ON WEAKLY ALMOST SQUARE BANACH SPACES

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Abstract We prove some results on weakly almost square Banach spaces and their relatives. On the one hand, we discuss weak almost squareness in the setting of Banach function spaces. More precisely, let (Ω, Σ) be a measurable space, let E be a Banach lattice and let $\nu : \Sigma \to E^+$ be a non-atomic countably additive measure having relatively norm compact range. Then the space $L_1(\nu)$ is weakly almost square. This result applies to some abstract Cesàro function spaces. Similar arguments show that the Lebesgue–Bochner space $L_1(\mu, Y)$ is weakly almost square for any Banach space Y and for any non-atomic finite measure μ. On the other hand, we make some progress on the open question of whether there exists a locally almost square Banach space, which fails the diameter two property. In this line, we prove that if X is any Banach space containing a complemented isomorphic copy of c_0, then for every $0 < \varepsilon < 1$, there exists an equivalent norm $|\cdot|$ on X satisfying the following: (i) every slice of the unit ball $B_{(X,|\cdot|)}$ has diameter 2; (ii) $B_{(X,|\cdot|)}$ contains non-empty relatively weakly open subsets of arbitrarily small diameter and (iii) $(X, |\cdot|)$ is (r, s)-SQ for all $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. The (closed) unit ball and the unit sphere of X are denoted by $B_{(X,\|\cdot\|)}$ and $S_{(X,\|\cdot\|)}$, respectively. If the norm does not need to be explicitly mentioned, we just write B_X and S_X instead. Given a bounded set $C \subseteq X$, a *slice* of C is a set of the form

$$S(C, x^*, \alpha) := \{x \in C : x^*(x) > \sup x^*(C) - \alpha\}$$

for some $x^* \in X^*$ (the topological dual of X) and $\alpha > 0$. Notice that $S(C, x^*, \alpha)$ is non-empty and relatively weakly open in C. A Banach space is said to have the *slice diameter two property (slice-D2P)* (respectively, *diameter two property – D2P, strong diameter*

two property – SD2P) if every slice (respectively, non-empty relatively weakly open subset, convex combination of slices) of the unit ball has diameter 2. D2Ps have attracted the attention of many researchers in the last 20 years (see, e.g., [1, 3, 19, 20, 25]) and have motivated the appearance of new properties of Banach spaces (almost squareness [2], symmetric strong D2Ps [24, 34], or diametral D2Ps [23]).

According to Abrahamsen et al. [2], a Banach space $(X, \|\cdot\|)$ is said to be

(i) *locally almost square (LASQ)* if for every $x \in S_{(X,\|\cdot\|)}$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $B_{(X,\|\cdot\|)}$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$;

(ii) *weakly almost square (WASQ)* if for every $x \in S_{(X,\|\cdot\|)}$, there exists a weakly null sequence $(y_n)_{n\in\mathbb{N}}$ in $B_{(X,\|\cdot\|)}$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$;

(iii) *almost square (ASQ)* if for every finite set $\{x_1, \ldots, x_k\} \subseteq S_{(X,\|\cdot\|)}$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $B_{(X,\|\cdot\|)}$ such that $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$ and $\|y_n\| \to 1$.

All these properties are isometric in nature, that is, they depend on the norm considered. For instance, the basic example of an ASQ space is c_0 with its usual norm, while every Banach space admits an equivalent norm failing the slice-D2P and so it cannot be LASQ (see, e.g., [22, Lemma 2.1]).

Even though the previous properties were introduced in [2], LASQ and WASQ spaces were implicitly used by Kubiak [31] to study the D2P in some Cesàro function spaces. Apart from being interesting by themselves, almost squareness properties have shown to be a powerful tool in order to study D2Ps in certain Banach spaces where there is no good description of the dual space. In this direction, let us mention, for instance, that in [27, Section 4], it is proved that if X is LASQ (respectively, ASQ), then any ultrapower $X_{\mathcal{U}}$ of X is LASQ (respectively, ASQ), and, in particular, $X_{\mathcal{U}}$ has the slice-D2P (respectively, SD2P). Observe that it is unclear whether $X_{\mathcal{U}}$ has the slice-D2P (respectively, SD2P) if X has the slice-D2P (respectively, SD2P). Another context in which these properties are useful are the projective symmetric tensor products. It is known that if X is WASQ and has the Dunford-Pettis property (respectively, X is ASQ), then all the projective symmetric tensor products $\widehat{\otimes}_{\pi,s,N} X$ have the slice-D2P [33, Proposition 3.6] (respectively, SD2P [21, Theorem 3.3]). Notice that it is unknown whether any of the D2Ps is stable by taking projective symmetric tensor products.

Among all the almost squareness properties introduced in [2], it is clear that ASQ has been studied in a more intensive way because it turns out to characterize the containment of c_0. More precisely, a Banach space admits an ASQ equivalent renorming if and only if it contains an isomorphic copy of c_0 (see [2, Lemma 2.6] and [21, Theorem 2.3]). The contribution to examples of LASQ and WASQ spaces has been more modest. In spite of that, we find several results in the literature about these properties in the context of function spaces. On the one hand, Kubiak proved in [31, Lemma 3.3] that the weighted Cesàro function spaces on an interval are WASQ. In particular, $L_1[0, 1]$ is WASQ. On the other hand, Hardtke proved in [28, Theorem 3.1] that the Köthe–Bochner space $E(X)$ is LASQ whenever the Banach space X is LASQ, for any Banach function space E.

The aim of this note is to deepen the understanding of WASQ and LASQ Banach spaces. The paper is organized as follows.

In § 2, we focus on certain Banach function spaces, which play an important role in Banach lattice and operator theory. Namely, we consider the space $L_1(\nu)$ of all real-valued functions that are integrable with respect to a countably additive vector measure ν (defined on a σ-algebra and taking values in a Banach space). Up to Banach lattice isometries, these spaces represent all order continuous Banach lattices having a weak order unit (see, e.g., [12, Theorem 8]). Therefore, there are reflexive (hence, having the Radon–Nikodým property and so failing the slice-D2P) Banach lattices within this class, like ℓ_p and $L_p[0,1]$ for $1 < p < \infty$. For detailed information on the L_1 space of a vector measure, see [38]. More recent references on this topic are [10, 14, 15, 37, 39]. Our main result in this section is the following:

Theorem 1.1. *Let (Ω, Σ) be a measurable space, let E be a Banach lattice and let $\nu : \Sigma \to E$ be a countably additive measure. If ν is non-atomic and the set*

$$\mathcal{R}(\nu) := \{\nu(A) : A \in \Sigma\}$$

(the range of ν) is a relatively norm compact subset of $E^+ := \{x \in E : x \geq 0\}$, then $L_1(\nu)$ is WASQ.

Clearly, Theorem 1.1 generalizes the fact that the classical space $L_1(\mu)$ of a non-atomic finite measure μ is WASQ. As an application of Theorem 1.1 and some results of Curbera and Ricker [13], it follows that if E is an order continuous rearrangement invariant Banach function space on $[0,1]$, then the abstract Cesàro function space $[\mathcal{C}, E]$ is WASQ (Corollary 2.3). This generalizes the aforementioned result by Kubiak in the case of the interval $[0,1]$. Abstract Cesàro function spaces have been widely studied in the literature (see, e.g., [6, 7, 13]).

The techniques of Theorem 1.1 allow us to show the Lebesgue–Bochner space $L_1(\mu, Y)$ is WASQ for any Banach space Y whenever μ is a non-atomic finite measure (Corollary 2.5). This result should be compared with the above mentioned result of [28] that the property of being LASQ passes from a Banach space Y to the Köthe–Bochner space $E(Y)$, for any Banach function space E. We finish § 2 with an example of a WASQ Banach space of the form $L_1(\nu)$ as in Theorem 1.1, which is not an $\mathcal{L}_1$-space (§ 2.3).

In § 3, we go a bit further in the analysis of the link between almost squareness and D2Ps. One of the main questions raised in [2] is whether there exists an LASQ Banach space that is not WASQ. Very recently, Kaasik and Veeorg proved in [29, Section 2] that the answer is negative and that an example can be found in the class of Lipschitz-free spaces over complete metric spaces. For such spaces, the properties SD2P, D2P, slice-D2P and LASQ are equivalent (combine [8, Theorem 1.5] and [26, Theorem 3.1]), so the above mentioned example satisfies the SD2P. Since the slice-D2P and the D2P are different properties [19], it is a natural question whether there exists an LASQ Banach space that fails the D2P. Within the framework of Banach lattices, a stronger version of the LASQ property that implies the D2P has been considered in [11]. Even though we do not know the answer to the previous question, we make some progress in this direction. Our main result in § 3 is the following:

Theorem 1.2. *Let X be a Banach space containing a complemented isomorphic copy of c_0. Then for any $0 < \varepsilon < 1$, there exists an equivalent norm $|\cdot|$ on X such that:*

(i) $(X, |\cdot|)$ *has the slice-D2P, that is, every slice of* $B_{(X,|\cdot|)}$ *has a diameter 2.*

(ii) *There are non-empty relatively weakly open subsets of* $B_{(X,|\cdot|)}$ *of arbitrarily small diameter.*

(iii) $(X, |\cdot|)$ *is* (r, s)-SQ *for all* $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$ *in the sense of [9, Section 6], that is, for every finite set* $\{x_1, \ldots, x_n\} \subseteq S_X$, *there exists* $y \in S_X$ *satisfying*

$$|rx_i \pm sy| \leq 1 \text{ for every } i \in \{1, \ldots, n\}.$$

Condition (iii) measures somehow how far is the norm from being ASQ. Notice that a Banach space is ASQ if and only if it is (r, s)-SQ for all $0 < r, s < 1$.

Theorem 1.2 applies to any separable Banach space containing an isomorphic copy of c_0, thanks to Sobczyk's theorem. The proof of Theorem 1.2 is inspired by the renorming technique developed by Becerra Guerrero et al. in [19, Theorem 2.4], which in turn uses ideas of the example of Argyros et al. [5] of a closed bounded convex subset of c_0 having the convex point of continuity property but failing the point of continuity property.

Terminology

We follow standard notation as can be found in [4, 17]. We will consider real Banach spaces only. By an *operator* we mean a continuous linear map between Banach spaces. By a *subspace* of a Banach space we mean a norm closed linear subspace. Let $(X, \|\cdot\|)$ be a Banach space. Given a set $C \subseteq X$, we denote by $\mathrm{conv}(C)$ (respectively, $\overline{\mathrm{conv}}(C)$) its convex hull (respectively, closed convex hull). The *diameter* of C is defined by $\mathrm{diam}_{\|\cdot\|}(C) := \sup\{\|x - x'\| : x, x' \in C\}$ and will be also denoted by $\mathrm{diam}(C)$ if no confusion arises.

Let (Ω, Σ, μ) be a finite measure space. A Banach space $(E, \|\cdot\|)$ is said to be a *Banach function space* on (Ω, Σ, μ) (or just over μ) if the following conditions hold:

(i) E is a (not necessarily closed) linear subspace of $L_1(\mu)$;

(ii) if $f \in L_0(\mu)$ and $|f| \leq |g|$ μ-a.e. for some $g \in E$, then $f \in E$ and $\|f\| \leq \|g\|$;

(iii) the characteristic function χ_A of each $A \in \Sigma$ belongs to E.

In this case, E is a Banach lattice when endowed with the μ-a.e. order, and the inclusion map from E to $L_1(\mu)$ is an operator. A set $H \subseteq E$ is called *uniformly μ-integrable* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\chi_A\| \leq \varepsilon$ for every $f \in H$ and for every $A \in \Sigma$ with $\mu(A) \leq \delta$. Suppose now that E is order continuous. Then every bounded uniformly μ-integrable subset of E is relatively weakly compact (see, e.g., [38, Proposition 2.39]), but the converse might fail, in contrast to the case of the classical L_1 space of a finite measure (for which the Dunford–Pettis theorem ensures the equivalence). Moreover, given $f, g \in E$ with $f \leq g$, the order interval $[f, g] \subseteq E$ is uniformly μ-integrable (see, e.g., [38, Lemma 2.37]) and weakly compact.

2. WASQ Banach function spaces

We begin this section with some preliminaries on the L_1 space of a vector measure (see [38, Chapter 3] for the basics on this topic). Let (Ω, Σ) be a measurable space, let X be

a Banach space and let $\nu : \Sigma \to X$ be a countably additive measure. A set $A \in \Sigma$ is said to be ν-null if $\nu(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$. The family of all ν-null sets is denoted by $\mathcal{N}(\nu)$. We say that a property holds ν-a.e. if it holds on some $A \in \Sigma$ such that $\Omega \setminus A \in \mathcal{N}(\nu)$. We say that a set $A \in \Sigma \setminus \mathcal{N}(\nu)$ is an atom of ν if for every $B \in \Sigma$ with $B \subseteq A$, we have either $B \in \mathcal{N}(\nu)$ or $A \setminus B \in \mathcal{N}(\nu)$. We say that ν is *non-atomic* if it has no atoms. By a *Rybakov control measure* of ν, we mean a finite measure of the form $\mu = |x_0^* \circ \nu|$ (the variation of the signed measure $x_0^* \circ \nu : \Sigma \to \mathbb{R}$) for some $x_0^* \in X^*$ such that $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ (see, e.g., [17, p. 268, Theorem 2] for a proof of the existence of Rybakov control measures).

A Σ-measurable function $f : \Omega \to \mathbb{R}$ is called ν-*integrable* if $f \in L_1(|x^* \circ \nu|)$ for all $x^* \in X^*$ and, for each $A \in \Sigma$, there is $\int_A f \, \mathrm{d}\nu \in X$ such that

$$x^* \left(\int_A f \, \mathrm{d}\nu \right) = \int_A f \, \mathrm{d}(x^* \circ \nu) \quad \text{for all } x^* \in X^*.$$

Identifying functions that coincide ν-a.e., the set $L_1(\nu)$ of all (equivalence classes of) ν-integrable functions is a Banach lattice with the ν-a.e. order and the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, \mathrm{d}|x^* \circ \nu|.$$

$L_1(\nu)$ is an order continuous Banach function space over any Rybakov control measure of ν. The (norm 1) operator $I_\nu : L_1(\nu) \to X$ defined by

$$I_\nu(f) := \int_\Omega f \, \mathrm{d}\nu \quad \text{for all } f \in L_1(\nu)$$

is called the *integration operator* of ν.

To provide a proof of Theorem 1.1, we need a couple of lemmata. The first one belongs to the folklore (cf. [4, Lemma 6.3.2] for the case of the unit interval):

Lemma 2.1. *Let (Ω, Σ, μ) be a non-atomic finite measure space. Then there is a sequence $(r_n)_{n \in \mathbb{N}}$ in $L_\infty(\mu)$ such that:*

(i) *$|r_n| = 1$ for all $n \in \mathbb{N}$; and*
(ii) *for each $f \in L_1(\mu)$, the sequence $(fr_n)_{n \in \mathbb{N}}$ is weakly null in $L_1(\mu)$.*

A sequence as in the previous lemma will be called a *Rademacher-type sequence* on (Ω, Σ, μ).

Lemma 2.2. *Let (Ω, Σ) be a measurable space, let X be a Banach space and let $\nu : \Sigma \to X$ be a non-atomic countably additive measure. Let μ be a Rybakov control measure of ν and let $(r_n)_{n \in \mathbb{N}}$ be a Rademacher-type sequence on (Ω, Σ, μ). Then for each $f \in L_1(\nu)$, the sequence $(fr_n)_{n \in \mathbb{N}}$ is weakly null in $L_1(\nu)$.*

Proof. Fix $f \in L_1(\nu)$ and take any $\varphi \in L_1(\nu)^*$. Since $L_1(\nu)$ is an order continuous Banach function space over μ, there is $g \in L_1(\mu)$ such that for each $h \in L_1(\nu)$, we have $hg \in L_1(\mu)$ and

$$\varphi(h) = \int_\Omega hg \, \mathrm{d}\mu$$

(see, e.g., [36, p. 29]). In particular, we have $fg \in L_1(\mu)$ and so $(fgr_n)_{n \in \mathbb{N}}$ is weakly null in $L_1(\mu)$. Hence, $\varphi(fr_n) = \int_\Omega fgr_n \, \mathrm{d}\mu \to 0$ as $n \to \infty$. $\qquad \square$

Proof of Theorem 1.1 The fact that $\mathcal{R}(\nu) = \{\nu(A) : A \in \Sigma\} \subseteq E^+$ ensures that

$$\|h\|_{L_1(\nu)} = \left\| \int_\Omega |h| \, \mathrm{d}\nu \right\|_E \quad \text{for all } h \in L_1(\nu) \tag{2.1}$$

(see, e.g., [38, Lemma 3.13]), where $\|\cdot\|_E$ denotes the norm of E. Let μ be a Rybakov control measure of ν and let $(r_n)_{n \in \mathbb{N}}$ be a Rademacher-type sequence on (Ω, Σ, μ).

Fix $f \in S_{L_1(\nu)}$. For each $n \in \mathbb{N}$, we have $fr_n \in L_1(\nu)$ and

$$\|fr_n\|_{L_1(\nu)} \overset{(2.1)}{=} \left\| \int_\Omega |fr_n| \, \mathrm{d}\nu \right\|_E = \left\| \int_\Omega |f| \, \mathrm{d}\nu \right\|_E \overset{(2.1)}{=} \|f\|_{L_1(\nu)} = 1.$$

Moreover, the sequence $(fr_n)_{n \in \mathbb{N}}$ is weakly null in $L_1(\nu)$ (by Lemma 2.2).

We claim that $\|f \pm fr_n\|_{L_1(\nu)} \to 1$ as $n \to \infty$. Indeed, for each $n \in \mathbb{N}$, we have $1 \pm r_n \geq 0$ and so

$$g_n^\pm := |f \pm fr_n| = |f|(1 \pm r_n) = |f| \pm |f|r_n.$$

Therefore, both sequences $(g_n^+)_{n \in \mathbb{N}}$ and $(g_n^-)_{n \in \mathbb{N}}$ converge weakly to $|f|$ in $L_1(\nu)$ (by Lemma 2.2 applied to $|f|$). Hence, $(I_\nu(g_n^+))_{n \in \mathbb{N}}$ and $(I_\nu(g_n^-))_{n \in \mathbb{N}}$ converge weakly to $I_\nu(|f|)$ in E, where $I_\nu : L_1(\nu) \to E$ denotes the integration operator of ν. Observe that each $g_n^\pm$ belongs to the order interval $K := [0, 2|f|] \subseteq L_1(\nu)$, which is uniformly μ-integrable and weakly compact.

Since $\mathcal{R}(\nu)$ is relatively norm compact, I_ν maps every bounded, uniformly μ-integrable subset of $L_1(\nu)$ to a relatively norm compact subset of E (see, e.g., [38, Proposition 3.56(I)]). Therefore, $I_\nu(K)$ is norm compact. It follows that both sequences $(I_\nu(g_n^+))_{n \in \mathbb{N}}$ and $(I_\nu(g_n^-))_{n \in \mathbb{N}}$ are norm convergent to $I_\nu(|f|)$, so

$$\|f \pm fr_n\|_{L_1(\nu)} \overset{(2.1)}{=} \left\| I_\nu(g_n^\pm) \right\|_E \to \left\| I_\nu(|f|) \right\|_E \overset{(2.1)}{=} \|f\|_{L_1(\nu)} = 1$$

as $n \to \infty$. The proof is completed. $\qquad \square$

The rest of this section is devoted to providing applications of Theorem 1.1.

2.1. Cesàro function spaces

The Cesàro "operator" is the map $f \mapsto \mathcal{C}(f)$ defined pointwise by $\mathcal{C}(f)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$ for any $f \in L_1[0,1]$. Given a rearrangement invariant Banach function space $(E, \|\cdot\|_E)$ on $[0,1]$, the Cesàro function space $[\mathcal{C}, E]$ is the Banach function space on $[0,1]$ consisting of all $f \in L_1[0,1]$ for which $\mathcal{C}(|f|) \in E$, equipped with the norm $\|f\|_{[\mathcal{C},E]} := \|\mathcal{C}(|f|)\|_E$.

Corollary 2.3. *Let E be an order continuous rearrangement invariant Banach function space on $[0,1]$. Then the Cesàro function space $[\mathcal{C}, E]$ is WASQ.*

Proof. By [13, Theorem 2.1], the formula $\nu(A) := \mathcal{C}(\chi_A)$ defines an E^+-valued countably additive measure on the Lebesgue σ-algebra of $[0,1]$ such that ν has the same null sets as the Lebesgue measure (hence, it is non-atomic) and the range of ν is relatively norm compact. Since E is order continuous, we have $[\mathcal{C}, E] = L_1(\nu)$ (see [13, Proposition 3.1]). The conclusion now follows from Theorem 1.1. □

We stress that the weighed Cesàro function space $C_{p,w}$ on $[0,1]$ considered in [31], for $1 \le p < \infty$ and a measurable positive function w, is equal to $[\mathcal{C}, E]$ for $E = L_p((xw(x))^p\,\mathrm{d}x)$. Thus, the previous corollary generalizes [31, Lemma 3.3] in the case of $[0,1]$.

2.2. Köthe–Bochner spaces

Let $(E, \|\cdot\|_E)$ be a Banach function space on a finite measure space (Ω, Σ, μ) and let $(Y, \|\cdot\|)$ be a Banach space. The Köthe–Bochner space $E(Y)$ is the Banach space of all (equivalence classes of) strongly μ-measurable functions $f : \Omega \to Y$ such that $\|f(\cdot)\| \in E$, with the norm $\|f\|_{E(Y)} := \|\|f(\cdot)\|\|_E$. Here $\|f(\cdot)\| : \Omega \to \mathbb{R}$ is the μ-measurable function given by $t \mapsto \|f(t)\|$.

The following result should be compared with [28, Theorem 3.1], where it is proved that $E(Y)$ is LASQ if Y is LASQ.

Theorem 2.4. *Let (Ω, Σ) be a measurable space, let X be a Banach lattice and let $\nu : \Sigma \to X$ be a non-atomic countably additive measure such that $\mathcal{R}(\nu)$ is a relatively norm compact subset of X^+. Let μ be a Rybakov control measure of ν and consider $E := L_1(\nu)$ as a Banach function space on (Ω, Σ, μ). Let Y be a Banach space. Then the Köthe–Bochner space $E(Y)$ is WASQ.*

Proof. Let $(r_n)_{n\in\mathbb{N}}$ be a Rademacher-type sequence on (Ω, Σ, μ) (see Lemma 2.1).

Fix $f \in S_{E(Y)}$. Then $fr_n \in S_{E(Y)}$ for every $n \in \mathbb{N}$, and we claim that $(fr_n)_{n\in\mathbb{N}}$ is weakly null in $E(Y)$. Indeed, given any $\varphi \in E(Y)^*$, the order continuity of E allows us to represent φ as a w^*-scalarly μ-measurable function $\varphi : \Omega \to Y^*$ such that $\|\varphi(\cdot)\| \in E^*$, the duality being given by

$$\varphi(h) = \int_\Omega \langle \varphi, h \rangle \,\mathrm{d}\mu \quad \text{for every } h \in E(Y)$$

(see, e.g., [35, Theorem 3.2.4]). Here we denote by $\|\cdot\|$ the norm of both Y and Y^*, while $\langle \varphi, h \rangle \in L_1(\mu)$ is defined by $t \mapsto \langle \varphi(t), h(t) \rangle$. Therefore, we have $\varphi(fr_n) =$

$\int_\Omega \langle \varphi, f \rangle r_n \, \mathrm{d}\mu \to 0$ as $n \to \infty$. This shows that $(fr_n)_{n\in\mathbb{N}}$ is weakly null in $E(Y)$, as claimed.

Moreover, we have $\|f(\cdot)\| \in S_E$ and so

$$\|f \pm fr_n\|_{E(Y)} = \|\|f(\cdot)\|(1 \pm r_n)\|_{L_1(\nu)} \to 1$$

as $n \to \infty$, by the proof of Theorem 1.1. □

To the best of our knowledge, the following corollary seems to be new:

Corollary 2.5. *Let (Ω, Σ, μ) be a non-atomic finite measure space and let Y be a Banach space. Then the Lebesgue–Bochner space $L_1(\mu, Y)$ is WASQ.*

Remark 2.6. The previous result is also interesting from the point of view of the identification of the space $L_1(\mu, Y)$ as the projective tensor product $L_1(\mu) \widehat{\otimes}_\pi Y$ (see, e.g., [17, p. 228, Example 10]). In general, given two Banach spaces X and Y, it is not known whether $X \widehat{\otimes}_\pi Y$ is WASQ if X is WASQ. It is even open if $X \widehat{\otimes}_\pi Y$ has the D2P if X has the D2P (see [33, Question 4.2]).

2.3. An example

The aim of this subsection is to give an example of a WASQ Banach function space as in Theorem 1.1, which is not an $\mathcal{L}_1$-space. To do so, we need to introduce some terminology first. Throughout this subsection, $(p_n)_{n\in\mathbb{N}}$ is a sequence in $(1, \infty)$.

The *Nakano sequence space* $\ell_{(p_n)}$ is the Banach lattice consisting of all sequences $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} |sa_n|^{p_n} < \infty$ for some $s > 0$, equipped with the coordinate-wise ordering and the norm

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell_{(p_n)}} := \inf \left\{ t > 0 : \sum_{n\in\mathbb{N}} \left| \frac{a_n}{t} \right|^{p_n} \leq 1 \right\}.$$

Given a sequence of Banach spaces $(X_n, \|\cdot\|_{X_n})_{n\in\mathbb{N}}$, its $\ell_{(p_n)}$-*sum* is the Banach space $\ell_{(p_n)}(X_n)$ consisting of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, such that $(\|x_n\|_{X_n})_{n\in\mathbb{N}} \in \ell_{(p_n)}$, with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell_{(p_n)}(X_n)} := \|(\|x_n\|_{X_n})_{n\in\mathbb{N}}\|_{\ell_{(p_n)}}.$$

If $(p_n)_{n\in\mathbb{N}}$ is bounded, then the unit vectors form an unconditional basis of $\ell_{(p_n)}$ (see, e.g., [41, Theorem 3.5]) and so [2, Proposition 5.2] applies to get:

Corollary 2.7. *Suppose that $(p_n)_{n\in\mathbb{N}}$ is bounded and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces which are WASQ. Then $\ell_{(p_n)}(X_n)$ is WASQ.*

We denote by λ the Lebesgue measure on the Lebesgue σ-algebra Σ of $[0, 1]$.

Proposition 2.8. *Let $(A_n)_{n\in\mathbb{N}}$ be a partition of $[0,1]$ such that $A_n \in \Sigma \setminus \mathcal{N}(\lambda)$ for all $n \in \mathbb{N}$. Then the map $\nu : \Sigma \to \ell_{(p_n)}$ given by*

$$\nu(A) := \big(\lambda(A \cap A_n)\big)_{n\in\mathbb{N}} \quad \text{for all } A \in \Sigma$$

is a well-defined countably additive measure. Moreover,

- (i) *ν is non-atomic and $\mathcal{R}(\nu)$ is relatively norm compact.*
- (ii) *$L_1(\nu)$ is WASQ.*
- (iii) *For each $n \in \mathbb{N}$, let λ_n be the restriction of λ to the σ-algebra on A_n given by $\Sigma_n := \{A \cap A_n : A \in \Sigma\}$. If $(p_n)_{n\in\mathbb{N}}$ is bounded, then the map*

$$\Phi : L_1(\nu) \to \ell_{(p_n)}(L_1(\lambda_n))$$

  given by

$$\Phi(f) := \big(f|_{A_n}\big)_{n\in\mathbb{N}} \quad \text{for all } f \in L_1(\nu)$$

  is a well-defined lattice isometry.

Proof. Define $\tilde{\nu} : \Sigma \to \ell_1$ by

$$\tilde{\nu}(A) := \big(\lambda(A \cap A_n)\big)_{n\in\mathbb{N}} \quad \text{for all } A \in \Sigma.$$

Note that $\tilde{\nu}$ is finitely additive and satisfies $\|\tilde{\nu}(A)\|_{\ell_1} = \lambda(A)$ for all $A \in \Sigma$; hence, $\tilde{\nu}$ is countably additive. Since the inclusion $\iota : \ell_1 \hookrightarrow \ell_{(p_n)}$ is a well-defined operator, the composition $\nu = \iota \circ \tilde{\nu} : \Sigma \to \ell_{(p_n)}$ is a countably additive measure.

(i) Clearly, we have $\mathcal{N}(\lambda) = \mathcal{N}(\nu)$, so ν is non-atomic. The range of any countably additive Banach space-valued measure is relatively weakly compact (see, e.g., [17, p. 14, Corollary 7]). Hence, by the Schur property of ℓ_1, the set $\mathcal{R}(\tilde{\nu})$ is relatively norm compact. Alternatively, this can also be deduced from the usual criterion of relative norm compactness in ℓ_1 (see, e.g., [16, p. 6, Exercise 6]). Therefore, $\mathcal{R}(\nu) = \iota(\mathcal{R}(\tilde{\nu}))$ is relatively norm compact as well.

(ii) Follows from (i) and Theorem 1.1 (note that ν takes values in $\ell_{(p_n)}^+$).

(iii) Fix $f \in L_1(\nu)$. For each $n \in \mathbb{N}$, let $\pi_n \in \ell_{(p_n)}^*$ be the nth-coordinate functional. Since $(\pi_n \circ \nu)(A) = \lambda(A \cap A_n)$ for all $A \in \Sigma$ and $f \in L_1(\pi_n \circ \nu)$, we have $f|_{A_n} \in L_1(\lambda_n)$ and

$$\pi_n\left(I_\nu(|f|)\right) = \int_{[0,1]} |f| \, \mathrm{d}(\pi_n \circ \nu) = \|f|_{A_n}\|_{L_1(\lambda_n)}.$$

Hence, $(\|f|_{A_n}\|_{L_1(\lambda_n)})_{n\in\mathbb{N}} = I_\nu(|f|) \in \ell_{(p_n)}$. Moreover, the fact that ν takes values in $\ell_{(p_n)}^+$ ensures that

$$\|f\|_{L_1(\nu)} = \big\|I_\nu(|f|)\big\|_{\ell_{(p_n)}} = \left\|\Big(\|f|_{A_n}\|_{L_1(\lambda_n)}\Big)_{n\in\mathbb{N}}\right\|_{\ell_{(p_n)}}$$

(see, e.g., [38, Lemma 3.13]). Thus, Φ is a well-defined isometric embedding. Clearly, Φ is a lattice homomorphism. It remains to check that Φ is surjective.

Let $(f_n)_{n \in \mathbb{N}} \in \ell_{(p_n)}(L_1(\lambda_n))$. Define $f \in L_0[0,1]$ by declaring $f|_{A_n} := f_n$ for all $n \in \mathbb{N}$. Since $(p_n)_{n \in \mathbb{N}}$ is bounded, the space $\ell_{(p_n)}$ contains no isomorphic copy of c_0 (see, e.g., [41, Theorem 3.5]). Therefore, in order to prove that $f \in L_1(\nu)$, it suffices to show that $f \in L_1(|\varphi \circ \nu|)$ for every $\varphi \in \ell^*_{(p_n)}$ (see, e.g., [30, p. 31, Theorem 1]). It is known that $\ell^*_{(p_n)} = \ell_{(q_n)}$, where $(q_n)_{n \in \mathbb{N}}$ is the sequence in $(1, \infty)$ defined by $1/p_n + 1/q_n = 1$ for all $n \in \mathbb{N}$, the duality being

$$\langle (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} a_n b_n \quad \text{for all } (a_n)_{n \in \mathbb{N}} \in \ell_{(p_n)} \text{ and } (b_n)_{n \in \mathbb{N}} \in \ell_{(q_n)}$$

(see, e.g., [41, Theorem 4.2]). Take any $\varphi = (b_n)_{n \in \mathbb{N}} \in \ell_{(q_n)}$. Then

$$(\varphi \circ \nu)(A) = \sum_{n \in \mathbb{N}} b_n \lambda(A \cap A_n) \quad \text{for all } A \in \Sigma$$

and so the variation of $\varphi \circ \nu$ is given by

$$|\varphi \circ \nu|(A) = \sum_{n \in \mathbb{N}} |b_n| \lambda(A \cap A_n) \quad \text{for all } A \in \Sigma.$$

Then

$$\int_{[0,1]} |f| \, \mathrm{d}|\varphi \circ \nu| = \sum_{n \in \mathbb{N}} \int_{A_n} |f| \, \mathrm{d}|\varphi \circ \nu|$$
$$= \sum_{n \in \mathbb{N}} |b_n| \int_{A_n} |f| \, \mathrm{d}\lambda = \sum_{n \in \mathbb{N}} |b_n| \|f_n\|_{L_1(\lambda_n)} < \infty,$$

because $\left(\|f_n\|_{L_1(\lambda_n)} \right)_{n \in \mathbb{N}} \in \ell_{(p_n)}$ and $\varphi \in \ell_{(q_n)}$. Thus, $f \in L_1(\nu)$, and we have $\Phi(f) = (f_n)_{n \in \mathbb{N}}$. The proof is completed. □

Remark 2.9. Note that each $L_1(\lambda_n)$ is WASQ (in fact, it is isometrically isomorphic to $L_1[0,1]$). Hence, when $(p_n)_{n \in \mathbb{N}}$ is bounded, the fact that $L_1(\nu)$ is WASQ can also be deduced from Corollary 2.7 and Proposition 2.8(iii).

Proposition 2.10. *Let ν be as in Proposition 2.8. If $(p_n)_{n \in \mathbb{N}}$ is bounded and $\frac{p_n}{(p_n - 1) \log n} \to 0$ as $n \to \infty$, then $L_1(\nu)$ is not an $\mathcal{L}_1$-space.*

Proof. Since $(p_n)_{n \in \mathbb{N}}$ is bounded, the space $\ell_{(p_n)}$ has an unconditional basis (see, e.g., [41, Theorem 3.5]). The additional condition on $(p_n)_{n \in \mathbb{N}}$ implies that $\ell_{(p_n)}$ is not isomorphic to ℓ_1, see [40, Lemma 4]. Therefore, $\ell_{(p_n)}$ cannot be isomorphic to a complemented subspace of an $\mathcal{L}_1$-space (see, e.g., [18, Theorem 3.13]).

Since $L_1(\nu)$ contains a complemented subspace isomorphic to $\ell_{(p_n)}$ (this can be deduced from Proposition 2.8(iii)), it follows that $L_1(\nu)$ is not an $\mathcal{L}_1$-space. □

For instance, the sequence $p_n := 1 + (\log(n+1))^{-1/2}$ satisfies the conditions of Proposition 2.10.

3. Proof of Theorem 1.2

The aim of this section is to provide a proof of Theorem 1.2. The first step is to prove the result for the space c_0, see Theorem 3.5 below. The proof of this particular case is based on the renorming technique of [19, Theorem 2.4], where it was shown that every Banach space containing an isomorphic copy of c_0 admits an equivalent norm so that its unit ball contains non-empty relatively weakly open subsets with arbitrarily small diameter, but every slice has a diameter 2.

The symbol $\mathbb{N}^{<\omega}$ stands for the *Baire tree*, i.e., the set of all *finite* sequences of positive integers. The empty sequence is included in $\mathbb{N}^{<\omega}$ as the root of the tree. The order on $\mathbb{N}^{<\omega}$ is defined by declaring that $\alpha \preceq \beta$ if and only if β extends α. Given $\alpha \in \mathbb{N}^{<\omega}$ and $p \in \mathbb{N}$, we denote by $\alpha \frown p \in \mathbb{N}^{<\omega}$ the sequence defined by $\alpha \frown p := (\alpha_1, \ldots, \alpha_n, p)$ if $\alpha = (\alpha_1, \ldots, \alpha_n)$ or $\alpha \frown p := (p)$ (a sequence with just one element) if $\alpha = \emptyset$. The following is standard (see, e.g., [19, p. 857]):

Lemma 3.1. *There exists a bijection* $\phi : \mathbb{N}^{<\omega} \to \mathbb{N}$ *such that:*

(i) $\phi(\emptyset) = 1$.
(ii) $\phi(\alpha) \leq \phi(\beta)$ *for all* $\alpha, \beta \in \mathbb{N}^{<\omega}$ *with* $\alpha \preceq \beta$.
(iii) $\phi(\alpha \frown j) < \phi(\alpha \frown k)$ *for every* $\alpha \in \mathbb{N}^{<\omega}$ *and for all* $j, k \in \mathbb{N}$ *with* $j < k$.

Let c be the subspace of ℓ_∞ consisting of all convergent sequences and let $c(\mathbb{N}^{<\omega})$ be the subspace of $\ell_\infty(\mathbb{N}^{<\omega})$ defined by

$$c(\mathbb{N}^{<\omega}) := \{x \in \ell_\infty(\mathbb{N}^{<\omega}) : x \circ \phi^{-1} \in c\}.$$

Clearly, $c(\mathbb{N}^{<\omega})$ and c are isometric; hence, $c(\mathbb{N}^{<\omega})$ is isomorphic to c_0. We denote by $\lim \in c(\mathbb{N}^{<\omega})^*$ the functional defined by

$$\lim x := \lim_{n \to \infty} x(\phi^{-1}(n)) \quad \text{for all } x \in c(\mathbb{N}^{<\omega}).$$

For each $\alpha \in \mathbb{N}^{<\omega}$, we denote by $e_\alpha^* \in c(\mathbb{N}^{<\omega})^*$ the functional defined by

$$e_\alpha^*(x) := x(\alpha) \quad \text{for all } x \in c(\mathbb{N}^{<\omega}).$$

Given $\alpha \in \mathbb{N}^{<\omega}$, we define $x_\alpha \in \ell_\infty(\mathbb{N}^{<\omega})$ by the formula

$$x_\alpha(\beta) := \begin{cases} 1 & \text{if } \beta \preceq \alpha \\ -1 & \text{otherwise} \end{cases}$$

so that $x_\alpha \in S_{c(\mathbb{N}^{<\omega})}$ and $\lim x_\alpha = -1$. Define

$$A := \{x_\alpha : \alpha \in \mathbb{N}^{<\omega}\} \subseteq S_{c(\mathbb{N}^{<\omega})} \quad \text{and} \quad K := \overline{\text{conv}}(A \cup -A) \subseteq B_{c(\mathbb{N}^{<\omega})}.$$

We will need the following result (see [19, Proposition 2.2]):

Lemma 3.2. *Let $n \in \mathbb{N}$ and $\rho > 0$. Define*

$$W_{n,\rho} := \left\{ x \in K : e^*_{\emptyset \frown i}(x) > \frac{2}{n} - 1 - 2\rho \quad \text{for all } i \in \{1, \dots, n\} \right.$$

$$\left. \text{and } \lim x < -1 + \rho \right\}.$$

Then $W_{n,\rho}$ is a non-empty relatively weakly open subset of K and $\operatorname{diam}(W_{n,\rho}) \to 0$ as $n \to \infty$ and $\rho \to 0$.

The following lemma is elementary and its proof will be omitted:

Lemma 3.3. *Let V be a linear space, let $A_1, \dots, A_m$ be subsets of V and let $v \in \operatorname{conv}(A_1 \cup \dots \cup A_m)$. Then there exist $v_i \in \operatorname{conv}(A_i)$ and $c_i \in [0, \infty)$ for $i \in \{1, \dots, m\}$ such that $\sum_{i=1}^m c_i v_i = v$ and $\sum_{i=1}^m c_i = 1$.*

Lemma 3.4. *Let $(X, \|\cdot\|)$ be a Banach space, let $S \subseteq S_{(X, \|\cdot\|)}$ be a dense set and let $0 < \delta < 1$. Suppose that for all $0 < r, s < \delta$ and for every finite set $\{x_1, \dots, x_n\} \subseteq S$, there exists $y \in S_{(X, \|\cdot\|)}$ satisfying*

$$\|rx_i \pm sy\| \le 1 \quad \text{for every } i \in \{1, \dots, n\}.$$

Then $(X, \|\cdot\|)$ is (r, s)-SQ for all $0 < r, s < \delta$.

Proof. Fix $0 < r, s < \delta$. Choose $r < r' < \delta$ such that $s' := s\frac{r'}{r} < \delta$ and then choose $\theta > 0$ such that $r\theta + \frac{r}{r'} \le 1$. Take any finite set $\{x_1, \dots, x_n\} \subseteq S_{(X, \|\cdot\|)}$. Since S is dense in $S_{(X, \|\cdot\|)}$, there exist $x'_1, \dots, x'_n \in S$ such that $\|x_i - x'_i\| \le \theta$ for every $i \in \{1, \dots, n\}$. By the assumption, we can find $y \in S_{(X, \|\cdot\|)}$ in such a way that $\|r'x'_i \pm s'y\| \le 1$ for every $i \in \{1, \dots, n\}$. Then

$$\|rx_i \pm sy\| \le r\|x_i - x'_i\| + \frac{r}{r'}\|r'x'_i \pm s'y\| \le r\theta + \frac{r}{r'} \le 1$$

for every $i \in \{1, \dots, n\}$. This shows that $(X, \|\cdot\|)$ is (r, s)-SQ. $\qquad\square$

Theorem 3.5. *Let $0 < \varepsilon < 1$. Then there exists an equivalent norm $|\cdot|$ on c_0 such that:*

(i) *$(c_0, |\cdot|)$ has the slice-D2P.*
(ii) *There are non-empty relatively weakly open subsets of $B_{(c_0, |\cdot|)}$ of arbitrarily small diameter.*
(iii) *$(c_0, |\cdot|)$ is (r, s)-SQ for all $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$.*

Proof. Let us denote by $\|\cdot\|_Z$ the norm of the Banach space $Z := c(\mathbb{N}^{<\omega}) \oplus_\infty c_0$. Since c_0 and Z are isomorphic, it suffices to prove the statement of the theorem for the space $(Z, \|\cdot\|_Z)$. Let $c_0(\mathbb{N}^{<\omega}) \subseteq c(\mathbb{N}^{<\omega})$ be the subspace of all $x \in c(\mathbb{N}^{<\omega})$ such that

$\lim x = 0$. For each $\alpha \in \mathbb{N}^{<\omega}$, we write e_α to denote the element of $S_{c_0(\mathbb{N}^{<\omega})}$ defined by $e_\alpha(\alpha) = 1$ and $e_\alpha(\beta) = 0$ for every $\beta \in \mathbb{N}^{<\omega} \setminus \{\alpha\}$.

Let $|\cdot|$ be the Minkowski functional of the closed convex symmetric set

$$B := \overline{\operatorname{conv}} \left((A \times \{0\}) \cup (-A \times \{0\}) \cup \left((1-\varepsilon)B_Z + \varepsilon B_{c_0(\mathbb{N}^{<\omega})} \times \{0\} \right) \right) \subseteq Z,$$

that is, $|z| := \inf\{t > 0 : z \in tB\}$ for all $z \in Z$. Since $(1-\varepsilon)B_Z \subseteq B \subseteq B_Z$, it follows that $|\cdot|$ is an equivalent norm on Z with unit ball $B_{(Z,|\cdot|)} = B$. We have

$$\|z\|_Z \le |z| \le \frac{1}{1-\varepsilon}\|z\|_Z \quad \text{for all } z \in Z \tag{3.1}$$

and

$$|(x,0)| = \|(x,0)\|_Z = \|x\|_\infty \quad \text{for all } x \in c_0(\mathbb{N}^{<\omega}) \tag{3.2}$$

because $B_{c_0(\mathbb{N}^{<\omega})} \times \{0\} \subseteq B$. We denote by $\|\cdot\|_{Z^*}$ and $|\cdot|_{Z^*}$ the equivalent norms on Z^* induced by $\|\cdot\|_Z$ and $|\cdot|$, respectively. We will check that $|\cdot|$ satisfies the required properties.

Proof of (i). Let $S \subseteq B$ be a slice of B. Since $B \setminus S$ is convex and closed, we have

$$S \cap \left((A \times \{0\}) \cup (-A \times \{0\}) \cup \left((1-\varepsilon)B_Z + \varepsilon B_{c_0(\mathbb{N}^{<\omega})} \times \{0\} \right) \right) \neq \emptyset.$$

We now distinguish several cases.

Case (a): $S \cap (A \times \{0\}) \neq \emptyset$. Then $(x_\alpha, 0) \in S$ for some $\alpha \in \mathbb{N}^{<\omega}$. Observe that the sequence $((x_{\alpha \frown n}, 0))_{n \in \mathbb{N}} = ((x_\alpha + 2e_{\alpha \frown n}, 0))_{n \in \mathbb{N}}$ converges weakly to $(x_\alpha, 0)$ in Z (because $(e_{\alpha \frown n})_{n \in \mathbb{N}}$ is weakly null in $c_0(\mathbb{N}^{<\omega})$). Since S is relatively weakly open in B and $(x_{\alpha \frown n}, 0) \in B$ for every $n \in \mathbb{N}$, we have $(x_{\alpha \frown n_0}, 0) \in S$ for large enough $n_0 \in \mathbb{N}$. Hence,

$$\operatorname{diam}_{|\cdot|}(S) \ge |(x_{\alpha \frown n_0}, 0) - (x_\alpha, 0)| = 2|(e_{\alpha \frown n_0}, 0)| \overset{(3.2)}{=} 2\|(e_{\alpha \frown n_0}, 0)\|_Z = 2$$

and, therefore, $\operatorname{diam}_{|\cdot|}(S) = 2$.

Case (b): $S \cap (-A \times \{0\}) \neq \emptyset$. The proof that $\operatorname{diam}_{|\cdot|}(S) = 2$ runs similarly as in (a).

Case (c): $S \cap ((1-\varepsilon)B_Z + \varepsilon B_{c_0(\mathbb{N}^{<\omega})} \times \{0\}) \neq \emptyset$. Then we can pick $(x, y) \in B_Z$ and $x' \in B_{c_0(\mathbb{N}^{<\omega})}$ in such a way that

$$z := (1-\varepsilon)(x,y) + \varepsilon(x',0) \in S.$$

We can assume without loss of generality that x' has finite support because the set of all finitely supported functions from $\mathbb{N}^{<\omega}$ to $[-1,1]$ is a norm dense subset of $B_{c_0(\mathbb{N}^{<\omega})}$.

Choose $\alpha \in \mathbb{N}^{<\omega}$ such that $x'(\alpha \frown n) = 0$ for every $n \in \mathbb{N}$. Observe that

$$x - x(\alpha \frown n)e_{\alpha \frown n} \pm e_{\alpha \frown n} \in B_{c(\mathbb{N}^{<\omega})} \quad \text{and} \quad x' \pm e_{\alpha \frown n} \in B_{c_0(\mathbb{N}^{<\omega})}$$

and so

$$\begin{aligned} z_n^{\pm} &:= (1-\varepsilon)\big(x - x(\alpha \frown n)e_{\alpha \frown n}, y\big) + \varepsilon(x', 0) \pm (e_{\alpha \frown n}, 0) \\ &= (1-\varepsilon)\big(x - x(\alpha \frown n)e_{\alpha \frown n} \pm e_{\alpha \frown n}, y\big) + \varepsilon(x' \pm e_{\alpha \frown n}, 0) \in B \end{aligned}$$

for every $n \in \mathbb{N}$. Since S is relatively weakly open in B and both sequences $(z_n^+)_{n \in \mathbb{N}}$ and $(z_n^-)_{n \in \mathbb{N}}$ converge weakly to z in Z, we can find $n_0 \in \mathbb{N}$ large enough so that both $z_{n_0}^+$ and $z_{n_0}^-$ belong to S. Hence,

$$\operatorname{diam}_{|\cdot|}(S) \geq |z_{n_0}^+ - z_{n_0}^-| = 2|(e_{\alpha \frown n_0}, 0)| \overset{(3.2)}{=} 2\|(e_{\alpha \frown n_0}, 0)\|_Z = 2$$

and so $\operatorname{diam}_{|\cdot|}(S) = 2$. This completes the proof of (i).

Proof of (ii). Fix $\theta > 0$. By Lemma 3.2, we can take $n \in \mathbb{N}$ and $\rho > 0$ such that

$$\operatorname{diam}(W_{n,\rho}) \leq \frac{(1-\varepsilon)\theta}{2} \quad \text{and} \quad \eta := \frac{2\rho}{5} \leq \frac{\theta}{16}. \tag{3.3}$$

Define

$$U := \Big\{ z \in B : (e_{\emptyset \frown i}^*, 0)(z) > \frac{2}{n} - 1 - \eta \quad \text{for all } i \in \{1, \dots, n\}$$
$$\text{and } (\lim, 0)(z) < -1 + \varepsilon\eta \Big\}.$$

It is clear that U is a relatively weakly open subset of B. To prove that $U \neq \emptyset$, we will check that the vector $z_0 := (\frac{1}{n} \sum_{j=1}^n x_{\emptyset \frown j}, 0) \in B$ belongs to U. Indeed, for each $i \in \{1, \dots, n\}$, we have

$$(e_{\emptyset \frown i}^*, 0)(z_0) = \frac{1}{n} \sum_{j=1}^n e_{\emptyset \frown i}^*(x_{\emptyset \frown j}) = \frac{1}{n}\big(1 - (n-1)\big) = \frac{2}{n} - 1 > \frac{2}{n} - 1 - \eta,$$

and we also have

$$(\lim, 0)(z_0) = \frac{1}{n} \sum_{j=1}^n \lim x_{\emptyset \frown j} = -1 < -1 + \varepsilon\eta.$$

Hence, $z_0 \in U$ and so $U \neq \emptyset$.

We will show that $\mathrm{diam}_{|\cdot|}(U) \leq \theta$. The key point is the following:
Claim. For every

$$z \in V := U \cap \mathrm{conv}\left((A \times \{0\}) \cup (-A \times \{0\}) \cup \left((1-\varepsilon)B_Z + \varepsilon B_{c_0(\mathbb{N}^{<\omega})} \times \{0\}\right)\right),$$

there is $z' \in W_{n,\rho} \times \{0\}$ such that $|z - z'| < 4\eta$.
Indeed, by Lemma 3.3, we can write

$$z = az_1 - bz_2 + c((1-\varepsilon)u + \varepsilon v) \tag{3.4}$$

for some $a, b, c \geq 0$ with $a + b + c = 1$ and

$$z_1, z_2 \in \mathrm{conv}(A \times \{0\}), \quad u \in B_Z \text{ and } v \in B_{c_0(\mathbb{N}^{<\omega})} \times \{0\}.$$

Observe that

$$(\lim, 0)(z_1) = (\lim, 0)(z_2) = -1$$

because $\lim x_\alpha = -1$ for all $\alpha \in \mathbb{N}^{<\omega}$. We also have

$$|(\lim, 0)(u)| \leq \|(\lim, 0)\|_{Z^*}\|u\|_Z \leq 1 \quad \text{and} \quad (\lim, 0)(v) = 0.$$

Thus,

$$\begin{aligned}
-1 + \varepsilon\eta > (\lim, 0)(z) &\overset{(3.4)}{=} -a + b + c(1-\varepsilon)(\lim, 0)(u) \\
&\geq -a + b - c(1-\varepsilon) = -a - b - c + 2b + \varepsilon c = -1 + 2b + \varepsilon c,
\end{aligned}$$

and so $2b + \varepsilon c < \varepsilon\eta$. This inequality implies that $b < \eta$ (bear in mind that $\varepsilon < 1$) and that $c < \eta$. Consequently,

$$\begin{aligned}
|z - z_1| &\overset{(3.4)}{=} |(a-1)z_1 - bz_2 + c((1-\varepsilon)u + \varepsilon v)| \\
&= |-b(z_1 + z_2) + c((1-\varepsilon)u + \varepsilon v - z_1)| \\
&\leq b|z_1| + b|z_2| + c|(1-\varepsilon)u + \varepsilon v| + c|z_1| \\
&\overset{(\star)}{\leq} 2b + 2c < 4\eta,
\end{aligned}$$

where inequality $(\star)$ follows from the fact that z_1, z_2 and $(1-\varepsilon)u + \varepsilon v$ belong to $B = B_{(Z,|\cdot|)}$. Hence, $|z - z_1| < 4\eta$.
We can write $z_1 = (x, 0)$ for some $x \in \mathrm{conv}(A)$. Then $\lim x = -1$ and for each $i \in \{1, \ldots, n\}$, we have

$$e^*_{\emptyset \frown i}(x) = (e^*_{\emptyset \frown i}, 0)(z_1) \geq (e^*_{\emptyset \frown i}, 0)(z) - |(e^*_{\emptyset \frown i}, 0)|_{Z^*}|z - z_1| > \frac{2}{n} - 1 - 5\eta$$

because $z \in U$ and $|(e^*_{\emptyset \frown i}, 0)|_{Z^*} \leq \|(e^*_{\emptyset \frown i}, 0)\|_{Z^*} = 1$ (by Equation (3.1)). This implies, with the notation of Lemma 3.2, that $x \in W_{n,\rho}$ (recall that $\eta = \frac{2}{5}\rho$). Therefore, the conclusion of the *Claim* holds taking $z' = z_1$.

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Finally, let $w_1, w_2 \in U$ and fix $s > 0$. Since U is relatively open in B, we can find $v_1, v_2 \in V$ such that $|w_1 - v_1| \leq s$ and $|w_2 - v_2| \leq s$. By the *Claim* above, there exist $v_1', v_2' \in W_{n,\rho} \times \{0\}$ such that $|v_1 - v_1'| < 4\eta$ and $|v_2 - v_2'| < 4\eta$. Then

$$|v_1' - v_2'| \overset{(3.1)}{\leq} \frac{1}{1-\varepsilon}\|v_1' - v_2'\|_Z \overset{(3.2)}{\leq} \frac{1}{1-\varepsilon}\operatorname{diam}(W_{n,\rho}) \overset{(3.3)}{\leq} \frac{\theta}{2}$$

and so

$$|w_1 - w_2| < 2s + 8\eta + \frac{\theta}{2} \overset{(3.3)}{\leq} 2s + \theta.$$

As $w_1, w_2 \in U$ and $s > 0$ are arbitrary, we conclude that $\operatorname{diam}_{|\cdot|}(U) \leq \theta$.

Proof of (iii). We will show that $(Z, |\cdot|)$ is (r,s)-SQ for any $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$ with the help of Lemma 3.4. Let H be the norm dense subset of B_{c_0} consisting of all finitely supported functions from $\mathbb{N}$ to $[-1,1]$. Then the set

$$S := S_{(Z,|\cdot|)} \cap \operatorname{conv}\left((A \times \{0\}) \cup (-A \times \{0\}) \cup \left((1-\varepsilon)(B_{c(\mathbb{N}^\omega)} \times H) + \varepsilon B_{c_0(\mathbb{N}^\omega)} \times \{0\}\right)\right)$$

is norm dense in $S_{(Z,|\cdot|)}$. Fix $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$ and take finitely many $z_1, \ldots, z_m \in S$. By Lemma 3.3, each z_i can be written as

$$z_i = a_i(x_i^1, 0) + b_i(-x_i^2, 0) + c_i\left((1-\varepsilon)(x_i, y_i) + \varepsilon(x_i^3, 0)\right)$$

for some $a_i, b_i, c_i \geq 0$ with $a_i + b_i + c_i = 1$ and

$$x_i^1, x_i^2 \in \operatorname{conv}(A), \ x_i \in B_{c(\mathbb{N}^\omega)}, \ y_i \in H \text{ and } x_i^3 \in B_{c_0(\mathbb{N}^{<\omega})}.$$

Let $(e_n)_{n \in \mathbb{N}}$ be the usual basis of c_0 and choose $n \in \mathbb{N}$ large enough such that $\|y_i \pm e_n\|_\infty \leq 1$ for all $i \in \{1, \ldots, m\}$.

Observe that for each $i \in \{1, \ldots, m\}$, we have

$$z_i \pm (0, e_n) = a_i(x_i^1, \pm e_n) + b_i(-x_i^2, \pm e_n) + c_i\left((1-\varepsilon)(x_i, y_i \pm e_n) + \varepsilon(x_i^3, \pm e_n)\right),$$

thus

$$\|z_i \pm (0, e_n)\|_Z \leq a_i\|(x_i^1, \pm e_n)\|_Z + b_i\|(x_i^2, \pm e_n)\|_Z + c_i\left((1-\varepsilon)\|(x_i, y_i \pm e_n)\|_Z + \varepsilon\| \right.$$
$$\left. (x_i^3, \pm e_n)\|_Z\right) \leq a_i + b_i + c_i((1-\varepsilon) + \varepsilon)) = 1,$$

which combined with Equation (3.1) yields

$$|z_i \pm (0, e_n)| \leq \frac{1}{1-\varepsilon}. \tag{3.5}$$

Another appeal to Equation (3.1) gives $1 = \|(0, e_n)\|_Z \leq |(0, e_n)| \leq \frac{1}{1-\varepsilon}$ and, therefore, the vector $z := \frac{1}{|(0,e_n)|}(0, e_n) \in S_{(Z,|\cdot|)}$ satisfies

$$|(0, e_n) - z| = \left| \left(1 - \frac{1}{|(0, e_n)|} \right) (0, e_n) \right| = |(0, e_n)| - 1 \leq \frac{1}{1 - \varepsilon} - 1 = \frac{\varepsilon}{1 - \varepsilon}.$$

This inequality and Equation (3.5) give

$$|z_i \pm z| \leq \frac{1}{1 - \varepsilon} + \frac{\varepsilon}{1 - \varepsilon} = \frac{1 + \varepsilon}{1 - \varepsilon} \quad \text{for every } i \in \{1, \ldots, m\}.$$

Since $r, s \leq \frac{1-\varepsilon}{1+\varepsilon}$, we can apply [9, Lemma 6.3] to conclude that

$$|rz_i \pm sz| \leq 1 \quad \text{for every } i \in \{1, \ldots, m\}.$$

From Lemma 3.4, it follows that $(Z, |\cdot|)$ is (r, s)-SQ for all $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$. The proof is completed. $\square$

We can now prove Theorem 1.2 in full generality.

Proof of Theorem 1.2. Let Z and W be subspaces of X such that Z is isomorphic to c_0 and $X = Z \oplus W$. Fix $0 < \varepsilon < 1$ and let $|\cdot|_Z$ be an equivalent norm on Z like in Theorem 3.5. Since any Banach space admits an equivalent norm for which the unit ball has slices of arbitrarily small diameter (see, e.g., [22, Lemma 2.1]), we can take an equivalent norm $|\cdot|_W$ on W satisfying that property. Let $|\cdot|$ be the equivalent norm on X defined by $|z + w| := \max\{|z|_Z, |w|_W\}$ for every $z \in Z$ and for every $w \in W$. We claim that $(X, |\cdot|)$ satisfies all the requirements.

(i) The ℓ_∞-sum of two Banach spaces has the slice-D2P whenever one of the factors has the slice-D2P (see, e.g., [32, Theorem 2.29]). Since $(Z, |\cdot|_Z)$ has the slice-D2P, we conclude that the same holds for $(X, |\cdot|)$.

(ii) Let us prove that $B_{(X,|\cdot|)}$ contains non-empty relatively weakly open subsets of arbitrarily small diameter. Fix $\eta > 0$. Then there exists a non-empty relatively weakly open set $U \subseteq B_{(Z,|\cdot|_Z)}$ with $\text{diam}_{|\cdot|_Z}(U) < \eta$. Now, take a slice S of $B_{(W,|\cdot|_W)}$ with $\text{diam}_{|\cdot|_W}(S) < \eta$. Since the map $\varphi : B_{(Z,|\cdot|_Z)} \times B_{(W,|\cdot|_W)} \to B_{(X,|\cdot|)}$ given by

$$\varphi(z, w) := z + w \quad \text{for all } (z, w) \in B_{(Z,|\cdot|_Z)} \times B_{(W,|\cdot|_W)}$$

is a homeomorphism when each of the balls is equipped with the restriction of the weak topology, it follows that $V := \varphi(U \times S)$ is a relatively weakly open subset of $B_{(X,|\cdot|)}$. Clearly, $V \neq \emptyset$ and $\text{diam}_{|\cdot|}(V) < \eta$, as desired.

(iii) The space $(X, |\cdot|)$ is (r, s)-SQ for arbitrary $0 < r, s < \frac{1-\varepsilon}{1+\varepsilon}$ because so is $(Z, |\cdot|_Z)$, and the ℓ_∞-sum of two Banach spaces is (r, s)-SQ whenever one of the factors is (r, s)-SQ (see [9, Proposition 6.6]). $\square$

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