Dynamics of heavy quarks in the Fock space

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This paper concerns a method of describing hadrons that starts with the canonical front form Hamiltonian of QCD. The method is developed in the relatively simple context of QCD with only heavy quarks. We regulate its canonical Hamiltonian by introducing a vanishingly small gluon mass m_g. For positive m_g, the small-x gluon divergences become ultraviolet and hence they are renormalized in the same way the ultraviolet transverse divergences are. This is done using the renormalization group procedure for effective particles. Up to the second order of expansion of the renormalized Hamiltonian in powers of the quark-gluon coupling constant g, only the quark mass-squared and gluon-exchange divergences require counterterms. In these circumstances, we calculate an effective potential between quarks in heavy quarkonia in an elementary way, replacing all the quarkonium-state components with gluons of mass m_g by only one component with just one gluon that is assigned a mass m_G, comparable to or exceeding the scale of typical relative momenta of bound quarks. In the limit of $m_g \to 0$ and large m_G two results are obtained. (1) While the color-singlet quarkonium mass eigenvalue stays finite and physically reasonable in that limit, the eigenvalues for single quarks and octet quarkonia are infinite. (2) Besides the coulomb terms, the effective quark-antiquark potential is quadratic as a function of the distance and spherically symmetric for typical separations between quarks but becomes logarithmic and no longer spherically symmetric for large separations. Our conclusion indicates how to systematically improve upon the approximations made in this paper.

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I. INTRODUCTION

Description of heavy-quark bound states in terms of their virtual Fock-space components is meant to be achievable through solving the Hamiltonian eigenvalue problem in QCD, which in the first approximation is limited to only involve quarks b and c. However, the canonical Hamiltonian of even so severely limited theory poses conceptual and computational problems. To begin with, the Hamiltonian needs regularization. The formal momentum cutoffs one imposes on the virtual Fock states of heavy quarks are much greater than the quark masses. Therefore, from the regularization point of view, the heavy quarks do not differ much from the light ones—their masses are negligible in comparison with the cutoffs. Further, the canonical gluon mass is zero, which is infinitely small in comparison to any nonzero quark mass. The key distinction between the heavy and light quarks is provided by the ratio of their masses to the parameter Λ_{QCD}. The latter results from dimensional transmutation [1–3]. However, such a parameter cannot be introduced in a precise way without a renormalization group procedure for Hamiltonians. One faces the difficulty that Hamiltonians in quantum field theory result from integrating Hamiltonian densities over a three-dimensional space.

The three-dimensional Hamiltonian density is a different concept from the four-dimensional Lagrangian density and the associated concepts of action, path integration, and diagrammatic techniques of perturbation theory.

In this paper, we address the issue of dynamics of heavy quarks using the renormalization technique called the renormalization group procedure for effective particles (RGPEP). The RGPEP is designed for Hamiltonians and it applies to the front form (FF) of dynamics [4,5]. One is motivated to use the FF instead of the more familiar instant form (IF) because of the desire to describe the quarkonia observed in motion as well as the quarkonia observed at rest and to include other moving particles with which the heavy quarks interact. The key feature of the FF of Hamiltonian dynamics is that the required boosts are kinematic; their generators do not involve interactions. This is not the case in the IF, where motion is associated with a dynamical change in the virtual Fock-space decomposition.

The RGPEP has been used for the purpose of describing heavy quarks before [6–8], including the effective potentials derived using the RGPEP that were used in other approaches for description of heavy tetraquarks [9]. The new element utilized in this article is the small gluon mass parameter m_g that regulates the singularities caused by gluons carrying small longitudinal momenta [10,11] or, in the parton-model language, those that carry a small x. The arbitrarily small parameter m_g, in combination with the running scale parameter, denoted by t, provides a lower bound on x. This bound is absent in the free part of the Hamiltonian. It only appears in the interaction terms. The reason is that the RGPEP does not integrate out large FF energies. Instead it integrates out large *changes* of the FF energies due to interactions. The free part of the Hamiltonian does not change the FF energy. In other words, instead of Wilsonian integrating out of large energies and hence limiting the range of momenta of field quanta in the Fock-space basis, we only integrate out the interactions that cause large changes of the FF energy. In this respect, our Hamiltonian approach differs from the extended literature on heavy-quark bound states, including the outstanding reviews [12,13]. Regarding boost invariance in Minkowski approaches, we wish to mention [14–18] and references therein, from which our Hamiltonian approach also differs in this respect.

As a consequence of the regularization used in this article, one circumvents the vacuum problem in the theory. Instead of involving the ground state in the dynamics and arguing that it somehow expels the colored states out from the spectrum, we find that the renormalized Hamiltonian is capable of producing infinite eigenvalues for colored states in the limit $m_g \to 0$. This result only follows under the assumption that an exchange of an effective gluon between effective quarks is blocked by the non-Abelian interactions when the RGPEP running scale parameter t is increased to the value that characterizes the formation of the quark bound states.

The simplest model of the blocking of effective gluons from being exchanged between effective quarks is defined by giving the gluons a large mass $m_G \gg \Lambda_{\rm QCD} \gg m_g$, where $\Lambda_{\rm QCD}$ is defined in the RGPEP scheme. One does not need to specify how m_G depends on t. It suffices to assume that it is larger than the momentum transfers between quarks involved in formation of bound states. That way the Fock-space dynamics, which *a priori* involves unlimited numbers of effective gluons, is drastically simplified because emission and absorption of heavy gluons is suppressed. In our calculation, we limit the number of heavy effective gluons involved in the dynamics to one. Since our FF Hamiltonian approach is only developed in gauge $A^+ = 0$, we ought to mention that an effective gluon mass has also been found useful in calculations using the Dyson-Schwinger equations in Landau gauge, see Refs. [19–22].

The results we report follow from the renormalized Hamiltonian that is computed in the limit $m_g \to 0$ using expansion in a series of powers of the coupling constant only up to second order. Although nowadays it may be not surprising that such low-order computations can point toward some mechanism of quark binding [23–25], the mechanism our calculation points to is surprisingly simple: the quark self-interaction tends to infinity as $|\log(tm_g^2)|$, but this logarithmic growth is canceled in colorless quarkonia by the effective interaction term computed using the RGPEP.

Before the limit $m_g \to 0$ is taken, a finite value of m_g converts the small-x divergences into the large FF energy divergences. It happens because the gluon minus momentum is

$$p_g^- = \frac{m_g^2 + p_g^{\perp 2}}{p_g^+}. \tag{1}$$

It becomes infinite when $p_g^+ \to 0$ no matter how small $p_g^\perp$ is. If m_g is zero, $p_g^\perp \sim \sqrt{x}$ or smaller would lead to finite or even vanishing p_g^-. However, for $m_g > 0$, the divergences due to small $x = p_g^+/P^+$, where P^+ is a momentum of a system under consideration, can be treated on an equal footing with the transverse UV divergences associated with $p_g^\perp \to \infty$, using the RGPEP.

It should be pointed out that the finite and phenomenologically reasonable eigenvalues we obtain for white quarkonia depend on the ratio of the quarkonium longitudinal momentum as a whole, P^+, to the running renormalization group scale-parameter t. This is so due to the approximations we are forced to make at this stage of developing the theory. Namely: instead of solving the RGPEP equation for scale-dependent, effective Hamiltonians H_t exactly, we use expansion in powers of g only up to g^2; we introduce the

hypothetical mass m_G for effective gluons, while we do not know yet how the effective gluon mass actually evolves with t; we limit the eigenvalue problem for H_t to a subspace spanned by two effective components, one with a quark and an antiquark and another one with a quark, an antiquark and a gluon; we approximate the dynamical contribution of the three-particle component to the eigenvalue equation for the two-particle component keeping only terms on the order of g^2; we adjust the value of t to bring the resulting mass eigenvalues close to data, because our calculations are not exact so that the resulting observables depend on t, and we extrapolate the coupling constant g to the value g_t that is in the ballpark of expectations based on the perturbative evolution of g_t, but we cannot estimate the magnitude of error caused by such extrapolation in calculations limited to second order. The effective quark-antiquark potential we thus obtain is rotationally symmetric only for relatively small quark-antiquark distances. Its asymmetry at large distances indicates that our approximations are too crude for a precise description of the excited states.

The paper is organized in the following way. Section II introduces the canonical FF Hamiltonian of QCD to which we add the gluon mass term with a small parameter m_g. In Sec. III we apply the RGPEP to compute and then supplement with a large mass m_G the effective Hamiltonian for a sizable t, keeping terms on the order of 1, g and g^2. The quarkonium eigenvalue problem for the resulting Hamiltonian H_t is examined in Sec. IV. The effective potential we obtain in the nonrelativistic limit of the eigenvalue problem is described in Sec. V, including comments concerning rotational symmetry. Appendix A contains details of derivation of the effective Hamiltonian up to the second order, while Appendix B shows that the gluon exchange counterterm ensures cancellation of small-x divergences due to exchange of a gluon. We often abbreviate the quark quantum number subscripts, momentum, isospin or flavor, spin and color in just one subscript. For example, instead of p_1, i_1 or f_1, σ_1 and c_1, we write only 1.

II. QCD WITH GLUON MASS m_g

The canonical FF Hamiltonian density for QCD is obtained from its classical Lagrangian through the well-known quantization procedure in gauge $A^+ = 0$, *e.g.* see Ref. [5]. We limit the theory to quarks c and b and supply the Hamiltonian density with a gluon mass term,

$$\mathcal{H} = \mathcal{H}_{\rm QCD} + \frac{1}{2} m_g^2 A^{\perp a} A^{\perp a}. \tag{2}$$

The mass m_g can be considered extremely small, so that its presence in a regulated quantum theory is not noticeable at the level of classical gauge symmetry and could not be detected by experiment (current upper limit on the gluon mass is on the order of a few MeV$/c^2$ [26]). After integration over x^- and $x^\perp$ and normal-ordering, one obtains the free Hamiltonian term for gluons in the form

$$H_{A^2} = \int_3 p_3^- a_3^\dagger a_3, \tag{3}$$

where a_3 is the gluon annihilation operator labeled by quantum numbers σ_3, c_3, p_3^+, and $p_3^\perp$ for polarization, color, longitudinal momentum, and transverse momentum, respectively, collectively denoted by 3.

$$p_3^- = \frac{m_g^2 + p_3^{\perp 2}}{p_3^+}, \tag{4}$$

and

$$\int_3 = \sum_{\sigma_3, c_3} \int \frac{dp_3^+ d^2 p_3^\perp}{16\pi^3 p_3^+}. \tag{5}$$

We use label 3 for gluons because we choose label 1 for quarks and label 2 for antiquarks. For example, labels $1'$ and $\tilde{1}$ will refer to quarks. Also, $\int_1 \int_2 = \int_{12}$ and phrase "pair 12" refers to a quark 1 and antiquark 2. The invariant mass squared of a quark-gluon pair 13 is $\mathscr{M}_{13}^2 = (p_1^+ + p_3^+)(p_1^- + p_3^-) - (p_1^\perp + p_3^\perp)^2$ with the minus components being eigenvalues of the free part of the canonical Hamiltonian.

Nonzero m_g implies that whenever p_3^+ approaches zero, p_3^- approaches infinity. In contrast, if $m_g = 0$, one can simultaneously set $p_3^+ \to 0$, and $p_3^\perp \to 0$ in such a way that p_3^- stays constant or vanishes. Therefore, for $m_g > 0$ all gluon modes with vanishing longitudinal momenta are FF high-energy (large p_3^-) modes, while for $m_g = 0$ such modes can also be FF small-energy modes. Using $m_g > 0$ one can simultaneously regulate transverse and small-x singularities by a cutoff on the invariant mass. Renormalization due to both singularities is discussed in Sec. III. Once the Hamiltonian is renormalized so that the cutoff dependence of its matrix elements between states of effective particles with finite momenta is removed, the resulting theory depends on m_g. We discuss its limit when $m_g \to 0$ in Sec. IV on the example of quarkonium eigenvalue problem.

III. RENORMALIZED HAMILTONIAN

To obtain the renormalized Hamiltonian we use the renormalization group procedure for effective particles (RGPEP). The procedure involves regularization, calculation of an effective Hamiltonian and determination of counterterms, if necessary.

Gluon operators a_3 (and $a_3^\dagger$), introduced in Eq. (3), annihilate (and create) gluons that can be characterized as bare or pointlike. Effective gluons are created by $a_{t3}^\dagger$ and

annihilated by a_{t3}, where $t \geq 0$ is the RGPEP parameter related to the size of effective particles. The canonical, point-like gluons correspond to $t = 0$. The same relationships hold for quark operators. So, the effective quarks and antiquarks are created by $b_{t1}^\dagger$ and $d_{t2}^\dagger$, respectively. The effective operators are related to the bare operators by means of a unitary transformation $\mathcal{U}_t$,

$$b_{t1} = \mathcal{U}_t b_1 \mathcal{U}_t^\dagger, \quad d_{t2} = \mathcal{U}_t d_2 \mathcal{U}_t^\dagger, \quad a_{t3} = \mathcal{U}_t a_3 \mathcal{U}_t^\dagger. \tag{6}$$

Dimension of the parameter t is the front form energy to power minus two. In accordance with the dimensional analysis of Ref. [27], we introduce two scale parameters, longitudinal momentum scale $\mathcal{P}$, and transverse momentum scale λ,

$$t = \frac{\mathcal{P}^2}{\lambda^4}. \tag{7}$$

The effective Hamiltonians, denoted by H_t, are linear combinations of products of creation and annihilation operators for effective particles. The coefficients in front of those products are functions of t as well as all quantum numbers labeling each particle operator involved. The effective Hamiltonians describe the same theory, thus $H_t = H_0$. Additionally, we define

$$\mathcal{H}_t = \mathcal{U}_t^\dagger H_t \mathcal{U}_t. \tag{8}$$

Whereas H_t is a linear combination of products of effective operators, $\mathcal{H}_t$ is the same linear combination (with the same coefficients) of products of bare particle operators. We are able now to implicitly define $\mathcal{U}_t$ by demanding that $\mathcal{H}_t$ is the solution of the following differential equation,

$$\frac{d\mathcal{H}_t}{dt} = [[\mathcal{H}_f, \mathcal{H}_t], \mathcal{H}_t], \tag{9}$$

where $\mathcal{H}_f$ is the free part of $\mathcal{H}_t$, i.e., the part that is obtained by setting the coupling constant g to 0. The relation between $\mathcal{U}_t$ and $\mathcal{H}_t$ can be recovered from Eqs. (8) and (9) remembering that $dH_t/dt = 0$. Effective particles essentially define a t-dependent basis in the space of states. Equation (9) is simpler than the corresponding equation in Ref. [28] and it leads to simplified computations in the cases we consider. We adopt the simplification because it readily yields the attractive results that are described in this paper.

The operator $H_{t>0}$ that solves Eq. (9) order-by-order in powers of the coupling constant g is "narrow" in terms of creation and annihilation operators for effective particles [28] in the sense similar to the narrowness of solutions to the Wegner equation for Hamiltonian matrices [29,30]. It means that the matrix elements of H_t between effective particle states with vastly different FF energies are negligibly small. More precisely, if the difference of FF energies considerably exceeds $t^{-1/2} = \lambda^2/\mathcal{P}$, then the matrix element is exponentially suppressed and equivalent to zero in our calculations. Therefore, the larger t the narrower the Hamiltonian. Consequently, one can apply to H_t the principles of the similarity renormalization group procedure [31].

A. Formulas for the Hamiltonian

Below we present the relevant part of the renormalized Hamiltonian obtained as an approximate solution of Eq. (9) with the regularized canonical Hamiltonian of heavy-flavor QCD as the initial condition. The solution is obtained using the power expansion in the coupling constant g_t for some finite t, although up to the second order $g_t = g$, where g is the coupling constant of the canonical Hamiltonian. Nevertheless, we use the notation g_t to indicate that the coupling constant will evolve with t in higher order calculations than the ones described in this paper.

We do not present the formulas for the pure canonical Hamiltonian. They can be recovered by putting $t = t_r = 0$ and omitting the counterterms.

Interaction vertices are regularized. For a three-leg vertex in which particle a is annihilated and particles b and c are created, and for its hermitian conjugate, see Fig. 1, the regulating factor is set to

$$f_{bc.a,t_r} = \exp\left[-t_r(p_b^- + p_c^- - p_a^-)^2\right] \tag{10}$$

$$= \exp\left[-t_r\left(\frac{\mathscr{M}_{bc}^2 - m_a^2}{p_a^+}\right)^2\right], \tag{11}$$

where $\mathscr{M}_{bc}^2$ is the invariant mass of the bc pair, m_a is the mass of particle a, and t_r is a positive regulating parameter. In the renormalized Hamiltonian, after cancellations of divergences are guaranteed, one can set t_r to zero. The Hamiltonian includes also the instantaneous interaction terms that graphically have four legs. Consider a term that changes particles 1' and 2' to 1 and 2, see Fig. 2. It is interpreted as composed of two three-leg vertices labeled by 1, 1', 3 and 2, 2', 3, respectively, and joined by the

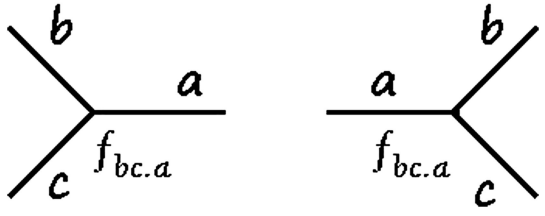

FIG. 1. First-order interaction vertices. The left vertex stands for annihilation of particle a and creation of particles b, c and the right vertex is for annihilation of particles b, c and creation of particle a. The regularization factors $f_{bc.a,t_r}$ in these vertices are the same.

common leg 3. The corresponding regularization factor is set to

$$r_{12.1'2'} = \theta(p_{1'}^+ - p_1^+) f_{13.1',t_r} f_{2'3.2,t_r} + \theta(p_1^+ - p_{1'}^+) f_{1'3.1,t_r} f_{23.2',t_r}. \quad (12)$$

The two terms correspond to two time orderings of the two three-leg vertices, and the ordering is uniquely determined by the sign of $p_{1'}^+ - p_1^+$. Using conservation of momentum components $+$ and $\perp$ in the vertices, this factor can be simplified to

$$r_{12.1'2'} = f_{1,t_r} f_{2,t_r}, \quad (13)$$

with

$$f_{i,t} = \exp\left[-t\left(\frac{m_g^2 - q_i^2}{q_i^+}\right)^2\right], \quad (14)$$

where

$$q_1^\mu = p_{1'}^\mu - p_1^\mu, \quad (15)$$

$$q_2^\mu = p_2^\mu - p_{2'}^\mu. \quad (16)$$

Due to momentum conservation, $q_1^+ = q_2^+ = q^+$, $q_1^\perp = q_2^\perp = q^\perp$.

Now, we write the effective Hamiltonian up to second order in g_t,

$$H_t = H_{tf} + g_t H_{t1} + g_t^2 H_{t2}. \quad (17)$$

The term H_{tf} is the free Hamiltonian,

$$H_{tf} = \int_1 p_1^- b_{t1}^\dagger b_{t1} + \int_2 p_2^- d_{t2}^\dagger d_{t2} + \int_3 p_3^- a_{t3}^\dagger a_{t3}. \quad (18)$$

The first-order interaction Hamiltonian is,

$$H_{t1} = \int_{131'} j_1^\mu t_{11'}^3 f_{1,t+t_r} b_{t1}^\dagger (\tilde{\delta}_{13.1'} \varepsilon_{3\mu}^* a_{t3}^\dagger + \tilde{\delta}_{1'3.1} \varepsilon_{3\mu} a_{t3}) b_{t1'} - \int_{232'} j_2^\mu t_{2'2}^3 f_{2,t+t_r} d_{t2}^\dagger (\tilde{\delta}_{23.2'} \varepsilon_{3\mu}^* a_{t3}^\dagger + \tilde{\delta}_{2'3.2} \varepsilon_{3\mu} a_{t3}) d_{t2'}, \quad (19)$$

with $j_1^\mu = \bar{u}_1 \gamma^\mu u_{1'}$, $j_2^\mu = \bar{v}_{2'} \gamma^\mu v_2 = \bar{u}_2 \gamma^\mu u_{2'}$, $t_{11'}^3 = \chi_{c_1}^\dagger T^{c_3} \chi_{c_{1'}}$, and $t_{2'2}^3 = \chi_{c_{2'}}^\dagger T^{c_3} \chi_{c_2}$, where T^{c_3} is half of the Gell-Mann matrix λ^{c_3}, $c_3 = 1, 2, \ldots, 8$, and $\chi_{c_i} = (\delta_{1,c_i}, \delta_{2,c_i}, \delta_{3,c_i})^T$ is the color vector of quark i, $c_i = 1, 2, 3$. This interaction is represented diagrammatically in Fig. 1. Note that $f_{i,t+t_r} = f_{i,t} f_{i,t_r}$, where f_{i,t_r} comes from regularization, while $f_{i,t}$ is a result of solving Eq. (9). The fact that they combine into $f_{i,t+t_r}$ is the motivation behind our choice of regularization, Eq. (11). Whenever FF energy changes in the interaction by more than $t^{-1/2}$ the form factor $f_{i,t}$ becomes very small, manifesting the narrowness of H_t.

The second-order interaction Hamiltonian contains the quark-antiquark interaction term and the quark and antiquark self-interaction terms,

$$H_{t2} = H_{Ut} + H_{\delta m}. \quad (20)$$

The quark-antiquark interaction term is

$$H_{Ut} = -\int_{121'2'} \tilde{\delta}_{12.1'2'} U_{t12.1'2'} t_{11'}^a t_{2'2}^a b_{t1}^\dagger d_{t2}^\dagger d_{t2'} b_{t1'}, \quad (21)$$

where the color superscript a is summed over and the interaction kernel $U_{t12.1'2'}$ comprises three terms,

$$U_{t12.1'2'} = U_C + U_H + U_X, \quad (22)$$

$$U_C = f_{1,t_r} f_{2,t_r} g_{\mu\nu} j_1^\mu j_2^\nu f_t \mathcal{F}, \quad (23)$$

$$U_H = -f_{1,t+t_r} f_{2,t+t_r} \left(\frac{q_1^2 + q_2^2}{2(q^+)^2} j_1^+ j_2^+ + g_{\mu\nu} j_1^\mu j_2^\nu\right) \mathcal{F}, \quad (24)$$

$$U_X = f_t f_{1,t_r} f_{2,t_r} \frac{j_1^+ j_2^+}{(q^+)^2} \left(1 + \frac{q_1^2 + q_2^2}{2} \mathcal{F}\right) - f_t X, \quad (25)$$

with

$$\mathcal{F} = \frac{1}{2}\left(\frac{1}{m_g^2 - q_1^2} + \frac{1}{m_g^2 - q_2^2}\right), \quad (26)$$

and

$$f_t = \exp\left[-t\left(\frac{\mathscr{M}_{12}^2 - \mathscr{M}_{1'2'}^2}{p_1^+ + p_2^+}\right)^2\right] \quad (27)$$

$$= \exp\left[-t\left(\frac{q_2^2 - q_1^2}{p_3^+}\right)^2\right]. \quad (28)$$

The kernel is illustrated in Fig. 2. Subscripts "12.1′2′," which indicate dependence of the kernel on quantum numbers of particles 1, 2, 1′, and 2′ are dropped for U_C, U_H, and U_X to simplify notation. Details of derivation of H_{Ut} are given in Appendix A 3. H_{Ut} does not include the terms in which the initial quark-antiquark pair annihilates into an octet of gluons and then is recreated from the gluons. The octet terms are not produced here because they yield zero acting on the color-singlet quark-antiquark states whose dynamics is the main focus of this article. The symbol X, introduced in Eq. (25), denotes the contribution of the gluon-exchange counterterm, discussed in Sec. III C. The self-interaction term $H_{\delta m}$ in Eq. (20) is discussed in Sec. III B.

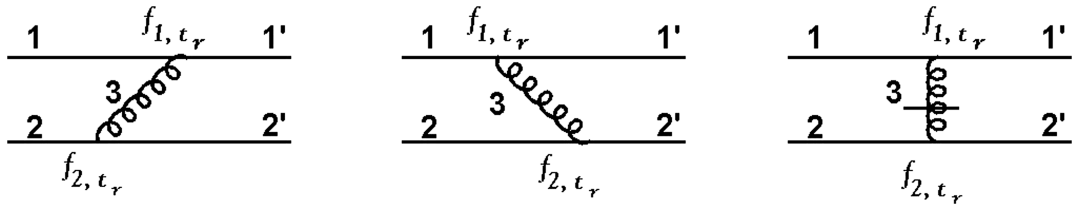

FIG. 2. Graphs left and middle illustrate the second-order gluon-exchange terms resulting from the product of two first-order interaction terms. Right graph illustrates the effective second-order instantaneous interaction that results from the unitary rotation of the instantaneous term in the canonical Hamiltonian, see Eq. (A20).

B. Self-interaction counterterm

Second-order quark self-interaction terms result from successive action of two 1st-order Hamiltonian interaction terms, in accordance with Eq. (A7) that is illustrated in Fig. 3,

$$H_{\delta m} = \int_1 \frac{\delta m_{1t}^2}{p_1^+} b_{t1}^\dagger b_{t1} + \int_2 \frac{\delta m_{2t}^2}{p_2^+} d_{t2}^\dagger d_{t2}. \tag{29}$$

Details of computing $H_{\delta m}$ are in Appendix A 2. The self-interaction shifts the free quark mass squared, m_i^2 in H_t, by $g_t^2 \delta m_{it}^2$, where

$$\delta m_{it}^2 = \delta m_{iX}^2 + I_i(t + t_r, m_g) - I_i(t_r, m_g), \tag{30}$$

with

$$I_i(t, m_g) = C_F \sum_{\sigma_{\tilde{i}}, \sigma_3} \int \frac{d^2 k_{3\tilde{i}} dx_{3\tilde{i}}}{16\pi^3 x_{3\tilde{i}} x_{\tilde{i}3}} \frac{f_{\tilde{i}3,i,t}^2}{\mathscr{M}_{\tilde{i}3}^2 - m_i^2} \bar{u}_i \not{\varepsilon}_3 u_{\tilde{i}} \bar{u}_{\tilde{i}} \not{\varepsilon}_3^* u_i, \tag{31}$$

and $i = 1, 2$, and $\tilde{i} = \tilde{1}, \tilde{2}$, respectively, see Fig. 3. The integrals $I_i(t + t_r, m_g)$ are finite for any finite $t > 0$, but $I_i(t_r, m_g)$ depends on t_r in a divergent way,

$$I_i(t_r, m_g) = \frac{C_F}{16\pi^2} \left\{ p_i^+ \sqrt{\frac{\pi}{2t_r}} \left[\log\left(\frac{p_i^{+2}}{8 m_g^4 t_r} \right) - \frac{7}{2} - \gamma \right] + \frac{1}{2} m_g^2 \log^2\left(\frac{p_i^{+2}}{2 m_i^4 t_r} \right) \right.$$
$$\left. - \left(3m_i^2 + 4m_g^2 \log\frac{m_g}{m_i} + 3 m_i m_g - \frac{3}{2} m_g^2 + \gamma m_g^2 \right) \log\left(\frac{p_i^{+2}}{2 m_i^4 t_r} \right) - 3m_i^2 + 3\gamma m_i^2 \right\} + o(1), \tag{32}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. The symbol $o(1)$ denotes the terms that tend to 0 when $t_r \to 0$ and then $m_g \to 0$. One needs the counterterm δm_{iX}^2 to remove the divergent part of $I_i(t_r, m_g)$. The finite part of the counterterm is discussed below.

We define the mass counterterm to have the form which removes "1" in "$1 - f^2$" in the Hamiltonian with finite t, see Eq. (A10), by which we mean that the counterterm is,

$$H_{t_r}^{\text{mass CT}} = g^2 \int_1 \frac{\delta m_{1X}^2}{p_1^+} b_1^\dagger b_1 + g^2 \int_2 \frac{\delta m_{2X}^2}{p_2^+} d_2^\dagger d_2, \tag{33}$$

where

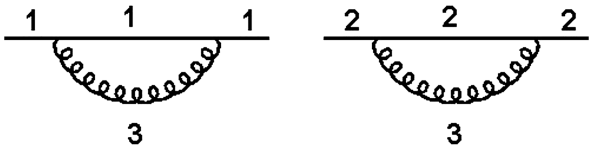

FIG. 3. Second-order quark and antiquark self-interaction terms resulting from the product of two first-order interaction terms.

$$\delta m_{iX}^2 = I_i(t_r, m_g). \tag{34}$$

On the one hand, this definition is motivated by the results it leads to in our computations of masses of heavy quarkonia [6–8]. Namely, the singlet quarkonium eigenvalue problem takes a simple and phenomenologically reasonable form. At the same time the single quark mass eigenvalue tends to infinity when $m_g \to 0$, see below. On the other hand, this counterterm removes the ultraviolet divergence from the quark self-interaction in the way that is analogous to how the electron self-interaction counterterm is defined in the FF Hamiltonian of QED. Our definition of the counterterm is also compatible with the coupling coherence, which in this case implies $\lim_{t\to\infty} \delta m_{it}^2 = 0$, see page 66 in Ref. [23]. At the current, crude-approximation stage of the theory development, the authors find the above reasons sufficient for adopting this choice of the quark self-interaction counterterm including its finite part.

C. Gluon exchange counterterm

The quark-antiquark interaction term H_{Ut} does not contain any loops. However, it leads to the divergent

regularization dependence due to the factor $1/(q^+)^2$ in U_H and U_X. As long as $t > 0$, U_H is regulated by $f_{1,t+t_r} f_{2,t+t_r}$. These factors vanish exponentially fast when $q^+ \to 0$, *cf.* Eq. (14). For finite $t > 0$ one can remove the regularization dependence by setting $t_r = 0$. However, in U_X there is only the regulating factor $f_{1,t_r} f_{2,t_r}$, which goes to 1 when $t_r \to 0$. The factor f_t in front of U_X does not regulate the singularity when $q^+ \to 0$. Moreover,

$$\frac{j_1^+ j_2^+}{(q^+)^2}\left(1+\frac{q_1^2+q_2^2}{2}\mathcal{F}\right)=\frac{j_1^+ j_2^+}{(q^+)^2}\frac{m_g^2}{m_g^2-q_1^2}+O\left(\frac{1}{q^+}\right). \tag{35}$$

Therefore, matrix elements of H_{Ut} diverge for $t_r \to 0$ whenever $q^+ \to 0$. More precisely, in the vicinity of $q^+ = 0$, $-q_1^2 \approx \Delta k^2 = (k_{12}^\perp - k_{1'2'}^\perp)^2$, and the regulator $f_{1,t_r} f_{2,t_r} \approx e^{-2t_r \frac{(\Delta k^2+m_g^2)^2}{(q^+)^2}}$. Integrating $f_{1,t_r} f_{2,t_r}/(q^+)^2$ over q^+ gives $\sqrt{\frac{\pi}{2t_r(\Delta k^2+m_g^2)^2}}$ as the part divergent when $t_r \to 0$. Moreover, $m_g^2/(m_g^2 - q_1^2) \approx m_g^2/(m_g^2 + \Delta k^2)$. Hence, to counter the divergence we add the gluon exchange counterterm, whose kernel is

$$X = \delta(p_{1'}^+ - p_1^+) j_1^+ j_2^+ \frac{m_g^2}{(\Delta k^2 + m_g^2)^2}\sqrt{\frac{\pi}{2t_r}}. \tag{36}$$

Demonstration of cancellation of divergences is presented in Appendix B. Since $m_g^2(\Delta k^2 + m_g^2)^{-2}$ tends to a two-dimensional Dirac δ-function of the transverse momentum, the counterterm is nonzero even in the limit $m_g \to 0$. It becomes diagonal in momentum and spin. We remind the reader the limit $t_r \to 0$ is performed before we consider the limit $m_g \to 0$. The latter is discussed in Sec. IV.

D. Renormalized Hamiltonian

With both mass and exchange counterterms included, one can remove the ultraviolet regulator, i.e., one can take the limit $t_r \to 0$. It is easily done in H_{t1}, U_C, and U_H. The renormalized mass terms contain $m_i^2 + g_t^2 \delta m_{it}^2 = m_i^2 + g_t^2 I_i(t, m_g)$. The limit of U_X when $t_r \to 0$ is described in Appendix B.

IV. QUARKONIUM IN HEAVY-FLAVOR QCD WITH GLUON MASS ANSATZ

In this section, we derive the quarkonium eigenvalue equation using expansion in powers of the coupling constant up to second order. Working at so low order of the expansion, we have to pay a price for not knowing what comes out from the non-Abelian interactions of gluons in orders higher than 2nd. In particular, these interactions prevent gluons from behaving like photons in QED. Also, the coupling constant g_t needs to be extrapolated to values larger than the charge e in QED. We assume that the emission and absorption of effective gluons by effective quarks and antiquarks is blocked for sizable t. We model the non-Abelian blocking by introducing the gluon-mass m_G for the effective gluons. The resulting eigenvalue equations for quarkonia and single quarks are then obtained keeping $m_g > 0$. Subsequently, we discuss these equations in the limit $m_g \to 0$.

A. Effective Hamiltonian

Our description of heavy quarkonium closely follows Ref. [6]. Here we focus on the main steps, *cf.* [32]. We consider the quarkonium eigenvalue problem assuming that a single quark-antiquark pair gives the dominant contribution. Other Fock sectors are included using expansion in powers of g_t. Up to the second order, we need two Fock sectors: the leading quark-antiquark sector $Q\bar{Q}$ and the quark-antiquark-gluon sector $Q\bar{Q}G$. The large letter G signifies the effective gluons whose mass is assigned a hypothetical value m_G. Namely, we modify the Hamiltonian limited to $Q\bar{Q}$ and $Q\bar{Q}G$ by adding to it a nonperturbative gluon mass term,

$$\hat{m}_G^2 = \int_{123} \frac{m_G^2}{p_3^+} b_{t1}^\dagger d_{t2}^\dagger a_{t3}^\dagger |0\rangle\langle 0| a_{t3} d_{t2} b_{t1}. \tag{37}$$

This operator acts only in the $Q\bar{Q}G$ sector. In principle, m_G could be a multiple of $\Lambda_{\rm QCD}$ in the RGPEP scheme and hence not expandable in powers of g_t. More about our gluon mass ansatz can be found in Refs. [6,25].

Perturbative computation of the effective Hamiltonian [33] in the $Q\bar{Q}$ sector yields $H_{\rm eff}$ whose matrix elements are

$$\langle L|H_{\rm eff}|R\rangle = \langle L|\left[H_{11} + \frac{1}{2}H_{12}\left(\frac{1}{E_L - H_{22} - \hat{m}_G^2} + \frac{1}{E_R - H_{22} - \hat{m}_G^2}\right)H_{21}\right]|R\rangle, \tag{38}$$

where $H_{ij} = P_i H_t P_j$, with P_1 and P_2 the projection operators onto the $Q\bar{Q}$ and $Q\bar{Q}G$ sectors, respectively. We keep only the free part of H_{22} in the denominators, because other terms in H_{22} are of order g^2 and contribute terms in $H_{\rm eff}$ of at least 4th order. The states

$$|L\rangle = \int_{12} P_L^+ \tilde{\delta}_{12.P_L} \frac{\delta_{c_1,c_2}}{\sqrt{N_c}} \psi_L(1,2) b_{t1}^\dagger d_{t2}^\dagger |0\rangle, \tag{39}$$

$$|R\rangle = \int_{1'2'} P_R^+ \tilde{\delta}_{1'2'.P_R} \frac{\delta_{c_{1'},c_{2'}}}{\sqrt{N_c}} \psi_R(1',2') b_{t1'}^\dagger d_{t2'}^\dagger |0\rangle, \tag{40}$$

are eigenstates of the free part of H_{11} with the eigenvalues $E_L = p_1^- + p_2^- = [\mathscr{M}_{12}^2 + (P_L^\perp)^2]/P_L^+$ and $E_R = p_{1'}^- + p_{2'}^- = [\mathscr{M}_{1'2'}^2 + (P_R^\perp)^2]/P_R^+$, respectively. Due to momentum

conservation only matrix elements with $P_L = P_R \equiv P$ are nonzero. The evaluation of the matrix elements gives

$$\langle L|H_{\rm eff}|R\rangle = P^+\tilde{\delta}_{P_L.P_R}\sum_{\sigma_1,\sigma_2}\int [12]P_L^+\tilde{\delta}_{12.P_L}\psi_L^*(1,2) \times (\mathcal{E}\psi_R)(1,2), \tag{41}$$

where

$$(\mathcal{E}\psi_R)(1,2) = \left(\frac{m_1^2+\mathscr{M}_1^2+p_1^{\perp 2}}{p_1^+}+\frac{m_2^2+\mathscr{M}_2^2+p_2^{\perp 2}}{p_2^+}\right)\psi_R(1,2) - C_F g_t^2\sum_{\sigma_{1'},\sigma_{2'}}\int [1'2']\tilde{\delta}_{12.1'2'}\tilde{U}_{t12.1'2'}\psi_R(1',2'). \tag{42}$$

The momentum integration measure for two particles, 1 and 2, is,

$$[12] = \frac{dp_1^+ d^2p_1^\perp}{16\pi^3 p_1^+}\frac{dp_2^+ d^2p_2^\perp}{16\pi^3 p_2^+}, \tag{43}$$

and analogously $[1'2']$ for particles $1'$ and $2'$. The self-interaction terms are

$$\mathscr{M}_i^2 = C_F g_t^2\sum_{\sigma_{\tilde{i}},\sigma_3}\int \frac{d^2k_{3\tilde{i}}dx_{3\tilde{i}}}{16\pi^3 x_{3\tilde{i}}x_{\tilde{i}3}}\frac{m_G^2}{x_{3\tilde{i}}} \times \frac{f_{\tilde{i}3.i,t}^2}{\left(\mathscr{M}_{\tilde{i}3}^2+\frac{m_G^2}{x_{3\tilde{i}}}-m_i^2\right)(\mathscr{M}_{\tilde{i}3}^2-m_i^2)}\bar{u}_i\not{\epsilon}_3 u_{\tilde{i}}\bar{u}_{\tilde{i}}\not{\epsilon}_3^* u_i, \tag{44}$$

and the effective $Q\bar{Q}$ interaction kernel, which one can call the $Q\bar{Q}$ potential, is

$$\tilde{U}_{t12.1'2'} = U_C + \tilde{U}_H + U_X, \tag{45}$$

where U_C and U_X are given in Eqs. (23) and (25), respectively, and

$$\tilde{U}_H = f_{1,t}f_{2,t}\left(\frac{q_1^2+q_2^2}{2(q^+)^2}j_1^+j_2^+ + g_{\mu\nu}j_1^\mu j_2^\nu\right) \times\left[\frac{1}{2}\left(\frac{1}{m_G^2+m_g^2-q_1^2}+\frac{1}{m_G^2+m_g^2-q_2^2}\right)-\mathcal{F}\right]. \tag{46}$$

B. Color-singlet quarkonium

Consider the eigenvalue problem,

$$(\mathcal{E}\psi)(1,2) = \frac{M^2+P^{\perp 2}}{P^+}\psi(1,2), \tag{47}$$

where M^2 is the mass of the bound state. There are two types of interaction terms in $\mathcal{E}$, self-interactions and potential terms. Both diverge logarithmically when $m_g \to 0$. However, the self-interactions diverge to the positive infinity, while the potential diverges to the negative infinity. In order to isolate the divergence of the potential terms we rewrite

$$(\mathcal{E}\psi)(1,2) = \left(\frac{m_1^2+\mathscr{M}_1^2+p_1^{\perp 2}}{p_1^+}+\frac{m_2^2+\mathscr{M}_2^2+p_2^{\perp 2}}{p_2^+}-\frac{\Delta}{p_1^++p_2^+}\right)\psi(1,2) - C_F g_t^2\sum_{\sigma_{1'},\sigma_{2'}}\int [1'2']\tilde{\delta}_{12.1'2'}\tilde{U}_H[\psi(1',2')-\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}}\psi(1,2)] - C_F g_t^2\sum_{\sigma_{1'},\sigma_{2'}}\int [1'2']\tilde{\delta}_{12.1'2'}(U_C+U_X)\psi(1',2'), \tag{48}$$

where

$$\Delta = C_F g_t^2\sum_{\sigma_{1'},\sigma_{2'}}\int [1'2'](p_1^++p_2^+)\tilde{\delta}_{12.1'2'}\tilde{U}_H\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}}, \tag{49}$$

results from subtracting and adding $\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}}\psi(1,2)$ to $\psi(1',2')$ under the integral $\int[1'2']$. The most singular part of the integrand appears in the vicinity of $q^+ = q_1^+ = q_2^+ = 0$ while $q^\perp = q_1^\perp = q_2^\perp = 0$. The singularity is regulated by m_g. Near that singular region the integrand is proportional to $(p_1^+ + p_2^+)f_{1,t}f_{2,t}/(q^+)^2$, where $f_{1,t} \approx f_{2,t} \approx e^{-t(p_3^-)^2} = \exp\{-t[m_g^2+q^{\perp 2}]^2/(q^+)^2\}$. Integration over q^+ from $-p_1^+$ to p_2^+ and over $q^\perp$ over the whole two-dimensional transverse plane gives,

$$\Delta = \frac{C_F g_t^2}{8\pi^2}\log\left(\frac{p_1^+p_2^+}{8m_g^4 t}\right)\sqrt{\frac{\pi}{2t}}(p_1^++p_2^+)+O(m_g^0), \tag{50}$$

where $O(m_g^0)$ denotes the terms that are finite in the limit $m_g \to 0$. Similarly, the integrand in the self-interaction of Eq. (44) near $p_3^+ = 0$ is approximately $p_i^+ f_{i,t}^2/(p_3^+)^2$,

where $f_{i,t} \approx \exp\{-t[m_g^2 + (p_3^\perp)^2]^2/(p_3^+)^2\}$ and $k_{3\tilde{i}}^\perp \approx p_3^\perp$. Integration over p_3^+ from 0 to p_i^+ and over $p_3^\perp$ over the whole two-dimensional plane gives,

$$\mathscr{M}_i^2 = \frac{C_F g_t^2}{16\pi^2} \log\left(\frac{p_i^{+2}}{8m_g^4 t}\right)\sqrt{\frac{\pi}{2t}}p_i^+ + O(m_g^0). \quad (51)$$

The logarithms of the gluon mass m_g obtained above cancel out in

$$\frac{\mathscr{M}_1^2}{p_1^+} + \frac{\mathscr{M}_2^2}{p_2^+} - \frac{\Delta}{p_1^+ + p_2^+} = O(m_g^0). \quad (52)$$

The eigenvalue problem has a finite limit for $m_g \to 0$. However, the quarkonium mass eigenvalue M^2 depends on $P^+ = p_1^+ + p_2^+$ of the state. The dependence comes mainly from the potential produced by the second term on the right-hand side of Eq. (48). That potential is confining, *cf.* Eq. (72) below. For small distances r between Q and $\bar{Q}$ it behaves as r^2. The associated oscillator frequency ω is proportional to $(t/P^{+2})^{-3/4}$. Therefore, it is natural to set the longitudinal momentum scale $\mathcal{P}$ of Eq. (7) to the quarkonium momentum, $\mathcal{P} = P^+$, which implies $\omega \sim \lambda^3$, see Sec. V.

C. Color-octet quarkonium and eigenquarks

For color-octet quark-antiquark states one can proceed by the same steps as for the color-singlet case. The quark-antiquark annihilation interaction needs to be included. Its contribution is finite in the limit $m_g \to 0$. Now, the color-octet wave functions of the states $|L\rangle$ and $|R\rangle$ lead to the different color factors in the potential term of the octet eigenvalue equation, $(2N_c)^{-1} = 1/6$ instead of $-C_F = -4/3$. The self-interactions $\mathscr{M}_i^2$ do not depend on the color wave function of quarkonium. Accordingly, the term Δ includes the factor $-(2N_c)^{-1}$. Therefore, instead of Eq. (52), we obtain for the octet states

$$\frac{\mathscr{M}_1^2}{p_1^+} + \frac{\mathscr{M}_2^2}{p_2^+} - \frac{\Delta}{p_1^+ + p_2^+} = N_c \frac{g_t^2}{16\pi^2} \log\left(\frac{p_1^+ p_2^+}{2m_g^4 t}\right)\sqrt{\frac{\pi}{2t}} + O(m_g^0). \quad (53)$$

The cancellation of $\log m_g$ is absent. The expectation value of $H_{\rm eff}$ in the color-octet states diverges to plus infinity in the limit $m_g \to 0$.

Similar noncancellation of $\log m_g$ appears in the eigenvalue equations for states with quantum numbers of a single quark, which we for brevity call eigenquarks. The eigenquark mass eigenvalue M_Q diverges to plus infinity when $m_g \to 0$. Assuming the same m_G as in the quarkonium case, we obtain

$$M_Q^2 = m_1^2 + C_F g_t^2 \sum_{\sigma_{\tilde{1}},\sigma_3} \int \frac{d^2k_{3\tilde{1}}dx_{3\tilde{1}}}{16\pi^3 x_{3\tilde{1}}x_{\tilde{1}3}} \frac{m_G^2}{x_{3\tilde{1}}} \frac{f_{\tilde{1}3.1,t}^2}{(\mathscr{M}_{\tilde{1}3}^2 + \frac{m_G^2}{x_{3\tilde{1}}} - m_1^2)(\mathscr{M}_{\tilde{1}3}^2 - m_1^2)} \bar{u}_1 \not{\epsilon}_3 u_{\tilde{1}} \bar{u}_{\tilde{1}} \not{\epsilon}_3^* u_1 \quad (54)$$

$$= m_1^2 + P^+ \frac{C_F g_t^2}{16\pi^2}\sqrt{\frac{\pi}{2t}}\left[\log\left(\frac{P^{+2}}{2m_g^4 t}\right) - 2 + \frac{1}{\sqrt{\pi}}\int_0^\infty ds'\, e^{-s'^2} f\left(s', \frac{m'^2}{P^+}, \frac{m_G'^2}{P^+}\right)\right] + o(m_g^0), \quad (55)$$

where P^+ is the longitudinal momentum of the eigenquark state, $m'^2 = \sqrt{2t}m_1^2$, and $m_G'^2 = \sqrt{2t}m_G^2$, and

$$f(a,b,c) = 4\log\left(\frac{a^2}{a+b}\right) + \frac{2c}{a+b} - 4c\left(\frac{1}{c} + \frac{1}{a} + \frac{b+c/2}{a^2}\right)\log\left(1 + \frac{a^2}{c(a+b)}\right). \quad (56)$$

Function $f(a,b,c)$ should not be confused with any of the form factors. Terms $o(m_g^0)$ vanish when $m_g \to 0$.

V. NONRELATIVISTIC APPROXIMATION OF THE EFFECTIVE POTENTIAL

The eigenvalue equation (47) has interesting properties. We exhibit them using the nonrelativistic (NR) limit in which the quark masses are considered very large. In that limit quarks have typical longitudinal momentum fractions $x_1 \approx x_{10} = m_1/(m_1+m_2)$ and $x_2 \approx x_{20} = m_2/(m_1+m_2)$. Information about the state resides in the wave function dependence on the deviation of x_1 from x_{10} and the quarks relative motion in the transverse directions. We introduce the quark three-dimensional relative momentum $\vec{k} = (k^x, k^y, k^z)$,

$$\vec{k} = \sqrt{\frac{m_1 m_2}{x_1 x_2}}\left(\frac{k_{12}^x}{m_1+m_2}, \frac{k_{12}^y}{m_1+m_2}, x_1 - x_{10}\right), \quad (57)$$

which generalizes the definition of $\vec{k}$ in Ref. [6] to the case of $m_1 \neq m_2$, both masses being large. The potential is a function of $\vec{k}$ and $\vec{k}'$. The NR limit is obtained assuming that $|\vec{k}|, |\vec{k}'| \ll m_1 + m_2$ and keeping only the leading terms.

After multiplying Eq. (47) by $P^+/(2m_1+2m_2)$, the NR limit of the eigenvalue equation reads

$$\left(\frac{\vec{k}^2}{2\mu}+\frac{\mathscr{M}_1^2}{2m_1}+\frac{\mathscr{M}_2^2}{2m_2}\right)\psi_{\sigma_1,\sigma_2}(\vec{k}) - C_F g_t^2\int\frac{d^3k'}{(2\pi)^3}(V_C+V_H+V_X)\psi_{\sigma_1,\sigma_2}(\vec{k}') = E\psi_{\sigma_1,\sigma_2}(\vec{k}), \tag{58}$$

where the reduced mass $\mu = m_1m_2/(m_1+m_2)$. The potentials V_C, V_H, and V_X are the nonrelativistic approximations of $U_C/(4m_1m_2)$, $\tilde{U}_H/(4m_1m_2)$, and $U_X/(4m_1m_2)$, respectively. We obtain

$$V_C = \frac{f_t}{\vec{q}^2+m_g^2}, \tag{59}$$

$$V_H = f_{1,t}f_{2,t}\left(\frac{1}{(q^z)^2}-\frac{1}{\vec{q}^2}\right)\frac{\vec{q}^2}{\vec{q}^2+m_g^2}\frac{m_G^2}{m_G^2+m_g^2+\vec{q}^2}, \tag{60}$$

$$V_X = 0, \tag{61}$$

where $\vec{q}=\vec{k}'-\vec{k}$. V_X is set to 0 because it is suppressed by the inverse of μ^2 in comparison to V_C and V_H, see Eq. (B20). The eigenvalue $E=[M^2-(m_1+m_2)^2]/[2(m_1+m_2)]\approx M-(m_1+m_2)$, since $M\approx m_1+m_2$. The RGPEP form factors are

$$f_{i,t} = \exp\left(-\frac{t(m_1+m_2)^2}{(p_1^++p_2^+)^2}\frac{(\vec{q}^2+m_g^2)^2}{(q^z)^2}\right), \tag{62}$$

$$f_t = \exp\left(-\frac{t(m_1+m_2)^4(\vec{k}^2-\vec{k}'^2)^2}{(p_1^++p_2^+)^2m_1^2m_2^2}\right). \tag{63}$$

The NR approximation for the self-interaction terms $\mathscr{M}_i^2$ is obtained using variables $\tilde{q}^{x,y}=k^{x,y}_{3\tilde{i}}$, $\tilde{q}^z=x_{3\tilde{i}}m_i$ that appear in the form factors. We find

$$\mathscr{M}_i^2 = m_iC_Fg_t^2\int\frac{d^3\tilde{q}}{(2\pi)^3}f_{i,t}^2\left(\frac{1}{(\tilde{q}^z)^2}-\frac{1}{\tilde{q}^2}\right)\times\frac{\tilde{q}^2}{\tilde{q}^2+m_g^2}\frac{m_G^2}{m_G^2+m_g^2+\tilde{q}^2} \tag{64}$$

$$= m_iC_Fg_t^2\int\frac{d^3q}{(2\pi)^3}V_H, \tag{65}$$

where $f_{i,t}$ is the same as in Eq. (62) except that $\vec{q}$ is replaced by $\tilde{\vec{q}}$. The second equality holds because in the NR approximation $f_{1,t}=f_{2,t}$. Therefore, the self-energy terms $\mathscr{M}_1^2$ and $\mathscr{M}_2^2$ can be combined with the potential term V_H. The quarkonium eigenvalue equation in the limit $m_g\to 0$ becomes

$$\frac{\vec{k}^2}{2\mu}\psi(\vec{k})+(V_{\rm conf}\psi)(\vec{k})-\int\frac{d^3k'}{(2\pi)^3}\frac{C_Fg_t^2f_t}{\vec{q}^2}\psi(\vec{k}')=E\psi(\vec{k}), \tag{66}$$

where the spin indices are omitted and, by definition,

$$(V_{\rm conf}\psi)(\vec{k}) = \lim_{m_g\to 0^+}(-C_Fg_t^2)\int\frac{d^3q}{(2\pi)^3}V_H[\psi(\vec{k}+\vec{q})-\psi(\vec{k})]. \tag{67}$$

The limit $m_g\to 0$ is well-defined, because the difference of wave functions regulates $1/(q^z)^2$ in V_H. We expand $\psi(\vec{k}+\vec{q})$ in a Taylor series in $\vec{q}$. Only terms with even powers of q^x, q^y, and q^z contribute to the result of integration over $\vec{q}$ because V_H is even in $\vec{q}$. Therefore,

$$(V_{\rm conf}\psi)(\vec{k}) = {\sum_{n,k}}'C_{n,k}\left(\frac{\partial^2}{(\partial k^x)^2}+\frac{\partial^2}{(\partial k^y)^2}\right)^n\left(\frac{\partial}{\partial k^z}\right)^{2k}\psi(\vec{k}), \tag{68}$$

where $\sum_{n,k}{}'$ is the sum over all non-negative n and k with the exception of the $n=k=0$ term, and

$$C_{n,k} = -\frac{C_Fg_t^2}{(2\pi)^2}\frac{2}{(2^nn!)^2(2k)!}\int_0^1 dw(1-w^2)^{n+1}w^{2k-2}\times\int_0^\infty dq\frac{m_G^2}{m_G^2+q^2}q^{2n+2k}e^{-\frac{t'q^2}{w^2}}, \tag{69}$$

where

$$t' = \frac{2t(m_1+m_2)^2}{(p_1^++p_2^+)^2} = \frac{2\mathcal{P}^2(m_1+m_2)^2}{\lambda^4(p_1^++p_2^+)^2}. \tag{70}$$

Coefficients $C_{n,k}$ can be evaluated explicitly for $m_G^2\to\infty$,

$$\lim_{m_G^2\to\infty}C_{n,k} = -\frac{C_Fg_t^2}{(2\pi)^2}\frac{\sqrt{\pi}}{2^{3n+k+1}}\frac{(n+1)t'^{-k-n-\frac{1}{2}}}{(2k+2n)!!(2k+2n+1)(2k+n)}\binom{n+2k}{n}. \tag{71}$$

The potential in position space is obtained by substituting, $\frac{\partial^2}{(\partial k^x)^2}+\frac{\partial^2}{(\partial k^y)^2}\to-\rho^2=-x^2-y^2$ and $(\frac{\partial}{\partial k^z})^2\to -z^2$. Consequently,

$$V_{\rm conf}(\vec{r}) = V_{\rm conf}(\rho,z) = {\sum_{n,k}}'C_{n,k}(-\rho^2)^n(-z^2)^k. \tag{72}$$

The double summation makes it difficult to obtain the potential as a function of $\vec{r}$. However, for small separations between the quark and antiquark only the quadratic terms in $\vec{r}$ count in $V_{\rm conf}$. Moreover, in the limit $m_G \to \infty$ one obtains $C_{1,0} = C_{0,1}$ and the potential is spherically symmetric as a function of $\vec{r}$,

$$V_{\rm conf}(\vec{r}) = \frac{C_F g_t^2}{192\pi^{3/2} t'^{3/2}} \vec{r}^2 + O\left(\frac{r^4}{t'^{5/2}}\right). \tag{73}$$

For arbitrary separations we found simple expressions along some special directions of $\vec{r}$ in the limit $m_G^2 \to \infty$.

$$V_{\rm conf}(\rho, 0) = \frac{C_F g_t^2}{8\pi^{3/2}} \left[\frac{2\sqrt{\pi}}{\rho} \mathrm{erf}\left(\frac{\rho}{4\sqrt{t'}}\right) + \frac{\log\left(\frac{\rho^2}{16t'}\right) + E_1\left(\frac{\rho^2}{16t'}\right) + \gamma - 1}{\sqrt{t'}} \right], \tag{74}$$

$$V_{\rm conf}(0, z) = \frac{C_F g_t^2}{16\pi^{3/2}} \left[\frac{2\sqrt{\pi}}{z} \mathrm{erf}\left(\frac{z}{2\sqrt{t'}}\right) + \frac{\log\left(\frac{z^2}{4t'}\right) + E_1\left(\frac{z^2}{4t'}\right) + \gamma - 2}{\sqrt{t'}} \right], \tag{75}$$

where erf is the error function and E_1 is the exponential integral function. For large ρ or z (or small t') the effective potential becomes logarithmic,

$$V_{\rm conf}(\rho, 0) = \frac{C_F g_t^2}{8\pi^{3/2}} \frac{\log\left(\frac{\rho^2}{16t'}\right) + \gamma - 1}{\sqrt{t'}} + \frac{C_F g_t^2}{4\pi\rho} + O\left(\frac{\sqrt{t'} e^{-\frac{\rho^2}{16t'}}}{\rho^2}\right), \tag{76}$$

$$V_{\rm conf}(0, z) = \frac{C_F g_t^2}{16\pi^{3/2}} \frac{\log\left(\frac{z^2}{4t'}\right) + \gamma - 2}{\sqrt{t'}} + \frac{C_F g_t^2}{8\pi z} + O\left(\frac{t'^{3/2} e^{-\frac{z^2}{4t'}}}{z^4}\right). \tag{77}$$

Our result agrees at large distances with the logarithmic potentials found in Ref. [23], if one sets $\frac{\Lambda^2}{\mathcal{P}^+} = \frac{1}{2}\sqrt{\frac{\pi}{2}} \frac{\lambda^2}{m_1+m_2} \frac{p_1^+ + p_2^+}{\mathcal{P}}$, where $\Lambda^2/\mathcal{P}^+$ is the p^- cutoff of Ref. [23]. Hence, our potential breaks rotational symmetry at large interquark distances in a similar way to the one discussed in Refs. [24,34] using coupling coherence. However, our potential is not constrained that way and can be improved using the RGPEP, see below.

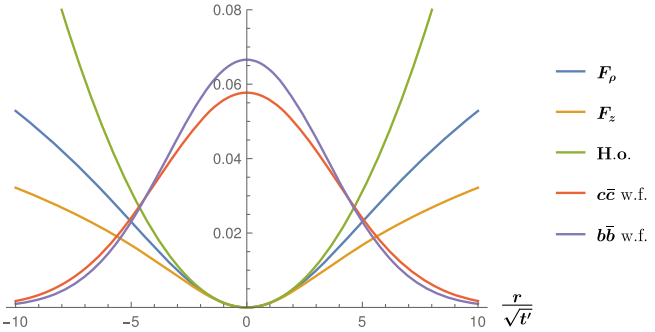

FIG. 4. Harmonic oscillator approximation compared to the full potential, as functions of the distance r between quark and antiquark in units of $\sqrt{t'}$, see Eq. (70). The ground state wave functions, $c\bar{c}$ w.f. and $b\bar{b}$ w.f., are obtained fitting the corresponding spectra. They are plotted for comparison with the potential. The harmonic oscillator H.o. is an extrapolation of the quadratic behavior of the potential near $r \sim 0$. Functions F_ρ and F_z are explained in the text below Eqs. (76) and (77).

Figure 4 shows the accuracy of the harmonic oscillator approximation for the potential. We first notice that $\frac{\sqrt{t'}}{g_t^2} V_{\rm conf}(\rho, 0) = F_\rho(\frac{\rho}{\sqrt{t'}})$ and $\frac{\sqrt{t'}}{g_t^2} V_{\rm conf}(0, z) = F_z(\frac{z}{\sqrt{t'}})$, where the functions F_ρ and F_z do not depend on the parameters of the theory. In the harmonic oscillator approximation, $F_\rho(r/\sqrt{t'}) = F_z(r/\sqrt{t'}) = r^2/144 t' \pi^{3/2}$. Typical separations between quarks in theoretical description of ground-states of heavy quarkonia can be determined by computing the quarkonium electromagnetic form factors [8]. Using the values for λ, g_t, and the quark masses fitted in Ref. [7], which imply results quoted in Table I, we obtain[1] that: For bottomonium $\sqrt{t'} \approx 0.37$ GeV$^{-1} \approx 0.073$ fm, while $r_{\rm EM} \sim 0.15$ fm, where $r_{\rm EM}$ is the radius extracted from the electromagnetic form factors [8]. The relative separation between quark and antiquark is $r \sim 2 r_{\rm EM}$. Therefore, typical $r/\sqrt{t'}$ for bottomonium is 4.1. For charmonium $\sqrt{t'} \approx 0.55$ GeV$^{-1} \approx 0.109$ fm, while $r_{\rm EM} \sim 0.25$ fm. Therefore, typical $r/\sqrt{t'} \sim 4.6$. Harmonic oscillator potential becomes twice too strong in the z direction for $r/\sqrt{t'} \approx 5.4$ at which point it is approximately 1.4 times too strong in the transverse direction.

In the excited states, an important weakening of the potential occurs due to the change of a rotation-symmetric quadratic behavior over to a logarithmic one that breaks rotational symmetry. In addition, the excited states are likely to be sensitive to the details of gluon components.

[1]Since Ref. [7] uses a different RGPEP generator, we have to rescale λ in order to obtain the same spectroscopy. If we define $\lambda_{\rm old}$ to be the value of λ used in Ref. [7] and $\lambda_{\rm new}$ to be the value of λ we are using here, then, in Ref. [7] $t' = (m_1^2 + m_2^2)/\lambda_{\rm old}^4$, while from Eq. (70) we have here $t' = 2(m_1 + m_2)^2/\lambda_{\rm new}^4$ (assuming $\mathcal{P} = p_1^+ + p_2^+$). Therefore, for equal quark masses, $m_1 = m_2$, we have $\lambda_{\rm new} = \sqrt{2}\lambda_{\rm old}$.

TABLE I. Examples of the quark masses m_Q in MeV obtained using the RGPEP scale λ from our fits [8] to measured charmonium and bottomonium mass spectra [26], together with the resulting values of harmonic oscillator frequency ω in MeV and electric charge radii r_{EM} in fm for the lowest 0^{-+} and 1^{--} states for the optimized m_Q and λ. For more details and more examples of radii computations, see Ref. [8]. For comparison, PDG particle listings [26] give the $\overline{\mathrm{MS}}$ running masses in GeV: $\bar{m}_c(\bar{m}_c) = 1.27(2)$ and $\bar{m}_b(\bar{m}_b) = 4.18(3)$, and the corresponding pole masses: $m_c = 1.67(7)$ and $m_b = 4.78(6)$. The lattice results for charmonium electric radii are on the order of 0.25 fm with accuracy on the order of a few percent exemplified in Ref. [35].

	m_Q	λ	ω	$r_{\mathrm{EM}}(0^{-+})$	$r_{\mathrm{EM}}(1^{--})$
$c\bar{c}$	1460(10)	2749(30)	321.6	0.249	0.257
$b\bar{b}$	4698(10)	6022(70)	268.8	0.1521	0.1535

Such details are not accounted for in any way by the gluon mass ansatz and the resulting oscillator. To derive these details one needs to solve the RGPEP equation to higher order than second and include effective components with more gluons than one, shifting the mass ansatz to sectors with the maximal number of gluons one includes in numerical computations. We hope that systematically proceeding along these lines will lead to reduction of rotational symmetry breaking.

VI. CONCLUSION

Once canonical FF Hamiltonian for heavy-quark QCD in gauge $A^+ = 0$ is supplied with a small gluon mass m_g and subjected to the RGPEP scale-evolution of second order in a weak-coupling expansion, a simple dynamical picture is obtained in terms of the resulting eigenvalue equations for quarkonia at the scale of quark masses, provided that the emission and absorption of the effective gluons by quarks is blocked by assigning to them a hypothetically large effective gluon mass m_G. Quark self-interactions diverge in the limit $m_g \to 0$ but the divergence is canceled by the effective quark-antiquark interaction in color-singlet states. In color-octet states the cancellation does not occur. As a consequence, they cannot have finite masses in that limit. Single quark mass also cannot be finite. The finite color-singlet quarkonium eigenvalue problem can be further analyzed using the nonrelativistic approximation. The effective quark-antiquark potential at small distances r between the quarks includes a Coulomb term and a spherically symmetric oscillator term. The latter turns at large distances into a logarithmic dependence on r with different strengths in the transverse and longitudinal directions, matching the confining potential obtained by Perry using coupling coherence. Previous calculations of white quarkonia masses with such potentials indicate that the effective dynamics is likely to explain the ground and low excited states when the effective quark and gluon dynamics is computed using the RGPEP for heavy quarks more accurately. However, inclusion of light quarks in the dynamics would initially require guessing effective masses for them in a similar way to how it is done here for gluons.

Systematic increase in accuracy of quarkonium dynamics may be sought using the RGPEP by computing the running of H_t for heavy quarks in orders higher than 2nd. The actual running of the coupling constant g_t shows up in the quarkonium dynamics first in 4th order. Increasing the order implies inclusion of Fock components with more than one effective gluon in the hadronic states that are described numerically by solving the nonperturbative eigenvalue problem of H_t. The ansatz m_G must be shifted to the sector with the highest number of gluons. One may hope that the ansatz is eventually eliminated when the gluon effects are saturated by increasing their number.

Blocking effective gluons from significant involvement in the dynamics of lowest-mass quarkonia using mass m_G may reasonably reflect the effective gluons behavior because the results summarized above do not depend qualitatively on the value of m_G when it exceeds the scale of relative momentum of effective quarks, $p \sim \sqrt{\mu\omega} = \sqrt{m\omega/2}$. Our mass spectrum fit for quarkonia, see Table I, yields for $c\bar{c}$ that $m_c \sim 1.46$ GeV, $\omega \sim 322$ MeV and $p_{c\bar{c}} \sim 0.5$ GeV. For $b\bar{b}$, we get $m_b \sim 4.7$ GeV, $\omega_{b\bar{b}} \sim 269$ MeV and $p_{b\bar{b}} \sim 0.8$ GeV. More accurate computations than ours can provide more precise estimates of m_G, if m_G turns out to properly grasp the mechanism of blocking gluons. It is not known yet if a universal m_G may result from the RGPEP evolution that includes some arbitrarily small m_g added to the canonical QCD Hamiltonian at $t = 0$.

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APPENDIX A: SECOND-ORDER RGPEP

1. General formulas

Our notation closely resembles the notation used in Ref. [28]. Letters a, b, and x denote configurations of particles, i.e. collections of quantum numbers that label particle operators. Each configuration can contain arbitrary number and kinds of particles. $\mathcal{H}_{tab}$ denotes the coefficient that multiplies the term in $\mathcal{H}_t$ in which particles in configuration b are annihilated and particles in configuration a are created. For arbitrary a and b, the RGPEP Eq. (9) gives

$$\mathcal{H}'_{tab} = -(P_a^- - P_b^-)^2\mathcal{H}_{Itab} + \sum_x (P_a^- + P_b^- - 2P_x^-)\mathcal{H}_{Itax}\mathcal{H}_{Itxb}, \quad \text{(A1)}$$

where $\sum_x$ denotes the sum over all possible configurations x and the sum or integration over all quantum numbers in each individual configuration, P_a^-, P_b^-, and P_x^- are sums of front-form energies of all particles in configurations a, b, and x, respectively, and $\mathcal{H}_{It} = \mathcal{H}_t - \mathcal{H}_f$. Factoring out the 1st-order form factors from vertices,

$$\mathcal{H}_{tab} = f_{a.b,t}\mathcal{G}_{tab}, \quad \text{(A2)}$$

where

$$f_{a.b,t} = e^{-t(P_a^- - P_b^-)^2}, \quad \text{(A3)}$$

we get differential equation for $\mathcal{G}_{tab}$:

$$\mathcal{G}'_{tab} = \sum_x (P_a^- + P_b^- - 2P_x^-)\frac{f_{a.x,t}f_{x.b,t}}{f_{a.b,t}}\mathcal{G}_{tax}\mathcal{G}_{tab}. \quad \text{(A4)}$$

One obtains $f_{a.x,t}f_{x.b,t}/f_{a.b,t} = \exp[-2t(P_a^- - P_x^-) \times (P_b^- - P_x^-)]$. Expanding $\mathcal{G}_t$ in a series in powers of the coupling constant g,

$$\mathcal{G}_{tab} = \mathcal{G}_{fab} + g\mathcal{G}_{t1ab} + g^2\mathcal{G}_{t2ab} + \ldots, \quad \text{(A5)}$$

and can solve Eq. (A4) order by order. $\mathcal{G}_f$ is independent of t, which is already reflected in the notation. Up to second order,

$$\mathcal{G}_{t1ab} = \mathcal{G}_{01ab}, \quad \text{(A6)}$$

$$\mathcal{G}_{t2ab} = \mathcal{G}_{02ab} + \sum_x B^{(123,0)}_{taxb}\mathcal{G}_{01ax}\mathcal{G}_{01xb}, \quad \text{(A7)}$$

where

$$B^{(123,0)}_{taxb} = \int_0^t d\tau (P_a^- + P_b^- - 2P_x^-)\frac{f_{a.x,\tau}f_{x.b,\tau}}{f_{a.b,\tau}}. \quad \text{(A8)}$$

After integration,

$$B^{(123,0)}_{taxb} = \begin{cases} (P_a^- + P_b^- - 2P_x^-)t & \text{when} \quad (P_a^- - P_x^-)(P_b^- - P_x^-) = 0, \\ \frac{1}{2}[(P_a^- - P_x^-)^{-1} + (P_b^- - P_x^-)^{-1}](1 - f_{a.x,t}f_{x.b,t}/f_{a.b,t}) & \text{otherwise.} \end{cases} \quad \text{(A9)}$$

2. Self-interaction terms

For mass terms we have $P_a^- = P_b^-$, hence, $f_{a.b,t} = 1$, and $f_{a.x,t} = f_{x.b,t}$. Equation (A7) becomes,

$$\mathcal{G}_{t2ab} = \mathcal{G}_{02ab} + \int_{\tilde{i}3}\frac{1}{P_a^- - P_x^-}(1 - f^2_{a.x,t})\mathcal{G}_{01ax}\mathcal{G}_{01xb}, \quad \text{(A10)}$$

where $\mathcal{G}_{01ax} = \bar{u}_i\gamma^\mu u_{\tilde{i}}t^3_{i\tilde{i}}f_{i.\tilde{i}3,t_r}\tilde{\delta}_{i.\tilde{i}3}\varepsilon_{3\mu}$, $\mathcal{G}_{01xb} = \bar{u}_{\tilde{i}}\gamma^\nu u_{i'}t^3_{\tilde{i}i'}f_{\tilde{i}3.i',t_r}\tilde{\delta}_{\tilde{i}3.i'}\varepsilon^*_{3\nu}$, and $P_a^- - P_x^- = (m_i^2 - \mathscr{M}^2_{\tilde{i}3})/p_i^+$. Therefore,

$$\mathcal{G}_{t2ab} = \mathcal{G}_{02ab} + \tilde{\delta}_{i.i'}\int_{\tilde{i}3} t^3_{i\tilde{i}}t^3_{\tilde{i}i'}\frac{p_i^+\tilde{\delta}_{\tilde{i}3.i}}{m_i^2 - \mathscr{M}^2_{\tilde{i}3}}\left(f^2_{\tilde{i}3.i,t_r} - f^2_{\tilde{i}3.i,t+t_r}\right)\bar{u}_i\not{\varepsilon}_3 u_{\tilde{i}}\bar{u}_{\tilde{i}}\not{\varepsilon}^*_3 u_{i'}. \quad \text{(A11)}$$

Now, using $\sum_{c_3} t^{c_3}_{i\tilde{i}}t^{c_3}_{\tilde{i}i'} = C_F\delta_{c_i,c_{i'}}$, and $\sum_{\sigma_{\tilde{i}}}\tilde{\delta}_{i.i'}\bar{u}_i\not{\varepsilon}_3 u_{\tilde{i}}\bar{u}_{\tilde{i}}\not{\varepsilon}^*_3 u_{i'} = \sum_{\sigma_{\tilde{i}}}\tilde{\delta}_{i.i'}\delta_{\sigma_i,\sigma_{i'}}\bar{u}_i\not{\varepsilon}_3 u_{\tilde{i}}\bar{u}_{\tilde{i}}\not{\varepsilon}^*_3 u_i$, we get,

$$\mathcal{G}_{t2ab} = \mathcal{G}_{02ab} + \tilde{\delta}_{i.i'}\delta_{\sigma_i,\sigma_{i'}}\delta_{c_i,c_{i'}}C_F\sum_{\sigma_{\tilde{i}},\sigma_3}\int[\tilde{i}3]p_i^+\tilde{\delta}_{\tilde{i}3.i}\frac{f^2_{\tilde{i}3.i,t+t_r} - f^2_{\tilde{i}3.i,t_r}}{\mathscr{M}^2_{\tilde{i}3} - m_i^2}\bar{u}_i\not{\varepsilon}_3 u_{\tilde{i}}\bar{u}_{\tilde{i}}\not{\varepsilon}^*_3 u_i \quad \text{(A12)}$$

$$= \mathcal{G}_{02ab} + \tilde{\delta}_{i.i'}\delta_{\sigma_i,\sigma_{i'}}\delta_{c_i,c_{i'}}[I_i(t + t_r, m_g) - I_i(t_r, m_g)]. \quad \text{(A13)}$$

Writing $\mathcal{G}_{02ab} = \tilde{\delta}_{i.i'}\delta_{\sigma_i,\sigma_{i'}}\delta_{c_i,c_{i'}}\delta m^2_{iX}$, we obtain $\mathcal{G}_{t2ab} = \tilde{\delta}_{i.i'}\delta_{\sigma_i,\sigma_{i'}}\delta_{c_i,c_{i'}}\delta m^2_{it}$, where δm^2_{it} is defined in Eq. (30). Finally,

$$H_{\delta m} = \int_{11'} \mathcal{G}_{t2ab} b^\dagger_{t1} b_{t1'} + \int_{22'} \mathcal{G}_{t2ab} d^\dagger_{t2} d_{t2'} \tag{A14}$$

$$= \int_1 \frac{\delta m^2_{1t}}{p^+_1} b^\dagger_{t1} b_{t1} + \int_2 \frac{\delta m^2_{2t}}{p^+_2} d^\dagger_{t2} d_{t2}. \tag{A15}$$

3. Gluon exchange terms

For gluon exchange potentials, if gluon is emitted from the quark, we have,

$$\mathcal{G}_{01ax} = -j^\mu_2 \, t^3_{2'2} \, f_{2,t_r} \, \tilde{\delta}_{2'3.2} \, \varepsilon_{3\mu}, \tag{A16}$$

$$\mathcal{G}_{01xb} = j^\nu_1 \, t^3_{11'} \, f_{1,t_r} \, \tilde{\delta}_{13.1'} \, \varepsilon^*_{3\nu}, \tag{A17}$$

$$B^{(123,0)}_{taxb} = \frac{1}{2}\left(\frac{1}{p^-_2 - p^-_{2'} - p^-_3} + \frac{1}{p^-_{1'} - p^-_1 - p^-_3}\right) \times \left(1 - \frac{f_{2.2'3,t} f_{13.1',t}}{f_{12.1'2',t}}\right) \tag{A18}$$

$$= \frac{1}{2}\left(\frac{p^+_3}{q^2_2 - m^2_g} + \frac{p^+_3}{q^2_1 - m^2_g}\right)\left(1 - \frac{f_{2,t} f_{1,t}}{f_t}\right), \tag{A19}$$

and

$$\mathcal{G}_{t2ab} = \mathcal{G}_{02ab} + \int_3 B^{(123,0)}_{taxb} \mathcal{G}_{01ax} \mathcal{G}_{01xb} \tag{A20}$$

$$= \mathcal{G}_{02ab} - f_{1,t_r} f_{2,t_r} \tilde{\delta}_{12.1'2'} \frac{1}{2}\left(\frac{1}{q^2_1 - m^2_g} + \frac{1}{q^2_2 - m^2_g}\right) \times \left(1 - \frac{f_{1,t} f_{2,t}}{f_t}\right) \sum_{\sigma_3} \varepsilon_{3\mu} \, \varepsilon^*_{3\nu} \, j^\nu_1 \, j^\mu_2 \, t^a_{11'} \, t^a_{2'2}. \tag{A21}$$

The initial condition, $\mathcal{G}_{02ab}$, includes the canonical instantaneous interaction (regularized) plus the counterterm,

$$\mathcal{G}_{02ab} = \tilde{\delta}_{12.1'2'}\left(-f_{1,t_r} f_{2,t_r} \frac{j^+_1 j^+_2}{(q^+)^2} + X\right) t^a_{11'} t^a_{2'2}. \tag{A22}$$

If gluon is emitted from the antiquark, we have,

$$\mathcal{G}_{01ax} = j^\mu_1 \, t^3_{11'} \, f_{1,t_r} \, \tilde{\delta}_{1'3.1} \, \varepsilon_{3\mu}, \tag{A23}$$

$$\mathcal{G}_{01xb} = -j^\nu_2 \, t^3_{2'2} \, f_{2,t_r} \, \tilde{\delta}_{23.2'} \, \varepsilon^*_{3\nu}, \tag{A24}$$

$$B^{(123,0)}_{taxb} = \frac{1}{2}\left(\frac{1}{p^-_1 - p^-_{1'} - p^-_3} + \frac{1}{p^-_{2'} - p^-_2 - p^-_3}\right) \times \left(1 - \frac{f_{1.1'3,t} f_{23.2',t}}{f_{12.1'2',t}}\right) \tag{A25}$$

$$= \frac{1}{2}\left(\frac{p^+_3}{q^2_1 - m^2_g} + \frac{p^+_3}{q^2_2 - m^2_g}\right)\left(1 - \frac{f_{1,t} f_{2,t}}{f_t}\right), \tag{A26}$$

and we arrive again at Eq. (A21). The final result,

$$H_{Ut} = \int_{121'2'} \mathcal{G}_{t2ab} \, b^\dagger_{t1} \, d^\dagger_{t2} \, d_{t2'} \, b_{t1'}, \tag{A27}$$

after simple manipulations gives Eq. (21).

APPENDIX B: GLUON EXCHANGE COUNTERTERM

If we split $H_{Ut} = \hat{U}_C + \hat{U}_H + \hat{U}_X$, in accordance with Eq. (22), then

$$\langle L|\hat{U}_X|R\rangle = -C_F g^2_t \tilde{\delta}_{P_L.P_R} \sum_{\sigma_1,\sigma_2} \int \frac{dx_1 d^2k^\perp_{12}}{16\pi^3 x_1 x_2} \psi^*_L(1,2) \times \int \frac{dx_{1'} d^2k^\perp_{1'2'}}{16\pi^3}(Y_1 + Y_2 + Y_3), \tag{B1}$$

where

$$Y_1 = f_{1,t_r} f_{2,t_r} [Z(x_{1'}) - Z(x_1)]/q^{+2}, \tag{B2}$$

$$Y_2 = \left[f_{1,t_r} f_{2,t_r} - e^{-2t_r \frac{(\Delta k^2 + m^2_g)^2}{P^{+2} x^2_3}}\right] Z(x_1)/q^{+2}, \tag{B3}$$

$$Y_3 = e^{-2t_r \frac{(\Delta k^2 + m^2_g)^2}{P^{+2} x^2_3}} Z(x_1)/q^{+2} - \frac{f_t X}{x_{1'} x_{2'}} \psi_R(x_{1'}, k_{1'2'}), \tag{B4}$$

and

$$Z(x_{1'}) = \left(1 + \frac{q^2_1 + q^2_2}{2}\mathcal{F}\right) \frac{f_t j^+_1 j^+_2}{x_{1'} x_{2'}} \psi_R(x_{1'}, k_{1'2'}), \tag{B5}$$

$$Z(x_1) = \lim_{x_{1'} \to x_1} Z(x_{1'}) = 4P^{+2} \tilde{f}_t \frac{m^2_g}{m^2_g + \Delta k^2} \psi_R(x_1, k_{1'2'}), \tag{B6}$$

with the dependence of Z on x_1, k_{12}, and $k_{1'2'}$ not indicated explicitly, while $\tilde{f}_t = \lim_{x_{1'} \to x_1} f_t$.

Lifting the regularization we obtain

$$\lim_{t_r \to 0^+} \int \frac{dx_{1'} d^2k^\perp_{1'2'}}{16\pi^3} Y_1 = \mathcal{P} \int dx_{1'} \int \frac{d^2k^\perp_{1'2'}}{16\pi^3} \lim_{t_r \to 0^+} Y_1. \tag{B7}$$

The singularity $1/(q^+)^2$ is removed due to the difference $Z(x_{1'}) - Z(x_1)$. The principal value $\mathcal{P}$ is obtained because the regulator is approximately symmetric in q^+ in the vicinity of $q^+ = 0$. We take the limit $m_g \to 0$,

$$\lim_{m_g\to 0^+} \mathcal{P}\int dx_{1'}\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\lim_{t_r\to 0^+} Y_1 = \mathcal{P}\int dx_{1'}\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\lim_{m_g\to 0^+}[Z(x_{1'}) - Z(x_1)]/q^{+2} \tag{B8}$$

$$= -\mathcal{P}\int dx_{1'}\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{1}{q^{+2}}\frac{(q_1^2 - q_2^2)^2}{4q_1^2q_2^2}\frac{f_t j_1^+ j_2^+}{x_{1'}x_{2'}}\psi_R(x_{1'}, k_{1'2'}) \tag{B9}$$

$$= -\int dx_{1'}\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{(\mathscr{M}_{12}^2 - \mathscr{M}_{1'2'}^2)^2}{4q_1^2q_2^2}\frac{f_t j_1^+ j_2^+}{p_{1'}^+ p_{2'}^+}\psi_R(x_{1'}, k_{1'2'}), \tag{B10}$$

where in the last equality the principal value turns out not needed anymore. The term Y_2 gives zero in the limit $t_r \to 0$. The term Y_3 is where the divergence resides and needs a counterterm. Below we present the demonstration that small-x divergences are canceled once the counterterm is added, which we denote by X. Using Eq. (36) and the definition of Z we rewrite Y_3,

$$Y_3 = e^{-2t_r\frac{(\Delta k^2 + m_g^2)^2}{P^{+2}x_3^2}} Z(x_1)/q^{+2} - \frac{1}{P^+}\delta(x_{1'} - x_1)\frac{1}{\Delta k^2 + m_g^2}\sqrt{\frac{\pi}{2t_r}}Z(x_1). \tag{B11}$$

The integral becomes,

$$\int \frac{dx_{1'}d^2k^\perp_{1'2'}}{16\pi^3}Y_3 = \int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{Z(x_1)}{P^{+2}}\int_0^1 dx_{1'}\left[\frac{e^{-\frac{2t_r(\Delta k^2 + m_g^2)^2}{P^{+2}(x_{1'}-x_1)^2}}}{(x_{1'}-x_1)^2} - \delta(x_{1'} - x_1)\frac{P^+}{\Delta k^2 + m_g^2}\sqrt{\frac{\pi}{2t_r}}\right]. \tag{B12}$$

The integral over $x_{1'}$ can be evaluated,

$$\int \frac{dx_{1'}d^2k^\perp_{1'2'}}{16\pi^3}Y_3 = \int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{Z(x_1)}{P^{+2}}\frac{P^+}{\Delta k^2 + m_g^2}\sqrt{\frac{\pi}{2t_r}}\left[\frac{1}{2}\mathrm{erfc}\left(\frac{\sqrt{2t_r}(\Delta k^2 + m_g^2)}{P^+x_2}\right) + \frac{1}{2}\mathrm{erfc}\left(\frac{\sqrt{2t_r}(\Delta k^2 + m_g^2)}{P^+x_1}\right) - 1\right]. \tag{B13}$$

Note that -1 in the square bracket comes from the counterterm. If it were absent, the bracket would be 1 for $t_r \to 0$ and the integral would be proportional to $t_r^{-1/2}$. Therefore, the matrix elements of $\hat{U}_X$ are divergent without the counterterm. Expansion of the square bracket for small t_r gives,

$$\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{Z(x_1)}{P^{+2}}\frac{P^+}{\Delta k^2 + m_g^2}\sqrt{\frac{\pi}{2t_r}}\left[-\sqrt{\frac{2t_r}{\pi}}\left(\frac{\Delta k^2 + m_g^2}{P^+x_2} + \frac{\Delta k^2 + m_g^2}{P^+x_1}\right) + O(t_r^{3/2})\right]. \tag{B14}$$

Hence,

$$\lim_{t_r\to 0^+}\int \frac{dx_{1'}d^2k^\perp_{1'2'}}{16\pi^3}Y_3 = -\int \frac{d^2k^\perp_{1'2'}}{16\pi^3}\frac{Z(x_1)}{p_1^+p_2^+}. \tag{B15}$$

Since $Z(x_1) \to 0$ when $m_g \to 0$,

$$\lim_{m_g\to 0^+}\lim_{t_r\to 0^+}\int \frac{dx_{1'}d^2k^\perp_{1'2'}}{16\pi^3}Y_3 = 0. \tag{B16}$$

Summarizing,

$$\lim_{m_g\to 0^+}\lim_{t_r\to 0^+}\langle L|\hat{U}_X|R\rangle = C_F g_r^2\tilde{\delta}_{P_L.P_R}\sum_{\sigma_1,\sigma_2}\sum_{\sigma_{1'},\sigma_{2'}}\int \frac{dx_1d^2k^\perp_{12}}{16\pi^3x_1x_2}\int \frac{dx_{1'}d^2k^\perp_{1'2'}}{16\pi^3x_{1'}x_{2'}}\psi_L^*(1,2)\frac{(\mathscr{M}_{12}^2 - \mathscr{M}_{1'2'}^2)^2}{4q_1^2q_2^2}\frac{f_t j_1^+ j_2^+}{P^{+2}}\psi_R(1',2'). \tag{B17}$$

This formula could be obtained by first taking the limit $t_r \to 0$ in U_X, neglecting the counterterm and the divergences, and then taking the limit $m_g \to 0$. Such procedure is valid only if the vicinity of $q^+ = 0$ is excluded. However, the above analysis shows that if one includes the counterterm the result is the same. Thus, we can write

$$\lim_{m_g\to 0^+}\lim_{t_r\to 0^+} U_X = -f_t\sqrt{x_1x_2x_{1'}x_{2'}}\frac{(\mathscr{M}^2_{12}-\mathscr{M}^2_{1'2'})^2}{q_1^2q_2^2}\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}}, \quad \text{(B18)}$$

and in the nonrelativistic limit,

$$V_X = \lim_{\mathrm{NR}}\lim_{m_g\to 0^+}\lim_{t_r\to 0^+}\frac{U_X}{4m_1m_2} \quad \text{(B19)}$$

$$= -f_t\frac{(m_1+m_2)^2}{4(m_1m_2)^2}\frac{(\vec{k}^2-\vec{k}'^2)^2}{(\vec{q}^2)^2}\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}} = O\left(\frac{1}{\mu^2}\right). \quad \text{(B20)}$$

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