to construct a tractor conformal bundle. We characterize when the induced connection of a such immersion defines a tractor connection and then, in this case, when this tractor conformal bundle with the induced connection is standard for the induced metric. Finally, the normality conditions for this tractor conformal bundle, endowed with the induced connection, are characterized in terms of a strong relationship between the intrinsic and the extrinsic geometry of the starting spacelike immersion.

5.1 Tractor conformal bundles induced via spacelike immersions

Let M be an $(n \geq 2)$-dimensional manifold and $(\widetilde{M}, \widetilde{g})$ be an $(n+2)$-dimensional Lorentzian manifold. Throughout this Chapter we are interested in working with an arbitrary codimension two spacelike immersion $\Psi : M \to (\widetilde{M}, \widetilde{g})$ simply assuming that it admits a lightlike normal vector field $\xi \in \mathfrak{X}^{\perp}(M)$ along Ψ. Under these assumptions, there is a natural choice of a tractor conformal bundle on M as follows, see Defintion 3.7. The vector bundle is $\mathcal{T} \to M$ where $\mathcal{T} = \Psi^*(T\widetilde{M})$, that is, $\mathcal{T} \to M$ is the pullback bundle (or induced bundle) via Ψ of the tangent bundle $T\widetilde{M} \to \widetilde{M}$. The Lorentzian bundle metric h is the metric $\widetilde{g}$ and the distinguished lightlike line subbundle $\mathcal{T}^1$ is given by $\mathrm{Span}(\xi)$. As usual, we denote the induced connection by $\widetilde{\nabla}$. It is well-known that $\widetilde{\nabla}\widetilde{g} = 0$. Taking into account that $\Psi : M \to (\widetilde{M}, \widetilde{g})$ is a codimension two spacelike immersion, there is $\omega \in \Omega^1(M, \mathbb{R})$ such that $\nabla^{\perp}\xi = \omega \otimes \xi$.

Proposition 5.1. *Let $\Psi : M \to (\widetilde{M}, \widetilde{g})$ be a spacelike immersion and consider $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ as above. Then, $\widetilde{\nabla}$ is a tractor connection (Definition 3.8) if and only if the Weingarten endomorphism A_ξ is non-singular at every point $x \in M$. In this case, the induced metric g on M by Ψ is in the conformal class of the Riemannian conformal structure deduced from the tractor connection $\widetilde{\nabla}$ if and only if there is a smooth function $\mu \in C^\infty(M)$ such that $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$ (where μ is a nonvanishing function on M). Under these circumstances, $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is a standard tractor conformal bundle for (M, c), where c is the conformal class of g.*

Proof. As was mentioned, the condition $\widetilde{\nabla}\widetilde{g} = 0$ always holds. In our case, by means of the Weingarten formula, the vector bundle homomorphism β given by (3.7) is determined by

$$\beta(V)(\xi) = \widetilde{\nabla}_V \xi + \mathcal{T}^1 = -A_\xi(V) + \omega(V)\xi + \mathcal{T}^1 = -A_\xi(V) + \mathcal{T}^1, \tag{5.1}$$

where $V \in \mathfrak{X}(M)$. That is, $\beta_\xi(V) = -A_\xi(V) + \mathcal{T}^1$. From (5.1), it is clear that β is an isomorphism of vector bundles if and only if A_ξ is non-singular at all points of M. In such case, from formulas (3.6) and (5.1), every section $\sigma = h\,\xi \in \Gamma(\mathcal{T}^1)$ (where h is a nonvanishing function on M) produces the Riemannian metric

$$h^2\, g\big(A_\xi(V), A_\xi(W)\big) = h^2\, g\big(A_\xi^2(V), W\big), \tag{5.2}$$

where $V, W \in \mathfrak{X}(M)$. Then, the metric given in (5.2) belongs to the conformal class of the induced metric g if and only if there is $\mu \in C^\infty\big(M, \mathbb{R} \setminus \{0\}\big)$ with $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$. □

Definition 5.2. *Let $\Psi : M \to (\widetilde{M}, \widetilde{g})$ be a spacelike immersion and $\xi \in \mathfrak{X}^\perp(M)$ a fixed lightlike normal vector field. Assume there is a nonvanishing smooth function $\mu \in C^\infty(M)$ such that $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$. The standard tractor conformal bundle $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ for (M, c), where $\mathcal{T} = \Psi^*(T\widetilde{M})$ and $\mathcal{T}^1 = \mathrm{Span}(\xi)$, is said to be the associated tractor to the pair $(\Psi : M \to \widetilde{M}, \xi)$.*

Remark 5.3. For associated tractors to codimension two spacelike immersions as above, the induced metric satisfies $g = \widetilde{g}^{\frac{1}{\mu}\xi}$, according to the notation in (3.6). Moreover, any metric $e^{2u}g$ in the conformal class c of g satisfies $e^{2u}g = \widetilde{g}^{\frac{e^u}{\mu}\xi}$.

Under the assumptions of Proposition 5.1 and taking into account that M is assumed to be connected, there are two mutually disjoint possibilities. Namely, ξ is an umbilic direction, that is, $A_\xi = \mu \cdot \mathrm{Id}$ on M or the tensor field $P := \frac{1}{\mu} \cdot A_\xi$ defines an almost product structure (with

$P \neq \pm \mathrm{Id}$) on the Riemannian manifold (M, g). That is, we have $P^2 = \mathrm{Id}$ and $g(PV, PW) = g(V, W)$ for any $V, W \in \mathfrak{X}(M)$.

Recall that Ric^g and S^g are the Ricci tensor and the scalar curvature on (M, g), respectively. For a fixed almost product structure (M, P, g) on a Riemannian manifold, the P-Ricci tensor Ric^* and the P-scalar curvature S^* are defined by (see [44])

$$\mathrm{Ric}^*(V, W) = \sum_{i=1}^{n} g\left(R(E_i, V)W, PE_i\right), \quad S^* = \sum_{i=1}^{n} \mathrm{Ric}^*(E_i, E_i), \quad V, W \in \mathfrak{X}(M),$$

where $(E_1, \ldots, E_n)$ is a local orthonormal frame.

In order to shorten the statement of the following result, we will write $P = \frac{1}{\mu} A_\xi$ in both cases, namely, when ξ is an umbilic direction (i.e., $P = \mathrm{Id}$) and for the (proper) almost product structure. Of course, when ξ is an umbilic direction, we have $\mathrm{Ric}^* = \mathrm{Ric}^g$ and $S^* = S^g$. From Theorem 3.22, we arrive to the following result.

Theorem 5.4. *Let $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ be the associated tractor to the pair $(\Psi : M \to \widetilde{M}, \xi)$ with $A_\xi^2 = \mu^2 \cdot \mathrm{Id}$, where μ is a nonvanishing function on M. Assume there is a lightlike normal vector field $\ell \in \mathfrak{X}^\perp(M)$ such that $\widetilde{g}(\xi, \ell) = -1$. Then, $\widetilde{\nabla}$ is a normal tractor connection (Theorem 3.22) if and only if the following conditions hold:*

1. *For every $V, W \in \mathfrak{X}(M)$, we have*

$$(\nabla_V A_\xi)(W) - (\nabla_W A_\xi)(V) = \omega(V) A_\xi W - \omega(W) A_\xi V. \tag{5.3}$$

2. *For every $V, W \in \mathfrak{X}(M)$, the P-Ricci tensor satisfies*

$$\mathrm{Ric}^*(V, W) = -\mathrm{trace}(A_\xi \circ A_\ell)\, g(V, PW) - (n-2)\mu\, g(V, A_\ell W). \tag{5.4}$$

Proof. A direct computation shows for the curvature tensor of $\widetilde{\nabla}$,

$$\begin{aligned}\widetilde{R}(V, W)\xi &= \widetilde{\nabla}_V \widetilde{\nabla}_W \xi - \widetilde{\nabla}_W \widetilde{\nabla}_V \xi - \widetilde{\nabla}_{[V,W]} \xi \\ &= d\omega(V, W)\xi - \nabla_V(A_\xi W) - \mathrm{II}(V, A_\xi W) - \omega(W) A_\xi V \\ &\quad + \nabla_W(A_\xi V) + \mathrm{II}(A_\xi V, W) + \omega(V) A_\xi W + A_\xi([V, W]),\end{aligned}$$

where $V, W \in \mathfrak{X}(M)$. Also we have

$$\widetilde{g}(-\mathrm{II}(V, A_\xi W) + \mathrm{II}(A_\xi V, W), \xi) = -g(A_\xi V, A_\xi W) + g(A_\xi V, A_\xi W) = 0.$$

Now, taking into account that M is a codimension two spacelike immersion, the term $-\mathrm{II}(V, A_\xi W)+\mathrm{II}(A_\xi V, W)$ must be collinear with ξ. Therefore, the first normality condition for $\widetilde{\nabla}$ reduces to

$$-\nabla_V(A_\xi W) - \omega(W)A_\xi V + \nabla_W(A_\xi V) + \omega(V)A_\xi W + A_\xi([V,W]) = 0$$

which is equivalently written as (5.3).

Assume now equation (5.3) holds. From (5.1) and Remark 5.3, the second normality condition for $\widetilde{\nabla}$ is equivalent to

$$\sum_{i=1}^n g\Big(\mathcal{W}(E_i,V)W, E_i\Big) = \frac{1}{\mu^2}\sum_{i=1}^n \widetilde{g}\Big(\widetilde{R}(E_i,V)A_\xi W, A_\xi E_i\Big) = 0,$$

where $(E_1,\dots,E_n)$ is a local orthonormal frame with respect to g. By means of the Gauss equation (2.3), we have

$$\begin{aligned}\sum_{i=1}^n \widetilde{g}\Big(\widetilde{R}(E_i,V)A_\xi W, A_\xi E_i\Big) = \sum_{i=1}^n g\Big(R(E_i,V)A_\xi W, A_\xi E_i\Big) + \widetilde{g}\Big(\mathrm{II}(E_i,A_\xi W), \mathrm{II}(V, A_\xi E_i)\Big)\\ - \widetilde{g}\Big(\mathrm{II}(E_i, A_\xi E_i), \mathrm{II}(V, A_\xi W)\Big).\end{aligned}$$

Now, taking into account (2.6), straightforward computations show that

$$\sum_{i=1}^n \widetilde{g}\Big(\mathrm{II}(E_i,A_\xi W), \mathrm{II}(V, A_\xi E_i)\Big) = -2\mu^2 g(V, A_\ell(A_\xi W))$$

and

$$\sum_{i=1}^n \widetilde{g}\Big(\mathrm{II}(E_i,A_\xi E_i), \mathrm{II}(V, A_\xi W)\Big) = -\mu^2\Big(\mathrm{trace}(A_\xi \circ A_\ell)g(V,W) + ng(V, A_\ell(A_\xi W))\Big).$$

On the other hand, for the curvature term we get

$$\sum_{i=1}^n g\Big(R(E_i,V)A_\xi W, A_\xi E_i\Big) = \mu^2\sum_{i=1}^n g\Big(R(E_i,V)PW, PE_i\Big) = \mu^2 \mathrm{Ric}^*(V, PW).$$

Therefore, the second normality condition for $\widetilde{\nabla}$ writes as follows

$$\mathrm{Ric}^*(V,PW) + \mathrm{trace}(A_\xi \circ A_\ell)g(V,W) + (n-2)g(V, A_\ell(A_\xi W)) = 0.$$

Since $P^2 = \mathrm{Id}$, one easily shows that this formula is equivalent to (5.4). □

Remark 5.5. Although the conditions (5.3) and (5.4) make sense for two dimensional Riemannian conformal structures (i.e., $n = 2$), in such a case, neither of the conditions provides uniqueness for the tractor connection.

Remark 5.6. For a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ of constant sectional curvature, the equation (5.3) is a direct consequence of the Codazzi equation. Therefore, for a codimension two spacelike immersion $\Psi : M \to (\widetilde{M}, \widetilde{g})$ with a fixed lightlike normal vector field $\xi \in \mathfrak{X}^{\perp}(M)$, where $(\widetilde{M}, \widetilde{g})$ has constant sectional curvature, we have that the associated tractor to the pair $(\Psi : M \to \widetilde{M}, \xi)$ has normal tractor connection if and only if (5.4) holds.

In the particular case that ξ is an umbilic direction with $A_\xi = \mu \cdot \mathrm{Id}$, we have $\mu = \widetilde{g}(\mathbf{H}, \xi)$ and then, Theorem 5.4 reads as follows.

Corollary 5.7. *Let $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ be the associated tractor to the pair $(\Psi : M \to \widetilde{M}, \xi)$ with ξ an umbilic direction with $A_\xi = \mu \cdot \mathrm{Id}$, where μ is a nonvanishing function on M. Assume there is a lightlike normal vector field $\ell \in \mathfrak{X}^{\perp}(M)$ such that $\widetilde{g}(\xi, \ell) = -1$. Then, $\widetilde{\nabla}$ is a normal tractor connection if and only if $\omega = \frac{1}{\mu} d\mu$ and for every $V, W \in \mathfrak{X}(M)$ the Ricci tensor of g satisfies*

$$\mathrm{Ric}^g(V, W) = \frac{n}{2} \|\mathbf{H}\|^2 g(V, W) - (n-2)\widetilde{g}(\mathbf{H}, \xi) g(V, A_\ell W). \tag{5.5}$$

Remark 5.8. Taking in mind formula (2.7) let us note that equation (5.5) implies that the scalar curvature of g is given by $S^g = n(n-1)\|\mathbf{H}\|^2$. This formula widely generalizes to [52, Cor. 4.5]. Moreover, for dimension $n = 2$, the equation reduces to $\mathrm{Ric}^g(V, W) = \|\mathbf{H}\|^2 g(V, W)$. Therefore, in this dimension the equation (5.5) holds if and only if $\|\mathbf{H}\|^2 = K^g$, where K^g is the Gauss curvature of the induced metric g.

Remark 5.9. We assume that we are in the hypotheses given in Corollary 5.7 and also that $\widetilde{\nabla}$ is normal. Then, there exists a smooth function h on M such that $\mathrm{Ric}^g = hg$ if and only if $\Psi : M \to \widetilde{M}$ is totally umbilical. In such a case, we have $\mathrm{Ric}^g = (n-1)\|\mathbf{H}\|^2\, g$. Furthermore, from Exercise 21 on [51, p. 96], for dimension $n \geq 3$ we have that $\|\mathbf{H}\|^2$ is a constant function and (M, g) must be Einstein.

Remark 5.10. Let us consider $\widetilde{M} = \mathbb{L}^{n+2}$, where $\mathbb{L}^{n+2}$ denotes the $(n+2)$-dimensional Minkowski spacetime. Then, the condition $d\omega = \frac{1}{\mu} d\mu$ with $\mu > 0$ is equivalent to $\Psi : M \to \widetilde{M}$ factors, up to a translation, through the lightlike cone Λ, see [56]. A classical result due to Brinkmann [12] states that a simply connected Riemannian manifold (M, g) with $n \geq 3$ is conformally flat if and only if can be isometrically immersed into $\mathbb{L}^{n+2}$ through the lightlike cone Λ (see [5] for a proof in a modern form). It is a direct consequence of the Uniformization Theorem that every two dimensional simply connected Riemannian manifold (M, g) can be isometrically immersed into $\mathbb{L}^4$ through the three dimensional lightlike cone Λ. For spacelike immersions $\Psi\colon M \to \mathbb{L}^{n+2}$ through the lightlike cone Λ, the position vector field $\Psi \in \mathfrak{X}^{\perp}(M)$ is lightlike with $A_{\Psi} = -\mathrm{Id}$. Therefore, for $\xi = -\Psi$ and from the Gauss equation, we have for the Ricci tensor of the induced metric g (see [52] and take into account that the normalization condition in [52] is $\widetilde{g}(\xi, \ell) = 1$),

$$\mathrm{Ric}^g(V, W) = ng(A_{\mathbf{H}}V, W) + 2g(V, A_{\ell}W). \tag{5.6}$$

Now, since $\mu = 1$, a direct computation shows that equations (5.5) and (5.6) are the same. Hence, for simply connected conformally flat Riemannian manifolds, we get a tractor normal connection by means of an isometric immersion through the lightlike cone of the Minkowski spacetime.

5.2 Application to generalized Schwarzschild spacetimes

In this Section we give an application to the generalized Schwarzschild spacetimes. Let us recall that this family is given by Lorentzian warped products $B \times_{\lambda} F$, where B is an open subset of $\mathbb{R}^2$ and (F, g_F) is a n-dimensional connected Riemannian manifold, Definition 1.1. The assumptions on the functions $\xi\lambda$ and $\eta\lambda$ in the statements of the following results are only required along Ψ. As a consequence of Proposition 5.1 and using Lemmas 4.21 and 4.22, we have.

Proposition 5.11. *Let $\Psi : M \to B \times_{\lambda} F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ (resp. $\mathcal{N}$) of D_{ξ} (resp. D_{η}). Then, $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ is the associated tractor to the pair $(\Psi : M \to B \times_{\lambda} F, \xi)$ $\big($resp. $(\Psi : M \to B \times_{\lambda} F, \eta)\big)$ if and only if $\xi\lambda \neq 0$ (resp. $\eta\lambda \neq 0$).*

Taking in mind the normal lightlike frames given in Lemmas 4.18 and 4.19, we particularize Corollary 5.7 for the generalized Schwarzschild spacetimes.

Theorem 5.12. *Assume $\xi\lambda \neq 0$ and let $\Psi : M \to B \times_{\lambda} F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_{ξ}. Let $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ be the associated tractor to the pair $(\Psi : M \to B \times_{\lambda} F, \xi)$. Then, $\widetilde{\nabla}$ is a normal tractor connection if and only if $\big(\xi(\xi\lambda) \circ \Psi_B\big)dv = 0$ at every point, where dv is the exterior derivative of $v := r \circ \Psi_B$, and,* *for every $V, W \in \mathfrak{X}(M)$, the Ricci tensor of g satisfies*

$$\mathrm{Ric}^g(V,W) = (n-1)\left[\|\mathbf{H}\|^2 + \left(\frac{2\xi\lambda}{n\lambda}\circ\Psi_B\right)\Delta v\right] g(V,W) - \left[\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)\Delta v\right] g(V,W) \\ - (n-2)\left(\frac{\xi\lambda}{\lambda}\circ\Psi_B\right)\mathrm{Hess}(v)(V,W).$$

Theorem 5.13. *Assume $\eta\lambda \neq 0$ and let $\Psi : M \to B \times_\lambda F$ be a codimension two spacelike immersion through an integral hypersurface $\mathcal{N}$ of D_η. Let $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ be the associated tractor to the pair $(\Psi : M \to B \times_\lambda F, \eta)$. Then, $\widetilde{\nabla}$ is a normal tractor connection if and only if $\big((\eta(\eta\lambda) + ff'\eta\lambda)\circ\Psi_B\big)dv = 0$ at every point and, for every $V, W \in \mathfrak{X}(M)$, the Ricci tensor of g satisfies*

$$\mathrm{Ric}^g(V,W) = (n-1)\left[\|\mathbf{H}\|^2 - \left(\frac{4\eta\lambda}{n\lambda f^2}\circ\Psi_B\right)\Delta v\right] g(V,W) + \left[\left(\frac{2\eta\lambda}{\lambda f^2}\circ\Psi_B\right)\Delta v\right] g(V,W) \\ + (n-2)\left(\frac{2\eta\lambda}{\lambda f^2}\circ\Psi_B\right)\mathrm{Hess}(v)(V,W).$$

Remark 5.14. Assuming that we are in the hypotheses given in Theorem 5.12 (resp. Theorem 5.13), when $\eta^\perp$ (resp. $\xi^\perp$) is an umbilic direction, the normalization condition given by the equation for the Ricci curvature reduces to $\mathrm{Ric}^g = (n-1)\|\mathbf{H}\|^2\, g$. This formula agrees with the given in Remark 5.9. Moreover, let us note that the hypotheses $\big(\xi(\xi\lambda)\circ\Psi_B\big)dv = 0$ and $\big((\eta(\eta\lambda) + ff'\eta\lambda)\circ\Psi_B\big)dv = 0$ are satisfied when $\lambda(t,r) = r$ or when M factors through a slice.

Although it is directly deduced from Remarks 4.54, 5.9 and 5.14, we will state the following result for clarity.

Corollary 5.15. *Assume $\lambda(t,r) = r$ and let $\Psi : M \to B \times_\lambda F$ be a totally umbilical codimension two spacelike immersion through an integral hypersurface $\mathcal{L}$ of D_ξ. Let $(\mathcal{T}, \mathcal{T}^1, \widetilde{g}, \widetilde{\nabla})$ be the associated tractor to the pair $(\Psi : M \to B \times_\lambda F, \xi)$. Then, $\widetilde{\nabla}$ is a normal tractor connection if and only if $\mathrm{Ric}^g = (n-1)\|\mathbf{H}\|^2\, g$. Furthermore, for dimension $n \geq 3$, the immersion Ψ has parallel mean curvature vector field.*

Similarly, this result can be adapted to the case of factoring through an integral hypersurface $\mathcal{N}$ of D_η.

Chapter 6

Möbius structures

Starting from a Riemannian conformal structure (M, c), we provide a method to construct a family of Lorentzian manifolds. The construction relies on the choice of a metric in the conformal class and a smooth 1-parameter family of self-adjoint tensor fields. Then, every metric in the conformal class corresponds to the induced metric on M by a codimension two spacelike immersion into these Lorentzian manifolds. Under suitable choices of the 1-parameter family of tensor fields, there exists a lightlike normal vector field along such spacelike immersions whose Weingarten endomorphism provide a Möbius structure on the Riemannian conformal structure. Conversely, every Möbius structure on a Riemannian conformal structure arises in this way. Flat Möbius structures are characterized in terms of the extrinsic geometry of the corresponding spacelike surfaces.

6.1 Möbius structures and pre-ambient spaces

Unless stated otherwise, we assume $n \geq 2$. Let us recall that a Möbius structure (Definition 3.24) on an n-dimensional manifold M is a triple (M, c, D), where (M, c) is a Riemannian conformal structure and

1. D is a map $D : c \to \mathcal{T}_{(0,2)}M$ such that for every $g \in c$, the tensor $D(g)$ is symmetric with
$$\mathrm{trace}_g D(g) = \frac{\mathrm{S}^g}{2(n-1)}.$$
2. Furthermore, D satisfies the following conformal transformation law
$$D(e^{2u}g) = D(g) - \frac{\|\nabla u\|^2}{2} g - \mathrm{Hess}(u) + du \otimes du.$$

We mean the map D as a Möbius structure for the conformal class c. As suggested in Section 3.3, for $(n \geq 3)$-dimensional Riemannian conformal structures (M, c) there is a preferred Möbius structure defined by means of the Schouten tensor given in (2.13). The conformal transformation law (2.14) for the Schouten tensor implies that $D(g) = P^g$ provides a Möbius structure for the conformal class c. Therefore, for conformal structures on $(n \geq 3)$-dimensional manifolds, the Schouten tensor gives a canonical Möbius structure.

Remark 6.1. The Uniformization Theorem states that a two dimensional Riemannian manifold (M, g) admits a metric g' conformal to g with constant Gauss curvature k. This fact leads to a choice of the Möbius structure determinated by $D(g') = (k/2)\, g'$ and the conformal transformation law. On the other hand, recall that for a connected oriented two dimensional manifold M, there is a well-known one-to-one correspondence between conformal classes and complex structures. A Riemann surface is a such two dimensional manifold endowed with a particular choice of conformal or complex structure. Thus, a Möbius structure on a connected oriented two dimensional manifold M is equivalent to specifying a complex structure and a "Schouten type-tensor" on M.

Remark 6.2. For $n \geq 3$ and taking into account $2\,\mathrm{div}\,(\mathrm{Ric}^g) = d\,\mathrm{S}^g$ (see for instance [51, Cor. 3.54]), one gets that $\mathrm{div}\,(\mathrm{P}^g) = \frac{1}{2(n-1)} d\,\mathrm{S}^g$. This property is not satisfied for Möbius structures, in general.

Let (M, c) be an n-dimensional Riemannian conformal structure. Let us consider the $\mathbb{R}_{>0}$-principal fiber bundle $\pi : \mathcal{Q} \to M$ defined as the ray fiber subbundle in the fiber bundle of Riemannian metrics given by metrics in the conformal class c. Thus, the fiber over $x \in M$ is formed by the values of g_x for all metrics $g \in c$. Every section of π provides a Riemannian metric in the conformal class c and the principal $\mathbb{R}_{>0}$-action on $\mathcal{Q}$ is given by $\varphi(\tau, g_x) = \tau^2 g_x$, $x \in M$. Let us denote by $Z_{\mathcal{Q}}$ the fundamental vector field for the action φ, that is,

$$Z_{\mathcal{Q}}(g_x) = \left.\frac{d}{dt}\right|_{t=0} \varphi(e^t, g_x) = \left.\frac{d}{dt}\right|_{t=0} (e^{2t} g_x).$$

The principal bundle $\pi : \mathcal{Q} \to M$ is called the scale bundle of (M, c).

Definition 6.3. ([31]) *A pre-ambient space for a Riemannian conformal structure (M, c) is an $(n+2)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ such that*

1. *There is a free $\mathbb{R}_{>0}$-action $\widetilde{\varphi}$ on $\widetilde{M}$ and an embedding $\iota : \mathcal{Q} \to \widetilde{M}$ such that the following*

diagram commutes

$$\begin{array}{ccc} \mathbb{R}_{>0} \times \mathcal{Q} & \xrightarrow{\mathrm{id}_{\mathbb{R}_{>0}} \times \iota} & \mathbb{R}_{>0} \times \widetilde{M} \\ {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\varphi}} \\ \mathcal{Q} & \xrightarrow[\iota]{} & \widetilde{M} \end{array} \tag{6.1}$$

Hence, the fundamental vector field $Z \in \mathfrak{X}(\widetilde{M})$ *for the action* $\widetilde{\varphi}$ *and the vector field* $Z_{\mathcal{Q}} \in \mathfrak{X}(\mathcal{Q})$ *are* ι*-related, i.e.,* $T_{g_x}\iota \cdot Z_{\mathcal{Q}}(g_x) = Z(\iota(g_x))$ *for all* $g_x \in \mathcal{Q}$*.*

2. *For* Z *we have* $\mathcal{L}_Z\widetilde{g} = 2\widetilde{g}$*, where* $\mathcal{L}$ *is the Lie derivative.*

3. *For any* $g_x \in \mathcal{Q}$ *and* $\xi, \eta \in T_{g_x}\mathcal{Q}$*, the following equality holds*

$$\iota^*(\widetilde{g})_{g_x}(\xi, \eta) = g_x(T_{g_x}\pi \cdot \xi, T_{g_x}\pi \cdot \eta).$$

In particular, we have $\iota^*(\widetilde{g})(Z_{\mathcal{Q}}, -) = 0$*.*

For a pre-ambient space $(\widetilde{M}, \widetilde{g})$ the metric $\widetilde{g}$ is called a pre-ambient metric. The condition $\mathcal{L}_Z\widetilde{g} = 2\widetilde{g}$ tells us that the vector field Z is homothetic with respect to the pre-ambient metric $\widetilde{g}$.

Remark 6.4. The notion of ambient metric in [31] satisfies a normalisation condition. In fact, in order to obtain the uniqueness of the ambient Lorentzian metric $\widetilde{g}$, the ambient metric by Fefferman and Graham imposes that the Ricci tensor of the metric $\widetilde{g}$ vanishes to a certain order (depending on the dimension) on $\mathcal{Q}$, see [31] for details. The pre-ambient space has been used by Čap and Gover in order to establish the relationship with the standard tractor conformal bundles, see [19].

We end this Section with several comments from the point of view of Lorentzian geometry of the notion of pre-ambient space. Let (M, c) be an n-dimensional Riemannian conformal structure and $(\widetilde{M}, \widetilde{g})$ a pre-ambient space for (M, c). Then, condition (3) in Definition 6.3 implies that $\iota\colon \mathcal{Q} \to \widetilde{M}$ is a lightlike hypersurface. Moreover, the induced lightlike metric $\bar{h} := \iota^*(\widetilde{g})$ does not depend on the particular pre-ambient metric $\widetilde{g}$. In the terminology of [31], the lightlike metric $\bar{h}$ is called the tautological tensor. The radical distribution $\mathrm{Rad}(\bar{h})$ is globally generated by the vector field $Z_{\mathcal{Q}}$.

Recall that every choice of a metric $g \in c$ provides a section of $\pi\colon \mathcal{Q} \to M$ and conversely. The following result is well-known. We include here a proof for the sake of completeness.

Lemma 6.5. *Let (M,c) be a Riemannian conformal structure and $(\widetilde{M},\widetilde{g})$ a pre-ambient space for (M,c). For every $g\in c$, the map*

$$\Psi^g := \iota\circ g\colon M\to(\widetilde{M},\widetilde{g})$$

is a codimension two spacelike immersion with induced metric $(\Psi^g)^*(\widetilde{g})=g$. Moreover, the vector field $\xi := Z|_{\Psi^g}$ is normal and lightlike along Ψ^g with $A_\xi=-\mathrm{Id}$.

Proof. For every $x\in M$ a direct computation gives

$$(\Psi^g)^*(\widetilde{g})_x = g^*(\iota^*(\widetilde{g})_{g_x}) = g^*(\pi^*(g)_{g_x}) = (\pi\circ g)^*(g)_x = g_x.$$

Taking into account that $\xi_x = Z(\Psi^g(x)) = T_{g_x}\iota\cdot Z_{\mathcal{Q}}(g_x)$, we get $\xi\in\mathfrak{X}^\perp(M)$ (for the immersion Ψ^g) and $\widetilde{g}(\xi,\xi)=0$. In order to see that $A_\xi=-\mathrm{Id}$, recall that the condition $\mathcal{L}_Z\widetilde{g}=2\widetilde{g}$ is equivalent to

$$\widetilde{g}(\widetilde{\nabla}_X Z,Y)+\widetilde{g}(X,\widetilde{\nabla}_Y Z)=2\widetilde{g}(X,Y),\quad X,Y\in\mathfrak{X}(\widetilde{M}).$$

Using the previous equation for $V,W\in\mathfrak{X}(M)$ and as consequence of the polarization identity we arrive to

$$\widetilde{g}\Big(\widetilde{\nabla}_V Z,W\Big)=\widetilde{g}(V,W).$$

We are in position to compute $\widetilde{\nabla}_V\xi$ as follows

$$\widetilde{\nabla}_V\xi=\widetilde{\nabla}_V(Z|_{\Psi^g})=(\widetilde{\nabla}_V(Z|_{\Psi^g}))^\top+(\widetilde{\nabla}_V(Z|_{\Psi^g}))^\perp=V+\nabla^\perp_V\xi$$

and now the assertion $A_\xi=-\mathrm{Id}$ is clear. $\square$

6.2 A method to construct pre-ambient spaces

Let (M,c) be an n-dimensional Riemannian conformal structure.

Definition 6.6. *A smooth 1-parameter family $\gamma\colon\mathbb{R}\to\mathcal{T}_{(1,1)}M$ is called admissible when*

1. *$\gamma(r)$ is a self-adjoint tensor field with respect to any representative $g\in c$,*
2. *$\gamma(0)=\mathrm{Id}$,*
3. *there is $\delta>0$ such that $\gamma(r)$ is not singular for $|r|<\delta$.*

Here, the smoothness of γ means that for every $V \in \mathfrak{X}(M)$ and $x \in M$, there exists

$$\dot{\gamma}(r)(V_x) = \lim_{\varepsilon \to 0} \frac{\gamma(r+\varepsilon)(V_x) - \gamma(r)(V_x)}{\varepsilon} \in T_x M.$$

In particular, we have $\dot{\gamma}(0) \in \mathcal{T}_{(1,1)}M$.

Remark 6.7. The condition 3 in the above definition can be deleted when M is compact and, at least locally, δ always exists in the general case.

Let us fix a metric $g \in c$ and an admissible smooth 1-parameter family $\gamma \colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$. For every $r \in \mathbb{R}$, we define the following symmetric tensor on M,

$$\langle V, W \rangle_r^g = g\Big(\gamma(r)(V), W\Big).$$

Clearly, $\langle\, , \rangle_0^g = g$ and so $\langle\, , \rangle_r^g$ can be seen as a 1-parameter deformation of the metric g. Moreover, $\langle\, , \rangle_r^g$ is positive definite on M for $|r| < \delta$. Henceforth, let us consider the manifold $\widetilde{M} := B \times M$, where $B := \mathbb{R}_{>0} \times (-\delta, +\delta)$ with coordinates (t, r). This manifold $\widetilde{M}$ can be endowed with the Lorentzian metric

$$\widetilde{g} = d(rt) \otimes dt + dt \otimes d(rt) + t^2 \langle -, - \rangle_r^g \tag{6.2}$$

and with the free $\mathbb{R}_{>0}$-action $\widetilde{\varphi}(\tau, (t, r, x)) = (\tau t, r, x)$. The choice of the metric $g \in c$ provides the global trivialization of $\pi \colon \mathcal{Q} \to M$ given by

$$t^2 g_x \in \mathcal{Q} \mapsto (t, x) \in \mathbb{R}_{>0} \times M$$

and the following embedding of $\mathcal{Q}$ in $\widetilde{M}$ at $r = 0$,

$$\iota_g \colon \mathcal{Q} \to \widetilde{M}, \quad t^2 g_x \mapsto (t, 0, x). \tag{6.3}$$

A direct computation shows that $\iota_g \circ \varphi(\tau, t^2 g_x) = \widetilde{\varphi} \circ (\mathrm{id}_{\mathbb{R}_{>0}} \times \iota_g)(\tau, t^2 g_x) = (\tau t, 0, x)$. On the other hand, the fundamental vector field $Z \in \mathfrak{X}(\widetilde{M})$ corresponding to the action $\widetilde{\varphi}$ is $Z = t\partial_t$ and one directly checks that $\mathcal{L}_Z \widetilde{g} = 2\widetilde{g}$. Finally, for $t^2 g_x \in \mathcal{Q}$ and $\xi, \eta \in T_{t^2 g_x}\mathcal{Q}$, we have

$$(\iota_g^* \widetilde{g})_{t^2 g_x}(\xi, \eta) = \widetilde{g}_{(t,0,x)}(T_{t^2 g_x}\iota_g \cdot \xi, T_{t^2 g_x}\iota_g \cdot \eta) = t^2 g_x(T_{t^2 g_x}\pi \cdot \xi, T_{t^2 g_x}\pi \cdot \eta).$$

Hence, $(\widetilde{M} = B \times M, \widetilde{g})$ where the metric $\widetilde{g}$ is given in (6.2) is a pre-ambient space for (M, c). We have thus led to the following result.

Proposition 6.8. *Let (M, c) be an n-dimensional Riemannian conformal structure. For every choice of a metric $g \in c$ and an admissible smooth 1-parameter family $\gamma \colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$, the manifold $\widetilde{M} = B \times M$ is a pre-ambient space for (M, c).*

Remark 6.9. In the particular case that $\gamma(r) = h^2(r)\,\mathrm{Id}$ with $h\colon (-\delta, +\delta) \to \mathbb{R}$, $h(0) = 1$ and $h > 0$, the pre-ambient space $(\widetilde{M}, \widetilde{g})$ with metric $\widetilde{g} = d(rt) \otimes dt + dt \otimes d(rt) + (th(r))^2 g$ is a warped product.

Remark 6.10. The one-form $\bar{\omega}$ metrically equivalent to the vector field Z is

$$\bar{\omega} = t^2 dr + 2trdt,$$

thus, we have $d\bar{\omega} = 0$.

As a Lorentzian manifold, the pre-ambient space $(\widetilde{M}, \widetilde{g})$ is timelike orientable, that is, there exists a globally defined timelike vector field, namely,

$$T := \frac{1}{t}\partial_t - \Big(1 + \frac{r}{t^2}\Big)\partial_r \in \mathfrak{X}(\widetilde{M}), \tag{6.4}$$

which satisfies $\widetilde{g}(T, T) = -2$. To be used later, we also introduce the spacelike vector field

$$E := \frac{1}{t}\partial_t + \Big(1 - \frac{r}{t^2}\Big)\partial_r \in \mathfrak{X}(\widetilde{M}), \tag{6.5}$$

with $\widetilde{g}(E, E) = 2$ and $\widetilde{g}(T, E) = 0$. The set of all natural lifts of vector fields $V \in \mathfrak{X}(M)$ to $\mathfrak{X}(\widetilde{M})$ is denoted by $\mathfrak{L}(M)$. For a vector field $V \in \mathfrak{X}(M)$, its lift is also denoted by V.

As was mentioned in Remark 6.9, the metrics $\widetilde{g}$ in (6.2) are not warped product metrics, in general. Hence, the formulas for the Levi-Civita connection of warped products metrics in (2.8) do not work.

Proposition 6.11. *The Levi-Civita connection $\widetilde{\nabla}$ of $(\widetilde{M}, \widetilde{g})$ satisfies*

$$\widetilde{\nabla}_{\partial_t}\partial_t = \widetilde{\nabla}_{\partial_r}\partial_r = 0, \qquad \widetilde{\nabla}_{\partial_t}\partial_r = \widetilde{\nabla}_{\partial_r}\partial_t = \frac{1}{t}\partial_r, \tag{6.6}$$

$$\widetilde{\nabla}_{\partial_t}V = \frac{1}{t}V, \quad \widetilde{\nabla}_{\partial_r}V = \frac{1}{2}\gamma(r)^{-1}(\dot{\gamma}(r)(V)), \tag{6.7}$$

$$\widetilde{\nabla}_V W|_{\iota_g(\mathcal{Q})} = -\frac{1}{2t}\widetilde{g}(\dot{\gamma}(0)(V), W)\partial_t - \frac{1}{t^2}\widetilde{g}(V, W)\partial_r + \nabla_V W, \tag{6.8}$$

where $V, W \in \mathfrak{L}(M)$.

Proof. A direct consequence of Koszul formula for the Levi-Civita connection of $(\widetilde{M}, \widetilde{g})$ shows $\widetilde{\nabla}_{\partial_t}\partial_t = \widetilde{\nabla}_{\partial_r}\partial_r = 0$ and $\nabla_{\partial_t}\partial_r = \dfrac{1}{t}\partial_r$. On the other hand, the Koszul formula also implies $\widetilde{g}(\widetilde{\nabla}_{\partial_t}V, \partial_t) = \widetilde{g}(\widetilde{\nabla}_{\partial_t}V, \partial_r) = 0$ and $2\widetilde{g}(\widetilde{\nabla}_{\partial_t}V, W) = \partial_t\widetilde{g}(V, W)$. By definition of the metric $\widetilde{g}$,

$$\partial_t\widetilde{g}(V, W) = 2tg(\gamma(r)(V), W) = \frac{2}{t}\widetilde{g}(V, W),$$

and then we get $\widetilde{\nabla}_{\partial_t} V = \dfrac{1}{t} V$. In the same manner, we compute

$$2\widetilde{g}(\widetilde{\nabla}_{\partial_r} V, W) = \partial_r\Big(t^2 g(\gamma(r)(V), W)\Big) = t^2 g(\dot{\gamma}(r)(V), W) = \widetilde{g}\Big(\gamma(r)^{-1}(\dot{\gamma}(r)(V)), W\Big).$$

From (6.7), it follows that

$$\widetilde{g}(\widetilde{\nabla}_V W, \partial_t) = -\widetilde{g}(\widetilde{\nabla}_V \partial_t, W) = -\frac{1}{t}\widetilde{g}(V, W), \quad \widetilde{g}(\widetilde{\nabla}_V W, \partial_r) = -\frac{1}{2}\widetilde{g}\Big(\gamma(r)^{-1}(\dot{\gamma}(r)(V)), W\Big).$$

In order to compute $\widetilde{g}(\widetilde{\nabla}_{V_p} W, U_p)$ for $U \in \mathfrak{L}(M)$ and $p = (t, 0, x) \in \iota_g(\mathcal{Q})$, we can assume $U, V, W \in \mathfrak{L}(M)$ so that all their brackets are zero at the point p. Then, the Koszul formula yields

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_{V_p} W, U_p) &= V_p\, \widetilde{g}(W, U) + W_p\, \widetilde{g}(V, U) - U_p\, \widetilde{g}(V, W) \\ &= t^2\Big(V_x\, g(W, U) + W_x\, g(V, U) - U_x\, g(V, W)\Big) \\ &= 2t^2 g(\nabla_{V_x} W, U_x) = 2\widetilde{g}\Big(\left(\nabla_V W\right)_p, U_p\Big). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \widetilde{\nabla}_V W|_{\iota_g(\mathcal{Q})} &= -\frac{1}{2}\widetilde{g}(\widetilde{\nabla}_V W, T)T + \frac{1}{2}\widetilde{g}(\widetilde{\nabla}_V W, E)E + \nabla_V W \\ &= -\frac{1}{2t}\widetilde{g}(\dot{\gamma}(0)(V), W)\partial_t - \frac{1}{t^2}\widetilde{g}(V, W)\partial_r + \nabla_V W. \end{aligned}$$

□

Remark 6.12. Let us fix $(t, r) \in B$ and consider the spacelike submanifold

$$\mathcal{F} := \{(t, r)\} \times M \subset \widetilde{M}.$$

The vector fields $T|_{\mathcal{F}}$ and $E|_{\mathcal{F}}$ span the normal bundle of $\mathcal{F}$ and Proposition 6.11 implies

$$\begin{aligned} \widetilde{\nabla}_V T|_{\mathcal{F}} &= \frac{1}{t^2}V - \frac{1}{2}\Big(1 + \frac{r}{t^2}\Big)\gamma(r)^{-1}(\dot{\gamma}(r)(V)) \text{ and} \\ \widetilde{\nabla}_V E|_{\mathcal{F}} &= \frac{1}{t^2}V + \frac{1}{2}\Big(1 - \frac{r}{t^2}\Big)\gamma(r)^{-1}(\dot{\gamma}(r)(V)), \end{aligned}$$

for every $V \in \mathcal{L}(M)$. Therefore, the second fundamental form $\mathrm{II}_{\mathcal{F}}$ for $\mathcal{F}$ is given by

$$\mathrm{II}_{\mathcal{F}}(V, W) = -\frac{1}{2t}\widetilde{g}\big(\gamma(r)^{-1}(\dot{\gamma}(r)(V)), W\big)\partial_t - \frac{1}{t^2}\Big(\widetilde{g}(V, W) - r\widetilde{g}\big(\gamma(r)^{-1}(\dot{\gamma}(r)(V)), W\big)\Big)\partial_r$$

where $V, W \in \mathfrak{X}(M)$. Thus, on the contrary to the warped products metrics, the slices $\mathcal{F}$ are not totally umbilical, in general. It is not difficult to show that for a fixed (t, r), the corresponding slice $\mathcal{F}$ is totally umbilical if and only if the endomorphism field $\gamma(r)^{-1} \circ \dot{\gamma}(r) = h \cdot \mathrm{Id}$ for some $h \in \mathcal{C}^\infty(M)$.

Remark 6.13. For $\gamma(r) = h^2(r)\,\mathrm{Id}$ with $h > 0$, the metric $\widetilde{g}$ is a warped metric with warping function $\lambda(t,r) = th(r)$. In this case, the formula given in Remark 6.12 reduces to

$$\mathrm{II}_{\mathcal{F}}(V,W) = -\frac{\widetilde{g}(V,W)}{th(r)}\Big(h'(r)\partial_t + \frac{h(r) - 2rh'(r)}{t}\partial_r\Big).$$

A direct computation shows that the above formula agrees with (2.8).

From [19], the Ricci tensor $\widetilde{\mathrm{Ric}}$ of any pre-ambient space $(\widetilde{M}, \widetilde{g})$ restricted to $\iota_g(\mathcal{Q})$ satisfies

$$\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(\partial_t, \partial_t) = \widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(\partial_t, V) = 0, \quad V \in \mathcal{L}(M) \tag{6.9}$$

if and only if $d\bar{\omega}|_{\iota_g(\mathcal{Q})} = 0$. As consequence of Remark 6.10, this formula (6.9) holds for the metric $\widetilde{g}$ in (6.2). The following result provides the other component of $\widetilde{\mathrm{Ric}}$ on $\iota_g(\mathcal{Q})$.

Corollary 6.14. *The Ricci tensor* $\widetilde{\mathrm{Ric}}$ *of* $(\widetilde{M}, \widetilde{g})$ *satisfies*

$$\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(V,W) = \mathrm{Ric}^g(V,W) - \frac{\mathrm{trace}(\dot{\gamma}(0))}{2} g(V,W) - \left(\frac{n-2}{2}\right) g\left(\dot{\gamma}(0)(V), W\right), \tag{6.10}$$

where $V, W \in \mathfrak{L}(M)$*. For* $\xi, \eta \in \mathfrak{X}(\mathcal{Q})$*, we have*

- *If* $n = 2$*,*
$$\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(T\iota_g \cdot \xi, T\iota_g \cdot \eta) = 0$$
if and only if $\mathrm{trace}(\dot{\gamma}(0)) = 2K^g$*, where* K^g *is the Gauss curvature of* g*.*

- *If* $n \geq 3$*,*
$$\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(T\iota_g \cdot \xi, T\iota_g \cdot \eta) = 0$$
if and only if $g(\dot{\gamma}(0)(-), -) = 2P^g$*, where* P^g *is the Schouten tensor of* g*.*

Proof. Let $(e_1, \dots, e_n)$ be an orthonormal local frame on (M, g) and consider the orthonormal local frame for $(\widetilde{M}, \widetilde{g})$ on $r = 0$ given by

$$\left(\frac{1}{\sqrt{2}}T, \frac{1}{\sqrt{2}}E, E_1, \dots, E_n\right),$$

where $E_i = \frac{1}{t}e_i$ and the vector fields T, E are given in (6.4) and (6.5), respectively. Then, we get

$$\begin{aligned}
\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(V,W) &= \sum_{i=1}^{n} \widetilde{g}\left(\widetilde{\mathrm{R}}(E_i, V)W, E_i\right) + \frac{1}{2}\widetilde{g}\left(\widetilde{\mathrm{R}}(E,V)W, E\right) - \frac{1}{2}\widetilde{g}\left(\widetilde{\mathrm{R}}(T,V)W,T\right) \\
&= \sum_{i=1}^{n} \widetilde{g}\left(\widetilde{\mathrm{R}}(E_i, V)W, E_i\right) + \frac{1}{t}\left(\widetilde{g}\left(\widetilde{\mathrm{R}}(\partial_t, V)W, \partial_r\right) + \widetilde{g}\left(\widetilde{\mathrm{R}}(\partial_r, V)W, \partial_t\right)\right).
\end{aligned}$$

For every vector field $X \in \mathfrak{X}(\widetilde{M})$, we have the following decomposition

$$\begin{aligned} X &= \sum_{i=1}^{n} f_i E_i + \frac{1}{2}\widetilde{g}\,(X, E)\, E - \frac{1}{2}\widetilde{g}\,(X, T)\, T \\ &= \sum_{i=1}^{n} f_i E_i + \frac{1}{t}\left(\widetilde{g}\,(X, \partial_t)\, \partial_r + \widetilde{g}\,(X, \partial_r)\, \partial_t\right) - \frac{2r}{t^2}\widetilde{g}\,(X, \partial_r)\, \partial_r \end{aligned}$$

where $f_i \in \mathcal{C}^\infty(\widetilde{M})$. Let us note that $f_i|_{r=0} = \widetilde{g}(X, E_i)$. Now, a straightforward computation from Proposition 6.11 gives

$$\widetilde{g}\left(\widetilde{\mathrm{R}}(\partial_t, V)W, \partial_r\right) + \widetilde{g}\left(\widetilde{\mathrm{R}}(\partial_r, V)W, \partial_t\right) = 0. \tag{6.11}$$

Finally, it is a standard computation, from Proposition 6.11 and (6.11), to check that

$$\begin{aligned} \widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(V, W) &= \sum_{i=1}^{n} g\left(\nabla_{e_i}\nabla_V W, e_i\right) - \sum_{i=1}^{n} g\left(\nabla_V \nabla_{e_i} W, e_i\right) - \sum_{i=1}^{n} g\left(\nabla_{[e_i, V]} W, e_i\right) \\ &- \frac{1}{2} g(V, W) \sum_{i=1}^{n} g(\dot{\gamma}(0)(e_i), e_i) - \frac{n}{2} g(\dot{\gamma}(0)(V), W) \\ &+ \frac{1}{2}\sum_{i=1}^{n} g(e_i, W) g(\dot{\gamma}(0)(V), e_i) + \frac{1}{2}\sum_{i=1}^{n} g(V, e_i) g(\dot{\gamma}(0)(e_i), W) \\ &= \mathrm{Ric}^g(V, W) - \frac{\mathrm{trace}(\dot{\gamma}(0))}{2} g(V, W) - \left(\frac{n-2}{2}\right) g(\dot{\gamma}(0)(V), W). \end{aligned}$$

The vanishing properties of the Ricci tensor on $\iota_g(\mathcal{Q})$ are direct consequences of (6.9) and (6.10). $\square$

6.3 Constructing Möbius structures from spacelike immersions

At the beginning of this Section it is important to distinguish when we view a vector field V as a lift along an immersion $V|_{\Psi^u}$ or when we view it simply as vector field $V \in \mathfrak{X}(M)$. Henceforth, we assume that (M, c) is an n-dimensional Riemannian conformal structure and we have fixed

1. a metric $g \in c$ and

2. an admissible smooth 1-parameter family $\gamma \colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$.

Thus, we have the pre-ambient space $(\widetilde{M}, \widetilde{g})$ as in Proposition 6.8. For every $u \in C^\infty(M)$, the spacelike immersion $\Psi^{e^{2u}g}$ in Lemma 6.5 satisfies

$$\Psi^{e^{2u}g} : M \to (\widetilde{M}, \widetilde{g}), \quad x \mapsto (e^{u(x)}, 0, x) \tag{6.12}$$

and $(\Psi^{e^{2u}g})^*(\widetilde{g}) = e^{2u}g$. For simplicity of notation, from now on, we write Ψ^u instead of $\Psi^{e^{2u}g}$. The differential map of Ψ^u is

$$T\Psi^u \cdot V = V(u)e^u \partial_t|_{\Psi^u} + V|_{\Psi^u}, \tag{6.13}$$

where $V \in \mathfrak{X}(M)$. A direct computation from (6.13) shows that the vector fields

$$\xi^u = e^u \partial_t|_{\Psi^u} \quad \text{and} \quad \ell^u = e^{-u}\frac{\|\nabla u\|^2}{2}\partial_t|_{\Psi^u} - e^{-2u}\partial_r|_{\Psi^u} + e^{-2u}\nabla u|_{\Psi^u} \tag{6.14}$$

span the normal bundle of Ψ^u and one easy checks that $\{\xi^u, \ell^u\}$ is a global lightlike normal frame.

Remark 6.15. *The lightlike normal vector field ξ^u agrees with $Z|_{\Psi^u}$ where $Z \in \mathfrak{X}(\widetilde{M})$ is the fundamental vector field corresponding to the action $\widetilde{\varphi}$.*

Lemma 6.16. *Let $\Psi^u : M \to (\widetilde{M}, \widetilde{g})$ be the immersion given in* (6.12). *For every $V \in \mathfrak{L}(M) \subset \overline{\mathfrak{X}}(M)$, the following formulas hold*

$$V^\top = V \quad \textit{and} \quad \partial_r^\top = \nabla u^\top.$$

Proof. From (6.13) and (6.14), it is easy to check that

$$\begin{aligned} T\Psi^u \cdot V^\top &= V|_{\Psi^u} + \widetilde{g}\Big(V|_{\Psi^u}, \xi^u\Big)\ell^u + \widetilde{g}\Big(V|_{\Psi^u}, \ell^u\Big)\xi^u \\ &= V|_{\Psi^u} + V(u)e^u\partial_t|_{\Psi^u} = T\Psi^u \cdot V. \end{aligned}$$

Note that V to the right of the equality is $V \in \mathfrak{X}(M)$. In a similar way we compute the tangent parts of $\partial_r|_{\Psi^u}$ and $\nabla u|_{\Psi^u}$. $\square$

Proposition 6.17. *Let A_{ξ^u}, A_{ℓ^u} be the Weingarten endomorphisms associated to the lightlike normal vector fields ξ^u, ℓ^u given in* (6.14)*, then $A_{\xi^u} = -\mathrm{Id}$ and*

$$A_{\ell^u} = e^{-2u}\left[\frac{\dot{\gamma}(0) - \|\nabla u\|^2\,\mathrm{Id}}{2} + g(\nabla u, \mathrm{Id})\nabla u \; - \nabla\nabla\, u\right], \tag{6.15}$$

where $\nabla\nabla\, u(V) := \nabla_V \nabla u$ for all $V \in \mathfrak{X}(M)$.

Proof. The first assertion is a direct consequence of Lemma 6.5. On the other hand, according again to (6.13) and Proposition 6.11, we have for $V \in \mathfrak{X}(M)$,

$$\widetilde{\nabla}_V\Big(e^{-u}\frac{\|\nabla u\|^2}{2}\partial_t|_{\Psi^u}\Big) = V\Big(e^{-u}\frac{\|\nabla u\|^2}{2}\Big)\partial_t|_{\Psi^u} + e^{-2u}\frac{\|\nabla u\|^2}{2}\,V|_{\Psi^u}, \tag{6.16}$$

$$\widetilde{\nabla}_V(e^{-2u}\partial_r|_{\Psi^u}) = -2e^{-2u}V(u)\,\partial_r|_{\Psi^u} + \frac{e^{-2u}}{2}\Big(\dot{\gamma}(0)(V)\Big)|_{\Psi^u} \tag{6.17}$$

and

$$\widetilde{\nabla}_V(e^{-2u}\nabla u|_{\Psi^u}) = -2e^{-2u}V(u)\,\nabla u|_{\Psi^u} + e^{-2u}\Big(\widetilde{\nabla}_V\nabla u\Big)|_{\Psi^u}. \tag{6.18}$$

We know that $\partial_t^\top = 0$ and, from Lemma 6.16, $\Big(\partial_r - \nabla u\Big)^\top = 0$. Then, from (6.16), (6.17) and (6.18), we arrive to

$$\Big(\widetilde{\nabla}_V\,\ell^u\Big)^\top = e^{-2u}\left[\frac{\|\nabla u\|^2}{2}V^\top - \frac{1}{2}\Big(\dot{\gamma}(0)\,(V)\Big)^\top + \Big(\widetilde{\nabla}_V\nabla u\Big)^\top\right].$$

Now, the proof ends by means of a straightforward computation from (6.8) and Lemma 6.16.

$\square$

Corollary 6.18. *Let* $\Psi^u : M \to (\widetilde{M}, \widetilde{g})$ *be the immersion given in* (6.12). *The normal vector fields* ξ^u *and* ℓ^u *are parallel with respect to the normal connection. In particular, the normal curvature tensor vanishes, that is,* $R^\perp(V,W) = 0$ *for every* $V, W \in \mathfrak{X}(M)$.

Proof. From Proposition 6.17, we know that $A_{\xi^u} = -\mathrm{Id}$. Then, the Weingarten formula reads as follows

$$\widetilde{\nabla}_V\xi^u = T\Psi^u\cdot V + \nabla^\perp_V\,\xi^u = V(u)e^u\partial_t|_{\Psi^u} + V|_{\Psi^u} + \nabla^\perp_V\,\xi^u.$$

On the other hand, from (6.7), we get

$$\widetilde{\nabla}_V\xi^u = \widetilde{\nabla}_V\,(e^u\partial_t|_{\Psi^u}) = V(u)e^u\partial_t|_{\Psi^u} + e^u e^{-u}\,V|_{\Psi^u} = V(u)e^u\partial_t|_{\Psi^u} + V|_{\Psi^u},$$

and therefore $\nabla^\perp_V\,\xi^u = 0$. Now, taking into account that $\{\xi^u, \ell^u\}$ is a global lightlike normal frame, we have $V\widetilde{g}(\xi^u, \ell^u) = \widetilde{g}(\xi^u, \nabla^\perp_V\ell^u) = 0$ for every $V \in \mathfrak{X}(M)$. Thus, since Ψ^u is a codimension two spacelike immersion, there is a smooth function $h \in C^\infty(M)$ such that $\nabla^\perp_V\ell^u = h\,\xi^u$ and then $0 = \widetilde{g}(\ell^u, \nabla^\perp_V\ell^u) = -h$ and so $\nabla^\perp_V\,\ell^u = 0$. $\square$

Remark 6.19. From Proposition 6.17 and formula (2.6), one obtains the second fundamental form II^u of Ψ^u as follows

$$\mathrm{II}^u(V,W) = -g\Big(\frac{\dot{\gamma}(0)(V) - \|\nabla u\|^2\,V}{2} + V(u)\nabla u\, - \nabla_V\nabla\,u, W\Big)\xi^u + e^{2u}g(V,W)\ell^u,$$

for every $V, W \in \mathfrak{X}(M)$. In particular, the corresponding mean curvature vector field is

$$\mathbf{H}^u = \frac{e^{-2u}}{n}\Big(\Delta u - \frac{\mathrm{trace}(\dot\gamma(0)) - (n-2)\|\nabla u\|^2}{2}\Big)\xi^u + \ell^u, \tag{6.19}$$

where Δ denotes the Laplace operator of the metric g.

Now, we are in position to state the main result of this Chapter. Assume (M, c) is an n-dimensional Riemannian conformal structure and $\gamma\colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$ is an admissible smooth 1-parameter family. By means of Proposition 6.17, we have $A_{\ell^0} = \frac{\dot\gamma(0)}{2}$ and then, for every $u \in \mathcal{C}^\infty(M)$,

$$A_{\ell^u} = e^{-2u}\left[A_{\ell^0} - \frac{1}{2}\|\nabla u\|^2\,\mathrm{Id} + g(\nabla u, \mathrm{Id})\nabla u \; - \nabla\nabla\, u\right].$$

Hence, for every $V, W \in \mathfrak{X}(M)$, we get

$$e^{2u} g\left(A_{\ell^u}(V), W\right) = g(A_{\ell^0}(V), W) - \frac{\|\nabla u\|^2}{2} g - \mathrm{Hess}(u) + du \otimes du.$$

In other words, the assignment

$$D : c \to \mathcal{T}_{(0,2)}M, \quad e^{2u} g \mapsto e^{2u} g\left(A_{\ell^u}(-), -\right), \tag{6.20}$$

satisfies the conformal transformation law (3.9). In addition, if we assume $\mathrm{trace}_g(A_{\ell^0}) = \frac{\mathrm{S}^g}{2(n-1)}$, the map D defines a Möbius structure for the conformal class c. Therefore, we have obtained the following result.

Theorem 6.20. *Let (M, c) be an n-dimensional Riemannian conformal structure. Assume the admissible smooth 1-parameter family $\gamma\colon \mathbb{R} \to \mathcal{T}_{(1,1)}M$ satisfies $\mathrm{trace}(\dot\gamma(0)) = \frac{\mathrm{S}^g}{n-1}$. Then, the assignment D given in (6.20) defines a Möbius structure for the conformal class c.*

Conversely, every Möbius structure (M, c, D) can be constructed (at least locally) from the above Theorem. In fact, fix $g \in c$ and consider

$$\gamma(r) = \mathrm{Id} + 2r\,\widehat{D}(g),$$

where $D(g)(V, W) = g(\widehat{D}(g)(V), W)$ for $V, W \in \mathfrak{X}(M)$. For any $x \in M$, there is an open subset $x \in \mathcal{O} \subset M$ such that γ is an admissible smooth 1-parameter family on $\mathcal{T}_{(1,1)}\mathcal{O}$. It is easily checked $(\mathcal{O}, c, D)$ is obtained from γ by means of Theorem 6.20. Note that $\gamma(r) = \mathrm{Id} + 2r\widehat{D}(g)$ can be replaced for any curve with $\gamma(0) = \mathrm{Id}$ and $\dot\gamma(0) = 2\widehat{D}(g)$.

Remark 6.21. If we remove the hypothesis $\mathrm{trace}(\dot{\gamma}(0)) = \frac{\mathrm{S}^g}{n-1}$ in Theorem 6.20, the assignment D given in (6.20) recovers (at least locally) all the "Schouten type-tensors" in the sense of Definition 3.12.

Remark 6.22. When M is compact, every Möbius structure (M, c, D) is globally recovered from suitable Weingarten endomorphisms as in Theorem 6.20. As a consequence of Remark 6.21, the same can be asserted for the "Schouten type-tensors".

Corollary 6.23. *Let (M, g) be an $(n \geq 3)$-dimensional Riemannian manifold. Then, the Schouten tensor P^g is given by $P^g = g(A(-), -)$ (at least locally) where A is the Weingarten endomorphism of a suitable isometric codimension two immersion of (M, g) in a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$.*

Remark 6.24. This result could be compared with the classical Brinkmann result [12] which is stated in Remark 5.10.

Remark 6.25. Since, there is no preferred Möbius structure on a two dimensional manifold, Theorem 6.20 provides an explicit method to construct such structures. Moreover, by means of Corollary 6.14, the condition $\mathrm{trace}(\dot{\gamma}(0)) = 2K^g$, where K^g is the Gauss curvature of the fixed metric g, implies that the Ricci tensor of $\widetilde{g}$ satisfies $\widetilde{\mathrm{Ric}}|_{\iota_g(\mathcal{Q})}(T\iota_g \cdot \xi, T\iota_g \cdot \eta) = 0$ for all $\xi, \eta \in \mathfrak{X}(\mathcal{Q})$.

Remark 6.26. Under the assumption $\mathrm{trace}(\dot{\gamma}(0)) = \frac{\mathrm{S}^g}{n-1}$ and by means of the relationship between the scalar curvature of conformally related metrics, formula (6.19) reduces to

$$\mathbf{H}^u = -\frac{1}{2n(n-1)}\mathrm{S}^{e^{2u}g}\,\xi^u + \ell^u,$$

and therefore, $\|\mathbf{H}^u\|^2 = \frac{\mathrm{S}^{e^{2u}g}}{n(n-1)}$. This formula widely generalizes [52, Cor. 4.5] and [56, Cor. 3.7]. Therefore, the causality of $\mathbf{H}^u$ is determined by the sign of $\mathrm{S}^{e^{2u}g}$. For two dimensional compact Riemannian conformal structures (M, c) and, as direct consequence of the Gauss-Bonnet theorem, we get

$$\int_M e^{2u}\|\mathbf{H}^u\|^2\, d\mu_g = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of the manifold M and $d\mu_g$ is the canonical measure associated to g. Also, from Corollary 6.18, the condition $\nabla^{\perp}\mathbf{H}^u = 0$ is equivalent to $\mathrm{S}^{e^{2u}g}$ being constant (compare with [56, Cor. 3.10]). The positive solution to the Yamabe problem (see [41]) states that on every $(n \geq 3)$-dimensional compact Riemannian conformal structure (M, c) there is a metric $g \in c$ with constant scalar curvature. Therefore, in the compact case, there exists an immersion Ψ^u as in (6.12) with parallel mean curvature vector field.

6.4 An Application

For a Möbius structure (M, c, D) on a two dimensional manifold M, the Cotton-York tensor for $g \in c$ has been introduced in [15] and [60] as follows

$$C(g)(U,V,W) = g\left(\left(\nabla_U \widehat{D}(g)\right)(V) - \left(\nabla_V \widehat{D}(g)\right)(U), W\right), \quad U,V,W \in \mathfrak{X}(M). \tag{6.21}$$

This definition formally agrees with the usual Cotton-York tensor defined from the Schouten tensor of an $(n \geq 3)$-dimensional Riemannian manifold (M, g). The Cotton-York tensor given in (6.21) for $n = 2$ satisfies $C(g) = C(e^{2u}g)$ (e.g., [60]).

In this Section, we assume (M, c, D) is a Möbius structure on a two dimensional manifold M which is achieved by means of Theorem 6.20.

Lemma 6.27. *Let (M, c, D) be a Möbius structure on a two dimensional manifold M. Then, the Cotton-York tensor satisfies*

$$C(g)(V,U,W)\xi^u = (\nabla_U \mathrm{II}^u)(V,W) - (\nabla_V \mathrm{II}^u)(U,W),$$

for Ψ^u as in (6.12). Hence, the Codazzi equation (2.5) reduces to

$$\left(\widetilde{R}(U,V)W\right)^{\perp} = C(g)(V,U,W)\xi^u.$$

Proof. According to Remark 6.19, the second fundamental form of Ψ^u is

$$\mathrm{II}^u(V,W) = -D(e^{2u}g)(V,W)\xi^u + e^{2u}g(V,W)\ell^u. \tag{6.22}$$

From Corollary 6.18, we have $\nabla^{\perp}_U \xi^u = \nabla^{\perp}_U \ell^u = 0$ and then, a direct computation gives

$$\nabla^{\perp}_U(\mathrm{II}^u(V,W)) = -e^{2u}g\left(\left(\nabla^{e^{2u}g}_U \widehat{D}(e^{2u}g)\right)(V), W\right)\xi^u.$$

where $\nabla^{e^{2u}g}$ is the Levi-Civita connection of $e^{2u}g$. Now, the derivative of the second fundamental form in (2.4) is easily computed. The proof ends by means of (6.21) and $C(g) = C(e^{2u}g)$ for $n = 2$. $\square$

Definition 6.28. ([15], [60]) *A Möbius structure (M, c, D) on a two dimensional manifold M is called flat when $C(g) = 0$ for every $g \in c$.*

As a direct consequence of Lemma 6.27, we have the following result.

Proposition 6.29. *A Möbius structure* (M, c, D) *on a two dimensional manifold* M *is flat if and only if for every immersion* $\Psi^u : M \to (\widetilde{M}, \widetilde{g})$ *as in* (6.12)*, the curvature tensor* $\widetilde{R}$ *of the pre-ambient manifold* $(\widetilde{M}, \widetilde{g})$ *satisfies*

$$\widetilde{R}(U, V)W \in \mathfrak{X}(M) \subset \overline{\mathfrak{X}}(M),$$

for all $U, V, W \in \mathfrak{X}(M)$.

Remark 6.30. For a flat Möbius structure (M, c, D), Proposition 6.29 states that tangent spaces of M along Ψ^u are invariant under the curvature tensor of $(\widetilde{M}, \widetilde{g})$. As far as we know, the theory of immersions satisfying this condition appeared for the first time in [49]. K. Ogiue called these immersions as invariant immersions. This condition generalizes properties of the immersions into manifolds of constant sectional curvature. The existence of curvature invariant tangent subspaces in a general Riemannian manifold is related with the existence of totally geodesic immersions (see [64] for more details).

Conclusiones

En esta tesis hemos explorado la intersección entre la geometría lorentziana y la geometría conforme riemanniana. La teoría de subvariedades ha demostrado ser una herramienta realmente potente para mostrar con claridad las conexiones tan profundas que tienen ambas subramas de la Geometría Diferencial. Para ser precisos, hemos visto cómo, por medio de inmersiones espaciales de codimensión dos que factoricen a través de hipersuperficies luminosas de ciertos espacio-tiempos, somos capaces de reconstruir objetos muy relevantes desde el punto de vista de la geometría conforme riemanniana. En resumen, esta investigación ha contribuido a ampliar nuestra visión sobre la geometría del espacio-tiempo y su conexión con la geometría conforme, destacando la riqueza y la profundidad de estos campos interrelacionados.

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