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Creation of Problems by Prospective Teachers to Develop Proportional and Algebraic Reasonings in a Probabilistic Context

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Abstract: To promote optimal learning in their students, mathematics teachers must be proficient in problem posing, making this skill a cornerstone in teacher training programs. This study presents a formative action in which pre-service teachers are required to create and analyze a problem involving proportional reasoning within a probabilistic context. For this problem, they must identify the objects and processes involved in its resolution, recognize the degree of algebraic reasoning implied and identify potential difficulties for students. Subsequently, they need to formulate and analyze a new problem with variation, which mobilizes higher-level algebraic activity. Results indicate that prospective teachers struggle to pose problems that engage proportional reasoning, as well as to identify in their analysis which elements of proportional and algebraic reasoning are present in their solutions. Despite this fact, a significant percentage of participants adequately modify the original problem to address higher levels of algebraic reasoning, identifying in these cases the new algebraic objects and potential difficulties that might arise as the degree of generalization required in the solution increases. The study concludes by underscoring the importance of training in problem posing to enhance the knowledge and competences of prospective teachers concerning proportional and algebraic reasoning.

Keywords: problem posing; algebraization levels; proportional reasoning; probability; teacher training

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1. Introduction

Numerous studies have highlighted the importance of problem creation for improving students' mathematical learning [1–5]. Problem posing emerges as a complementary activity to problem solving [6,7], which promotes the understanding of mathematical concepts and relationships, reasoning and the development of problem-solving skills, and self-perception and confidence in mathematics [8,9]. To design problem-posing tasks that develop students' mathematical skills, teachers must be familiar with problem creation. Moreover, the need to consider problem posing in teacher training programs is supported by research showing its close connection with teaching competencies [10]. Mathematics teachers should be able to select, modify, or create problems suitable for a didactic purpose [11,12], as well as analyze and evaluate the mathematical activity they promote in students [11,13,14]. However, without specific training in problem posing, teachers create problems that are ambiguous, incomplete, predictable, not adapted to the educational level, or simply unsolvable [15–17]. Therefore, it is necessary to design, implement, and evaluate formative actions that develop problem posing competence in mathematics teacher training programs.

With this purpose, this work describes and analyzes the results of a formative intervention with prospective primary education teachers aimed at promoting the development of competence to pose mathematical problems for didactic purposes. Since problem posing is not only a goal but also a means to develop knowledge [18], we focus on posing problems that involve proportional and algebraic reasoning in a probabilistic context.

Proportional reasoning, understood as the ability to establish multiplicative relationships between two quantities and to extend this relationship to another pair of quantities, is a goal present in the primary education curriculum, integrating the various interpretations of the rational number, and involving a sense of covariation and multiplicative relative comparisons [19]. Probabilistic reasoning requires, among other skills: understanding the fundamental probabilistic ideas of variability, randomness, independence, or predictability/uncertainty; calculating or estimating probabilities of events in random situations; using the language of chance appropriately; and employing arguments to prove the truth of a probabilistic statement or the validity of the solution to a problem [20]. Given the characteristics of proportional reasoning, on the one hand, it is considered a basic component of probabilistic reasoning, as it is part of the analysis of the sample space, the proportional quantification of probabilities, and the understanding and use of correlations [21], and, on the other hand, it allows access to central practices of early algebraic reasoning [22].

Despite its importance, several studies point out that pre-service teachers have difficulties teaching concepts related to proportionality [23–25]. Specifically, the lack of proportional reasoning contributes to biases in probabilistic reasoning and the difficulties of teachers, both in training and in practice, to teach probability [26–29]. On the other hand, the few studies concerned with specifying what kind of knowledge future teachers need to teach early algebra indicate that the training they receive in this regard is insufficient and limited to connecting arithmetic with algebra [30,31].

Regarding problem posing, works such as those by Burgos and Chaverri-Hernández [32,33] or Şengül and Katranci [34,35] reveal that limited knowledge of content related to proportionality causes difficulties for pre-service teachers when creating tasks to work on ratio and proportionality in an arithmetic context. Similarly, the results of Zapatera and Quevedo [31], show that limited algebraic knowledge prevents pre-service teachers from proposing appropriate tasks to develop algebraic reasoning in primary students based on open situations. The authors emphasize the importance of including in teacher training programs experiences that allow them to design tasks to detect and promote algebraic thinking in their future students.

In research works such as Burgos and Chaverri-Hernández [32] and Burgos and Godino [23], the competence to create problems of proportionality in an arithmetic context is related to the analysis of the mathematical practices involved in their resolution, using theoretical–methodological tools of the Onto-semiotic Approach (OSA) of mathematical knowledge and instruction [36]. These works highlight the difficulties of future teachers to recognize the mathematical objects involved in their own solutions to proportionality problems, as well as to detect the potential difficulties that students may encounter in solving these problems. Moreover, in Burgos and Godino [23], it is observed that, although future teachers are able to identify the algebraic activity involved in their solutions to proportionality problems, they have difficulties modifying these problems to vary the emerging algebraic reasoning level. We believe it is necessary for teachers to identify the potential for unintentionally algebraic tasks and be able to modify them so that solutions involve objects and processes of an algebraic nature [37].

To the best of our knowledge, no studies have yet analyzed how pre-service teachers pose problems to promote proportional an algebraic reasonings in probabilistic contexts. Thus, in this work, we aim to answer the following research questions:

- RQ1—How do prospective teachers pose problems in a probabilistic context to promote proportional reasoning?
- RQ2—What objects and processes related to proportional reasoning involved in their solutions do prospective teachers identify?
- RQ3—What potential difficulties do prospective teachers identify after analyzing the posed problems, and how do they relate to the analysis of objects and processes?

- RQ4—How do prospective teachers transform proportionality problems in a probabilistic context to increase the level of algebraic reasoning involved in their solutions?

To answer these questions, in the experience described in this work, the prospective teachers must pose a problem (Pre-problem) involving proportional reasoning based on a given situation in a context of urns. Participants must identify, based on the mathematical objects and processes involved in their solution, the potential difficulties these problems might pose to students and the involved algebraic reasoning level (understood as a degree of generality and formalization according to Godino et al. [37]). Afterwards, pre-service teachers must pose, with variation of the Pre-problem, one or several Post-problems, whose solution implies higher degrees of algebraic reasoning, and again analyze the mathematical objects and processes present in their solutions and the potential difficulties of these problems.

2. Theoretical Framework

In this study, we will apply some theoretical tools from the OSA [36], in particular the elementary algebraic reasoning model for analyzing mathematical activity [37], and the model of didactic-mathematical knowledge and competence of mathematics teacher [38].

2.1. Pragmatic Meaning and Levels of Algebraization

From the anthropological conception of mathematics assumed by the OSA [36], the notion of practice occupies a central place in the analysis of mathematical activity. Mathematical practice is considered to be any (verbal, graphical, symbolical, or expressed in any other language) action or expression performed by someone to solve mathematical problems, communicate the solution to others, validate it, or generalize it to other contexts. The tangible or intangible entities involved in mathematical practice, underpinning and regulating its realization, are primary mathematical objects, classified according to their nature and function into: problem situations, languages, concepts, propositions, procedures, and arguments [36]. These primary objects emerge from practice systems through the respective processes of problematization, communication, definition, enunciation, algorithmization, and argumentation. Problem solving, modeling, or problem posing are considered as mega-processes, as they may involve several of these elementary processes.

From the OSA perspective, elementary algebraic reasoning (EAR) is understood as the system of operative and discursive practices used in solving mathematical tasks approachable from primary education in which algebraic objects and processes are involved [37]. Algebraic objects are: (a) binary relations of equivalence or order and their respective properties (which are used to obtain new mathematical objects); (b) operations and their properties, performed on elements of sets of various objects; (c) functions, their components, types, operations and properties; (d) structures (semigroup, group, ring, field, vector space), their types, and properties.

As indicated by Godino et al. [37], in the case of algebraic practices, the processes of particularization–generalization have special importance, given the role of generalization as one of the characteristic features of algebraic reasoning. As a result of a generalization process, we obtain a type of mathematical object which in the OSA, is called intensive (the rule that generates the class); through the reverse process of particularization, we obtain an extensive object, that is, a particular object. In Godino et al. [37], an algebraic reasoning model for primary education is proposed, establishing criteria that allow identifying purely arithmetic mathematical activity (level 0 of EAR) and distinguishing it from the progressive levels of EAR. The criteria for defining the different levels are based on the class of mathematical objects and processes involved: types of representations that are used, generalization processes involved, and the analytic calculation that is brought into play in the corresponding mathematical activity.

- *Level 0.* Operations are carried out with intensive objects of the first degree of generality, using natural, numeric, iconic, or gestural languages.
- *Level 1.* Intensive objects of the second degree of generality are used, properties of the algebraic structure of natural numbers, and equality as equivalence.
- *Level 2.* Symbolic–literal representations are used to refer to recognized intensive objects, which are linked to spatial, temporal, and contextual information. Equations of the form $Ax + B = C$ are solved.
- *Level 3.* Symbols are used analytically, without referring to contextual information. Operations are performed with unknowns. Equations of the form $Ax + B = Cx + D$ are solved.

Although the initial intention of the EAR levels model proposed in [37] is the description of the type of algebraic reasoning used in solving specific mathematical tasks from an epistemic viewpoint, it is possible to apply this model to analyze the nature of the mathematical activity expected or achieved by students when solving mathematical problems in different contexts. In Burgos and Godino [39] and Burgos et al. [40], the EAR model is applied to identify the types of objects and processes put into play in solutions of a selection of problems involving proportional and probabilistic reasoning, respectively, showing the progression of mathematical activity from arithmetic and proto-algebraic levels to higher levels of formalization.

2.2. Teacher's Didactic-Mathematical Knowledge and Competence Model

The teacher's Didactic-Mathematical Knowledge and Competence (DMKC) model, developed within the OSA framework, articulates the categories of didactic-mathematical knowledge and competence of the mathematics teacher through the facets and components of the mathematical study processes considered in that framework [13,38]. Thus, it is accepted that the teacher must have *mathematical knowledge per se*, which allows them to solve the problems and tasks proposed in the curriculum of the educational level where they teach, and link it with higher levels [13,38,41,42]. Moreover, as some mathematical content comes into play, the teacher must have *didactic-mathematical knowledge* of the different facets that affect the educational process: epistemic (didactic-mathematical knowledge about the content itself), ecological (relationships of mathematical content with other disciplines and both curricular and socio-professional factors that condition the processes of mathematical instruction), cognitive (how students reason mathematics and how they progress in their learning), affective (affective, emotional and attitudinal aspects and belief of students concerning mathematical objects and their study), mediational (technological, material, and temporal resources suitable to enhance learning), and interactional (knowledge about the teaching of mathematics, selection and organization of tasks, addressing student difficulties, and management of interactions in the classroom) [13,38,41,42].

In addition to having this knowledge, the DMKC model proposes that the teacher must be competent to address the basic didactic problems present in teaching and learning processes. In particular, the *competence of didactic analysis and intervention* allows the teacher to analyze the mathematical activity involved in solving the problems they propose to their students, with the aim of designing, managing, and evaluating the implementation of appropriate teaching–learning situations [37]. This competence includes, among others, the identification of problem situations and the operative and discursive practices involved in their resolution (subcompetence of *analysis of global meanings*), and the recognition of the configuration of objects and processes intervening and emerging from mathematical practices (*onto-semiotic analysis of practices*), in particular, those of an algebraic nature [38,41,42].

2.3. Problem Posing

As it is pointed out by Silver [43], problem posing essentially involves both the enunciation of new situations and the modification of pre-existing problems. This author considers problem posing as a component of problem solving and establishes a classification of problem-posing tasks based on the moment at which they occur: before solving (posing a problem based on a given situation or experience), during solving (modifying the goals or conditions of a problem to improve its understanding), and after solving (posing a problem by varying the initial problem, applying the knowledge obtained from its solution) [43].

Stoyanova and Ellerton propose a new categorization of problem-posing tasks [44], differentiating between free situations (posing problems with no restrictions), semi-structured (posing problems from an open situation), and structured (posing problems from a pre-existing one). In this sense, Baummans and Rott [45] point out the difficulty in distinguishing between free and semi-structured situations, so they propose only two categories: structured and unstructured.

According to Malaspina et al. [46], problem posing is a process by which a new problem is obtained, which is determined by four fundamental elements: the *information*, that is, the quantitative or relational data provided in the problem; the *requirement*, that is, what is asked to be found, examined, or concluded, which can be quantitative or qualitative, including graphs and demonstrations; the *context*, which can be intra-mathematical or extra-mathematical; and the *mathematical environment*, where the mathematical objects that intervene or can intervene to solve the problem are located. For these authors, the creation of new problems can occur through two procedures:

- The *variation* of a given problem, by which a new problem is posed by modifying one or more of the four elements of an initial problem.
- The *elaboration* of a new problem, which can be performed *freely*, *from a* (given or configured by the author) *situation*, or *from a specific requirement*, which can have a mathematical or didactic emphasis. Didactic-mathematical knowledge about the content, in our case proportionality, is especially relevant to adequately respond to the requirement.

The mathematical and analysis and didactic intervention competences of the teacher are fundamental in problem posing for didactic purposes [38]. Conversely, problem posing serves as a means to develop them, as it requires: reflecting on the overall structure of the problem, the objectives it pursues, whether the information provided is sufficient to solve the problem and how it can be approached; analyzing the mathematical objects and processes that intervene and how they relate to solve the proposed problem; and recognizing the possible difficulties that students might encounter and how to address them in the presentation of new situations.

3. Method

This work presents a teaching experiment conceived within the framework of design research applying the OSA as the base theory [47]. For space reasons, we only show the results of the evaluation of a first experimentation cycle, applying a mixed quantitative and qualitative method [48].

The population on which the research focuses is sixth-semester students of primary education at the University of Granada (Spain). One of the main focuses of the course in which the experience is developed is the analysis, design, and sequencing of mathematical tasks according to specific content and certain learning expectations. A total of 17 prospective teachers (PTs, henceforth) participated in the experience.

Before the intervention, the PTs had received specific preparation (three theoretical sessions lasting two hours each) on the notions of mathematical tasks, practices and mathematical objects, and the EAR levels model. The importance of the teacher selecting, designing, and sequencing tasks that effectively promote student learning was emphasized, presenting elements that guide the selection and modification of tasks: (a) mathematical content (mathematical objects and processes, contexts), (b) purpose (specific learning objectives and competences) that allows to develop, and (c) learning limitations (possible difficulties and errors that may arise in its resolution). The analysis of tasks, some of which were about proportionality, was shown, recalling both the configurations that emerge from practices associated with different contexts and some of the difficulties and errors related to the content previously identified in didactic mathematics research. The assignment of EAR levels to mathematical activity was also exemplified from both epistemic (expert resolutions) and cognitive (student resolutions) perspectives.

In the following practical session (one hour long), the PTs worked with their regular work teams on the analysis of a mathematical task following the described elements. The next theoretical session (two hours long) focused on introducing the PTs to the methodology for problem posing described in the previous section. Some proportionality tasks were also used, among others, to show problem posing with variation or elaboration, recalling again the necessary knowledge about proportional reasoning. The next practical session (one hour long) was devoted to the PTs working as a team in the creation of mathematical tasks (variation and elaboration) to develop proportional and algebraic reasoning.

The task (Figure 1) was proposed as part of the voluntary classroom activities in that context.

Create a problem from the following situation in the context of probability that involves proportional reasoning in its solution.

> **Situation**
> *In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls.*

We will call this problem the Pre-problem. Solve it, identify the mathematical objects and processes, and the potential difficulties your students might encounter when solving it. Justifiably assign the level of algebraic reasoning involved.

Based on the Pre-problem you have created, justifiably propose and solve new problems (Post-problem 1, Post-problem 2,..) whose solutions imply higher levels of algebraic reasoning. Identify the algebraic objects and processes involved, as well as the potential difficulties your students might encounter when solving it.

Figure 1. Proposed task on problem posing to the PTs. Own elaboration.

The analysis of Pre-problems (elaboration from the situation) and Post-problems (variations of the Pre-problem) proposed by the PTs was carried out taking into account both their significance and the extent to which they respond to the established didactic-mathematical purpose, allowing us to establish a categorization based on their degree of relevance. To ensure the validity and reliability of the analysis, the researchers discussed and agreed a priori on the categories of relevance according to the didactic-mathematical requirements of the evaluation task. They then independently analyzed the reports of the PTs and contrasted the results, examining discrepancies (Figure 2).

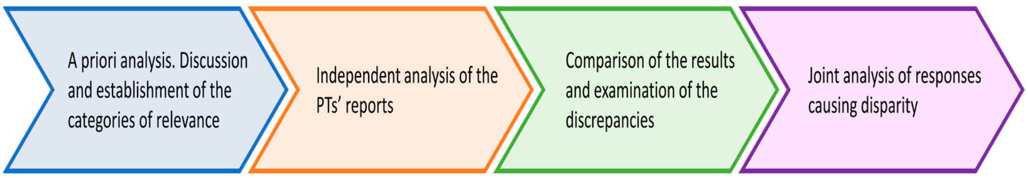

Figure 2. Scheme of the analysis process.

A Pre- or Post-problem is significant if its wording is clear, unambiguously establishes the requirement, the solution is not implicit, and the information is adequate for it to be solved. If it lacks any of these features, it is considered a non-significant problem (therefore, not relevant).

For the significant Pre-problems, the categories of relevance are:

- Relevant (PreR). The problem stems from the proposed situation, and its solution involves proportional reasoning (Figures 3 and 4).
- Partially relevant (PrePR). Although it is significant, one of the following errors is made:
    - ○ The proposed situation is modified (information error, IE).
    - ○ Its solution does not involve proportional reasoning (mathematical environment error, EE).

- Non-relevant (PreNR). The proposed situation is modified, and its solution does not involve proportional reasoning.

Pre-problem
In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. In which box am I more likely to draw a white ball?

Solution
The solution would be box A. This is because if we calculate the percentage chance of drawing a white ball from each box, we find a 40% chance in box A and a 33.3% chance in box B. We arrive at this result by performing the following calculations:

- In box A, knowing that there are 5 balls in total (which represents 100%), what would be the percentage for 2 balls? We multiply 100 × 2 and then divide by 5, giving us a result of 40%.
- In box B, knowing there are 11 balls in total (which represents 100%), what would be the percentage for 4 balls? We multiply 100 × 4 and then divide by 11, giving us a result of 33.3%.

Figure 3. Relevant Pre-problem and resolution (PT9). CMP category.

Pre-problem
In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. If you had to blindly choose a white ball, which box would you choose to draw the ball from?

Solution
In box A, we find 2 white balls out of the 3 black balls that are in the box, and in box B, we find 4 white balls (which is double that of the previous box) and 7 black balls (which is more than double of the previous box, which would be 6), so there is a higher probability of getting a white ball from box A since 2/3 is a larger number than 4/7.

Figure 4. Relevant Pre-problem and resolution (PT5). SEL category.

For the significant Post-problems, the following categories of relevance are considered:

- Relevant (PosR). The Post-problem is a variation of the Pre-problem, and its solution involves a higher EAR level than the actual one involved in the solution of the Pre-problem.
- Partially relevant (PosPR). Although it is significant, one of the following situations occurs:
    - It is not a variation of the Pre-problem (NoPre).
    - It does not involve in its solution a higher EAR level than the one involved in the solution of the Pre-problem (NoEAR).
- Non-relevant (PosNR). It is not a variation of the Pre-problem, nor does its solution imply a higher EAR level than the corresponding Pre-problem solution.

Let us mention that, to assign a category of relevance, the actual EAR levels involved in resolutions proposed by the PTs to the Pre- and Post-problems they pose are considered, not those assigned by the PTs.

Finally, the following categories of relevance are considered in recognizing the difficulties that their problems might pose to potential students:

- Relevant difficulty (RD). The difficulty is appropriate to the posed problem and refers to knowledge or skills involved in the proposed solution.
- Partially relevant difficulty (PRD). The difficulty is appropriate to the posed problem, but it does not refer to knowledge or skills involved in the proposed solution.
- Non-relevant difficulty (NRD). The difficulty is not appropriate to the posed problem.

4. Results

4.1. Pre-Problem

All the PTs proposed a significant Pre-problem. Table 1 lists the different categories of problems posed by the PTs and the frequency with which these appear according to their degree of relevance. Next, prototypical problem examples in each category are described.

Table 1. Types of Pre-problems and frequency according to the degree of relevance (n = 17).

Category	Fr.			
	PreR	PrePR		PreNR
		IE	EE	
CLP. Explicit calculation of the probability of drawing one or more balls in one or two boxes.	0	0	6	1
CMP. Explicit comparison of the probability of success in both boxes.	4	1	0	0
SEL. Selection of a box to perform a draw leading to a prize.	4	0	0	0
CCP. Explicit calculation and comparison of the probability of success in both boxes.	0	1	0	0

Nine PTs posed a Pre-problem whose solution required comparing the probabilities of drawing a ball of a specific color in both boxes. Five of them explicitly asked about the box in which there is a higher probability of drawing a white or black ball (CMP category); this was relevant in four of the cases (Figure 3).

On the other hand, four PTs posed a Pre-problem (all relevant) that implicitly required comparing probabilities, describing a situation where one must decide from which box to draw (without looking) a ball of a given color to win a prize (SEL category) (Figure 4).

Seven of the PTs proposed a problem where it is explicitly asked about the probability of drawing one or more balls of a color from one or both boxes. In six cases, a problem was posed with a simple experiment (drawing a single ball), while in one of them, a problem was posed with a compound experiment (drawing two consecutive balls without replacement).

The problems in the CLP category were partially relevant in six cases, since, even starting from the proposed situation, they did not require the comparison of probability, thus not involving proportional reasoning (Figure 5). The problems were irrelevant in one case because, in addition, the proposed starting situation was modified (Figure 6).

Pre-problem

In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. Given this situation, what is the probability of drawing a white ball from box A? And what is the probability of drawing a black one? In box B, what is the probability of drawing a white ball? And a black one?

Figure 5. Partially relevant Pre-problem (PT11). CLP category.

Pre-problem

In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. What is the probability of drawing a white ball from box A? Next, calculate the probability of drawing the white ball from box A in the following table:

Number of white balls (5 balls in the box in total)	1	2	3	4
Percentage		40%	60%	

Figure 6. Non-relevant Pre-problem (PT2). CLP category.

Finally, only one of the PTs proposed a (partially relevant) problem with several questions combining the explicit calculation of the probability and the explicit comparison of the success probability in both boxes (CCP category).

Once the Pre-problem was formulated, the PTs had to solve it and identify the mathematical objects and processes involved in their solution. The majority (thirteen PTs) correctly solved their proposed problem. Those who did not, forgot part of the requirement, confused the data, or wrongly argued based on the total number of balls and not on the ratio between favorable and possible cases. However, the PTs had difficulties in adequately identifying the mathematical objects present in their solutions, a situation that has been observed in previous research [23,32,49]. Although they frequently recognized languages (especially natural and symbolic) and concepts (not always appropriate), they hardly identified procedures, propositions, and arguments, and when they did, they were usually incorrect.

Given the aim of this work, we focus on Pre-problems that involved proportional reasoning, analyzing what types of solution strategies they proposed and how such strategies are related to the objects they recognized as involved. In this regard, five of the ten PTs who posed a problem requiring the comparison of probabilities solved it by calculating the success probability of both boxes and comparing them using their expression as a percentage (Figure 3). However, the PTs did not make explicit the connection of proportionality with percentages nor its relation to the comparison of probabilities. Another three PTs solved their Pre-problem using the multiplicative relationship between the number of favorable and unfavorable cases (Figure 4). These PTs connected the comparison of the ratios of white to black balls with the comparison of the probability of drawing a white ball from each box (although they did not explicitly indicate that the extraction probability is determined as the number of favorable cases among possible ones). Lastly, two PTs expressed the success probabilities of each box as a fraction and reasoned which of them was greater by comparing both fractions; although, only one of them pointed out that the fraction favorable cases/possible cases indicates the probability of drawing a ball of a specific color.

Among the ten PTs whose Pre-problems involved proportional reasoning, two of them did not identify any object related to proportionality. Four PTs indicated the concept "proportionality", even though in their solution, the proportional relationship is not explicitly stated. Among the concepts related to proportional reasoning identified by the eight PTs, we could find "proportionality", "fraction", "percentage", and "ratio", while among the procedures we could find "calculating percentage from fractions" and "comparing fractions". Properties or arguments related to proportionality had not been adequately identified, except for the property: "the product of the extremes is equal to the product of the means", which PT7 indicates as necessary to argue their solution (Figure 3). On the other hand, objects typical of probabilistic reasoning were also identified, such as the concepts "probability", "favorable cases", "possible cases" and "chance", the procedures "counting favorable and possible cases" and "applying the Laplace's rule", as well as the property "Laplace's rule" (referred to as "formula for favorable cases/possible cases" or "probability calculation formula").

Regarding the analysis of the mathematical processes involved in their solution, four of the PTs did not identify any processes, while in two of the cases, all the processes indicated were incorrect. The processes correctly identified most frequently were "interpretation/signification" of the different elements of the formulation, "particularization of the Laplace's rule", and "representation" in tabular or symbolic form. Two PTs indicated the process "particularization of proportionality properties", referring to properties of proportions they had indicated as objects, although they did not appear explicitly in their solutions. It is also significant that, despite not having identified arguments as objects, seven PTs indicated "argumentation of the solution" as a process.

After identifying the objects and processes, the PTs pointed out the difficulties that students might encounter during the resolution of their Pre-problem. They indicated a total of 37 potential difficulties, which are categorized and ranked according to their relevance in Table 2.

The difficulties associated with probabilistic reasoning (D-PBR) included those related to the identification and understanding of stochastic phenomena ("not understanding the random phenomenon and chance", PT6; "difficulty in using probability knowledge to interpret, analyze, and produce information about such situations", PT8), the identification of favorable or unfavorable cases ("not correctly identifying favorable cases", PT16), and Laplace's rule ("lack of knowledge about the formulation of Laplace's rule", PT1). The difficulties related to proportional reasoning (D-PPR) expressed limitations regarding the understanding and application of proportionality ("not correctly identifying the proportionality relationship", PT8) or the use of additive strategies to solve the problem ("mistaken belief that it is easier to draw a white ball from the box containing more white balls, ignoring the number of black balls", PT3). On the other hand, PTs pointed out difficulties (D-RN) related to understanding fractions and their terms ("not knowing the meaning of the terms of the fraction", PT5) and converting percentages to fractions and vice versa ("difficulty converting from fraction to percentage", PT10). In other cases, generic difficulties (D-G) were found, that is, difficulties related to understanding the formulation of the problem and interpreting the data ("lack of understanding of the formulation", PT8, "difficulty in interpreting the data", PT4) or to arithmetic non-specific procedures of the problem.

Table 2. Frequency of difficulties in each category according to the degree of relevance (n = 37).

Type of Difficulties	Fr.			
	RD	**PRD**	**NRD**	**Total**
D-PBR. Associated to probabilistic reasoning.	11	1	1	13
D-PPR. Related to proportional reasoning.	7	1	2	10
D-RN. Associated to the rational number.	3	1	2	6
D-G. Of a generic nature.	0	6	2	8

We observe that the most frequent type of difficulty (35.13% of the total difficulties identified) are those related to probabilistic reasoning, followed by those explicitly linked to proportional reasoning (27.02% of the total); although, they were only detected in half of the problems that involve proportionality. Furthermore, we notice that almost all PTs that detect difficulties related to probabilistic and proportional reasoning had adequately identified objects related to each of these reasonings, respectively.

Finally, PTs had to analyze the level of EAR involved in their resolution. All resolutions corresponded to arithmetic practices (level 0, in seven solutions) or incipient proto-algebraic practices (level 1, in ten solutions). More than half of the PTs were not able to correctly identify the algebraization level of their own problem resolution. Moreover, only eight of them justified (either correctly or incorrectly) their assignment. Three PTs correctly identified the absence of algebraic activity in their practices, based on the involvement of only specific numbers and arithmetic operations with them (see Figure 7). In contrast, four PTs indicated a proto-algebraic character in solutions that did not present algebraic activity, without justifying in any case the assigned level.

Solution

Considering the problem formulation, the easiest way to solve this problem using algebraic reasoning would be to approach it through proportional reasoning. First, we need to determine the total number of balls in each box.

Box A = 2 white balls + 3 black balls = 5 balls. Box B = 4 white balls + 7 black balls = 11 balls.

From here, to determine the probability for each type of ball, we will set up the proportion using fractions. Thus:

Box A:

White balls = 2 white balls out of 5 total balls → $\frac{2}{5}$.

Black balls = 3 black balls out of 5 total balls → $\frac{3}{5}$.

Box B:

White balls = 4 white balls out of 11 total balls → $\frac{4}{11}$.

Black balls = 7 black balls out of 11 total balls → $\frac{7}{11}$.

With this in mind, we can say that the probability of drawing a white ball from box A is 2/5, and the probability of drawing a black ball is 3/5. Meanwhile, in box B, the probability of drawing a white ball is 4/11, and that of drawing a black ball is 7/11.

EAR level assignment

In solving this type of problem, there would have been various possibilities. If the student solves it this way, which would be most likely, it would be entirely correct. In this reasoning, we see that the student exclusively and completely uses specific numbers, from which through basic operations we quickly and easily arrive at the solution. This kind of reasoning can be executed with any language, be it arithmetic, iconic or even involving symbols. The entire procedure is based on arithmetic operations with specific numbers.

In this problem, the EAR level 0 has been used.

Figure 7. Solution of the Pre-problem shown in Figure 5 and correct assignment of EAR level 0 (PT11).

Six PTs correctly identified a level 1 proto-algebraic activity in their resolutions. Although four of them justified their assignment, only two did so properly by indicating that equality with a relational meaning is involved (specifically, they refer to the equivalence relation of fractions). Lastly, four PTs assigned an incorrect level to level 1 resolutions. In one case, he assigned (without justification) an algebraization level 3; the other three considered the activity as purely arithmetic. These PTs indicated that only specific numbers intervene, ignoring that the equivalence and order relations between the involved fractions suppose a proto-algebraic character of the mathematical activity. For example, PT5 believes that in his solution (Figure 4), "the involved algebraic reasoning level is 0 because specific numbers are used, and no unknowns are reflected".

4.2. Post-Problem

To pose the Post-problems, PTs had to modify the Pre-problem so that its solution involved a mathematical activity with a higher level of EAR.

PTs proposed a total of 31 Post-problems, 5 of which were considered non-significant because the provided data did not allow solving the proposed question. The results we present are derived from the analysis of the 26 significant Post-problems.

Table 3 lists the different categories of Post-problems found and their frequency according to their relevance. As can be seen, 16 of the proposed Post-problems were relevant (that is, significant, a variation of the Pre-problem, and in such a way that their resolution implied a level of EAR higher than original Pre-problem). Prototypical examples of these categories are included below.

Table 3. Types of Post-problems and frequencies according to the degree of relevance (n = 26).

Category	Fr.			
	PosR	PosPR		PosNR
		NoPre	NoEAR	
DCP. Determination of the composition of a box from a known probability.	7	0	1	0
DCR. Determination of the composition of boxes knowing the ratio of white to black (or white to total) balls.	3	0	3	0
CAP. Explicit calculation of the simple or compound probability.	5	0	2	0
COP. Comparison of probabilities of drawing a ball from the two boxes.	1	2	1	1

In the DCP category of Post-problems, a situation was presented in which either the distribution of balls in a new box must be determined or the composition of one of the original boxes was modified and the new distribution of balls was asked, knowing the probability of drawing balls of a given color. For example, in Figure 8, the Post-problem posed by PT5 (his Pre-problem appears in Figure 4), its solution, and assignment of the EAR level are shown.

In this case, PT5 proposed a solution involving the parameter "n" that takes values in the set of natural numbers to create the classes of fractions equivalent to each of the fractions that determine the probabilities of success in boxes A and B. This mathematical activity corresponds to a consolidated algebraic level [37], although PT5 only attributed it an incipient proto-algebraic character, given that he did not distinguish the role of the literal symbol as a parameter.

In the DCR category of Post-problems, the probabilistic environment was eliminated, and the new distribution of balls must be determined knowing the ratio of white balls to black balls or to total balls (Figure 9).

Post-problem

In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. Is it possible to create a box C that has the same probability of drawing a white ball as box A? And as box B? Box C cannot have the same number of balls as the previous ones.

Solution

To solve this problem, it is necessary to understand the concept of equivalent fractions. Therefore, students can establish the relation 2/3 = 2n/3n where n can be any number, but it should be multiplied both in the numerator and the denominator. This can be directly applied to box B with 4/7 = 4n/7n. Hence, yes, box C can have the same probability of drawing a white ball with the total number of balls also multiplied by n.

EAR level assignment

The level of algebraic reasoning used is 1 since it involves the relational meaning of equality as equivalence and literal symbols as unknowns, but without operating with them.

Figure 8. Relevant Post-problem in DCP category (PT5).

Post-problem

Now we know that in box A every 2 white balls there are 3 black balls, and in box B every 4 white balls there are 7 black ones. We know that the total number of balls in box A is 20 balls, and in box B is 22 balls. Given this, how many balls of each color are there in each box?

Figure 9. Relevant Post-problem in DCR category. Variation of the Pre-problem shown in Figure 5 (PT11).

As observed in Figure 9, PT11 modified his Pre-problem (Figure 5), abandoning the probabilistic environment. In the Post-problem, the total number of balls of each type in each box is unknown, but the ratio of white balls to black balls and the total number of balls in each of them is known. In their resolution of the Pre-problem, PT11 did not involve any algebraic object (only operations with natural numbers), so he appropriately assigned a level 0 of EAR (Figure 7). To solve their Post-problem, PT11 proposed various equations established by the proportionality relation between the number of white or black balls and the total number of balls in each box, isolating the unknown for their resolutions. PT11 solved equations of the type $Ax + B = C$, which corresponds to an EAR level 2, correctly identified by the PT.

Post-problems in which it was required to explicitly calculate the probability of a given event (CAP Category) were relevant in most cases. Although this category was identified in the analysis of Pre-problems, we found a higher number of Post-problems that introduced compound experiments consisting of either the successive extraction of balls without replacement or the extraction of balls from both boxes as a way to increase their EAR level. This was the case of the Post-problem proposed by PT17 (Figure 10) as a variant of his Pre-problem in which it was required to determine the probability of drawing a white ball in A (CLP Category).

Post-problem

In box A, there are 2 white balls and 3 black balls. In box B, there are 4 white balls and 7 black balls. A die is rolled, and if an even number comes up, two balls are drawn from box A, one after the other, without replacement. On the other hand, if an odd number comes up, two balls are drawn from box B, also one after the other, without replacement. What is the probability of drawing exactly two white balls?

Solution

(a) Probability of drawing two white balls from box A is: P(WW/BoxA) = 2/5 × 1/4 .

(b) Probability of drawing two white balls from box B is: P(WW/BoxB) = 4/11 × 3/10 .

(c) When rolling a die, the probability of even and odd is the same: P(even) = P(odd) = 1/2 .

Therefore, using the total probability expression, the probability of drawing exactly two white balls is:

P(WW) = P(even) × P(WW/BoxA) + P(odd) × P(WW/BoxB) = 1/2 × 2/5 × 1/4 + 1/2 × 4/11 × 3/10 = 920/8800 = 23/220.

Figure 10. Relevant Post-problem (CAP category) and resolution (PT17).

The type of Post-problems proposed less frequently aimed to compare the probability of drawing a ball of a given color in both boxes. The COP category was the only one in which Post-problems not entailing a variation of the Pre-problem were identified (three out of the five proposed).

We observed that the nine PTs who correctly identified some objects related to probabilistic reasoning in their analysis of the Pre-problems posed relevant Post-problems involving probability (two COP problems, six DCP problems, eight CAP problems). On the other hand, four PTs who did not identify objects associated with probability created DCR-type problems in a non-probabilistic context.

Most of the Post-problems were solved correctly. Table 4 shows the EAR levels involved in the resolutions of Post-problems and their frequency, according to whether or not they represented an increase from the actual EAR level of the Pre-problem resolution, and whether or not they were correctly identified by PTs. As observed, most of the proposed Post-problems involved an algebraic activity of a higher level than the original Pre-problem. However, similarly to what happened with the Pre-problems, the level assignment was only justified in seven of the Post-problems.

Table 4. EAR levels of Post-problems resolutions.

EAR Level	Fr.			
	Increase		Decrease	
	Correct Id.	Incorrect Id.	Correct Id.	Incorrect Id.
Level 0.	0	0	0	4
Level 1.	4	4	1	3
Level 2.	4	2	0	0
Level 3.	0	2	0	0
Higher levels.	0	2	0	0
Total.	18		8	

The PTs most often proposed level 1 solutions to Post-problems. These solutions appeared in Post-problems where it was requested to determine either the explicit calculation of probability (CAP) or the modification of the composition of a box knowing the probability of success (DCP). To solve those Post-problems, PTs performed operations with rational numbers in fractional or percentage representation, or solutions were proposed in which equality had a relational meaning. In all cases where this EAR level was not correctly identified, the PTs assigned a higher algebraization level (in three cases they assigned level 3, in the other four cases they assigned level 2).

All Post-problems whose solutions involved algebraic activity above level 1 increased the level of algebraic reasoning put into play in the solutions of the respective Pre-problems. Six Post-problems implied a resolution of EAR level 2. The four PTs that correctly identified this level 2 had posed problems that required determining the composition of a box from the ratio of white balls to black balls or to totals, justifying their assignment based on the setting and resolution of equations of the type $Ax + B = C$ (equations arising from a proportion or equality between ratios). The PTs that posed Post-problems implying consolidated algebraic activity (level 3 or higher) either set and solved equations of the type $Ax + B = Cx + D$ or systems of linear equations to determine the composition of a box given the ratio between balls of different colors, or proposed solutions involving the use of parameters (Figure 8).

As was the case with the onto-semiotic analysis of the solutions to the Pre-problems, languages and concepts were once again the objects most frequently identified in the solutions to the Post-problems, while procedures, propositions, and arguments were rarely identified, typically in an incorrect manner. Concerning mathematical processes, the situation was also similar to the one observed in Pre-problems. In six Post-problems no process was identified, and in four of them the identification was incorrect. Again, the most common processes were "interpretation/meaning", "particularization", and "argumentation". However, unlike what happened in the analysis of the Pre-problems, PTs identified several objects and processes of an algebraic nature in their analysis of the Post-problem solutions. Among the objects are concepts like "equation", "systems of equations", and "unknowns", and procedures like "operating with unknowns" and "translating between representations". On the other hand, "particularization" related to both the application of Laplace's rule and the application of general properties related to proportionality, and "representation using symbolic language", were also mentioned.

Regarding the potential difficulties pointed out by the PTs in the Post-problems, these can be classified according to categories similar to those of the Pre-problems. However, a new category of difficulties associated with algebraic reasoning (D-A) arises. A total of 82 difficulties were identified, 13 of which were not significant. The categorization of the degree of relevance is the same as in the case of the potential difficulties related to the Pre-problem. Table 5 shows the frequency with which significant difficulties are identified in each category based on their relevance.

Table 5. Frequencies of difficulties in each category according to the degree of relevance (*n* = 69).

Type of Difficulties	Fr. (%)			
	RD	PRD	NRD	Total
D-A.	14 (20.29)	0 (0)	0 (0)	14 (20.29)
D-PBR.	13 (18.84)	0 (0)	2 (2.90)	15 (21.74)
D-PPR.	13 (18.84)	0 (0)	3 (4.35)	16 (23.19)
D-RN.	10 (14.49)	2 (2.90)	3 (4.35)	15 (21.74)
D-G.	0 (0)	2 (2.90)	7 (10.14)	9 (13.04)

In general, the difficulties pointed out by the PTs in categories not referring to algebraic objects and processes were similar to those already indicated in the Pre-problems. Even so, the number of relevant difficulties associated with rational numbers (D-RN) and proportional reasoning (D-PPR) increased compared to the analysis of the Pre-problems, predominating in Post-problems that required determining the composition of a box knowing the probability or the ratio of white to black/total balls, respectively. The number of difficulties associated with probabilistic reasoning (D-PBR) was similar, although these difficulties had only been identified in Post-problems where the explicit calculation of the success probability was required. The PTs also pointed out a smaller percentage of general difficulties (D-G).

The difficulties associated with algebraic reasoning that PTs highlighted (category with the highest relevance) were related to translating from natural language to symbolic language and vice versa ("translating symbolic language", PT1), with the formulation and resolution of equations ("difficulty in setting up the equation, [. . .] difficulty in solving the proposed equation", PT4), and with syntactic calculation ("when performing operations with unknowns, students can make mistakes when carrying them out", PT11). These difficulties were also the most often identified in a relevant way, appearing in half of the problems of creating or modifying boxes, whether or not they were framed in a probabilistic context.

5. Discussion

Mathematics teachers must be competent in selecting, creating, or modifying problems that promote meaningful learning for their students [11,12,14]. Specifically, they should consider the opportunities of a task to develop and articulate different types of reasoning, recognizing the relationship between the elements of the problem, that is, context, information, requirements, and the mathematical environment, with the mathematical activity it motivates [11,13,14]. In this way, they will be able to identify and manage the potential or actual difficulties of their students when creating or modifying such situations.

In this study, we have presented the analysis of PTs' responses to a problem-posing task at two different times: before solving (starting from a given situation) to develop proportional reasoning in a probabilistic context, and after solving (modifying a pre-existing problem) to promote higher levels of algebraic reasoning. Less than half of the PTs were able to pose relevant problems where, starting from the situation in a probability context, proportional reasoning was mobilized. These results are consistent with what has been observed in previous research [33,34] in arithmetic contexts. Our findings, therefore, show that this difficulty extends to the context of probability, with which future teachers are less familiar and have more limited mathematical and didactic knowledge [50,51].

Consistent with previous research on proportional reasoning in the arithmetic context [23,34], PTs had difficulties identifying mathematical objects in their practices, especially propositions and arguments. In these previous studies, participants were not asked to pre-identify mathematical processes, something considered in this study. Despite their limited response in this regard, analyzing the objects and processes allows us to understand more about students' thinking than what simply the written solution to the problem shows. In this analysis, they reflect, for example, on the properties they used to solve the problem and which they did not detail in their answer. It also allows us to better understand what they consider to be proportional reasoning and, therefore, to understand what may hinder them when posing problems that involve proportional reasoning.

The results related to identifying the algebraic reasoning level involved in their resolutions of both the Pre- and Post-problems are slightly worse than those obtained by Burgos and Godino on proportionality in the arithmetic context [23]. The difficulty in distinguishing arithmetic practices from proto-algebraic ones relates to the fact that during the analysis of objects and processes emerging in the solutions to the Pre-problems, the PTs barely pointed out algebraic elements. Despite this fact, the PTs succeed in modifying the task to address higher levels of EAR, as more than half of the proposed Post-problems imply in their resolution an algebraic activity of a higher level than the respective Pre-problem. Therefore, it seems that they have some knowledge about the hierarchical nature of algebraic activity, although they lack the professional discourse to adequately describe and justify the emerging algebraic reasoning features in mathematical activity.

As Osterman points out [52], identifying potential difficulties when solving a task is essential for selecting or creating tasks that adapt to students' abilities. While most of the difficulties indicated by the PTs in our study have been considered to be relevant, it is important to note that only half of the PTs who posed Pre-problems involving proportional reasoning were able to identify potential difficulties related to proportionality. This result aligns with research by Burgos and collaborators [32,49]. However, this percentage improved in the Post-problems. Since increasing the EAR level caused the PTs to use mathematical objects with a higher level of generality or abstraction, such as operating with fractions and percentages or setting up and solving proportional equations, the results obtained seem to indicate that the PTs become more aware of the difficulties these elements may originate.

Authors like Supply et al. [53] highlight the importance of context in solving problems involving proportional reasoning. Given that, as the authors show, the probabilistic context is where they find the most difficulty, it is not surprising that this also occurs when it comes to problem posing. In future research, we propose to expand the instrument to include the creation of proportionality tasks in other contexts and analyze whether there are differences between the difficulties that prospective teachers could find.

Although our study provides original information on the articulation of proportional, probabilistic, and algebraic reasoning through problem posing in teacher training, we understand that a clear limitation of our research is the size and nature of the sample. In subsequent studies, it will be necessary to increase the number of participants, also considering prospective secondary education teachers, to contrast the results derived from the study, as well as expand the study to new situations and probabilistic approaches.

6. Conclusions

In this work, we present what, to our knowledge, constitutes the first study on how PTs pose and analyze mathematical problems to promote proportional and algebraic reasoning in a probabilistic context. We observed that, in response to our first research question RQ1, PTs who posed problems involving proportional reasoning in a given probability context, two urns of known composition, included as a requirement the comparison of probabilities of drawing a ball of a given color. However, this occurs in less than half of the cases, highlighting the difficulties that PTs encountered when posing relevant problems involving proportional reasoning in a probabilistic setting. Regarding research question RQ2, PTs also showed significant difficulties in identifying the objects and processes related to proportional reasoning in their solutions, with the most frequent being concepts of proportionality, fraction, and percentage, as well as procedures related to changing the representation from fraction to percentage and vice versa. In relation to research question RQ3, the most frequently found difficulties were related to probabilistic reasoning (understanding the stochastic phenomenon, identifying favorable cases, or using Laplace's rule) and proportional reasoning (understanding and applying proportionality or using additive strategies) in the analysis of their Pre-problems. However, when analyzing Post-problems, the difficulties most frequently pointed out by the PTs were associated to algebraic reasoning (change of representation from verbal to symbolic language, setting up equations, or syntactic calculation). Finally, in response to research question RQ4, although the PTs did not adequately identify objects or processes of an algebraic nature in their Pre-problems, they mostly posed Post-problems that involved a higher level of algebraic reasoning. We conclude the importance of fostering the creation of problems in teacher training to improve the knowledge and competences of prospective teachers in relation to proportional and algebraic reasoning, highlighting the role of the probabilistic context.

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Institutional Review Board Statement: According to the current regulations on ethical principles to be respected in research (https://www.ugr.es/sites/default/files/2017-12/comisio%CC%81n%20e%CC%81tica.pdf accessed on 1 November 2023), and after having previously consulted with the Vice-Rectorate for Research and Transfer of our University, University of Granada, Spain, we were informed that it was not necessary to request evaluation by the ethics committee since we do not deal with human research in terms of the health system, clinical data, or biological samples. Moreover, the responses of the participants are always anonymized, with no possibility of identifying the source subject.

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