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Bernstein–Jacobi-type operators preserving derivatives

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Abstract
A general frame for Bernstein-type operators that preserve derivatives is given. We introduce Bernstein-type operators based in the weighted classical Jacobi inner product on the interval [0, 1] that extend the well known Bernstein–Durrmeyer operator as well as some other types of Bernstein operators that appear in the literature. Apart from standard results, we deduce properties about the preservation of derivatives and prove that classical Jacobi orthogonal polynomials on [0, 1] are the eigenfunctions of these operators. We also study the limit cases when one of the parameters of the Jacobi polynomials is a negative integer. Finally, we study several numerical examples.

Keywords Bernstein-type operators · Classical Jacobi polynomials · Extended Jacobi polynomials

Mathematics Subject Classification Primary 33C50 · 42C05

1 Introduction

In 1912, Bernstein [5] provided a constructive proof of the Weierstrass Approximation Theorem, that states that every continuous function defined over a closed interval can be uniformly approximated by polynomials. In fact, Bernstein introduced the so-called (classical) Bernstein polynomials as

$$\mathcal{B}_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \tag{1.1}$$

for $f \in \mathcal{C}[0, 1]$ and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. The above expression can be seen as an operator transforming continuous functions into polynomials of limited degree, and can be extended to a wide class of functions defined on [0, 1].

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According to Lorentz (1986), *Bernstein operators are the most important and interesting concrete operators on a space of continuous functions*. Many properties about this kind of polynomial approximants were established (see, for instance Lorentz 1986), and Bernstein polynomials have been a fundamental pillar in Approximation Theory since then. The eigenstructure of the classical Bernstein operator was studied in Cooper and Waldron (2000), but it depends on the index n.

Very soon, several authors tried to extend and/or modify the Bernstein polynomials in different directions to improve or obtain some particular properties. On the one hand, several papers have devoted their study to the extension of this kind of polynomials to non-closed intervals (such as the Szász–Mirakyan operators, see for instance, Szász (1950), Berdysheva and Al-Aidarous (2016)), or to the extension to several variables (see Derriennic 1985; Waldron 2006, among others). Another kind of modifications are given by substituting the values of the function in (1.1) for other mean values (Kantorovitch 1930; Durrmeyer 1967; Derriennic 1981; Berens and Xu 1991; Gupta et al. 2009; Berdysheva 2015, and many other papers). In both cases, the main properties of the classical Bernstein operator are inherited by the modifications.

As far as we know, modifications by means of piecewise integrals of the function were introduced in the pioneering paper by Kantorovitch (1930). Later, Durrmeyer (1967) defined the modified Bernstein operator given by

$$\mathcal{M}_n(f,x) = (n+1)\sum_{k=0}^{n}\int_0^1 f(t)p_{n,k}(t)\mathrm{d}t\; p_{n,k}(x), \tag{1.2}$$

for integrable functions, and was deeply studied by Derriennic in Derriennic (1981). In that paper, several properties analogous to the properties of the classical Bernstein operator were proved. Observing the expression (1.2), the mean values of the function can be read as

$$(n+1)\int_0^1 f(t)p_{n,k}(t)\mathrm{d}t = \frac{\langle f, p_{n,k}\rangle}{\langle 1, p_{n,k}\rangle},$$

where $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ denotes the classical Legendre inner product. Unlike the classical Bernstein operator, the author obtained a complete set of eigenfunctions independent of n, given by the classical Legendre polynomials.

Later, Sablonnière (1981) extended Durrmeyer's operator introducing the classical Jacobi weight function as

$$\mathcal{B}_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n}\frac{\langle f, p_{n,k}\rangle_{\alpha,\beta}}{\langle 1, p_{n,k}\rangle_{\alpha,\beta}}\, p_{n,k}(x), \tag{1.3}$$

where $\langle f, g\rangle_{\alpha,\beta} = \int_0^1 f(t)\,g(t)\,t^{\alpha}\,(1-t)^{\beta}\mathrm{d}t$, for $\alpha, \beta > -1$. For $\alpha = \beta = 0$, the Durrmeyer operator appears. This time, classical Jacobi orthogonal polynomials on $[0, 1]$ are the eigenfunctions of the operator (1.3). Moreover, a new property appears, the preservation of the derivatives, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{B}_n^{(\alpha,\beta)}(f,x) = \mathcal{B}_n^{(\alpha+1,\beta+1)}\Big(\frac{\mathrm{d}}{\mathrm{d}x}f, x\Big),$$

for a differentiable function f. In Gupta et al. (2009), the authors studied simultaneous approximation by a type of Bernstein-Durrmeyer operator that preserves the derivatives.

This work intends to provide a general frame for Bernstein-type operators related to classical Jacobi polynomials and preserving derivatives in the above sense. In particular, we define an operator based on the Jacobi inner product, that comprises a much wider class of operators than those studied by Durrmeyer and Derriennic, Sablonnière, and Gupta, among others. Apart from the standard properties satisfied by a Bernstein-type operator (uniform convergence, conservative properties, Voronowskaja type theorem, etc.), our new operator has two important properties: it admits a complete set of eigenfunctions independent of n that are the Jacobi orthogonal polynomials on [0, 1], and preserves the derivative of the function. Moreover, although the standard Jacobi parameters are given by $\alpha, \beta > -1$ to assure the convergence of the integrals, we will extend the definition of the Bernstein-type operators for non-standard values of $\alpha = -l$, for $l = 1, 2, \ldots$, obtaining that the so-called generalized Jacobi polynomials (Szegő 1975, p. 64) are the eigenfunctions of the operator. In this paper, we collect some useful properties satisfied by Jacobi polynomials transforming expressions in Abramowitz and Stegun (1972) and Szegő (1975) from the interval $[-1, 1]$ to $[0, 1]$.

This paper is organized as follows: In Sect. 2, we define the Bernstein-Jacobi-type operator, and we include the first properties, and the convergence results. The eigenfunctions of the operator are analysed in Sect. 3. In Sect. 4, we study the derivative properties, proving the preservation of the derivatives. Sect. 5 deals with the relation of the Bernstein–Jacobi-type operator with the classical Durrmeyer-Derriennic operators. Section 6 focuses on the study of the limit case $\alpha = -1$, introducing non-standard values of the Jacobi parameters. In this section, we will prove that the operator introduced in Gupta et al. (2009) is a particular case of our Bernstein–Jacobi-type operator. The case $\beta = -1$ is also considered. Moreover, we show that generalized Jacobi polynomials are the eigenfunctions of the Bernstein–Jacobi-type operator, and that this operator also preserves the derivative properties.

In Sect. 7, we analyse the extension of our results to the general non-standard case $\alpha = -l$, for $l = 1, 2, \ldots$, dealing with a non-standard Bernstein-Jacobi-type operator. Finally, numerical experiments for test functions contained in Surjanovic and Bingham (2013) are analysed.

2 Bernstein–Jacobi-type operators

In this paper we will work with the classical Jacobi inner product that we will review here. Let $w^{\alpha,\beta}(x) = x^{\alpha}(1-x)^{\beta}$, $x \in (0, 1)$, $\alpha, \beta > -1$, be the Jacobi weight function on (0, 1), and let

$$\langle f, g\rangle_{\alpha,\beta} = \int_0^1 f(t)\, g(t)\, w^{\alpha,\beta}(t)\, \mathrm{d}t, \tag{2.1}$$

be the corresponding Jacobi inner product, for $f, g \in L^2_{w^{\alpha,\beta}}[0, 1] = L^2_{\alpha,\beta}[0, 1]$. When the involvement of the parameters α and β is clear from context, we will omit them.

For $0 \le k \le n$, the basic Bernstein polynomials

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, \ldots, n,$$

where $x \in [0, 1]$. For mathematical convenience, we will consider $p_{n,k}(x) = 0$, for $k < 0$ or $k > n$. The set of basic Bernstein polynomials of degree n,

$$\{p_{n,k}(x) : 0 \le k \le n\},$$

forms a basis of Π_n, the linear space of polynomials with real coefficients of degree less than or equal to n. Moreover, $0 \le p_{n,k}(x) \le 1$ and

$$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad x \in [0,1], \quad n \ge 0.$$

Several useful properties will be collect in the next lemma.

Lemma 2.1 *The following formulas hold,*

(1) *For* $0 \le k \le n$,

$$\langle 1, p_{n,k}\rangle_{\alpha,\beta} = \binom{n}{k} \frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}. \tag{2.2}$$

(2) *For* $0 \le r \le n$,

$$D^r p_{n,k}(x) = \frac{n!}{(n-r)!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} p_{n-r,k-j}(x), \tag{2.3}$$

where D^r *means the standard* r*-th derivative.*

(3) *For* $0 \le r \le n$,

$$\begin{aligned} D^r[p_{n,k}(x)w(x)] =& \frac{n!}{(n-r)!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{(n-k-r+j+\beta+1)_{r-j}}{(n-k-r+j+1)_{r-j}} \\ &\times \frac{(k-j+\alpha+1)_j}{(k-j+1)_j} p_{n-r,k-j}(x)w(x). \end{aligned} \tag{2.4}$$

For $0 \le r \le n$, and $\alpha, \beta > -1$, we define

$$\lambda_{n,r}^{(\alpha,\beta)} = \frac{n!}{(n-r)!} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+r+\alpha+\beta+2)} = \frac{(n-r+1)_r}{(n+\alpha+\beta+2)_r}, \tag{2.5}$$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$, $a \in \mathbb{R}$, $n \ge 0$, denotes, as usual, the Pochhammer symbol.

Notice that $0 \le \lambda_{n,r}^{(\alpha,\beta)} \le 1$; $\lambda_{n,r}^{(\alpha,\beta)} = 0$ for $r > n$; $\lambda_{n,0}^{(\alpha,\beta)} = 1$ for $n \ge 0$, and

$$\lim_{n\to+\infty} \lambda_{n,r}^{(\alpha,\beta)} = 1,$$

for $0 \le r \le n$. Moreover, for $\alpha = -1$ and/or $\beta = -1$, expression (2.5) is also well defined.

An inductive reasoning allows us to prove that

$$\lim_{n\to+\infty} n\left[\lambda_{n,r}^{(\alpha,\beta)} - 1\right] = -r(r+\alpha+\beta+1), \quad r = 0, \ldots, n.$$

Now, we define the Bernstein–Jacobi-type operator.

Definition 2.2 For $0 \le r \le n$, and $f \in L^2_{\alpha,\beta}[0,1]$, the Bernstein–Jacobi-type operator is defined as follows

$$\begin{aligned} \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) &= \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \frac{\langle f, p_{n+r,k+r}\rangle_{\alpha,\beta}}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\ &= \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \frac{\int_0^1 f(t) p_{n+r,k+r}(t) w^{\alpha,\beta}(t)\,\mathrm{d}t}{\int_0^1 p_{n+r,k+r}(t) w^{\alpha,\beta}(t)\,\mathrm{d}t} p_{n-r,k}(x), \end{aligned} \tag{2.6}$$

where $\lambda_{n,r}^{(\alpha,\beta)}$ is defined in (2.5).

The Bernstein–Jacobi-type operator (2.6) can be written as

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) = \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \mu_{n+r,k+r}^{(\alpha,\beta)}(f)\, p_{n-r,k}(x).$$

where we define the constants

$$\mu_{n,k}^{(\alpha,\beta)}(f) = \frac{\langle f, p_{n,k}\rangle_{\alpha,\beta}}{\langle 1, p_{n,k}\rangle_{\alpha,\beta}} = \frac{\int_0^1 f(t)p_{n,k}(t)w^{\alpha,\beta}(t)\,\mathrm{d}t}{\int_0^1 p_{n,k}(t)w^{\alpha,\beta}(t)\,\mathrm{d}t} = \frac{\int_0^1 f(t)t^{k+\alpha}(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k+\alpha}(1-t)^{n-k+\beta}\,\mathrm{d}t}, \tag{2.7}$$

for $k = 0, \ldots, n$. We must observe that $\mu_{n,k}^{(\alpha,\beta)}(f)$ is well defined for $k + \alpha > -1$, and $n - k + \beta > -1$.

The above operator is linear, positive and transforms integrable functions into polynomials of degree less than or equal to $n - r$.

Analogously to the classical Bernstein operators, we prove that the Bernstein–Jacobi-type operator preserves the degree of the polynomials, and we can give its explicit expression in this case.

Lemma 2.3 *For $m \geq 0$, we get*

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^m, x) = \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \frac{(k+r+\alpha+1)_m}{(n+r+\alpha+\beta+2)_m}\, p_{n-r,k}(x), \tag{2.8}$$

and

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^m, x) = \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{m} \binom{m}{k} \frac{(n-r-k+1)_k(k+r+\alpha+1)_{m-k}}{(n+r+\alpha+\beta+2)_m}\, x^k. \tag{2.9}$$

As a consequence, the Bernstein–Jacobi-type operator preserves the degree. Moreover,

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(1, x) = \lambda_{n,r}^{(\alpha,\beta)}, \tag{2.10}$$

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(x, x) = \lambda_{n,r}^{(\alpha,\beta)} \frac{(n-r)x + r + \alpha + 1}{n + r + \alpha + \beta + 2}, \tag{2.11}$$

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^2, x) = \lambda_{n,r}^{(\alpha,\beta)} \frac{(n-r-1)_2 x^2 + 2(n-r)(r+\alpha+2)x + (r+\alpha+1)_2}{(n+r+\alpha+\beta+2)_2} \tag{2.12}$$

We must remark that expressions (2.8)–(2.9) are also valid when $\alpha = -1$ and $\beta = -1$.

Proof Expression (2.8) is a direct consequence of the Beta function and the Pochhammer symbol. A straightforward induction on m allows us to prove

$$\sum_{k=0}^{n-r} (k+a)_m\, p_{n-r,k}(x) = \sum_{k=0}^{m} \binom{m}{k} (n-r-k+1)_k (k+a)_{m-k}\, x^k,$$

and the result follows. Computing directly on the explicit expressions, we can get (2.10), (2.11) and (2.12). □

When $f \in \mathcal{C}[0,1]$ is a continuous function we prove the uniform convergence by using the Korovkin Theorem (Lorentz 1986) and Lemma 2.3, since $\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^m,x)$ and $\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^m,x)/\lambda_{n,r}^{(\alpha,\beta)}$ converge uniformly to x^m for $m=0,1,2$.

Theorem 2.4 *For* $r \geq 0$ *and* $f \in \mathcal{C}[0,1]$,

$$\lim_{n\to+\infty} \|\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) - f(x)\|_\infty = 0,$$

and

$$\lim_{n\to+\infty} \left\| \frac{1}{\lambda_{n,r}^{(\alpha,\beta)}} \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) - f(x) \right\|_\infty = 0.$$

Following the proof given in Sablonnière (1981), we stablish the order of convergence in terms of the modulus of continuity $\omega(f,\delta)$, for $f \in \mathcal{C}[0,1]$. Given $\delta > 0$ the modulus of continuity is given by

$$\omega(f,\delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|, \quad x, y \in [0,1].$$

Theorem 2.5 *For* $f \in \mathcal{C}[0,1]$, *there exists* $n(r,\alpha,\beta) \in \mathbb{N}$ *such that for* $n \geq n(r,\alpha,\beta)$ *we have*

$$\left\| \frac{1}{\lambda_{n,r}^{(\alpha,\beta)}} \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) - f(x) \right\|_\infty \leq 2\,\omega\left(f, \frac{1}{\sqrt{n}}\right).$$

Proof On the one hand, for all $\delta > 0$, we get the following property

$$|f(t) - f(x)| \leq \omega(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega(f,\delta). \tag{2.13}$$

On the other hand, by the convexity of $x \longmapsto x^2$ and the Cauchy–Schwarz inequality we have

$$\left[\sum_{k=0}^{n-r} \frac{\int_0^1 |t-x| p_{n+r,k+r}(t) w(t)\,\mathrm{d}t}{\int_0^1 p_{n+r,k+r}(t) w(t)\,\mathrm{d}t} p_{n-r,k}(x)\right]^2$$
$$\leq \sum_{k=0}^{n-r} \frac{\int_0^1 (t-x)^2 p_{n+r,k+r}(t) w(t)\,\mathrm{d}t}{\int_0^1 p_{n+r,k+r}(t) w(t)\,\mathrm{d}t} p_{n-r,k}(x) = T_{n,r,2}^{(\alpha,\beta)}(x),$$

where

$$T_{n,r,2}^{(\alpha,\beta)}(x) = \frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\left[\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^2,x) - 2x\,\mathscr{L}_{n,r}^{(\alpha,\beta)}(x,x) + x^2\right], \tag{2.14}$$

is a polynomial of degree less than or equal to 2, where (2.14) was obtained using (2.10), (2.11), and (2.12).

Therefore, using the explicit expression of the operator, and the fact that

$$f(x) = f(x)\sum_{k=0}^{n-r} p_{n-r,k}(x) = f(x)\sum_{k=0}^{n-r} \frac{\int_0^1 p_{n+r,k+r}(t) w(t)\,\mathrm{d}t}{\int_0^1 p_{n+r,k+r}(t) w(t)\,\mathrm{d}t} p_{n-r,k}(x),$$

we get

$$\left|\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x)-f(x)\right| \leq \sum_{k=0}^{n-r}\frac{\int_0^1|f(t)-f(x)|p_{n+r,k+r}(t)w(t)\,\mathrm{d}t}{\int_0^1 p_{n+r,k+r}(t)w(t)\,\mathrm{d}t}p_{n-r,k}(x)$$
$$\leq\left[1+\sqrt{nT_{n,r,2}^{(\alpha,\beta)}(x)}\right]\omega\left(f,\frac{1}{\sqrt{n}}\right),$$

where we substituted (2.13) with $\delta=\frac{1}{\sqrt{n}}$.

From the explicit expression of $T_{n,r,2}^{(\alpha,\beta)}(x)$, it can be verified that

$$x_{n,r}^{(\alpha,\beta)}=\frac{n-(2r+\alpha+\beta+3)(r+\alpha+1)+r}{2n-4r^2+(4r+\alpha+\beta+2)(\alpha+\beta+3)},$$

is a maximum of $T_{n,r,2}^{(\alpha,\beta)}(x)$, and

$$nT_{n,r,2}^{(\alpha,\beta)}(x_{n,r}^{(\alpha,\beta)})\longrightarrow\frac{1}{2}.$$

Therefore, there exists $n(r,\alpha,\beta)$ such that $nT_{n,r,2}^{(\alpha,\beta)}(x)\leq 1$ for $n\geq n(r,\alpha,\beta)$. □

Finally, as Voronowskaja did for the classical Bernstein operator, an asymptotic formula for the Bernstein–Jacobi-type operator can be proved.

Theorem 2.6 (Asymptotic formula) *Let $f\in L^2_{\alpha,\beta}[0,1]$, and suppose that the second derivative $f''(x)$ exists for $x\in[0,1]$. Then,*

$$\lim_{n\to+\infty} n\left[\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x)-f(x)\right]$$
$$=x(1-x)f''(x)+[r+\alpha+1-(2r+\alpha+\beta+2)x]\,f'(x). \tag{2.15}$$

Proof The Taylor formula of second order of f at the point x is given by

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+h(t-x)(t-x)^2, \tag{2.16}$$

where h is an integrable and bounded function on $[-x,1-x]$, and verifies that $h(u)\to 0$ when $u\to 0$. Applying the Bernstein–Jacobi-type operator to (2.16) we have

$$\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x)=f(x)+f'(x)T_{n,r,1}^{(\alpha,\beta)}(x)+\frac{1}{2}f''(x)T_{n,r,2}^{(\alpha,\beta)}(x)+\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\mathscr{L}_{n,r}^{(\alpha,\beta)}(g,x),$$

where $g(t,x)=h(t-x)(t-x)^2$ and by (2.10), (2.11), and (2.12),

$$T_{n,r,1}^{(\alpha,\beta)}(x)=\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\left[\mathscr{L}_{n,r}^{(\alpha,\beta)}(x,x)-x\right]=\frac{r+\alpha+1-(2r+\alpha+\beta+2)x}{n+r+\alpha+\beta+2},$$
$$T_{n,r,2}^{(\alpha,\beta)}(x)=\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}\left[\mathscr{L}_{n,r}^{(\alpha,\beta)}(x^2,x)-2x\,\mathscr{L}_{n,r}^{(\alpha,\beta)}(x,x)+x^2\right].$$

Multiplying the above explicit expressions by n, and taking the limit when $n\to+\infty$, we get (2.15) plus the term

$$\lim_{n\to+\infty}\left[\frac{n}{\lambda_{n,r}^{(\alpha,\beta)}}\mathscr{L}_{n,r}^{(\alpha,\beta)}(g,x)\right].$$

We are going to prove that this term vanishes. Fixed $\epsilon > 0$, there is some $\delta > 0$ such that if $|u| < \delta$, then $|h(u)| < \epsilon$. We divide the interval in two parts

$$A = \{t \in [-x, 1-x] : |x - t| < \delta\} \cap [0, 1],$$

and

$$B = \{t \in [-x, 1-x] : |x - t| \geq \delta\} \cap [0, 1].$$

Then,

$$\begin{aligned}
\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}} \mathscr{L}_{n,r}^{(\alpha,\beta)}(g, x) &= \sum_{k=0}^{n-r} \frac{\int_A h(t-x)(t-x)^2 p_{n+r,k+r}(t) w(t)\, \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\
&\quad + \sum_{k=0}^{n-r} \frac{\int_B h(t-x)(t-x)^2 p_{n+r,k+r}(t) w(t)\, \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\
&\leq \epsilon \sum_{k=0}^{n-r} \frac{\int_0^1 (t-x)^2 p_{n+r,k+r}(t) w(t)\, \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\
&\quad + \frac{M}{\delta^2} \sum_{k=0}^{n-r} \frac{\int_0^1 (t-x)^4 p_{n+r,k+r}(t) w(t)\, \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\
&= \epsilon T_{n,r,2}^{(\alpha,\beta)}(x) + \frac{M}{\delta^2} T_{n,r,4}^{(\alpha,\beta)}(x),
\end{aligned}$$

where

$$T_{n,r,4}^{(\alpha,\beta)}(x) = \frac{1}{\lambda_{n,r}^{(\alpha,\beta)}} \sum_{k=0}^{4} (-1)^k \binom{4}{k} x^k \mathscr{L}_{n,r}^{(\alpha,\beta)}(x^{4-k}, x),$$

and M is the bound of the function h. A straightforward computation allows us to show that

$$T_{n,r,4}^{(\alpha,\beta)}(x) = \mathcal{O}(n^{-2}),$$

and using (2.14), we conclude

$$\lim_{n\to+\infty} \left[\frac{n}{\lambda_{n,r}^{(\alpha,\beta)}} \mathscr{L}_{n,r}^{(\alpha,\beta)}(g, x) \right] \leq \epsilon \lim_{n\to+\infty} \left[n T_{n,r,2}^{(\alpha,\beta)}(x) \right] + \frac{M}{\delta^2} \lim_{n\to+\infty} \left[n T_{n,r,4}^{(\alpha,\beta)}(x) \right] \leq \frac{\epsilon}{2}.$$

□

We must remark that the classical Jacobi polynomial shifted to the interval $[0, 1]$, $P_n^{(\alpha,\beta)}$, is a solution for the second order differential equation

$$x\,(1-x)\,y'' + [(\alpha+1) - (\alpha+\beta+2)x]\,y' + n\,(n+\alpha+\beta+1)\,y = 0. \tag{2.17}$$

Thus, the Voronowskaja-type formula is related to classical Jacobi polynomials.

3 Eigenfunctions

In this section, we prove that the Bernstein–Jacobi-type operator (2.6) admits a complete set of eigenfunctions. These eigenfunctions are the classical Jacobi polynomials shifted to the interval [0, 1], writing by $P_n^{(\alpha,\beta)}(t) = \widehat{P}_n^{(\alpha,\beta)}(2t-1)$ as the classical Jacobi polynomials on $[-1, 1]$, orthogonal with respect to the inner product (2.1) such that

$$h_n^{(\alpha,\beta)} = \int_0^1 [P_n^{(\alpha,\beta)}(t)]^2\, w^{\alpha,\beta}(t)\,\mathrm{d}t = \frac{1}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}. \tag{3.1}$$

To prove that classical Jacobi polynomials are eigenfunctions of the Bernstein–Jacobi-type operator, we need some lemmas.

Lemma 3.1 *For $0 \le k \le n-r$, and $\alpha, \beta > -1$, we get*

$$\mu_{n+r,k+r}^{(\alpha,\beta)}(f) = \mu_{n-r,k}^{(\alpha+r,\beta+r)}(f).$$

Proof Observe that

$$\begin{aligned}\mu_{n+r,k+r}^{(\alpha,\beta)}(f) &= \frac{\langle f, p_{n+r,k+r}\rangle_{\alpha,\beta}}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} = \frac{\int_0^1 f(t)t^{k+r+\alpha}(1-t)^{n-k+\beta}\mathrm{d}t}{\int_0^1 t^{k+r+\alpha}(1-t)^{n-k+\beta}\mathrm{d}t}\\ &= \frac{\langle f, p_{n-r,k}\rangle_{\alpha+r,\beta+r}}{\langle 1, p_{n-r,k}\rangle_{\alpha+r,\beta+r}} = \mu_{n-r,k}^{(\alpha+r,\beta+r)}(f).\end{aligned}$$

□

Lemma 3.2 *The Bernstein–Jacobi-type operator (2.6) is symmetric with respect to the Jacobi inner product $\langle\cdot,\cdot\rangle_{\alpha+r,\beta+r}$, that is,*

$$\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x), g\rangle_{\alpha+r,\beta+r} = \langle f, \mathscr{L}_{n,r}^{(\alpha,\beta)}(g,x)\rangle_{\alpha+r,\beta+r}.$$

Proof Using the definition (2.6) and Lemma 3.1, we compute

$$\begin{aligned}\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x), g\rangle_{\alpha+r,\beta+r} &= \lambda_{n,r}^{(\alpha,\beta)}\sum_{k=0}^{n-r}\frac{\langle f, p_{n+r,k+r}\rangle_{\alpha,\beta}}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}}\langle p_{n-r,k}, g\rangle_{\alpha+r,\beta+r}\\ &= \lambda_{n,r}^{(\alpha,\beta)}\sum_{k=0}^{n-r}\frac{\langle f, p_{n-r,k}\rangle_{\alpha+r,\beta+r}}{\langle 1, p_{n-r,k}\rangle_{\alpha+r,\beta+r}}\langle p_{n-r,k}, g\rangle_{\alpha+r,\beta+r}\\ &= \lambda_{n,r}^{(\alpha,\beta)}\sum_{k=0}^{n-r}\langle f, p_{n-r,k}\rangle_{\alpha+r,\beta+r}\frac{\langle p_{n+r,k+r}, g\rangle_{\alpha,\beta}}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}}\\ &= \langle f, \mathscr{L}_{n,r}^{(\alpha,\beta)}(g,x)\rangle_{\alpha+r,\beta+r}.\end{aligned}$$

□

Theorem 3.3 *For $n, r \ge 0$, the eigenfunctions of the Bernstein–Jacobi-type operator are the classical Jacobi polynomials $\{P_m^{(\alpha+r,\beta+r)}\}_{m\ge 0}$. Moreover,*

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(P_m^{(\alpha+r,\beta+r)}, x) = \lambda_{n,r+m}^{(\alpha,\beta)} P_m^{(\alpha+r,\beta+r)}(x),$$

where

$$\lambda_{n,r+m}^{(\alpha,\beta)} = \begin{cases}\dfrac{(n-r-m+1)_{r+m}}{(n+\alpha+\beta+2)_{r+m}} & \text{if} \quad m \le n,\\ 0 & \text{if} \quad m > n.\end{cases}$$

Proof Since the Bernstein–Jacobi-type operator preserves the degree by Lemma 2.3, there exist constants a_i^m such that

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(P_m^{(\alpha+r,\beta+r)}, x) = \sum_{i=0}^{m} a_i^m \, P_i^{(\alpha+r,\beta+r)}(x),$$

where

$$a_i^m = \frac{\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(P_m^{(\alpha+r,\beta+r)}, x), P_i^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}}{\langle P_i^{(\alpha+r,\beta+r)}, P_i^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}}.$$

By Lemma 3.2, the Bernstein–Jacobi-type operator is symmetric, and by Lemma 2.3 preserves the degree. Therefore, by the orthogonality of the Jacobi polynomials we get

$$\begin{aligned}
&\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(P_m^{(\alpha+r,\beta+r)}, x), P_i^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r} \\
&\quad = \langle P_m^{(\alpha+r,\beta+r)}, \mathscr{L}_{n,r}^{(\alpha,\beta)}(P_i^{(\alpha+r,\beta+r)}, x)\rangle_{\alpha+r,\beta+r} = 0,
\end{aligned}$$

for $i = 0, 1, \ldots m-1$. Therefore,

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(P_m^{(\alpha+r,\beta+r)}, x) = a_m^m \, P_m^{(\alpha+r,\beta+r)}(x),$$

and the value of a_m^m can be deduce from (2.9). $\square$

Using the eigenfunctions, we can express the operator in terms of the Jacobi polynomials.

Corollary 3.4 *Let $f \in L^2_{\alpha+r,\beta+r}[0,1]$. Then*

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x) = \sum_{k=0}^{n-r} \lambda_{n,r+k}^{(\alpha,\beta)} \frac{\langle f, P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}}{h_k^{(\alpha+r,\beta+r)}} P_k^{(\alpha+r,\beta+r)}(x),$$

where $h_k^{(\alpha+r,\beta+r)} = \langle P_k^{(\alpha+r,\beta+r)}, P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}$ was defined in (3.1).

Proof Since $\mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x) \in \Pi_{n-r}$, we can express it in terms of Jacobi polynomials as

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x) = \sum_{k=0}^{n-r} \gamma_{n-r,k}(f) P_k^{(\alpha+r,\beta+r)}(x),$$

where

$$\gamma_{n-r,k}(f) = \frac{\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x), P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}}{\langle P_k^{(\alpha+r,\beta+r)}, P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r}}.$$

From Lemma 3.2 and Theorem 3.3, we deduce that

$$\begin{aligned}
\langle \mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x), P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r} &= \langle f, \mathscr{L}_{n,r}^{(\alpha,\beta)}(P_k^{(\alpha+r,\beta+r)}, x)\rangle_{\alpha+r,\beta+r} \\
&= \lambda_{n,r+k}^{(\alpha,\beta)} \langle f, P_k^{(\alpha+r,\beta+r)}\rangle_{\alpha+r,\beta+r},
\end{aligned}$$

and we get the result. $\square$

Using the expression of the operator in terms of the eigenfunctions given in Lemma 3.3, and a similar reasoning as in Theorem 3 in Sablonnière (1981) the convergence when f is a integrable function holds.

Theorem 3.5 *Let $f \in L^p_{\alpha+r,\beta+r}[0,1]$. Then $\mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x)$ converges to f on $L^p_{\alpha+r,\beta+r}[0,1]$ if $1 \leq p \leq \infty$.*

4 Derivative properties

In this section, we study the derivative properties of the Bernstein–Jacobi-type operator (2.6). We denote $Df(x) = d/dx f(x) = f'(x)$, $x \in [0, 1]$.

Lemma 4.1 *For $n \geq 0$ and a differentiable function $f(x)$, we have*

$$\mu_{n,k}^{(\alpha,\beta)}(f) - \mu_{n,k-1}^{(\alpha,\beta)}(f) = \frac{1}{n+\alpha+\beta+2}\mu_{n+1,k}^{(\alpha,\beta)}(Df), \quad k = 1, \ldots, n. \tag{4.1}$$

Proof We write

$$\mu_{n,k}^{(\alpha,\beta)}(f) - \mu_{n,k-1}^{(\alpha,\beta)}(f) = \frac{\langle 1, p_{n,k-1}\rangle\langle f, p_{n,k}\rangle - \langle 1, p_{n,k}\rangle\langle f, p_{n,k-1}\rangle}{\langle 1, p_{n,k}\rangle\langle 1, p_{n,k-1}\rangle},$$

and we compute the numerator $N(n, k, f)$ for $k = 1, \ldots, n$. Using (2.2), we get

$$\begin{aligned} N(n,k,f) =& \frac{n!}{(k-1)!(n-k)!}\frac{\Gamma(k+\alpha)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \\ & \times \int_0^1 f(t)\left[\frac{n-k+\beta+1}{n-k+1}p_{n,k}(t) - \frac{k+\alpha}{k}p_{n,k-1}(t)\right]w(t)\,\mathrm{d}t \\ =& -\frac{1}{(n+1)}\frac{n!}{(k-1)!(n-k)!}\frac{\Gamma(k+\alpha)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \\ & \times \int_0^1 f(t)D\left[p_{n+1,k}(t)w(t)\right]\,\mathrm{d}t, \end{aligned}$$

where in the last equality we have used (2.4). Integrating by parts we have that

$$\int_0^1 f(t)D\left[p_{n+1,k}(t)w(t)\right]\,\mathrm{d}t = -\int_0^1 Df(t)p_{n+1,k}(t)w(t)\,\mathrm{d}t,$$

because $p_{n+1,k}(t)w(t)\big|_0^1 = 0$. Therefore, we get

$$\mu_{n,k}^{(\alpha,\beta)}(f) - \mu_{n,k-1}^{(\alpha,\beta)}(f) = \frac{1}{n+\alpha+\beta+2}\mu_{n+1,k}^{(\alpha,\beta)}(Df),$$

and the result follows. □

Theorem 4.2 *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function such that $D^s f(x)$ exists, $\forall x \in [0,1]$ and $s \geq 1$. Then for each $n, r \in \mathbb{N}$ such that $n \geq r+s$ and $x \in [0,1]$, we have*

$$D^s \mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) = \mathscr{L}_{n,r+s}^{(\alpha,\beta)}(D^s f, x). \tag{4.2}$$

Proof We will first prove the identity (4.2) for $r = 0$, fixing $n \geq 1$, and by induction on s. For $s = 1$, using (2.3) and (4.1), we get

$$\begin{aligned} D\mathscr{L}_{n,0}^{(\alpha,\beta)}(f,x) &= \sum_{k=0}^{n}\mu_{n,k}^{(\alpha,\beta)}(f)Dp_{n,k}(x) \\ &= n\left[\sum_{k=1}^{n}\mu_{n,k}^{(\alpha,\beta)}(f)p_{n-1,k-1}(x) - \sum_{k=0}^{n-1}\mu_{n,k}^{(\alpha,\beta)}(f)p_{n-1,k}(x)\right] \\ &= n\sum_{k=1}^{n}\left[\mu_{n,k}^{(\alpha,\beta)}(f) - \mu_{n,k-1}^{(\alpha,\beta)}(f)\right]p_{n-1,k-1}(x) \end{aligned}$$

$$= \frac{n}{n+\alpha+\beta+2}\sum_{k=1}^{n}\mu_{n+1,k}^{(\alpha,\beta)}(Df)p_{n-1,k-1}(x)$$
$$= \frac{n}{n+\alpha+\beta+2}\sum_{k=0}^{n-1}\mu_{n+1,k+1}^{(\alpha,\beta)}(Df)p_{n-1,k}(x)$$
$$= \mathscr{L}_{n,1}^{(\alpha,\beta)}(Df,x),$$

which means that the identity is satisfied for $s=1$. Let us suppose now that the result holds for s, i.e.,

$$D^s\mathscr{L}_{n,0}^{(\alpha,\beta)}(f,x) = \mathscr{L}_{n,s}^{(\alpha,\beta)}(D^sf,x),$$

such that $s\geq 1$ and $n\geq s+1$ and we will prove it for $s+1$. In fact, using again (2.3) and (4.1),

$$\begin{aligned}
D^{s+1}\mathscr{L}_{n,0}^{(\alpha,\beta)}(f,x) &= D\left[D^s\mathscr{L}_{n,0}^{(\alpha,\beta)}(f,x)\right] = D\mathscr{L}_{n,s}^{(\alpha,\beta)}(D^sf,x)\\
&= (n-s)\lambda_{n,s}^{(\alpha,\beta)}\sum_{k=1}^{n-s}\left[\mu_{n+s,k+s}^{(\alpha,\beta)}(D^sf)-\mu_{n+s,k+s-1}^{(\alpha,\beta)}(D^sf)\right]p_{n-s-1,k-1}(x)\\
&= \frac{n-s}{n+s+\alpha+\beta+2}\lambda_{n,s}^{(\alpha,\beta)}\sum_{k=1}^{n-s}\mu_{n+s+1,k+s}^{(\alpha,\beta)}(D^{s+1}f)p_{n-s-1,k-1}(x)\\
&= \lambda_{n,s+1}^{(\alpha,\beta)}\sum_{k=0}^{n-s-1}\mu_{n+s+1,k+s+1}^{(\alpha,\beta)}(D^{s+1}f)p_{n-s-1,k}(x) = \mathscr{L}_{n,s+1}^{(\alpha,\beta)}(D^{s+1}f,x).
\end{aligned}$$

In this way, we complete induction over s.

Finally, if $g:[0,1]\longrightarrow\mathbb{R}$ such that $D^rg(x)=f(x)$, $\forall x\in[0,1]$, $r\geq 0$, and $r+s\leq n$, then

$$\begin{aligned}
D^s\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) &= D^s\mathscr{L}_{n,r}^{(\alpha,\beta)}(D^rg,x) = D^sD^r\mathscr{L}_{n,0}^{(\alpha,\beta)}(g,x)\\
&= D^{r+s}\mathscr{L}_{n,0}^{(\alpha,\beta)}(g,x) = \mathscr{L}_{n,r+s}^{(\alpha,\beta)}(D^{r+s}g,x) = \mathscr{L}_{n,r+s}^{(\alpha,\beta)}(D^sf,x).
\end{aligned}$$

□

Definition 4.3 Let $f\in\mathcal{C}^s[0,1]$, for $s\geq 0$. We will say that f is s-convex if $D^sf(x)\geq 0$, $\forall x\in[0,1]$.

Notice that the concept of s-convex functions generalize the increasing and convex functions for the cases $s=1$ and $s=2$, respectively. Using the derivative property (4.2) and the fact that $\mathscr{L}_{n,r}^{(\alpha,\beta)}$ is a positive operator, the preservation of the s-convexity holds.

Corollary 4.4 *Let $f:[0,1]\longrightarrow\mathbb{R}$ be a function. For $s\geq 1$, if $f\in\mathcal{C}^s[0,1]$ and s-convex, then $\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x)$ is s-convex.*

Using (4.2) and Theorem 2.5, we deduce the uniform convergence for the derivatives.

Corollary 4.5 *Let $f\in\mathcal{C}^s[0,1]$, for $s\geq 1$. Then for each $r\in\mathbb{N}$, we have*

$$\lim_{n\to+\infty}\left\|\frac{1}{\lambda_{n,r}^{(\alpha,\beta)}}D^s\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x)-D^sf(x)\right\|_\infty = 0.$$

5 Relation with the Durrmeyer–Derriennic operators

The Bernstein–Jacobi-type operator defined in (2.6) generalizes a wide class of Bernstein-type operators. In this section we will analyse two types of Bernstein operators based on Jacobi inner products that appeared in the literature.

First of all, we analyse the so-called Durrmeyer–Derriennic operator (Durrmeyer 1967; Derriennic 1981). This operator was defined as

$$\mathcal{M}_n(f,x) = (n+1)\sum_{k=0}^{n}\int_0^1 f(t)p_{n,k}(t)\mathrm{d}t\; p_{n,k}(x), \tag{5.1}$$

for $f \in L^2[0,1]$. Several properties satisfied by this operator were deduced in Durrmeyer (1967) and Derriennic (1981). From the definition (2.6), we must observe that

$$\mathcal{M}_n(f,x) = \mathscr{L}_{n,0}^{(0,0)}(f,x),$$

since $\int_0^1 p_{n,k}(t)\mathrm{d}t = 1/(n+1)$.

In 1981, P. Sablonnière extended the above operator by introducing the Jacobi weight function, $w^{\alpha,\beta}(t) = t^\alpha(1-t)^\beta$, for $\alpha,\beta > -1$, and $f \in L^2_{\alpha,\beta}[0,1]$. He defined the Bernstein-Jacobi operator as

$$\mathcal{B}_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n}\frac{\int_0^1 f(t)p_{n,k}(t)t^\alpha(1-t)^\beta\mathrm{d}t}{\int_0^1 p_{n,k}(t)t^\alpha(1-t)^\beta\mathrm{d}t}\; p_{n,k}(x). \tag{5.2}$$

Observe that Bernstein–Jacobi and Durrmeyer–Derriennic operators are related by

$$\mathcal{M}_n(f,x) = \mathcal{B}_n^{(0,0)}(f,x).$$

Moreover, Bernstein–Jacobi operator is a particular case of the Bernstein–Jacobi-type operator (2.6) since

$$\mathcal{B}_n^{(\alpha,\beta)}(f,x) = \mathscr{L}_{n,0}^{(\alpha,\beta)}(f,x),$$

and then, the Durremeyer–Derriennic operator is also related in the form

$$\mathcal{M}_n(f,x) = \mathcal{B}_n^{(0,0)}(f,x) = \mathscr{L}_{n,0}^{(0,0)}(f,x).$$

As a consequence, a complete set of eigenfunctions can be obtained from Theorem 3.3, as was obtained in Derriennic (1981) and Sablonnière (1981).

Corollary 5.1 *The classical Jacobi polynomials* $\{P_m^{(\alpha,\beta)}\}_{m\geq 0}$ *are the eigenfunctions of the Bernstein–Jacobi operator, that is,*

$$\mathcal{B}_n^{(\alpha,\beta)}(P_m^{(\alpha,\beta)},x) = \lambda_{n,m}^{(\alpha,\beta)}\, P_m^{(\alpha,\beta)}(x),$$

and the Legendre polynomials $\{P_m\}_{m\geq 0} = \{P_m^{(0,0)}\}_{m\geq 0}$ *on* $[0,1]$ *are eigenfunctions of the Durrmeyer-Derriennic operator; that is,*

$$\mathcal{M}_n(P_m,x) = \lambda_{n,m}^{(0,0)}\, P_m(x).$$

From Lemma 3.1, another relation between the Bernstein–Jacobi and the Bernstein–Jacobi-type operators can be established.

Proposition 5.2 *For $0 \le r \le n$, we have*

$$\mathscr{L}_{n,r}^{(\alpha,\beta)}(f,x) = \lambda_{n,r}^{(\alpha,\beta)}\, \mathcal{B}_{n-r}^{(\alpha+r,\beta+r)}(f,x).$$

Using the above relations, we can recover the properties of the Durrmeyer–Derriennic (5.1) and the Bernstein–Jacobi operators (5.2). In addition, new properties satisfied by the derivatives can be proved.

Lemma 5.3 *Let $f \in L^2_{\alpha,\beta}[0,1]$ such that $D^r f$ exists for $r \ge 1$. Then,*

$$D^r \mathcal{B}_n^{(\alpha,\beta)}(f,x) = \mathscr{L}_{n,r}^{(\alpha,\beta)}(D^r f, x).$$

Proof Using (2.3), (2.2), and (2.4), we get

$$\begin{aligned} D^r \mathcal{B}_n^{(\alpha,\beta)}(f,x) &= \sum_{k=0}^{n} \mu_{n,k}^{(\alpha,\beta)}(f) D^r p_{n,k}(x) \\ &= \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \mu_{n,k+j}^{(\alpha,\beta)}(f) \\ &= (-1)^r \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \frac{\int_0^1 f(t) D^r \left[p_{n+r,k+r}(t) w(t)\right] \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x). \end{aligned}$$

Integrating by parts r times we deduce

$$\int_0^1 f(t) D^r \left[p_{n+r,k+r}(t) w(t)\right] \mathrm{d}t = (-1)^r \int_0^1 D^r f(t) p_{n+r,k+r}(t) w(t)\, \mathrm{d}t.$$

and thus

$$\begin{aligned} D^r \mathcal{B}_n^{(\alpha,\beta)}(f,x) &= \lambda_{n,r}^{(\alpha,\beta)} \sum_{k=0}^{n-r} \frac{\int_0^1 D^r f(t) p_{n+r,k+r}(t) w(t)\, \mathrm{d}t}{\langle 1, p_{n+r,k+r}\rangle_{\alpha,\beta}} p_{n-r,k}(x) \\ &= \mathscr{L}_{n,r}^{(\alpha,\beta)}(D^r f, x). \end{aligned} \tag{5.3}$$

□

From (5.3) in Lemma 3.1, we deduce the following derivative property satisfied by the Bernstein-Jacobi operator and then, for the Durrmeyer–Derriennic operator (5.1).

Theorem 5.4 *The following relation hold*

$$D^r\, \mathcal{B}_n^{(\alpha,\beta)}(f,x) = \lambda_{n,r}^{(\alpha,\beta)}\, \mathcal{B}_{n-r}^{(\alpha+r,\beta+r)}(D^r f, x), \quad 0 \le r \le n.$$

As a consequence,

$$D^r \mathcal{M}_n(f,x) = \lambda_{n,r}^{(0,0)}\, \mathcal{B}_{n-r}^{(r,r)}(D^r f, x).$$

6 The case $\alpha = -1$

In this section, we focus on the study of the limit case $\alpha = -1$ and $\beta > -1$, introducing non-standard values of the Jacobi parameters. We will prove that the operator introduced in Gupta et al. (2009) is a particular case of our Bernstein-Jacobi-type operator. The case $\beta = -1$ and

$\alpha > -1$ is also considered. Moreover, we show that generalized Jacobi polynomials are the eigenfunctions of the Bernstein–Jacobi-type operator, and that this operator also preserves the derivative properties. We point out that the function $w^{-1,\beta}(t) = t^{-1}(1-t)^{\beta}$, $\beta > -1$, $t \in (0,1)$, does not define an inner product.

For $0 \le r \le n$, and $\beta > -1$, let

$$\lambda_{n,r}^{(-1,\beta)} = \frac{n!}{(n-r)!}\frac{\Gamma(n+\beta+1)}{\Gamma(n+r+\beta+1)} = \frac{(n-r+1)_r}{(n+\beta+1)_r},$$

that is well defined for all $n, r \ge 0$ being integers, and $\beta > -1$. As before, $\lambda_{n,r}^{(-1,\beta)} > 0$,

$$\lim_{n\to+\infty} \lambda_{n,r}^{(-1,\beta)} = 1,$$

for $0 \le r \le n$, and $\lambda_{n,0}^{(-1,\beta)} = 1$, $n \ge 0$.

Using the explicit expression (2.7), for $r \ge 1$ and $0 \le k \le n-r$, the coefficient

$$\mu_{n+r,k+r}^{(-1,\beta)}(f) = \frac{\int_0^1 f(t)\, t^{k+r-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k+r-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t} = \mu_{n+r-1,k+r-1}^{(0,\beta)}(f),$$

exists since the involved integrals are convergent, and, similarly, the following also exists

$$\mu_{n,k}^{(-1,\beta)}(f) = \frac{\int_0^1 f(t)\, t^{k-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t} = \mu_{n-1,k-1}^{(0,\beta)}(f), \quad 1 \le k \le n.$$

Definition 6.1 For $0 \le r \le n$, and $f \in L^2_{0,\beta}[0,1]$, we define the following Bernstein–Jacobi-type operator

$$\mathscr{L}_{n,r}^{(-1,\beta)}(f,x) = \begin{cases} f(0)\,p_{n,0}(x) + \displaystyle\sum_{k=1}^{n} \frac{\int_0^1 f(t)\, t^{k-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}\, p_{n,k}(x), & r = 0, \\ \lambda_{n,r}^{(-1,\beta)} \displaystyle\sum_{k=0}^{n-r} \frac{\int_0^1 f(t)\, t^{k+r-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k+r-1}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}\, p_{n-r,k}(x), & r > 0. \end{cases} \tag{6.1}$$

Using the Jacobi inner product, we can express the above operator in the form

$$\mathscr{L}_{n,0}^{(-1,\beta)}(f,x) = f(0)\,p_{n,0}(x) + \sum_{k=1}^{n} \frac{\langle f, p_{n-1,k-1}\rangle_{0,\beta}}{\langle 1, p_{n-1,k-1}\rangle_{0,\beta}}\, p_{n,k}(x),$$

$$\mathscr{L}_{n,r}^{(-1,\beta)}(f,x) = \lambda_{n,r}^{(-1,\beta)} \sum_{k=0}^{n-r} \frac{\langle f, p_{n+r-1,k+r-1}\rangle_{0,\beta}}{\langle 1, p_{n+r-1,k+r-1}\rangle_{0,\beta}}\, p_{n-r,k}(x),$$

and defining $\mu_{n,0}^{(-1,\beta)}(f) = f(0)$, (6.1) can be written as the compact form

$$\mathscr{L}_{n,r}^{(-1,\beta)}(f,x) = \lambda_{n,r}^{(-1,\beta)} \sum_{k=0}^{n-r} \mu_{n+r,k+r}^{(-1,\beta)}(f)\, p_{n-r,k}(x), \quad r \ge 0. \tag{6.2}$$

This operator is linear and positive, and, a direct computation as in Lemma 2.3 shows that the Bernstein–Jacobi-type operator $\mathscr{L}_{n,r}^{(-1,\beta)}(f,x)$, for $0 \le r \le n$, preserves the degree of polynomials, and expressions (2.8)–(2.9) hold by taking $\alpha \to -1$.

The Bernstein–Jacobi-type operator with $\alpha = -1$ also preserves derivatives.

Theorem 6.2 *Let f be defined on $[0,1]$ and $s \geq 0$ such that $D^s f(x)$ exists $\forall x \in [0,1]$. For $r \geq 0$ such that $n \geq r+s$, we have*

$$D^s \mathscr{L}_{n,r}^{(-1,\beta)}(f,x) = \mathscr{L}_{n,r+s}^{(-1,\beta)}(D^s f,x). \tag{6.3}$$

Proof The result for $s=0$ is trivial. Now, the conditions $n \geq r+s$ and $s \geq 1$ imply that $n \geq 1$. For $r \geq 1$, Theorem 4.2 holds by taking $\alpha \to -1$ in formula (4.2). Therefore, we need to prove the result for $r=0$.

$$D^s \mathscr{L}_{n,0}^{(-1,\beta)}(f,x) = \mathscr{L}_{n,s}^{(-1,\beta)}(D^s f,x),$$

for fixed $n \geq 1$ and induction on s. For $s=1$, we use (2.3), obtaining

$$\begin{aligned} D\mathscr{L}_{n,0}^{(-1,\beta)}(f,x) &= -nf(0)p_{n-1,0}(x) + n\sum_{k=1}^{n} \mu_{n,k}^{(-1,\beta)}(f)[p_{n-1,k-1}(x) - p_{n-1,k}(x)] \\ &= n[\mu_{n,1}^{(-1,\beta)}(f) - f(0)]p_{n-1,0}(x) \\ &\quad + n\sum_{k=1}^{n-1}[\mu_{n,k+1}^{(-1,\beta)}(f) - \mu_{n,k}^{(-1,\beta)}(f)]p_{n-1,k}(x). \end{aligned}$$

We can use expression (4.1) in this case, since the involved integrals are convergent, obtaining

$$\begin{aligned} D\mathscr{L}_{n,0}^{(-1,\beta)}(f,x) = n\left[\mu_{n,1}^{(-1,\beta)}(f) - f(0)\right] p_{n-1,0}(x) \\ + \frac{n}{n+\beta+1}\sum_{k=1}^{n-1} \mu_{n+1,k+1}^{(-1,\beta)}(Df)p_{n-1,k}(x). \end{aligned}$$

Using integration by parts, and the explicit expression for the Beta function, we compute

$$\begin{aligned} \mu_{n,1}^{(-1,\beta)}(f) - \frac{1}{n+\beta+1}\mu_{n+1,1}^{(-1,\beta)}(Df) =& \frac{\int_0^1 f(t)p_{n,1}(t)t^{-1}(1-t)^\beta \mathrm{d}t}{\int_0^1 p_{n,1}(t)t^{-1}(1-t)^\beta \mathrm{d}t} \\ & - \frac{1}{n+\beta+1}\frac{\int_0^1 f'(t)p_{n+1,1}(t)t^{-1}(1-t)^\beta \mathrm{d}t}{\int_0^1 p_{n+1,1}(t)t^{-1}(1-t)^\beta \mathrm{d}t} \\ =&(n+\beta)\int_0^1 f(t)(1-t)^{n+\beta-1}\mathrm{d}t \\ & - \int_0^1 f'(t)(1-t)^{n+\beta}\mathrm{d}t \\ =& f(0), \end{aligned}$$

which means that the expression is satisfied for $s=1$. For $s \geq 1$, let us suppose that (6.3) holds for s, such that $n \geq s+1$, and we will prove it for $s+1$. In fact, by the induction hypothesis and (6.3) for $r=0$, we get

$$\begin{aligned} D^{s+1}\mathscr{L}_{n,0}^{(-1,\beta)}(f,x) =& D[D^s\mathscr{L}_{n,0}^{(-1,\beta)}(f,x)] = D\mathscr{L}_{n,s}^{(-1,\beta)}(D^s f,x) \\ =& \mathscr{L}_{n,s}^{(-1,\beta)}(D(D^s f),x) = \mathscr{L}_{n,s+1}^{(-1,\beta)}(D^{s+1}f,x). \end{aligned}$$

□

The Bernstein–Jacobi-type operator with $\alpha = -1$ also admits a complete set of eigenfunctions. For $r \geq 1$, Theorem 3.3 can be applied.

Corollary 6.3 *For $n \geq r \geq 1$, the eigenfunctions of the operator are the classical Jacobi polynomials* $\{P_m^{(r-1,r+\beta)}\}_{m\geq 0}$

$$\mathscr{L}_{n,r}^{(-1,\beta)}(P_m^{(r-1,r+\beta)}, x) = \lambda_{n,r+m}^{(-1,\beta)} P_m^{(r-1,r+\beta)}(x),$$

where

$$\lambda_{n,r+m}^{(-1,\beta)} = \begin{cases} \dfrac{(n-r-m+1)_{r+m}}{(n+\beta+1)_{r+m}} & \text{if} \quad m \leq n, \\ 0 & \text{if} \quad m > n. \end{cases}$$

In the case $r = 0$ and $\alpha = -1$, we can not use the classical Jacobi polynomials nor the above results. In this case, the method will be different from that used in the Corollary 6.3.

Following Szegő (1975), the explicit expression of the classical Jacobi polynomials over [0,1] is given by

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} x^{n-k} (x-1)^k. \tag{6.4}$$

For $n \geq 1$, expression (6.4) defines a polynomial of exact degree n when $\alpha = -1$, and satisfies

$$P_n^{(-1,\beta)}(x) = \frac{n+\beta}{n} x\, P_{n-1}^{(1,\beta)}(x), \tag{6.5}$$

for $n \geq 1$ and $P_0^{(-1,\beta)}(x) = 1$. The family of polynomials $\{P_n^{(-1,\beta)}\}_{n\geq 0}$, called *generalized Jacobi polynomials on* [0, 1], are not orthogonal with respect to the inner product (2.1), and appear, for instance, in Szegő (1975, p. 64).

Now, we compute in this case the application of the operator to x^m.

Lemma 6.4 *For $n \geq 0$, $\beta > -1$, and $\alpha = -1$,*

$$\begin{aligned} \mathscr{L}_{n,0}^{(-1,\beta)}(1, x) =& 1, \\ \mathscr{L}_{n,0}^{(-1,\beta)}(x^m, x) =& x\, \mathscr{L}_{n,1}^{(0,\beta-1)}(x^{m-1}, x), \quad m \geq 1. \end{aligned}$$

Proof Using the definition (6.1), we get

$$\mathscr{L}_{n,0}^{(-1,\beta)}(1, x) = 1.$$

For $n, m \geq 1$, $r = 0$, and $\alpha = -1$, expression (2.9) is also valid, and we can compute

$$\begin{aligned} \mathscr{L}_{n,0}^{(-1,\beta)}(x^m, x) &= \sum_{k=1}^{m} \binom{m}{k} \frac{(n-k+1)_k\, (k)_{m-k}}{(n+\beta+1)_m} x^k \\ &= x \sum_{k=0}^{m-1} \binom{m}{k+1} \frac{(n-(k+1)+1)_{k+1}\, (k+1)_{m-(k+1)}}{(n+\beta+1)_m} x^k \\ &= x \sum_{k=0}^{m-1} \frac{m!}{(k+1)!(m-1-k)!} \frac{(n-k)_{k+1}\, (k+1)_{m-1-k}}{(n+\beta+1)_m} x^k \\ &= x \frac{n}{n+\beta+1} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(n-k)_k\, (k+2)_{m-1-k}}{(n+\beta+2)_{m-1}} x^k \end{aligned}$$

$$= x\,\mathscr{L}_{n,1}^{(0,\beta-1)}(x^{m-1}, x).$$

□

Using the above lemma, we obtain the eigenfunctions of the Bernstein–Jacobi-type operator for $r = 0$ and $\alpha = -1$.

Theorem 6.5 *For $n \geq 0$, $\beta > -1$, and $\alpha = -1$, the eigenfunctions of the Bernstein-Jacobi-type operator $\mathscr{L}_{n,0}^{(-1,\beta)}$ are the generalized Jacobi polynomials $\{P_m^{(-1,\beta)}\}_{m\geq 0}$ on $[0, 1]$. Moreover,*

$$\mathscr{L}_{n,0}^{(-1,\beta)}(P_m^{(-1,\beta)}, x) = \lambda_{n,m}^{(-1,\beta)}\, P_m^{(-1,\beta)}(x),$$

where

$$\lambda_{n,m}^{(-1,\beta)} = \begin{cases} \dfrac{(n-m+1)_m}{(n+\beta+1)_m} & \text{if} \quad m \leq n, \\ 0 & \text{if} \quad m > n. \end{cases}$$

Proof Let suppose that the explicit expression of the generalized Jacobi polynomials in terms of the monomials is given by

$$P_m^{(-1,\beta)}(x) = \sum_{k=1}^{m} a_{m,k}^{(-1,\beta)}\, x^k,$$

where $a_{m,k}^{(-1,\beta)}$ are real numbers with $a_{m,m}^{(-1,\beta)} \neq 0$, and, in the same way,

$$P_m^{(1,\beta)}(x) = \sum_{k=0}^{m} a_{m,k}^{(1,\beta)}\, x^k, \quad a_{m,m}^{(1,\beta)} \neq 0.$$

From (6.5), we know that the first sum starts for $k = 1$, and the coefficients are related by

$$a_{m,k}^{(-1,\beta)} = \frac{m+\beta}{m}\, a_{m-1,k-1}^{(1,\beta)}, \quad k = 1, 2, \ldots m.$$

Using the linearity of the operator, and Lemma 6.4, we get

$$\begin{aligned}
\mathscr{L}_{n,0}^{(-1,\beta)}(P_m^{(-1,\beta)}, x) &= \sum_{k=1}^{m} a_{m,k}^{(-1,\beta)}\, \mathscr{L}_{n,0}^{(-1,\beta)}(x^k, x) \\
&= x \sum_{k=1}^{m} a_{m,k}^{(-1,\beta)}\, \mathscr{L}_{n,1}^{(0,\beta-1)}(x^{k-1}, x) \\
&= x \sum_{k=1}^{m} \frac{m+\beta}{m}\, a_{m-1,k-1}^{(1,\beta)}\, \mathscr{L}_{n,1}^{(0,\beta-1)}(x^{k-1}, x) \\
&= x\, \frac{m+\beta}{m}\, \mathscr{L}_{n,1}^{(0,\beta-1)}(P_{m-1}^{(1,\beta)}(x), x) \\
&= x\, \frac{m+\beta}{m}\, \lambda_{n,m}^{(0,\beta-1)}\, P_{m-1}^{(1,\beta)}(x),
\end{aligned}$$

where the last equality is justified by Theorem 3.3. Now, using (6.5) again, we get

$$\mathscr{L}_{n,0}^{(-1,\beta)}(P_m^{(-1,\beta)}, x) = \lambda_{n,m}^{(0,\beta-1)}\, P_m^{(-1,\beta)}(x),$$

and $\lambda_{n,m}^{(0,\beta-1)} = \lambda_{n,m}^{(-1,\beta)}$. □

As a consequence,

Proposition 6.6 (i) *For* $f \in L^2_{r-1,r+\beta}[0,1]$,

$$\mathscr{L}_{n,r}^{(-1,\beta)}(f,x) = \sum_{k=0}^{n-r} \frac{\langle f, P_k^{(r-1,r+\beta)}\rangle_{r-1,r+\beta}}{h_k^{(r-1,r+\beta)}} P_k^{(r-1,r+\beta)}(x),$$

where $h_k^{(r-1,r+\beta)}$ *is given by (3.1).*

(ii) *For* $p \geq 1$, $r \geq 1$, *and* $f \in L^p_{r-1,r+\beta}[0,1]$, *then* $\mathscr{L}_{n,r}^{(-1,\beta)}(f,x)$ *converges to* f *on* $L^p_{r-1,r+\beta}[0,1]$.

Similar proofs as in the standard case allow us to deduce the uniform convergence and a Voronowskaja-type formula.

Proposition 6.7 (i) *For* $f \in \mathcal{C}[0,1]$, $\mathscr{L}_{n,r}^{(-1,\beta)}(f,x)$ *converges uniformly to* f.

(ii) *There exists* $n_\beta \in \mathbb{N}$ *such that for* $n \geq n_\beta$ *we have*

$$\left\| \frac{1}{\lambda_{n,r}^{(-1,\beta)}} \mathscr{L}_{n,r}^{(-1,\beta)}(f,x) - f(x) \right\|_\infty \leq 2\omega\left(f, \frac{1}{\sqrt{n}}\right).$$

(iii) *For* $f \in \mathcal{C}^s[0,1]$, *we get*

$$\lim_{n\to+\infty} \left\| \frac{1}{\lambda_{n,r}^{(-1,\beta)}} D^s \mathscr{L}_{n,r}^{(-1,\beta)}(f,x) - D^s f(x) \right\|_\infty = 0.$$

(iv) *For* $f \in L^2_{0,\beta}[0,1]$ *such that* $f''(x)$ *exists for a fixed* $x \in [0,1]$, *then,*

$$\lim_{n\to+\infty} n \left[\frac{1}{\lambda_{n,r}^{(-1,\beta)}} \mathscr{L}_{n,r}^{(-1,\beta)}(f,x) - f(x) \right] = x(1-x)f''(x) \\ +[r-(2r+\beta+1)x]f'(x).$$

As above, we must point out that using the explicit expression of the generalized Jacobi polynomials, it can be proved that they are solutions of the second order differential (2.17) with $\alpha = -1$.

Finally, we remark that the Bernstein–Durrmeyer operator defined in Gupta et al. (2009) given by

$$\mathcal{P}_{n,r}(f,x) = \begin{cases} p_{n,0}(x)\, f(0) + n \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)\, f(t)\mathrm{d}t, & r = 0, \\ \frac{(n-r+1)_r}{(n+1)_r} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+r-1,k+r-1}(t)\, f(t)\mathrm{d}t, & r > 0, \end{cases}$$

is a particular case of the Bernstein–Jacobi-type operator (6.2) since

$$\mathcal{P}_{n,r}(f,x) = \mathscr{L}_{n,r}^{(-1,0)}(f,x).$$

An analogous analysis can be done for $\beta = -1$, defining the Bernstein–Jacobi-type operator in the form

$$\mathscr{L}_{n,r}^{(\alpha,-1)}(f,x) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \frac{\int_0^1 f(t)\, t^{k+\alpha}\,(1-t)^{n-k-1}\,\mathrm{d}t}{\int_0^1 t^{k+\alpha}\,(1-t)^{n-k-1}\,\mathrm{d}t}\, p_{n,k}(x) + f(1)\, p_{n,n}(x), & r = 0, \\ \displaystyle\lambda_{n,r}^{(\alpha,-1)} \sum_{k=0}^{n-r} \frac{\int_0^1 f(t)\, t^{k+r+\alpha}\,(1-t)^{n-k-1}\,\mathrm{d}t}{\int_0^1 t^{k+r+\alpha}\,(1-t)^{n-k-1}\,\mathrm{d}t}\, p_{n-r,k}(x), & r > 0, \end{cases}$$

for $0 \le r \le n$, $f \in L^2_{\alpha,0}[0,1]$, and $\lambda_{n,r}^{(\alpha,-1)}$ defined in (2.5).

The results for $\beta = -1$ are analogous to the previous case $\alpha = -1$. Apart from the convergence results, this operator also preserves the derivatives, and has the generalised Jacobi polynomials $\{P_m^{(r+\alpha,r-1)}\}_{m\ge 0}$ as eigenfunctions.

7 The general case $\alpha = -l$, $l \in \mathbb{Z}^+$

In this section, we study Bernstein–Jacobi-type operators for the general non-standard case $\alpha = -l$, $l \in \mathbb{Z}^+$, and $\beta > -1$. In this case, derivative properties as described in the above sections are satisfied, and also the Bernstein-Jacobi-type operators have the generalized Jacobi polynomials as eigenfunctions. The case $\alpha > -1$ and $\beta = -l$, $l \in \mathbb{Z}^+$, will be analogous.

For $0 \le l \le n$ being integers, $r \ge 0$, and $\beta > -1$, we consider

$$\lambda_{n,r}^{(-l,\beta)} = \frac{n!}{(n-r)!}\,\frac{\Gamma(n-l+\beta+2)}{\Gamma(n+r-l+\beta+2)} = \frac{(n-r+1)_r}{(n-l+\beta+2)_r},$$

that was defined in (2.5).

Using the explicit expression (2.7) for $\max\{0, l-r\} \le k \le n-r$, the coefficient

$$\mu_{n+r,k+r}^{(-l,\beta)}(f) = \frac{\int_0^1 f(t)\, t^{k+r-l}\,(1-t)^{n-k+\beta}\,\mathrm{d}t}{\int_0^1 t^{k+r-l}\,(1-t)^{n-k+\beta}\,\mathrm{d}t} = \mu_{n+r-l,k+r-l}^{(0,\beta)}(f),$$

exists since the involved integrals are convergent.

Looking at the form of $\mathscr{L}_{n,r}^{(-1,\beta)}$, we can deduce that the generic operator should have $l+1$ pieces in its definition, for $0 \le r \le l$, and the derivative property should be

$$D^s \mathscr{L}_{n,r}^{(-l,\beta)}(f,x) = \mathscr{L}_{n,r+s}^{(-l,\beta)}(D^s f, x), \tag{7.1}$$

for adequate functions.

We define the generic Bernstein–Jacobi-type operator $\mathscr{L}_{n,r}^{(-l,\beta)}$ in a compact form, with constants that we will determinate later.

Definition 7.1 For $n, l \in \mathbb{N}$, $0 \le l \le n$, and $f \in L^2_{0,\beta}[0,1]$, we define

$$\begin{aligned} \mathscr{L}_{n,r}^{(-l,\beta)}(f,x) = & \sum_{k=0}^{l-r-1} A_{r,k}(f)\, p_{n-r,k}(x) \\ & + A_{r,l-r}(f) \sum_{k=l-r}^{n-r} \mu_{n+r-l,k+r-l}^{(0,\beta)}(f)\, p_{n-r,k}(x), \qquad 0 \le r \le l-1, \\ \mathscr{L}_{n,r}^{(-l,\beta)}(f,x) = & A_{r,0}(f) \sum_{k=0}^{n-r} \mu_{n+r-l,k+r-l}^{(0,\beta)}(f)\, p_{n-r,k}(x), \qquad l \le r \le n, \end{aligned}$$

(7.2)

where the constants $A_{r,k}(f)$ depends on n, l, r, k, β and f.

Now, we will find the values of the constants involved in (7.2) by imposing the derivative property.

Lemma 7.2 *The constants $A_{r,k}(f)$ are given by*

$$A_{r,k}(f) = A_{r,l-r}(f) \sum_{i=0}^{l-r-k-1} (-1)^i \binom{l-r-k-1}{i} \frac{D^i f(0)}{(n-l+r+\beta+2)_i},$$

for $0 \leq r+k < l$. Choosing $A_{0,l}(f) = 1$, then

$$A_{r,l-r}(f) = \lambda_{n,r}^{(-l,\beta)}, \quad r = 0, \ldots, l.$$

Proof We have to impose the following l conditions

$$D\mathscr{L}_{n,r}^{(-l,\beta)}(f,x) = \mathscr{L}_{n,r+1}^{(-l,\beta)}(Df,x), \quad r = 0, \ldots, l-1. \tag{7.3}$$

First, we compute the left side of (7.3) using (2.3),

$$\begin{aligned}
D\mathscr{L}_{n,r}^{(-l,\beta)}(f,x) &= (n-r) \sum_{k=0}^{l-r-1} A_{r,k}(f) \left[p_{n-r-1,k-1}(x) - p_{n-r-1,k}(x)\right] \\
&\quad + (n-r) A_{r,l-r}(f) \sum_{k=l-r}^{n-r} \mu_{n+r-l,k+r-l}^{(0,\beta)}(f) \\
&\quad \left[p_{n-r-1,k-1}(x) - p_{n-r-1,k}(x)\right] \\
&= (n-r) \sum_{k=0}^{l-r-2} \left[A_{r,k+1}(f) - A_{r,k}(f)\right] p_{n-r-1,k}(x) \\
&\quad + (n-r) \left[A_{r,l-r}(f) \mu_{n+r-l,0}^{(0,\beta)}(f) - A_{r,l-r-1}(f)\right] p_{n-r-1,l-r-1}(x) \\
&\quad + \frac{n-r}{n-l+r+\beta+2} A_{r,l-r}(f) \\
&\quad \sum_{k=l-r}^{n-r-1} \mu_{n+r-l+1,k+r-l+1}^{(0,\beta)}(Df) p_{n-r-1,k}(x),
\end{aligned}$$

and we write the right hand side of the (7.3),

$$
\begin{aligned}
\mathscr{L}_{n,r+1}^{(-l,\beta)}(Df,x) = & \sum_{k=0}^{l-r-2} A_{r+1,k}(Df)p_{n-r-1,k}(x) \\
& + A_{r+1,l-r-1}(Df)\sum_{k=l-r-1}^{n-r-1} \mu_{n+r-l+1,k+r-l+1}^{(0,\beta)}(Df)p_{n-r-1,k}(x).
\end{aligned}
$$

Imposing (7.3) and since $\{p_{n-r-1,k}(x) : k = 0, \ldots, n-r-1\}$ is a basis of Π_{n-r-1} we get

$$
A_{r+1,k}(Df) = (n-r)\left[A_{r,k+1}(f) - A_{r,k}(f)\right], \quad k = 0, \ldots, l-r-2, \tag{7.4}
$$

$$
A_{r+1,l-r-1}(Df)\mu_{n+r-l+1,0}^{(0,\beta)}(Df) = (n-r)\left[A_{r,l-r}(f)\mu_{n+r-l,0}^{(0,\beta)}(f) - A_{r,l-r-1}(f)\right], \tag{7.5}
$$

$$
A_{r+1,l-r-1}(Df) = \frac{n-r}{n-l+r+\beta+2}A_{r,l-r}(f), \tag{7.6}
$$

for $0 \le r \le l-1$. If we substitute (7.6) in (7.5) and compute $A_{r,l-r-1}(f)$,

$$
\begin{aligned}
A_{r,l-r-1}(f) = & \left[\mu_{n+r-l,0}^{(0,\beta)}(f) - \frac{1}{n-l+r+\beta+2}\mu_{n+r-l+1,0}^{(0,\beta)}(Df)\right]A_{r,l-r}(f) \\
= & f(0)A_{r,l-r}(f).
\end{aligned} \tag{7.7}
$$

Now, we substitute $A_{r,l-r-2}(f)$ in (7.4) for $k = l-r-2$ and compute (7.7),

$$
\begin{aligned}
A_{r,l-r-2}(f) &= A_{r,l-r-1}(f) - \frac{1}{n-r}A_{r+1,l-r-2}(Df) \\
&= f(0)A_{r,l-r}(f) - \frac{Df(0)}{n-r}A_{r+1,l-r-1}(Df) \\
&= \left[f(0) - \frac{Df(0)}{n-l+r+\beta+2}\right]A_{r,l-r}(f).
\end{aligned}
$$

We are going to prove by induction on d the following equality

$$
A_{r,l-r-d}(f) = A_{r,l-r}(f)\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}, \tag{7.8}
$$

for $d = 1, \ldots, l$ and $r = 0, \ldots, l-d$. For $d = 1, 2$, we have already proved it. We assume that (7.8) is holds for $d-1$ and we compute $A_{r,l-r-d-1}(f)$. Since $d \ge 1$, we can apply the formula (7.4) and, in addition, we will use (7.8) and (7.6) respectively,

$$
\begin{aligned}
A_{r,l-r-d-1}(f) = & A_{r,l-r-d}(f) - \frac{1}{n-r}A_{r+1,l-r-d-1}(Df) \\
= & A_{r,l-r}(f)\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i} \\
& - \frac{1}{n-r}A_{r+1,l-r-1}(Df)\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^{i+1} f(0)}{(n-l+r+\beta+3)_i} \\
= & A_{r,l-r}(f)\left[\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{n-l+r+\beta+2}\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^{i+1}f(0)}{(n-l+r+\beta+3)_i}\Bigg]\\
&=A_{r,l-r}(f)\left[\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}\right.\\
&\quad\left.+\sum_{i=0}^{d-1}(-1)^{i+1}\binom{d-1}{i}\frac{D^{i+1}f(0)}{(n-l+r+\beta+2)_{i+1}}\right]\\
&=A_{r,l-r}(f)\left[\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}\right.\\
&\quad\left.+\sum_{i=1}^{d}(-1)^i\binom{d-1}{i-1}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}\right]\\
&=A_{r,l-r}(f)\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}.
\end{aligned}
$$

Thus, we finish the induction on d and we have proved (7.8). If we denote $k=l-r-d$, we have $d=l-r-k$ from where

$$
A_{r,k}(f)=A_{r,l-r}(f)\sum_{i=0}^{l-r-k-1}(-1)^i\binom{l-r-k-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i},
$$

for $0\le r+k<l$. Let's observe that all coefficients depend on the main diagonal $A_{r,l-r}(f)$. We can choose the coefficient $A_{0,l}(f)$ so that it is independent of f, that is, $A_{0,l}(f)\equiv A_{0,l}$, and then, using the formula (7.6),

$$
A_{r,l-r}(f)=\lambda_{n,r}^{(-l,\beta)}A_{0,l},\quad r=0,\dots,l.
$$

In particular, if $A_{0,l}=1$ we obtain the desired result. □

Notice that in order to define this operator, the existence of $D^i f(0)$ for $i=0,\dots,l-1$ for $f\in L^2_{0,\beta}[0,1]$ is required.

We will denote by $H^l_\beta[0,1]$ the linear subspace of $L^2_{0,\beta}[0,1]$ of functions whose first l successive derivatives exist at $x=0$.

Theorem 7.3 *Let $n,l\in\mathbb{N}$, $0\le l\le n$, and $f\in H^l_\beta[0,1]$. There is a unique operator of the form (7.2), which satisfies the derivative property (7.1) and with $A_{0,l}(f)=1$ is given by*

$$
\begin{aligned}
\frac{1}{\lambda_{n,r}^{(-l,\beta)}}\mathscr{L}_{n,r}^{(-l,\beta)}(f,x)&=\sum_{k=0}^{l-r-1}\left[\sum_{i=0}^{l-r-k-1}(-1)^i\binom{l-r-k-1}{i}\frac{D^i f(0)}{(n-l+r+\beta+2)_i}\right]\\
&\quad\times p_{n-r,k}(x)+\sum_{k=l-r}^{n-r}\mu^{(0,\beta)}_{n+r-l,k+r-l}(f)p_{n-r,k}(x),\quad 0\le r<l,\\
\frac{1}{\lambda_{n,r}^{(-l,\beta)}}\mathscr{L}_{n,r}^{(-l,\beta)}(f,x)&=\sum_{k=0}^{n-r}\mu^{(0,\beta)}_{n+r-l,k+r-l}(f)p_{n-r,k}(x),\qquad l\le r\le n.
\end{aligned}
\tag{7.9}
$$

7.1 Eigenfunctions

Our next objective is to prove that the eigenfunctions of (7.9) are the generalized Jacobi polynomials with parameter $\alpha = -l$. When $r \geq l$, we use Theorem 3.3. On the other cases, we prove that formula (2.9) also holds for (7.9).

Proposition 7.4 *For $l \leq r \leq n$, the eigenfunctions of the operator are the classical Jacobi polynomials $\{P_m^{(r-l,r+\beta)}\}_{m\geq 0}$,*

$$\mathscr{L}_{n,r}^{(-l,\beta)}(P_m^{(r-l,r+\beta)}, x) = \lambda_{n,r+m}^{(-l,\beta)}\, P_m^{(r-l,r+\beta)}(x).$$

Now, we study the cases $r = 0, \ldots, l-1$. Following the reasoning of (6.5), we have for $0 \leq r \leq l$,

$$P_m^{(-(l-r),r+\beta)}(x) = \frac{(m-(l-r)+r+\beta+1)_{l-r}}{(m-(l-r)+1)_{l-r}} x^{l-r} P_{m-(l-r)}^{(l-r,r+\beta)}(x). \tag{7.10}$$

We compute the application of the operator to x^m.

Lemma 7.5 *For $n, l, r, m \in \mathbb{N}$, $0 \leq r < l \leq n$, $m \geq l-r$ and $\beta > -1$, we have*

$$\mathscr{L}_{n,r}^{(-l,\beta)}(x^m, x) = \lambda_{n,r}^{(-l,\beta)} \sum_{k=0}^{m} \binom{m}{k} \frac{(n-r-k+1)_k (k+r-l+1)_{m-k}}{(n+r-l+\beta+2)_m} x^k. \tag{7.11}$$

Proof We let $f_m(x) = x^m$ for $x \in [0,1]$. Then if $i \geq 0$, we have

$$D^i f_m(0) = \begin{cases} 1 \; if \;\; m = i, \\ 0 \; if \;\; m \neq i. \end{cases}$$

If $i = 0, \ldots, l-r-k-1$, then $m \geq l > l-1 \geq i$ and for these reason $D^i f_m(0)$, for $i = 0, \ldots, l-r-k-1$. Thus,

$$\mathscr{L}_{n,r}^{(-l,\beta)}(x^m, x) = \lambda_{n,r}^{(-l,\beta)} \sum_{k=l-r}^{n-r} \mu_{n+r-l,k+r-l}^{(0,\beta)}(f_m) p_{n-r,k}(x),$$

and in order to prove

$$\sum_{k=l-r}^{n-r} (k+r-l+1)_m\, p_{n-r,k}(x) = \sum_{k=0}^{m} \binom{m}{k} (n-r-k+1)_k (k+r-l+1)_{m-k}\, x^k,$$

we just have to take into account that $(k+r-l+1)_m = 0$ if $k = 0, \ldots, l-r-1$ and follow the same reasoning as in (2.3) with $a = r-l+1$. □

Lemma 7.6 *For $0 \leq r < l \leq n$, we have*

$$\mathscr{L}_{n,r}^{(-l,\beta)}(x^m, x) = x^{l-r} \mathscr{L}_{n,l}^{(-r,\beta+r-l)}(x^{m-(l-r)}, x), \quad m \geq l-r.$$

Proof For $\alpha = -l$, the expression (7.11) is also valid, and we can compute

$$\begin{aligned}\mathscr{L}_{n,r}^{(-l,\beta)}(x^m, x) &= \lambda_{n,r}^{(-l,\beta)} \sum_{k=0}^{m} \binom{m}{k} \frac{(n-r-k+1)_k (k-(l-r)+1)_{m-k}}{(n-(l-r)+\beta+2)_m} x^k \\ &= \lambda_{n,r}^{(-l,\beta)} \sum_{k=l-r}^{m} \binom{m}{k} \frac{(n-r-k+1)_k (k-(l-r)+1)_{m-k}}{(n-(l-r)+\beta+2)_m} x^k\end{aligned}$$

$$
\begin{aligned}
&= x^{l-r}\lambda_{n,r}^{(-l,\beta)} \\
&\quad\times \sum_{k=0}^{m-(l-r)} \binom{m}{k+l-r} \frac{(n-l-k+1)_{k+l-r}(k+1)_{m-(l-r)-k}}{(n-(l-r)+\beta+2)_m} x^k \\
&= x^{l-r}\lambda_{n,r}^{(-l,\beta)} \frac{(n-l+1)_{l-r}}{(n-(l-r)+\beta+2)_{l-r}} \\
&\quad\times \sum_{k=0}^{m-(l-r)} \binom{m-(l-r)}{k} \frac{(n-l-k+1)_k (k+l-r+1)_{m-(l-r)-k}}{(n+\beta+2)_{m-(l-r)}} x^k \\
&= x^{l-r}\mathscr{L}_{n,l}^{(-r,\beta+r-l)}(x^{m+r-l}, x),
\end{aligned}
$$

because $\lambda_{n,r}^{(-l,\beta)} \frac{(n-l+1)_{l-r}}{(n+r-l+\beta+2)_{l-r}} = \lambda_{n,l}^{(-r,\beta+r-l)}$. ◻

Using the above lemma, we can deduce the eigenfunctions of the Bernstein–Jacobi-type operator for $r = 0, \dots, l-1$ and $\alpha = -l$.

Theorem 7.7 *For $n \geq 0$ and $r = 0, \dots, l-1$, the eigenfunctions of the operator are the classical Jacobi polynomials $\{P_m^{(r-l,r+\beta)}\}_{m\geq 0}$,*

$$
\mathscr{L}_{n,r}^{(-l,\beta)}(P_m^{(r-l,r+\beta)}, x) = \lambda_{n,r+m}^{(-l,\beta)}\, P_m^{(r-l,r+\beta)}(x).
$$

Proof Let suppose that the explicit expression of the generalized Jacobi polynomials in terms of the monomials is given by

$$
P_m^{(r-l,r+\beta)}(x) = \sum_{k=l-r}^{m} a_{m,k}^{(r-l,r+\beta)}\, x^k,
$$

where $a_{m,k}^{(r-l,r+\beta)}$ are real numbers with $a_{m,m}^{(r-l,r+\beta)} \neq 0$. From (7.10), we know that the first sum starts with $k = l-r$, and the coefficients are related by

$$
a_{m,k}^{(r-l,r+\beta)} = \frac{(m+2r-l+\beta+1)_{l-r}}{(m+r-l+1)_{l-r}}\, a_{m+r-l,k+r-l}^{(l-r,r+\beta)}, \quad k = l-r, \dots, m.
$$

Using the linearity of the Bernstein-Jacobi-type operator, and Lemma 7.6, we get

$$
\begin{aligned}
\mathscr{L}_{n,r}^{(-l,\beta)}(P_m^{(r-l,r+\beta)}, x) &= \sum_{k=l-r}^{m} a_{m,k}^{(r-l,r+\beta)} \mathscr{L}_{n,r}^{(-l,\beta)}(x^k, x) \\
&= x^{l-r} \sum_{k=l-r}^{m} a_{m,k}^{(r-l,r+\beta)}\, \mathscr{L}_{n,l}^{(-r,\beta+r-l)}(x^{k+r-l}, x) \\
&= x^{l-r} \frac{(m+2r-l+\beta+1)_{l-r}}{(m+r-l+1)_{l-r}} \\
&\quad\times \sum_{k=l-r}^{m} a_{m+r-l,k+r-l}^{(l-r,r+\beta)}\, \mathscr{L}_{n,l}^{(-r,\beta+r-l)}(x^{k+r-l}, x) \\
&= xx^{l-r} \frac{(m+2r-l+\beta+1)_{l-r}}{(m+r-l+1)_{l-r}} \\
&\quad\times \sum_{k=0}^{m+r-l} a_{m+r-l,k}^{(l-r,r+\beta)}\, \mathscr{L}_{n,l}^{(-r,\beta+r-l)}(x^k, x)
\end{aligned}
$$

$$= x^{l-r} \frac{(m+2r-l+\beta+1)_{l-r}}{(m+r-l+1)_{l-r}} \mathscr{L}_{n,l}^{(-r,\beta+r-l)}(P_{m+r-l}^{(l-r,r+\beta)}, x)$$
$$= x^{l-r} \frac{(m+2r-l+\beta+1)_{l-r}}{(m+r-l+1)_{l-r}} \lambda_{n,r+m}^{(-r,\beta+r-l)} P_{m+r-l}^{(l-r,r+\beta)}(x),$$

where we have used the Theorem 3.3 since $r \le l$, and then, we can apply induction. Now, using again (7.10), we get

$$\mathscr{L}_{n,r}^{(-l,\beta)}(P_m^{(r-l,r+\beta)}, x) = \lambda_{n,r+m}^{(-r,\beta+r-l)} P_m^{(r-l,r+\beta)}(x),$$

and $\lambda_{n,r+m}^{(-r,\beta+r-l)} = \lambda_{n,r+m}^{(-l,\beta)}$. □

8 Numerical experiments

In this section, we present numerical experiments where we approximate some functions with $\mathscr{L}_{n,r}^{(\alpha,\beta)}$. We measure the accuracy of this approximation through the Root Mean Square Error (RMSE) associated with a partition P of $[0, 1]$. We choose the usual partition $P_N := \{x_i = \frac{i}{N} : i = 0, \ldots, N\}$, for $N \ge 0$. As usual,

$$RMSE(f, N) = \sqrt{\sum_{i=0}^{N} \frac{[f(x_i) - \mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x_i)]^2}{N+1}}.$$

Let us observe that the higher the parameter r, the lower the degree of the polynomial $\mathscr{L}_{n,r}^{(\alpha,\beta)}(f, x)$ and, therefore, the worse is the approximation to the function f. For this reason, in the following examples we are going to choose small values of r.

8.1 Example 1

Let $f_1(x) = \exp(-x^2)$ defined on $[0, 1]$. Figure 1 shows the comparison of the plots of the function $f_1(x)$, represented in blue, and the Bernstein-Jacobi-type operator $\mathscr{L}_{n,r}^{(\alpha,\beta)}(f_1, x)$, represented in orange. We take $r = 0$, $\alpha = 2$ and $\beta = 1.5$ as parameters.

Next, we show how the first and second derivative of the operator converge to the first and second derivative of the function, respectively, by using Theorem 4.2. Figure 2 presents the comparison of the plots of the function $Df_1(x) = -2x \exp(-x^2)$, drawn in blue, and the operator $D\mathscr{L}_{n,0}^{(2,1.5)}(f_1, x) = \mathscr{L}_{n,1}^{(2,1.5)}(Df_1, x)$, represented in orange, in the first row of the table. In the second row, $D^2 f_1(x) = (4x^2 - 2) \exp(-x^2)$ is represented in blue, and $D^2\mathscr{L}_{n,0}^{(2,1.5)}(f_1, x) = \mathscr{L}_{n,2}^{(2,1.5)}(D^2 f_1, x)$, is represented in orange.

We observe that the graph of $\mathscr{L}_{n,0}^{(2,1.5)}(f_1, x)$ and its derivatives approach the function $f_1(x)$ and its derivatives, respectively.

8.2 Example 2

We take $f_2(x) = 10 \exp(-0.2x) - \exp(\cos(10\pi x))$, an Ackley function (Surjanovic and Bingham 2013), and we choose the parameters $r = 1$, $\alpha = 1$ and $\beta = -0.9$ for the Bernstein-Jacobi-type operator.

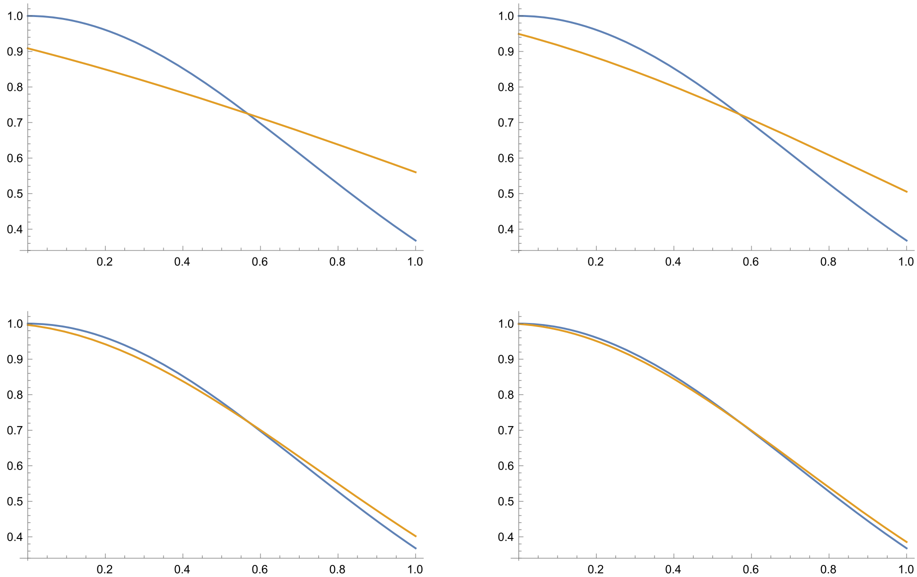

Fig. 1 Graphs of $f_1(x)$ and $\mathscr{L}_{n,0}^{(2,1.5)}(f_1, x)$ for $n = 5, 10, 50, 100$

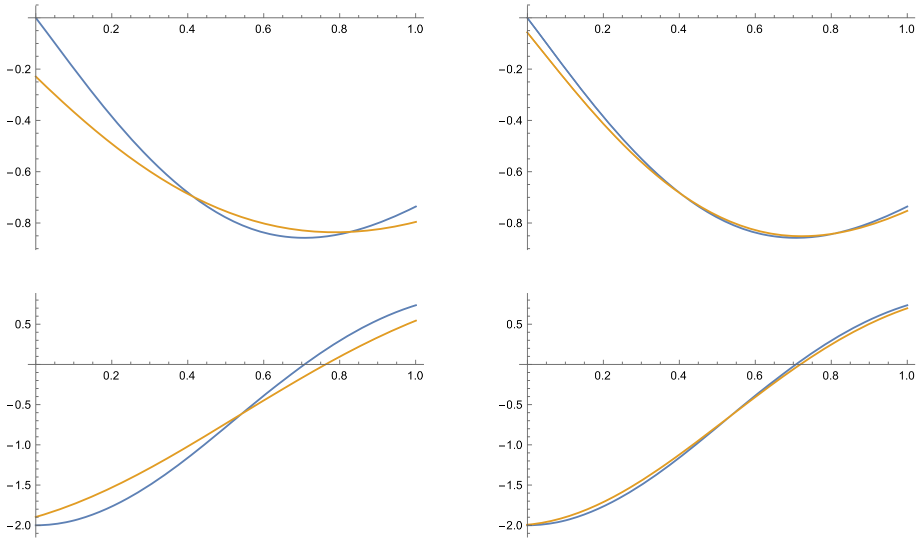

Fig. 2 Graphs of $Df_1(x)$ and $\mathscr{L}_{n,1}^{(2,1.5)}(Df_1, x)$ in the first row, and $D^2 f_1(x)$ and $\mathscr{L}_{n,2}^{(2,1.5)}(D^2 f_1, x)$ in the second row, for $n = 20, 100$

Due to the oscillations of the Ackley function, the graph of $\mathscr{L}_{n,1}^{(1,-0.9)}(f_2, x)$ approaches $f_2(x)$ more slowly than in the previous case. The graphs plotted in Table 3 show that the worse approximation occurs at the local maximum and minimum points of the function.

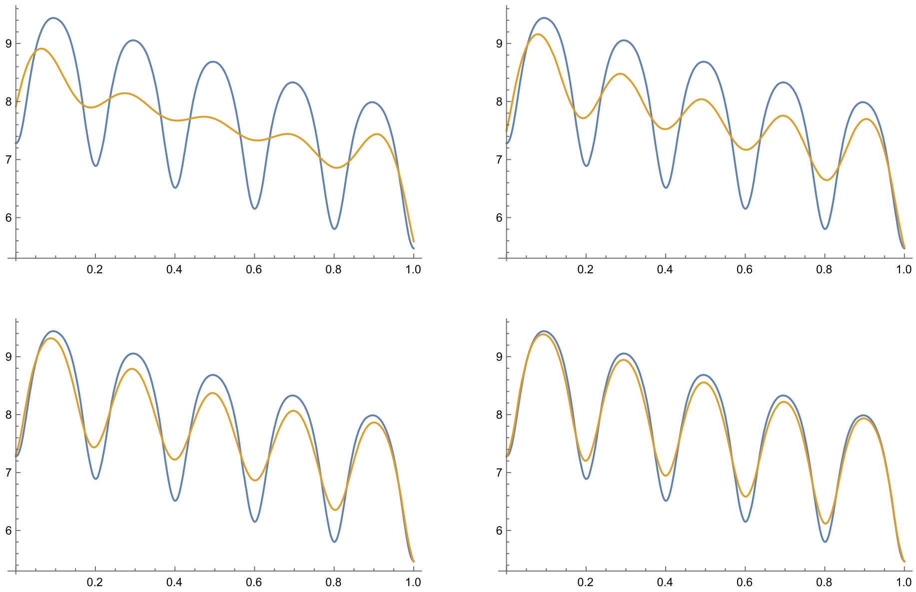

Fig. 3 Plots of $f_2(x)$ and $\mathscr{L}_{n,1}^{(1,-0.9)}(f_2, x)$ for $n = 100, 200, 500, 1000$

8.3 Example 3

Finally, we study the behavior of the operator applied to the discontinuous function

$$f_3(x) = \begin{cases} -1 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

We choose $r = 3$, $\alpha = -0.5$ and $\beta = 0.5$ as parameters, and we get

In Fig. 4 we observe that $\mathscr{L}_{n,3}^{(-0.5,0.5)}(f_3, x)$ approximates well in the continuous parts of $f_3(x)$ and, at the discontinuity point $x_0 = \frac{1}{2}$, the sequence of Bernstein–Jacobi-type operators $\{\mathscr{L}_{n,3}^{(-0.5,0.5)}(f_3, x_0)\}_{n\geq 0}$ approximate the midpoint.

Interestingly, there is no apparent presence of the Gibbs phenomenon in this example, even for n large enough, as we can see in the plots. The experimental results suggest that the Gibbs phenomenon does not occur, but we must not forget that it is a particular case for certain choices of the parameters. We must comment that in all the studied numerical experiments the Gibbs phenomena was not appreciated in any of the cases.

Finally, we quantify the approach of the operator to each of the three previous functions through the $RMSE$ using $N = 50$ points. In Fig. 5, the $RMSE$ of $f_1(x)$, $f_2(x)$ and $f_3(x)$ is represented in blue, orange and green respectively. We observe that the error is ordered according to the properties of the approximated function.

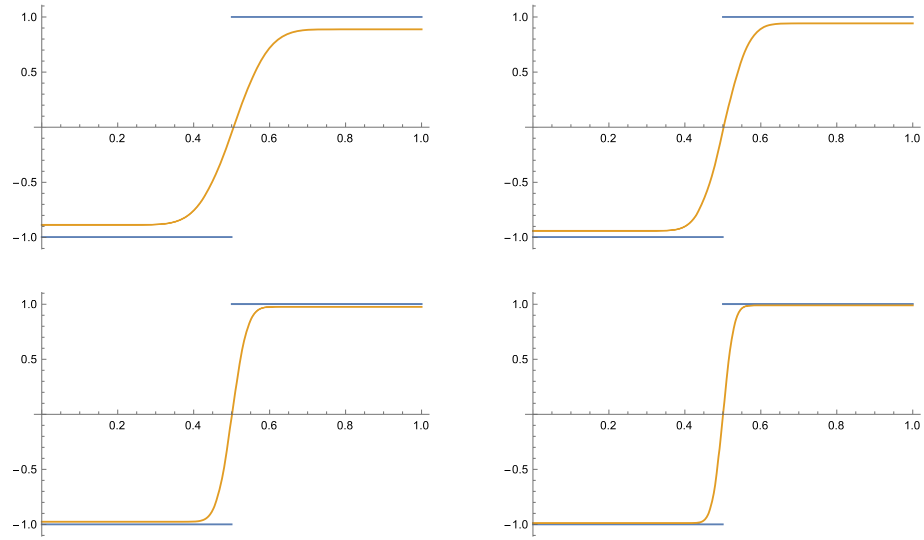

Fig. 4 Graphs of $f_3(x)$ and $\mathscr{L}_{n,3}^{(-0.5,0.5)}(f_3, x)$ for $n = 100, 200, 500, 1000$

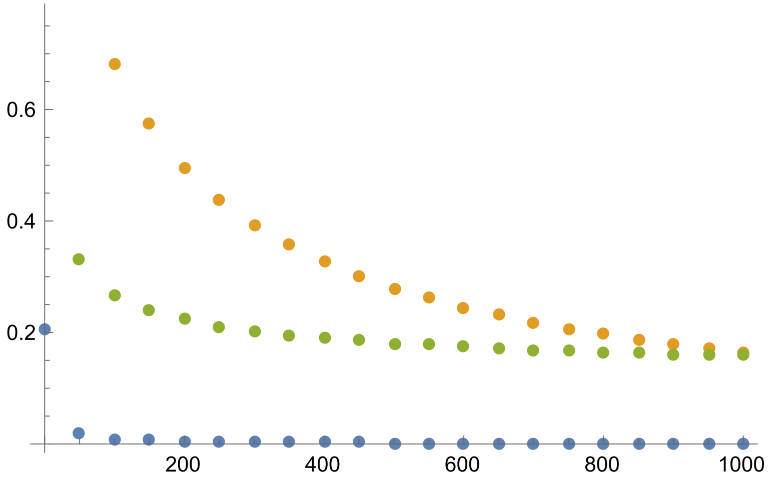

Fig. 5 $RSME$ of $f_1(x)$, $f_2(x)$ and $f_3(x)$ for $n = 50\,k$, $k = 0, \ldots, 20$

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