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Rotationally invariant translators of the mean curvature flow in Einstein's static universe

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ABSTRACT

In this paper, we deal with non-degenerate translators of the mean curvature flow in the well-known Einstein's static universe. We focus on the rotationally invariant translators, that is, those invariant by a natural isometric action of the special orthogonal group on the ambient space. In the classification list, there are three space-like cases and five time-like cases. All of them, except a totally geodesic example, have one or two conic singularities. Also, we show a uniqueness result based on the behaviour of the translator on its boundary. As an application, we extend an isometry of the sphere to the whole translator under simple conditions. This leads to a characterization of a bowl-like example.

1. Introduction

Translating solitons or *translators* are the well-known solutions to the mean curvature flow which are invariant by translations of the ambient space. The geometric idea is that the flow is the displacement of a specific hypersurface in the direction of a suitable Killing vector field on the ambient space. See [8] for a recent survey on translators in Euclidean Space $\mathbb{R}^3$, and also [2], [6], [13] and the references therein. The basic idea is to reduce the mean curvature flow to the equation

$$\mathbf{H} = v^{\perp}, \tag{1}$$

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where $\mathbf{H}$ is the mean curvature vector of the immersion, v is a Killing vector and $v^{\perp}$ is the orthogonal projection of v into the normal bundle of the hypersurface.

In [12], de Lira and Martín made a generalization for Riemannian products $M \times \mathbb{R}$ by taking $v = \partial_t$. Some other authors simplified equation (1) by considering rotationally invariant hypersurfaces and also, from a broader perspective, a cohomogeneity 1-action on M, which is the action by isometries of some Lie group in such a way that the orbits are constant mean curvature hypersurfaces except at most two of them (see [1] for more details). This is the case of Bueno in [4] and [5], where the product of the real hyperbolic plane and a real line $\mathbb{H}^2 \times \mathbb{R}$ is considered. In [11], de Lima and Pipoli classified invariant translators of a family of curvature flows (including the classical mean curvature flow), where $\mathbb{H}^n \times \mathbb{R}$, $n \geq 1$, is one of the ambient spaces. In [9], Kim studied the action of some groups in Minkowski space only for space-like hypersurfaces. In [10], Lawn and the first author dealt with the action of the special orthogonal and the orthocronal groups in Minkowski space by studying both space-like and time-like hypersurfaces. In addition, Batista and de Lima obtained some rotationally invariant space-like translators in Lorentzian products $\mathbb{P} \times_{-1} \mathbb{R}$ in [3] when $\mathbb{P}$ has non-positive sectional curvature, but they did not exhibit a complete classification. On the other hand, Pipoli focused on the solvable group $\mathbf{Sol}_3$ in [15] and the Heisenberg group in [16].

In this paper, we obtain new translators in the important Lorentzian manifold $\mathbb{S}^n \times_{-1} \mathbb{R}$, $n \geq 2$, well-known as either the *Einstein's (static) universe* or the *Einstein's (static) space-time* of arbitrary dimension, [17], which we denote by ESU. Thus, we pay attention to an important case avoided in [3]. We use a similar definition of translator to the one given in [3] and [12]. It can be easily seen that it is a solution to the PDE

$$\mathbf{H} = \partial_t^{\perp},$$

where $\mathbf{H}$ is the mean curvature vector, ∂_t is the time-like unit vector field in the direction of the real line, and $\partial_t^{\perp}$ represents the orthogonal projection of ∂_t in the normal bundle of the hypersurface. In order to obtain specific solutions, we focus on the *rotationally invariant* translators. More precisely, we study those which are invariant by a natural extension of the action of the Lie group $SO(n)$ on $\mathbb{S}^n$ to $\mathbb{S}^n \times_{-1} \mathbb{R}$. Later, we characterize a compact piece of one of these examples.

This paper is structured as follows: In Section 2, we remind some basic tools. We point out that a graphical translator is determined by a function which satisfies PDE (3). Section 3 is devoted to specifying the action of $SO(n)$, its naturally associated projection τ and a few consequences. Indeed, Proposition 3.1 states that any $SO(n)$-invariant graphical translator is the graph of the composition of a solution to ODE (7) with the map τ.

A detailed study of ODE (7) is carried out in Sections 4 and 5. We classify the space-like $SO(n)$-invariant translators in ESU in Theorem 4.1, obtaining a specific bowl-like example A and two families B and C. Unsurprisingly, the time-like setting is richer. We classify the time-like $SO(n)$-invariant translators in ESU in Theorem 5.1, obtaining five cases in total, namely, the family D, the examples E and F, the family of bi-graphs G and a non-graphical totally geodesic example. Except the totally geodesic one, all of them have one or two conic singularities, i.e., they are tangent to the vertices of light-like cones.

In Section 6, we obtain some uniqueness results for space-like translators. Since any of them is graphical, we reduce the study to a suitable function u and its domain, which will be the closure of an open and connected subset $\Omega \subsetneq \mathbb{S}^n$, namely $\overline{\Omega}$. In Theorem 6.1, we show that if two space-like translators coincide on the boundary $\partial\Omega$, then they are globally equal in $\overline{\Omega}$. As a first consequence, in Theorem 6.2, when the boundary $\partial\Omega$ and the function u are invariant by some isometry σ of $\mathbb{S}^n$, then the whole translator is invariant by $\sigma \times id_{\mathbb{R}}$, where $id_{\mathbb{R}}$ is the identity map of $\mathbb{R}$. As a result, in Corollary 6.2, when the domain is a ball without the centre, and the function u is constant on the boundary of the ball, then the translator is rotationally invariant. When the map u is also smooth in the whole ball, the translator has to be a compact piece of example A.

2. Setup

Let $\mathbb{R}^{n+1}$, $n \geq 2$, be the standard $(n+1)$-dimensional Euclidean space with its usual flat metric $\langle , \rangle$. If $p, q \in \mathbb{R}^{n+1}$, then $\langle p, q\rangle = p\, q^t$, where p and q are regarded as row-matrices and q^t is the transpose of q. We denote by $|w|^2 = \langle w, w\rangle$ the squared $\langle , \rangle$-norm of any $w \in \mathbb{R}^{n+1}$. The usual round hypersphere is $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} : \langle p, p\rangle = 1\}$.

We consider the Einstein's static universe (ESU), namely, the product $\widehat{M} = \mathbb{S}^n \times \mathbb{R}$ with Lorentzian metric $g = \langle , \rangle - dt^2$ (cf. [14], [17].) Take $(p, t) \in \mathbb{S}^n \times \mathbb{R}$. Given an open subset M of $\mathbb{S}^n$, we consider a function $u \in C^2(M, \mathbb{R})$ and construct its graph map $\Gamma : M \to \widehat{M}$, where $\Gamma(p) = (p, u(p))$. Given the metric $\gamma = F^* \langle , \rangle$ on M, we assume that $F : (M, \gamma) \to (\widehat{M}, g)$ is a non-degenerate hypersurface. Under the usual identifications, for each $X \in TM$, we have

$$dF(X) = (X, du(X)) = (X, \langle \nabla u, X\rangle),$$

where ∇u is the $\langle , \rangle$-gradient of u. The upward normal vector field is

$$\nu = \frac{1}{W}(\nabla u, 1), \quad W = +\sqrt{\varepsilon\big(|\nabla u|^2 - 1\big)}, \tag{2}$$

where $\varepsilon := \operatorname{sign}\big(|\nabla u|^2 - 1\big) = \pm 1$ is a constant function on the whole M. Note that $g(\nu, \nu) = \varepsilon$. We will use the following definitions of the mean curvature vector and the mean curvature function. If II_Γ is the second fundamental form of Γ, the mean curvature vector $\mathbf{H}_\Gamma$ is

$$\mathbf{H}_\Gamma = \operatorname{trace}_{\langle , \rangle}(II_\Gamma) = \varepsilon H \nu,$$

where $H = \operatorname{trace}(A_\nu)$ is the mean curvature function, i.e., the trace of the shape operator A_ν. The following proposition is known (see for example [10]):

Proposition 2.1. *Under the previous setting, Γ is a graphical translator if, and only if, function u satisfies the quasilinear PDE*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{\varepsilon\big(|\nabla u|^2 - 1\big)}}\right) = \frac{1}{\sqrt{\varepsilon\big(|\nabla u|^2 - 1\big)}} = H. \tag{3}$$

Remark 2.1. By Corollary 2.1 in [10], since $\mathbb{S}^n$ is a closed manifold (compact, orientable without boundary), there are no entire translators in our setting. Thus, M can be smooth at most on $\mathbb{S}^n$ without a point.

Remark 2.2. It is important to keep in mind that any space-like hypersurface in ESU is the graph of a function u over a piece of a slice $\mathbb{S}^n \times \{t_0\}$, $t_0 \in \mathbb{R}$, satisfying $|\nabla u|^2 < 1$. If necessary, we will write a space-like translator as Γ_u, remarking the map u.

We note that if Γ is a graphical translator, then its image by an isometry is also a translator.

Lemma 2.1. *Let Γ be a graphical translator in $\mathbb{S}^n \times_{-1} \mathbb{R}$. Given an isometry $F : \mathbb{S}^n \to \mathbb{S}^n$, we extend it to* $\mathbf{F} : \mathbb{S}^n \times_{-1} \mathbb{R} \to \mathbb{S}^n \times_{-1} \mathbb{R}$, $(p, t) \mapsto (F(p), t)$. *Then, $\mathbf{F}(\Gamma)$ is also a graphical translator.*

Proof. By Proposition 2.1, Γ is determined by a certain function $u : \Omega \to \mathbb{R}$ that satisfies (3). A straightforward computation shows that $\hat{u} := u \circ F^{-1} : F(\Omega) \to \mathbb{R}$ also satisfies (3). From this, the associated graph to $\hat{u}$ is also a translator, namely $\mathbf{F}(\Gamma)$. □

3. The action of the Lie group $SO(n)$

We recall the Lie group $SO(n) = \{A \in \mathcal{M}_n(\mathbb{R}) : AA^t = I_n, \det(A) = 1\}$, of orthogonal $n \times n$ matrices whose determinants are 1, where A^t is the transpose of A and I_n is the identity matrix. This group acts by isometries on $\mathbb{S}^n$, namely

$$\Psi : SO(n) \times \mathbb{S}^n \to \mathbb{S}^n, \quad (A, p) \mapsto p \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right). \tag{4}$$

This is a well-known cohomogeneity 1-action, whose non-singular orbits are totally umbilical $(n-1)$-dimensional hyperspheres, and its singular orbits are the North pole $\mathbf{n} = (0, \dots, 0, 1)$ and the South pole $\mathbf{s} = (0, \dots, 0, -1)$ of $\mathbb{S}^n$. The orbits of the action Ψ coincide with the level sets of the map

$$\tau : \mathbb{S}^n \to [-\pi/2, \pi/2], \ \tau(p) = \arcsin(\langle p, \mathbf{n} \rangle). \tag{5}$$

A straightforward computation shows that the map τ is a Riemannian submersion. Thus, we can use $-\nabla\tau$ as a globally defined unit normal vector field to the level sets of τ. Given $s \in (-\pi/2, \pi/2)$ and $p \in \tau^{-1}\{s\} \subset \mathbb{S}^n \backslash \{\mathbf{n}, \mathbf{s}\}$, a simple computation gives the mean curvature function with respect to $-\nabla\tau$:

$$\mathrm{trace}(A_{-\nabla\tau})_p = (1-n)\tan(s).$$

Then, we take $h : (-\pi/2, \pi/2) \to \mathbb{R}$, $h(s) = (1-n)\tan(s)$. However, the action by isometries that we really need is the following extension:

$$SO(n) \times \widehat{M} \to \widehat{M}, \quad (A, (p, t)) \mapsto \big(\Psi(A, p), t\big). \tag{6}$$

Assume that a graphical translator Γ is invariant by the action (6). Since Γ is described in terms of an open subset $M \subset \mathbb{S}^n$ and a function $u \in C^2(M, \mathbb{R})$, both M and u must be also invariant by $SO(n)$. This implies that there exists $f \in C^2(I, \mathbb{R})$ such that $I = \tau(M) \subset [-\pi/2, \pi/2]$, $u = f \circ \tau_{|M}$. By a straightforward application of Theorem 3.5 in [10], we obtain the following result:

Proposition 3.1. *Let $u \in C^2(M, \mathbb{R})$ be an invariant function by action (4) such that $u = f \circ \tau$. The graph map Γ of u is an $SO(n)$-invariant translator if, and only if, function f satisfies the following ODE:*

$$f''(s) = \big(1 - (f'(s))^2\big)\big(1 + (n-1)\tan(s) f'(s)\big), \ \forall s \in I. \tag{7}$$

Our next target is to study the solutions to (7). For this aim, we take $v = f'$ in (7) and deal with the following ODE

$$v'(s) = \big(1 - v(s)^2\big)\big(1 + (n-1)\tan(s) v(s)\big). \tag{8}$$

Remark 3.1. In the following sections, we find all solutions to equation (8). We note that for each solution $v \in C^1(I)$, we compute a primitive $f = \int v$. Thus, we obtain an $SO(n)$-invariant translator according to Proposition 3.1. Namely, $M = \tau^{-1}(I) \subset \mathbb{S}^n$, $\Gamma : M \to \widehat{M}$, $\Gamma(p) = (p, (f \circ \tau)(p))$ for each $p \in M$.

Recall that $n \geq 2$ is a natural number. There are two immediate solutions to (8),

$$\mathbf{v}_+, \ \mathbf{v}_- : \mathbb{R} \to \mathbb{R}, \quad \mathbf{v}_+(s) = 1, \ \mathbf{v}_-(s) = -1.$$

These will behave as *barrier* solutions. We will use well-known tools and results, which can be found in the book [18], especially in Section 3.2. We define the function

$$\Theta : \mathbb{R}^2 \to \mathbb{R}, \quad (s,x) \mapsto (1-x^2)\big(\cos(s) + (n-1)\sin(s)x\big),$$

and the vector field, or rather, the *dynamical system*

$$X : \mathbb{R}^2 \to \mathbb{R}^2, \quad X(s,x) = (\cos(s), \Theta(s,x)).$$

The following two results justify the election of X. First, from almost any integral curve of X, we obtain a solution to (8). Second, from a solution, we obtain an integral curve.

Lemma 3.1. *Let $\alpha : J_o \subset \mathbb{R} \to \mathbb{R}^2$, $\alpha(r) = (s(r), x(r))$, be an integral curve of X such that $s : J_o \to K_o \subset \mathbb{R}$ is bijective. Then, $v : K_o \to \mathbb{R}$, $v := x \circ s^{-1}$, is a solution to (8). In addition, for $y \in K_o$,* $\operatorname{sign}(v'(y)) = \operatorname{sign}(\Theta(s^{-1}(y), v(y)))$.

Proof. Since α is an integral curve of X (that is, $X(\alpha(r)) = \alpha'(r)$), then

$$s'(r) = \cos(s(r)), \quad x'(r) = (1 - x(r)^2)\big(\cos(s(r)) + (n-1)\sin(s(r))x(r)\big).$$

As s is bijective, then $s'(s^{-1}(y)) = \cos(y)$ for any $y \in K_o$. Therefore,

$$\begin{aligned} v'(y) =& \frac{x'(s^{-1}(y))}{s'(s^{-1}(y))} = \frac{\big(1 - (x(s^{-1}(y)))^2\big)\big(\cos(y) + (n-1)\sin(y)x(s^{-1}(y))\big)}{\cos(y)} \\ =& \frac{\Theta\big(y, v(y)\big)}{\cos(y)} = (1 - v(y)^2)\big(1 + (n-1)\tan(y)v(y)\big). \end{aligned}$$

This readily finishes the proof. □

Lemma 3.2. *Given a solution $v : J \subset I \to \mathbb{R}$ to (8), there exists a smooth function $s = s(r)$ defined on a suitable interval such that $\alpha(r) = (s(r), v(s(r)))$ is an integral curve of X.*

Proof. Take the curve $\alpha(r) = (s(r), v(s(r)))$ for some smooth function s (to determine). Since v is a solution to (8), we obtain

$$\begin{aligned} X(\alpha(r)) =& \big(\cos(s(r)), (1 - v(s(r))^2)(\cos(s(r)) + (n-1)\sin(s(r))v(s(r))\big) \\ =& \big(\cos(s(r)), \cos(s(r))v'(s(r))\big). \end{aligned}$$

Solving the ODE $s'(r) = \cos(s(r))$ provides the desired reparametrization. □

The geometrical interpretation of function τ is *the angle or the position vector $p \in \mathbb{S}^n$ with respect to the horizontal hyperplane.* Then, we can reduce our study to the subset

$$\mathcal{S} := [-\pi/2, \pi/2] \times \mathbb{R}.$$

The zeros of X in $[-\pi/2, \pi/2] \times \mathbb{R}$ are the points

$$\begin{aligned} p_0 = (-\pi/2, -1),\ p_1 = (-\pi/2, 0),\ p_2 = (-\pi/2, +1), \\ q_0 = (\pi/2, -1),\ q_1 = (\pi/2, 0),\ q_2 = (\pi/2, +1). \end{aligned} \tag{9}$$

We denote the differential of X at p by $DX(p)$. We classify the points p_0, p_1, p_2, q_0, q_1 and q_2 according to the eigenvalues of $DX(p)$:

$$DX(p_0) = \begin{pmatrix} 1 & 0 \\ 0 & 2(n-1) \end{pmatrix}, \; DX(p_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1-n \end{pmatrix},$$
$$DX(p_2) = \begin{pmatrix} 1 & 0 \\ 0 & 2(n-1) \end{pmatrix}, \; DX(q_0) = \begin{pmatrix} -1 & 0 \\ 0 & -2(n-1) \end{pmatrix},$$
$$DX(q_1) = \begin{pmatrix} -1 & 0 \\ -1 & n-1 \end{pmatrix}, \; DX(q_2) = \begin{pmatrix} -1 & 0 \\ 0 & -2(n-1) \end{pmatrix}.$$

Clearly, p_0 and p_2 are *sources* because both eigenvalues are positive. q_0 and q_2 are *sinks* as both eigenvalues are negative. But p_1 and q_1 are *saddle points* since two eigenvalues are of different sign for each of these points. A straightforward computation proves the following result:

Lemma 3.3. *For each solution to ODE $x'(r) = \pm(n-1)(1-x(r)^2)x(r)$, the curves $\beta_\pm(r) = (\pm\pi/2, x(r))$ are integral curves of X.*

We now restrict the function Θ to the set $\mathcal{S}$. The set $\mathcal{Z}_0 = \{(s,x) \in \mathcal{S} : \Theta(s,x) = 0\}$ can be written in terms of the lines $x = \pm 1$ and the disconnected curve $\mathcal{C}$ given by the implicit equation $\cos(s) + (n-1)\sin(s)x = 0$, where $s \in [-\pi/2, \pi/2]$. We identify the set of points satisfying $\Theta(s,x) > 0$ and $\Theta(s,x) < 0$ as follows:

- $A_1 = \{(s,x) \in (-\pi/2, 0) \times [1, +\infty) : x > -\cot(s)/(n-1)\}$,
- $A_2 = \{(s,x) \in (-\pi/2, 0) \times [1, +\infty) : x < -\cot(s)/(n-1)\} \cup ([0, \pi/2) \times (1, +\infty))$,
- $A_3 = \{(s,x) \in (-\pi/2, 0) \times (0, 1) : x > -\cot(s)/n - 1\}$,
- $A_5 = \{(s,x) \in (0, \pi/2) \times (-1, 0) : x < -\cot(s)/(n-1)\}$,
- $A_6 = ((-\pi/2, 0] \times (-\infty, -1]) \cup \{(s,x) \in (0, \pi/2) \times (-\infty, -1) : x > -\cot(s)/(n-1)\}$,
- $A_7 = \{(s,x) \in (0, \pi/2) \times (-\infty, -1) : x < -\cot(s)/(n-1)\}$,
- $A_4 = ((-\pi/2, \pi/2) \times (-1, 1)) \backslash (\overline{A_3 \cup A_5})$.

Note that $\Theta_{|A_1 \cup A_4 \cup A_7} > 0$ and $\Theta_{|A_2 \cup A_3 \cup A_5 \cup A_6} < 0$. Also, $\mathcal{Z}_+ = \{(s,x) \in \mathcal{S} : \Theta(s,x) > 0\} = A_1 \cup A_4 \cup A_7$, $\mathcal{Z}_- = \{(s,x) \in \mathcal{S} : \Theta(s,x) < 0\} = A_2 \cup A_3 \cup A_5 \cup A_6$. According to Lemma 3.1, the monotony of the solutions will be indicated by the sign of Θ along their graphs. The following result can be proved by a very simple computation.

Proposition 3.2. *Given $(s_0, x_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$, let $v : (a,b) \to \mathbb{R}$ be the solution to (8) such that $v(s_0) = x_0$. Then, $w : (-b, -a) \to \mathbb{R}$ defined by $w(s) = -v(-s)$ is also a solution to (8) satisfying $w(-s_0) = -x_0$. In particular, their graphs are symmetric with respect to the point $(0,0)$.*

Remark 3.2. Proposition 3.2 simplifies the rest of the computations. Indeed, given an $SO(n)$-invariant translator associated with v, a certain rotation angle π provides another one whose associated solution is $w(s) = -v(-s)$. We call them rotated sisters.

4. The space-like case

Space-like graphical translating solitons are those satisfying $\varepsilon = -1 = \text{sign}(|\nabla u|^2 - 1)$, that is to say, $|\nabla u|^2 < 1$. Since $u = f \circ \tau$, by Proposition 3.1 and Remark 3.1, it becomes $(f')^2 < 1$. Finding all solutions to (8) such that $v^2 < 1$ is equivalent to classifying all space-like $SO(n)$-invariant translators in ESU. To do

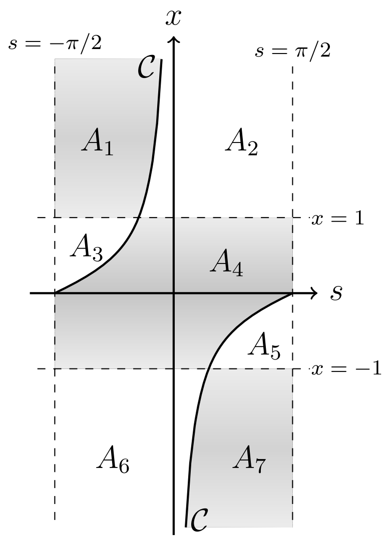

Fig. 1. Regions for $\Theta > 0$ in grey, and for $\Theta < 0$ in white.

so (recall Fig. 1), we study those solutions whose graphs are contained in $[-\pi/2, \pi/2] \times [-1, 1]$. We start by showing that any such solution can be globally extended.

Lemma 4.1. *Given $(s_0, x_0) \in (-\pi/2, \pi/2) \times (-1, 1)$, the solution v to (8) with $v(s_0) = x_0$ can be extended (as solution) to $v : (-\pi/2, \pi/2) \to [-1, 1]$. Also, $\lim_{s\to\pm\pi/2} v(s) \in \{-1, 0, 1\}$.*

Proof. We take a local solution v to (8) such that $-1 < v(s) < +1$, which is bounded by the constant solutions $\mathbf{v}_\pm(s) = \pm 1$. Therefore, v can be extended to $v : (-\pi/2, \pi/2) \to [-1, 1]$. By Lemma 3.2, we construct an integral curve $\alpha_v : [-\pi/2, \pi/2] \to [-1, 1]$ of X from v. By Lemma 3.3, α_v and $\beta_\pm$ can only coincide at some point p such that $X(p) = 0$. But this only holds when $p \in \{p_0, p_1, p_2, q_0, q_1, q_2\}$, namely, when $\lim_{s\to\pm\pi/2} v(s) \in \{-1, 0, 1\}$. □

Lemma 4.2. *We find the solutions to two boundary problems as follows:*

a) *There exists a unique $\mathbf{w}_- \in C^0[-\pi/2, \pi/2] \cap C^\infty[-\pi/2, \pi/2)$ solution to (8) such that $\mathbf{w}_-(-\pi/2) = 0$. Also, $\mathbf{w}_-(\pi/2) = 1$.*
b) *There exists a unique $\mathbf{w}_+ \in C^0[-\pi/2, \pi/2] \cap C^\infty(-\pi/2, \pi/2]$ solution to (8) such that $\mathbf{w}_+(\pi/2) = 0$. Also, $\mathbf{w}_+(-\pi/2) = -1$.*

Proof. As $p_1 = (-\pi/2, 0)$ is a saddle point, according to Theorem 3.2.1 of [18], there are two 1-dimensional submanifolds passing through p_1 such that their tangent vectors are the eigenvectors of $DX(p_1)$. One of them is parallel to $(0, 1)$, so it does not generate a solution as in Lemma 3.1. However, the other 1-dimensional submanifold $\alpha : [0, \delta) \to [-\pi/2, \pi/2] \times \mathbb{R}$ can be parametrized by $\alpha(r) = (s(r), x(r))$, so that $\alpha(0) = (-\pi/2, 0)$ and it provides a solution as in Lemma 3.1, namely, $\mathbf{w}_- : [-\pi/2, \hat{\delta}) \to \mathbb{R}$, $\mathbf{w}_-(y) = x(s^{-1}(y))$. In particular, $\lim_{y\to-\pi/2} \mathbf{w}_-(y) = 0$. This means that $\mathbf{w}_-$ is a solution to (8) with the boundary condition $\mathbf{w}_-(-\pi/2) = 0$. The uniqueness of the (local) integral curves of X implies that $\mathbf{w}_-$ is unique. Moreover, the graph of $\mathbf{w}_-$ is included in the region A_4, so that $\mathbf{w}'_-(s) > 0$ for any $s > -\pi/2$ and also $\mathbf{w}_-(-\pi/2) = 0$. By Lemma 4.1, we can extend $\mathbf{w}_- : [-\pi/2, \pi/2] \to \mathbb{R}$ and $\lim_{y\to\pi/2} \mathbf{w}_-(y) = 1$. Finally, since X is smooth, then $\mathbf{w}_-$ is also smooth on $[-\pi/2, \pi/2)$.

Similar computations hold for the point q_1, obtaining $\mathbf{w}_+$. Also, $\mathbf{w}_+$ is the rotated sister of $\mathbf{w}_-$, as in Remark 3.2. □

The region $(-\pi/2, \pi/2) \times [-1, 1]$ is split in three parts by the graphs of $\mathbf{w}_\pm$. In Lemma 4.3, we will find the solutions to (8) between the graph of $\mathbf{w}_-$ and the constant $+1$. Later, we will study the solutions between the graphs of $\mathbf{w}_\pm$ in Lemma 4.4. Finally, between $\mathbf{w}_+$ and -1, we resort to Proposition 3.2. By this way, we find all solutions included in this region.

Lemma 4.3. *For each $(s_0, x_0) \in (-\pi/2, \pi/2) \times (-1, 1)$ such that $\mathbf{w}_-(s_0) < x_0$, the unique solution v to (8) with initial condition $v(s_0) = x_0$, can be extended to $v \in C^0[-\pi/2, \pi/2] \cap C^\infty(-\pi/2, \pi/2)$, with $v(-\pi/2) = v(\pi/2) = 1$, $|v| \le 1$. For some $(s_1, x_1) \in \mathcal{C}$, $s_1 < 0$, then $v'(s_1) = 0$. Moreover, $v(s_1)$ is its unique absolute minimum.*

Proof. By Lemma 4.1, the associated solution to (8) can be extended to $v : (-\pi/2, \pi/2) \to [-1, 1]$. At one point $(s_1, x_1) \in \mathcal{C}$, $v(s_1) = x_1$. From (8), $v'(s_1) = 0$. Then, for any $s < s_1$, the graph of v is included in the region A_3, so that $v'(s) < 0$. For any $s > s_0$, the graph of v is included in the region A_4, so that $v'(s) > 0$. In particular, s_1 is the global minimum of v. Since $0 \le \mathbf{w}_-(s_0) < v(s_0) < 1$ and $\lim_{s\to\pi/2} \mathbf{w}_-(s) = 1$, $\lim_{s\to\pi/2} v(s) = 1$. If $\lim_{s\to-\pi/2} v(s) = 0$, then the uniqueness given in Lemma 4.2 implies that for each s, $v(s) = \mathbf{w}_-(s)$ which is a contradiction. Finally, Lemma 4.1 yields $v(-\pi/2) = 1$. □

Lemma 4.4. *For each $(s_0, x_0) \in (-\pi/2, \pi/2) \times (-1, 1)$ such that $\mathbf{w}_+(s_0) < x_0 < \mathbf{w}_-(s_0)$, the unique solution v to (8) with initial condition $v(s_0) = x_0$, can be extended to $v \in C^0[-\pi/2, \pi/2] \cap C^\infty(-\pi/2, \pi/2)$, with $v(-\pi/2) = -1$, $v(\pi/2) = 1$ and $v' > 0$.*

Proof. By Lemma 4.1, the associated solution to (8) can be extended to $v : [-\pi/2, \pi/2] \to [-1, 1]$. But now, the graph of v is between the graphs of $\mathbf{w}_\pm$, so that $v'(s) > 0$ for any $s \in (-\pi/2, \pi/2)$. Thus, v has no critical points. The uniqueness of $\mathbf{w}_\pm$ and Lemma 4.1 implies that $\lim_{s\to-\pi/2} v(s) = -1$ and $\lim_{s\to\pi/2} v(s) = 1$. □

Now, from the previous lemmas and Remark 3.1, we construct the corresponding translators. For each function $v : [-\pi/2, \pi/2] \to \mathbb{R}$ in Lemmas 4.2, 4.3 and 4.4, we consider a primitive $f = \int v$. One advantage of translators is that the integration constants do not matter, so we do not write them down. Each translator Γ is the graph map of the composition $u = f \circ \tau$, $u : \mathbb{S}^n \to \mathbb{R}$. In all cases, $u \in C^0(\mathbb{S}^n) \cap C^\infty(\mathbb{S}^n \backslash \{\mathbf{s}, \mathbf{n}\})$. We recall that $\mathbf{n}$ and $\mathbf{s}$ are the North and South poles, respectively. In the following list, we give names and study Γ at the points $\mathbf{s}$ and $\mathbf{n}$.

(A) Take $v = \mathbf{w}_-$ of Lemma 4.2. Since $\mathbf{w}'_-(0) = 0$, the tangent plane to the graph of $\mathbf{u}_-$ at this point is orthogonal to the rotation axis. That is, this translator is smooth at the point $\mathbf{s}$. We call this point *the main point* of the translator. On the other hand, since $\mathbf{w}'_-(\pi/2) = +1$, then the hypersurface *hits* the rotation axis with an angle of $\pi/4$, which implies a conic singularity at $\mathbf{n}$. Similarly, we can construct $\mathbf{u}_+$ from $\mathbf{w}_+$. Note that $\mathbf{u}_+$ is the rotated sister of $\mathbf{u}_-$ (Remark 3.2.)

(B) We take a solution v of Lemma 4.3. Since $f'(\pm\pi/2) = v(\pm\pi/2) = \pm 1$, they *hit* the rotation axis with an angle of $\pm\pi/4$, so that there are two conic singularities at $\mathbf{s}$ and $\mathbf{n}$.

(C) We take a solution v of Lemma 4.4. Similarly to case B, there are two conic singularities at $\mathbf{s}$ and $\mathbf{n}$.

With the aid of wxMaxima [19], we show numerical approximations of solutions $\mathbf{w}_-$ and of Lemmas 4.3 and 4.4 in Fig. 2. Now, we make use of Theorem 3.5 in [10], obtaining the following result.

Theorem 4.1. *Up to direct isometries, any space-like $SO(n)$-invariant translator in $\mathbb{S}^n \times_{-1} \mathbb{R}$, $n \ge 2$, is an open subset of a translator of case either A, B or C.*

Proof. Take an $SO(n)$-invariant space-like translator $\Gamma : M \to \mathbb{S}^n \times_{-1} \mathbb{R}$, $n \ge 2$. Since it is space-like, it is the graph of a function $u : M \to \mathbb{R}$, $\Gamma(p) = (p, u(p))$ with $|\nabla u| < 1$. We consider the associated function

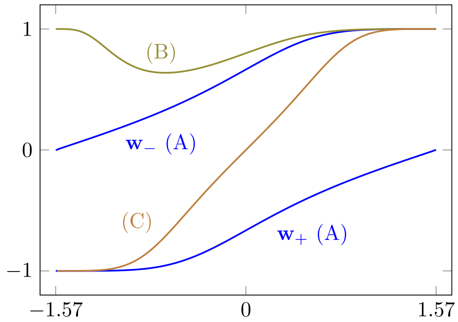

Fig. 2. Numerical approximations of the main types of solutions to (8).

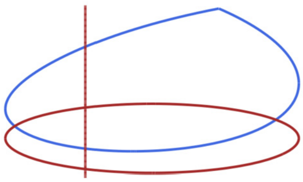

Fig. 3. Case A.

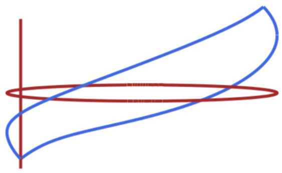

Fig. 4. Case B.

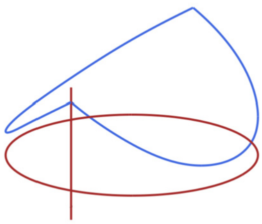

Fig. 5. Case C.

f as in Proposition 3.1 such that $u = f \circ \tau$, and therefore $(f')^2 < 1$. Taking $v = f'$, then v is one of the solutions to (8) among Lemmas 4.2, 4.3 and 4.4. The primitives of such solutions provide the different space-like translators, namely cases A, B and C. □

Figs. 3–5 show numerical approximations of functions f in the cases A, B and C, respectively, when $n = 3$. The circle represents $\mathbb{S}^n$, while the vertical line is the time-like axis.

5. The time-like case

We now deal with the case $(f')^2 > 1$. By Proposition 3.2, the case $f' < -1$ can be reduced to $f' > 1$. Then, we focus on the region $\mathcal{R} = [-\pi/2, \pi/2] \times [1, +\infty) = \overline{A_1 \cup A_2}$. We need the curve $\mathcal{C}$ given by the implicit equation $0 = \cos(s) + (n-1)\sin(s)x$, where $-\pi/2 \le s \le \pi/2$.

We summarize how to obtain all solutions to (8) included in $\overline{A_1 \cup A_2}$. First, we show that those touching the curve $\mathcal{C}$ can be extended to $[-\pi/2, \pi/2]$ in Lemma 5.2. Next, for each $s_0 \in (-\pi/2, \pi/2)$, $s_0 \neq 0$, there is a solution whose limit at s_0 is $+\infty$ in Lemma 5.3. Finally, we will obtain two solutions that are asymptotic to each side of the axis $s = 0$ in Lemmas 5.4 and 5.5. The behaviour of the different primitives will play an important role later.

Lemma 5.1.

1. *Each solution v to (8) included in region A_1 can be extended to $v \in C^0[-\pi/2, s_0) \cap C^\infty(-\pi/2, s_0)$ for some $s_0 \in (-\pi/2, 0)$. Also, $v(-\pi/2) = 1$.*
2. *Each solution v to (8) included in region A_2 can be extended to $v \in C^0(s_0, \pi/2] \cap C^\infty(s_0, \pi/2)$ for some $s_0 \in (-\arctan(1/(n-1)), \pi/2)$. Also, $v(\pi/2) = 1$.*

Proof. By Lemma 3.2, we take the integral curve α of X associated with v. Then, $v'(s) = \Theta(s, v(s))/\cos(s) > 0$. The constant solution $\mathbf{v}_+(s) = 1$ is a lower bound, so that we can extend to $v : [-\pi/2, s_0) \to \mathbb{R}$ with $\lim_{s\to-\pi/2} v(s) = x_0 \in [1, +\infty)$. By Lemma 3.3, a reparametrization of the line $s = -\pi/2$ is another integral curve of X. By uniqueness of integral curves, the curve α and the vertical line can only coincide at a point p such that $X(p) = 0$. Then, $p = (-\pi/2, 1)$. In particular, $v(-\pi/2) = 1$. This shows item a).

A similar reasoning holds for the solutions included in region A_2. To do so, remark that $\Theta_{|A_2} < 0$ and $\mathcal{C} \cap \{x = 1\} = \{(-\arctan(1/(n-1)), 1)\}$. $\square$

By recalling that $\Theta_{|A_1} > 0$ and $\Theta_{|A_2} < 0$, a straightforward application of Lemma 5.1 readily proves the following result:

Lemma 5.2. *Each solution v with initial condition $(s_0, x_0) \in \mathcal{C}$, can be extended to $v \in C^0[-\pi/2, \pi/2] \cap C^\infty(-\pi/2, \pi/2)$. In addition, $v(\pm\pi/2) = 1$, and $v'(s_0) = 0$, where $v(s_0)$ is its unique absolute maximum.*

Our next target is to look for solutions with finite time blow-ups.

Lemma 5.3.

a) *For each $(s_0, x_0) \in A_1$, there exist two solutions $f_\pm : (-\pi/2,\ s_0] \to \mathbb{R}$ to (7) such that $f_\pm(s_0) = x_0$. In addition, $v_\pm = f'_\pm$ are solutions to (8), where $\lim_{s\to s_0} v_\pm(s) = \pm\infty$, $v_+(-\pi/2) = 1$ and $v_-(-\pi/2) = -1$.*
b) *For each $s_0 \in (0, \pi/2)$ and $x_0 > 1$, there exists two solutions $f_\pm : [s_0, \pi/2] \to \mathbb{R}$ to (7) such that $f_\pm(s_0) = x_0$. In addition, $v_\pm = f'_\pm$ are solutions to (8), where $\lim_{s\to s_0} v_\pm(s) = \pm\infty$, $v_+(\pi/2) = 1$ and $v_-(\pi/2) = -1$.*

Proof. First, we study case a). Due to $\Theta_{|A_1} > 0$, we can assume that both f and $u = f'$ are injective around each point $(s_0, x_0) \in A_1$. Next, we write $w = f^{-1}$ around (s_0, x_0). Then, as in the proof of Corollary 3.7 in [10], (7) becomes

$$w''(x) = (1 - w'(x)^2)\big((n-1)\tan(w(x)) + w'(x)\big). \tag{10}$$

Initial conditions $w'(x_0) = 0$ and $w(x_0) = s_0$ provide a local solution $w : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$. Note that $w''(x_0) = (n-1)\tan(s_0) < 0$, so that w is not injective on a neighbourhood of x_0. Then, we define $f_+ := (w_{|(x_0-\delta,x_0)})^{-1} : (s_0 - \hat{\delta}, s_0) \to \mathbb{R}$ and we can extend $v = f'_+ : [-\pi/2, s_0) \to [1, +\infty)$ due to Lemma 5.1. Clearly, we can extend $f_+ : [-\pi/2, s_0) \to \mathbb{R}$, where $v(-\pi/2) = f'_+(-\pi/2) = 1$. Also, $\lim_{s\to s_0} v(s) = \lim_{s\to s_0} 1/\alpha'(f(s)) = +\infty$. We now consider the other half $f_- = (w_{|(x_0,x_0+\delta)})^{-1} : (s_0 - \hat{\delta}, s_0) \to \mathbb{R}$ with similar properties to f_+, although its graph is contained in A_6. Thus, we can extend $f_- : (-\pi/2, s_0) \to \mathbb{R}$ and compute $v = f'_- : (-\pi/2, s_0) \to (-\infty, -1]$. But now, $f'_- < -1$ and $\lim_{s\to-\pi/2} f'_-(s) = -\infty$.

Case b) can be checked by Proposition 3.2 and case a). $\square$

The remaining two solutions are asymptotic to the x-axis.

Lemma 5.4. *There exists a solution* $\xi \in C^0[-\pi/2, 0) \cap C^\infty(-\pi/2, 0)$ *to* (8) *such that it is strictly increasing,* $\xi(-\pi/2) = 1$, $\lim_{s\to 0} \xi(s) = +\infty$ *and any primitive* $\Xi = \int \xi$ *satisfies* $\lim_{s\to 0} \Xi(s) = +\infty$.

Proof. We consider the set

$$\mathfrak{I}_1 = \{x \geq 1 : \exists v : (-\pi/2, \pi/2) \to [1, +\infty) \text{solution to (8)}, \ v(-\pi/4) = x\}.$$

By Lemma 5.2, $\mathfrak{I}_1$ is not empty. By Lemma 5.3, $\mathfrak{I}_1$ is bounded from above. Take $\hat{x}_1 := \sup \mathfrak{I}_1$. Let $\xi : (s_0, s_1) \to [1, +\infty)$, $-\pi/2 \leq s_0 < -\pi/4 < s_1$, be the local solution to (8) such that $\xi(-\pi/4) = \hat{x}_1$. Note that the graph of ξ is included in region A_1. By Lemma 5.1, it can be extended to $\xi : [-\pi/2, s_1) \to [1, +\infty)$ with $\xi(-\pi/2) = 1$.

Assume $\lim_{s\to s_2} \xi(s) = +\infty$, for some $s_2 \in [s_1, 0)$. By Lemma 5.3, given $s_3 = s_2/2$, there exists another solution $v : [-\pi/2, s_3) \to [1, +\infty)$ such that $\lim_{s\to s_3} v(s) = +\infty$. Since $v(s_2) < +\infty$, then it holds $v(s) < \xi(s)$ for any $s < s_2$. In particular, $\hat{x}_1 = \xi(-\pi/4) > v(-\pi/4) > 1$. This contradicts that $\hat{x}_1 = \sup \mathfrak{I}_1$.

Next, assume that $s_1 > 0$. Then, for $x_1 = \xi(0)$, by Lemma 5.3, ξ is defined on $(-\pi/2, \pi/2)$. In particular, there exists another solution $\hat{\xi} : (-\pi/2, \pi/2) \to [1, +\infty)$ such that $\hat{\xi}(0) > \xi(0)$, so that $\hat{\xi}(-\pi/4) > \xi(-\pi/4) = \hat{x}_1$. This is a contradiction.

The conclusion is that $\xi : [-\pi/2, 0) \to [1, +\infty)$ has a finite-time blow-up at zero, namely, $\lim_{s\to 0} \xi(s) = +\infty$.

We consider a primitive $\Xi : [-\pi/2, 0) \to \mathbb{R}$, $\Xi = \int \xi$. Note that Ξ is strictly increasing since $\xi > 0$. By contradiction, we assume that $\lim_{s\to 0} \Xi(s) = \Xi_0 \in \mathbb{R}$. Then, we call $w = \Xi^{-1}$ and recall (10). Also, $w'(\Xi_0) = 1/ \lim_{s\to 0} \xi(s) = 0$. But the only solution to (10) with $w(\Xi_0) = 0$ and $w'(\Xi_0) = 0$ is the constant solution $w(x) = 0$. This contradicts the injectivity of w. Thus, $\lim_{s\to 0} \Xi(s) = +\infty$. □

Quite similar computations prove the existence of another solution which is asymptotic to $s = 0$.

Lemma 5.5. *There exists a solution* $\psi \in C^0(0, \pi/2] \cap C^\infty(0, \pi/2)$ *to* (8) *such that it is strictly decreasing,* $\psi(\pi/2) = 1$, $\lim_{s\to 0} \psi(s) = +\infty$ *and any primitive* $\Psi = \int \psi$ *satisfies* $\lim_{s\to 0} \Psi(s) = +\infty$.

We have obtained all solutions to (8) included in $\overline{A_1 \cup A_2}$. Next, for each solution v of Lemmas 5.2, 5.3, 5.4 and 5.5, we consider a primitive $f = \int v$ (see Fig. 6). Its associated $SO(n)$-invariant translator is just the graph of the function $f \circ \tau$ as in Proposition 3.1 and Remark 3.1. Similarly to the space-like translators, when v is defined at $-\pi/2$ or $\pi/2$, the translator has a conic singularity at $\mathbf{s}$ or $\mathbf{n}$. We also recall their rotated sisters, according to Remark 3.2.

From now on, we describe them and provide some names:

(D) Given a solution $v : [-\pi/2, \pi/2] \to \mathbb{R}$ of Lemma 5.2, the translator is the graph of $u : \mathbb{S}^n \to \mathbb{R}$ given by $u = f \circ \tau$. Similarly to cases B and C, all these examples have two conic singularities at $\mathbf{n}$ and $\mathbf{s}$, and they are smooth on $\mathbb{S}^n \backslash \{\mathbf{s}, \mathbf{n}\}$.

(E) Constructed from function Ξ in Lemma 5.4, this is a graph over a half-hypersphere with one conic singularity at the point $\mathbf{s}$. It explodes to infinity as the graph approaches to the equator.

(F) Constructed from Ψ of Lemma 5.5, its geometrical description is quite similar to case E. If we recall function $\Theta(s, x) = (1 - x^2)(\cos(s) + (n - 1) \sin(s)x)$, it is clear that $\Theta(-s, x)/ \cos(-s) \neq \Theta(s, x)/ \cos(s)$ for $x \notin \{0, 1, -1\}$ and $s \neq 0$. This justifies that cases E and F are indeed different.

(G) By a careful reading of Lemma 5.3, we can *glue* two translators, because there are two functions $f_\pm$ that provide the upper and the lower parts. This union becomes a smooth bi-graph with tangent planes parallel to ∂_t at the points of contact of both graphs. The whole translator covers less than half of the

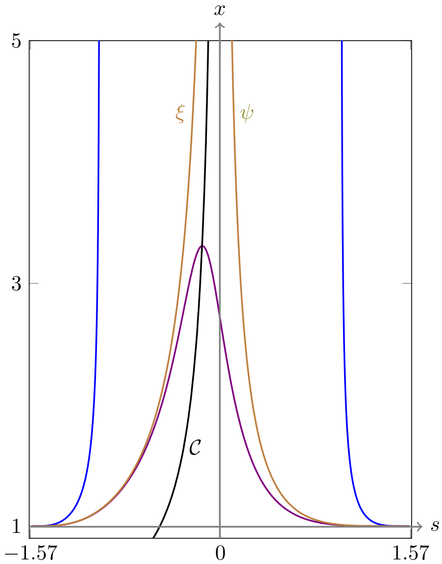

Fig. 6. Numerical approximations of the solutions to (8) in Lemmas 5.2, 5.3, 5.4 and 5.5.

hypersphere, having two conic singularities. From one conic point, the hypersurface grows gradually to a hypersphere of maximum radius, and later it shrinks gradually to the other conic singularity. Both singularities are projected to the same point of $\mathbb{S}^n$.

Finally, we construct a family of non-graphical time-like translators from minimal hypersurfaces in $\mathbb{S}^n$.

Example 5.1. Given a minimal or totally geodesic hypersurface $\sigma : M^{n-1} \to \mathbb{S}^n$, $n \geq 2$, consider $\widetilde{M}^n = M^{n-1} \times \mathbb{R}$ and the immersion $\overline{\sigma} : \widetilde{M}^n \to \mathbb{S}^n \times_{-1} \mathbb{R}$ given by $\overline{\sigma}(p,t) = (\sigma(p), t)$. A simple computation shows that the mean curvature vector field of $\overline{\sigma}$ is zero everywhere. As ∂_t is tangent to $\widetilde{M}^n$, then $\partial_t^\perp = 0$. Thus, $\overline{\sigma}$ is a translator. □

By recalling Proposition 3.2 and Remark 3.2, we obtain the following classification:

Theorem 5.1. *Up to direct isometries, any $SO(n)$-invariant, time-like translator in $\mathbb{S}^n \times_{-1} \mathbb{R}$, $n \geq 2$, is an open subset of cases D, E, F or G, or it is a non-graphical totally geodesic immersion $\mathbb{S}^{n-1} \times_{-1} \mathbb{R} \to \mathbb{S}^n \times_{-1} \mathbb{R}$.*

Proof. Take an $SO(n)$-invariant time-like translator $\Gamma : M \to \mathbb{S}^n \times_{-1} \mathbb{R}$, $n \geq 2$. We consider two cases:

Case 1: Assume that it is graphical with associated function $u : M \to \mathbb{R}$, $\Gamma(p) = (p, u(p))$ and $|\nabla u| > 1$. We consider the associated function f as in Proposition 3.1 such that $u = f \circ \tau$, with $(f')^2 > 1$. Take $v = f'$. By Proposition 3.2, it is enough to consider the solutions v to (8) among Lemmas 5.2, 5.3, 5.4 and 5.5. Then, Γ is an open subset of cases D, E, F or G.

Case 2: Assume that ∂_t is tangent to the translator on an open subset Ω. This allows the following local description. There exist an open subset $K \subset \mathbb{R}$, a smooth manifold M^{n-1} and a map $\widehat{\Gamma} : M^{n-1} \to \mathbb{S}^n$ such that

$$\Gamma : M^{n-1} \times K \to \mathbb{S}^n \times_{-1} \mathbb{R}, \quad \Gamma(p,t) = (\widehat{\Gamma}(p), t).$$

As ∂_t is tangent to the translator, $0 = {\partial_t}^\perp = \vec{H}_\Gamma$. Hence, the mean curvature vector field of $\widehat{\Gamma}$ is also 0. Since Γ is $SO(n)$-invariant, then M^{n-1} has to be a union of orbits. But the only minimal orbit is totally geodesic. This finishes the proof. □

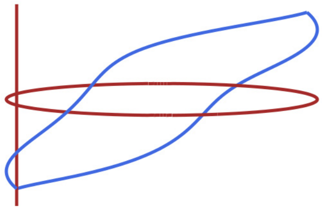

Fig. 7. Case D.

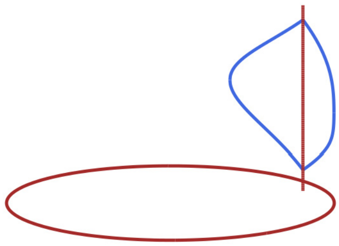

Fig. 8. Case E.

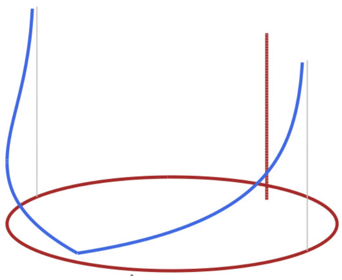

Fig. 9. Case F.

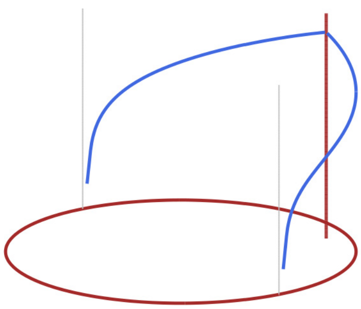

Fig. 10. Case G.

Remark 5.1. Due to the conic singularities, there do not exist any complete graphical examples, either space-like or time-like.

We show numerical approximations of function f in cases D, E, F and G when $n = 3$ (see Figs. 7–10). The circle represents $\mathbb{S}^3$, while the vertical line is the time-like axis.

6. Uniqueness results

Along all this section, we use the beginning of Chapter 10 of book [7]. We recall that $n \geq 2$ is a natural number.

Lemma 6.1. *When $\varepsilon = -1$, PDE (3) behaves as a quasilinear elliptic operator. Moreover, it is locally uniformly bounded.*

Proof. We consider the inverse map of the classical stereographical map, namely

$$\Phi : \mathbb{R}^n \to \mathbb{S}^n \backslash \{\mathbf{n}\} \subset \mathbb{R}^n \times \mathbb{R}, \quad \Phi(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{1+|x|^2} \right).$$

It is well-known that this is a conformal map. A straightforward computation shows that the induced metric on $\mathbb{R}^n$ is

$$\tilde{g} = \frac{4}{(1+|x|^2)^2} \langle , \rangle,$$

where $\langle , \rangle$ denotes the standard metric on $\mathbb{R}^n$. We use coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and the partial derivatives are $\partial_i = \partial/\partial x_i$, $i = 1, \ldots, n$. Given any function f, we write $f_i = \partial_i f$. Let Ω be open and connected such that $\overline{\Omega} \subsetneq \mathbb{S}^n \backslash \{\mathbf{n}\}$. We put $\Lambda = \Phi^{-1}(\Omega)$, which is open, connected and bounded in $\mathbb{R}^n$. Obviously, $\partial \Lambda = \Phi^{-1}(\partial \Omega)$. For simplicity, we identify $u \equiv u \circ \Phi : \overline{\Lambda} \to \mathbb{R}$. Let ∇ be the Levi-Civita connection of $(\mathbb{R}^n, \tilde{g})$. If we put $\lambda : \mathbb{R}^n \to \mathbb{R}$, $\lambda(x) = (1 + \sum_{i=1}^n x_i^2)/2$, then $\tilde{g} = \langle , \rangle / \lambda^2$. In addition, $\nabla u = \lambda^2 \sum_{i=1}^n u_i \partial_i$. Taking $\varepsilon = -1$, if we recall $W = \sqrt{1 - |\nabla u|^2}$, we obtain $W^2 = 1 - \lambda^2 \sum_k u_k^2$. We need the auxiliary functions $\Gamma_{ij} : \Lambda \to \mathbb{R}$, $\Gamma_{ij} = \tilde{g}\left(\nabla_{\partial_i} \partial_j, \partial_i\right)$, $i, j \in \{1, \ldots, n\}$ After a straightforward computation, by using this coordinate system, the PDE (3) can be written as follows:

$$\begin{aligned} 0 =& W^2 \mathrm{div}_{\tilde{g}}(\nabla u) - W \, \tilde{g}(\nabla u, \nabla W) - W^2 \\ =& \sum_{i,j=1}^n \lambda^2 \left(W^2 \delta_{ij} + \lambda^2 u_i u_j \right) u_{ij} + 2\lambda W^2 \sum_{i=1}^n x_i u_i \\ &+ \lambda^4 W^2 \sum_{i,j=1}^n u_j \Gamma_{ij} + \lambda^3 \sum_{i,j=1}^n x_i u_i u_j^2 - W^2. \end{aligned}$$

As usual, δ_{ij} is the Kronecker's delta. By taking $(x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, we introduce the corresponding functions $\widehat{W} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $a^{ij}, b : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\begin{aligned} \lambda(x) =& (1 + \langle x, x \rangle)/2, \quad \widehat{W}(x,p) = 1 - \lambda(x)^2 \langle p, p \rangle, \\ a^{ij}(x,z,p) =& \lambda(x)^2 \left(\widehat{W}(x,p) \delta_{ij} + p_i p_j \right), \\ b(x,z,p) =& 2\lambda(x) \widehat{W}(x,p) \langle x, p \rangle + \lambda(x)^4 \widehat{W}(x,p) \sum_{i,j=1}^n p_j \Gamma_{ij}(x) \\ &+ \lambda(x)^3 \sum_{i,j=1}^n x_i p_i p_j^2 - \widehat{W}(x,p). \end{aligned}$$

The associated quasilinear operator is

$$Qu = \sum_{i,j=1}^{n} a^{ij}(x, u, \nabla u)u_{ij} + b(x, u, \nabla u).$$

Let us show that the matrix $A = (a^{ij})$ is positive definite on a suitable subset $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We rewrite $A = \lambda^2 \widehat{W} I_n + \lambda^2 p^t p$, where I_n is the identity matrix. It is simple to see that for $q \perp p$, we have $Ap^t = \lambda^2(\widehat{W} + |p|^2)p^t$ and $Aq^t = \lambda^2 \widehat{W} q^t$. In particular, the eigenvalues of A are $\lambda_1 = \lambda^2(\widehat{W} + |p|^2)$ and $\lambda_2 = \lambda^2 \widehat{W}$. These two functions are strictly positive on

$$\mathcal{U} = \left\{ (x, z, p) \in \Lambda \times \mathbb{R} \times \mathbb{R}^n : 1 - \lambda(x)^2 \sum_{k=1}^{n} p_k > 0 \right\},$$

and bounded on compact subsets included in $\mathcal{U}$. The condition defining $\mathcal{U}$ can be regarded as $|\nabla u| < 1$. □

Theorem 6.1. *Let Ω be open, connected and $\overline{\Omega} \subsetneq \mathbb{S}^n$, $n \geq 2$. Let $u, \hat{u} \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that Γ_u and $\Gamma_{\hat{u}}$ are space-like translators in $\mathbb{S}^n \times_{-1} \mathbb{R}$, and $u = \hat{u}$ on $\partial\Omega$. Then, $\Gamma_u = \Gamma_{\hat{u}}$.*

Proof. Up to an isometry, we can assume that $\overline{\Omega} \subset \mathbb{S}^n \backslash \{\mathbf{n}\}$. By Lemma 6.1, we use the bounded open domain $\Lambda = \Phi^{-1}(\Omega) \subset \mathbb{R}^n$, and its boundary $\partial\Lambda = \Phi^{-1}(\partial\Omega)$. Define $\mathbf{u} := u \circ \Phi$ and $\hat{\mathbf{u}} := \hat{u} \circ \Phi : \overline{\Lambda} \to \mathbb{R}$. Both $\mathbf{u}$ and $\hat{\mathbf{u}}$ satisfy $Q\mathbf{u} = 0 = Q\hat{\mathbf{u}}$ on Λ, they are elliptic in Λ, and $\mathbf{u} = \hat{\mathbf{u}}$ on $\partial\Lambda$. In addition, the coefficients of the operator Q are of class C^∞ and independent of the variable z. By Theorem 10.2 in the book [7], we conclude that $\mathbf{u} = \hat{\mathbf{u}}$ on Λ. □

Theorem 6.2. *Let Ω be open, connected and $\overline{\Omega} \subsetneq \mathbb{S}^n$, $n \geq 2$. Let $\sigma : \mathbb{S}^n \to \mathbb{S}^n$ be an isometry such that $\sigma(\overline{\Omega}) = \overline{\Omega}$. Consider $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that Γ_u is a space-like translator in $\mathbb{S}^n \times_{-1} \mathbb{R}$, and $u \circ \sigma = u$ on $\partial\Omega$. Then, Γ_u is also invariant by $\sigma \times id_{\mathbb{R}}$.*

Proof. We take $\hat{u} := u \circ \sigma \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. By hypothesis, $\hat{u} = u$ on $\partial\Omega$. By Lemma 2.1, $\Gamma_{\hat{u}}$ is also a space-like translator. We use now Theorem 6.1. □

Corollary 6.1. *Let Ω be open, connected and $\overline{\Omega} \subsetneq \mathbb{S}^n$, $n \geq 2$. If $\overline{\Omega}$ is invariant by a subgroup Σ of isometries of $\mathbb{S}^n$ and $u \circ \sigma = u$ on $\partial\Omega$ for any $\sigma \in \Sigma$, then Γ_u is also invariant by $\Sigma \times \{id_{\mathbb{R}}\}$.*

As usual, we denote by $B(p, r) \subset \mathbb{S}^n$ the ball centred at $p \in \mathbb{S}^n$ and radius $r \in (0, 2\pi)$, and by $B^*(p, r) = B(p, r)\backslash\{p\}$.

Corollary 6.2. *Let $\Omega = B^*(p, r)$, $r \in (0, 2\pi)$ and $p \in \mathbb{S}^n$. Assume that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ provides a space-like translator Γ_u, and for some $c \in \mathbb{R}$, $u \equiv c$ on $\partial B(p, r)$. Then, up to isometries,*

1. *Γ_u is $SO(n)$ invariant;*
2. *In addition, if Γ_u is smooth at p, then it is a compact piece of case A.*

Proof. Clearly, $\overline{\Omega}$ is $SO(n)$-invariant. By Corollary 6.1, Γ_u is $SO(n)$-invariant. Then, we recall Theorem 4.1. In addition, if we can extend Γ_u smoothly to p, it can only be a smooth compact piece of case A. □

Declaration of competing interest

The authors declare that they have no interests/competing interests. No AI has been used to prepare this document. All pictures made by the authors of this paper with the aid of Maxima [19].

Data availability

No data was used for the research described in the article.

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