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ORIGINAL ARTICLE

A model for problem creation: implications for teacher training

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Abstract
The invention of problems is a fundamental competence that enhances the didactic-mathematical knowledge of mathematics teachers and therefore should be an objective in teacher training plans. In this paper, we revise different proposals for categorizing problem-creation activities and propose a theoretical model for problem posing that, based on the assumptions of the Onto-Semiotic Approach, considers both the elements that characterize a problem and a categorization of different types of problem-posing tasks. In addition, the model proposes a description of the mathematical processes that occur during the sequence of actions carried out when a new problem is created. The model is illustrated by its application to analyze the practices developed by pre-service teachers in three problem-posing tasks aimed at specific didactic-mathematical purposes (mobilizing certain mathematical knowledge or reasoning, contributing to achieving learning goals, or addressing students' difficulties). We conclude discussing the potential of our model to analyze the mathematical processes involved in problem creation from the perspective of teacher education.

Keywords Problem creation · Teacher training · Didactic-mathematical knowledge · Onto-semiotic approach

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Introduction

Although mathematical problem posing has been at the forefront of discussion for the past few decades (Cai et al., 2022), the recent intensification in research activity is reflected in the wide range of topics studied in relation to problem posing (Baumanns & Rott, 2021a; Cai et al., 2022; Cai & Huang, 2020; Cai & Leikin, 2020; Liljedahl & Cai, 2021, among other synthesis works). According to Cai and Leikin, problem posing in Mathematics Education can be considered in step with activity theory (Leont'ev, 1978), which "draws connections between goals, actions, conditions, and tools in any human activity" (Cai & Leikin, 2020, p. 287). The authors suggest that literature in this field can be categorized according to how problem posing is viewed.

Research viewing problem posing as a tool for teaching mathematics considers this activity as a tool which can be used by teachers to help their students learn mathematics. Research in this area examines how students' learning of mathematics (measured in terms of cognitive and non-cognitive aspects) can be improved by involving them in problem-posing activities (Cai & Hwang, 2022; Chen & Cai, 2020; Koichu, 2020; Silber & Cai, 2021; Xu et al., 2020).

The literature considering problem posing as a goal of mathematics teaching focuses on how the ability to pose good problems develops. This includes the study of creating new problems as part of other types of mathematical activities such as problem-solving, proof, and inquiry (Guo et al., 2021; Leikin & Elgrably, 2020; Ponte & Henriques, 2013).

Problem posing appears as a research tool in works focusing on other aspects of learning, thinking (Erdogan, 2020), reasoning, and creativity of students (Elgrably & Leikin, 2021; Leikin & Elgrably, 2020; Singer & Voica, 2015) or on the effects of its inclusion in curricula (Cai et al., 2013). It is emphasized that problem posing could be seen as a complementary proposal to problem-solving (Pino-Fan et al., 2020; Silver, 2013) as it improves students' perception of mathematics, enhances their reasoning and problem-solving skills, improves their attitudes and confidence in mathematics, and fosters a broader understanding of mathematical concepts, properties, and procedures (Ayllón et al., 2016; Christou et al., 2005; Fernández & Carrillo, 2020).

Finally, studies that consider problem creation as an object of research focus on understanding the nature of problem posing itself, including the analysis and evaluation of the typology, variety, and quality of posed problems (Ellerton, 2013; Guo et al., 2021), as well as the competences, strategies, and other factors enable effective problem formulation (Leikin & Elgrably, 2020).

Regardless of how problem invention is perceived, a large number of researchers in Mathematics Education have emphasized the importance of incorporating problem posing in teacher education programs (Grundmeier, 2015; Malaspina et al., 2019; Singer et al., 2013). Without specific training, teachers pose predictable, undemanding, poorly formulated, or unsolvable problems, or those more focused on procedural exercise than on reasoning and conceptual understanding, so it is important to broaden the knowledge of teachers in problem posing (Crespo, 2003; Crespo & Sinclair, 2008; Lee et al., 2018; Stein et al., 2000; Xie & Masingila, 2017). On one hand, it is stressed that specific teacher training is the only way to firmly integrate problem posing into mathematics curricula and classroom practices (Ellerton, 2013). On the other hand, problem posing is considered an appropriate way to introduce pre-service teachers to mathematics teaching, which allows them to explore mathematical content in depth and become aware of their possible deficiencies (Tichá & Hošpesová, 2013; Yao et al., 2021). To this regard, research on the creation of mathematics problems for educational purposes explicitly mentions their close connection with teaching competencies (Ellerton, 2013; Malaspina et al., 2015; Mallart et al., 2018; Silver, 2013), particularly with the capacity for didactic analysis (Malaspina et al., 2019). Consequently, problem posing in teacher education should be seen:

> both as a means of instruction (meant to engage students in genuine learning activities that produce deep understanding of mathematics concepts and procedures) and as an object of instruction (focused on developing students' proficiency in identifying and formulating problems from unstructured situations). (Singer et al., 2013, p. 5)

While criteria for categorizing problem-posing activities have largely depended on individual preferences of researchers (Lee et al., 2018), traditionally, studies on problem invention have primarily focused on posed problems as a product (Baumanns & Root, 2022; Bicer et al., 2020; Van Harpen & Presmeg, 2013). However, analyzing the processes of problem posing is equally relevant because "it is in the processes that problem posers come up with ideas for new problems, evaluate those ideas, and develop or reject them" (Cai et al., 2022, p. 123). This has led to an increasing consideration of problem creation as a process in recent studies (Baumanns & Rott, 2022; Cai & Leikin, 2020; Christou et al., 2005; Crespo & Harper, 2020; Headrick et al., 2020; Koichu & Kontorovich, 2013; Patáková, 2014; Ponte & Henriques, 2013). Nevertheless, these works mainly focus on the cognitive processes that students (Christou et al., 2005) or pre-service and in-service teachers (Cruz, 2006; Kontorovich & Koichu, 2016; Pelczer & Gamboa, 2009) activate in problem creation (Kontorovich, 2023). In other words, these studies analyzed problem posing from the point of view of the mathematical thinking that accompanies it, seeking to understand "the complex relationship between problem posing and problem solving related to the cognitive processes of problem posing" (Zhang et al., 2022, p. 498).

The processes involved in problem posing have their own (epistemic) nature independently of the individual learner's thinking, so that studies from a purely cognitive point of view do not allow us to address the entirety of the educational issue. In fact, many of the difficulties observed in the teaching and learning of mathematics, not only in problem creation or resolution, are related to the fact that institutional mathematical objects exhibit a complexity of inherently mathematical character (Font et al., 2013; Godino et al., 2007).

From this perspective, we believe that the theoretical and methodological tools developed by the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino et al., 2007) enable us to tackle the inherent difficulty in researching problem posing by analyzing it in onto-semiotic terms, that is, by means of the mathematical entities and processes involved and their relations. Additionally, to analyze the professional knowledge and competencies required by teachers to create problems with an educational purpose, we adopt the teacher's Didactic-Mathematical Knowledge and Competencies (DMKC) model (Breda et al., 2017; Godino et al., 2017), based on the analytical tools of mathematical and didactic activity developed within the core of OSA (Pino-Fan et al., 2015).

To address the study of problem-posing processes beyond the cognitive dimension, in this manuscript, we develop a theoretical model for problem posing which, by articulating mathematical and didactic dimensions, focuses the analysis on the involved mathematical (epistemic) processes. To support this model, we first provide a review of proposals for categorizing problem creation activities, particularly those aimed at teacher education, identifying their common characteristics and limitations. Next, we introduce the essential elements of OSA and DMKC frameworks on which our proposal is based. Subsequently, we describe the model and illustrate it by applying it to analyze practices of pre-service teachers in three problem creation tasks aimed at specific didactic-mathematical goals (mobilizing certain mathematical knowledge or reasoning, contributing to learning objectives, or addressing student difficulties). It is intended to show how the analysis based on the proposed model catches the complexity of problem posing through a microscopic analysis of the mathematical activity involved. The article concludes with a discussion of the purpose and potential of our model.

State of the art: problem creation as a product and as a process

Although different authors have given different names to the activity of creating problems, in essence, it involves both formulating new situations and reformulating given problems (English, 1998; Silver, 1994; Silver & Cai, 1996). In this section, we summarize some of the multiple categories and methodological proposals on problem creation found in previous literature.

For Silver (1994), problem posing is an important component of problem-solving, which leads him to assume a classification of problem creation in terms of the moment in the problem-solving process: before (pre-solution), during (within the solution), and after (post-solution). The goal of problem creation before its resolution is not to solve the problem but to pose a mathematical problem based on a situation or experience previously given to the student. In this case, problems are generated from a concrete stimulus, such as a story, an image, and a representation. When problems are created within the problem-solving process, they are reformulated, for example, by changing objectives and conditions to facilitate their understanding and resolution. In problem creation after resolution, new related problems are generated once properties or characteristics of the initial situation have been analyzed, modifying conditions, objectives, or questions. In this latter category, experiences from the problem-solving context are applied to new situations (Christou et al., 2005).

Stoyanova and Ellerton (1996) assume that problem creation processes can take place under three categories: free, semi-structured, or structured situations. In a *free situation*, students create problems without any kind of restriction, based on their experiences inside and outside of school (Stoyanova, 1998). For example, “create a difficult problem,” “create a money problem,” or “there are 10 girls and 10 boys standing in a line. Invent as many problems as you can that use this information in some way” (Van Harpen & Presmeg, 2013).

In the *semi-structured category*, students are given an open situation and are requested to explore its structure and complete it by applying knowledge, skills, and relationships derived from their previous mathematical experiences. Visual representations in which an image, a graph, or a table is presented; open verbal stories, among others, are frequently used in studies of semi-structured problem posing (English, 1998; Silver & Cai, 2005). Thus, within the category of semi-structured problem creation, it is possible to consider subcategories according to the starting situation (Akay & Boz, 2010): mathematical, open, modelling, and unknown data semi-structured situation. In the *mathematical semi-structured situation*, an appropriate problem related to the mathematical situation is required to be posed (for example, students must create a problem related to a graph in which the region bounded by the graphical representation of two functions is observed). In an *open semi-structured situation*, there are no restrictions on the mathematical components involved in a real-life scenario, and students can pose problems according to the structure and conditions they prefer (for example, posing an integral problem to calculate the volume of a solid figure that is not a surface of revolution). A *modelling semi-structured situation* leads students to concretize and anticipate real-life situations that can be solved through mathematics (for example, posing a problem about the design and cost calculation of a ring). Finally, in an *unknown data semi-structured situation*, the problem is created by adding the missing data to a situation in which some structures conflict.

In a *structured situation*, the objective can be determined by all the given elements and relationships. The problem is elaborated based on a previous one, reformulating the given situation or changing its conditions or questions (Van Harpen & Presmeg, 2013). In this process, students establish problems taking into account strategies and situations that may be limited by their teachers (English & Watson, 2015).

Authors such as Baumanns and Rott (2021a) encountered difficulties in distinguishing between free and semi-structured situations, leading them to differentiate between unstructured and structured situations based on the degree of given information (Baumanns & Rott, 2021a, b, 2022). The former form a spectrum of situations without an initial problem, where information ranges from almost none (“create a problem for a math olympiad”) to open situations rich in information where exploring the structure using mathematical knowledge is necessary. In structured situations, new problems based on a specific problem need to be posed, for example, by varying its conditions.

Models such as Silver’s (1994) which categorize problem-posing activities by their role in problem-solving, or those of Stoyanova and Ellerton (1996) or derivatives, which classify problem-posing tasks according to the degree of information or restrictions enabling the elaboration of a new problem, focus essentially on the product of problem creation. However, Christou et al. (2005) consider that these categories allow for the classification of students' thinking processes in problem posing (studying the product to discover the process). Thus, Christou et al. (2005) classified problem creation through four different processes: editing, selection, comprehension, and translation of quantitative information. The *editing processes* refer to tasks requiring students to pose a problem without any restrictions regarding provided information, stories, or instructions. The process of *selecting quantitative information* is associated with tasks that aim to have students pose appropriate problems or questions for specific, predetermined answers. The known response functions as a restriction, requiring students to focus on the structural context and relationships between the provided information, making selection more difficult than editing. In the processes of *understanding and organizing quantitative information*, students engage in problem-posing activities based on given equations or mathematical calculations. Understanding requires "understanding the meaning of the operations and students usually follow an algorithmic process focusing on the operational and not the semantic structure of the problems" (Christou et al., 2005, p. 151). Finally, the *translation of quantitative information* occurs when students pose problems or questions appropriate to graphs, diagrams, or tables. According Christou et al. (2005), translation is more demanding than understanding, as it requires interpreting different representations of mathematical relationships.

To approach the procedural study of problem posing, several authors (Baumann & Rott, 2022; Cruz, 2006; Koichu & Kontorovich, 2013; Pelczer & Gamboa, 2009) develop descriptive models (showing how subjects actually create problems) of phases (actions) from the numerous processes involved.

Cruz (2006) model, contextualized in a professional development program, aims to guide teachers in the stages of the "macrostructure of the process of elaborating new problems" (p. 83) in teaching and learning situations. It considers that elaboration, formulation, and posing are differentiated activities in the "metaproblem":

> Problem elaboration will be referred to a complex cognitive activity that the teacher acts (at a macro level); problem formulation constitutes a substructure of such activity, which is at the same time made of several different actions (at a meso level), while the problem statement will be related to a final operation of the formulation (at a micro level). (Cruz, 2006, p. 83)

For Cruz (2006), the process of formulating a problem begins with the *selection* of the mathematical object, which may be conditioned by didactic needs and purposes. Once selected, a *classification* (analysis) of the components of the mathematical object follows, in which information is obtained and organized according to various criteria. This analysis may result in *transforming* the object into a generalization of itself or using analogies. After transforming or not transforming the object, the subsequent action involves *associating* the elements obtained during the classification that are abstracted to relate to mathematical concepts, through their properties or relationships. The *search for dependence* between the selected concepts allows for making necessary decisions to pose the questions of the problem.

Pelczer and Gamboa (2009) developed a descriptive model for problem posing, which distinguishes five actions: *configuration* (reflection on the context of the given situation and the necessary knowledge to understand it), *transformation* (analysis of the problem's conditions and identification and assessment of possible modifications to execute them), *formulation* (exploration of problem formulations and possible alterations), *evaluation* (to verify if it satisfies the initial conditions or if it needs modifications), and *final assessment* (reflection on the entire process).

Koichu and Kontorovich (2013) also developed a descriptive model in which, based on two activities of future mathematics teachers called "success stories," they identified four phases of problem posing. In the first phase, *warming up*, the given task is associated with certain prototypical or familiar types of problems, and spontaneous ideas are discussed. In the next phase, *searching for an interesting mathematical phenomenon*, the initial problems serve as material to be critically considered, based on mathematical aspects that will underlie the next interesting problem. Then, problem creators *hide the problem-posing process* in the formulation of the problems, with the intention that the processes they go through in posing and solving the intermediate problems are not visible to solvers. In the final *revision* phase, the potential and relevance of the posed problems are evaluated and tested with peers.

For Baumann and Rott (2022), there is a conceptual and empirical need for a more generally applicable model for problem-posing research than previous ones. In this regard, they propose a new descriptive phase model based on the general structure, i.e., the sequence of distinguishable actions of the observed problems. The authors consider that problem posing for a structured situation follows the following sequence: (1) *analysis* of the situation, in which simple or multiple conditions of the initial task are collected, identifying which ones and to what extent they are suitable for creating a new task by variation or generation; (2) *variation* (some of the individual or multiple conditions of the initial task are changed or omitted and the new task is formulated or written); (3) *generation* of the new task by elaborating one or several new conditions, which are then formulated and written; (4) *solution* of the posed problem; and (5) *evaluation*, in which the posed problem is assessed as to whether it is solvable, well-defined, similar to the initial task, appropriate or interesting for a specific objective or group of students. Based on this evaluation, the created problem is accepted or rejected.

The four models presented above describe, through phases, the cognitive actions that unfold in the process of problem creation. These proposals consider that problem posing is part of a complex system in which the teacher acts, for example, to select the mathematical content according to the didactic need or purpose (Cruz, 2006) or to evaluate the relevance of the problems (Baumann & Rott, 2022; Koichu & Kontorovich, 2013; Pelczer & Gamboa, 2009). However, in the description of problem creation processes as a sequence of actions, the boundary between mathematical and didactical aspects is not always clear, which hinders a micro-level analysis of the activity.

Kontorovich et al. (2012) introduce an exploratory framework to address the complexity of mathematical problem posing, moving away from a purely cognitive approach. The framework integrates the following facets based on previous research:

(1) task organization (compound of both the specific problem-posing task and the didactic specifications taken by the teacher when planning the activity), (2) knowledge (and competences) base (concepts, properties, routine procedures, system of prototypical problems (Kontorovich & Koichu, 2016); competence for mathematical discourse and writing), (3) problem-posing heuristics and schemes (examples of problem-posing strategies) and (4) dynamics and social interactions. Furthermore, they introduce a new facet, (5) aptness (suitability or appropriateness) to oneself, to the potential evaluators of the product or to its potential solvers. The authors consider that to better understanding the problem-posing product, the problem-posing process must be analyzed through the modular structure offered by these facets.

In models like the ones proposed by Contreras (2007) or Grundmeier (2015), designed to assist future mathematics teachers in generating mathematical problems, actions indicate on which attributes of a base problem one acts and how to create a new problem. According to Contreras (2007), some problems can be modified to generate new problems by applying the following fundamental mathematical processes: *proof*, *inversion*, *particularization*, *generalization*, and *extension*. By applying these processes, the following types of problems are generated: *proof problems* (which ask for the proof of a certain property included among the attributes of the base problem), *inverse problems* (where a known attribute of the base problem is replaced by an unknown one and vice versa, adding additional conditions or constraints if necessary), *particular problems* (obtained by substituting a mathematical object from the base problem for a particular example or case of the original mathematical object), *general problems* (obtained by substituting a mathematical object from the initial problem for another one for which the original is an example), and *extended problems* (obtained by substituting a mathematical object from the base problem for another one that is similar or analogous, without being a particular case of the previous one). In any case, the didactic purpose of generating new problems is to achieve a deeper understanding of the properties of the mathematical objects involved (Contreras, 2007). As we can see, Contreras model clearly distinguishes between the process of modifying the original problem and the product (resulting problem) in problem creation.

Grundmeier's (2015) study incorporates problem posing in a mathematics content course from two perspectives: problem reformulation and problem generation. In the *problem reformulation process*, which focuses on a problem that has already been or is being solved, the following techniques are considered: (a) exchanging the given and required information (the same context as the original problem with the given and required information exchanged), (b) changing the context (same structure, but with the context changed), (c) changing the given information (same context and structure of the problem, but with the given information changed), (d) changing the required information (same context and structure of the problem, but changing what the question asks), (e) extending the given problem to more general situations, f) adding information (same context and structure of the problem with added information), and (g) modifying the wording of the problem. In the process of generating a problem from certain given information, additional information can be included, but it must be related to the original set of information (Grundmeier, 2015). For this author, adding information, changing what is given, changing what is required, or modifying the wording of the problem are "superficial techniques," as they do not require a change in the structure of the problem, as opposed to creating a problem by exchanging the given (information) and the required (requirement), modifying the context or extending it. Reformulating the structure involves more creativity and a better understanding of the mathematical content by the problem creator.

To determine what is involved in creating a problem, Malaspina (2013) begins by specifying the elements characterizing a mathematical problem, namely: information, requirement, context and mathematical environment. *Information* refers to the quantitative or relational data given in the problem. The *requirement* is what is asked to be found, examined or concluded, which can be quantitative or qualitative, including graphs and demonstrations. The *context* can be intra-mathematical or extra-mathematical determining the environment or setting that gives rise to the mathematical activity. The *mathematical environment* is the global framework in which the mathematical concepts that intervene or may intervene to solve the problem, their properties and relationships are located, that is, the mathematical structure (Grundmeier, 2015). For Malaspina (2013), problem creation is a process by which a new problem is obtained. He distinguishes between variation (of a problem) and elaboration. In a *variation*, a new problem is constructed by modifying one or more of the four elements (information, requirement, context, environment) of an initial problem. In the process of *elaboration* of a problem, a new problem is constructed in an open form, starting from a situation (given or configured by the author), or with a specific requirement or purpose, which may have a mathematical or didactic emphasis. In the elaboration of a problem from a situation, the context originates in the situation, the information is obtained by selection or by modification of that which is perceived in such situation, the requirement is a consequence of the relationships between the elements of the information implicit in the statement, and the mathematical environment can be determined by the author or by the ways of solving the problem. In this model, problem-posing tasks are linked, on the one hand, to the modifiable attributes in problems (thus overcoming the unclear boundaries between different problem-posing categories) and, on the other hand, to the presence or absence of an educational purpose that motivates them.

Theoretical framework

Problem solving in the construction of meaning

From the anthropological and pragmatic perspective of mathematics adopted by OSA (Godino et al., 2007), the activity of individuals in problem-solving occupies a central place in the construction of mathematical knowledge. Mathematical practice is considered any action or expression carried out by someone to solve mathematical problems, communicate their solution, validate it, or generalize it to other situations (Godino et al., 2007).

OSA assumes that the solution of mathematics education epistemological, cognitive, and instructional problems previously requires addressing the ontological problem, i.e., clarifying the nature and types of mathematical objects that appear in mathematics teaching and learning activities such as problem creation (Font et al., 2013; Godino et al., 2007).

The term *object* is used in a broad sense in the OSA to refer to any entity that is involved in some way in mathematical practice and that can be separated or individualized. A typology of primary mathematical objects is proposed: *problem-situations* (exercises and problems understood as tasks that induce mathematical activity), *languages* (mathematical terms and expressions), *concepts* (mathematical entities that can be introduced by description), *propositions* (statements about concepts, properties), *procedures* (calculation techniques, operations, algorithms), and *arguments* (statements required to demonstrate propositions or explain procedures). The consideration of an object as primary is not an absolute matter, as it pertains to functional entities relative to language games (institutional frameworks, communities of practice, and usage contexts) in which they participate. Primary objects involved in mathematical practices and their emergents, according to the language game they participate in, can be considered from five dual facets that dialectically complement each other (Fig. 1). These facets are regarded as attributes that, when applied to different primary objects, result in the following typology of secondary objects: *ostensive* (public, material, perceptible)—*non-ostensive* (abstract, ideal, immaterial); *extensive* (particular)—*intensive* (general); *personal* (relative to individual subjects)—*institutional* (shared in an institution or community of practices); *significant* (expression)—*meaning* (content); *unitary* (objects considered globally as a previously known whole)—*systemic* (objects as systems formed by structured components). These objects and their relationships (configurations), both in the personal and institutional facet, are constituted over time through mathematical processes.

Although OSA does not attempt to give a definition of "process" upfront, since there are many different types of processes (such as processes as series of practices, cognitive processes, metacognitive processes, instructional processes), an effort is made to establish certain characteristics of what a mathematical process is (Font et al., 2008; Godino et al., 2009). A mathematical process is considered to be any sequence of actions activated or developed over a certain time to achieve an objective, usually the response to a proposed task subject to mathematical or metamathematical rules.

Both primary and secondary objects (derived from the application of dualities) can be considered from the process–product perspective, which provides criteria for distinguishing types of primary and secondary mathematical processes (Fig. 1). On the one hand, primary objects emerge from systems of practices through the primary processes of problematization, communication, definition, enunciation, algorithmization-description, and argumentation, respectively. On the other hand, dualities give rise to the following secondary processes: institutionalization-personalization; materialization-idealization; representation-signification (or interpretation); particularization-generalization; unitarization (or reification)-decomposition (or splitting), from which secondary objects emerge (Font et al., 2013; Godino et al., 2007). The distinction between particularization and generalization processes with respect to idealization and materialization processes (Font & Contreras, 2008) and between these and unitarization and

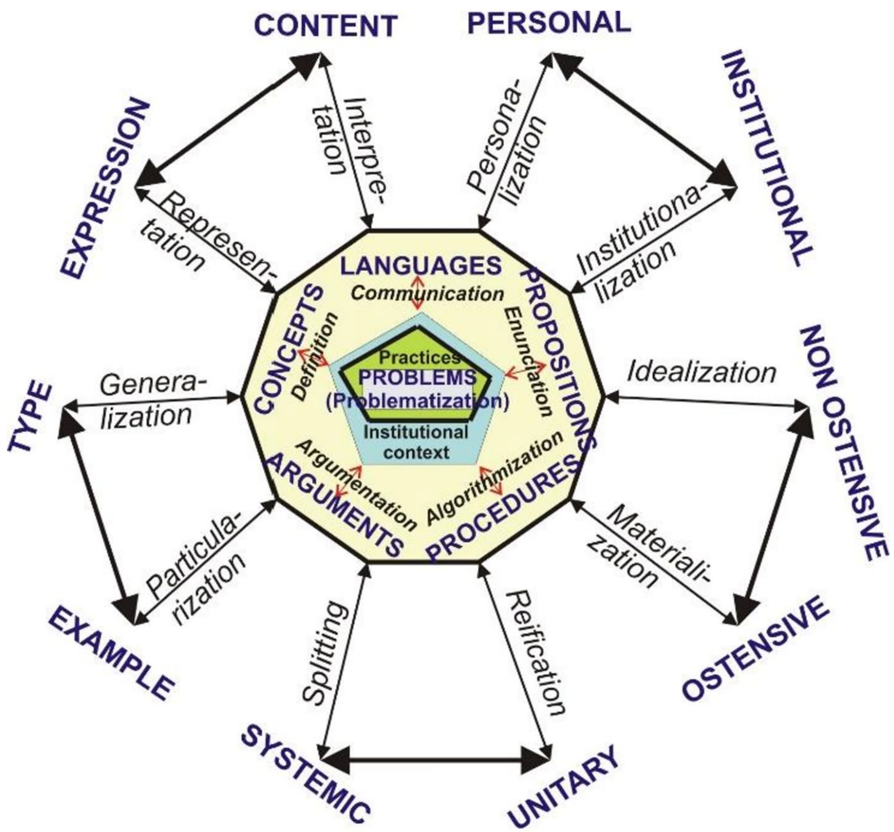


Fig. 1 Onto-semiotic configuration of practice, objects, and processes. Source: Godino et al., (2020, p. 7)

decomposition processes allows for a more detailed analysis of each of these processes and their combined presence in mathematical activity.

Problem-solving or modelling is considered in the OSA framework as hyper or mega-processes (Godino et al., 2009) in the sense that they involve some or several of the aforementioned processes.

From the OSA perspective, the (epistemic) analysis of mathematical practices, objects, and processes constitutes a first level of didactic analysis of any mathematical activity, particularly that which revolves around the creation of mathematical problems (Font et al., 2013; Godino et al., 2007). When problem creation takes place under the guidance of a teacher and in interaction with other students, didactic analysis must progress from the study of the epistemic configuration to the analysis of the didactic configuration, which also considers, in addition to the system of mathematical practices, the sequence of actions and means used by the teacher for this activity.

Mathematics teacher knowledge and competencies model

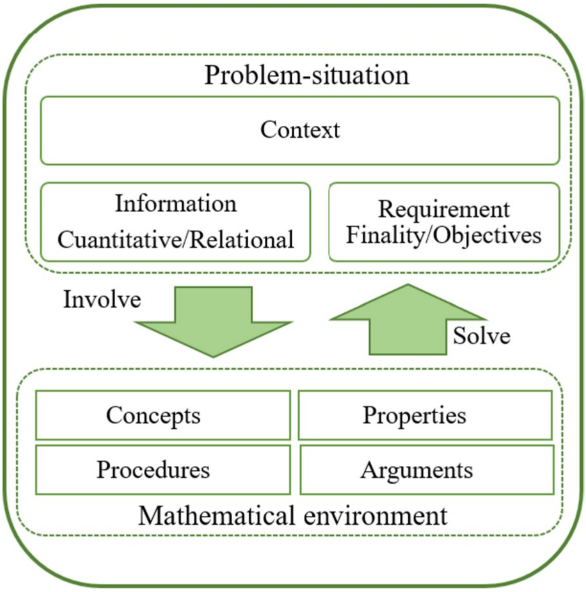
The DMKC model articulates categories of didactic-mathematical knowledge and competences of mathematics teachers, through the facets that condition instructional processes (Godino et al., 2017). In that way, it is accepted that the teacher must have *mathematical knowledge* per se, that is, *common* and relative to the educational level where they teach, as well as *expanded knowledge* allowing them to articulate it with higher levels. In addition, the teacher must have specialized or didactic-mathematical knowledge of the different facets that affect the educational process: *epistemic* (didactic-mathematical knowledge about the content itself, institutional reference meanings), *ecological* (relationships between mathematical contents and other disciplines, curricular factors and socio-professional factors that condition mathematical instruction processes), *cognitive* (how students reason and understand mathematics, and how they progress in their learning), *affective* (knowledge of the affective, emotional, attitudinal and belief aspects of students regarding mathematical objects and their study), *mediational* (technological, material, and temporal resources), and *interactional* (knowledge about mathematics teaching, selection and organization of tasks, resolution of student difficulties and classroom interaction management). These dimensions interact in every process aimed at teaching and learning mathematical content.

In addition to having this knowledge, the DMKC model proposes that the teacher should be competent in addressing the basic didactic problems present in instructional processes. The *general competence of didactic analysis and intervention* allows the teacher to describe, explain, and judge what has happened in the study process and make improvement proposals for future implementations (Godino et al., 2017). The theoretical and methodological tools of OSA allow the development of this competence, which encompasses, among other sub-competences, the identification and description of operational and discursive practices implied in mathematical activity (*analysis of global meanings*) and the recognition of the configuration of objects and processes emerging from mathematical practices (*ontosemiotic analysis*). Both skills are fundamental in the creation of problems to respond to certain requirements. But, reciprocally, the creation of problems with didactic purposes serves as a means to develop them, since this process requires: reflecting on the overall structure of the problem (what it seeks and whether the information provided is sufficient to solve it); analyzing both the possible ways of solving it and the mathematical objects and processes involved and how they are related, and recognizing possible difficulties that students may encounter and how to address them in the approach to new situations.

Aim

The literature review reveals the existence of (1) models considering the attributes or elements characterizing a problem (Contreras, 2007; Grundmeier, 2015; Malaspina, 2013), (2) models categorizing different tasks in problem creation (Akay & Boz,

Fig. 2 Components of a structured problem-situation



2010; Stoyanova & Ellerton, 1996; Van Harpen & Presmeg, 2013), and (3) models describing the problem creation process (Baumanns & Rott, 2022; Christou et al., 2005; Cruz, 2006; Koichu & Kontorovich, 2013; Pelczer & Gamboa, 2009).[1] However, these three aspects are not articulated within a framework for problem posing. Moreover, when these models describe processes observed during problem creation, they tend to focus on cognitive processes that occur afterward. In our model, we start from the OSA categorization of the types of mathematical processes (primary and secondary) involved in and modelling mathematical practice and identify those that are present in problem creation.

Hence, the aim of our research is to present a theoretical framework that integrates these three aspects, bridging the mathematical dimension (epistemic analysis of processes involved in the mega-process of problem posing) with the didactic dimension (categorization of problem-posing tasks based on problem components and facets of didactic-mathematical knowledge involved).

A framework for problem creation

Based on Malaspina's (2013) elements of a problem (Fig. 2) and considering previous models, we propose the following categories for problem creation:

[1] The comparison and articulation of the models described in the "State of the art: problem creation as a product and as a process" section can be significant given that, as we have seen, they employ similar or at least compatible terms, concepts, and principles.

- *Free or unstructured elaboration*. There is no starting (structured) problem-situation, and no indications, guidelines, or restrictions are included on any of the components (context, environment, information, and requirement) of the new problem. For example, "create an easy problem." The context-information-requirement field is completely empty and thus is the environment.
- *Semi-structured elaboration*. There is no base problem situation, but information or restrictions on elements of the new problem are included. That is, some of the context-information-requirement components are given. For example, it may contain context and information (Van Harper & Presmeg, 2013) that must be completed with the objective, or data (information) must be completed to answer a given question (requirement) (Cai & Jiang, 2017; Espinoza et al., 2018; Grundmeier, 2015). On other occasions, the restriction may be given as part of the mathematical environment, often as a strategy or certain calculation that must lead to the solution (Christou et al., 2005).
- *Structured elaboration or variation*. A problem is posed based on a previous one (Stoyanova & Ellerton, 1996; Van Harpen & Presmeg, 2013), so that the different elements (context, information, requirement and mathematical environment) are known. The variation may be partial if some of these components are modified but not all of them, for example, quantitative or relational data (information) or questions (requirement) (Van Harpen & Presmeg, 2013), or complete (if the four components are modified). The exchange between information and requirement, adding new information or asking new questions (Cai & Jiang, 2017), is also considered within the category of structured elaboration.

We consider that the elaboration of a problem to respond to a mathematical purpose or demand, such as employing a certain strategy or arithmetic calculations (English & Watson, 2015), or to a didactic-mathematical one (Malaspina, 2016; Malaspina & Vallejo, 2014), in the case of teachers, such as involving certain concepts or properties (epistemic dimension), or meeting a certain level of cognitive demand or diagnosing certain expected difficulties (cognitive dimension), can occur both in the semi-structured and structured (variation) case.

In the same way that problem-solving or modelling, OSA considers problem posing as a mega-process in that it may involve several primary (communication, enunciation, problematization) and secondary (representation-signification, materialization-idealization, particularization-generalization, decomposition-unitarization) processes. The elementary processes involved depend on the creative task that motivates them, i.e., what is the product expected to be obtained; reciprocally, certain processes determine different problem-products.

Let us take as an example a semi-structured elaboration situation in which some of the components of context-information-requirement are given (shown in grey in Fig. 3). To create the problem, it is necessary to determine the remaining elements. First, this involves separating into semiotic units (decomposition process) the given components, giving meaning (signification process) to each of the linguistic objects (in their various registers) in the situation (upper center part of Fig. 3). Next, information and requirement must be linked through the mathematical structure. Mathematical objects are abstract entities, that is, immaterial (non-ostensive) and general

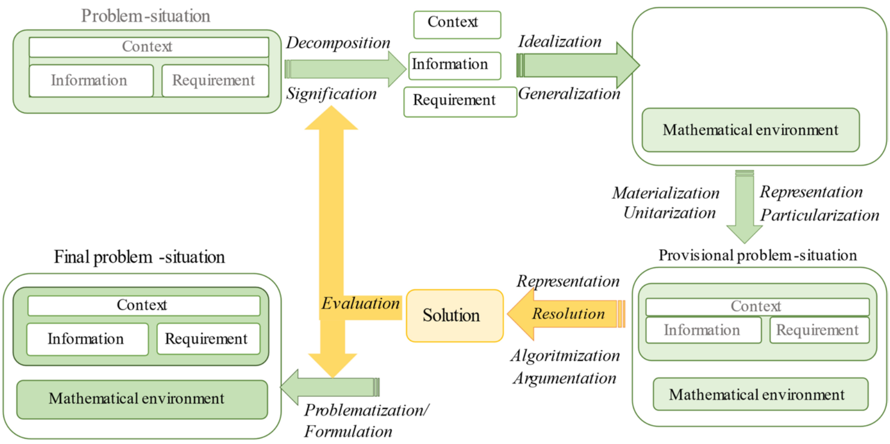


Fig. 3 The mega-process of problem posing. The case of a semi-structured elaboration

(intensive), so idealization and generalization processes occur in their articulation with the mathematical environment, i.e., a configuration of concepts, properties, and procedures, is established that enables giving meaning to the requirement based on the information or vice versa. In this way, all the elements that determine the problem are available (upper right part of Fig. 3). Analysis of mathematical objects involves classification, association, and search for relationships (Cruz, 2006). The previously identified semiotic units must be composed to acquire unitary entity (unitarization), so abstract objects that make up the mathematical structure of the problem (the environment) must be particularized, materialized, and represented in the enunciation of the new problem (central right part of Fig. 3). Additionally, as part of the problem creation process, it is possible to consider the (mega-process) of solving it (involving other primary processes such as representation, algorithmization, and argumentation of the solution), as occurs in the models of Baumanns and Rott (2022), Cruz (2006), and Pelczer and Gamboa (2009), within the evaluation phase. If the solution evaluation is not satisfactory, it may be necessary to resignify the components of the problem and reanalyze the mathematical objects to obtain new problem formulations (lower left part of Fig. 3). When problem posing serves a didactic purpose, it is not only the solution to this intermediate problem that is evaluated; the aptness of the provisional product is also assessed (Kontorovich et al., 2012), meaning its quality or the extent to which it aligns with the intended educational objectives.

As depicted in Fig. 4, we consider a dual dimension in problem creation: mathematical and didactic. In the mathematical dimension, problem creation is an activity shared by students and teachers, modelled in terms of mathematical processes and their emergent objects (see Fig. 3). In the didactic dimension, problem creation involves professional knowledge in various facets some of which are related to task organization, problem-posing heuristics, and schemes suggested by Kontorovich et al. (2012). Indeed, when creating a problem, the teacher should be aware of and employ a variety of problem-solving strategies that are appropriate from an instructional point of view

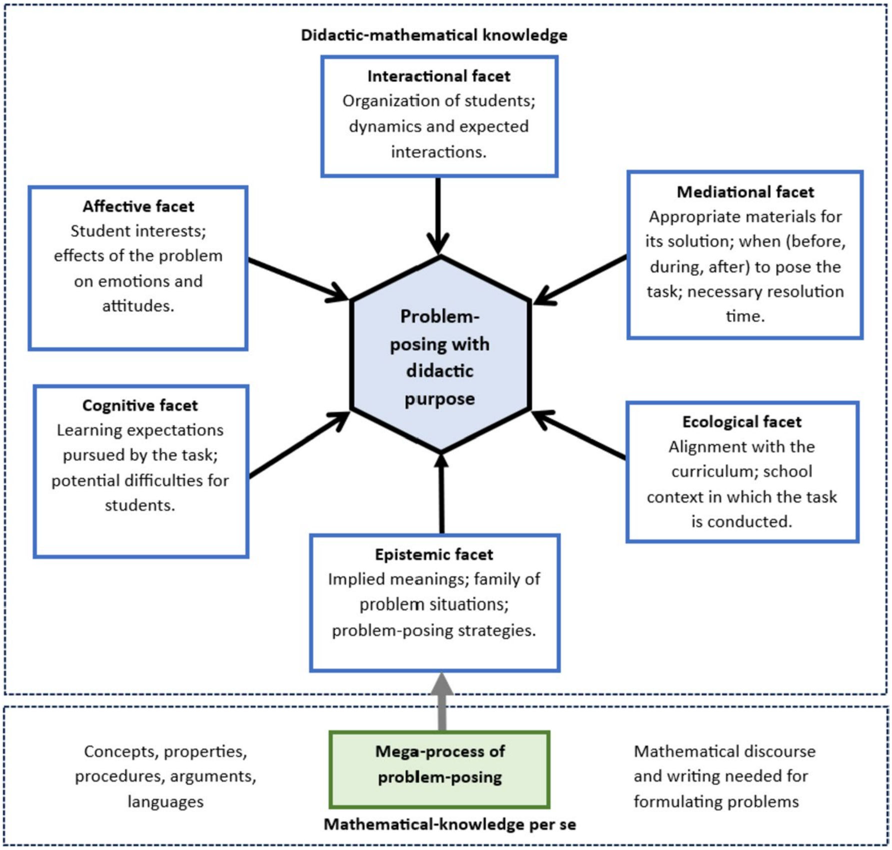


Fig. 4 Facets of didactic-mathematical knowledge in problem posing

(interactional facet) recognizing the material resources that could be helpful and the temporal requirements (mediational facet). Teacher should also identify the knowledge involved in its solution (epistemic facet) and evaluate its adequacy to the educational level and curriculum (ecological facet). This analysis will allow them to be aware of the complexity of the situations proposed to students, responding to their knowledge and (potential or encountered) difficulties (cognitive facet) and catering to their interests (affective facet). The deployment of these knowledge is determined by the didactic purpose of the problem-posing task, which can in turn have an epistemic, ecological, cognitive, etc. character.

Create from the following situation a problem in which students have to distinguish proportional situations from additive situations. Solve it by clearly identifying the magnitudes that are related in a proportional way and those that are related in an additive way.

> *Raquel, Juan, and Daniel are planting flowers. Raquel and Juan started at the same time. Daniel started before his classmates. When Raquel has planted 4 flowers, Juan has planted 12 flowers and Daniel has planted 8.*

Then, modify it to create a new problem in which not all the ratios involved in the proportionality relationships are integers. Indicate which elements you modify in your first problem, and which are the non-integer ratios.

Fig. 5 Problem creation according to a didactic-mathematical purpose. Source: Own elaboration

Illustrating the model's implementation

In this section, we aim to illustrate and operationalize the problem-posing framework proposed in the previous section. To do so, we describe some tasks implemented in a primary education teacher training course aimed at developing the competence to create problems with a didactic purpose.[2] This represents a particular case of the formative experiences we are conducting in the specific field of proportional reasoning focused on the epistemic and cognitive facets of didactic-mathematical knowledge (Burgos, et al., 2018; Burgos & Chaverri, 2022, 2023; Burgos & Godino, 2022a, b). In this experience, prospective teachers receive specialized training on the notions of practices and pragmatic meaning, the nature and typology of mathematical objects and processes (Godino et al., 2007), the elements that characterize a mathematical problem (Malaspina, 2013), as well as specific instruction on proportional reasoning and the difficulties it poses for students (Hilton & Hilton, 2019).

The responses of the trainee teachers (chosen, in each case, to be representative of the productions of the rest of the participants[3]) are analyzed from the point of view of the mathematical processes involved and the emergent objects in the mega-process of problem posing, as shown in Fig. 3. These responses include both the new problem and the (mathematical and didactical) practices on which they are based to address the intended educational purpose.

Problem creation with an epistemic didactic-mathematical purpose

In the first part of the situation shown in Fig. 5, a semi-structured elaboration task with a didactic-mathematical purpose of epistemic nature is proposed. The context of the problem to be elaborated, the information, and the mathematical environment in which it should be framed are provided. The teacher in training must recognize from the information provided which magnitudes can be related in an additive way,

[2] The aim of this section is not to describe the results of these empirical studies per se, but rather to exemplify our model through its implementation in a real teacher training context.

[3] The formative experience from which the examples are taken was carried out with a group of 66 students for teacher from the third year of the Primary Education degree at a Spanish university.

Problem 1: Raquel, Juan and Daniel are planting flowers. Raquel and Juan started at the same time, but Juan is faster than Raquel. Raquel and Daniel plant at the same speed, but Daniel started earlier than his companions. When Raquel has planted 4 flowers, Juan has planted 12 flowers and Daniel 8. If Raquel has planted 20 flowers, how many did Juan and Daniel plant?

In this case, the magnitudes involved are the number of flowers that Raquel, Juan, and Daniel plant. The relationship between the number of flowers Raquel and Daniel plant is additive, as the statement "Raquel and Daniel plant at the same speed, but Daniel started earlier. When Raquel has planted 4, Daniel has planted 8" describes an additive relationship between the quantities. If we let " x " be the number of flowers that Daniel has planted and " y" be the number of flowers that Raquel has planted, we obtain the relationship $y = x + k$, $k = 4$, since this is the difference between the number of flowers that Raquel and Daniel plant when Raquel has planted 4 flowers. However, the number of flowers that Raquel and Juan plant is directly proportional, as the statement "They started at the same time, but Juan is faster. When Raquel has planted 4 flowers, Juan has planted 12 flowers" describes a multiplicative relationship between the quantities (4 flowers of Raquel correspond to 12 flowers of Juan, and this relationship is always the same from the beginning). Therefore, at any time, the number of flowers that Juan has planted is 3 times the number that Raquel has planted (12 = 4 × 3). If we let "x" be the number of flowers that Raquel has planted and " y" be the number of flowers that Juan has planted, we obtain the relationship $y = kx$, where k is 3 in this case, therefore $y = kx$. If $x = 20$, then y = 3 × 20 = 60. Thus, when Raquel has planted 20 flowers, Juan has planted 60 flowers.

Fig. 6 Problem created by Laura to solve the task given in Fig. 5

which in a proportional way, complete it and establish the requirement so that solving the problem requires differentiating these relationships (epistemic facet of didactic-mathematical knowledge).

Figure 6 shows the problem created by a trainee teacher, Laura (a fictitious name), and how she justifies the relevance of her proposal.

Laura decomposes the context and information given in the proposed situation into semiotic units. Considering the stated objective, Laura selects the magnitudes that can potentially be related in an additive or multiplicative way, based on the data and their links. The process of idealization and generalization leads to identify the need to add regularity conditions and relations between the magnitudes in the situation so that they fit the models $y = x + k$ (additive relation) and $y = kx$ (proportional relation). Finally, Laura particularizes these relations ("Juan is faster than Raquel," "Raquel and Daniel plant at the same speed," "Daniel started before his classmates") and provides a value of the magnitude "number of flowers planted by Raquel" to ask for the corresponding values of the magnitudes "number of flowers planted by Juan" and "number of flowers planted by Daniel." Laura solves the problem, giving as final her formulation of the problem (Fig. 7).

The second part of the task (Fig. 5) entails a structured elaboration, as the teacher-in-training must vary her problem with an epistemic purpose: not all the ratios involved should be integers. This implies idealizing and generalizing the

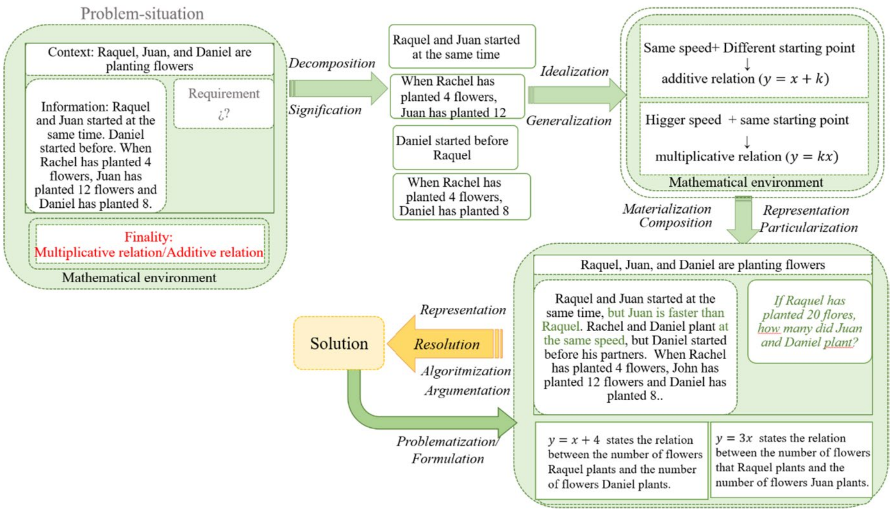


Fig. 7 Mega-process of problem posing (semi-structured elaboration) followed by Laura

information (data and relations) in the created problem and connecting with the mathematical environment provided by the didactic objective: multiplicative relation or not between antecedent and consequent of a ratio. Then, the particularization and materialization allows to decide which elements of the ratios given in the problem are changed to formulate the problem with the established purpose.

As shown in Fig. 8, Laura interprets the didactic pursue provided in the second part of the task to decide on which elements of the problem she can act on and in what manner. Thus, she keeps the requirement, but modifies the information by changing the number of flowers planted by Juan and Daniel, so that, although the ratio in the number of flowers planted by Raquel, 4 to 20, is still integer, the ratio between the number of flowers planted by Raquel and Juan, 4 to 21, is not.

> Problem 2. Raquel, Juan, and Daniel are planting flowers. Raquel and Juan started at the same time, but Juan is faster than Raquel. Raquel and Daniel plant at the same speed, but Daniel started before his partners. When Raquel has planted 4 flowers, Juan has planted 21 flowers, and Daniel has planted 19. If Raquel has planted 20 flowers, how many did Juan plant? And Daniel?
>
> We deduce from the statement that there is still a direct proportionality relationship between the number of flowers Juan and Raquel plant, but the proportionality constant 'k' changes, where k = 21/4, which is not an integer number. The elements I have modified in my problem are the number of plants for Juan and Daniel, introducing fractions to obtain non-integer ratios.

Fig. 8 Problem created by variation of problem 1 (Fig. 6) by Laura

The 5th grade teacher proposes the following problem to his students:

> *Alexa saved 2 out of 5 penalties taken, David saved 3 out of 4 and Lola saved 11 out of 20. Who saved the best penalties? And the worst?*

The task is solved in class and students can talk to their classmates and discuss the solution. While the students try to solve the problem in class, some of them comment:

> Luis: *It's easy... Lola is the one who stopped the most and Alex is the one who stopped the least.*
> Maria then corrects Luis.
> María: *That doesn't mean that Lola is the best and Alexa the worst at saving penalties... David saved almost all of them, he's the best.*
> The teacher interrupts in the face of Maria's silence:
> Teacher: *María, and would you know which student has been the worst at saving penalties?*
> To which María answers:
> María: *I don't know teacher, but I think Lola because from 2 to 5 there are less than from 11 to 20.*

a) How do you interpret the students' answers? What difficulties have they encountered?
b) What learning objectives or expectations do you think the problem pursues?
c) Create, by variation, a problem that contributes to achieving these objectives, as well as facilitating student understanding. Indicate in each case the elements that you have varied in the statement and in what way it allows to achieve the learning objectives and overcome the difficulties.

Fig. 9 Structured elaboration task with a cognitive purpose. Source: Own elaboration

Problem creation with a cognitive didactic-mathematical purpose

Figure 9 depicts a (fictional) mathematics classroom situation in which a problem is presented, and a conversation takes place between two students and the teacher during its resolution. In this case, it is intended that teachers in training analyze the educational objective behind the base problem and that they can vary (structured elaboration) it when the result of its implementation is not the intended one from the point of view of expectations and learning achievements (cognitive facet of the didactic-mathematical knowledge).

In this case, the task organization (Kontorovich et al., 2012) does not only consider the didactic specifications or the situation itself but also the students' discursive mathematical practices regarding its solution. The trainee teacher must analyze these practices to decide on which elements of the base problem to act upon. That is, the processes of decomposition, signification, idealization, and generalization stem not only from the problem itself but also from the mathematical activity of the students.

Figure 10 shows the answer given by a trainee teacher. Daniela (fictitious name) identifies and interprets units of meaning (decomposition, signification) in the pupils' answers: "Lola is the one who stopped more, Alex who stopped less," "David saved almost all of them," "from 2 to 5 there are less than from 11 to 20." She recognizes the incorrect additive strategy in the answer and situates (idealization,

a) Some of the difficulties they may have encountered are: incorrect use of an additive strategy; not identifying the whole over which the fraction is considered; not recognizing the relationships between fractions; not distinguishing the different terms of a fraction (numerator and denominator).

b) Learning Objectives or Expectations: Encourage questioning and construct meaning; Identify the terms of a fraction; Write and read fractions; Compare fractions with different denominators; Make use of the least common multiple; Take into account the relationship between numerator and denominator, not only in quantitative data.

c) Problem. Alexa saved 2 out of 5 penalties taken, David saved 3 out of 4 and Lola 11 out of 20. Which student did better in proportion to the total number of balls taken? And which one did worse? Explain your answer and reflect each child's data in a graph.

In this problem, the element that I have changed for better understanding is the requirement since it says implicitly that they must pay attention to the proportion for the resolution, in this way the objective of the problem is clearer. Finally, a new request is made, which is the explanation of their answer and the realization of three graphs for their better understanding, since visually they can understand it better and settle their knowledge.

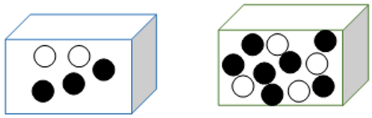
Fig. 10 Analysis and creation of problem by variation of the problem proposed in the episode (Fig. 9) developed by Daniela

generalisation) the mathematical environment in the proportionality relationship (highlighting the role of the comparison of fractions). Although the objectives she specifies are more curricular than didactic in nature, Daniela describes objects (concepts, procedures, properties) expected in the solution to the mathematical problem. In the variation, she modifies the requirement to make it clearer to the student the need to compare proportionally, preserving context, information, and mathematical environment. In addition, she considers that the graphical representation (without specifying the type) helps in the resolution, so she also includes it as a requirement in the problem. Daniela idealizes and generalizes the information (penalties saved by each goalkeeper and total penalties taken by each one) and the requirement (question about who saved the penalties better) of the initial problem to articulate the objects of the mathematical environment of the problem, identified as proportional relations and comparison of fractions. After that, she particularizes these abstract objects in the given context (penalty kicks) and represents them through natural language to modify the requirement of the problem (“in proportion to the total number of balls taken”).

Creating problems to develop algebraic reasoning

From research, it is highlighted that creating problems enables the development of mathematical reasoning and fosters a deeper understanding of concepts, properties, and procedures. This not only occurs with students (Ayllón et al., 2016) but also with prospective teachers (Malaspina et al., 2019). Moreover, it is emphasized that

Create from the following situation a problem in the probabilistic context that involves proportional reasoning in its solution.



We will call this problem the Pre problem. Solve it, identify the mathematical objects and processes and the potential difficulties that your students might encounter when solving it. Justifiably assign the level of algebraic reasoning involved.

From the Pre Problem you have elaborated, justifiably propose and solve new problems (Pos Problem) whose solution involves higher levels of algebraic reasoning. Identify the algebraic objects and processes involved.

Fig. 11 Creation of problems to promote algebraic and proportional reasoning in the probabilistic context. Source: Own elaboration

prospective teachers should identify the potential of tasks that are not intentionally algebraic and to modify them so that objects and processes of an algebraic nature come into play during their solution (Burgos & Godino, 2022a; Godino et al., 2014). Problem-posing tasks, like the one we illustrate below (Fig. 11), aim to encourage prospective teachers to create problems that articulate different types of reasoning, including proportional, probabilistic, and algebraic reasoning.[4]

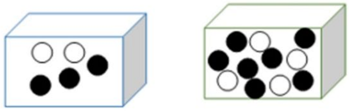
In the first part of the task shown in Fig. 11 (semi-structured situation), the trainee teachers have to create a problem in the probabilistic context, taking into account that proportional reasoning must be used (epistemic purpose). The result of this process is the Pre problem. Therefore, the context (urns), the information (number of balls in each urn, by means of graphic/pictorial language), and the environment (proportional reasoning) are provided, leaving the requirement open. Figure 12 shows the problem created by David (fictitious name) from the situation presented.

In his problem-posing process (Fig. 13), David gives meaning (signification) to the image in the probabilistic context (unrs), considering two boxes (A and B) with different numbers of black and white balls. As it must involve proportional reasoning, it is placed in the comparison of probabilities (ideal, abstract mathematical structure), through the correspondence with the comparison of proportions between the number of balls, delimiting the mathematical environment (idealization and generalization). The context of chance is clearly materialized in the Pre problem, specifying that the drawing is done blind.

[4] Proportional reasoning is an integral part of the components of the sample space analysis and quantification of probabilities (Bryant & Nunes, 2012). It is also an essential component of early algebraic thinking (Van Dooren et al., 2018).

Pre Problem

In box A, 2 white balls and 3 black balls have been put. In box B there are 4 white balls and 7 black balls. If you had to pick a white ball blindly, which box would you choose to draw the ball?



Solution

In box A we found 2 white balls and 3 black balls and in box B we found 4 white balls (which would be twice as many as in the previous one) and 7 black balls (which is more than twice as many as in the previous one, which would be 6), so there is a higher probability of getting the white ball in box A since 2/3 is greater than 4/7.

Fig. 12 Pre problem created by David in the environment of comparison of probabilities in urns

The didactical goal pursue by the second part of the task (Fig. 11) involves specific didactical-mathematical knowledge about algebraic reasoning. The trainee teacher must make a complex microscopic analysis of the mathematical structure involved in the solution of the Pre problem, in order to decide which variation would imply a higher level of algebraic reasoning, i.e., which elements of the Pre problem should be modified so that objects with a higher degree of intension and generalisation and new algebraic processes are involved in its solution (Godino et al., 2014). David appreciates that there is no algebraic activity in the solution to the Pre problem, "because particular numbers are used, and no unknowns are reflected." In addition, he identifies the process of "particularization of the proportionality relation" to the case of the composition of the boxes. This reflection leads him to propose the Pos problem included in Fig. 14.

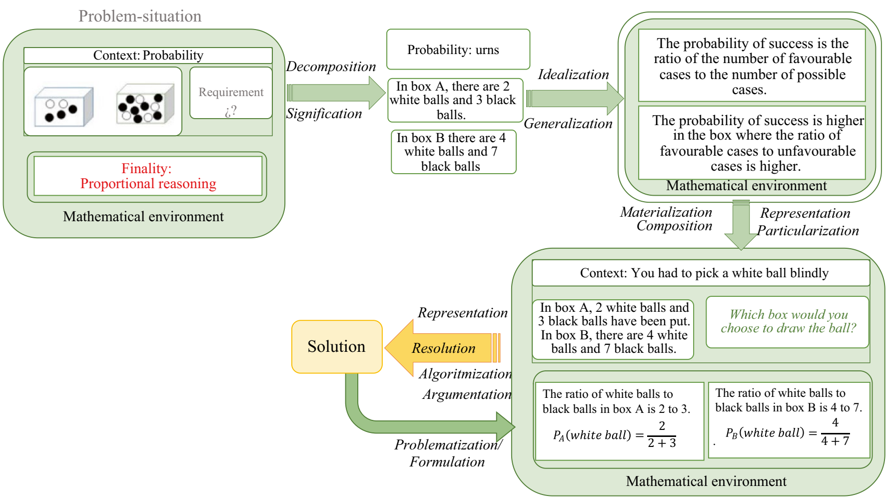


Fig. 13 Problem-posing process followed by David

Pos Problem
In box A 2 white balls and 3 black balls have been placed. In box B there are 4 white balls and 7 black balls. Could a box C be formed that has the same probability of getting a white ball as box A? And box B? Box C cannot have the same number of balls as the previous ones.
Solution
To solve this problem, it is necessary to know the concept of equivalence of fractions, so that the student can establish the relation 2/3=2n/3n where n can be any number, but that multiplies both numerator and denominator. This can be applied directly to box B, with 4/7=4n/7n, so that, yes, it is possible that box C has the same probability of obtaining a white ball when the total number of balls is also multiplied by n.

Fig. 14 Pos Problem created by David to increase the algebraic reasoning required

In this case, David maintains the context, information, and mathematical environment of the problem (comparison of probabilities) but modifies the requirement making use of a third box whose composition must be determined. This requirement mobilizes higher level objects and processes (relations and properties of rational numbers, parameters; generalisation, symbolic representation) in its resolution. The idealization process carried out by David to create the Pos problem is observed when he justifies what makes the level of algebraic reasoning higher in this case. He considers the implication of "properties of equivalent fractions [to] be able to know if it is the same proportion despite having different numbers, thus finding a generalization by knowing that $a/b = (na)/(nb)$," the use of the "relational meaning of equality as equivalence, and employing literal symbols as unknowns, but without operating with them."

Final reflection

The creation of problems, not only in the design and planning of lessons but whenever necessary during their implementation, to encourage students to use specific mathematical knowledge, employ certain competencies, overcome difficulties encountered with a prior problem, or achieve specific learning outcomes, is one of the features of a teacher's competence in analysis and didactic intervention. Reciprocally, the creation of problems for didactic purposes serves as a means to enhance and integrate other didactic-mathematical competencies and knowledge of the teacher: it requires reflection on the elements characterizing the problem, different ways it can be solved, analyzing the mathematical practices, objects, and processes involved, and identifying potential difficulties that students may encounter (Burgos & Chaverri, 2023; Calle et al., 2023; Godino et al., 2017). This justifies the interest in incorporating problem posing into teacher education (Grundmeier, 2015; Singer et al., 2013) and developing theoretical and methodological tools to guide them in a complex task they may not be familiar with (Ellerton, 2013; Mallart et al., 2018). With this commitment, this work has presented (founded and illustrated) a theoretical model for problem posing integrating the epistemic analysis of the involved processes, categorization of problem creation tasks based on the components of a problem, and the types of didactic-mathematical knowledge they involve.

Firstly, our model allows us to address the complexity of problem posing from the perspective of the mathematical activity it entails. In this sense, our framework provides a description of the mathematical processes and objects (as detailed in Fig. 1) involved in the sequence of actions carried out when posing a problem. This point of view complements previous problem-posing models focusing on describing cognitive processes that intervene when posing a problem (Baumann & Rott, 2022; Cruz, 2006; Koichu & Kontorovich, 2013; Pelczer & Gamboa, 2009), analyzing the mathematical actions performed to create a problem (Contreras, 2007; Grundmeier, 2015), or characterizing the facets involved in problem posing (Kontorovich et al., 2012). Secondly, based on Malaspina's (2013) description of the elements of a problem, our model structures a problem (Fig. 2) as an entity determined by context, information, requirements, and mathematical environment. Thirdly, this characterization of problem's attributes leads us to establish three categories of problem-posing tasks which groups previous categories of problem creation (Akay & Boz, 2010; Baumann & Rott, 2021a; Cai & Jiang, 2017; Stoyanova & Ellerton, 1996; Van Harper & Presmeg, 2013) as didactic decisions made by the teacher (Kontorovich et al., 2012): free or unstructured, semi-structured, and structured elaborations. Once the type of problem-posing task is determined based on the known components in the situation and the didactic instructions that determine the rest, the application of the model reveals different sequences of emerging processes and objects (Fig. 3). Up to our knowledge, we presented for the first time a model which articulates these three aspects of problem posing (components of a problem, categorization of problem-posing tasks, and description of mathematical processes involved when posing a problem) into a single framework.

The range of applicability of our model encompass two different dimensions. When focusing on teacher training, using the model allows teacher educators to design a priori problem-posing activities taking into account the mathematical practices, objects, and processes involved, as well as the actions (didactic practices) coming into play, their sequencing (didactic processes), and the emerging entities of such actions (didactic objects) determining the didactic dimension of problem creation (Fig. 4). Afterward, it enables the assessment of the achieved competence and the identification of processes (mathematical or didactic) that may have caused difficulties in some cases, guiding instruction towards the development of these capabilities. Problem creation in this case is part of a macrostructure (Kontorovich et al., 2012) organized around the different facets of the teacher's specialized knowledge (Godino et al., 2017), which broadens the perspective of previous research (Baumann & Rott, 2022; Contreras, 2007; Cruz, 2006; Grundmeier, 2015; Koichu & Kontorovich, 2013; Pelczer & Gamboa, 2009). When the problem-posing task is proposed to students, the model allows the researcher or teacher to reflect a priori on the adequacy of the information provided to the student for creating the problem, as well as to become aware of the intrinsic complexity of the mathematical processes leading to the problem as the final product. Afterward, it enables a microscopic analysis of the mathematical activity that led to this result.

In this work, we illustrated the applicability of our model by studying the responses of trainee teachers to different problem-posing tasks. In future research, we will apply our framework to analyze and categorize the mathematical processes in which prospective teachers encounter difficulties when creating problems with a didactic-mathematical purpose, not only in epistemic and cognitive but also affective, ecological, interactional, or mediational facets. Additionally, we will investigate how prospective teachers utilize the model to analyze students' productions in response to various problem-posing tasks, interpreting their difficulties in terms of the underlying processes. Finally, our model could be extended to encompass the didactic-mathematical knowledge and competencies of teacher educators concerning the design of problem-posing tasks.

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Declarations

Ethics approval and consent to participate The study does not require any specific approval by the ethics committee, as we were informed by our University, since no human and/or animal data are shown.

Competing interests The authors declare no competing interests.

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