Adv. Nonlinear Stud. 2024; 24(1): 206–221 **DE GRUYTER**

Research Article
Special Issue: In honor of Joel Spruck

William H. Meeks III and Joaquín Pérez*

Geometry of branched minimal surfaces of finite index

https://doi.org/10.1515/ans-2023-0118
Received November 9, 2022; accepted December 9, 2023; published online March 12, 2024

Abstract: Given $I, B \in \mathbb{N} \cup \{0\}$, we investigate the existence and geometry of complete finitely branched minimal surfaces M in $\mathbb{R}^3$ with Morse index at most I and total branching order at most B. Previous works of Fischer-Colbrie ("On complete minimal surfaces with finite Morse index in 3-manifolds," *Invent. Math.*, vol. 82, pp. 121–132, 1985) and Ros ("One-sided complete stable minimal surfaces," *J. Differ. Geom.*, vol. 74, pp. 69–92, 2006) explain that such surfaces are precisely the complete minimal surfaces in $\mathbb{R}^3$ of finite total curvature and finite total branching order. Among other things, we derive scale-invariant weak chord-arc type results for such an M with estimates that are given in terms of I and B. In order to obtain some of our main results for these special surfaces, we obtain general intrinsic monotonicity of area formulas for m-dimensional submanifolds Σ of an n-dimensional Riemannian manifold X, where these area estimates depend on the geometry of X and upper bounds on the lengths of the mean curvature vectors of Σ. We also describe a family of complete, finitely branched minimal surfaces in $\mathbb{R}^3$ that are stable and non-orientable; these examples generalize the classical Henneberg minimal surface.

Keywords: constant mean curvature; finite index H-surfaces; intrinsic monotonicity of area formula; area estimates for constant mean curvature surfaces; branched minimal surfaces of finite index; weak chord-arc results for minimal surfaces

Mathematics Subject Classification: Primary: 53A10; Secondary: 49Q05; 53C42

1 Introduction

Let X be a complete Riemannian 3-manifold with positive injectivity radius $\mathrm{Inj}(X)$. Let M be a complete immersed surface in X of constant mean curvature (CMC). The Jacobi operator of M is the Schrödinger operator

$$L = \Delta + |A_M|^2 + \mathrm{Ric}(N),$$

where Δ is the Laplace–Beltrami operator on M, $|A_M|^2$ is the square of the norm of its second fundamental form and $\mathrm{Ric}(N)$ denotes the Ricci curvature of X in the direction of the unit normal vector N to M; the index of M is the index of L,

***Corresponding author: Joaquín Pérez**, Department of Geometry and Topology and Institute of Mathematics (IMAG), University of Granada, 18071, Granada, Spain, E-mail: jperez@ugr.es
William H. Meeks III, Mathematics Department, University of Massachusetts, Amherst, MA 01003, USA, E-mail: profmeeks@gmail.com

$$\mathrm{Index}(M) = \lim_{R\to\infty} \mathrm{Index}(B_M(p,R)),$$

where $B_M(p,R)$ is the intrinsic metric ball in M of radius $R > 0$ centered at a point $p \in M$, and $\mathrm{Index}(B_M(p,R))$ is the number of negative eigenvalues of L on $B_M(p,R)$ with Dirichlet boundary conditions. Here, we have assumed that the immersion is two-sided (which is the case when the constant value H of the mean curvature of M is not zero). In the case, the immersion is one-sided, then the index is defined in a similar manner using compactly supported variations in the normal bundle; see Definition 3.2 for details.

Given $I, B \in \mathbb{N} \cup \{0\}$, we investigate the existence and geometry of complete finitely branched minimal surfaces M in $\mathbb{R}^3$ with index at most I and total branching order at most B; let $\mathcal{M}(I,B)$ be the space of such examples. Works of Fischer-Colbrie [1] and Ros [2] ensure that the surfaces

$$M \in \bigcup_{I,B\in\mathbb{N}\cup\{0\}} \mathcal{M}(I,B)$$

are precisely the complete minimal surfaces in $\mathbb{R}^3$ of finite total curvature and finite total branching order. One goal of this paper is to derive certain scale-invariant weak chord-arc type results for surfaces in $\mathcal{M}(I,B)$ with explicit estimates given in terms of I and B; see Proposition 4.1 for these estimates. We also describe some interesting new examples of non-orientable surfaces in $\mathcal{M}(0,B)$, $B \geq 2$. These new examples of complete stable branched minimal surfaces generalize the classical Henneberg surface of finite total curvature -2π that has two simple branch points; these surfaces are described analytically and geometrically at the end of Section 3. In Section 3 we also explain how to extend to $\mathcal{M}(I,B)$ the geometric and topological lower bound estimates for the index of complete unbranched minimal surfaces with finite total curvature due to Chodosh and Maximo [3].

In Section 2 we study the area of intrinsic balls $B_M(x,R)$ of an n-dimensional submanifold of a Riemannian m-manifold M, where $x \in M$ and $0 < R \leq \mathrm{Inj}(X)$. In particular, we derive explicit upper bounds for the area growth of $B_M(x,R)$ as a function of $R \in (0, \mathrm{Inj}(X)]$, that depend on upper bounds for the sectional curvature of the extrinsic geodesic ball $B_X(x,R)$ and for the length of the mean curvature vector of M restricted to $B_M(x,R)$. In Section 4 we will apply this intrinsic area estimate to obtain certain scale-invariant weak chord-arc bounds for any surface $M \in \mathcal{M}(I,B)$; see Proposition 4.1.

The intrinsic area estimates in Section 2 will also be applied in our papers [4], [5] to study CMC surfaces of bounded index in spaces X of dimension three. The monotonicity-of-area type formulae in Proposition 2.4, the weak chord-arc results given in Proposition 4.1 and other theoretical results in Section 3, such as the aforementioned extension of the Chodosh and Maximo lower bound estimates for the index of surfaces in $\mathcal{M}(I,B)$, have important applications to the proof to the Hierarchy Structure Theorem 1.1 in Ref. [5]; this theorem is a fundamental result that describes the structure of complete CMC surfaces of finite index in a 3-dimensional X with $\mathrm{Inj}(X) > \delta > 0$ and having a fixed an upper bound on its absolute sectional curvature function, and it was our main motivation for developing the results in the present paper.

2 Volume growth of intrinsic balls in submanifolds of bounded mean curvature vector

Let M be an immersed n-dimensional submanifold in a geodesic ball $B_X(x_0, R_1)$ of an m-dimensional manifold (X, g), with $x_0 \in M$ and R_1 less or than equal to the injectivity radius function $\mathrm{Inj}_X(x_0)$ of X at x_0. In this section we will find lower bounds for the n-dimensional volume $A(r)$ of $B_M(x_0, r)$, as a function of $r \in (0, \mathrm{Inj}_X(x_0)]$; see Proposition 2.4 below for a precise description.

Let us denote by $\overline{\Delta}, \Delta$ the Laplacians in X and M, respectively. Analogously, $\overline{\nabla}, \nabla$ will stand for the Levi-Civita connections and gradient operators. Let $N_{n+1}, \dots, N_m$ be a local orthonormal basis of the normal bundle to M, and let $\vec{H}$ be the mean curvature vector of M. We start with a well-known formula.

Lemma 2.1. *Given $f \in C^\infty(X)$, $(\overline{\Delta} f)|_M = \Delta(f|_M) - n\,\vec{H}(f) + \sum_{j=n+1}^m g(\overline{\nabla}_{N_j}\overline{\nabla} f, N_j)$.*

Proof. Let $\{v_1, \dots, v_n\}$ be a local orthonormal basis for TM.

$$
\begin{aligned}
(\overline{\Delta} f)|_M &= \sum_{i=1}^{n} g(\overline{\nabla}_{v_i}\overline{\nabla} f, v_i) + \sum_{j=n+1}^{m} g(\overline{\nabla}_{N_j}\overline{\nabla} f, N_j) \\
&= \sum_{i=1}^{n} g\left(\overline{\nabla}_{v_i}\left(\nabla f + \sum_{j=n+1}^{m} N_j(f) N_j\right), v_i\right) + \sum_{j=n+1}^{m} g(\overline{\nabla}_{N_j}\overline{\nabla} f, N_j) \\
&= \sum_{i=1}^{n} g(\nabla_{v_i}\nabla f, v_i) + \sum_{j=n+1}^{m} N_j(f)\sum_{i=1}^{n} g(\overline{\nabla}_{v_i} N_j, v_i) + \sum_{j=n+1}^{m} g(\overline{\nabla}_{N_j}\overline{\nabla} f, N_j) \\
&= \Delta(f|_M) - n\,\vec{H}(f) + \sum_{j=n+1}^{m} g(\overline{\nabla}_{N_j}\overline{\nabla} f, N_j).
\end{aligned}
$$

□

Given $a \in \mathbb{R}$, let $s_a(t)$ be the unique solution of $x''(t) + a\,x(t) = 0$, $x(0) = 0$, $x'(0) = 1$. We will denote by I_a the interval $[0, \pi/\sqrt{a})$ when $a > 0$, and $I_a = [0, \infty)$ if $a \le 0$. Thus, $s_a(t) > 0$ for all $t \in I_a \setminus \{0\}$. Let $f_a\colon I_a \to \mathbb{R}$ be the smooth function given by

$$
f_a(t) = \frac{1}{t^2}\left(1 - t\frac{s_a'(t)}{s_a(t)}\right), \quad t \in I_a. \tag{2.1}
$$

A direct computation gives that

$$
f_a(t) = \begin{cases} \frac{1}{t^2}\left(1 - t\sqrt{a}\cot(\sqrt{a}t)\right) & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ \frac{1}{t^2}\left(1 - t\sqrt{-a}\coth(\sqrt{-a}t)\right) & \text{if } a < 0. \end{cases} \tag{2.2}
$$

The last equality implies that $f_a(t)$ is smooth at $t = 0$, with value $f_a(0) = a/3$.

Lemma 2.2. *Let $R\colon B_X(x_0, R_1) \to [0, R_1)$ denote the extrinsic Riemannian distance function in X to x_0.*

1. *The intrinsic Laplacian of the restriction of R^2 to M is*

$$
\Delta((R^2)|_M) = 2(m-1)R\,H^{S(R)} + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2 - 2R\sum_{j=n+1}^{m} \mathrm{II}^{S(R)}\left(N_j^T, N_j^T\right),
$$

   where $H^{S(R)}$ denotes the mean curvature of the geodesic sphere $S(R) = \partial B_X(x_0, R)$ with respect to the unit normal $-\overline{\nabla}R$, $N_j^T = N_j - N_j(R)\overline{\nabla}R$ is the projection of N_j tangent to $S(R)$, and $\mathrm{II}^{S(R)}$ is the second fundamental form of $S(R)$ with respect to $-\overline{\nabla}R$.

2. *If the sectional curvature of X satisfies $K_{\sec} \le a$ for some $a \in \mathbb{R}$, then*

$$
\Delta((R^2)|_M) \ge 2n + 2nR\,\vec{H}(R) - 2R^2 f_a(R)\left(n - |\nabla(R|_M)|^2\right), \tag{2.3}
$$

   and equality holds in (2.3) if $K_{\sec} = a$. In particular if X is flat, then

$$
\Delta((R^2)|_M) = 2n + 2nR\,\vec{H}(R). \tag{2.4}
$$

Remark 2.3. For $a = 0$, Equation (2.4) generalizes the well-known formula $\Delta((R^2)|_M) = 2n$ for minimal submanifolds of Euclidean space. Similarly, if we assume $K_{\sec} \le 0$, inequality (2.3) generalizes the inequality $\Delta((R^2)|_M) \ge 2n$ for minimal submanifolds given by Yau in Ref. [6, Equation (7.1)].

Proof. Lemma 2.1 applied to R^2 gives

$$\Delta((R^2)|_M) = (\overline{\Delta}(R^2))|_M + 2nR\,\vec{H}(R) - \sum_{j=n+1}^{m} g(\overline{\nabla}_{N_j}\overline{\nabla}(R^2), N_j). \tag{2.5}$$

We now compute the first and third terms of the last RHS. On the one hand, since $|\overline{\nabla}R| = 1$,

$$\overline{\Delta}(R^2) = 2 + 2R\,\overline{\Delta}R. \tag{2.6}$$

As $\overline{\nabla}R$ is unitary and orthogonal to the geodesic spheres centered at x_0, we can take an orthonormal basis of TX of the form $\{E_1, \dots, E_{m-1}, \overline{\nabla}R\}$ where $E_1, \dots, E_{m-1}$ is an orthonormal basis of the tangent space to $S(R)$. Thus,

$$\overline{\Delta}R = \sum_{i=1}^{m-1} g(\overline{\nabla}_{E_i}\overline{\nabla}R, E_i) + g(\overline{\nabla}_{\overline{\nabla}R}\overline{\nabla}R, \overline{\nabla}R).$$

The first term in the last RHS equals $(m-1)H^{S(R)}$, and the second term clearly vanishes. Thus,

$$\overline{\Delta}R = (m-1)H^{S(R)}, \tag{2.7}$$

and

$$\overline{\Delta}(R^2) \overset{(2.6)}{=} 2 + 2(m-1)R\,H^{S(R)}. \tag{2.8}$$

On the other hand,

$$g(\overline{\nabla}_{N_j}\overline{\nabla}(R^2), N_j) = 2g(\overline{\nabla}_{N_j}(R\overline{\nabla}R), N_j) = 2N_j(R)^2 + 2R\,g(\overline{\nabla}_{N_j}\overline{\nabla}R, N_j). \tag{2.9}$$

Decomposing $N_j = N_j^T + N_j(R)\overline{\nabla}R$ where N_j^T is tangent to $S(R)$, the bilinearity of the second term of the last RHS with respect to N_j allows us to write

$$g(\overline{\nabla}_{N_j}\overline{\nabla}R, N_j) = g\left(\overline{\nabla}_{N_j^T}\overline{\nabla}R, N_j^T\right) = \mathrm{II}^{S(R)}\left(N_j^T, N_j^T\right), \tag{2.10}$$

where we have used that $g(\overline{\nabla}_{\overline{\nabla}R}\overline{\nabla}R, \overline{\nabla}R) = 0$ and that $g\left(\overline{\nabla}_{N_j^T}\overline{\nabla}R, \overline{\nabla}R\right) = 0$ because $\overline{\nabla}R$ has constant length. From (2.5) and (2.8)–(2.10) we have

$$\Delta((R^2)|_M) = 2 + 2(m-1)R\,H^{S(R)} + 2nR\,\vec{H}(R) - 2\sum_{j=n+1}^{m} N_j(R)^2 - 2R\sum_{j=n+1}^{m} \mathrm{II}^{S(R)}\left(N_j^T, N_j^T\right). \tag{2.11}$$

Since $\overline{\nabla}R = \nabla(R|_M) + \sum_j N_j(R)N_j$, then

$$1 = |\overline{\nabla}R|^2 = |\nabla(R|_M)|^2 + \sum_{j=n+1}^{m} N_j(R)^2. \tag{2.12}$$

Plugging (2.12) into (2.11) we obtain item 1 of the lemma.

As for item 2, we will assume that $K_{\sec} \leq a$ for some $a \in \mathbb{R}$. Let $e_1, \dots, e_{m-1}$ be an orthonormal basis of principal directions of $T_xS(R)$, with respective principal curvatures $\lambda_1, \dots, \lambda_{m-1}$ with respect to the unit normal $-\overline{\nabla}R$ to $S(R)$. For each $j = n+1, \dots, m$ we can write $N_j^T = \sum_{i=1}^{m-1} a_{ij}e_i$ where $a_{ij} = g\left(e_i, N_j^T\right) = g(e_i, N_j) \in \mathbb{R}$. Thus,

$$(m-1)H^{S(R)} = \sum_{i=1}^{m-1} \lambda_i \qquad \text{and} \qquad \mathrm{II}^{S(R)}\left(N_j^T, N_j^T\right) = \sum_{i=1}^{m-1} \lambda_i a_{ij}^2.$$

Hence, we can write the formula in item 1 of the lemma as

$$\Delta((R^2)|_M) = 2R\sum_{i=1}^{m-1} \lambda_i\left(1 - \sum_{j=n+1}^{m} a_{ij}^2\right) + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2. \tag{2.13}$$

Observe that given any tangent vector v to $S(R)$,

$$\mathrm{II}^{S(R)}(v,v) = g(\overline{\nabla}_v \overline{\nabla} R, v) = (\overline{\nabla}^2 R)(v,v), \tag{2.14}$$

where $\overline{\nabla}^2 R$ denotes the hessian of R. Since $K_{\mathrm{sec}} \le a$, standard comparison results (see e.g. Ref. [7, Theorem 27]) give

$$\frac{s_a'(R)}{s_a(R)}\, g_R \le \overline{\nabla}^2 R, \tag{2.15}$$

where g_R is the induced metric by g on $S(R)$. Evaluating (2.15) at the principal directions e_i, we have

$$\frac{s_a'(R)}{s_a(R)} \le \lambda_i, \qquad \text{for all } i = 1, \dots, m-1. \tag{2.16}$$

Given $i = 1, \dots, m-1$, we decompose e_i in its tangent and normal components to M as

$$e_i = e_i^{T,M} + \sum_{j=n+1}^{m} g(e_i, N_j) N_j = e_i^{T,M} + \sum_{j=n+1}^{m} a_{ij} N_j,$$

from where

$$1 = |e_i|^2 \ge \left| \sum_{j=n+1}^{m} a_{ij} N_j \right|^2 = \sum_{j=n+1}^{m} a_{ij}^2.$$

This last inequality together with (2.13) and (2.16), give

$$\begin{aligned}
\Delta((R^2)|_M) &\ge 2R\frac{s_a'(R)}{s_a(R)} \sum_{i=1}^{m-1} \left(1 - \sum_{j=n+1}^{m} a_{ij}^2\right) + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2 \\
&= 2R\frac{s_a'(R)}{s_a(R)} \left(m - 1 - \sum_{j=n+1}^{m} |N_j^T|^2\right) + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2 \\
&\overset{(2.1)}{=} 2\big(1 - R^2 f_a(R)\big)\left(m - 1 - \sum_{j=n+1}^{m} |N_j^T|^2\right) + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2 \\
&= 2(m-1) - 2\sum_{j=n+1}^{m} |N_j^T|^2 - 2R^2 f_a(R)\left(m - 1 - \sum_{j=n+1}^{m} |N_j^T|^2\right) \\
&\quad + 2nR\,\vec{H}(R) + 2|\nabla(R|_M)|^2 \\
&\overset{(*)}{=} 2n - 2R^2 f_a(R)\big(n - |\nabla(R|_M)|^2\big) + 2nR\,\vec{H}(R),
\end{aligned}$$

where in (*) we have used that

$$1 - |\nabla(R|_M)|^2 + \sum_{j=n+1}^{m} |N_j^T|^2 \overset{(2.12)}{=} \sum_{j=n+1}^{m} N_j(R)^2 + \sum_{j=n+1}^{m} |N_j^T|^2 = \sum_{j=n+1}^{m} |N_j|^2 = m - n.$$

Now inequality (2.3) is proved. If $K_{\mathrm{sec}} = a$, then both (2.15) and (2.16) are equalities, and the above argument shows that (2.3) is also an equality. In the case X is flat, then $a = 0$ and $f_0(t) = 0$, which gives (2.4). □

The next result generalizes the classical monotonicity of area formula of Allard [8, Section 5.1] for hypersurfaces of bounded mean curvature, in part since it does not require the hypersurface to be proper in the ambient space. Proposition 2.4 is motivated by the calculations in the last two pages of Yau [6], where he derived the lower bound area estimate given in (2.18) when $a \leq 0, H_0 = 0$.

Proposition 2.4. (Intrinsic monotonicity of area formula). *Let $\overline{B}_X(x_0, R_1)$ denote a closed geodesic ball in an m-dimensional manifold (X, g), where $0 < R_1 \leq \mathrm{Inj}_X(x_0)$, and suppose that $K_{\mathrm{sec}} \leq a$ on $B_X(x_0, R_1)$ for some $a \in \mathbb{R}$. Given $H_0 \geq 0$, define*

$$R_0(a, H_0) = \begin{cases} \dfrac{1}{\sqrt{a}} arc\ \cot\left(\dfrac{H_0}{\sqrt{a}}\right) & \text{if } a > 0, \\ 1/H_0 & \text{if } a = 0 \quad (\text{if } H_0 = 0 \text{ we take } R_0(0,0) = \infty) \\ \dfrac{1}{\sqrt{-a}} arc\ \coth\left(\dfrac{H_0}{\sqrt{-a}}\right), & \text{if } a < 0 \quad \left(\text{if } \dfrac{H_0}{\sqrt{-a}} \geq 1 \text{ we take } R_0(a, H_0) = \infty\right), \end{cases} \tag{2.17}$$

and let $r_1 = r_1(R_1, a, H_0) = \min\{R_1, R_0(a, H_0)\}$.

Suppose M is a complete, immersed, connected n-dimensional submanifold of X and $x_0 \in M$ is a point such that when $\partial M \neq \emptyset$, $d_M(x_0, \partial M) \geq R_1$ and the length of the mean curvature vector $\vec{H}$ of M restricted to $\overline{B}_X(x_0, R_1)$ is bounded from above by H_0. Then:

1. *If M is compact without boundary, then there exists $y \in M$ such that the extrinsic distance from x_0 to y is greater than or equal to r_1.*
2. *The n-dimensional volume $A(r)$ of $B_M(x_0, r)$ is a strictly increasing function of $r \in (0, r_1]$.*
3. *For all $r \in (0, r_1]$ when $r_1 \neq \infty$ or otherwise, for all $r \in (0, \infty)$:*

$$A(r) \geq \begin{cases} \omega_n\, r^n e^{-nH_0 r} & \text{if } a \leq 0, \\ \omega_n\, r^n e^{-nr\left(H_0 + \frac{1}{2} f_a(r_1) r\right)} & \text{if } a > 0, \end{cases} \tag{2.18}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and the function f_a is defined in (2.2).

Proof. Let $\mathbb{M}^m(a)$ denote the m-dimensional, simply-connected space form of constant sectional curvature $a \in \mathbb{R}$. Recall that the number $R_0(a, H_0)$ represents the radius of a geodesic sphere in $\mathbb{M}^m(a)$ with constant mean curvature H_0, and that geodesic spheres in $\mathbb{M}^m(a)$ of radii less than $R_0(a, H_0)$ have mean curvature greater than H_0.

We first prove item 1 of the lemma. Fix a point $x_0 \in M$ and let $r \in (0, r_1)$. Suppose M is compact with empty boundary and suppose that there does not exist a point $y \in M$ such that the extrinsic distance from x_0 to y is greater than r_1; in this case we have $M \subset \overline{B}_X(x_0, r_1)$. Since $r_1 \leq R_1 \leq \mathrm{Inj}_X(x_0)$, all the distance spheres $\partial B_X(x_0, r)$ with $r \in (0, r_1)$ are geodesic spheres. Since the absolute sectional curvature of X is bounded by a, comparison results imply that $\partial B_X(x_0, r)$ has normal curvatures greater than H_0 because $r < R_0(a, H_0)$ in this case. Assume for the moment that M is contained in $\overline{B}_X(x_0, r)$. As M is closed, there exists a largest $r_2 \in (0, r]$ such that $M \subset \overline{B}_X(x_0, r_2)$, and by compactness of M there exists a point $x \in M \cap \partial B_X(x_0, r_2)$. Therefore, all the normal curvatures of M at x are greater than H_0, which implies that the length of the mean curvature vector of M is greater than H_0, thereby contradicting one of the hypotheses on M. This contradiction proves that $M(x_0)$ cannot be contained in $\overline{B}_X(x_0, r)$. Since this non-containment equation holds for every $r \in (0, r_1)$ and M is compact, we conclude that M cannot be contained in $B_X(x_0, r_1)$. Item 1 is now proved.

To see that item 2 holds, consider two values $r_2 < r_3$ in $[0, r_1]$. By item 1 and the hypotheses on M, then $B_M(x_0, r_3) \backslash \overline{B}_M(x_0, r_2)$ is a non-empty open subset of M, hence $A(r_2) < A(r_3)$.

It remains to prove the lower bound estimates for $A(r)$ given in item 3. In what follows, we only consider values $r \in (0, r_1]$. By Stokes' Theorem,

$$\int_{B_M(x_0,r)} \Delta((R^2)|_M) \leq \int_{\partial B_M(x_0,r)} |\nabla(R^2)| = 2 \int_{\partial B_M(x_0,r)} R|\nabla R| \leq 2r\, l(r), \tag{2.19}$$

where $l(r) = \text{Volume}(\partial B_M(x_0, r)) = A'(r)$ is the $(n-1)$-dimensional volume of $\partial B_M(x_0, r)$.

Since $K_{\sec} \leq a$, inequality (2.3) implies that

$$\int_{B_M(x_0,r)} \Delta((R^2)|_M) \geq 2n\, A(r) + 2n \int_{B_M(x_0,r)} R\, \vec{H}(R) - 2 \int_{B_M(x_0,r)} R^2 f_a(R)\big(n - |\nabla(R|_M)|^2\big). \tag{2.20}$$

Since $R \leq r$ and $|\vec{H}(R)| = |\langle \vec{H}, \overline{\nabla} R\rangle| \leq |\vec{H}| \leq H_0$, we have $R\, \vec{H}(R) \geq -H_0 r$, and thus,

$$\int_{B_M(x_0,r)} R\, \vec{H}(R) \geq -H_0 r\, A(r). \tag{2.21}$$

Next we analyze the second integral in the RHS of (2.20).

If $a = 0$, then $f_a \equiv 0$ and the second integral in the RHS of (2.20) vanishes. In this case, (2.19)–(2.21) give

$$2r\, A'(r) = 2r\, l(r) \geq \int_{B_M(x_0,r)} \Delta((R^2)|_M) \geq 2n\, A(r) - 2nH_0 r\, A(r), \tag{2.22}$$

that is,

$$\frac{A'(r)}{A(r)} \geq \frac{n}{r} - nH_0 \qquad \forall r \in (0, r_1],$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{A(r)}{r^n e^{-nH_0 r}}\right) \geq 0,$$

hence the function

$$r \mapsto \frac{A(r)}{r^n e^{-nH_0 r}}$$

is non-decreasing for $r \in (0, r_1]$. Since the limit as $r \to 0^+$ of this last function is ω_n, we deduce:

$$\text{If } a = 0, \text{ then } A(r) \geq \omega_n r^n e^{-nH_0 r} \quad \text{for all } r \in (0, r_1].$$

In fact, the last estimate holds if $a \leq 0$, because $f_a \leq 0$ in $(0, \infty)$ in this case, and hence,

$$-2 \int_{B_M(x_0,r)} R^2 f_a(R)\big(n - |\nabla(R|_M)|^2\big) \geq 0 \tag{2.23}$$

so the same computations of case $a = 0$ are valid for $a \leq 0$.

We next study the case $a > 0$. Now $f_a(t)$ is strictly positive, increasing in the interval $I_a = [0, \pi/\sqrt{a})$ and limits to $a/3 > 0$ as $t \to 0^+$ and to $+\infty$ as $t \to (\pi/\sqrt{a})^-$. Whenever $r \in (0, r_1]$,

$$\begin{aligned} 2r\, A'(r) &\overset{(2.19)}{\geq} \int_{B_M(x_0,r)} \Delta((R^2)|_M) \\ &\overset{(2.20),(2.21)}{\geq} 2n\, A(r) - 2nH_0 r\, A(r) - 2 \int_{B_M(x_0,r)} R^2 f_a(R)\big(n - |\nabla(R|_M)|^2\big) \\ &\overset{(A)}{\geq} 2n\big[1 - r^2 f_a(r_1)\big] A(r) - 2nrH_0\, A(r) \end{aligned} \tag{2.24}$$

where in (A) we have bounded $R \le r$, $f_a(R) \le f_a(r_1)$ and $n - |\nabla(R|_M)|^2 \le n$.

Finally, (2.24) implies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{A(r)}{r^n e^{-nr\left(H_0+\frac{1}{2}f_a(r_1)r\right)}}\right) \ge 0;$$

hence the function

$$r \mapsto \frac{A(r)}{r^n e^{-nr\left(H_0+\frac{1}{2}f_a(r_1)r\right)}}$$

is non-decreasing for $r \in [0, r_1]$. Since the limit as $r \to 0^+$ of this last function is ω_n, we deduce the inequality (2.18) in the case where $a > 0$; thus, the proposition is proved. □

Remark 2.5. 1. Proposition 2.4 holds regardless whether or not the normal bundle of the submanifold M is trivial, since item 2 of Lemma 2.2 does not depend on whether or not the normal bundle of M admits a global trivialization. (note: Please re-number items 1,2,3 below as 2,3,4, and indent all four items equally)

1. The proof of Proposition 2.4 shows that if M has local density $k \in \mathbb{N}$ at x_0, then the RHS in (2.18) can be replaced by k times the same expression.
2. In the case $a > 0$, it holds that $A(r) \ge \omega_n r^n e^{-nr(H_0+\frac{1}{2}f_a(r)r)}$ for every $r \in (0, r_1]$. This can be proved by following the same proof for values $r \le r_1'$ where r_1' is any number less than or equal to r_1.
3. If $H_0 \ne 0$ or $a \ne 0$, the inequality (2.18) is strict.

Corollary 2.6. *Let $R_1 > 0$, $a \in \mathbb{R}$ and $H_0 \ge 0$, and suppose that X is a complete Riemannian m-dimensional manifold with injectivity radius at least $R_1 > 0$ and $K_{\mathrm{sec}} \le a$. If $M \looparrowright X$ is a complete, non-compact immersed n-dimensional submanifold with empty boundary and the mean curvature vector $\vec{H}$ of M satisfies $|\vec{H}| \le H_0$, then M has infinite volume.*

Proof. Let $r_1 > 0$ be the number given by Proposition 2.4. Observe that by Proposition 2.4, the n-dimensional volume of each component of M is at least $A(r_1) > 0$. Therefore, if M has infinitely many components, then M has infinite volume. So assume that M has a finite number of components. Since M is non-compact, then we can replace M by a non-compact component. Take a point $x_0 \in M$ and let $\gamma\colon [0,\infty) \to M$ a length-minimizing ray starting at x_0 and parameterized by arc length. Consider the pairwise disjoint intrinsic balls $B_M(\gamma(2kr_1), r_1)$, $k \in \mathbb{N}$. Since each of these balls has volume at least $A(r_1)$ by Proposition 2.4, then we conclude that M has infinite volume. □

Proposition 2.7. *Given $R_1 > 0$, $a \in \mathbb{R}$ and $H_0 \ge 0$, there exists $r_2 = r_2(R_1, a, H_0) \in (0, r_1]$ (here r_1 is given by Proposition 2.4) such that if X is a complete Riemannian 3-manifold with injectivity radius at least $R_1 > 0$ and $K_{\mathrm{sec}} \le a$, and if $M \looparrowright X$ is a complete, connected immersed surface with boundary, whose mean curvature vector $\vec{H}$ satisfies $|\vec{H}| \le H_0$, then for all $p \in \mathrm{Int}(M)$ we have*

$$\mathrm{Area}[B_M(p,r)] \ge 3r^2, \quad \textit{whenever } 0 < r \le \min\{r_2, d_M(p,\partial M)\}. \tag{2.25}$$

Furthermore, given $\varepsilon_0 > 0$ define $C_A = \min\left\{\varepsilon_0, \frac{r_2^2}{\varepsilon_0}\right\}$. If $p \in M$ satisfies $d_M(p,\partial M) \ge \varepsilon_0$, then

$$\mathrm{Area}[B_M(p, d_M(p,\partial M))] \ge C_A\, d_M(p,\partial M) \tag{2.26}$$

and

$$\mathrm{Area}[B_M(p,\varepsilon_0)] \ge C_A\, \varepsilon_0, \tag{2.27}$$

Proof. First suppose that $a > 0$. By (2.18), we have that whenever $0 < r \le \min\{r_1, d_M(p,\partial M)\}$,

$$\mathrm{Area}[B_M(p,r)] \ge \pi\, r^2 e^{-2r\left(H_0+\frac{1}{2}f_a(r_1)r\right)} = \phi(r) r^2, \tag{2.28}$$

where $\phi(r) = \pi e^{-2r(H_0+\frac{1}{2}f_a(r_1)r)}$ for all $r > 0$. Choose $r_2 = r_2(R_1, a, H_0) \in (0, r_1]$ such that $\phi(r_2) \geq 3$, which can be done since ϕ is continuous and $\phi(0) = \pi$. As $r > 0 \mapsto \phi(r)$ is decreasing, we have that if $0 < r \leq (0, \min\{r_2, d_M(p, \partial M)\}$, then

$$\text{Area}[B_M(p,r)] \overset{(2.28)}{\geq} \phi(r)r^2 \geq \phi(r_2)r^2 \geq 3r^2,$$

which proves (2.25) assuming $a > 0$. The proof of (2.25) when $a \leq 0$ is similar and we leave it for the reader.

Next assume that $p \in M$ satisfies $d_M(p, \partial M) \geq \varepsilon_0$, and we will show that (2.26) and (2.27) hold. Let $\gamma\colon [0, d_M(p, \partial M)) \to M$ be a minimizing geodesic from p to ∂M, parameterized by arc length. Choose the largest $k \in \mathbb{N}$ such that

$$(2k-1)\varepsilon_0 \leq d_M(p, \partial M) < (2k+1)\varepsilon_0 \leq 3k\varepsilon_0. \tag{2.29}$$

By the triangle inequality, the collection $\mathcal{B} = \{B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)\}_{i=1}^k$ is pairwise disjoint and $\cup\mathcal{B}$ is contained in $B_M(p, d_M(p, \partial M))$; hence,

$$\text{Area}[B_M(p, d_M(p, \partial M))] \geq \sum_{i=1}^{k} \text{Area}\big(B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)\big). \tag{2.30}$$

Also observe that given $i \in \{1, \dots, k\}$, (2.29) implies

$$\varepsilon_0 \leq d_M(\gamma(2(i-1)\varepsilon_0), \partial M). \tag{2.31}$$

We next prove (2.26) and (2.27) by consideration of two cases.

- Suppose $\varepsilon_0 \leq r_2$. By (2.31), for each $i \in \{1, \dots, k\}$ we have

$$\varepsilon_0 \leq \min\{r_2, d_M(\gamma(2(i-1)\varepsilon_0), \partial M)\}.$$

  The last inequality allows us to use (2.25) to conclude that

$$\text{Area}[B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)] \geq 3\varepsilon_0^2. \tag{2.32}$$

  Note that $C_A = \varepsilon_0$ in this case. Taking $i = 1$ in (2.32), we have $\text{Area}[B_M(p, \varepsilon_0)] \geq 3\varepsilon_0^2 > \varepsilon_0^2 = C_A\varepsilon_0$, hence (2.27) holds. As the collection $\mathcal{B}$ is pairwise disjoint, (2.30) and (2.32) imply

$$\text{Area}[B_M(p, d_M(p, \partial M))] \geq 3k\varepsilon_0^2 = 3kC_A\varepsilon_0 \overset{(2.29)}{\geq} C_A\, d_M(p, \partial M),$$

  hence (2.26) also holds in this case.
- Suppose $\varepsilon_0 > r_2$. By (2.31), for each $i \in \{1, \dots, k\}$ we have $r_2 < d_M(\gamma(2(i-1)\varepsilon_0), \partial M)$; hence (2.25) implies that

$$\text{Area}[B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)] \geq 3r_2^2. \tag{2.33}$$

  Since $d_M(p, \partial M) < 3k\varepsilon_0$ and $C_A = \frac{r_2^2}{\varepsilon_0}$ in this case,

$$\text{Area}[B_M(p, d_M(p, \partial M))] \geq 3kr_2^2 = 3kC_A\varepsilon_0 \overset{(2.29)}{>} C_A d_M(p, \partial M),$$

  which proves that (2.26). The inequality (2.27) follows from (2.26) after replacing M by the closure of $B_M(p, \varepsilon_0)$.

□

Remark 2.8. A straightforward adaptation of the proof of Proposition 2.7 gives a related statement and proof for any n-dimensional submanifold M, with a fixed upper bound on the length of its mean curvature vector field, in a Riemannian m-manifold X which has injectivity radius at least $R_1 > 0$ and sectional curvature bounded from above by some $a \in \mathbb{R}$; in this setting, $3r^2$ in (2.25) is replaced by $c_n r^n$, where c_n is any positive number less than ω_n.

3 Index of finitely branched minimal surfaces in $\mathbb{R}^3$

Definition 3.1. Let Σ be a smooth surface endowed with a conformal class of metrics. We say that a harmonic map $f: \Sigma \to \mathbb{R}^3$ is a (possibly non-orientable) branched minimal surface if it is a conformal immersion outside of a locally finite set of points $\mathcal{B}_\Sigma \subset \Sigma$, where f fails to be an immersion. Points in $\mathcal{B}_\Sigma$ are called branch points of f. It is well-known (see e.g. Micallef and White [9, Theorem 1.4]) that given $p \in \mathcal{B}_\Sigma$, there exist a conformal coordinate $(\overline{\mathbb{D}}, z)$ for Σ centered at p (here $\overline{\mathbb{D}}$ is the closed unit disk in the plane), a diffeomorphism u of $\overline{\mathbb{D}}$ and a rotation ϕ of $\mathbb{R}^3$ such that $\phi \circ f \circ u$ has the form

$$z \mapsto (z^q, x(z)) \in \mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3$$

for z near 0, where $q \in \mathbb{N}$, $q \geq 2$, x is of class C^2, and $x(z) = o(|z|^q)$. The branching order $B(p) \in \mathbb{N}$ is defined to be $q - 1$. The total branching order of f is

$$B(\Sigma) := \sum_{p \in \mathcal{B}_\Sigma} B(p).$$

Definition 3.2. Given a 1-sided minimal immersion $F: M \looparrowright X$, let $\widetilde{M} \to M$ be the two-sided cover of M and let $\tau: \widetilde{M} \to \widetilde{M}$ be the associated deck transformation of order 2. Denote by $\widetilde{\Delta}$, $|\widetilde{A}|^2$ the Laplacian and squared norm of the second fundamental form of $\widetilde{M}$, and let $N: \widetilde{M} \to TX$ be a unitary normal vector field. The index of F is defined as the number of negative eigenvalues of the elliptic, self-adjoint operator $\widetilde{\Delta} + |\widetilde{A}|^2 + \mathrm{Ric}(N, N)$ defined over the space of compactly supported smooth functions $\phi: \widetilde{M} \to \mathbb{R}$ such that $\phi \circ \tau = -\phi$.

We next recall a fundamental lower bound for the index $I(f)$ of a connected, complete, possibly finitely branched minimal surface $f: \Sigma \looparrowright \mathbb{R}^3$ with finite total curvature, which is due to Chodosh and Maximo [3], and to Karpukhin [10]:

$$3I(f) \geq \begin{cases} 2g(\Sigma) + 2\sum_{j=1}^{e}(d_j + 1) - 2B - 5 & \text{if } \Sigma \text{ is orientable,} \\ g(\widetilde{\Sigma}) + 2\sum_{j=1}^{e}(d_j + 1) - 2B - 4 & \text{if } \Sigma \text{ is non} - \text{orientable,} \end{cases} \tag{3.1}$$

where $g(\Sigma)$ is the genus of Σ if Σ is orientable (resp. $g(\widetilde{\Sigma})$ is the genus of the orientable cover $\widetilde{\Sigma}$ of Σ if Σ is not orientable[1]), e and B are respectively the number of ends and the total branching order of Σ, and for each end E_j of Σ, d_j is the multiplicity of E_j as a multi-graph over the limiting tangent plane of E_j.

Inequality (3.1) has not been explicitly stated in the literature, so an explanation is in order. Ros [2] proved that $3I(f) \geq 2g(\Sigma)$ using harmonic square integrable 1-forms on Σ for a minimal immersion $f: \Sigma \looparrowright \mathbb{R}^3$ with finite total curvature, in order to produce test functions for the index operator of f. Chodosh and Maximo [3, Theorem 1] improved Ros' technique with an enlarged space of harmonic 1-forms which admit certain singularities at the ends of Σ that take care of the spinning (multiplicity) of each end of such an immersion f, obtaining a simplified version of (3.1) without the term $-2B$. Finally, Karpukhin [10, Proposition 2.3 and Remark 2.4] included the study of branch points although he made use of the original space of $L^2(\Sigma)$ harmonic 1-forms considered by Ros. Equation (3.1) is the combined inequality that one can deduce from Refs. [3], [10].

The class of complete, non-flat, finitely branched, stable minimal surfaces in $\mathbb{R}^3$ contains an interesting non-trivial family of surfaces, as we explain next.

1 If Σ is a compact non-orientable surface and $\widehat{\Sigma} \xrightarrow{2:1} \Sigma$ denotes the oriented cover of Σ, then the genus of $\widehat{\Sigma}$ plus 1 equals the number of cross-caps in Σ.

1. Any non-orientable, complete, finitely branched minimal surface $f\colon\Sigma \nrightarrow \mathbb{R}^3$ with finite total curvature, whose extended unoriented Gauss map $G\colon\mathbb{P}^2 \to \mathbb{P}^2$ is a diffeomorphism, is stable (observe that the conformal compactification of Σ must be $\mathbb{P}^2$). We prove this property by contradiction: if f is not stable, then the first eigenvalue λ_1 of the Jacobi operator on Σ is negative, which implies that there exists an eigenfunction $\phi\colon\mathbb{S}^2 \to \mathbb{R}$ of the lifted Jacobi operator on the orientable cover $\pi\colon\widetilde{\Sigma} \to \Sigma$ of Σ so that $\phi\circ\tau = -\phi$ and $L\phi + \lambda\phi = 0$ on $\widetilde{\Sigma}$, where $\lambda < 0$ and $\tau\colon\mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map. Let Ω be a component of $\phi^{-1}(0,\infty)$. As ϕ is odd, $\tau(\Omega) \subset \phi^{-1}(-\infty,0)$ and so, $\pi|_\Omega\colon\ \Omega \to \pi(\Omega)$ is a diffeomorphism. In particular, $\pi(\Omega)$ is an orientable domain in Σ. Since G is also a diffeomorphism, $G(\pi(\Omega))$ is an orientable domain in $\mathbb{P}^2$. Thus, $G(\pi(\Omega))$ lifts to two disjoint diffeomorphic domains in $\mathbb{S}^2$ of the form $g(\Omega)$, $(g\circ\tau)(\Omega)$ (here $g\colon\widetilde{\Sigma} \to \mathbb{S}^2$ is the Gauss map of $\widetilde{\Sigma}$). In particular, $\mathrm{Area}((g\circ\tau)(\Omega)) = \mathrm{Area}(g(\Omega)) \le 2\pi$, which implies that the first eigenvalue of the Jacobi operator L on Ω is non-negative. This is a contradiction, as the first Dirichlet eigenvalue of L on Ω (defined as the supremum of the first Dirichlet eigenvalues of L on a increasing sequence of compact smooth domains $\Omega_i \nearrow \Omega$) is $\lambda < 0$. This contradiction proves that Σ is stable.
2. Using the Weierstrass representation for non-orientable minimal surfaces in Ref. [11], the classical Henneberg minimal surface given by the Weierstrass data[2] on its oriented covering $\mathbb{C}\backslash\{0\}$

$$g(z) = z, \quad \omega = z^{-4}(z^4 - 1)\,\mathrm{d}z,$$

is a non-orientable, complete branched minimal surface $f\colon\mathbb{P}^2\backslash\{0,\infty\} \nrightarrow \mathbb{R}^3$ with two branch points of order 1 at $\{1,-1\}, \{i,-i\} \in \mathbb{P}^2$ and a single end of spinning 3 at $\{0,\infty\}$. Since its extended Gauss map is a diffeomorphism from $\mathbb{P}^2$ to $\mathbb{P}^2$, then the Henneberg minimal surface $H_1 = f(\mathbb{P}^2\backslash\{0,\infty\})$ is stable. After translating the surface in $\mathbb{R}^3$ so that $f(\mathrm{e}^{\mathrm{i}\pi/4}) = \vec{0}$, the branch points are mapped by f into $\pm(0,0,1)$.

Henneberg's surface can be generalized as follows. Given an odd integer $m \in \mathbb{N}$, consider the following Weierstrass data on $\mathbb{C}\backslash\{0\}$,

$$g(z) = z, \quad \omega = z^{-(3+m)}(z^{2m+2} - 1)\,\mathrm{d}z,$$

which produces a two-sheeted cover of a complete minimal Mobius strip $f\colon\mathbb{P}^2\backslash\{0,\infty\} \nrightarrow \mathbb{R}^3$ which is stable with $m+1$ branch points of order 1 at the $(m+1)$ pairs of antipodal $(2m+2)$-roots of unity and a single end of spinning $m+2$ at $\{0,\infty\}$. Henneberg's minimal surface corresponds to the case $m = 1$. After translating the surface $H_m = f(\mathbb{P}^2\backslash\{0,\infty\})$ in $\mathbb{R}^3$ so that $f(\mathrm{e}^{\mathrm{i}\frac{\pi}{2(m+1)}}) = \vec{0}$, the branch points of H_m are located at $(0,0,\pm\frac{2}{m+1})$, and a parameterization of H_m in polar coordinates is

$$f(re^{i\theta}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{r^m\cos(m\theta)}{m} + \frac{\cos((m+2)\theta)}{(m+2)r^{m+2}}\right) - \frac{1}{2}\left(\frac{r^{m+2}\cos((m+2)\theta)}{m+2} + \frac{\cos(m\theta)}{mr^m}\right) \\ \frac{1}{2}\left(\frac{\sin((m+2)\theta)}{(m+2)r^{m+2}} - \frac{r^m\sin(m\theta)}{m}\right) - \frac{1}{2}\left(\frac{r^{m+2}\sin((m+2)\theta)}{m+2} - \frac{\sin(m\theta)}{mr^m}\right) \\ \frac{1}{m+1}\left(r^{m+1} + \frac{1}{r^{m+1}}\right)\cos((m+1)\theta) \end{pmatrix}.$$

We note that f maps each of the $m+1$ pairs of opposite half-lines

$$\left\{l_j = \left\{re^{\mathrm{i}\frac{\pi j}{2(m+1)}} \mid r > 0\right\}, -l_j\right\}$$

(for each j odd) into a horizontal line of $\mathbb{R}^3$ that passes through $\vec{0}$, and the union L of these $m+1$ horizontal lines forms an equiangular system contained in H_m. Therefore, the reflection in $\mathbb{C}\backslash\{0\}$ about $l_j \cup (-l_j)$

2 This means that $f(z) = \mathrm{Re}\left(\int^z \left(\frac{1}{2}(1-g^2)\omega, \frac{i}{2}(1+g^2)\omega, g\omega\right)\right)$ parameterizes the surface.

induces a symmetry of H_m. Reflections in the $m+1$ vertical planes that bisect each of the angles between the lines in L are planes of symmetry of H_m. Rotations of angle π about each of the lines in L together with these $m+1$ planar reflections form the group of isometries of H_m (all of which extend to ambient isometries), which, when considered to be a subgroup of $O(3)$, is the antiprismatic group $A_{2(m+1)}$. In fact, every intrinsic isometry of H_m extends to an extrinsic isometry, since such an intrinsic isometry produces a conformal diffeomorphism of $\mathbb{C}\backslash\{0\}$ into itself that preserves the set of $(2m+2)$-roots of unity.

For odd $m \geq 3$, these generalized Henneberg surfaces H_m can be deformed to less symmetric examples of non-orientable, complete finitely branched stable minimal surfaces in $\mathbb{R}^3$ whose branch locus consists of $m+1$ pairs of antipodal points in $\mathbb{C}\backslash\{0, \infty\}$ (H_1 can be proven to be the unique such surface for $m = 1$); see Ref. [12] for a description and special properties of these deformed Henneberg-type examples.

4 Scale invariant weak chord-arc type estimates for branched minimal surfaces of finite index in $\mathbb{R}^3$

Proposition 4.1. *Given $I, B \in \mathbb{N} \cup \{0\}$, let $f\colon (\Sigma, p_0) \looparrowright (\mathbb{R}^3, \vec{0})$ be a complete, connected, pointed branched minimal surface with index at most I and total branching order at most B. Given $R > 0$, let Ω_R denote the component of $f^{-1}(\overline{\mathbb{B}}(R))$ that contains p_0. Then, the following scale-invariant estimates hold and depend only on I, B:*

1. *For any $p \in \Omega_R$,*
$$d_{\Omega_R}(p, \partial\Omega_R) < \widehat{L}R, \tag{4.1}$$
where $\widehat{L} = \sqrt{\frac{1}{2}(3I + 2B + 3)}$.
2. *If f is injective with image a plane, then the distance between any two points of Ω_R is less than or equal to $2R$. Otherwise, given points p, q in Ω_R,*
$$d_{\Omega_{2R}}(p, q) < \widehat{C}R, \tag{4.2}$$
where $\widehat{C} = \widehat{C}(I, B) = 8\widehat{L}^3 + 2\pi\widehat{L}^2 - 20\widehat{L} - \frac{\pi}{2}$. In particular, $\Omega_R \subset B_\Sigma(p, \widehat{C}R)$ for every $p \in \Omega_R$.

Proof. Since (4.1) and (4.2) are invariant under re-scaling, we do not lose generality by assuming $R = 1$. Let $f\colon (\Sigma, p_0) \looparrowright (\mathbb{R}^3, \vec{0})$ be a complete, connected, pointed branched minimal surface in $\mathbb{R}^3$ with index $I(f) \leq I$ and total branching order $B(\Sigma) \leq B$. Observe that such an f has finite total curvature [1], [2], [13]. Thus, f is proper and Ω_1 is compact with non-empty boundary $\partial\Omega_1$. Given a point $p \in \text{Int}(\Omega_1)$, let $L = d_{\Omega_1}(p, \partial\Omega_1)$ and consider a length minimizing geodesic arc parameterized by arc length $\gamma\colon [0, L] \to \Sigma$ joining $\gamma(0) = p$ to $\partial\Omega_1$. Observe that the intrinsic ball of center p and radius L satisfies $\overline{B}_\Sigma(p, L) \subset \Omega_1$. The intrinsic version of the monotonicity formula for minimal surfaces described in Proposition 2.4 applied to the particular case $m = 3, a = 0, R_1 = \infty, n = 2$ and $H_0 = 0$, gives that $\text{Area}[B_\Sigma(p, L)] \geq \pi L^2$ (with the notation of Proposition 2.4, the number $r_1 = r_1(R_1, a, H_0)$ equals ∞ in this case; observe that the proof of Proposition 2.4 works for branched minimal surfaces; also see the last page of Yau [6] for the special case $a \leq 0, H_0 = 0$ in inequality (2.18)). Hence,
$$\text{Area}(\Omega_1) \geq \text{Area}[B_\Sigma(p, L)] \geq \pi L^2. \tag{4.3}$$

Next we deduce an upper bound for $\text{Area}(\Omega_1)$. Inequality (3.1) implies that regardless of the orientability character of Σ, we have $3I \geq 3I(f) \geq 2S + 2e - 2B(\Sigma) - 5$, where e is the number of ends of Σ, and S is the total spinning of the ends. Hence,
$$2S \leq 3I - 2e + 2B(\Sigma) + 5 \leq 3I - 2e + 2B + 5. \tag{4.4}$$

As $e \geq 1$, we have
$$\text{Area}(\Omega_1) \leq \text{Area}[f^{-1}(\overline{\mathbb{B}}(1))] \overset{(\star)}{\leq} \pi S \overset{(4.4)}{\leq} \frac{\pi}{2}(3I + 2B + 3) = \pi\widehat{L}^2, \tag{4.5}$$

where in ($\star$) we have used that the asymptotic area growth of Σ in balls of large radius R is πSR^2 (see e.g. Ref. [14]) and the classical (extrinsic) monotonicity formula. Now, (4.3) and (4.5) give

$$d_{\Omega_1}(p, \partial\Omega_1) = L \leq \widehat{L} \tag{4.6}$$

for any $p \in \mathrm{Int}(\Omega_1)$, which implies that $d_{\Omega_R}(p, \partial\Omega_R) \leq \widehat{L}R$; notice that this last inequality is strict (otherwise $f(\Sigma)$ is a possibly branched plane passing through the origin by the extrinsic monotonicity formula, in which case the first inequality in (4.4) is strict). This implies that the inequality (4.1) is strict, and item 1 of Proposition 4.1 is proven.

In order to obtain item 2, we will need the following auxiliary property: If f is not an embedded plane, then

$$d_{\Omega_R}(p, q) \leq 2\widehat{L}(3I + 2B - 1)R + \frac{1}{2}\mathrm{Length}(\partial\Omega_R). \tag{4.7}$$

Observe that (4.7) is invariant under re-scaling. We will divide the proof of (4.7) into four claims.

Claim 4.2. For any $p, q \in \Omega_1$,

$$d_{\Omega_1}(p, q) \leq \sup_{p', q' \in \Omega_1} d_{\Omega_1}(p', q') = \lim_{r \searrow 1} \sup_{p'', q'' \in \Omega_r} d_{\Omega_r}(p'', q''). \tag{4.8}$$

Proof. The first inequality in (4.8) holds by definition of supremum and so Claim 4.2 reduces to checking that the equality part of (4.8) holds. For each $r \in (1, 2]$, let p_r, q_r be points of Ω_r such that

$$L_r := d_{\Omega_r}(p_r, q_r) = \sup_{p'', q'' \in \Omega_r} d_{\Omega_r}(p'', q'')$$

and let $\alpha_r\colon [0, L_r] \to \Omega_r \subset \Omega_2$ be a Lipschitz curve contained in Ω_r with Lipschitz constant 1 that realizes the minimum distance L_r in Ω_r between p_r, q_r. Taking a sequence $r_j \searrow 1$, after passing to a subsequence we obtain a limit Lipschitz curve α_1 of the α_{r_j} with Lipschitz constant 1 joining points $p_1, q_1 \in \Omega_1$ of positive length $L_1 := \lim_{r_j \searrow 1} L_{r_j}$. It straightforward to check that $L_1 = d_{\Omega_1}(p_1, q_1) = \sup_{p', q' \in \Omega_1} d_{\Omega_1}(p', q')$, which shows the equality part on the RHS of (4.8). □

Claim 4.3. If (4.7) holds whenever Ω_R is transverse to $\mathbb{S}^2(R)$ along its boundary, then (4.7) holds for $R = 1$ (and thus, it also holds for any $R > 0$).

Proof. This is a direct consequence of Claim 4.2 since almost all spheres centered at $\vec{0}$ are transverse to f by Sard's theorem. □

By Claim 4.3, we can reduce the proof of (4.7) to the case that $R = 1$ and f is transverse to $\mathbb{S}^2(1)$ along $\partial\Omega_1$. This transversality assumption implies that Ω_1 is a smooth, connected, compact surface with a finite set $\{\partial_1, \dots, \partial_b\}$ of boundary components, $b \in \mathbb{N}$.

Claim 4.4. For any $p, q \in \Omega_1$, $d_{\Omega_1}(p, q) \leq 2b\widehat{L} + \frac{1}{2}\mathrm{Length}(\partial\Omega_1)$.

Proof. Assuming $b > 1$, there is a geodesic arc $\alpha_1 \subset \Omega_1$ that minimizes the distance from ∂_1 to the set $\cup_{i=2}^{b}\partial_i$, and, possibly after re-indexing, we may assume that α_1 joins ∂_1 to ∂_2. Notice that the distance from the midpoint of α_1 to $\partial\Omega_1$ is half the length of α_1, and so (4.1) implies that the length of α_1 is less than $2\widehat{L}$. Assuming that $b > 2$, let α_2 be a minimizing geodesic in Ω_1 from $\partial_1 \cup \partial_2$ to the set $\cup_{i=3}^{b}\partial_i$, which also has length less than $2\widehat{L}$ by similar reasoning as in the case of α_1; again after possibly re-indexing, we can assume that the end point of α_2 which does not lie in $\partial_1 \cup \partial_2$ lies in ∂_3. Continuing inductively, we obtain a collection of arcs $\{\alpha_1, \alpha_2, \dots \alpha_{b-1}\}$ in Ω_1, each with length less than $2\widehat{L}$ and the set

$$C := \partial\Omega_1 \cup \alpha_1 \cup \ldots \cup \alpha_{b-1}$$

is path connected. Note that if $b = 1$, then $C = \partial\Omega_1 = \partial_1$.

For any pair of points $p', q' \in C$, the intrinsic distance $d_C(p', q')$ measured in C can be realized as the length of an embedded piecewise smooth arc in C consisting of arcs alternating between arcs in components of $\partial\Omega_1$ and arcs in $\alpha_1 \cup \ldots \cup \alpha_{b-1}$. In particular,

$$d_{\Omega_1}(p', q') \leq d_C(p', q') \leq 2(b-1)\widehat{L} + \frac{1}{2}\text{Length}(\partial\Omega_1). \tag{4.9}$$

Let p, q be points in Ω_1. Let $p', q' \in \partial\Omega_1$ be the end points of respective length-minimizing geodesics in Ω_1 joining p and q to $\partial\Omega_1$. Applying (4.1) to p and q together with the estimate in (4.9), we have

$$d_{\Omega_1}(p, q) \leq d_{\Omega_1}(p, \partial\Omega_1) + d_{\Omega_1}(q, \partial\Omega_1) + d_C(p', q') \leq 2b\widehat{L} + \frac{1}{2}\text{Length}(\partial\Omega_1),$$

which proves Claim 4.4. □

Claim 4.5. Inequality (4.7) holds.

Proof. Since (4.7) is invariant under re-scaling, it suffices to prove it for $R = 1$. By Claim 4.4, we have that (4.7) will follow by proving that

$$b \leq 3I(f) + 2B(\Sigma) - 1. \tag{4.10}$$

Recall that $f|_{\Omega_1}$ is transverse to $\partial\mathbb{B}(1)$ and that $\partial\Omega_1 = \{\partial_1, \ldots, \partial_b\}$. Each ∂_i is a simple closed curve in Σ, and ∂_i admits a small tubular neighborhood U_i in Σ which is topologically an annulus.

Assume for the moment that Σ is orientable, and we will prove that $b \leq g(\Sigma) + e$. Let $\Delta = \{\Delta_1, \ldots, \Delta_k\}$ denote the set of components of $\Sigma \setminus \text{Int}(\Omega_1)$ and since each of these components has at least one end, then $k \leq e$. Given $i \in \{1, \ldots, k\}$, let A_i denote the set of components of $\partial\Delta_i$ with one of the components arbitrarily removed; in particular the number of components in $A := \cup_{i=1}^k A_i$ is $b - k$. Note that for each component $\beta \in A$, there is a simple closed curve γ_β in Σ that intersects A transversely in a single point of β, where γ_β consists of an arc in Ω_1 together with an arc in the component $\Delta_j \in \Delta$ that has β in its boundary. It follows that the collection of simple closed curves A does not separate Σ and so, by the definition of genus, the number of elements in A, which is $b - k$, is less than or equal to $g(\Sigma)$. Since $k \leq e$, then $b \leq g(\Sigma) + e$, which proves the desired inequality when Σ is orientable.

When Σ is non-orientable, then Σ is the connected sum of $g(\widetilde{\Sigma}) + 1$ projective planes punctured in e points, where $\widetilde{\Sigma}$ is the oriented cover of Σ, and a similar argument just carried out in the orientable case shows that $b \leq g(\widetilde{\Sigma}) + e + 1$.

According to the hypothesis stated for inequality (4.7), f is assumed not to be an embedded plane. Thus the total spinning S of f satisfies $S \geq 2$. If $S = 2$, then the extrinsic monotonicity formula for minimal surfaces implies that either f has one end with multiplicity 2 (in this case $f(\Sigma)$ is a plane, $B(\Sigma) = b = 1$ and $I(f) = 0$, so (4.10) is an equality in this case), or f is injective and has two ends. In this last case, $f(\Sigma)$ is a catenoid by Schoen [15], $b \leq 2$, $B(\Sigma) = 0$ and $I(f) = 1$, which implies that (4.10) holds in this case.

If $S \geq 3$ and Σ is orientable, then

$$\begin{aligned} b &\leq g(\Sigma) + e \\ &\leq 2g(\Sigma) + e && \text{(because } g(\Sigma) \geq 0) \\ &\leq 3I(f) - 2S - e + 2B(\Sigma) + 5 && \text{(by (3.1))} \\ &\leq 3I(f) + 2B(\Sigma) - 2 && \text{(because } S \geq 3 \text{ and } e \geq 1), \end{aligned}$$

hence (4.10) holds. Finally, if $S \geq 3$ and Σ is non-orientable, then

$$\begin{aligned} b &\leq g(\widetilde{\Sigma}) + e + 1 \\ &\leq 3I(f) - 2S - e + 2B(\Sigma) + 5 && \text{(by (3.1))} \\ &\leq 3I(f) + 2B(\Sigma) - 2 && \text{(because } S \geq 3 \text{ and } e \geq 1), \end{aligned}$$

hence (4.10) again holds. Therefore, inequality (4.10) holds in every case, and as observed above, this suffices to finish the proof of Claim 4.5. □

With the auxiliary property (4.7) at hand, we next prove item 2 of Proposition 4.1. The first statement for f injective with image a plane is obvious. Assume f is not in this case and we will prove (4.2) for $R = 1$.

First suppose that $f\colon (\Sigma, p_0) \to (\mathbb{R}^3, \vec{0})$ is injective with image a catenoid C. After a possible rotation of C fixing the origin, we can assume that the (x_1, x_3)-plane P is a plane of symmetry of C and the axis of C is parallel to the x_3-axis. As we observed previously, for estimating distances between pairs of points in Ω_1, we may assume that the boundary sphere $\partial\mathbb{B}(1)$ is transverse to C. Then $C \cap P \cap \overline{\mathbb{B}}(1)$ contains a component arc Γ with non-vanishing curvature passing through the origin. By convexity, Γ has length less than the length of the boundary circle of the disk $P \cap \overline{\mathbb{B}}(1)$, and so, length$(\Gamma) < 2\pi$. As the axis of C is parallel to the x_3-axis, we deduce that Γ can be parameterized by its third coordinate as $\Gamma = \{(x_1(t), 0, t) | t \in [a, b]\}$ for some $-1 \le a < 0 < b \le 1$. Let $C(1) = C \cap \{(x_1, x_2, x_3) \mid a \le x_3 \le b\}$; clearly $\Omega_1 \subset C(1)$ and $\Omega_1 \cap \partial C(1) = \{(x_1(a), 0, a), (x_1(b), 0, b)\}$. Similar comparison estimates also prove that each horizontal disk $\{x_3 = t\} \cap \overline{\mathbb{B}}(1)$ with $t \in [a, b]$ intersects Ω_1 in a connected component $\Lambda(t)$ passing through $(x(t), 0, t) \in \Gamma$, and $\Lambda(t)$ is invariant under reflection across P. $\Lambda(t)$ is either a horizontal circle of radius less than 1, a circular arc of length less than 2π or just the point $(x_1(t), 0, t)$ when $t = a$ or $t = b$. In particular, for any pair of points $p, q \in \Omega_1$ there exists a piecewise smooth path in Ω_1 joining p and q, which consists of a pair of horizontal circular arcs that join p and q to Γ together with an arc in Γ joining the end points of these two horizontal arcs. It follows that the distance $d_{\Omega_1}(p, q) < 4\pi$. Direct substitution of $I = 1$ and $B = 0$ in the RHS of (4.2) shows that the inequality (4.2) holds in this case that $f\colon (\Sigma, p_0) \to (\mathbb{R}^3, \vec{0})$ is injective with $f(\Sigma) = C$.

If $S = 2$, then the arguments in the fifth paragraph of the proof of Claim 4.5 show that either f is injective with $f(\Sigma)$ being a catenoid (hence (4.2) holds by the last paragraph), or else $f(\Sigma)$ is a plane passing through the origin with $B(\Sigma) = 1$; in this last case the intrinsic distance between any two points of Ω_1 is less than or equal to 4, and so, (4.2) is also seen to hold.

It remains to show that (4.2) holds if $S \ge 3$. Assume now that $S \ge 3$. We proved in the sixth paragraph of the proof of Claim 4.5 that if $S \ge 3$, then $b \le 3I(f) + 2B(\Sigma) - 2$. Plugging this estimate of b into the inequality in Claim 4.4 and using the scale invariance of this inequality, we get the following estimate for all points $p, q \in \Omega_R$ and for all $R > 0$:

$$d_{\Omega_R}(p, q) \le 2(3I(f) + 2B(\Sigma) - 2)\widehat{L}R + \frac{1}{2}\text{Length}(\partial\Omega_R). \tag{4.11}$$

By the extrinsic monotonicity formula, $\pi R^2 < \text{Area}[f^{-1}(\overline{\mathbb{B}}(R))] \le \pi S R^2$ for each $R > 0$, where the strict inequality holds since $f(\Sigma)$ is assumed not to be injective with image a plane passing through the origin. Taking $R = 1$ in the first of these inequalities and $R = 2$ in the second one, we deduce that

$$\text{Area}[f^{-1}(\overline{\mathbb{B}}(2) - \mathbb{B}(1))] < 4\pi S - \pi. \tag{4.12}$$

By the co-area formula,

$$\min_{r\in[1,2]} \text{Length}[f^{-1}(\partial\mathbb{B}(r))] \le \text{Area}[f^{-1}(\overline{\mathbb{B}}(2) - \mathbb{B}(1))]. \tag{4.13}$$

Let $\rho \in [1, 2]$ be such that Length$[f^{-1}(\partial\mathbb{B}(\rho))]$ equals the minimum in the LHS of (4.13). Given $p, q \in \Omega_1$,

$$\begin{aligned}
d_{\Omega_\rho}(p, q) &\le 2(3I + 2B - 2)\widehat{L}\rho + \frac{1}{2}\text{Length}(\partial\Omega_\rho) && \text{(by (4.11))} \\
&\le 2(3I + 2B - 2)\widehat{L}\rho + \frac{1}{2}\text{Length}[f^{-1}(\partial\mathbb{B}(\rho))] && \text{(because } \partial\Omega_\rho \subset f^{-1}(\partial\mathbb{B}(\rho))) \\
&< 2(3I + 2B - 2)\widehat{L}\rho + 2\pi S - \frac{\pi}{2} && \text{(by (4.12) and (4.13))} \\
&\le 4(3I + 2B - 2)\widehat{L} + \pi(3I - 2e + 2B + 5) - \frac{\pi}{2} && \text{(by (4.4) and } \rho \le 2) \\
&\le 4(3I + 2B - 2)\widehat{L} + \pi(3I + 2B + 3) - \frac{\pi}{2} && \text{(because } e \ge 1)
\end{aligned}$$

Since $\rho \leq 2$ and $3I + 2B = 2\widehat{L}^2 - 3$, then $d_{\Omega_2}(p,q) \leq d_{\Omega_\rho}(p,q) < 8\widehat{L}^3 + 2\pi\widehat{L}^2 - 20\widehat{L} - \frac{\pi}{2}$, which proves (4.2) holds. This completes the proof of Proposition 4.1. □

Acknowledgments: We thank Otis Chodosh for explaining to us his work [3] with Davi Maximo on lower bounds for the index of a complete branched minimal surface in $\mathbb{R}^3$ of finite total curvature, in terms of its genus and number of ends counted with multiplicity, and how the analysis by Karpukhin [10] can be used to allow finitely many branch points.

Research ethics: Not applicable.

Author contributions: The authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Competing interests: The authors state no conflict of interest.

Research funding: William H. Meeks funding information: This research was partially supported by CNPq - Brazil, grant no. 400966/2014-0. Joaquín Pérez funding information: Research of both authors was partially supported by MINECO/MICINN/FEDER grant no. PID2020-117868GB-I00, regional grant P18-FR-4049, and by the "Maria de Maeztu" Excellence Unit IMAG, reference CEX2020-001105- M, funded by MCINN/AEI/10.13039/501100011033/ CEX2020-001105-M.

Data availability: Not applicable.

References

[1] D. Fischer-Colbrie, "On complete minimal surfaces with finite Morse index in 3-manifolds," *Invent. Math.*, vol. 82, no. 1, pp. 121—132, 1985.

[2] A. Ros, "One-sided complete stable minimal surfaces," *J. Differ. Geom.*, vol. 74, no. 1, pp. 69—92, 2006.

[3] O. Chodosh and D. Maximo, "On the topology and index of minimal surfaces II," *J. Differ. Geom.*, vol. 123, no. 3, pp. 431—459, 2023.

[4] W. H. Meeks, III and J. Pérez, "Geometry of CMC surfaces of finite index," *Adv. Nonlinear Stud.*, vol. 23, no. 1, p. 20, 2023.

[5] W. H. Meeks, III and J. Pérez, "Hierarchy structures in finite index CMC surfaces," *Adv. Calc. Var.*, 2023. https://doi.org/10.1515/acv-2022-0113.

[6] S. T. Yau, "Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold," *Ann. Sci. Ecole Norm. Superieure*, vol. 8, no. 4, pp. 487—507, 1975.

[7] P. Petersen, *Riemannian Geometry, Volume 171 of Graduate Texts in Mathematics*, 3rd ed. Cham, Springer, 2016.

[8] W. K. Allard, "On the first variation of a varifold," *Ann. Math.*, vol. 95, no. 2, pp. 417—491, 1972.

[9] M. Micallef and B. White, "The structure of branch points in minimal surfaces and in pseudoholomorphic curves," *Ann. Math.*, vol. 141, no. 1, pp. 35—85, 1995.

[10] M. Karpukhin, "On the Yang-Yau inequality for the first Laplace eigenvalue," *Geom. Funct. Anal.*, vol. 29, no. 6, pp. 1864—1885, 2019.

[11] W. H. Meeks III, "The classification of complete minimal surfaces with total curvature greater than -8π," *Duke Math. J.*, vol. 48, no. 3, pp. 523—535, 1981.

[12] D. Moya and J. Pérez, "Generalized Henneberg minimal surfaces," *Results Math.*, vol. 78, no. 2, p. 25, 2023.

[13] D. Fischer-Colbrie and R. Schoen, "The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature," *Commun. Pure Appl. Math.*, vol. 33, no. 2, pp. 199—211, 1980.

[14] L. Jorge and W. H. Meeks, III, "The topology of complete minimal surfaces of finite total Gaussian curvature," *Topology*, vol. 22, no. 2, pp. 203—221, 1983.

[15] R. Schoen, "Uniqueness, symmetry, and embeddedness of minimal surfaces," *J. Differ. Geom.*, vol. 18, no. 4, pp. 791—809, 1983.