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The Bernstein problem for (X, Y)-Lipschitz surfaces in three-dimensional sub-Finsler Heisenberg groups

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In the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure, an (X, Y)-Lipschitz surface which is complete, oriented, connected and stable must be a vertical plane. In particular, the result holds for entire intrinsic graphs of Euclidean Lipschitz functions.

Keywords: Heisenberg group; area-stationary surfaces; sub-Finsler structure; stable surfaces; Bernstein problem; sub-Finsler perimeter.

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1. Introduction

Variational problems related to the sub-Riemannian perimeter introduced by Capogna *et al.* [6] (see also Garofalo and Nhieu [28] and Franchi *et al.* [20]) have received great interest recently, specially in the Heisenberg groups $\mathbb{H}^n$. In particular, Bernstein-type problems, either for stable intrinsic graphs or for stable surfaces without singular points, have been specially considered. We refer the reader to the introduction in [26] for an account of recent results, including

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[8, 37, 13, 2, 14, 31, 23, 39, 24, 35, 11, 10, 32, 5]. The monograph [7] provides a quite complete survey of progress on the subject.

In the last years, a left-invariant sub-Finsler perimeter has been considered on the Heisenberg groups, see [40, 34, 19, 29, 33]. A quite natural question is whether Bernstein-type results similar to the sub-Riemannian ones hold for the sub-Finsler perimeter.

The main result in this paper is Theorem 6.3, where we prove that in the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure, a complete, oriented, connected and stable (X,Y)-Lipschitz surface is a vertical plane. Roughly speaking an (X,Y)-Lipschitz surface is locally the intrinsic graph of a Euclidean Lipschitz function. Theorem 6.3 is a generalization of the corresponding sub-Riemannian result for graphs obtained by Nicolussi and Serra Cassano in [32]. Recently, Young [44] proved that a *ruled* area-minimizing entire intrinsic graph in $\mathbb{H}^1$ is a vertical plane by introducing a family of deformations of graphical strips based on variations of a vertical curve.

A sub-Finsler structure is obtained from a left-invariant asymmetric norm $||\cdot||$ in the horizontal distribution of $\mathbb{H}^1$. Such a norm can be obtained from a convex set K contained in the horizontal plane at the origin in $\mathbb{H}^1$. The associated K-perimeter is defined by

$$|\partial E|_K(V) = \sup\left\{\int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leq 1\right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in the open set V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. The integral is computed with respect to the Riemannian measure $d\mathbb{H}^1$ of a fixed left-invariant Riemannian metric g in $\mathbb{H}^1$, while the divergence is the one associated to this Riemannian metric. When $K = D$, the closed unit disk centered at the origin of $\mathbb{R}^2$, the K-perimeter coincides with the classical sub-Riemannian perimeter.

The first variation formula of the sub-Finsler perimeter was computed in [34, §3] for surfaces of class C^2 without singular points under the hypothesis that K is a convex set of class C^2_+. This means that ∂K is of class C^2 and has strictly positive sectional curvature. The following formula was obtained:

$$\left.\frac{d}{ds}\right|_{s=0} A_K(\varphi_s(S)) = \int_S u H_K\, dS, \tag{*}$$

where $\{\varphi_s\}_{s\in\mathbb{R}}$ is the flow (i.e. the one-parameter group of diffeomorphisms) associated to a vector field U with compact support in the regular part of S, the function u is equal to the normal component $\langle U, N\rangle$ of the variation (N is a unit normal for the Riemannian metric g), and dS is the Riemannian area element. The function H_K appearing in (*) is the K-mean curvature

$$H_K = \langle D_Z \pi_K(\nu_h), Z\rangle,$$

where Z is a unit horizontal vector field in S, ν_h is the horizontal unit normal obtained by rotating Z by 90°, D is the Levi-Civita connection associated to the

Riemannian metric g and π_K is the inverse of the normal map of ∂K. Hence, formula (*) has sense whenever the horizontal curves in S are of class C^2. However, the computations in [34] require to take one derivative of the normal to the surface and so they are not valid for surfaces with less regularity.

In [29], also under the assumption that $K \in C^2_+$, the authors proved that a Euclidean Lipschitz and $\mathbb{H}$-regular surface with prescribed continuous mean curvature has horizontal (characteristic) curves of class C^2, extending the corresponding sub-Riemannian result in [25]. Hence, the K-mean curvature can be computed along the characteristic curves in this type of surfaces. Our main task in Sec. 3 is to compute the first variation for (X, Y)-Lipschitz surfaces and to check that the first variation formula (*) also holds for these surfaces with lower regularity. Of course the proof is different from the one in [34] and makes use of a Jacobian of horizontal type introduced by Galli in his Ph.D. thesis [22], see also [23]. In particular, for area-stationary surfaces we get $H_K = 0$ on S. Following the arguments in [25, 29] we prove that an area-stationary surface S is foliated by horizontal straight lines and following [32] we show that S is $\mathbb{H}$-regular.

In Sec. 4, we show that for an area-stationary surface S the function $y = \langle N, T\rangle/|N_h|$ satisfies the differential equation

$$y'' - 6y'y + 4y^3 = 0$$

along almost every horizontal curve in S. Here, N is a Riemannian unit normal to S, N_h the orthogonal projection to the horizontal distribution and T the Reeb vector field on $\mathbb{H}^1$. The function $D = 1/y$ was proven to satisfy the equivalent equation

$$DD'' = 2(D' + 1)(D' + 2)$$

for C^1 surfaces by Cheng *et al.* [9].

Both equations play an important role in the study of the singular set for C^1 surfaces. Moreover, the regularity of $\langle N, T\rangle/|N_h|$ along the horizontal (characteristic) curves in S is crucial to compute the second variation formula. The function $\langle N, T\rangle/|N_h|$ appears frequently in the sub-Riemannian theory of hypersurfaces in the Heisenberg groups $\mathbb{H}^n$. For instance, it is the curvature of a length-minimizing geodesic realizing the distance between a hypersurface to a given point [36].

In Sec. 5, we compute the second variation formula of the area for *horizontal* vector fields with compact support. The second variation formula, which is formally similar to the one obtained in the sub-Riemannian case, is given by

$$\frac{d^2}{ds^2}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_S (Z(u)^2 + qu^2)\,\frac{|N_h|}{\kappa(\pi_K(\nu_h))},$$

where $\{\varphi_s\}_{s\in\mathbb{R}}$ is the flow associated to a horizontal vector field U with compact support, $u = \langle U, N\rangle$ is the normal component and q is the function defined by

$$\frac{q}{4} = Z\left(\frac{\langle N, T\rangle}{|N_h|}\right) - \frac{\langle N, T\rangle^2}{|N_h|^2}.$$

The function κ is the geodesic curvature of ∂K. In the sub-Riemannian case, where K is the unit disc, we have $\kappa = 1$. In our case, the vector ν_h is constant along horizontal lines in S, so that $\kappa(\pi_K(\nu_h))$ is constant on horizontal lines. The computation of this second variation follows the lines of [26], where the second variation of the sub-Riemannian area was computed for stable C^1 surfaces to solve the Bernstein problem. There is a slight difference in the definition of the function q with respect to [26] that is related to the choice of Z as $J(\nu_h)$ or $-J(\nu_h)$. We also use some ideas from Nicolussi and Serra Cassano [32], who proved Bernstein's theorem in the sub-Riemannian setting when S is the intrinsic graph of a Euclidean Lipschitz function.

Finally, in Sec. 6, we prove in our main result, Theorem 6.3, that a complete, oriented, connected and stable (X, Y)-Lipschitz surface is a vertical plane. We emphasize that Nicolussi and Serra Cassano showed that this result is optimal in the sub-Riemannian setting, exhibiting two counterexamples when the Euclidean Lipschitz regularity assumption is missing, see [32, Theorems 7.1 and 8.1].

2. Preliminaries

2.1. *The Heisenberg group*

We denote by $\mathbb{H}^1$ the *first Heisenberg group*, defined as the three-dimensional Euclidean space $\mathbb{R}^3$ with the product

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x'y - xy').$$

A basis of left-invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

For $p \in \mathbb{H}^1$, the *left translation by* p is the diffeomorphism $L_p(q) = p * q$. The *horizontal distribution* $\mathcal{H}$ is the planar distribution generated by X and Y, which coincides with the kernel of the contact one-form $\omega = dt - ydx + xdy$. The distribution $\mathcal{H}$ is completely non-integrable.

We shall consider on $\mathbb{H}^1$ the auxiliary left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$, so that $\{X, Y, T\}$ is an orthonormal basis at every point. Let D be the Levi-Civita connection associated to the Riemannian metric g. The following relations can be easily computed:

$$\begin{aligned} &D_X X = 0, &&D_Y Y = 0, &&D_T T = 0,\\ &D_X Y = -T, &&D_X T = Y, &&D_Y T = -X,\\ &D_Y X = T, &&D_T X = Y, &&D_T Y = -X. \end{aligned} \tag{2.1}$$

Setting $J(U) = D_U T$ for any vector field U in $\mathbb{H}^1$ we get $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Therefore, $-J^2$ coincides with the identity when restricted to the horizontal distribution. The Riemannian volume of a set E is, up to a constant, the Haar measure of the group and is denoted by $|E|$. The integral of a function f with respect to the Riemannian measure is denoted by $\int f \, d\mathbb{H}^1$.

2.2. *The pseudo-Hermitian connection*

The pseudo-Hermitian connection ∇ is the only affine connection satisfying the following properties:

(1) ∇ is a metric connection and
(2) $\mathrm{Tor}(U, V) = 2\langle J(U), V\rangle T$ for all vector fields $U, V \in \mathfrak{X}(\mathbb{H}^1)$.

We recall that a metric connection must satisfy

$$U(g(V, W)) = g(\nabla_U V, W) + g(V, \nabla_U W)$$

for vector fields $U, V, W \in \mathfrak{X}(\mathbb{H}^1)$. The torsion tensor associated to ∇ is defined by

$$\mathrm{Tor}(U, V) = \nabla_U V - \nabla_V U - [U, V]$$

for all $U, V \in \mathfrak{X}(\mathbb{H}^1)$. From this definition and Koszul formula, see [15, formula (9), Proof of Theorem 3.6], it follows easily that $\nabla X = \nabla Y = 0$ and $\nabla J = 0$. For a general discussion about the pseudo-Hermitian connection see for instance [16, Sec. 1.2]. Given a curve $\gamma : I \to \mathbb{H}^1$ we denote by ∇/ds the covariant derivative induced by the pseudo-Hermitian connection along γ.

2.3. *Sub-Finsler norms*

Given a convex set $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$ and associated asymmetric norm $||\cdot||$ in $\mathbb{R}^2$, we define a left-invariant norm $||\cdot||_K$ on the horizontal distribution of $\mathbb{H}^1$ by means of the equality

$$(||fX + gY||_K)(p) = ||(f(p), g(p))||,$$

for any $p \in \mathbb{H}^1$. The dual norm is denoted by $||\cdot||_{K,*}$.

If the boundary of K is of class C^ℓ, $\ell \geq 2$, and the geodesic curvature of ∂K is strictly positive, we say that K is of class C^ℓ_+. When K is of class C^2_+, the outer Gauss map N_K is a diffeomorphism from ∂K to $\mathbb{S}^1$ and the map

$$\pi_K(fX + gY) = N_K^{-1}\left(\frac{(f, g)}{\sqrt{f^2 + g^2}}\right),$$

defined for nowhere vanishing horizontal vector fields $U = fX + gY$, satisfies

$$||U||_{K,*} = \langle U, \pi_K(U)\rangle.$$

See [34, Sec. 2.3].

2.4. *Sub-Finsler perimeter*

Here, we summarize some of the results contained in [34, Sec. 2.4].

Given a compact convex set $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$, the norm $||\cdot||_K$ defines a perimeter functional: given a measurable set $E \subset \mathbb{H}^1$ and an open subset $\Omega \subset \mathbb{H}^1$, we say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup\left\{\int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \le 1\right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. The integral is computed with respect to the Riemannian measure $d\mathbb{H}^1$ of the left-invariant Riemannian metric g, and the divergence is the one associated to g. When $K = D$, the closed unit disk centered at the origin of $\mathbb{R}^2$, the K-perimeter coincides with classical sub-Riemannian perimeter.

If K, K' are bounded convex bodies containing 0 in its interior then there exist constants $\alpha, \beta > 0$ such that

$$\alpha||x||_{K'} \le ||x||_K \le \beta||x||_{K'}, \quad \text{for all } x \in \mathbb{R}^2,$$

and it is not difficult to prove that

$$\beta^{-1}|\partial E|_{K'}(V) \le |\partial E|_K(V) \le \alpha^{-1}|\partial E|_{K'}(V).$$

Then E has locally finite K-perimeter if and only if it has locally finite K'-perimeter. In particular, any set with locally finite K-perimeter has locally finite sub-Riemannian perimeter.

Riesz Representation Theorem implies the existence of a $|\partial E|_K$-measurable vector field ν_K so that for any horizontal vector field U with compact support of class C^1 we have

$$\int_\Omega \operatorname{div}(U)\, d\mathbb{H}^1 = \int_\Omega \langle U, \nu_K\rangle\, d|\partial E|_K.$$

In addition, ν_K satisfies $|\partial E|_K$-a.e. the equality $||\nu_K||_{K,*} = 1$, where $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$.

Given two convex sets $K, K' \subset \mathbb{R}^2$ containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure $|\partial E|_K$ and the vector field ν_K

$$|\partial E|_K = ||\nu_{K'}||_{K,*}|\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}.$$

Indeed, for the closed unit disk $D \subset \mathbb{R}^2$ centered at 0 we know that in the Euclidean Lipschitz case $\nu_D = \nu_h$ and $|N_h| = ||N_h||_{D,*}$ where N is the *outer* unit normal with respect to the Riemannian metric $g = \langle\cdot,\cdot\rangle$ and $N_h = N - \langle N, T\rangle T$ is the orthogonal projection of N onto the horizontal distribution $\mathcal{H}$. Hence, we have

$$|\partial E|_K = ||\nu_h||_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}.$$

Here, $|\partial E|_D$ is the standard sub-Riemannian measure. Moreover, $\nu_h = N_h/|N_h|$ and $|N_h|^{-1} d|\partial E|_D = dS$, where dS is the standard Riemannian measure on S.

Note that, if K is not a disk, ν_K is not the standard horizontal unit normal. Hence we get, for a set E with Euclidean Lipschitz boundary S

$$|\partial E|_K(\Omega) = \int_{S\cap\Omega} ||N_h||_{K,*}\, dS, \tag{2.2}$$

where dS is the Riemannian measure on S, obtained from the area formula using a local Lipschitz parametrization of S, see [20, Proposition 2.14]. It coincides with the two-dimensional Hausdorff measure associated to the Riemannian distance induced by g. We stress that here N is the *outer* unit normal. This choice is important because of the lack of symmetry of $||\cdot||_K$ and $||\cdot||_{K,*}$. Moreover when $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface the K-perimeter coincides with the area functional

$$A_K(S) = \int_S ||N_h||_{K,*}\, dS.$$

2.5. *Surfaces in* $\mathbb{H}^1$

Following [1, 20], we provide the following definition.

Definition 2.1 ($\mathbb{H}$-regular surface). A real continuous function f defined on an open set $\Omega \subset \mathbb{H}^1$ is of class $C^1_{\mathbb{H}}(\Omega)$ if the distributional derivative $\nabla_{\mathbb{H}} f = (Xf, Yf)$ is represented by a continuous vector field on Ω.

We say that $S \subset \mathbb{H}^1$ is an $\mathbb{H}$-regular surface if for each $p \in \mathbb{H}^1$ there exist an open set U containing p and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ on U and $S \cap U = \{f = 0\}$. Under such conditions, the horizontal unit normal ν_h on $S \cap U$ is defined as the restriction of the non-vanishing continuous vector field

$$\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|},$$

defined on all of U.

Given a vertical plane $P \subset \mathbb{H}^1$, and a function u defined on a domain $D \subset P$, we denote by $\mathrm{Gr}(u)$ the *intrinsic graph* of u, defined as the Riemannian normal graph of the function u. Since the Riemannian unit normal to P is the restriction of a unitary left-invariant vector field X_P, the intrinsic graph of u is given by

$$\mathrm{Gr}(u) = \{\exp_p(u(p)\, X_P(p)) : p \in D\},$$

where exp is the exponential map on the Riemannian manifold $(\mathbb{H}^1, g)$. Using Euclidean rotations about the vertical axis $x = y = 0$, that are isometries of the Riemannian metric g, we may assume that P is the plane $\{y = 0\}$. Since in this case $X_P = Y$, the intrinsic graph $\mathrm{Gr}(u)$ can be parametrized by the map

$$\Phi^u(x,t) = (x, u(x,t), t - xu(x,t)),$$

for $(x, 0, t) \in D$. Note that $\Phi^u(x,t) = (x,0,t) * (0, u(x,t), 0)$, where $*$ is the Heisenberg product defined in Sec. 2.1. For further details, we refer the reader to [21]. Note also that u measures the signed distance of $\Phi^u(x,t)$ to the plane P, see [36].

Given the intrinsic graph $\mathrm{Gr}(u)$ of a Euclidean Lipschitz function defined on some domain D of the vertical plane P, we know by Rademacher's Theorem that u is $\mathcal{H}^2$-a.e. differentiable on D, where $\mathcal{H}^2$ is the two-dimensional Euclidean Hausdorff measure on D. Assuming $P = \{y = 0\}$, and given a differentiability point $(x_0, 0, t_0)$ of u, the tangent plane of $\mathrm{Gr}(u)$ is well defined at $\Phi^u(x_0, t_0)$ and so it is the normal vector field N. Hence, N is defined $\mathcal{H}^2$-a.e. on $\mathrm{Gr}(u)$. Moreover,

$$N = \frac{(u_x + 2uu_t)X - Y + u_t T}{\sqrt{1 + u_t^2 + (u_x + 2uu_t)^2}}, \tag{2.3}$$

see the computations in [27, Sec. 4]. Hence, N is never vertical. At differentiability points of $\mathrm{Gr}(u)$ we define

$$\nu_h = \frac{N_h}{|N_h|} = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}},$$

and the vector field Z by

$$Z = -J(\nu_h),$$

which is tangent to S and horizontal. An orthonormal basis at the tangent space of $\mathrm{Gr}(u)$ at the differentiable point is obtained by adding to Z the vector

$$E = \langle N, T\rangle \nu_h - |N_h| T. \tag{2.4}$$

Following [43], we provide the following definition.

Definition 2.2. A set $S \subset \mathbb{H}^1$ is a (X,Y)-Lipschitz surface if for each $p \in S$ there exist an open neighborhood $U_p \subset \mathbb{H}^1$ of p, and a Lipschitz function $f : U \to \mathbb{R}$ such that

$$S \cap U = \{f = 0\}$$

and

$$Xf \geq l \quad \text{a.e. on } U \quad \text{or} \quad Yf \geq l \quad \text{a.e. on } U$$

for a suitable $l > 0$.

We stress that the following result was previously obtained by [43, Theorem 3.2], using the notion of homogeneous cone. Here, we provide a different proof.

Theorem 2.3. *A set $S \subset \mathbb{H}^1$ is a (X,Y)-Lipschitz surface if and only if S is locally the intrinsic graph of a Euclidean Lipschitz function.*

Proof. Assume that S is an (X,Y)-Lipschitz surface. Given $p \in S$ there exist an open ball $B_r(p) \subset \mathbb{H}^1$ and a Euclidean Lipschitz function f defined on $B_r(p)$ such that

$$S \cap B_r(p) = \{(x,y,t) \in B_r(p) : f(x,y,t) = 0\}.$$

Since S is (X,Y)-Lipschitz, after a rotation about the vertical axis we may assume the existence of $l > 0$ such that $Yf(q) \geq l > 0$ for every point of differentiability of

f close enough to p. In particular, the convex hull of

$$\left\{\lim_{i\to\infty} Yf(q_i) : \lim_{i\to\infty} q_i = p, q_i \text{ differentiability point of } f\right\}$$

does not contain 0. Let us consider the C^∞ diffeomorphism $H(x,y,t) = (x,y,t-xy)$ on $\mathbb{H}^1$. Then the function $f \circ H$ is Lipschitz and

$$\frac{\partial(f\circ H)}{\partial y}(q) = \left(\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial t}\right)(q) = Yf(q)$$

for each point q of differentiability of f. Therefore by the Implicit Function Theorem for Lipschitz functions [12, p. 255] there exists an open neighborhood $D \subset \{y=0\}$ of the projection of p on $\{y=0\}$ and a Euclidean Lipschitz function $u : D \to \mathbb{R}$ such that $f(x, u(x,t), t - xu(x,t)) = 0$. In other words, the surface S is locally an intrinsic graph of a Lipschitz function over the vertical plane $\{y=0\}$.

Assume now that S is locally the intrinsic graph of a Euclidean Lipschitz function u. Let p in S and assume that $S \cap B_r(p) = \Phi^u(D)$ where $\Phi^u(x,t) = (x, u(x,t), t - xu(x,t))$ and $u : D \to \mathbb{R}$ is a Euclidean Lipschitz function. Setting

$$f(x,y,t) = y - u(x, t+yx),$$

we clearly have that f is a Euclidean Lipschitz function defined in an open neighborhood of p. Eventually reducing the radius $r > 0$ we get that $S \cap B_r(p) = \{f = 0\}$ and $Y(f) = 1 > 0$ a.e. on $B_r(p)$. Therefore, S is a (X,Y)-Lipschitz surface. □

Remark 2.4. Note that

(1) An (X,Y)-Lipschitz surface is an embedded surface by Theorem 2.3.
(2) If a Euclidean Lipschitz function f defined on an open domain of $\mathbb{H}^1$ is $C^1_{\mathbb{H}}$ then their level sets are (X,Y)-Lipschitz. Indeed, since the horizontal gradient $\nabla_{\mathbb{H}} f = (Xf, Yf)$ is a never vanishing *continuous* vector field, we obtain that locally $Xf \geq l > 0$ or $Y(f) \geq l > 0$ eventually replacing f by $-f$.

Definition 2.5. Let $S \subset \mathbb{H}^1$ be a C^1 surface. We say that $p \in S$ belongs to the *singular set* S_0 of S if the tangent space T_pS coincides with the horizontal distribution $\mathcal{H}_p$.

The following result, whose proof can be found in [23], will be used to compute the first and second variation of a surface.

Proposition 2.6. *Let S be an oriented immersed C^2 surface in $\mathbb{H}^1$ with singular set $S_0 = \emptyset$ and let $f \in C^1(S)$. Then*

$$\operatorname{div}_S(fZ) = Z(f) - (\langle N,T\rangle\theta(E) + 2\langle N,T\rangle|N_h|)f$$

and

$$\operatorname{div}_S(fE) = E(f) + \langle N,T\rangle\theta(Z)f,$$

where we have set $\theta(W) = \langle\nabla_W \nu_h, Z\rangle$ for each vector field W.

3. The First Variation Formula

We start this section computing the first variation formula for (X, Y)-Lipschitz surfaces which are twice differentiable in the horizontal directions. We start by proving some technical lemmas. The Riemannian version of the following one can be found in [38].

Lemma 3.1. *Let U be a smooth vector field on $\mathbb{H}^1$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ be the flow associated to U. Let $p \in \mathbb{H}^1$ and $e \in T_p\mathbb{H}^1$. Define the smooth curve $\beta(s) = \varphi_s(p)$ and the smooth vector field $E(s) = (d\varphi_s)_p(e)$ along β. Then we have*

$$\frac{\nabla}{ds}\bigg|_{s=0} E(s) = \nabla_e U + 2\langle J(U_p), e\rangle\, T_p. \tag{3.1}$$

Proof. Let us rename the standard coordinates (x, y, t) as (x_1, x_2, x_3). Let $\varphi_s = (\varphi_1, \varphi_2, \varphi_3)$, and $e = (e_1, e_2, e_3)$, and $U = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$. Then

$$(d\varphi_s)_p = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial x_1} & \dfrac{\partial\varphi_1}{\partial x_2} & \dfrac{\partial\varphi_1}{\partial x_3} \\ \dfrac{\partial\varphi_2}{\partial x_1} & \dfrac{\partial\varphi_2}{\partial x_2} & \dfrac{\partial\varphi_2}{\partial x_3} \\ \dfrac{\partial\varphi_3}{\partial x_1} & \dfrac{\partial\varphi_3}{\partial x_2} & \dfrac{\partial\varphi_3}{\partial x_3} \end{pmatrix}$$

and

$$E(s) = (d\varphi_s)_p(e) = \sum_{i=1}^3\sum_{j=1}^3 e_j \frac{\partial\varphi_i}{\partial x_j}\left(\frac{\partial}{\partial x_i}\right)_{\beta(s)} = \sum_{i=1}^3 g_i(s)\left(\frac{\partial}{\partial x_i}\right)_{\beta(s)},$$

where $g_i(s) = \sum_{j=1}^3 e_j \frac{\partial\varphi_i}{\partial x_j}$. Therefore,

$$\frac{\nabla}{ds}\bigg|_{s=0} E(s) = \sum_{i=1}^3 g_i'(0)\left(\frac{\partial}{\partial x_i}\right)_p + \sum_{i=1}^3 g_i(0)\nabla_{U_p}\frac{\partial}{\partial x_i}.$$

Since $g_i(0) = e_i$ and $g_i'(0) = e(f_i)$ we have

$$\frac{\nabla}{ds}\bigg|_{s=0} E(s) = \sum_{i=1}^3 e(f_i)\left(\frac{\partial}{\partial x_i}\right)_p + e_i\nabla_{U_p}\frac{\partial}{\partial x_i}.$$

On the other hand,

$$\nabla_e U = \sum_{i=1}^3 e(f_i)\left(\frac{\partial}{\partial x_i}\right)_p + f_i\nabla_e\frac{\partial}{\partial x_i}.$$

Since

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} &= \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + \mathrm{Tor}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + 2\left\langle J\left(\frac{\partial}{\partial x}\right), \frac{\partial}{\partial y}\right\rangle T \\ &= \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + 2\left\langle J(X) - yJ(T), Y\right\rangle T + 2x\left\langle J(X), T\right\rangle T \\ &= \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + 2T\end{aligned}$$

and

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i},$$

for $\{i, j\} \neq \{1, 2\}$ there follows, evaluating at p,

$$\begin{aligned}\sum_{i=1}^{3} e_i \nabla_{U_p} \frac{\partial}{\partial x_i} &= \sum_{i,j=1}^{3} e_i\, f_j\, \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = e_1 f_2 \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + e_2 f_1 \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} \\ &\quad + \sum_{i,j=1,\{i,j\}\neq\{1,2\}}^{3} e_i\, f_j\, \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \\ &= e_1 f_2 \left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} - 2T\right) + e_2 f_1 \left(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} + 2T\right) \\ &\quad + \sum_{i,j=1,\{i,j\}\neq\{1,2\}}^{3} e_i\, f_j\, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \\ &= 2(e_2 f_1 - e_1 f_2)\, T + \sum_{j=1}^{3} f_j\, \nabla_e \frac{\partial}{\partial x_j} \\ &= 2(e_2 f_1 - e_1 f_2)\, T + \sum_{i=1}^{3} f_i\, \nabla_e \frac{\partial}{\partial x_i}.\end{aligned}$$

Hence, we obtain (3.1). □

Let $S \subset \mathbb{H}^1$ be an (X, Y)-Lipschitz surface and $p \in S$ a point of differentiability of S. Consider a C^1 vector field U with compact support on $\mathbb{H}^1$ and let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the associated flow. A basis of the tangent space to $\varphi_s(S)$ at $\varphi_s(p)$ is given by $E_1(s) = (d\varphi_s)_p(Z_p)$ and $E_2(s) = (d\varphi_s)_p(E_p)$, where E is defined in (2.4). Let N_s be the unit normal to $\varphi_s(S)$. Then

$$[(N_s)_h]_{\varphi_s(p)} = \frac{[E_1(s) \times E_2(s)]_h}{|E_1(s) \times E_2(s)|},$$

where $\times$ denotes the cross product in $(\mathbb{H}^1, g)$. Observe that $\mathrm{Jac}(\varphi_s)(p) = |E_1(s) \times E_2(s)|$. Setting $V(s, p) = [E_1(s) \times E_2(s)]_h$ we have

$$V(s, p) = \langle E_1, T\rangle\, T \times E_2 + \langle E_2, T\rangle\, E_1 \times T. \tag{3.2}$$

Since $\{Z, \nu_h, T\}$ is positively oriented, we have $V(0, \cdot) = |N_h|\,\nu_h$. The area functional A_K of $\varphi_s(S)$ is then given by

$$\begin{aligned} A_K(\varphi_s(S)) &= \int_S \langle \pi_K([E_1(s) \times E_2(s)]_h), \frac{[E_1(s) \times E_2(s)]_h}{|E_1(s) \times E_2(s)|} \rangle |E_1(s) \times E_2(s)|\, dS \\ &= \int_S \langle \pi_K(V(s,p)), V(s,p) \rangle \, dS(p) \\ &= \int_S ||V(s,p)||_* \, dS(p). \end{aligned} \tag{3.3}$$

Lemma 3.2. *Let S be an (X,Y)-Lipschitz surface, and $\{\varphi_s\}_{s\in\mathbb{R}}$ the flow associated to a compactly supported C^1 vector field U in $\mathbb{H}^1$. Let $L = supp(U) \cap S$. Then there exist positive constants l, s_0 such that $|V(s,p)| \geq \frac{l}{2}$ for a.e. $p \in L$ and any $s \in (-s_0, s_0)$.*

Proof. Theorem 2.3 implies that, after a rotation about the vertical axis, there exists $r > 0$ and a Euclidean Lipschitz function $u : D \to \mathbb{R}$, defined on an open set D of the vertical plane $y = 0$, such that $S \cap B_r(p) = \Phi^u(D)$. Let $B \subset D$ be the set of points of differentiability of u. Since u is Euclidean Lipschitz, Eq. (2.3) implies

$$|N_h| \geq \frac{1}{\sqrt{1 + (u_t)^2 + (u_x + 2uu_t)^2}} \geq l_{B_r(p)} > 0$$

for some constant $l_{B_r(p)}$ depending on a Lipschitz constant of u on $B_r(p)$. Therefore by the compactness of $\operatorname{supp}(U)$, there exists $l > 0$ such that $|N_h| \geq l > 0$ for a.e. $q \in L$.

Assume by contradiction the existence of a measurable set $A \subset L$ such that $\mathcal{H}^2(A) > 0$ and of a sequence $\{s_j\}_{j\in\mathbb{N}}$ converging to 0 such that $|V(s_j, q)| < \frac{l}{2}$ for each $q \in A$. Then

$$\lim_{j\to\infty} \int_A |V(s_j, q)|\, dS(q) \leq \frac{l}{2}\, \mathcal{H}^2(A). \tag{3.4}$$

On the other hand, since $|E_1(s_j)| = |d\varphi_{s_j}(Z)| \leq C'$, $|E_2(s_j)| = |d\varphi_{s_j}(E)| \leq C'$ for some $C' > 0$, we obtain from (3.2) the existence of $C > 0$ such that $|V(s_j)| \leq C$ on A. By the continuity of $V(s,q)$ in s we have $\lim_{j\to\infty} |V(s_j)| = |N_h|$ a.e. in A. By dominated convergence

$$\lim_{j\to\infty} \int_A |V(s_j, q)|\, dS(q) = \int_A |N_h|\, dS \geq l\, \mathcal{H}^2(A).$$

Therefore, since $\mathcal{H}^2(A) > 0$ we get a contradiction to (3.4). □

Now, we compute the first variation of the sub-Finsler area.

Proposition 3.3. *Let $K \in C^2_+$ be a convex body with $0 \in \operatorname{int}(K)$, and $S \subset \mathbb{H}^1$ an oriented (X,Y)-Lipschitz surface. Then the first variation of the sub-Finsler area* *induced by a* C^1 *vector field* U *with compact support in* $\mathbb{H}^1$, *with* $\partial S \cap supp(U) = \emptyset$, *is given by*

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_S [-\langle N,T\rangle Z(\langle U,T\rangle)\pi_Z - E(\langle U,T\rangle)\pi_\nu \\ - 2\langle N,T\rangle\langle J(U),\pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)),\nabla_Z U\rangle]\, dS, \quad (3.5)$$

where $\{\varphi_s\}_{s\in\mathbb{R}}$ *is the flow associated to* U.

Moreover, if we assume that the derivative in the Z*-direction of* ν_h *and* Z *exists and is continuous, then we have*

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_S u\,\langle \nabla_Z \pi_K(\nu_h), Z\rangle dS, \quad (3.6)$$

where $u = \langle N, U\rangle$.

Proof. We denote $V(s,\cdot)$ by $V(s)$ for simplicity. Let us prove first that

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_S \frac{d}{ds}\bigg|_{s=0} ||V(s)||_*\, dS. \quad (3.7)$$

This means we can differentiate under the integral sign.

By Lemma 3.2 the norm of $V(s)$ is strictly positive a.e. in $\operatorname{supp}(U)\cap S$. So, we have

$$\frac{d}{ds}||V(s)||_* = \left\langle \pi_K(V(s)), \frac{\nabla}{ds}V(s)\right\rangle \le ||\pi_K(V(s))||_K \left\|\frac{\nabla}{ds}V(s)\right\|_{K,*} \\ = \left\|\frac{\nabla}{ds}V(s)\right\|_* \le \beta\left|\frac{\nabla}{ds}V(s)\right|, \quad (3.8)$$

for a.e. in $\operatorname{supp} U\cap S$, where β is the positive constant defined in Sec. 2.4 taking K' equal to the Euclidean ball centered at zero. By Lemma 3.1 there holds

$$\frac{\nabla}{ds}V(s) = \langle \nabla_{E_1(s)}U, T\rangle T\times E_2(s) + 2\langle J(U), E_1(s)\rangle\, T\times E_2(s) \\ + \langle E_1(s), T\rangle T\times \nabla_{E_2(s)}U + 2\langle J(U), E_2(s)\rangle\, E_1(s)\times T \\ + \langle \nabla_{E_2(s)}U, T\rangle E_1(s)\times T + \langle E_2(s), T\rangle \nabla_{E_1(s)}U\times T.$$

Since $|E_1(s)| = |d\varphi_s(Z)| \le C$ and $|E_2(s)| = |d\varphi_s(E)| \le C$ for $s\in(-s_0,s_0)$ where $C>0$ is independent of s. Then, writing the covariant derivative $\nabla_{E_i(s)}U$ in standard coordinates, we obtain

$$\left|\frac{\nabla}{ds}V(s)\right| \le \tilde{C}\,\|U\|_{C^1}$$

a.e. in $\operatorname{supp} U\cap S$ for a suitable constant $\tilde{C}>0$. Here, $\|U\|_{C^1}$ denotes the standard C^1 norm of U. Since $\operatorname{supp} U\cap S$ is compact, dominated convergence implies (3.7).

Let us compute now

$$\left.\frac{d}{ds}\right|_{s=0} ||V(s)||_* = \left.\frac{d}{ds}\right|_{s=0} \langle \pi_K(V(s)), V(s)\rangle$$

at a point p of differentiability of S. By [34, Remark 3.3],

$$\left.\frac{d}{ds}\right|_{s=0} \langle \pi_K(V(s)), V(s)\rangle = \langle \pi_K(V(0)), \left.\frac{\nabla}{ds}\right|_{s=0} V(s)\rangle = \langle \pi_K((\nu_h)_p), \left.\frac{\nabla}{ds}\right|_{s=0} V(s)\rangle.$$

Since T is parallel with respect to the pseudo-Hermitian connection ∇ and $\langle Z, T\rangle = 0$, we have

$$\begin{aligned}\left.\frac{\nabla}{ds}\right|_{s=0} V(s) = \left\langle \left.\frac{\nabla}{ds}\right|_{s=0} E_1(s), T\right\rangle T \times S + \left\langle \left.\frac{\nabla}{ds}\right|_{s=0} E_2(s), T\right\rangle Z \times T \\ + \langle E, T\rangle \left.\frac{\nabla}{ds}\right|_{s=0} E_1(s) \times T.\end{aligned}$$

Lemma 3.1 implies

$$\left.\frac{\nabla}{ds}\right|_{s=0} E_1(s) = \nabla_Z U + 2\langle J(U), Z\rangle\, T$$

and

$$\left.\frac{\nabla}{ds}\right|_{s=0} E_2(s) = \nabla_E U + 2\langle J(U), E\rangle\, T.$$

Therefore, evaluating at p but omitting it for clarity,

$$\begin{aligned}\left.\frac{\nabla}{ds}\right|_{s=0} V(s) &= (\langle \nabla_Z U, T\rangle + 2\langle J(U), Z\rangle)\, T \times E \\ &\quad + (\langle \nabla_E U, T\rangle + 2\langle J(U), E\rangle)\, Z \times T + \langle E, T\rangle\, \nabla_Z\, U \times T \\ &= \langle N, T\rangle (Z(\langle U, T\rangle) + 2\langle J(U), Z\rangle)\, T \times \nu_h \\ &\quad + (E(\langle U, T\rangle) + 2\langle J(U), E\rangle)\, Z \times T - |N_h|\, \nabla_Z U \times T. \end{aligned} \tag{3.9}$$

We set $\pi_K(\nu_h) = \pi_Z Z + \pi_\nu \nu_h$, where $\pi_Z = \langle \pi(\nu_h), Z\rangle$ and $\pi_\nu = \langle \pi(\nu_h), \nu_h\rangle$. Note that $T \times \nu_h = -Z$, $Z \times T = -\nu_h$ and $\langle \pi(\nu_h), \nabla_Z U \times T\rangle = \langle J(\pi(\nu_h)), \nabla_Z U\rangle$, then we obtain

$$\begin{aligned}\langle \pi_K(\nu_h), \left.\frac{\nabla}{ds}\right|_{s=0} V(s)\rangle &= -\langle N, T\rangle (Z(\langle U, T\rangle) + 2\langle J(U), Z\rangle)\pi_Z \\ &\quad - (E(\langle U, T\rangle) + 2\langle J(U), E\rangle)\pi_\nu - |N_h| \langle J(\pi_K(\nu_h)), \nabla_Z U\rangle \\ &= -\langle N, T\rangle Z(\langle U, T\rangle)\pi_Z - E(\langle U, T\rangle)\pi_\nu \\ &\quad - 2\langle N, T\rangle \langle J(U), \pi_K(\nu_h)\rangle - |N_h| \langle J(\pi_K(\nu_h)), \nabla_Z U\rangle. \end{aligned} \tag{3.10}$$

This implies (3.5).

Let us finally prove (3.6). We show in Lemma 3.4 that (3.6) holds for C^2 surfaces. The general result follows by approximation. Following [18, Proposition 1.20] or [23, Remark 6.1], we approximate the (X,Y)-Lipschitz surface $S=\{p\in\mathbb{H}^1 : f(p)=0\}$ by a family of smooth surfaces $S_j=\{p\in\mathbb{H}^1 : f_j(p)=0\}$, where $f_j=\rho_j * f$ and ρ_j are the standard Friedrichs' mollifiers. Since S is (X,Y)-Lipschitz we gain that $(S_j)_0=\emptyset$. Hence, S_j converges to S on compact subsets in S. Given $j\geq 1$, let Z^j be the characteristic vector field of the regular part of S_j and ν_h^j be the horizontal unit norm to S_j. Then we have

$$Z=\lim_{j\to\infty} Z^j,\quad \nu_h=\lim_{j\to\infty}\nu_h^j,\quad \lim_{j\to\infty}\nabla_{Z^j}\pi_K(\nu_h^j)=\nabla_Z\pi_K(\nu).$$

Since by assumption all the terms are continuous we have that the convergence is uniformly on each compact set of S. On the other hand, since E and N are only L^∞ (thus in particular L^1_{loc}), we have that E^j and N^j converge to E and N a.e. in S. Applying Lemma 3.4 to the smooth surface S_j and Lebesgue's dominated convergence theorem we obtain from (3.10)

$$\begin{aligned}
&\int_S \left\langle \pi(\nu_h), \frac{\nabla}{ds}\bigg|_{s=0} V(s)\right\rangle dS\\
&\quad=\int_S[-\langle N,T\rangle Z(\langle U,T\rangle)\pi_Z - E(\langle U,T\rangle)\pi_\nu\\
&\qquad -2\langle N,T\rangle\langle J(U),\pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)),\nabla_Z U\rangle]dS\\
&\quad=\lim_{j\to\infty}\int_{S_j}[-\langle N^j,T\rangle Z^j(\langle U,T\rangle)\pi_{Z^j} - E^j(\langle U,T\rangle)\pi_{\nu^j}\\
&\qquad -2\langle N^j,T\rangle\langle J(U),\pi(\nu_h^j)\rangle dS_j - |N_h^j|\langle J(\pi(\nu_h^j)),\nabla_{Z^j}U\rangle]dS_j\\
&\quad=\lim_{j\to\infty}\int_{S^j} u\langle\nabla_{Z^j}\pi(\nu_h^j), Z^j\rangle dS_j\\
&\quad=\int_S u\langle\nabla_Z\pi(\nu_h),Z\rangle dS,
\end{aligned}$$

since π_K is C^1. This concludes the proof. ◻

Lemma 3.4. *Let $S\subset\mathbb{H}^1$ be an oriented C^2 surface with $S_0=\emptyset$. Let U be a C^1 vector field with compact support and normal component $u=\langle U,N\rangle\in C^1_0(S)$. Then we have*

$$\begin{aligned}
\int_S u\langle\nabla_Z\pi(\nu_h),Z\rangle dS &= \int_S[-\langle N,T\rangle Z(\langle U,T\rangle)\pi_Z - E(\langle U,T\rangle)\pi_\nu\\
&\quad -2\langle N,T\rangle\langle J(U),\pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)),\nabla_Z U\rangle]dS,
\end{aligned}\tag{3.11}$$

where $\pi_Z=\langle\pi_K,Z\rangle$ and $\pi_\nu=\langle\pi_K,\nu_h\rangle$.

Proof. Since both sides of (3.11) depend linearly on U, we consider the horizontal $U_h=\langle U,Z\rangle Z+\langle U,\nu_h\rangle\nu_h$ and vertical $U_v=\langle U,T\rangle T$ components of U.

We start considering the horizontal component U_h and we denote by I_h the corresponding right-hand side of (3.11). Thus, we obtain

$$\begin{aligned}I_h &= \int_S \{-2\langle N,T\rangle\langle J(U_h),\pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)),\nabla_Z U_h\rangle\}dS \\ &= \int_S -2\langle N,T\rangle\langle J(U_h),\pi(\nu_h)\rangle\, dS + \int_S \{|N_h|Z\langle \pi(\nu_h),J(U_h)\rangle \\ &\quad + |N_h|\langle \nabla_Z\pi(\nu_h),J(U_h)\rangle\}\, dS,\end{aligned}$$

since $\nabla_W J(V) = J(\nabla_W V)$ (as $\nabla_W J = 0$) for each pair of vector fields W, V. By Proposition 2.6, we have

$$\begin{aligned}\int_S |N_h|Z(\langle\pi(\nu_h),J(U_h)\rangle)dS = &-\int_S \langle\pi(\nu_h),J(U_h)\rangle Z(|N_h|)dS \\ &-\int_S |N_h|\langle\pi(\nu_h),J(U_h)\rangle\operatorname{div}_S(Z)dS,\end{aligned}$$

where $\operatorname{div}_S(Z) = \langle D_E Z, E\rangle = -\langle N,T\rangle\theta(E) - 2\langle N,T\rangle|N_h|$. Moreover, we have

$$\langle\nabla_Z\pi(\nu_h),J(U_h)\rangle = -\langle U,\nu_h\rangle\langle\nabla_Z\pi(\nu_h),Z\rangle,$$

since $\langle\nabla_Z\pi(\nu_h),\nu_h\rangle = 0$, by [34, Remark 3.3]. Hence, we have

$$\begin{aligned}I_h &= \langle J(U_h),\pi(\nu_h)\rangle(-2\langle N,T\rangle^3 - Z(|N_h|) + |N_h|\langle N,T\rangle\theta(E)) \\ &\quad + |N_h|\langle U,\nu_h\rangle\langle\nabla_Z\pi(\nu_h),Z\rangle. \end{aligned}\tag{3.12}$$

Now, we consider the vertical component of the variational vector field $U_v = \langle U,T\rangle T$. We denote by I_v the left-hand side of (3.11). Thus,

$$I_v = \int_S -\langle N,T\rangle\pi_Z Z(\langle U,T\rangle) - E(\langle U,T\rangle)\pi_\nu\, dS. \tag{3.13}$$

By Proposition 2.6, we get

$$\begin{aligned}-\int_S \langle N,T\rangle\pi_Z Z(\langle U,T\rangle)dS &= \int_S (Z(\langle N,T\rangle)\pi_Z + \langle N,T\rangle Z(\pi_Z))\langle U,T\rangle dS \\ &\quad - \int_S \langle U,T\rangle\langle N,T\rangle\pi_Z(\langle N,T\rangle\theta(E) + 2\langle N,T\rangle|N_h|)dS \\ &= \int_S \langle U,T\rangle Z(\langle N,T\rangle)\pi_Z dS \\ &\quad + \int_S \langle N,T\rangle\langle U,T\rangle(\langle\nabla_Z\pi(\nu_h),Z\rangle - \pi_\nu\theta(Z))dS \\ &\quad - \int_S \langle U,T\rangle\langle N,T\rangle\pi_Z(\langle N,T\rangle\theta(E)) + 2\langle N,T\rangle|N_h|)dS\end{aligned}$$

and

$$-\int_S \pi_\nu E(\langle U,T\rangle)dS = \int_S E(\pi_\nu)\langle U,T\rangle dS + \int_S \langle U,T\rangle\langle N,T\rangle \pi_\nu \theta(Z)dS$$
$$= \int_S \pi_Z \theta(E)\langle U,T\rangle dS + \int_S \langle U,T\rangle\langle N,T\rangle \pi_\nu \theta(Z)dS.$$

Adding the previous terms we obtain

$$I_v = \int_S \langle U,T\rangle Z(\langle N,T\rangle)\pi_Z dS + \int_S \langle N,T\rangle\langle U,T\rangle(\langle \nabla_Z \pi(\nu_h), Z\rangle)dS$$
$$- \int_S 2\langle U,T\rangle \pi_Z \langle N,T\rangle^2 |N_h| dS + \int_S \pi_Z |N_h|^2 \theta(E)\langle U,T\rangle dS. \qquad (3.14)$$

Since tangential variations do not change the first variation formula, we consider a normal variational vector field $U = uN$ with $u \in C_0^1(S)$ so that $\langle U,Z\rangle = 0$, $\langle U,\nu_h\rangle = u|N_h|$ and $\langle U,T\rangle = u\langle N,T\rangle$. Then adding the integral of the horizontal term (3.12) and the vertical term (3.14) we obtain

$$I_h + I_v = \int_S -u\,\pi_Z |N_h|(-2\langle N,T\rangle^3 - Z(|N_h|) + |N_h|\langle N,T\rangle\theta(E))dS$$
$$+ \int_S u\,\pi_Z(\langle N,T\rangle Z(\langle N,T\rangle) - 2|N_h|\langle N,T\rangle^3 + |N_h|^2\langle N,T\rangle\theta(E))dS$$
$$+ \int_S u(\langle N,T\rangle^2 + |N_h|^2)\langle \nabla_Z \pi(\nu_h), Z\rangle. \qquad (3.15)$$

Since $|N_h|Z(|N_h|) + \langle N,T\rangle Z(\langle N,T\rangle) = 0$ we obtain

$$I_h + I_v = \int_S u\langle \nabla_Z \pi(\nu_h), Z\rangle$$

thus proving (3.11). □

To obtain the first variation formula (3.6) we had to assume that the derivatives in the Z-direction of the vector fields ν_h and Z exist and are continuous functions on S. Let us see that the area-stationary property of S implies this regularity property. We follow the arguments in [29, 32]. This result was first proven in the sub-Riemannian case by Nicolussi and Serra Cassano [32].

Definition 3.5. Let $S \subset \mathbb{H}^1$ be a (X,Y)-Lipschitz surface with boundary ∂S. We say S is area-stationary if, for any C^1 vector field U with compact support such that $\operatorname{supp}(U) \cap \partial S = \emptyset$, and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$, we have

$$\left.\frac{d}{ds}\right|_{s=0} A_K(\varphi_s(S)) = 0.$$

Remark 3.6. Let D be a domain in the vertical plane $\{y = 0\}$ and let $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function. Since the vector field Y is a unit normal to the plane, the intrinsic graph $\mathrm{Gr}(u)$ is parametrized by

$$(x,t) \to (x, u(x,t), t - xu(x,t)).$$

Let $\gamma(s) = (x,t)(s)$ be a Lipschitz curve in D. Its lifting

$$\Gamma(s) = (x, u(x,t), t - xu(x,t))(s) \subset \mathrm{Gr}(u)$$

is also Lipschitz and

$$\Gamma'(s) = x'X + (x'u_x + t'u_t)Y + (t' - 2ux')T$$

a.e. in s. In particular horizontal curves in $\mathrm{Gr}(u)$ satisfy the ordinary differential equation

$$t' = 2u(x,t)\,x'. \tag{3.16}$$

Theorem 3.7. *Let $K \in C^2_+$ be a convex body with $0 \in \mathrm{int}(K)$. Let $S \subset \mathbb{H}^1$ be an area-stationary (X,Y)-Lipschitz surface. Then S is an $\mathbb{H}$-regular surface foliated by horizontal straight lines.*

Proof. Let p in S. Since S is (X,Y)-Lipschitz, by Theorem 2.3, there exist an open ball $B_r(p)$ and a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$ where $\mathrm{Gr}(u) = \{(x, u(x,y), t - xu(x,t)) \in \mathbb{H}^1 : (x,t) \in D\}$. Setting $\pi_K = (\pi_1, \pi_2)$ the area functional is given by

$$A(\mathrm{Gr}(u)) = \int_D (u_x + 2uu_t)\pi_1(u_x + 2uu_t, -1) - \pi_2(u_x + 2uu_t, -1)\,dxdt.$$

Given $v \in C_0^\infty(D)$, a straightforward computation shows that

$$\frac{d}{ds}\bigg|_{s=0} A(\mathrm{Gr}(u + sv)) = \int_D (v_x + 2vu_t + 2uv_t) M\,dxdt, \tag{3.17}$$

where

$$M = F(u_x + 2uu_t), \tag{3.18}$$

and F is the function

$$F(x) = \pi_1(x,-1) + x\frac{\partial \pi_1}{\partial x}(x,-1) - \frac{\partial \pi_2}{\partial x}(x,-1). \tag{3.19}$$

Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane. By composition with a left-translation we may assume that $(0,0) \in D$ is the projection of p to the xt-plane. We parametrize γ by $s \to (s, t(s))$. By Remark 3.6, the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$. For ε small enough, Picard–Lindelöf's theorem implies the existence of $r > 0$ and a solution $t_\varepsilon :]-r, r[\to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(0) = \varepsilon. \end{cases} \tag{3.20}$$

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. By Lemma 3.9, we obtain that $G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi))$ is a bi-Lipschitz homeomorphisms where the determinant of the Jacobian of G is given by $\partial t_\varepsilon(s)/\partial\varepsilon > C > 0$ for each $s \in]-r, r[$ and a.e. in ε.

Any function φ defined on D can be considered as a function of the variables (ξ, ε) by making $\tilde{\varphi}(\xi, \varepsilon) = \varphi(\xi, t_\varepsilon(\xi))$. Since the function G is C^1 with respect to ξ we have

$$\frac{\partial \tilde{\varphi}}{\partial \xi} = \varphi_x + t'_\varepsilon \, \varphi_t = \varphi_x + 2u\varphi_t.$$

Furthermore, by [17, Theorem 2 in Sec. 3.3.3] or [30, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of v is contained in a sufficiently small neighborhood of $(0,0)$, we can express the integral (3.17) as

$$\int_I \left(\int_{-r}^{r} \left(\frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v}\, \tilde{u}_t \right) \tilde{M} \, d\xi \right) d\varepsilon = 0, \tag{3.21}$$

where I is a small interval containing 0. In Eq. (3.21), we used the area-stationary assumption. Instead of $\tilde{v}$ in (3.21) we consider the function $\tilde{v}h/(t_{\varepsilon+h} - t_\varepsilon)$, where h is a small enough parameter. Then we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi} \left(\frac{\tilde{v}h}{t_{\varepsilon+h} - t_\varepsilon} \right) &= \frac{\partial \tilde{v}}{\partial \xi} \, \frac{h}{t_{\varepsilon+h} - t_\varepsilon} - \tilde{v}h \frac{t'_{\varepsilon+h} - t'_\varepsilon}{(t_{\varepsilon+h} - t_\varepsilon)^2} \\ &= \frac{\partial \tilde{v}}{\partial \xi} \, \frac{h}{t_{\varepsilon+h} - t_\varepsilon} - 2\tilde{v}h \frac{u(\xi, t_{\varepsilon+h}(\xi)) - u(\xi, t_\varepsilon(\xi))}{(t_{\varepsilon+h} - t_\varepsilon)^2}, \end{aligned}$$

that tends to

$$\left(\frac{\partial t_\varepsilon}{\partial \varepsilon} \right)^{-1} \left(\frac{\partial \tilde{v}}{\partial \xi} - 2\tilde{v}\tilde{u}_t \right) \quad \text{a.e. in } \varepsilon,$$

when h goes to 0. Putting $\tilde{v}h/(t_{\varepsilon+h} - t_\varepsilon)$ in (3.21) instead of $\tilde{v}$ we get

$$\int_I \left(\int_{-r}^{r} \frac{h \frac{\partial t_\varepsilon}{\partial \varepsilon}}{t_{\varepsilon+h} - t_\varepsilon} \left(\frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v} \left(\tilde{u}_t - \frac{\tilde{u}(\xi, \varepsilon + h) - \tilde{u}(\xi, \varepsilon)}{(t_{\varepsilon+h} - t_\varepsilon)} \right) \right) \tilde{M} \, d\xi \right) d\varepsilon = 0.$$

Using Lebesgue's dominated convergence theorem and letting $h \to 0$ we have

$$\int_I \left(\int_{-r}^{r} \frac{\partial \tilde{v}}{\partial \xi} \tilde{M} \, d\xi \right) d\varepsilon = 0. \tag{3.22}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a positive function compactly supported in I and for $\rho > 0$ we consider the family $\eta_\rho(x) = \rho^{-1}\eta(\frac{x - \varepsilon_0}{\rho})$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(\varepsilon)\psi(\xi)$ in (3.22) and letting $\rho \to 0$ we get

$$\int_{-r}^{r} \psi'(\xi) \tilde{M}(\xi, \varepsilon_0) \, d\xi = 0, \tag{3.23}$$

for each $\psi \in C_0^\infty((-r,r))$ for a.e. ε_0 in I. Since F is C^1 and the distributional derivatives of a Lipschitz function belongs L^∞ we gain that M defined in (3.18) is $L^\infty(D)$. In particular, we have that M belongs $L^1_{\text{loc}}(D)$, thus by Fubini's Theorem also $\tilde{M}(\cdot, \varepsilon_0)$ belongs to $L^1_{\text{loc}}((-r,r))$ for a.e. ε_0 in I. By Eq. (3.23), we gain that $\tilde{M}(\cdot, \varepsilon_0)$ belongs to $W^{1,1}((-r,r))$ with $\partial_\xi \tilde{M} = 0$ a.e. in $(-r,r)$. Then by [3, Theorem

8.2], we gain that $\tilde{M}(\cdot,\varepsilon_0)$ is absolutely continuous and $\partial_\xi \tilde{M} = 0$ a.e. in $(-r,r)$ thus $\tilde{M}(\cdot,\varepsilon_0)$ is constant in ξ for a.e. $\varepsilon_0 \in I$. Therefore, M is constant along $\gamma_{\varepsilon_0}(s) = (s, t_{\varepsilon_0}(s))$ for a.e. ε_0 in I. By [29, Lemma 3.2], F is a C^1 invertible function, therefore also $g(s) = (u_x + 2uu_t)_{\gamma_{\varepsilon_0}(s)} = F^{-1}(M)$ is constant in s for a.e. ε_0 in I. This shows that horizontal normal given by

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}} \tag{3.24}$$

is constant along the characteristic curves, thus also $Z = -J(\nu_h)$ is constant. Hence, the characteristic curves of S are straight lines. Here, we follow the approach developed by [32]. Moreover, since $2g(s) = 2(u_x + 2uu_t)_{\gamma_{\varepsilon_0}(s)} = t''_\varepsilon(s)$ is constant in s we have that $t_\varepsilon(s)$ is a polynomial of the second order given by

$$t_\varepsilon(s) = \varepsilon + a(\varepsilon)s + b(\varepsilon)s^2,$$

where $a(\varepsilon) = 2u(0,\varepsilon)$ that is Lipschitz continuous and $b(\varepsilon) = (u_x + 2uu_t)(0,\varepsilon) = (u_x + 2uu_t)(s,\varepsilon)$. Furthermore, choosing $s > 0$ we can easily prove that $b(\varepsilon)$ is also a Lipschitz function in ε. Hence in particular the horizontal normal ν_h given by (3.24) is continuous, then the surface is an $\mathbb{H}$-regular surface. □

Remark 3.8. Note that an (X,Y)-Lipschitz area-stationary surface is $\mathbb{H}$-regular and its horizontal normal ν_h is C^∞ in the Z-direction.

Lemma 3.9. *With the previous notation, there exists a bi-Lipschitz homeomorphism $G(\xi,\varepsilon) = (\xi, t_\varepsilon(\xi))$. Moreover, there exists a constant $C > 0$ such that $\partial t_\varepsilon(s)/\partial\varepsilon > C$ for each $s \in]-r, r[$ and a.e. in ε.*

Proof. Here, we exploit an argument similar to the one developed in [32]. By [42, Theorem 2.8], we gain that t_ε is Lipschitz with respect to ε with Lipschitz constant less than or equal to e^{Lr}. Fix $s \in]-r, +r[$, the inverse of the function $\varepsilon \to t_\varepsilon(s)$ is given by $\bar{\chi}_t(-s) = \chi_t(-s)$ where χ_t is the unique solution of the following Cauchy problem:

$$\begin{cases} \chi_t'(\tau) = 2u(\tau, \chi_t(\tau)), \\ \chi_t(s) = t. \end{cases} \tag{3.25}$$

Again by [42, Theorem 2.8], we have that $\bar{\chi}_t$ is Lipschitz continuous with respect to t, thus the function $\varepsilon \to t_\varepsilon$ is a locally bi-Lipschitz homeomorphisms.

We consider the following Lipschitz coordinates:

$$G(\xi,\varepsilon) = (\xi, t_\varepsilon(\xi)) = (s,t) \tag{3.26}$$

around the characteristic curve passing through $(0,0)$. Note that by the uniqueness result for (4.5), G is injective. Given (s,t) in the image of G and using the inverse function $\bar{\chi}_t$ defined in (3.25) we find ε such that $t_\varepsilon(s) = t$, therefore G is surjective. By the Invariance of Domain Theorem [4], G is a homeomorphism. By the uniqueness result of the Cauchy problem (4.5) we get that the map $\varepsilon \to t_\varepsilon(s)$ is not decreasing in ε, then we have

$$\frac{\partial t_\varepsilon(s)}{\partial \varepsilon} \geq 0 \tag{3.27}$$

for a.e. in ε. The differential of G is defined by

$$DG = \begin{pmatrix} 1 & 0 \\ t'_\varepsilon & \dfrac{\partial t_\varepsilon}{\partial \varepsilon} \end{pmatrix} \tag{3.28}$$

almost everywhere in ε and the differential of G^{-1} is given by

$$DG^{-1} = \left(\frac{\partial t_\varepsilon}{\partial \varepsilon}\right)^{-1} \begin{pmatrix} \dfrac{\partial t_\varepsilon}{\partial \varepsilon} & 0 \\ -t'_\varepsilon & 1 \end{pmatrix} \tag{3.29}$$

almost everywhere in ε. Since G^{-1} is Lipschitz we gain that there exists a constant $C > 0$ such that

$$|\mathbf{J}_{G^{-1}}| = |\det(DG^{-1})| = \left|\frac{\partial t_\varepsilon}{\partial \varepsilon}\right|^{-1} \leq \frac{1}{C}$$

a.e. in ε. Thus, by (3.27) we deduce that $\partial t_\varepsilon/\partial\varepsilon > C > 0$ a.e. in ε. □

4. A Codazzi-like Equation for (X, Y)-Lipschitz Minimal Surfaces

In this section, we shall show that, given an area-stationary surface S, the function $\langle N, T\rangle/|N_h|$ satisfies a differential equation along almost every characteristic curve on S.

We first prove a technical result similar to [26, Lemma 4.2]. We include the proof, with only slight differences, for completeness.

Lemma 4.1. *Given $a, b \in \mathbb{R}$, the only solution of equation*

$$y'' - 6y'y + 4y^3 = 0 \tag{4.1}$$

about the origin with initial conditions $y(0) = a$, $y'(0) = b$, is

$$y_{a,b}(s) = \frac{a - (2a^2 - b)s}{1 - 2as + (2a^2 - b)s^2}. \tag{4.2}$$

Moreover, we have

$$y^2_{a,b}(s) - y'_{a,b}(s) = \frac{a^2 - b}{(1 - 2as + (2a^2 - b)s^2)^2}. \tag{4.3}$$

If $y_{a,b}$ is defined for every $s \in \mathbb{R}$ then either $a^2 - b > 0$ or $y_{a,b} \equiv 0$.

Proof. By the uniqueness of solutions for ordinary differential equations we know that there exists a unique solution (4.1). Since we have

$$y'_{a,b} = \frac{b - 2as(2a^2 - b) + (2a^2 - b)^2 s^2}{(1 - 2as + (2a^2 - b)s^2)^2}$$

and

$$y''_{a,b} = \frac{2(a - (2a^2 - b)s)(3b - 2as(2a^2 - b) + (2a^2 - b)^2 s^2 - 2a^2)}{(1 - 2as + (2a^2 - b)s^2)^3},$$

a straightforward computation shows that $y_{a,b}(s)$ solves (4.1) and satisfies (4.3).

Let us write $y_{a,b} = p(s)/r(s)$ where $p(s) = a - (2a^2 - b)s$ and $r(s) = 1 - 2as + (2a^2 - b)s^2$. If $y_{a,b}$ is defined for every $s \in \mathbb{R}$, then there are two possibilities: $r(s)$ has no real zeroes or $r(s)$ has at least a zero at $s_0 \in \mathbb{R}$. In the first case the discriminant is $4(b - a^2)$ is negative. In the second case $r(s_0) = 0$ we must also have $p(s_0) = 0$ in order to have $y_{a,b}(s)$ well defined at s_0. Hence, $y_{a,b}(s)$ can be expressed as the quotient of a constant over a degree one polynomial. Then by (4.2) we get that $y_{a,b} = a(1 - 2as)^{-1}$ which has a pole unless $a = 0$, hence $y_{a,b}(s) \equiv 0$. □

Remark 4.2. If $f(s)$ is a solution of (4.1), then for each positive constant λ the function $f_\lambda(s) = \lambda^{-1} f(\frac{s}{\lambda})$ is still a solution of (4.1).

Remark 4.3. Let f be a solution of (4.1), then f belongs to C^∞ class. Indeed setting $y_1 = f$ and $y_2 = y'_1$ we have that (4.1) is equivalent to

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = F(y_1, y_2) = \begin{pmatrix} y_2 \\ -6y_1 y_2 - 4y_1^3 \end{pmatrix}.$$

Since F is C^∞ we obtain that $y_1 = f$ is smooth.

Proposition 4.4. *Let S be a complete oriented area-stationary (X,Y)-Lipschitz surface. Then along any arc-length parametrized geodesic $\bar\gamma_\varepsilon(s)$ in S, the function $\langle N, T\rangle/|N_h|(\bar\gamma_\varepsilon(s))$ satisfies the ordinary differential equation (4.1) for a.e. ε. Furthermore, $\langle N, T\rangle/|N_h|(\bar\gamma_\varepsilon(s))$ is smooth in s for a.e. ε.*

Proof. Let p in S. Since S is (X,Y)-Lipschitz, Theorem 2.3 implies the existence of an open ball $B_r(p)$ and of a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$. Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane, and $(0,0) \in D$ the projection of p to the xt-plane. We parametrize γ by $s \to (s, t(s))$. By Remark 3.6 (see also [29, Remark

2.5]) the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$ and

$$\Gamma'(s) = X + (u_x + 2uu_t)Y.$$

As computed in [29, Sec. 2.6], at smooth points of the graph of u, the unit normal can be computed as $N = \tilde{N}/|\tilde{N}|$, where

$$\tilde{N} = (u_x + 2uu_t) - Y + u_t T.$$

The quantity $|\tilde{N}|$ is the Riemannian Jacobian of the parametrization of the intrinsic graph of u by coordinates (x, t). So, we have

$$|N_h|\, dS = \sqrt{1 + (u_x + 2uu_t)^2}\, dx\, dt. \tag{4.4}$$

Since we have

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}} \quad \text{and} \quad Z = -J(\nu_h)$$

we get $Z = -\Gamma'(s)/|\Gamma(s)|$. For ε small enough, Picard–Lindelöf's theorem implies the existence of $r > 0$ and a solution $t_\varepsilon :]-r, r[\to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(0) = \varepsilon. \end{cases} \tag{4.5}$$

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. Since S is area-stationary we have that $(u_x + 2uu_t)$ is constant along $\gamma_\varepsilon(s)$. Moreover,

$$t''_\varepsilon(s) = 2(u_x + 2uu_t)(\gamma_\varepsilon(s)) = 2b(\varepsilon) = 2(u_x + 2uu_t)(0, \varepsilon)$$

is constant as a function of s. Thus, we have

$$t_\varepsilon(s) = \varepsilon + a(\varepsilon)s + b(\varepsilon)s^2, \tag{4.6}$$

where $a(\varepsilon) = 2u(0, \varepsilon)$. Choosing $s > 0$ in (4.6) we can easily prove that $b(\varepsilon)$, that *a priori* is only continuous, is also a Lipschitz function. By [32, Eq. (7), Theorem 3.7], we have

$$\frac{\partial}{\partial \varepsilon}\frac{\partial}{\partial s} t_\varepsilon(s) = \frac{\partial}{\partial s}\frac{\partial}{\partial \varepsilon} t_\varepsilon(s) \tag{4.7}$$

a.e. in ε, where the equality has to be interpreted in the sense of distributions. Putting (4.5) in the left-hand side of (4.7) and applying the chain rule for Lipschitz functions (see [32, Remark 3.6]) we get

$$2u_t(s,t_\varepsilon(s))(1+a'(\varepsilon)s+b'(\varepsilon)s^2)=(a'(\varepsilon)+2b'(\varepsilon)s)$$

a.e. in ε. Therefore, we get

$$u_t(s,t_\varepsilon(s))=\frac{\frac{a'(\varepsilon)}{2}+b'(\varepsilon)s}{(1+a'(\varepsilon)s+b'(\varepsilon)s^2)},$$

a.e. in ε, since by Lemma 3.9 we have $\partial t_\varepsilon/\partial\varepsilon>0$ a.e. in ε. Since we have $Z=-\Gamma'(s)/|\Gamma(s)|$ we consider $\tilde{\gamma}_\varepsilon(s)=\gamma_\varepsilon(-s)$. Then we have that

$$u_t(\tilde{\gamma}_\varepsilon(s))=\frac{\frac{a'(\varepsilon)}{2}-b'(\varepsilon)s}{(1-a'(\varepsilon)s+b'(\varepsilon)s^2)}$$

solves Eq. (4.1) with initial condition $y(0)=a'(\varepsilon)/2$ and $y'(0)=\frac{a'(\varepsilon)^2}{2}-b'(\varepsilon)$ for a.e. ε. Moreover, we have

$$t_\varepsilon(-s)=\varepsilon-a(\varepsilon)s+b(\varepsilon)s^2.$$

For each ε fixed we have $b(\varepsilon)=(u_x+2uu_t)(\tilde{\gamma}_\varepsilon)$ is constant, let

$$\bar{\gamma}_\varepsilon(s)=\tilde{\gamma}_\varepsilon\left(\frac{s}{\sqrt{1+b(\varepsilon)^2}}\right)$$

be an arc-length parametrization of $\tilde{\gamma}_\varepsilon$. Then Remark 4.2 shows that also

$$\langle N,T\rangle/|N_h|(\bar{\gamma}_\varepsilon)=\frac{u_t}{\sqrt{1+(u_x+2uu_t)^2}}(\bar{\gamma}_\varepsilon)$$

is a solution of (4.1) a.e. in ε. □

5. Second Variation Formula

In this section, we compute the second variation formula. First of all we need the following lemma.

Lemma 5.1. *Let U be a C^2 horizontal vector field in $\mathbb{H}^1$ with associated flow $\{\varphi_s\}_{s\in\mathbb{R}}$. Let $p\in\mathbb{H}^1$ and $e\in T_p\mathbb{H}^1$. Define the smooth curve $\beta(s)=\varphi_s(p)$ and the smooth vector field $E(s)=(d\varphi_s)_p(e)$ along β. Then we have*

$$\left.\frac{\nabla^2}{ds^2}\right|_{s=0}E(s)=\nabla_e\nabla_U U+2\langle J(\nabla_{U_p}U),e\rangle T_p+2\langle J(U_p),\nabla_e U\rangle T_p.$$

Proof. From (3.1), we get

$$\frac{\nabla^2}{ds^2}\bigg|_{s=0} E(s) = \frac{\nabla}{ds}\bigg|_{s=0} (\nabla_{E(s)} U_s + 2\langle J(U_s), E(s)\rangle\, T).$$

On the one hand, we have

$$\begin{aligned}\frac{\nabla}{ds}\bigg|_{s=0} \nabla_{E(s)} U_s &= \frac{\nabla}{ds}\bigg|_{s=0} \sum_{i=1}^{3} g_i(s) \nabla_{(\frac{\partial}{\partial x_i})_{\beta(s)}} U_s \\ &= \sum_{i=1}^{3} g_i'(0) \nabla_{(\frac{\partial}{\partial x_i})_p} U + g_i(0)\, \frac{\nabla}{ds}\bigg|_{s=0} \nabla_{(\frac{\partial}{\partial x_i})_p} U \\ &= \sum_{i=1}^{3} e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + e_i \nabla_U \nabla_{(\frac{\partial}{\partial x_i})_p} U.\end{aligned}$$

Note that

$$\begin{aligned}\nabla_e(\nabla_U U) = \nabla_e \nabla_{\sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}} U &= \sum_{i=1}^{3} e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + f_i(p)\, \nabla_e \nabla_{(\frac{\partial}{\partial x_i})} U \\ &= \sum_{i=1}^{3} e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i,j=1}^{3} f_i(p)\, e_j \nabla_{(\frac{\partial}{\partial x_j})} \nabla_{(\frac{\partial}{\partial x_i})} U \\ &= \sum_{i=1}^{3} e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i,j=1}^{3} f_i(p)\, e_j \nabla_{(\frac{\partial}{\partial x_i})} \nabla_{(\frac{\partial}{\partial x_j})} U \\ &= \sum_{i=1}^{3} e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i=1}^{3} e_i \nabla_U \nabla_{(\frac{\partial}{\partial x_i})} U,\end{aligned}$$

where we use that the Riemann tensor of ∇ vanishes

$$0 = R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right) U = \nabla_{(\frac{\partial}{\partial x_i})} \nabla_{(\frac{\partial}{\partial x_j})} U - \nabla_{(\frac{\partial}{\partial x_j})} \nabla_{(\frac{\partial}{\partial x_i})} U.$$

Therefore,

$$\frac{\nabla}{ds}\bigg|_{s=0} \nabla_{E(s)} U_s = \nabla_e(\nabla_U U) + \nabla_e U'. \tag{5.1}$$

On the other hand, since $\nabla J = 0$ we have

$$\begin{aligned}\frac{\nabla}{ds}\bigg|_{s=0} 2\langle J(U_s), E(s)\rangle\, T &= 2\langle J(\nabla_U U), e\rangle\, T + 2\langle J(U), \frac{\nabla}{ds}\bigg|_{s=0} E(s)\rangle\, T \\ &= 2\langle J(\nabla_U U), e\rangle\, T + 2\langle J(U), \nabla_e U\rangle\, T, \end{aligned} \tag{5.2}$$

where we have used once again Lemma 3.1 and the fact that $J(U)$ is horizontal. Finally, adding (5.1) and (5.2) we get the result. □

Now, we compute the second variation formula.

Theorem 5.2. *Let $K \in C^2_+$ be a convex body with $0 \in \operatorname{int}(K)$. Let $S \subset \mathbb{H}^1$ be an area-stationary (X,Y)-Lipschitz surface. Let U be an horizontal C^2 vector field compactly supported in $\mathbb{H}^1$, with $\partial S \cap supp(U) = \emptyset$, and associated flow $\{\varphi_s\}_{s\in\mathbb{R}}$. Then the second variation of the sub-Finsler area induced by U is given by*

$$\frac{d^2}{ds^2}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_S (Z(f)^2 + qf^2)\frac{|N_h|}{\kappa(\pi_K(\nu_h))}dS, \tag{5.3}$$

where

$$q = 4\left\{Z\left(\frac{\langle N,T\rangle}{|N_h|}\right) - \frac{\langle N,T\rangle^2}{|N_h|^2}\right\},$$

κ is the positive curvature of the boundary ∂K and $f = \langle U, \nu_h\rangle$.

Remark 5.3. As noticed in Sec. 1, there is a slight difference in this second variation formula with respect to the sub-Riemannian one computed in [26], due to the definition of Z as $-J(\nu_h)$ in this paper, instead of $J(\nu_h)$ as in [26]. This change was introduced in [34] as the most convenient way of dealing with the lack of symmetry of the sub-Finsler norm.

Proof of Theorem 5.2. First of all we notice that $\langle N,T\rangle/|N_h|$ is smooth in the Z-direction by Proposition 4.4, and so q is well defined. Moreover, by Theorem 3.7 an area-stationary (X,Y)-Lipschitz surface is $\mathbb{H}$-regular. The area functional is given by

$$A_K(\varphi_s(S)) = \int_S \langle \pi(V(s)), V(s)\rangle\, dS,$$

where

$$V(s) = \langle E_1(s), T\rangle\, T \times E_2(s) + \langle E_2(s), T\rangle\, E_1(s) \times T,$$

and dS is the Riemannian area element. At a regular point $p \in S$, a basis of tangent vectors to $\varphi_s(S)$ at $\varphi_s(p)$ is given by $E_1(s) = (d\varphi_s)_p(Z_p)$ and $E_2(s) = (d\varphi_s)_p(E_p)$.

Since by Lemma 3.2 the norm of $V(s)$ is strictly positive a.e. in $\operatorname{supp}(U) \cap S$, we have

$$\begin{aligned}\frac{d^2}{ds^2}||V(s)||_* &= \frac{d}{ds}\left\langle \pi_K(V(s)), \frac{\nabla}{ds}V(s)\right\rangle \\ &= \left\langle d\pi(V(s))\frac{\nabla}{ds}V(s), \frac{\nabla}{ds}V(s)\right\rangle + \left\langle \pi_K(V(s)), \frac{\nabla^2}{ds^2}V(s)\right\rangle \\ &\leq \frac{1}{\bar\kappa}\left|\frac{\nabla}{ds}V(s)\right|^2 + \beta\left|\frac{\nabla^2}{ds^2}V(s)\right|, \end{aligned} \tag{5.4}$$

a.e. in $\operatorname{supp}(U) \cap S$, where $\bar\kappa = \min_{||v||_*=1}\kappa(v)$ and β is the positive constant defined in Sec. 2.4 with K' equal to the Euclidean ball centered at the origin.

Setting $\tilde{N}(s) = E_1(s) \times E_2(s)$ we get

$$\frac{\nabla^2}{ds^2}\tilde{N}(s) = \frac{\nabla^2}{ds^2}E_1(s) \times E_2(s) + 2\frac{\nabla}{ds}E_1(s) \times \frac{\nabla}{ds}E_2(s) + E_1(s) \times \frac{\nabla^2}{ds^2}E_2(s).$$

Then Lemmas 3.1 and 5.1 imply

$$\frac{\nabla}{ds}E_i(s) = \nabla_{E_i(s)}U + 2\langle J(U_p), E_i(s)\rangle\, T,$$
$$\frac{\nabla^2}{ds^2}E_i(s) = \nabla_{E_i(s)}\nabla_U U + 2\,\langle J(\nabla_U U), E_i(s)\rangle T + 2\,\langle J(U_p), \nabla_{E_i(s)}U\rangle\, T$$

and $|E_1(s)| = |d\varphi_s(Z)| \leq C'$ and $|E_1(s)| = |d\varphi_s(E)| \leq C'$ for $s \in (-s_0, s_0)$, where the constant C' is independent of s. Then, writing the covariant derivative $\nabla_{E_i(s)}U$ and $\nabla_{E_i(s)}\nabla_U U$ in standard coordinates, we obtain

$$\left|\frac{\nabla^2}{ds^2}\tilde{N}(s)\right| \leq \tilde{C}\,\|U\|_{C^2}$$

a.e. in $\mathrm{supp}(U) \cap S$ for a suitable constant $\tilde{C} > 0$ and where $\|U\|_{C^2}$ denotes the standard C^2 norm of U. Then, since $V(s) = \tilde{N}(s) - \langle\tilde{N}(s), T\rangle\, T$ and thus $\frac{\nabla^2}{ds^2}V(s) = \frac{\nabla^2}{ds^2}\tilde{N}(s) - \langle\frac{\nabla^2}{ds^2}\tilde{N}(s), T\rangle\, T$, we have

$$\left|\frac{\nabla^2}{ds^2}V(s)\right| \leq 2\left|\frac{\nabla^2}{ds^2}\tilde{N}(s)\right| \leq 2\,\tilde{C}\,\|U\|_{C^2}$$

a.e. in $\mathrm{supp}(U) \cap S$. Then, since $\mathrm{supp}(U) \cap S$ is compact, Lebesgue's dominated convergence theorem yields

$$\left.\frac{d^2}{ds^2}\right|_{s=0} A_K(\varphi_s(S)) = \int_S \left.\frac{d^2}{ds^2}\right|_{s=0} \langle\pi(V(s)), V(s)\rangle\, dS.$$

By [34, Remark 3.3], we get

$$\frac{d}{ds}\left\langle\pi(V(s)), \frac{\nabla}{ds}V(s)\right\rangle = \left\langle\frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s)\right\rangle + \left\langle\pi(V(s)), \frac{\nabla}{ds}\frac{\nabla}{ds}V(s)\right\rangle. \tag{5.5}$$

Again by [34, Remark 3.3], we have

$$0 = \frac{d}{ds}\left\langle\frac{\nabla}{ds}\pi(V(s)), V(s)\right\rangle = \left\langle\frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s)\right\rangle + \left\langle\frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s)\right\rangle.$$

Then we gain

$$\left\langle\frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s)\right\rangle = -\left\langle\frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s)\right\rangle. \tag{5.6}$$

Therefore, substituting (5.6) in (5.5) we obtain

$$\frac{d}{ds}\left\langle\pi(V(s)), \frac{\nabla}{ds}V(s)\right\rangle = \left\langle\pi(V(s)), \frac{\nabla}{ds}\frac{\nabla}{ds}V(s)\right\rangle - \left\langle\frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s)\right\rangle.$$

Evaluating the previous equality at $s=0$ we get

$$\frac{d^2}{ds^2}\Big|_{s=0}\langle\pi(V(s)),V(s)\rangle=\left\langle\pi(\nu_h),\,\frac{\nabla^2}{ds^2}\Big|_{s=0}V(s)\right\rangle-\left\langle\frac{\nabla^2}{ds^2}\Big|_{s=0}\pi(V(s)),\nu_h\right\rangle$$
$$=\mathbf{I}+\mathbf{II},\tag{5.7}$$

since $V(0)=|N_h|\nu_h$.

As

$$\frac{\nabla}{ds}V(s)=\left\langle\frac{\nabla}{ds}E_1(s),T\right\rangle T\times E_2(s)+\langle E_1(s),T\rangle T\times\frac{\nabla}{ds}E_2(s)$$
$$+\left\langle\frac{\nabla}{ds}E_2(s),T\right\rangle E_1(s)\times T+\langle E_2(s),T\rangle\frac{\nabla}{ds}E_1(s)\times T,$$

we obtain

$$\begin{aligned}\frac{\nabla^2}{ds^2}\Big|_{s=0}V(s)=&\left\langle\frac{\nabla^2}{ds^2}\Big|_{s=0}E_1(s),T\right\rangle T\times E_2(0)\\&+2\left\langle\frac{\nabla}{ds}\Big|_{s=0}E_1(s),T\right\rangle T\times\frac{\nabla}{ds}\Big|_{s=0}E_2(s)\\&+\langle E_1(0),T\rangle T\times\frac{\nabla^2}{ds^2}\Big|_{s=0}E_2(s)+\left\langle\frac{\nabla^2}{ds^2}\Big|_{s=0}E_2(s),T\right\rangle E_1(0)\times T\\&+2\left\langle\frac{\nabla}{ds}\Big|_{s=0}E_2(s),T\right\rangle\frac{\nabla}{ds}\Big|_{s=0}E_1(s)\times T+\langle E_2(0),T\rangle\frac{\nabla^2}{ds^2}\Big|_{s=0}\\&\times E_1(s)\times T.\end{aligned}$$

By Lemma 3.1,

$$\frac{\nabla}{ds}\Big|_{s=0}E_i(s)=\nabla_{E_i(0)}U+2\langle J(U),E_i(0)\rangle T,$$

for $i=1,2$. By Lemma 5.1, we gain

$$\frac{\nabla^2}{ds^2}\Big|_{s=0}E_i(s)=\nabla_{E_i(0)}(\nabla_UU)+2\langle J(\nabla_UU),E_i(0)\rangle T+2\langle J(U),\nabla_{E_i(0)}U\rangle T.$$

Noting that ∇_ZU is horizontal we get

$$\begin{aligned}\frac{\nabla^2}{ds^2}\Big|_{s=0}V(s)=&(\langle\nabla_Z(\nabla_UU),T\rangle+2\langle J(\nabla_UU),Z\rangle+2\langle J(U),\nabla_ZU\rangle)T\times E\\&+4\langle J(U),Z\rangle T\times\nabla_EU+(\langle\nabla_E(\nabla_UU),T\rangle+2\langle J(\nabla_UU),E\rangle\\&+2\langle J(U),\nabla_EU\rangle)Z\times T+2(\langle\nabla_EU,T\rangle+2\langle J(U),E\rangle)\nabla_ZU\times T\\&-|N_h|\nabla_Z(\nabla_UU)\times T.\end{aligned}$$

We set $\pi_K(\nu_h)=\pi_ZZ+\pi_\nu\nu_h$, where $\pi_Z=\langle\pi(\nu_h),Z\rangle$ and $\pi_\nu=\langle\pi(\nu_h),\nu_h\rangle$. Note that $T\times\nu_h=-Z$, $Z\times T=-\nu_h$ and $\langle\pi(\nu_h),W\times T\rangle=\langle J(\pi(\nu_h)),W\rangle$ for each vector field W, then a straightforward computation shows that

$$\mathbf{I} = \langle \pi(\nu_h), \frac{\nabla^2}{ds^2}\Big|_{s=0} V(s)\rangle = \mathbf{A} + \mathbf{B}, \tag{5.8}$$

where

$$\begin{aligned}\mathbf{A} = &-2\langle N, T\rangle\langle J(U), \nabla_Z U\rangle \pi_Z - 4\langle J(U), Z\rangle\langle J(\pi(\nu_h)), \nabla_E U\rangle \\ &- 2\pi_\nu\langle J(U), \nabla_E U\rangle + 4\langle J(\pi(\nu_h)), \nabla_Z U\rangle\langle J(U), E\rangle\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\mathbf{B} = &-\langle N, T\rangle Z(\langle \nabla_U U, T\rangle)\pi_Z - E(\langle \nabla_U U, T\rangle)\pi_\nu \\ &- 2\langle N, T\rangle\langle J(\nabla_U U), \pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)), \nabla_Z \nabla_U U\rangle.\end{aligned} \tag{5.10}$$

Since S is area-stationary, by Eq. (3.5) in Proposition 3.3 we have that

$$\begin{aligned}0 = \frac{d}{ds}\Big|_{s=0} A_K(\varphi_s(S)) = \int_S [&-\langle N, T\rangle Z(\langle U, T\rangle)\pi_Z - E(\langle U, T\rangle)\pi_\nu \\ &- 2\langle N, T\rangle\langle J(U), \pi(\nu_h)\rangle - |N_h|\langle J(\pi(\nu_h)), \nabla_Z U\rangle]dS,\end{aligned} \tag{5.11}$$

for every U compactly supported C^1 vector field. Taking into account the first variation formula (5.11) induced by the C^1 horizontal vector field $\nabla_U U$ we get

$$\int_S \mathbf{B} = 0.$$

Thus, we obtain

$$\int_S \mathbf{I} = \int_S \mathbf{A}. \tag{5.12}$$

On the other hand, by Lemma 5.4 and by Eq. (5.17) in Remark 5.5, we obtain

$$\mathbf{II} = -\left\langle \frac{\nabla^2}{ds^2}\Big|_{s=0} \pi(V(s)), \nu_h \right\rangle = \left\langle \frac{\nabla}{ds}\Big|_{s=0} \left(\frac{V(s)}{|V(s)|}\right), (d\pi)^*_{\nu_h} Z \right\rangle \left\langle \frac{\nabla}{ds}\Big|_{s=0} V(s), Z \right\rangle.$$

Then [29, Lemma 4.3] yields

$$\mathbf{II} = \frac{1}{\kappa\,|N_h|} \left\langle \frac{\nabla}{ds}\Big|_{s=0} V(s), Z \right\rangle^2,$$

where $\kappa = \kappa(\pi_K(\nu_h))$ is the positive constant curvature of ∂K evaluated at $\pi_K(\nu_h)$, that is constant along the characteristic curves by Theorem 3.7. Thanks to (3.9) we have

$$\left\langle \frac{\nabla}{ds}\bigg|_{s=0} V(s), Z \right\rangle = -2\langle N,T\rangle\langle J(U),Z\rangle - |N_h|\langle \nabla_Z U, \nu_h\rangle$$
$$= 2\langle N,T\rangle\langle U,\nu_h\rangle - |N_h|\langle \nabla_Z U, \nu_h\rangle.$$

Setting $f = \langle U,\nu_h\rangle$ and $g = \langle U, Z\rangle$ we get that $\mathbf{A} + \mathbf{II}$ is equal to

$$2\langle N,T\rangle\pi_Z(gZ(f)+fZ(g)) + 4f\pi_Z(E(f)-g\theta(E)) - 2f\pi_\nu(E(g)+f\theta(E))$$
$$- 2g\pi_\nu(E(f)-g\theta(E)) - 4g\langle N,T\rangle\pi_\nu Z(g)$$
$$+ \frac{1}{\kappa\,|N_h|}(2\langle N,T\rangle f - |N_h|Z(f))^2.$$

Note that

$$\mathbf{II} = \frac{1}{\kappa\,|N_h|}(2\langle N,T\rangle f - |N_h|Z(f))^2$$
$$= \frac{1}{\kappa\,|N_h|}(4\langle N,T\rangle^2 f^2 - 4\langle N,T\rangle|N_h|fZ(f) + |N_h|^2 Z(f)^2). \tag{5.13}$$

The second term of (5.13) can be written as

$$4\frac{\langle N,T\rangle|N_h|}{\kappa\,|N_h|}fZ(f) = 2|N_h|\frac{\langle N,T\rangle}{\kappa\,|N_h|}Z(f^2)$$
$$= 2|N_h|\left(Z\left(\frac{\langle N,T\rangle}{\kappa\,|N_h|}f^2\right) - Z\left(\frac{\langle N,T\rangle}{\kappa\,|N_h|}\right)f^2\right).$$

Then, setting $h = \frac{\langle N,T\rangle f^2}{\kappa|N_h|}$ in Lemma 5.6, we obtain that the integrals of the first and second term in (5.13) are equal to

$$\int_S \left(4\,\frac{\langle N,T\rangle^2}{\kappa|N_h|^2}f^2 - 2\frac{\langle N,T\rangle}{\kappa\,|N_h|}Z(f^2)\right)|N_h|\,dS = 2\int_S Z\left(\frac{\langle N,T\rangle}{\kappa\,|N_h|}\right)f^2\;|N_h|\,dS.$$

The integral of the third summand in (5.13) is equal to

$$\int_S Z(f)^2\,\frac{|N_h|}{\kappa}\,dS.$$

Hence, we obtain

$$\int_S \mathbf{II}\,dS = \int_S Z(f)^2\,\frac{|N_h|}{\kappa}\,dS + 2\int_S Z\left(\frac{\langle N,T\rangle}{\kappa\,|N_h|}\right)f^2\;|N_h|\,dS. \tag{5.14}$$

Finally, we deal with

$$\mathbf{A} = 2\langle N,T\rangle\pi_Z(gZ(f)+fZ(g)) + 4f\pi_Z(E(f)-g\theta(E)) - 2f\pi_\nu(E(g)+f\theta(E))$$
$$- 2g\pi_\nu(E(f)-g\theta(E)) - 4g\langle N,T\rangle\pi_\nu Z(g).$$

By Eq. (5.19) in Lemma 5.6, we have that

$$\begin{aligned}
&2\pi_\nu(g^2\theta(E) - 2\langle N,T\rangle gZ(g)) \\
&\quad = 2\pi_\nu |N_h| \left(g^2 \left(2\frac{\langle N,T\rangle^2}{|N_h|^2} - Z\left(\frac{\langle N,T\rangle}{|N_h|}\right)\right) - \frac{\langle N,T\rangle}{|N_h|} Z(g^2)\right) \\
&\quad = 2||N_h||_* \left(2g^2 \frac{\langle N,T\rangle^2}{|N_h|^2} - Z\left(\frac{\langle N,T\rangle}{|N_h|} g^2\right)\right)
\end{aligned}$$

a.e. in S. Then, by Lemma 5.7 the integral over S of the previous term is equal to zero. Therefore, we have

$$\mathbf{A} = 2\langle N,T\rangle \pi_Z Z(gf) + 4f\pi_Z(E(f) - g\theta(E)) - 2f^2\pi_\nu\theta(E) - 2\pi_\nu E(gf).$$

On the one hand, we have

$$\int_S \pi_\nu E(gf) + \pi_Z\theta(E)gf = 0,$$

by Eq. (5.21) in Lemma 5.7. On the other hand using Eq. (5.20) in Lemma 5.7 and Eq. (5.19) in Lemma 5.6 we have

$$\begin{aligned}
&2\int_S \pi_Z(\langle N,T\rangle Z(gf) - gf\theta(E)) \\
&\quad = 2\int_S \left(Z\left(\frac{\langle N,T\rangle}{|N_h|}\pi_Z gf\right) - 2\frac{\langle N,T\rangle^2}{|N_h|^2}\pi_Z gf\right)|N_h| dS = 0.
\end{aligned}$$

Hence, we gain

$$\begin{aligned}
\mathbf{A} &= 2\pi_Z E(f^2) - 2f^2\pi_\nu\theta(E) = -2f^2(E(\pi_Z) + \pi_\nu\theta(E)) \\
&= -2f^2(\langle \nabla_E \pi(\nu_h), Z\rangle - \pi_\nu\theta(E) + \pi_\nu\theta(E)) \\
&= -2f^2\langle (d\pi)_{\nu_h}(\nabla_E \nu_h), Z\rangle = -2f^2\frac{\theta(E)}{\kappa(\pi(\nu_h))}, \qquad (5.15)
\end{aligned}$$

a.e. in S, where the last inequality follows from [29, Lemma 4.3]. Finally, by (5.19) in Lemma 5.6 and (5.12), (5.14) and (5.15) we obtain

$$\begin{aligned}
\int_S \mathbf{I} + \mathbf{II} = \int_S \mathbf{A} + \mathbf{II} = \int_S &\left(Z(f)^2 + 4\left(Z\left(\frac{\langle N,T\rangle}{|N_h|}\right) - \frac{\langle N,T\rangle^2}{|N_h|^2}\right) f^2\right) \\
&\times \frac{|N_h|}{\kappa(\pi(\nu_h))} dS. \qquad (5.16)
\end{aligned}$$

In the last equation, we use that $Z(\pi(\nu_h)) = 0$, therefore $\kappa(\pi(\nu_h))$ is constant along the characteristic curves. □

Lemma 5.4. *Following the previous notation we have*

$$\left\langle \frac{\nabla}{ds}\frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)), V(s) \right\rangle = -\left\langle \frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)), Z \right\rangle \left\langle \frac{\nabla}{ds}\bigg|_{s=0} V(s), Z \right\rangle.$$

Proof. By [34, Remark 3.3], we have $\langle \frac{\nabla}{ds}\pi(V(s)), V(s)\rangle = 0$. Then

$$\frac{\nabla}{ds}\pi(V(s)) = f(s)J\left(\frac{V(s)}{|V(s)|}\right),$$

where $f(s) = \langle \frac{\nabla}{ds}\pi(V(s)), J(\frac{V(s)}{|V(s)|})\rangle$. Since

$$\frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)) = \frac{d}{ds}f(s)J\left(\frac{V(s)}{|V(s)|}\right) + f(s)\frac{\nabla}{ds}J\left(\frac{V(s)}{|V(s)|}\right)$$

and $\nabla J = 0$ we obtain

$$\begin{aligned}\left\langle \frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s) \right\rangle &= f(s)\left\langle \frac{\nabla}{ds}J\left(\frac{V(s)}{|V(s)|}\right), V(s) \right\rangle \\ &= -f(s)\left\langle \frac{\nabla}{ds}\left(\frac{V(s)}{|V(s)|}\right), J(V(s)) \right\rangle.\end{aligned}$$

Evaluating at $s = 0$ we gain

$$\left\langle \frac{\nabla}{ds}\frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)), V(s) \right\rangle = -\left\langle \frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)), Z \right\rangle \left\langle \frac{\nabla}{ds}\bigg|_{s=0} V(s), Z \right\rangle,$$

since $V(0) = |N_h|\nu_h$. □

Remark 5.5. Letting

$$\pi(V(s)) = \pi_1(V(s))X_{\gamma(s)} + \pi_2(V(s))Y_{\gamma(s)},$$

and noticing that $\nabla X = \nabla Y = 0$ we gain

$$\frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)) = \frac{d}{ds}\bigg|_{s=0} \pi_1(V(s))X_{\gamma(0)} + \frac{d}{ds}\bigg|_{s=0} \pi_2(V(s))Y_{\gamma(0)}.$$

Setting $\nu_h = aX + bY$ we obtain

$$\frac{\nabla}{ds}\bigg|_{s=0} \pi(V(s)) = (d\pi)_{(a,b)}\left(\frac{\nabla}{ds}\bigg|_{s=0} \frac{V(s)}{|V(s)|}\right), \tag{5.17}$$

where

$$(d\pi)_{(a,b)} = \begin{pmatrix} \dfrac{\partial \pi_1}{\partial a}(a,b) & \dfrac{\partial \pi_1}{\partial b}(a,b) \\ \dfrac{\partial \pi_2}{\partial a}(a,b) & \dfrac{\partial \pi_2}{\partial b}(a,b) \end{pmatrix}.$$

By [41, Corollary 1.7.3] $\pi_K = \nabla h$ where h is a C^2 function, thus by Schwarz's theorem the Hessian $\mathrm{Hess}_{(a,b)}(h) = (d\pi)_{(a,b)}$ is symmetric.

Lemma 5.6. *Let $S \subset \mathbb{H}^1$ be a (X,Y)-Lipschitz surface. Let h be a compactly supported function on S, differentiable in the Z-direction. Then we have*

$$\int_S \left(Z(h) - 2\frac{\langle N,T\rangle}{|N_h|}h \right) |N_h| dS = 0 \tag{5.18}$$

and

$$\theta(E) = -|N_h| Z\left(\frac{\langle N,T\rangle}{|N_h|}\right) + 2|N_h|\frac{\langle N,T\rangle^2}{|N_h|^2} \tag{5.19}$$

a.e. in S, where $\theta(E) = \langle \nabla_E \nu_h, Z\rangle$.

Proof. Following [18, Proposition 1.20] or [23, Remark 6.1], we approximate the (X,Y)-Lipschitz surface $S = \{p \in \mathbb{H}^1 : f(p) = 0\}$ by a family of smooth surfaces $S_j = \{p \in \mathbb{H}^1 : f_j(p) = 0\}$, where $f_j = \rho_j * f$ and ρ_j are the standard Friedrichs' mollifiers, that converges to S on compact subsets of S. Let Z^j, N^j and E^j relative to S_j. Then we have

$$\operatorname{div}(|N_h^j| h Z^j) = |N_h^j| Z^j(h) - \langle N^j, T\rangle h \left(-\frac{Z^j(|N_h^j|)}{\langle N^j, T\rangle} + |N_h^j|\theta(E^j) + 2|N_h^j|^2 \right).$$

Using $-|N_h^j|^{-1} Z^j(\langle N^j,T\rangle) = \langle N^j,T\rangle^{-1} Z^j(|N_h^j|)$, $|N_h^j|^{-1} Z^j(\langle N^j,T\rangle) - 2\langle N^j,T\rangle^2 + |N_h^j|\theta(E^j) = 0$ and the divergence theorem we get

$$\int_{S_j} |N_h^j| Z^j(h) - 2\langle N^j,T\rangle h \, dS_j = 0.$$

Then, passing to the limit when $j \to +\infty$ we obtain (5.18). Since S_j are smooth a straightforward computation shows that

$$\theta(E^j) = -|N_h^j| Z^j\left(\frac{\langle N^j,T\rangle}{|N_h^j|}\right) + 2|N_h^j|\frac{\langle N^j,T\rangle^2}{|N_h^j|^2}.$$

Passing to the limit when $j \to +\infty$ we have $E^j \to E$ a.e. in S, since S is Euclidean Lipschitz. Therefore, we obtain that (5.19) holds a.e. in S. □

Lemma 5.7. *Let $S \subset \mathbb{H}^1$ be an area-stationary (X,Y)-Lipschitz surface. Let h be a compactly supported function in S, differentiable in the Z-direction then we have*

$$\int_S \left(Z(h) - 2\frac{\langle N,T\rangle}{|N_h|}h \right) ||N_h||_* dS = 0. \tag{5.20}$$

Moreover, there holds

$$\int_S \pi_\nu E(h) + \pi_Z \theta(E) h = 0. \tag{5.21}$$

Proof. Let $\pi_\nu = \langle \pi(\nu_h), \nu_h\rangle = ||\nu_h||_*$, then $||N_h||_* = |N_h|\pi_\nu$. Since S is an area-stationary surface, Theorem 3.7 implies that ν_h is constant in the Z-direction, thus in particular we have $Z(\pi_\nu) = 0$. Therefore, applying the same divergent argument of the proof of Lemma 5.6 we obtain

$$\int_S \left(Z(h) - 2\frac{\langle N, T\rangle}{|N_h|} h \right) ||N_h||_* dS = 0.$$

Always following [18, Proposition 1.20] or [23, Remark 6.1] we approximate the (X,Y)-Lipschitz surface $S = \{p \in \mathbb{H}^1 : f(p) = 0\}$ by a family of smooth surfaces $S_j = \{p \in \mathbb{H}^1 : f_j(p) = 0\}$, where $f_j = \rho_j * f$ and ρ_j are the standard Friedrichs' mollifiers, that converges to S on compact subsets of S. Let Z^j, N^j and E^j relative to S_j. Using [34, Remark 3.3] it is easy to prove that $E(\pi_{\nu^j}) = \pi_{Z^j}\theta(E^j)$. Thus, by Proposition 2.6 we gain

$$\int_{S_j} \pi_{\nu^j} E^j(h) + \pi_{Z^j}\theta(E^j) h \; dS_j = -\int_{S_j} \langle N^j, T\rangle \theta(Z^j) h \; dS_j. \tag{5.22}$$

Since S is area-stationary, we get $H_K = 0$ and [29, Proposition 4.2] implies $H_D = \langle \nabla_Z \nu_h, Z\rangle = 0$. Then $\theta(Z^j) = \langle \nabla_{Z^j} \nu_h^j, Z^j\rangle \to \langle \nabla_Z \nu_h, Z\rangle = 0$ and, passing to the limit in (5.22) when $j \to +\infty$, we obtain (5.21). □

6. The Bernstein Problem for (X,Y)-Lipschitz Surfaces

We say that a (X,Y)-Lipschitz surface S is *complete* if it is complete in $(\mathbb{H}^1, g)$. This means that the surface S equipped with the intrinsic distance associated to the restriction of $g = \langle \cdot, \cdot \rangle$ to S is a complete metric space.

Definition 6.1. We say that a complete oriented area-stationary (X,Y)-Lipschitz surface $S \subset \mathbb{H}^1$ is stable if inequality

$$\int_S \left(Z(f)^2 + 4\left(Z\left(\frac{\langle N, T\rangle}{|N_h|} \right) - \frac{\langle N, T\rangle^2}{|N_h|^2} \right) f^2 \right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS \geq 0 \tag{6.1}$$

holds for any continuous function f on S with compact support such that $Z(f)$ exists and is continuous.

The following lemma is proven in [2, p. 45].

Lemma 6.2. *Let $A, B \in \mathbb{R}$ be such that $A^2 \leq 2B$ and set $h(s) := 1 + As + Bs^2/2$. If*

$$\int_{\mathbb{R}} \phi'(s)^2 h(s) ds \geq (2B - A^2) \int_{\mathbb{R}} \phi(s)^2 \frac{1}{h(s)} ds$$

for each $\phi \in C_0^1(\mathbb{R})$ then $2B = A^2$.

Theorem 6.3 (The Bernstein Problem). *Let $K \in C^2_+$ be a convex body with $0 \in \mathrm{int}(K)$. Let $S \subset \mathbb{H}^1$ be a complete, connected and stable (X,Y)-Lipschitz surface. Then S is a vertical plane.*

Proof. First of all we have that S is an $\mathbb{H}$-regular surface by Theorem 3.7. Let p in S. Since S is (X,Y)-Lipschitz, by Theorem 2.3, there exist an open ball $B_r(p)$

and a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$ where $\mathrm{Gr}(u) = \{(x, u(x,y), t - xu(x,t)) \in \mathbb{H}^1 \ : \ (x,t) \in D\}$. Let $(0,0) \in D$ be the projection of p to the xt-plane. On D we consider the coordinates around $(0,0)$ furnished by $G(s,\varepsilon)$ defined in Lemma 3.9. Let I be a small interval containing 0, then $\varepsilon \in I$ and $s \in]-r,r[$. Since S is complete by the Hopf–Rinow Theorem each geodesic (in particular the straight lines in the Z-direction) can be indefinitely extended along any direction, thus the open interval $]-r,r[$ extend to $\mathbb{R}$. Note that $\bar{\gamma}_\varepsilon(s)$ is the integral curve of Z, thus $Z(f) = \partial_s(f)$. Hence, taking into account that $(u_x + 2uu_t)(s)$ is constant along $\bar{\gamma}_\varepsilon$ and equal to $b(\varepsilon)$, the stability condition (6.1) is equivalent to

$$\int_I \int_{\mathbb{R}} \left((\partial_s f)^2 - 4\left(\frac{\langle N,T\rangle^2}{|N_h|^2} - \partial_s\left(\frac{\langle N,T\rangle}{|N_h|}\right)\right) f^2 \right) \frac{\partial t_\varepsilon}{\partial \varepsilon} \frac{\sqrt{1+b(\varepsilon)^2}}{\kappa(\pi(\nu_h))}\, ds\, d\varepsilon \geq 0, \quad (6.2)$$

for any continuous function f on S with compact support such that $Z(f)$ exists and is continuous.

Since $\langle N,T\rangle/|N_h|$ solves Eq. (4.1) with initial condition $y(0) = a'(\varepsilon)/2$ and $y'(0) = a'(\varepsilon)^2/2 - b'(\varepsilon)$, by (4.3) we get

$$\frac{\langle N,T\rangle^2}{|N_h|^2} - \left(\frac{\langle N,T\rangle}{|N_h|}\right)' = \frac{b'(\varepsilon) - \frac{a'(\varepsilon)^2}{4}}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)^2}.$$

Therefore, computing $\partial t_\varepsilon/\partial\varepsilon$ from (4.6), we obtain that (6.2) is equivalent to

$$\int_I \int_{\mathbb{R}} \left((1 - a'(\varepsilon)s + b'(\varepsilon)s^2)(\partial_s f)^2 - \frac{4b'(\varepsilon) - a'(\varepsilon)^2}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)} f^2 \right) \times \frac{\sqrt{1+b(\varepsilon)^2}}{\kappa(\pi(\nu_h))}\, ds\, d\varepsilon \geq 0.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a positive function compactly supported in $\mathbb{R}$ and for $\rho > 0$ we consider the family $\eta_\rho(x) = \rho^{-1}\eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(x - \varepsilon)\psi(s)$, where $\psi \in C_0^1(\mathbb{R})$, in the previous equation and letting $\rho \to 0$ we get

$$\int_{\mathbb{R}} (1 - a'(\varepsilon)s + b'(\varepsilon)s^2)(\psi'(s))^2 ds \geq (4b'(\varepsilon) - a'(\varepsilon)^2) \int_{\mathbb{R}} \frac{\psi(s)^2}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)}\, ds,$$

for a.e. ε since $\kappa(\pi(\nu_h))$ is a positive constant along the horizontal straight lines for each ε (since ν_h is constant along such horizontal straight lines) and $\sqrt{1+b(\varepsilon)^2}$ is a positive constant on $\bar{\gamma}_\varepsilon$.

Setting $A = -a'(\varepsilon)$, $B = 2b'(\varepsilon)$ and $h(s) := 1 + As + Bs^2/2$, we obtain

$$\int_{\mathbb{R}} h(s)\psi'(s)^2 ds \geq (2B - A^2) \int_{\mathbb{R}} \frac{\psi^2(s)}{h(s)} ds$$

for each $\psi \in C_0^1(\mathbb{R})$. Assume that $2B - A^2 \geq 0$ then by Lemma 6.2 we get that $2B = A^2$, then $4b'(\varepsilon) - a'(\varepsilon)^2 = 0$. Therefore by Lemma 4.1, we obtain $\langle N,T\rangle \equiv 0$, $a'(\varepsilon) = b'(\varepsilon) = 0$ a.e. in ε. On the other hand, if $2B - A^2 < 0$ then directly by

Lemma 4.1 we obtain $\langle N, T\rangle \equiv 0$, $a'(\varepsilon) = b'(\varepsilon) = 0$ a.e. in ε. Hence, $a(\varepsilon)$ and $b(\varepsilon)$ are constant functions in ε and

$$t_\varepsilon(s) = \varepsilon + as + bs^2,$$

for some constant $a, b \in \mathbb{R}$. Since $t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)) = 2\tilde{u}(s, \varepsilon)$ we get $\tilde{u}(s, \varepsilon) = a/2 + bs$, thus $\tilde{u}$ is an affine function. Hence, S is locally a strip contained in a vertical plane. A standard connectedness argument implies that each connected component of S is a vertical plane. □

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