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Ratio-Covarieties of Numerical Semigroups

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Abstract: In this work, we will introduce the concept of ratio-covariety, as a family $\mathscr{R}$ of numerical semigroups that has a minimum, denoted by $\min(\mathscr{R})$, is closed under intersection, and if $S \in \mathscr{R}$ and $S \neq \min(\mathscr{R})$, then $S \backslash \{\mathrm{r}(S)\} \in \mathscr{R}$, where $\mathrm{r}(S)$ denotes the ratio of S. The notion of ratio-covariety will allow us to: (1) describe an algorithmic procedure to compute $\mathscr{R}$; (2) prove the existence of the smallest element of $\mathscr{R}$ that contains a set of positive integers; and (3) talk about the smallest ratio-covariety that contains a finite set of numerical semigroups. In addition, in this paper we will apply the previous results to the study of the ratio-covariety $\mathscr{R}(F, m) = \{S \mid S \text{ is a numerical semigroup with Frobenius number F and multiplicity } m\}$.

Keywords: numerical semigroup; ratio-covariety; Frobenius number; genus; ratio; algorithm

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S$ is finite. The set $\mathbb{N} \backslash S$ is known as the set of *gaps* of S, and its cardinality is called the *genus* of S, denoted by $\mathrm{g}(S)$. The largest integer not belonging to S is the *Frobenius number* of S, and it will be denoted by $\mathrm{F}(S)$. For instance, $\mathrm{F}(\mathbb{N}) = -1$. Let $\{a_1, \ldots, a_n\} \subseteq \mathbb{N}$ such that $\gcd(a_1, \ldots, a_n) = 1$. Then, $\langle a_1, \ldots, a_n \rangle = \left\{ \sum_{j=1}^{n} \alpha_j a_j \mid \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{N} \right\}$ is a numerical semigroup, and every numerical semigroup has this form (see [1], Lemma 2.1). The set $\{a_1, \ldots, a_n\}$ is called a *system of generators* of S, and we write $S = \langle a_1, \ldots, a_n \rangle$. A system of generators of a numerical semigroup is called the *minimal system of generators* if none of its proper subsets generates the numerical semigroup. In ([1], Corollary 2.8) is proven that every numerical semigroup has a unique minimal system of generators which, in addition, is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of a numerical semigroup S. Its cardinality is called the *embedding dimension*, and it will be denoted by $\mathrm{e}(S)$. The *multiplicity* of S is another invariant which we will use in this work. It is denoted by $\mathrm{m}(S)$, and it is the minimum of $S \backslash \{0\}$. It is verified that $\mathrm{e}(S) \leq \mathrm{m}(S)$ (see [1], Proposition 2.10).

Definition 1. *A numerical semigroup S is said to have a maximal embedding dimension if* $\mathrm{e}(S) = \mathrm{m}(S)$*; from now on, we will call such an* MED-*semigroup.*

In the literature, one can find a long list of works dealing with the study of one dimensional analytically irreducible local domains via their value semigroups. One of the properties studied for this kind of ring using this approach is that of the maximal embedding dimension (see [2–5]). The characterization of rings with the maximal embedding dimension via their value semigroup gave rise to the notion of the MED-semigroup.

The Frobenius problem (see [6]) for numerical semigroups is a classical mathematical problem. It consists of obtaining formulas for calculating the Frobenius number and the genus of a numerical semigroup in terms of its minimal system of generators. The solution to the problem was found for numerical semigroups with embedding dimension two in [7]. Since then, many researchers have tried to solve this problem for numerical semigroups with an embedding dimension greater than or equal to three. However, the problem is still open. Furthermore, in this case, the problem becomes NP-hard (see [8]).

The main aim of this paper is to introduce the concept of ratio-covariety and justify its study.

For integers a and b, we say that *a divides b* if there exists an integer c, such that $b = ca$, and we denote this by $a \mid b$. Otherwise, *a does not divide b*, and we denote this by $a \nmid b$. Let S be a numerical semigroup such that $S \neq \mathbb{N}$, the *ratio* of S, is defined $\mathrm{r}(S) := \min(\mathrm{msg}(S) \backslash \{\mathrm{m}(S)\})$. Note that $\mathrm{r}(S) = \min\{s \in S \mid \mathrm{m}(S) \nmid s\}$.

Definition 2. *We say that a nonempty family $\mathscr{R}$ of numerical semigroups is a ratio-covariety if it verifies the following conditions:*

(1) There is the minimum (with respect to set inclusion) of $\mathscr{R}$, denoted by $\min(\mathscr{R})$.
(2) If $\{S, T\} \subseteq \mathscr{R}$, then $S \cap T \in \mathscr{R}$.
(3) If $S \in \mathscr{R}$ and $S \neq \min(\mathscr{R})$, then $S \backslash \{\mathrm{r}(S)\} \in \mathscr{R}$.

The most trivial example of a ratio-covariety is the singleton $\{S\}$ with any numerical semigroup S. There are another families of numerical semigroups satisfying these conditions. For instance, the next two families of numerical semigroups are ratio-covarieties: $\mathscr{R}(F, m) = \{S \mid S \text{ is a numerical semigroup}, \mathrm{F}(S) = F \text{ and } \mathrm{m}(S) = m\}$ and $\mathrm{MED}(F, m) = \{S \mid S \text{ is a numerical semigroup with maximal embedding dimension}, \mathrm{F}(S) = F \text{and} \mathrm{m}(S) = m\}$. Section 2 is devoted to proving that every ratio-covariety is finite and its elements can be ordered in a rooted tree. Moreover, we will describe the children of an arbitrary vertex of the tree. As a consequence, we obtain an algorithmic procedure to compute all the elements of a ratio-covariety.

Definition 3. *Let $\mathscr{R}$ be a ratio-covariety. A subset $X \subseteq \mathbb{N}$ will be call an $\mathscr{R}$-set if $X \cap \min(\mathscr{R}) = \emptyset$ and there exists $S \in \mathscr{R}$ with $X \subseteq S$.*

We will investigate, in Section 5, the smallest element of $\mathscr{R}$ containing an $\mathscr{R}$-*set*, X. This element will be denoted by $\mathscr{R}[X]$, and we will say that it is the element of $\mathscr{R}$ generated by X.

Definition 4. *If $S = \mathscr{R}[X]$, X is called an $\mathscr{R}$-system of generators of S. Moreover, if $S \neq \mathscr{R}[Y]$ for all $Y \subsetneq X$, then X will be called a minimal $\mathscr{R}$-system of generators of S.*

In Section 5, we will show an example of ratio-covariety $\mathscr{R}$, in which the minimal $\mathscr{R}$-system of generators is not unique.

Definition 5. *Let $\mathscr{R}$ be a ratio-covariety and $S \in \mathscr{R}$, then the $\mathscr{R}$-*rank *of S is* $\mathscr{R}\mathrm{rank}(S) = \min\{\#X \mid X \textit{ is } \mathscr{R}\textit{-set and } \mathscr{R}[X] = S\}$ *(where $\#Y$ denotes the cardinality of a set Y).*

We will characterize the elements of $\mathscr{R}$ with $\mathscr{R}$-rank 0 and 1. In Section 6, we will show that if $S_1, \cdots, S_k$ are numerical semigroups with multiplicity m and $F = \max\{\mathrm{F}(S_1), \cdots, \mathrm{F}(S_k)\}$, then there is the smallest ratio-covariety containing the set $\{S_1, \cdots, S_k\}$, and with $\Delta(F, m) = \langle m \rangle \cup \{F + 1, \rightarrow\}$ as its minimum (the symbol $\rightarrow$ means that every integer greater than $F + 1$ belongs to the set). This ratio-covariety is denoted by $\langle S_1, \cdots, S_k \rangle$, and we present an algorithm which computes all its elements.

Along with this work and with the aim of giving examples of general results, we will use the ratio-covariety $\mathscr{R}(F, m)$. In particular, we will obtain an alternative algorithm to the one presented in [9] to compute all the elements of $\mathscr{R}(F, m)$.

Throughout this paper, some examples are shown to illustrate the proven results. For the development of these examples, the GAP (see [10]) package `numericalsgps` (see [11]) can be used.

2. Basic Properties and Examples

If T is a numerical semigroup, we have the following result.

Proposition 1. *Every ratio-covariety has a finite cardinality.*

Proof. As $\mathbb{N}\backslash T$ is a finite set, the set $\{S \mid S \text{ is a numerical semigroup and } T \subseteq S\}$ is also finite, and every ratio-covariety has a minimum. $\square$

Throughout this work, m and F denote positive integers such that $m < F$ and $m \nmid F$. Recall that $\mathscr{R}(F,m) = \{S \mid S \text{ is a numerical semigroup, } \mathrm{F}(S) = F \text{ and } \mathrm{m}(S) = m\}$ and $\mathrm{MED}(F,m) = \{S \mid S \text{ is a numerical semigroup with maximal embedding dimension, } \mathrm{F}(S) = F \text{ and } \mathrm{m}(S) = m\}$ are examples of ratio-covarieties.

Definition 6. *A numerical semigroup S is Strong if $x + y - \mathrm{m}(S) \in S$ for every $(x,y) \in (S\backslash\{0\})^2$ such that $x \not\equiv y (\mathrm{mod}\, \mathrm{m}(S))$.*

The set $\mathrm{ST}(F,m) = \{S \mid S \text{ is a Strong numerical semigroup, } \mathrm{F}(S) = F \text{ and } \mathrm{m}(S) = m\}$ is another example of ratio-covariety.

The following result has an easy proof.

Lemma 1. *With the above notation, we have that*

$$\Delta(F,m) = \langle m \rangle \cup \{F+1, \rightarrow\}$$

is the minimum of $\mathscr{R}(F,m)$.

Proof. Obviously $\Delta(F,m) \in \mathscr{R}(F,m)$, and it is clear that every numerical semigroup with multiplicity m and whose Frobenius number is F, then it contains $\Delta(F,m)$. $\square$

The next Lemma is known, and its proof is not difficult.

Lemma 2. *Let S and T be numerical semigroups and $x \in S$. Then, the following conditions hold:*

1. *$S \cap T$ is a numerical semigroup and $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.*
2. *$S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

Lemmas 1 and 2 immediately imply the following result.

Proposition 2. *$\mathscr{R}(F,m)$ is a ratio-covariety.*

3. The Tree Associated with a Ratio-Covariety

A *graph* G is a pair (V,E), where V is a nonempty set and E is a subset of $\{(x,y) \in V \times V \mid x \neq y\}$. The elements of V and E are called *vertices* and *edges*, respectively. A *path, of length n*, connecting the vertices u and v of G, is a sequence of different edges of the form $(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n)$, such that $x_0 = u$ and $x_n = v$.

A graph G is *a rooted tree* if there is a vertex r (known as *the root* of G), such that for any other vertex x of G, there exists a unique path linking x and r. We say that x is a *son* of y, if (x,y) is an edge of the rooted tree G.

Let $\mathscr{R}$ be a ratio-covariety and $S \in \mathscr{R}$. Recursively define the following sequence of elements of $\mathscr{R}$:

- $S_0 = S$,

- $S_{n+1} = \begin{cases} S_n \backslash \{\mathrm{r}(S_n)\} & \text{if } S_n \neq \min(\mathscr{R}), \\ \min(\mathscr{R}) & \text{otherwise.} \end{cases}$

By observing the definition from the above sequence, the following result is immediately clear.

Lemma 3. *Let $\mathscr{R}$ be a ratio-covariety, $S \in \mathscr{R}$ and let $\{S_n\}_{n\in\mathbb{N}}$ be the sequence of elements of $\mathscr{R}$ defined above. Then, there exists $k \in \mathbb{N}$ such that $\min(\mathscr{R}) = S_k \subsetneq S_{k-1} \subsetneq \cdots \subsetneq S_0 = S$. Moreover, $\#(S_i \backslash S_{i+1}) = 1$ for all $i \in \{0, 1, \cdots, k-1\}$.*

If $\mathscr{R}$ is a ratio-covariety, then we define the graph $\mathrm{G}(\mathscr{R})$ as follows: $\mathscr{R}$ is the set of its vertices and $(S, T) \in \mathscr{R} \times \mathscr{R}$ is an edge of $\mathrm{G}(\mathscr{R})$, if and only if $T = S \backslash \{\mathrm{r}(S)\}$.

Note that in the sequence $\{S_n\}_{n\in\mathbb{N}}$ of elements of $\mathscr{R}$ defined above, the pair (S_n, S_{n+1}) is an edge of the graph $\mathrm{G}(\mathscr{R})$. As a consequence from Lemma 3, we have the following result.

Proposition 3. *Let $\mathscr{R}$ be a ratio-covariety. Then, $\mathrm{G}(\mathscr{R})$ is a tree with root $\min(\mathscr{R})$.*

A tree can be built recursively starting from the root and connecting, through an edge, the vertices already built with their children. Hence, it is very interesting to characterize the children of an arbitrary vertex in the tree.

Definition 7. *Let S be a numerical semigroup. An integer z is called a pseudo-Frobenius number of S, if $z \notin S$ and $z + s \in S$ for all $s \in S \backslash \{0\}$.*

We denote by $\mathrm{PF}(S)$ the set formed by the pseudo-Frobenius numbers of S. The elements of the set $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \in S\}$ are called the *special gaps* of a numerical semigroup S. The following result is [1], Proposition 4.33.

Lemma 4. *If S is a numerical semigroup and $x \in \mathbb{N} \backslash S$, then $x \in \mathrm{SG}(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

Example 1. *Let $S = \langle 3, 5, 7 \rangle = \{0, 3, 5, \rightarrow\}$, then it is clear that $\mathrm{PF}(S) = \{2, 4\}$ and $\mathrm{SG}(S) = \{4\}$.*

Proposition 4. *Let $\mathscr{R}$ be a ratio-covariety and $S \in \mathscr{R}$. Then, the set formed by the children of S in the tree $\mathrm{G}(\mathscr{R})$ is*

$$\{S \cup \{x\} \mid x \in \mathrm{SG}(S),\ \mathrm{m}(S) < x < \mathrm{r}(S) \text{ and } S \cup \{x\} \in \mathscr{R}\}.$$

Proof. If T is a child of S, then $T \in \mathscr{R}$ and $T \backslash \{\mathrm{r}(T)\} = S$. Therefore, $T = S \cup \{\mathrm{r}(T)\} \in \mathscr{R}$, $\mathrm{r}(T) \in \mathrm{SG}(S)$ and $\mathrm{m}(S) < \mathrm{r}(T) < \mathrm{r}(S)$.
if $\mathrm{m}(S) < x < \mathrm{r}(S)$, then $\mathrm{r}(S \cup \{x\}) = x$. Hence, $(S \cup \{x\}) \backslash \{\mathrm{r}(S \cup \{x\})\} = S$ and, consequently, $S \cup \{x\} \in \mathscr{R}$ is a child of S in the tree $\mathrm{G}(\mathscr{R})$. □

As a consequence of Propositions 2 and 4, we can characterize the children of the tree $\mathrm{G}(\mathscr{R}(F, m))$.

Proposition 5. *If $S \in \mathscr{R}(F, m)$, then the children of S in $\mathrm{G}(\mathscr{R}(F, m))$ form the set $\{S \cup \{x\} \mid x \in \mathrm{SG}(S),\ m < x < \mathrm{r}(S) \text{ and } x \neq F\}$.*

Given a numerical semigroup S and $n \in S \backslash \{0\}$, the *Apéry set* of n in S (in honor of [12]) is the set $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. This set has n elements, one for every congruence class modulo n. That is, $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n, for all $i \in \{0, \ldots, n-1\}$ (see [1], Lemma 2.4).

Let S be a numerical semigroup. We define an order relation on $\mathbb{Z}$ as follows: $a \leq_S b$ if $b - a \in S$.

Definition 8. *Let $A \subseteq \mathbb{Z}$. An element $a \in A$ is called maximal with respect to $\leq_S$, if there exists no $b \in A$ such that $a \leq_S b$ and $b \neq a$. Denote by $\mathrm{Maximals}_{\leq_S} A$, the maximal elements of a set A with respect to the ordering $\leq_S$.*

The following result is Proposition 2.20 from [1].

Lemma 5. *If S is a numerical semigroup and $n \in S \backslash \{0\}$, then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, n)\}.$$

Lemma 6. *Let S be a numerical semigroup, $n \in S \backslash \{0\}$ and $w \in \mathrm{Ap}(S, n)$. Then, $w \in \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, n))$ if and only if $w + w' \notin \mathrm{Ap}(S, n)$ for all $w' \in \mathrm{Ap}(S, n) \backslash \{0\}$.*

Proof. *Necessity.* As $w \leq_S w + w'$, then $w + w' \notin \mathrm{Ap}(S, n)$.
Sufficiency. Let $w' \in \mathrm{Ap}(S, n)$ such that $w <_S w'$ then $w' - w \in \mathrm{Ap}(S, n) \backslash \{0\}$. Consequently, we have that $w' = w + (w' - w) \in \mathrm{Ap}(S, n)$, a contradiction. □

It is straightforward to prove the following result, since if $x \in \mathrm{PF}(S)$, then either $2x \in \mathrm{PF}(S)$ or $2x \in S$.

Lemma 7. *Let S be a numerical semigroup such that $S \neq \mathbb{N}$. Then,*

$$\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \notin \mathrm{PF}(S)\}.$$

Note 1. *As a consequence of Lemmas 5–7, observe that if S is a numerical semigroup, and we know $\mathrm{Ap}(S, n)$ for some $n \in S \backslash \{0\}$, then we can compute easily the set $\mathrm{SG}(S)$.*

We will explain the above note with an example.

Example 2. *If $S = \langle 11, 12, 25, 41 \rangle$, then $\mathrm{Ap}(S, 11) = \{0, 12, 24, 25, 37, 41, 49, 50, 53, 62, 65\}$. By applying Lemma 6, we have that $\mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, 11)) = \{49, 62, 65\}$. Then, Lemma 5, asserts that $\mathrm{PF}(S) = \{38, 51, 54\}$. Finally, by using Lemma 7, we have that $\mathrm{SG}(S) = \{38, 51, 54\}$.*

The following result has an easy proof.

Lemma 8. *Let S be a numerical semigroup, $n \in S \backslash \{0\}$ and $x \in \mathrm{SG}(S)$. Then, $x + n \in \mathrm{Ap}(S, n)$. Moreover, $\mathrm{Ap}(S \cup \{x\}, n) = (\mathrm{Ap}(S, n) \backslash \{x + n\}) \cup \{x\}$.*

Note 2. *Observe that as a consequence of Lemma 8, if we know $\mathrm{Ap}(S, n)$, then we can easily calculate $\mathrm{Ap}(S \cup \{x\}, n)$. In particular, if $\mathscr{R}$ is a ratio-covariety and $S \in \mathscr{R}$, then Lemma 8 allows us to compute the set $\mathrm{Ap}(T, n)$ from $\mathrm{Ap}(S, n)$, for every child T of S in the tree $\mathrm{G}(\mathscr{R})$ (see Proposition 4).*

The next example illustrates the previous note.

Example 3. *Let $S = \{0, 7, \rightarrow\}$ be a numerical semigroup. We have that $\mathrm{Ap}(S, 7) = \{0, 8, 9, 10, 11, 12, 13\}$. By Example 2, we know that $5 \in \mathrm{SG}(S)$. If $T = S \cup \{5\}$, then Lemma 8 asserts that*

$$\mathrm{Ap}(T, 7) = (\{0, 8, 9, 10, 11, 12, 13\} \backslash \{5 + 7\}) \cup \{5\} = \{0, 5, 8, 9, 10, 11, 13\}.$$

We already have all the necessary tools to present the algorithm that gives title to this section. That is, we start by computing the root $\Delta(F, m)$, using Lemma 1, of the tree and, by using Proposition 5 and Lemma 8, we determine the children of each vertex. Namely we obtain $\mathscr{R}(F, m)$.

Algorithm 1 Computation of $\mathscr{R}(F,m)$

INPUT: Two positive integers F and m, such that $m < F$ and $m \nmid F$.
OUTPUT: $\mathscr{R}(F,m)$.

(1) Compute $\mathrm{Ap}(\Delta(F,m),m)$.
(2) $R(F,m) = \{\Delta(F,m)\}$ and $B = \{\Delta(F,m)\}$.
(3) For all $S \in B$ compute $\theta(S) = \{x \in \mathrm{SG}(S) \mid m < x < \mathrm{r}(S) \text{ and } x \neq F\}$.
(4) If $\bigcup_{S\in B} \theta(S) = \emptyset$, then return $R(F,m)$.
(5) $C = \bigcup_{S\in B} \{S \cup \{x\} \mid x \in \theta(S)\}$.
(6) $R(F,m) = R(F,m) \cup C$ and $B = C$.
(7) For every $S \in B$, compute $\mathrm{Ap}(S,m)$ and go to Step (3).

Next, we show how this algorithm works.

Example 4. *We compute $\mathscr{R}(7,4)$ by using Algorithm 1.*

- $\Delta(7,4) = \{0,4,8,\rightarrow\}$ *and* $\mathrm{Ap}(\Delta(7,4),4) = \{0,9,10,11\}$.
- $R(7,4) = \{\Delta(7,4)\}$ *and* $B = \{\Delta(7,4)\}$.
- $\theta(\Delta(7,4)) = \{5,6\}$.
- $C = \{\Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$.
- $R(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$ *and* $B = \{\Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$.
- $\mathrm{Ap}(\Delta(7,4) \cup \{5\},4) = \{0,5,10,11\}$ *and* $\mathrm{Ap}(\Delta(7,4) \cup \{6\},4) = \{0,6,9,11\}$.
- $\theta(\Delta(7,4) \cup \{5\}) = \emptyset$ *and* $\theta(\Delta(7,4) \cup \{6\}) = \{5\}$.
- $C = \{\Delta(7,4) \cup \{5,6\}\}$.
- $R(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}, \Delta(7,4) \cup \{5,6\}\}$ *and* $B = \{\Delta(7,4) \cup \{5,6\}\}$.
- $\mathrm{Ap}(\Delta(7,4) \cup \{5,6\},4) = \{0,5,6,11\}$.
- $\theta(\Delta(7,4) \cup \{5,6\}) = \emptyset$.
- *The Algorithm returns*

$$\mathscr{R}(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}, \Delta(7,4) \cup \{5,6\}\}.$$

4. The Elements of $\mathscr{R}(F,m)$ with a Fixed Genus

Let F, m and g be positive integers. Denote by $\mathscr{R}(F,m,g) = \{S \in \mathscr{R}(F,m) \mid \mathrm{g}(S) = g\}$. From [1] (Lemma 2.14), and the fact that $\mathrm{g}(S) \leq \mathrm{F}(S)$, we can deduce the following result.

Proposition 6. *If S is a numerical semigroup, then* $\frac{\mathrm{F}(S)+1}{2} \leq \mathrm{g}(S) \leq \mathrm{F}(S)$.

The following notion appears in [1].

Definition 9. *A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.*

In [1], Theorem 4.2, the next result is proven.

Lemma 9. *Let S be a numerical semigroup. Then, S is irreducible if and only if S is a maximal element in the set $\{T \mid T$ is a numerical semigroup with $\mathrm{F}(T) = \mathrm{F}(S)\}$.*

The irreducible numerical semigroups are very interesting because, from [3,13], we know that a numerical semigroup is irreducible if and only if it is symmetric or pseudo-symmetric. This kind of numerical semigroup has been widely studied in the literature, because a one dimensional analytically irreducible local ring is Gorenstein (respectively

Kunz), if and only if its value semigroup is symmetric (respectively pseudo-symmetric), as it can be seen in [3,14].

From ([1], Corollary 4.5), we have the following characterization.

Lemma 10. *Let S be a numerical semigroup. Then, the following conditions hold.*

1. *S is symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+1}{2}$.
2. *S is pseudo-symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+2}{2}$.
3. *S is irreducible if and only if* $\mathrm{g}(S) = \left\lceil \dfrac{\mathrm{F}(S)+1}{2} \right\rceil$*, where $\lceil\ \rceil$ denotes the ceiling operator.*

Denote by $\mathrm{Max}(\mathscr{R}(F,m))$ the set formed by the maximal elements of $\mathscr{R}(F,m)$. The following result is the same as [15], Proposition 6, with a bit different notation.

Proposition 7. *With the above notation, we have the following.*

1. *If $F = m-1$, then* $\mathrm{Max}(\mathscr{R}(F,m)) = \mathscr{R}(F,m) = \{\{0, m, \rightarrow\}\}$.
2. *If $m < F < 2m$, then* $\mathrm{Max}(\mathscr{R}(F,m)) = \{\{0, m, \rightarrow\} \backslash \{F\}\}$.
3. *If $F > 2m$, then* $\mathrm{Max}(\mathscr{R}(F,m)) = \{S \in \mathscr{R}(F,m) \mid S \text{ is irreducible}\}$.

As a consequence of Lemma 10 and Proposition 7, we have the following result. The proof can be obtained as a direct application of them.

Corollary 1. *Let $S \in \mathscr{R}(F,m)$. Then, $S \in \mathrm{Max}(\mathscr{R}(F,m))$, if and only if one of the following conditions are satisfied.*

1. $F = m-1$.
2. *$m < F < 2m$ and $\mathrm{g}(S) = m$.*
3. *$F > 2m$ and* $\mathrm{g}(S) = \left\lceil \dfrac{F+1}{2} \right\rceil$.

For a numerical semigroup $S \neq \mathbb{N}$, the *ratio-sequence associated* to S is recursively defined as: $S_0 = S$ and $S_{n+1} = S_n \backslash \{\mathrm{r}(S_n)\}$ for all $n \in \mathbb{N}$.

For a numerical semigroup S, we set $\mathrm{A}(S) = \{x \in S \mid x < \mathrm{F}(S) \text{ and } \mathrm{m}(S) \nmid x\}$. The cardinality of $\mathrm{A}(S)$ will be denoted by $\mathrm{a}(S)$.

Let S be a numerical semigroup and let $\{S_n\}_{n\in\mathbb{N}}$ be the ratio-sequence associated with S, then the set $\text{Rat-Cad}(S) = \{S_0, S_1, \ldots, S_{\mathrm{a}(S)}\}$ is called the *ratio-chain associated* to S. It is clear that $S_{\mathrm{a}(S)} = \Delta(\mathrm{F}(S), \mathrm{m}(S))$.

Lemma 11. *With the above notation, it is verified that* $\mathrm{g}(\Delta(F,m)) = F - \left\lfloor \dfrac{F}{m} \right\rfloor$*, where $\lfloor\ \rfloor$ denotes the floor operator.*

Proof. It is enough to observe that

$$\Delta(F,m) = \{0, m, 2m, \cdots, \left\lfloor \frac{F}{m} \right\rfloor m, F+1, \rightarrow\}.$$

□

The following result has an easy proof.

Lemma 12. *If $m < F < 2m$, then*

$$\mathscr{R}(F,m) = \{A \cup \{0, m, F+1, \rightarrow\} \mid A \subseteq \{m+1, \cdots, F-1\}\}.$$

Proof. In order to see this, it is enough to observe that, by Lemma 1, $\Delta(F,m) = \langle m \rangle \cup \{F+1, \rightarrow\}$ is the minimum (with respect to inclusion set) of $\mathscr{R}(F,m)$. $\square$

By applying the previous results, we can easily deduce the following proposition.

Proposition 8. *With the above notation, the following holds:*

1. *If* $F = m-1$, *then* $\{\mathrm{g}(S) \mid S \in \mathscr{R}(F,m)\} = \{m-1\}$.
2. *If* $m < F < 2m$, *then* $\{\mathrm{g}(S) \mid S \in \mathscr{R}(F,m)\} = \{m, m+1, \cdots, F-1\}$.
3. *If* $F > 2m$, *then* $\{\mathrm{g}(S) \mid S \in \mathscr{R}(F,m)\} = \left\{ \left\lceil \frac{F+1}{2} \right\rceil, \cdots, F - \left\lfloor \frac{F}{m} \right\rfloor \right\}$.

We have now all the ingredients needed to give an algorithmic procedure to compute $\mathscr{R}(F,m,g)$, where $F > 2m$, $m \nmid F$ and $\left\lceil \frac{F+1}{2} \right\rceil \leq g \leq F - \left\lfloor \frac{F}{m} \right\rfloor$.

That is, by using Lemma 11 and Proposition 8, we compute the genus of each vertex of the tree until we achieve the given genus.

Algorithm 2 Computation of $\mathscr{R}(F,m,g)$

INPUT: Positive integers F, m and g, such that $F > 2m$, $m \nmid F$ and $\left\lceil \frac{F+1}{2} \right\rceil \leq g \leq F - \left\lfloor \frac{F}{m} \right\rfloor$.
OUTPUT: $\mathscr{R}(F,m,g)$.

(1) $H = \{\Delta(F,m)\}$, $i = F - \left\lfloor \frac{F}{m} \right\rfloor$.
(2) If $i = g$, return H.
(3) For all $S \in H$ compute $\theta(S) = \{x \in \mathrm{SG}(S) \mid m < x < \mathrm{r}(S) \text{ and } x \neq F\}$.
(5) $H := \bigcup_{S \in H} \{S \cup \{x\} \mid x \in \theta(S)\}$, $i := i-1$ and go to Step(2).

We are going to see how the previous algorithm works in the following example.

Example 5. *By using Algorithm 2, we will compute the set* $\mathscr{R}(12,5,8)$.

- $H = \{\Delta(12,5)\}$, $i = 10$.
- $\theta(\Delta(12,5)) = \{8,9,11\}$.
- $H = \{\Delta(12,5) \cup \{8\}, \Delta(12,5) \cup \{9\}, \Delta(12,5) \cup \{11\}\}$, $i = 9$.
- $\theta(\Delta(12,5) \cup \{8\}) = \emptyset$, $\theta(\Delta(12,5) \cup \{9\}) = \{8\}$ *and* $\theta(\Delta(12,5) \cup \{11\}) = \{8,9\}$.
- $H = \{\Delta(12,5) \cup \{8,9\}, \Delta(12,5) \cup \{8,11\}, \Delta(12,5) \cup \{9,11\}\}$, $i = 8$.
- *The algorithm returns*

$$\mathscr{R}(12,5,8) = \{\Delta(12,5) \cup \{8,9\}, \Delta(12,5) \cup \{8,11\}, \Delta(12,5) \cup \{9,11\}\}.$$

We finish this section, noting that in [9], an equivalence relation $\sim$ is defined on $\mathscr{R}(F,m)$ such that the set

$$\frac{\mathscr{R}(F,m)}{\sim} = \{[S] \mid S \in \mathrm{Max}(\mathscr{R}(F,m))\}$$

is the quotient set of $\mathscr{R}(F,m)$ by $\sim$.

As $\frac{\mathscr{R}(F,m)}{\sim}$ is a partition of $\mathscr{R}(F,m)$, to compute all the elements of $\mathscr{R}(F,m)$, it is enough to have:

1. An algorithm which computes $\mathrm{Max}(\mathscr{R}(F,m))$.
2. Another algorithm which computes $[S]$ when S belongs to $\mathrm{Max}(\mathscr{R}(F,m))$.

This idea is applied in [9] to obtain an algorithm that calculates $\mathscr{R}(F,m)$.

5. $\mathscr{R}$-System of Generators

Throughout this section, $\mathscr{R}$ will denote a ratio-covariety. Recall that if X is an $\mathscr{R}$-set, then we will denote by $\mathscr{R}[X]$ the intersection of all elements of $\mathscr{R}$ containing X. By Proposition 1, we know that $\mathscr{R}$ is a finite set, and so the intersection of elements of $\mathscr{R}$ containing X is an element of $\mathscr{R}$. Therefore, $\mathscr{R}[X]$ is the smallest element of $\mathscr{R}$, with respect to set inclusion, containing X.

Recall that if X is an $\mathscr{R}$-set and $S = \mathscr{R}[X]$, we will call that X is an *$\mathscr{R}$-system of generators* of S. Moreover, if $S \neq \mathscr{R}[Y]$ for all $Y \subsetneq X$, then X will be called a *minimal $\mathscr{R}$-system of generators* of S. Observe that $X = \emptyset$ is the unique minimal $\mathscr{R}$-system of generators for $\min(\mathscr{R})$.

In general, the minimal $\mathscr{R}$-system of generators is not unique, as we can show in the following example.

Example 6. *Let* $\mathscr{R} = \{S \mid S$ *is a numerical semigroup,* $\mathrm{m}(S) = 8$, $\mathrm{r}(S) \geq 10$ *and* $\mathrm{F}(S) \leq 15\} \cup \{S_1 = \{0, 8, 9, 10, 11, 15, \rightarrow\}, S_2 = \{0, 8, 9, 13, 15, \rightarrow\}, S_3 = \{0, 8, 9, 15, \rightarrow\}\}$.

It is easy to see that $\mathscr{R}$ *is a ratio-covariety with* $\min(\mathscr{R}) = \{0, 8, 16, \rightarrow\}$, $\mathscr{R}[\{9, 10\}] = \mathscr{R}[\{9, 11\}] = S_1$, $\mathscr{R}[\{9\}] = S_3$, $\mathscr{R}[\{10\}] = \{0, 8, 10, 15, \rightarrow\}$ *and* $\mathscr{R}[\{11\}] = \{0, 8, 11, 15, \rightarrow\}$. *Therefore,* $\{9, 10\}$ *and* $\{9, 11\}$ *are minimal $\mathscr{R}$-system of generators of* S_1.

Our next aim in this section will be to prove that every element of $\mathscr{R}(F, m)$ admits a unique minimal $\mathscr{R}(F, m)$-system of generators.

Lemma 13. *If* $S \in \mathscr{R}$, *then* $X = \{x \in \mathrm{msg}(S) \mid x \notin \min(\mathscr{R})\}$ *is an $\mathscr{R}$-set and* $\mathscr{R}[X] = S$.

Proof. It is clear that X is an $\mathscr{R}$-set. As $S \in \mathscr{R}$ and $X \subseteq S$, then $\mathscr{R}[X] \subseteq S$. We are going to show the reverse inclusion. Let $T \in \mathscr{R}$ such that $X \subseteq T$. Then, $X \cup \min(\mathscr{R}) \subseteq T$ and so $\mathrm{msg}(S) \subseteq T$. Hence, $S \subseteq T$ and, consequently, $S \subseteq \mathscr{R}[X]$. □

Proposition 9. *If* $S \in \mathscr{R}(F, m)$, *then* $X = \{x \in \mathrm{msg}(S) \mid x \notin \Delta(F, m)\}$ *is the unique minimal $\mathscr{R}(F, m)$-system of generators of S.*

Proof. By Lemma 13, we know that X is an $\mathscr{R}(F, m)$-set and $\mathscr{R}(F, m)[X] = S$. To conclude the proof, it remains to show that if Y is an $\mathscr{R}(F, m)$-set and $\mathscr{R}(F, m)[Y] = S$, then $X \subseteq Y$. In fact, if $X \nsubseteq Y$, then there is $x \in X \backslash Y$. Note that $x \in \mathrm{msg}(S)$ and $x \notin \Delta(F, m)$, then $m < x < F$. By applying Lemma 2, we deduce that $S \backslash \{x\} \in \mathscr{R}(F, m)$ and $Y \subseteq S \backslash \{x\}$. Therefore, $S = \mathscr{R}(F, m)[Y] \subseteq S \backslash \{x\}$, a contradiction. □

If $\mathscr{R}$ is a ratio-covariety and $S \in \mathscr{R}$, then we define the *$\mathscr{R}$-rank* of S as

$$\mathscr{R}\mathrm{rank}(S) = \min\{\#X \mid X \text{ is an } \mathscr{R}\text{-set and } \mathscr{R}[X] = S\}.$$

As an immediate consequence of Lemma 13, in the following result, we show the relation between the $\mathscr{R}$-rank and the embedding dimension of a numerical semigroup.

Proposition 10. *If* $\mathscr{R}$ *is a ratio-covariety and* $S \in \mathscr{R}$, *then* $\mathscr{R}\mathrm{rank}(S) \leq \mathrm{e}(S)$.

The following result has an immediate proof.

Lemma 14. *Let* $\mathscr{R}$ *be a ratio-covariety and* $S \in \mathscr{R}$. *Then,* $\mathscr{R}\mathrm{rank}(S) = 0$ *if and only if* $S = \min(\mathscr{R})$.

Lemma 15. *If* $\mathscr{R}$ *is a ratio-covariety,* $S \in \mathscr{R}$, $S \neq \min(\mathscr{R})$ *and* X *is an $\mathscr{R}$-set such that* $S = \mathscr{R}[X]$, *then* $\mathrm{r}(S) \in X$.

Proof. If $\mathrm{r}(S) \notin X$, then $X \subseteq S \backslash \{\mathrm{r}(S)\}$. As $S \backslash \{\mathrm{r}(S)\} \in \mathscr{R}$, then $S = \mathscr{R}[X] \subseteq S \backslash \{\mathrm{r}(S)\}$, which is absurd. $\square$

As a consequence of preceding two results, in the following result we characterize the numerical semigroups $S \in \mathscr{R}$ such that $\mathscr{R}\mathrm{rank}(S) = 1$.

Proposition 11. *Let $\mathscr{R}$ be a ratio-covariety and $S \in \mathscr{R}$. Then, $\mathscr{R}\mathrm{rank}(S) = 1$ if and only if $S = \mathscr{R}[\{\mathrm{r}(S)\}]$.*

As a consequence of Propositions 9 and 11, we have the following result.

Corollary 2. *Let $m < r < F$ be positive integers, such that $m \nmid r$ and $F \notin \langle m, r \rangle$. Then, $\langle m, r \rangle \cup \{F+1, \rightarrow\}$ belongs to $\mathscr{R}(F, m)$ with $\mathscr{R}(F, m)$-rank equal to 1. Moreover, every element of $\mathscr{R}(F, m)$ with $\mathscr{R}(F, m)$-rank equal to 1, has this form.*

6. The Elements of $\mathscr{R}(F, m)$ with Maximum $\mathscr{R}(F, m)$-Rank

Our first goal in this section will be to show that the maximum of the set $\{\mathscr{R}(F, m)\mathrm{rank}(S) \mid S \in \mathscr{R}(F, m)\}$ does not exceed $m - 2$. To this end, we need to introduce some results.

In ([1], Corollary 3.2) appears the following result.

Lemma 16. *If S is an* MED*-semigroup and $S \neq \mathbb{N}$, then $\mathrm{F}(S) = \max(\mathrm{msg}(S)) - \mathrm{m}(S)$.*

Proposition 12. *If $S \in \mathscr{R}(F, m)$, then $\mathscr{R}(F, m)\mathrm{rank}(S) \leq m - 2$.*

Proof. By Proposition 9, we know that $\mathscr{R}(F, m)\mathrm{rank}(S)$ is the cardinality of the set $X = \{x \in \mathrm{msg}(S) \mid x \notin \Delta(F, m)\}$. As $m \in \mathrm{msg}(S)$ and $m \in \Delta(F, m)$, then $\mathscr{R}(F, m)\mathrm{rank}(S) \leq \mathrm{e}(S) - 1$. Now, by applying that $\mathrm{e}(S) \leq \mathrm{m}(S)$, we have that $\mathscr{R}(F, m)\mathrm{rank}(S) \leq \mathrm{m}(S) - 1$. To finish the proof, we will show that the case $\mathscr{R}(F, m)\mathrm{rank}(S) = \mathrm{m}(S) - 1$ is impossible. Indeed, if $\mathscr{R}(F, m)\mathrm{rank}(S) = \mathrm{m}(S) - 1$, then we deduce that $\mathrm{e}(S) = m$, and so S is an MED-semigroup. But then, Lemma 16 implies $\max(\mathrm{msg}(S)) \in \Delta(F, m)$. Since $\mathrm{e}(S) \leq \mathrm{m}(S)$, this implies $\mathscr{R}(F, m)\mathrm{rank}(S) \leq \mathrm{e}(S) - 2 \leq m - 2$. $\square$

From Proposition 9, the following result is easily deduced.

Proposition 13. *If $\{m = a_1 < a_2 < \cdots < a_k < a_{k+1} = F\} \subseteq \mathbb{N}$ and $a_{i+1} \notin \langle a_1, \cdots, a_i \rangle$ for all $i \in \{1, \cdots, k\}$, then $\langle a_1, \cdots, a_k \rangle \cup \{F+1, \rightarrow\}$ is an element of $\mathscr{R}(F, m)$ with $\mathscr{R}(F, m)$-rank equal to $k - 1$. Moreover, every element of $\mathscr{R}(F, m)$ with $\mathscr{R}(F, m)$-rank equal to $k - 1$ has this form.*

Corollary 3. *If $S \in \mathscr{R}(F, m)$ and $\mathscr{R}(F, m)\mathrm{rank}(S) = m - 2$, then $F \geq 2m - 1$. Moreover, if $F = 2m - 1$, then $S = \{0, m, m+1, \cdots, 2m-2, 2m, \rightarrow\}$.*

Proof. By Proposition 13, it is straightforward to see that $F \geq 2m - 1$. By now applying Lemma 12 and Proposition 13, we have that $S = \{0, m, m+1, \cdots, 2m-2, 2m, \rightarrow\}$. $\square$

Proposition 14. *If $F > 2m$, then the set $\{S \in \mathscr{R}(F, m) \mid \mathscr{R}(F, m)\mathrm{rank}(S) = m - 2\} \neq \emptyset$.*

Proof. Let $S = \langle m, F-(m-1), F-(m-2), \cdots, F-1 \rangle \cup \{F+1, \rightarrow\}$. As $F > 2m$, then $F - (m-1) > m$. As $m \nmid F$, then there is a unique $i \in \{1, \cdots, m-1\}$, such that $m \mid F - i$. Then, we easily deduce that $\{x \in \mathrm{msg}(S) \mid x \notin \Delta(F, m)\} = \{F-(m-1), \cdots, F-1\} \backslash \{F-i\}$. By now applying Proposition 9, we have that $\mathscr{R}(F, m)\mathrm{rank}(S) = m - 2$. $\square$

The above proposition allows us to define a new kind of semigroup.

Definition 10. *A numerical semigroup S is said to have maximal rank (hereinafter* MR-semigroup*) if $\mathrm{F}(S) > 2\mathrm{m}(S)$ and $\mathscr{R}(\mathrm{F}(S), \mathrm{m}(S))\mathrm{rank}(S) = \mathrm{m}(S) - 2$.*

The existence of MR-semigroups is assured by Proposition 13. Now, we present a characterization of these semigroups, which is also a direct consequence of Proposition 13.

Proposition 15. *Let $S \in \mathscr{R}(F,m)$ with $F > 2m$. Then, S is an* MR-semigroup *if and only if one of the following statements is true:*

- $\mathrm{e}(S) = m-1$ *and* $\mathrm{msg}(S) \subseteq \{1,\cdots,F-1\}$.
- $\mathrm{e}(S) = m$ *and* $\#\{x \in \mathrm{msg}(S) \mid x > F\} = 1$.

Next, we illustrate this characterization with an example.

Example 7.

1. *Let $S = \langle 5,7,9,11\rangle$. Then, $S \in \mathscr{R}(13,5)$, $\mathrm{e}(S) = 4$ and $\mathrm{msg}(S) \subseteq \{1,\cdots,13\}$, so S is an* MR-*semigroup by Proposition 15.*
2. *If $S = \langle 5,12,13,14,21\rangle$, then $S \in \mathscr{R}(16,5)$, $\mathrm{e}(S) = 5$ and $\{x \in \mathrm{msg}(S) \mid x > 16\} = \{21\}$. Consequently, by Proposition 15, we can assert that S is an* MR-*semigroup.*

7. The Ratio-Covariety Generated by a Finite Family of Numerical Semigroups

In general, the intersection of ratio-covarieties is not a ratio-covariety. Actually, if S and T are numerical semigroups, such that $S \neq T$, then $\mathscr{R}_1 = \{S\}$ and $\mathscr{R}_2 = \{T\}$ are ratio-covarieties. However, $\mathscr{R}_1 \cap \mathscr{R}_2 = \emptyset$, which is not a ratio-covariety.

The following result has an immediate proof.

Lemma 17. *If $\{\mathscr{R}_i\}_{i\in I}$ is a family of ratio-covarieties and Δ is the minimum of $\mathscr{R}_i$ for every $i \in I$, then $\cap_{i\in I}\mathscr{R}_i$ is a ratio-covariety.*

Let m and F be positive integers such that $m < F$ and $m \nmid F$, then define by $\mathscr{A}(F,m) = \{S \mid S \text{ is a numerical semigroup, } \mathrm{F}(S) \le F \text{ and } \mathrm{m}(S) = m\}$.

Remark 1. *The set $\mathscr{A}(F,m)$ is obviously a ratio-covariety with minimum $\Delta(F,m)$.*

Remark 2. *Let $S_1,\cdots,S_k$ be numerical semigroups with multiplicity m, and let $F = \max\{\mathrm{F}(S_1),\cdots,\mathrm{F}(S_k)\}$. Then, $\{S_1,\cdots,S_k\} \subseteq \mathscr{A}(F,m)$.*

Let $\{S_1,\cdots,S_k\}$ be a finite set of numerical semigroups with multiplicity m and $F = \max\{\mathrm{F}(S_1),\cdots,\mathrm{F}(S_k)\}$. Then, we denote by $\langle S_1,\cdots,S_k\rangle$ the intersection of all the ratio-covarieties which contain the set $\{S_1,\cdots,S_k\}$, and have the set $\Delta(F,m)$ as minimum.

Note that $\mathscr{A}(F,m)$ is a ratio-covariety with the above features. Next, by applying Lemma 17, we obtain the following result.

Proposition 16. *If $S_1,\cdots,S_k$ are numerical semigroups with multiplicity m and $F = \max\{\mathrm{F}(S_1),\cdots,\mathrm{F}(S_k)\}$, then $\langle S_1,\cdots,S_k\rangle$ is the smallest (with respect to set inclusion) ratio-covariety containing the set $\{S_1,\cdots,S_k\}$ and having the set $\Delta(F,m)$ as minimum.*

We will call $\langle S_1,\cdots,S_k\rangle$ the *ratio-covariety generated* by $\{S_1,\cdots,S_k\}$. Our next aim will be to present an algorithm which allows us to compute all the elements of $\langle S_1,\cdots,S_k\rangle$.

For every $i \in \{1,\cdots,k\}$, define the following sequence recursively:

- $S_i^0 = S_i$,
- $S_i^{n+1} = \begin{cases} S_i^n \backslash \{\mathrm{r}(S_i^n)\} & \text{if } S_i^n \neq \Delta(F,m), \\ \Delta(F,m) & \text{otherwise.} \end{cases}$

The following result has an immediate proof.

Lemma 18. *For every $i \in \{1,\cdots,k\}$ there exists $P_i = \min\{n \in \mathbb{N} \mid S_i^n = \Delta(F,m)\}$.*

For all $i \in \{1, \cdots, k\}$, set $\Omega(S_i) = \{S_i^0, \cdots, S_i^{P_i}\}$. Observe that $\Delta(F, m) = S_i^{P_i} \subsetneq S_i^{P_i - 1} \subsetneq \cdots \subsetneq S_i^0 = S_i$ and $\#(S_i^n \backslash S_i^{n+1}) = 1$ for each $n \in \{0, \cdots, P_i - 1\}$.

In the next proposition, we give the previously announced algorithmic procedure.

Proposition 17. *Let $S_1, \cdots, S_k$ be numerical semigroups with multiplicity m and $F = \max\{\mathrm{F}(S_1), \cdots, \mathrm{F}(S_k)\}$. Then, $\langle S_1, \cdots, S_k \rangle = \{\bigcap_{b \in B} T_b \mid \emptyset \neq B \subseteq \{1, \cdots, k\}$ and $T_b \in \Omega(S_b)$ for all $b \in B\}$.*

Proof. To prove the proposition, it will be enough to see that

$$\mathscr{R} = \{\bigcap_{b \in B} T_b \mid \emptyset \neq B \subseteq \{1, \cdots, k\} \text{ and } T_b \in \Omega(S_b) \text{ for all } b \in B\}$$

is a ratio-covariety with minimum $\Delta(F, m)$. We also have to show $S_i \in \mathscr{R}$ for all $i \in \{1, \cdots, k\}$ (which is easy, of course, just choose $B = \{i\}$.)

It is clear that $\min(\mathscr{R}) = \Delta(F, m)$. Also, it is easy to demonstrate that the intersection of two elements belonging to $\Omega(S_i)$ is also an element of $\Omega(S_i)$. Therefore, the intersection of two elements of $\mathscr{R}$ is again an element of $\mathscr{R}$. We prove next that if $T \in \mathscr{R}$ and $T \neq \Delta(F, m)$, then $T \backslash \{\mathrm{r}(T)\} \in \mathscr{R}$. In fact, if $T \in \mathscr{R}$, then there is $\emptyset \neq B \subseteq \{1, \cdots, k\}$ and there exists $T_b \in \Omega(S_b)$ for all $b \in B$ such that $T = \bigcap_{b \in B} T_b$. As $\mathrm{r}(T) \in T$, then $\mathrm{r}(T) \in T_b$ for all $b \in B$. For every $b \in B$, denote by $T'_b = \Delta(F, m) \cup \{x \in T_b \mid x > \mathrm{r}(T)\}$. As $T_b \in \Omega(S_b)$ and T'_b is obtained by removing recurrently $\mathrm{r}(T)$ from T_b, then $T'_b \in \Omega(S_b)$ for all $b \in B$ and $T \backslash \{\mathrm{r}(T)\} = \bigcap_{b \in B} T'_b$. Therefore, $T \backslash \{\mathrm{r}(T)\} \in \mathscr{R}$. $\square$

This result immediately implies:

Corollary 4. *If S is a numerical semigroup, then $\langle S \rangle = \text{Rat-Cad}(S)$.*

We end up illustrating the content of Proposition 17 with an example.

Example 8. *Let $S_1 = \langle 5, 7, 9 \rangle = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ and $S_2 = \langle 5, 6, 8 \rangle = \{0, 5, 6, 8, 10, \rightarrow\}$. Then, $13 = \max\{\mathrm{F}(S_1), \mathrm{F}(S_2)\}$ and so $\Omega(S_1) = \{S_1, S_1 \backslash \{7\}, S_1 \backslash \{7, 9\}, S_1 \backslash \{7, 9, 12\}\}$ and $\Omega(S_2) = \{S_2, S_2 \backslash \{6\}, S_2 \backslash \{6, 8\}, S_2 \backslash \{6, 8, 11\}, S_2 \backslash \{6, 8, 11, 12\}, S_2 \backslash \{6, 8, 11, 12, 13\}\}$. By applying Proposition 17, we have that*

$$\langle S_1, S_2 \rangle = \Omega(S_1) \cup \Omega(S_2) \cup \{T_1 \cap T_2 \mid T_1 \in \Omega(S_1) \text{ and } T_2 \in \Omega(S_2)\}.$$

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