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Article

Certain Bounds of Formulas in Free Temporal Algebras

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Abstract: In this paper, we give a basic structure theorem based on the study of extreme cases for the value of $\prec$ (the classical precedence relation between ultrafilters), i.e., $\prec = \emptyset$ and no isolated element in $\prec$. This gives rise, respectively, to the temporal varieties O and W, with the result that O generates a variety of temporal algebras. We also characterize the simple temporal algebras by means of arithmetical properties related to basical temporal operators; we conclude that the simplicity of the temporal algebra lies in being able to make 0 any element less than 1 by repeated application to it of the L operator. We then present an algebraic construction similar to a product but in which the temporal operations are not defined componentwise. This new "product" is shown to be useful in the study of algebra order and finding of bounds by means of something similar to a lifting process. Finally, we give an alternative proof of an already known result on atoms counting in free temporal algebras.

Keywords: temporal logic; free temporal algebras; polymodal algebras

MSC: 03B44; 03G25; 06E25

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1. Introduction

Temporal logic was born with the intention of allowing the representation of temporal information in the logical framework. It was also intended to address the modal logic approach introduced around 1960 by A. Prior under the name of Tense Logic. As is well known, this field caught the subsequent attention not only of logicians but also of computer specialists.

Technical applications have been derived from it, but let us not forget the use of Tense Logic as a formalism to elucidate philosophical questions related to time, especially those arising from temporal expressions in natural language, as a language to encode temporal information in applications to A.I. development and, finally, as a tool to manipulate the execution of programs in their temporal aspects. The most modern treatment we know of these aspects is chapter 10 of [1] written by Y. de Venema and entitled "Temporal Logic".

Among the precursors is H. Reichenbach (see [2]). He explains how the function of each tense is focused on specifying relations between three tenses related to the utterance: the time of conversation (S), the time of reference (R), and the time of the event (E). Prior disagreed with this simplification and thus opened up a great avenue of evolution in analysis; out of it emerged a great variety of temporal logics. A good collection of these logics can be found in [3].

Applications of temporal logic to artificial intelligence are poorly documented in published works. Almost all of them are found in internal company documents or in untested writings. The last academic news we had on this topic was contained in [4,5]. The problem was focused on the properties of the world that change with the results of other events or actions and those that do not change; at that time, it all remained very much tied to automatic demonstration.

The modal style of temporal logic has reigned supreme in applications to computing that deal with program specification and verification, especially in concurrent programming with several processors working in parallel. The central problem here is to guarantee synchronicity to ensure the integrity of the information shared between processors. The nondeterminism imprinted on computer applications has led to branching temporal models. The treatment of this can be traced back to [6].

The treatment of logics has sought the use of algebra as a great ally. Whenever a logic has turned out to be algebraizable, algebra has been not only the most elegant and expressive language for its treatment but also the ideal means for devising algorithms. That is why in this work on temporal logic we resort exactly to the study of free temporal algebra. Given the many philosophical nuances appreciated in the literature, in studying temporal algebra the course of research will drift towards varieties of temporal algebras. To see the overwhelming dimension of the problem, we can consult [7,8].

As is well known, the study of algebraic structures is founded whenever possible on the study of order relations; the case of Boolean algebra is an archetypal example of this. Classical temporal algebras are built on the platform of a Boolean algebra, and their tradition has been fully consolidated since the last quarter of the twentieth century. In this case, to the formal operations of the Boolean algebra are added the temporal ones, g and h. This addition is made, as is logical, by making these operators interact harmoniously with the Boolean operations, which leads them to play a role in the underlying order. Intuition dictates that certain values, such as $g0$ and $h0$, will play a relevant role in this context; to found and analyze it from the algebraic point of view is the essential motivation of this article. The problem of atom identification is a classical one in the study of varieties of temporal algebras (see, for example, [7,8]). By reading both papers, we confirm our intuition that the problem rests on the study of $g0 \wedge h0$. However, neither of these papers nor the rest of those consulted isolate in one study the role played by $g0 \wedge h0$ within the order of the temporal algebra; we have begun this task, and it has inevitably led us to the study of simple temporal algebras, without which we believe the problem cannot be understood in its full depth. The study we present here does not pretend to be exhaustive but an indispensable first step.

The problem of capturing atoms can be approached by understanding it as a boundedness search problem. With the purpose of contributing to possible generalizations up to temporal and polymodal varieties in general, the authors of the present paper have searched for a method of bounding formulas in free temporal algebras, using exclusively their operations, valuations, filtrations (see [9]), quotients, and products. The study should proceed by exhibiting non-zero lower bounds of a sufficient number of formulas that may have them, so that the atoms would be among the rest and could be finally selected. The essential idea of the procedure would be, as so often is the case in algebra and calculus, to descend from the temporary free algebra to a certain finite algebra carefully chosen to fit the given formula, then to gather there the necessary information, and finally to ascend again to the free algebra with the information necessary to prove what is desired. Finding that finite algebra to descend to is the task we consider central to the paper. To achieve this, a construction has been introduced that we believe to be novel: a "product" of temporal algebras in which the temporal operations are not defined componentwise but by taking into account the conditions on the adjacent components. The definitions of g, h given in Definition 6, and the consequence thereof on their dual temporal operator f and p (see Lemma 11), tentatively suggest for such a new product the name "Temporal Algebra Skew-Product" (or, for short in this context, simply "skew-product").

In our approach, the use of these skew-products has been shown to be versatile with very good algebraic qualities: it is finite by definition, it preserves the simplicity of skew-factors, and its terminal "factor" can be chosen on purpose to pick up or produce effects. It could even become a skew–product again; examples of this are shown and analyzed here. In short, we believe that this study exemplifies the application of a working technique that could be generalized to suit the search for formulae quantifications in polymodal algebras –when necessary– and, in particular, to the study of atomicity in them. On the other hand, it would suggest to researchers an algebraic object whose study may in itself be of sufficient interest.

Although this paper has been inspired by the initiative put forward in others such as [10] and even more [11], we have not overlooked classical works on finitely generated free temporal and modal algebras such as [7,12], where excellent studies of atomicity touching on varieties of temporal algebras and modal algebras are exposed. In order to constrast our views, we have devoted the last section to giving an alternative proof of Theorem 1.1 in [7].

Section 2 contains a summarized compilation of the preliminary basic concepts and language used in this paper. Section 3 is devoted to the study of extremal situations on the relation $\prec$ over ultrafilters, i.e., we study the classes of temporal algebras in which $\prec$ is empty, O, and those in which $\prec$ has only a connected component, W. Both classes turn out to be varieties, and a structure theorem in the classical sense of universal algebra arises. Once again in this study it is glimpsed, and here it all began for us, that $g0 \wedge h0$ plays an interesting role in all this. Theorem 7 is the final and central product of the section. Indeed, this theorem exposes with crystal clarity the relevance of the study of extreme cases for $\prec$ in that it will establish that any temporal algebra is isomorphic to a product of two algebras, one in O and the other in W, which will a priori separate the atoms of the algebra into two sets. This simple observation will turn out to be transcendental in the treatment of the final example of the article when we understand that one factor contributes all the atoms and the other, none. Section 4 is devoted to a classical topic in universal algebra; it provides a practical characterization of simple temporal algebras in their maximum generality. In it, we resort to the operators L and M for their outstanding expressive capacity. From now on, these operators will be essential for our exposition. The most outstanding result of this section is Theorem 9, which, to our knowledge, has not been reported so far. It is fully satisfactory from the algebraic point of view. It will therefore be an important tool in the rest of the paper. In Section 5, we present a skew-product construction and describe its general and technical properties. Of course the key piece is Definition 6; the subsequent results prove that it could be very appropriate for the study of order-related phenomena. In it, we highlight Lemma 15, which relates the process similar to classical "lifting" that we have suggested above. Also of interest are Corollary 8 and Lemma 16, where we establish the behavior of the construction with respect to the simplicity of the skew–factors. In Section 6, we find lower bounds of non-atomic formulas in the free temporal algebra. In this section, we highlight Theorem 14 and Theorem 16 because of the lower bound it provides. But the central result is Corollary 10; it establishes the upper bound of any atom of the free temporal algebra. Finally, as an application, Section 7 is devoted to providing an alternative and new proof of Theorem 1.1 [7] (p. 61) about the number of atoms of $\mathbf{F}_t(X)$, whenever X is of cardinality n. Specifically, the technique of proof is to give a particular bijection between the set of atoms under investigation and another well known finite set that we will specify. Of course, the result highlighted in the section is Theorem 17, although Corollary 11 is also of interest.

As for the related bibliography, we include papers with prospective value and others with algebraic subject matter. Those with prospective content are divided into two groups: those that are like [1–3,13] have been useful for us to be able to capture the fine nuances of astute thinkers and try to imprint them on our algebraic approaches, but their influence on the work lies in the background of our research training. The second group of prospective papers are those that open up fields of technological applications beyond the scope of the present work; they are of the [4,5,14] type. They indicate that the theoretical mathematics we elaborate could underlie certain applications. Closer to our work are [7,8,12]. The first deals with certain general varieties of temporal algebras inspired by the classical literature represented by [15,16]. While F. Bellissima uses the language of temporal structures, we choose the one provided by Universal Algebra, which opens us to totally different methods, language and subject matter. In the case of free temporal algebra, which is the one we are concerned with, in [7] we do not find structure theorems or allusions to congruences or simplicity. Although it speaks of atoms, the problem of bounds is not raised. In common with [7], our work only has the statement of Theorem 1.1, to which we give a very different proof. Ref. [8] does have a language and subject matter closer to ours, but while we focus on temporal algebra at full generality, T. Kowalski descends to the varieties $\mathcal{D}_n$, which are interesting for their nuance to the deduction theorem and their ingenious conceptual relation with Hilbert Algebra. Moreover, we will emphasize that T. Kowalski has understood the interest of the $g0 \wedge h0$ element, although his treatment is very different from ours. In Section 8, we detail how [8] suggests a field of possible applications of our work. Finally, [11,12,14,17–19] have served us only to intuit a second field of applications or evolution of the techniques developed in this work. Finally, [9,20] are instrumental works where instruments are developed that we now use here, as indicated in the appropriate places; that was their purpose and that is what they were written for.

2. Preliminaries

The paper deals with temporal algebras. A *temporal algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1\rangle$ of type $\langle 2,2,1,1,1,0\rangle$ such that:

(T.1) $\langle A, \wedge, \vee, \neg, 1\rangle$ is a Boolean algebra.
(T.2) Both g and h are $\wedge$-morphisms (i.e. $k(a \wedge b) = ka \wedge kb$, for $k \in \{g, h\}$).
(T.3) The equivalence $[ga \vee b = 1$ if and only if $a \vee hb = 1]$ is satisfied.
(T.4) $g1 = h1 = 1$.

As usual, we also consider in A the operators $p = \neg h \neg$ and $f = \neg g \neg$, as well as the unary operators L and M defined, respectively, by $Lx = hx \wedge x \wedge gx$ and $Mx = px \vee x \vee fx$.

The class T of temporal algebras is a variety. In the sequel, we will denote by $\mathbf{F}(X)$ (resp. $\mathbf{F}_t(X)$) the algebra of terms of type $\langle 2,2,1,1,1,0\rangle$ over (resp. the free temporal algebra freely generated by) the set X. The universe of $\mathbf{F}(X)$ (resp. $\mathbf{F}_t(X)$) is denoted by $F(X)$ (resp. $F_t(X)$). $\mathbf{F}_t(X)$ is a quotient of $\mathbf{F}(X)$ by certain well known congruence $\theta_t(X)$ or simply θ (see [21]). Hence, the elements of $F_t(X)$ are the quotient classes α/θ, with $\alpha \in F(X)$. If we represent by π_θ, or simply π, the epimorphic projection of $\mathbf{F}(X)$ onto $\mathbf{F}_t(X)$ and i is the inclusion map of X in $F(X)$, it is well known that for all temporal algebras $\mathbf{A}$ and for all mappings $v : X \longrightarrow A$ there are unique morphisms $\bar{v} : \mathbf{F}(X) \longrightarrow \mathbf{A}$ and $\tilde{v} : \mathbf{F}_t(X) \longrightarrow \mathbf{A}$ such that $v = \bar{v} \circ i$ and $\bar{v} = \tilde{v} \circ \pi$. If $\alpha \in F(X)$, which we will write indistinctly $\pi(\alpha)$ or α/θ. For all temporal algebra $\mathbf{A}$, $Atm(\mathbf{A})$ is the set of atoms of de underliying Boolean algebra.

In this article, we shall use ω (resp. ω^*) to represent the set of natural numbers (resp. non-zero natural numbers). For any set X, here $\mathcal{P}(X)$ (resp. $\mathcal{P}_\omega(X)$) stands for the power set of X (resp. finite parts of X, i.e., finite elements of $\mathcal{P}(X)$); moreover, $\mathcal{P}(X)^*$ (resp. $\mathcal{P}_\omega(X)^*$), by definition, stands for $\mathcal{P}(X) \setminus \{\emptyset\}$ (resp. $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$).

It is possible to construct temporal algebras in a standard form as follows. A structure is a pair $\langle T, \sigma\rangle$, where T is a non-empty set and $\sigma \subseteq T^2$. Given a structure $\langle T, \sigma\rangle$, the temporal algebra $\langle T, \sigma\rangle^+$ is, by definition, the algebra $\langle \mathcal{P}(T), \wedge, \vee, \neg, g_\sigma, h_\sigma, T\rangle$, where $\wedge$,$\vee$, and $\neg$ are the Boolean operations $\cap$, $\cup$, and complementation over $\mathcal{P}(T)$, respectively. As for g_σ and h_σ, these operators are defined by $g_\sigma(X) = \{a \in A\colon$ for all $b \in T$, if $a\sigma b$ then $b \in X\}$ and $h_\sigma(X) = \{a \in A\colon$ for all $b \in T$, if $b\sigma a$ then $b \in X\}$.

Let $\mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1\rangle$ be a temporal algebra. The concept of *filter*, *ideal*, and *ultrafilter* is the proper of the underlying Boolean algebra, i.e., $F \in \mathcal{P}(A)$ is a filter iff, by definition: $1 \in F$, $a \wedge b \in F$ whenever $a, b \in F$, and $b \in F$ whenever $a \in F$ and $a \leq b$ (see [21] (p. 127)); the filter F is an ultrafilter iff, by definition, F is maximal with respect to the property that $0 \notin F$ (see [21] (p. 132)). The symbol $Sp(\mathbf{A})$ (resp. $Ult(\mathbf{A})$) will denote the set of filters (resp. ultrafilters) of the temporal algebra $\mathbf{A}$, i.e., the Boolean algebra $\langle A, \wedge, \vee, \neg, 1\rangle$. For all map $k\colon X \longrightarrow Y$ and $B \subseteq Y$ (resp. $A \subseteq X$), the set $k^*(B)$ (resp. $k_*(A)$) is, by definition, the set $\{x \in X\colon g(x) \in B\}$ (resp. $\{g(a) \in Y\colon a \in A\}$); $k^*(X)$ (resp. $k_*(X)$), which represents the reciprocal (resp. direct) image of X by the map k. The relation of precedence induced by **A** in $Ult(\mathbf{A})$, in symbols $\prec$, is defined as follows:

$$D \prec D' \text{ if and only if } g^*(D) \subseteq D'$$

or using any equivalent condition as $p_*(D) \subseteq D'$, $f_*(D') \subseteq D$, or $h^*(D') \subseteq D$. It is well known that for any temporal algebra there is a homomorphic inclusion of **A** into $\langle Ult(\mathbf{A}), \prec \rangle^+$.

Here, we will use the following result proved in [9]: let $\alpha \in F(X)$ such that $\pi(\alpha) \neq 1$; then, there exists a temporal valuation w_α over a finite temporal algebra $\mathbf{A}_\alpha$ such that $\tilde{w}_\alpha \neq 1$.

Let $\mathbf{A} = \mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1 \rangle$ be a temporal algebra. A filter F (resp. an ideal I) of $\langle A, \wedge, \vee, \neg, 1 \rangle$ is a *temporal filter* (resp. *ideal*) iff, by definition, $Lx \in F$ (resp. $Mx \in I$) whenever $x \in F$ (resp. $x \in I$). The symbol $Tsp(\mathbf{A})$ will represent the set of temporal filters of **A**. A temporal filter of **A** is a maximal temporal filter iff, by definition, it is distinct from A and it is maximal in the set of temporal filters distinct from A. The set of maximal temporal filters of **A** will be denoted by $Mtsp(\mathbf{A})$; the set $\bigcap SptM(A)$ will be denoted by $Rad_{mt}(A)$.

If $X \subseteq A$, $D_t(X)$ will denote the smallest temporal filter of **A** including X, that is

$$D_t(X) = \bigcap \{D \in Tsp(\mathbf{A}) \colon X \subseteq D\}$$

If $a \in A$, we henceforth will represent the set $\{x \in A \colon a \leq x\}$ (resp. $\{x \in A \colon x \leq a\}$) by $[a, 1]$ (resp. $[0, a]$). For all $0 \leq i \leq k$, here $\pi_i(\langle y_0, \ldots, y_i, \ldots, y_k \rangle) = y_i$ is the projection map on the ith coordinate of the usual product of Universal Algebra.

According to the notion of Universal Algebra, a temporal algebra **A** is simple iff, by def., it has no nontrivial congruence relations or, equivalently, if every homomorphism with domain A is either injective or constant.

The following results are well known.

Lemma 1. *Let* $\mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1 \rangle$ *be a temporal algebra. Then, f,p,g, and h are increasing functions.*

Lemma 2. *For all temporal algebra* **A** *and for all* $x \in A$:

1. $pgx \leq x$.
2. $fhx \leq x$.
3. $x \leq gpx$.
4. $x \leq hfx$.

where $\leq$ *is the Boolean partial order.*

Lemma 3. *Let* **A** *be a finite temporal algebra. For all* $a, b \in Atm(\mathbf{A})$, *the following assertions are equivalent:*

1. $b \leq pa$,
2. $a \leq fb$,
3. *if* $a \leq gx$, *then* $b \leq x$,
4. *if* $b \leq hx$, *then* $a \leq x$.

3. A Basic Structure Theorem for Temporal Algebras

In this section, we will prove that any temporal algebra is isomorphic to a product of two others, one of which meets the condition $g0 = 0$; the problem is knowing how to obtain the two factors. To achieve this, we will consider elements which are, from a temporal point of view, similar to the element 0. First, we will define the following class of temporal algebras denoted by O.

Definition 1. *Let* O *be the class of temporal algebras* $\mathbf{A}$ *satisfying* $g0 = 1$*.*

Since the above definition of temporal algebras is equational, O is a variety. We will illustrate the above definition by giving two useful examples of temporal algebras. Actually, the second is a particular case of the first, and it is given using the standard method to construct a temporal algebra from a structure.

Example 1. *Let* $\langle B, \wedge, \vee, \neg, 1\rangle$ *be a Boolean algebra and* $k : B \longrightarrow B$ *a map defined by* $k(x) = 1$*, for all* $x \in B$*. The algebra* $\langle B, \wedge, \vee, \neg, k, k, 1\rangle$ *is an element of* O*. In the particular case that* $B = \{0, 1\}$*, we will write* $\mathbf{E_0}$ *to represent the simple temporal algebra* $\langle B, \wedge, \vee, \neg, k, k, 1\rangle$*.*

Example 2. *Let us consider the structure* $\langle T, \sigma\rangle$*, where* $T \neq \emptyset$ *and* $\sigma = \emptyset$*, and the temporal algebra* $\langle T, \sigma\rangle^+$*. It is clear, from the definition of* $g_\sigma(X)$*, that* $g_\sigma(\emptyset) = T$*. Hence,* $\langle T, \sigma\rangle^+$ *is an element of* O*.*

From property *T.3*, it is straightforward to show that the equality $g0 = 1$ holds if and only if any of the equalities $f1 = 0$, $h0 = 1$, and $p1 = 0$ hold. This brings three new equivalent definitions of O. Furthermore, if $\mathbf{A}$ is a temporal algebra and $g0 = 1$, then g is constant since it is increasing (see Lemma 1) and 0 is the minimum of the lattice $\langle A, \wedge, \vee, \neg, 1\rangle$; obviously, the converse holds too. It is easy to show from the axioms of temporal algebras that g is constant if and only if any function in the set $\{h, f, p\}$ is constant. So, we have four new definitions of O.

In any elements of O, it occurs that the relation $\prec$ is equal to $\emptyset$. Furthermore, this gives a characterization of elements in O involving $\prec$.

Theorem 1. *Let* $\mathbf{A}$ *be a temporal algebra.* $\mathbf{A} \in \mathsf{O}$ *if and only if* $\prec = \emptyset$*.*

Proof. Let us assume that $\mathbf{A} \in \mathsf{O}$, and let $D, D' \in Ult(\mathbf{A})$. If $D \prec D'$, then $p_*(D) \subseteq D'$. Since $p1 = 0$, it follows that $0 \in D'$, which is impossible since $D \neq A$. Conversely, let us assume that $\prec = \emptyset$. From Example 2, $\langle Ult(\mathbf{A}), \emptyset\rangle^+$ is in O. Moreover, since $\mathbf{A}$ is a subalgebra of $\langle Ult(\mathbf{A}), \emptyset\rangle^+$ then $\mathbf{A} \in \mathsf{O}$. □

For every temporal algebra in O, both g and h are constantly 1; then, L is the identity map in A. It follows that each filter of each temporal algebra $\mathbf{A}$ in O is closed for the map L, and so $Tsp(\mathbf{A})$ coincides with $Sp(\mathbf{A})$. This is not a characterization of algebras in O; nevertheless, O is a distinguished subclass of the class (actually a subvariety of the variety) of temporal algebras $\mathbf{A}$ verifying the property $Sp(\mathbf{A}) = Tsp(\mathbf{A})$.

From the equality $Tsp(\mathbf{A}) = Sp(\mathbf{A})$, it follows at once that $Mtsp(\mathbf{A}) = Ult(\mathbf{A})$ and so $Rad_{mt}(\mathbf{A}) = \{1\}$. Moreover, the equality $Mtsp(\mathbf{A}) = Ult(\mathbf{A})$ implies that, for all $D \in Mtsp(\mathbf{A})$, $\mathbf{A}/D = \mathbf{E_0}$. Therefore, any temporal algebra in O is a subdirected power of the algebra $\mathbf{E_0}$. So, the variety O is semisimple and generated by $\mathbf{E_0}$.

Corollary 1. *The variety* O *is minimal in the lattice of subvarieties of* T*.*

Proof. Let us assume that X is another subvariety of T such that $\mathsf{X} \subseteq \mathsf{D}$. Let $\mathbf{A} \in \mathsf{X}$ and $D \in Mtsp(\mathbf{A})$. As $\mathbf{A}/D$ is equal to $\mathbf{E_0}$ and X is a variety, it follows that $\mathbf{E_0} \in \mathsf{X}$. Therefore, X contains the variety generated by $\mathbf{E_0}$, that is, it includes D. It follows immediately that $\mathsf{X} = \mathsf{D}$. □

Definition 2. *Let* $\mathbf{A}$ *be a temporal algebra. The set* $O(\mathbf{A})$ *is defined by the following equality:*

$$O(\mathbf{A}) = \{x \in A \colon px = fx = 0\}$$

Given a temporal algebra $\mathbf{A}$ and an element $a \in A$, $Ult(\mathbf{A})$ is the disjoint union of two sets, namely, that of ultrafilters containing a, U_a, and that of ultrafilters containing $\neg a$, $U_{\neg a}$.

When $a \in O(\mathbf{A})$, the following theorem holds (of course $\prec\mid U_a$ represents the restriction of $\prec$ to the set U_a).

Theorem 2. *Let* $\mathbf{A}$ *be a temporal algebra and* $a \in O(\mathbf{A})$*. Then,* $\prec\mid U_a = \emptyset$*.*

Proof. Let a be an element of $O(\mathbf{A})$, D an ultrafilter such that $a \in D$, and $D' \in Ult(\mathbf{A})$. If $D \prec D'$, then $p_*(D) \subseteq D'$, so $0 = pa \in D'$ and this is impossible if D' is maximal. If $D' \prec D$, we again obtain the above contradiction, now using f instead of p. So, the theorem follows. $\square$

By means of elements of $O(\mathbf{A})$, it is possible to distinguish subsets of isolated ultrafilters with regard to $\prec$. In the following for all $a, b \in A$, $[a, b] = \{x \in A \colon a \leq x \leq b\}$.

Lemma 4. *Let* $\mathbf{A}$ *be a temporal algebra and* $a \in O(\mathbf{A})$*. The set* $[0, a]$ *is a temporal ideal.*

Proof. Obviously, $[0, a]$ is an ideal. Furthermore, $[0, a]$ is temporal since if $x \leq a$ then $px \leq pa = 0$, i.e., $px = 0 \in [0, a]$. Analogously, $fx = 0 \in [0, a]$ and so $Mx \in [0, a]$. $\square$

Theorem 3. *For any temporal algebra* $\mathbf{A}$*,* $O(\mathbf{A}) = [0, g0 \wedge h0]$*.*

Proof. The proof is given by double-inclusion. Let us take $x \in A$ such that $x \leq g0 \wedge h0$. Hence, $x \leq g0$ and $x \leq h0$. Since p and f are both two increasing functions (see Lemma 1), we have $px \leq pg0 \leq 0$ and $fx \leq fh0 \leq 0$. So, $fx = px = 0$. Conversely, let us suppose that $x \in O(\mathbf{A})$, i.e., $fx = px = 0$. By Lemma 2, we have $x \leq hfx \wedge gpx$. Therefore, $x \in [0, g0 \wedge h0]$. $\square$

Corollary 2. *Let* $\mathbf{A}$ *be a temporal algebra. Then,* $O(\mathbf{A})$ *is a temporal ideal.*

Proof. It is an inmediate consequence of Lemma 4 and the fact that $g0 \wedge h0 \in O(\mathbf{A})$. $\square$

Definition 3. *For all temporal algebra* $\mathbf{A}$*, we define* F_0 *and* F_1 *by the following equalities:*

$$F_0 = [g0 \wedge h0, 1] \qquad \text{and} \qquad F_1 = [f1 \vee p1, 1]$$

Theorem 3 gives a very concrete description of the set $O(\mathbf{A})$. It is indeed the principal ideal generated by $g0 \wedge h0$. Lemma 5 follows immediately from this remark and Corollary 2 (note that $F_1 = \neg O(\mathbf{A})$).

Lemma 5. *Let* $\mathbf{A}$ *be a temporal algebra. Then,* F_1 *is a temporal filter of* $\mathbf{A}$*.*

Given a temporal algebra $\mathbf{A}$, we can consider the temporal ideal $O(\mathbf{A})$. Our inmediate aim is to study the algebras $\mathbf{A}$ with extreme value of $O(\mathbf{A})$. Obviously, a temporal algebra $\mathbf{A}$ for which $O(\mathbf{A}) = A$ belongs to O.

Definition 4. *Let* W *be the class of temporal algebras* $\mathbf{A}$ *satisfying* $O(\mathbf{A}) = \{0\}$*.*

Theorem 4. W *is a variety.*

Proof. For all temporal algebra $\mathbf{A}$, $\mathbf{A} \in \mathsf{W}$ if and only if $g0 \wedge h0 = 0$; then, the members of W admit an equational definition, so W is a variety. $\square$

Theorem 5. *Let* $\mathbf{A}$ *be a temporal algebra. The following statements are equivalent:*

1. $\mathbf{A} \in \mathsf{W}$.
2. *For all* $D \in Ult(\mathbf{A})$ *there is* $D' \in Ult(\mathbf{A})$ *such that* $D \prec D'$ *or* $D' \prec D$*.*

Proof. First, let us assume that $\mathbf{A} \in \mathsf{W}$, i.e., $g0 \wedge h0 = 0$. As $\mathbf{A}$ is isomorphic to a subalgebra of $\langle Ult(\mathbf{A}), \prec\rangle^+$, then $g_\prec\emptyset \cap h_\prec\emptyset = \emptyset$. If $D \in Ult(\mathbf{A})$ and there is no $D' \in Ult(\mathbf{A})$ such that $D \prec D'$ or $D' \prec D$ then $D \in \emptyset$, and this is impossible. So the second statement follows. The converse can be showed analogously. $\square$

Now, given a temporal algebra $\mathbf{A}$ we can consider two subsets of temporal filters: one made up by temporal filters D such that $\mathbf{A}/D \in \mathsf{O}$ and one made up by temporal filters such that $\mathbf{A}/D \in \mathsf{W}$. As a consequence, we dispose of two radicals, understanding this concept as in ring theory.

Definition 5. *For all temporal algebra $\mathbf{A}$, we define the spectrum $Osp(\mathbf{A})$ and the radical $Rad_o(\mathbf{A})$ by the following equalities:*

$$Osp(\mathbf{A}) = \{D \in Tsp(\mathbf{A}) : \mathbf{A}/D \in \mathsf{O}\} \text{ and } Rad_o(\mathbf{A}) = \bigcap Osp(\mathbf{A})$$

Similarly, we define the spectrum $Wsp(\mathbf{A})$ and the radical $Rad_w(\mathbf{A})$ by the equalities:

$$Wsp(\mathbf{A}) = \{D \in Tsp(\mathbf{A}) : \mathbf{A}/D \in \mathsf{W}\} \text{ and } Rad_w(\mathbf{A}) = \bigcap Wsp(\mathbf{A})$$

Lemma 6. *Let $\mathbf{A}$ be a temporal algebra. Then,*

$$g_*(A) \cup h_*(A) \subseteq F_0$$

Proof. Let $x \in A$. Since $0 \leq x$ and g are increasing (see Lemma 1), then we have that $g0 \leq gx$, so $h0 \wedge g0 \leq g0 \leq gx$. Analogously, we have $h0 \wedge g0 \leq h0 \leq hx$. $\square$

Corollary 3. *In any temporal algebra $\mathbf{A}$, the following statements hold:*

1. $F_0 \in Osp(\mathbf{A})$,
2. $F_1 \in Wsp(\mathbf{A})$.

Proof. First, from Lemma 6 it follows that $F_0 \in Tsp(\mathbf{A})$. Next, we will prove that $\mathbf{A}/F_0 \in \mathsf{O}$. Since $fhx \leq x$, for all $x \in A$, it follows that $fh0 = 0$. Moreover, since f is monotone we have that $f(h0 \wedge g0) \leq fh0$; hence, $f(h0 \wedge g0) = 0$ and so $f(1/F_0) = 0/F_0$. This proves the first statement since $1/F_0 = (h0 \wedge g0)/F_0$.

Moreover, it is clear that $(g0 \wedge h0)/F_1 = 0/F_1$. So, $g(0/F_1) \wedge h(0/F_1) = 0/F_1$. This implies that $\mathbf{A}/F_1 \in \mathsf{W}$. $\square$

Lemma 7. *Let $\mathbf{A}$ be a temporal algebra and $D \in Tsp(\mathbf{A})$. Then,*

1. *If $D \in Osp(\mathbf{A})$ then $h0 \wedge g0 \in D$.*
2. *If $D \in Wsp(\mathbf{A})$ then $p1 \vee f1 \in D$.*

Proof. If $D \in Osp(\mathbf{A})$, then $g(0/D) = h(0/D) = 1/D$ and so $(g0 \wedge h0)/D = g(0/D) \wedge h(0/D) = 1/D$. This implies that $g0 \wedge h0 \in D$. Moreover, if $D \in Wsp(\mathbf{A})$ then $\mathbf{A}/D \in (O)$. It follows that $g(0/D) \wedge h(0/D) = 0/D$, i.e., $g0 \wedge h0 \in \neg D$, equivalently $f1 \vee p1 \in D$. $\square$

Theorem 6. *Let $\mathbf{A}$ be a temporal algebra. Then, $Rad_o(\mathbf{A}) = F_0$ and $Rad_w(\mathbf{A}) = F_1$.*

Proof. It follows from Corollary 3 that $Rad_o(\mathbf{A}) \subseteq F_0$ and $Rad_w(\mathbf{A}) \subseteq F_1$. The converse inclusions are straightforward from Lemma 7. $\square$

Remark 1. *We have, from Theorem 6 and Lemma 5, a clear relation between $O(\mathbf{A})$ and $Rad_w(\mathbf{A})$, namely, $\neg O(\mathbf{A}) = Rad_w(\mathbf{A})$. In the following, $\mathbf{A}/Rad_o(\mathbf{A})$ (resp. $\mathbf{A}/Rad_w(\mathbf{A})$) will be represented by $\mathbf{A_o}$ (resp. $\mathbf{A_w}$).*

Theorem 7 and (its) Corollary 4 are the central applications/results of this section. In the Theorem, we establish that every temporal algebra is the product of two, one of which satisfies $g0 = 1$ and the other satisfies $g0 \wedge h0 = 0$; it is in fact the announced structure theorem. This result will be used of the Section 7.

Theorem 7. *For all temporal algebra* $\mathbf{A}$, $\mathbf{A} \cong \mathbf{A_o} \times \mathbf{A_w}$.

Proof. Since any two congruences of a Boolean algebra permute, it is enough to show that the temporal filter generated by $Rad_o(\mathbf{A}) \cup Rad_w(\mathbf{A})$ is A and that $Rad_o(\mathbf{A}) \cap Rad_w(\mathbf{A}) = \{1\}$. Both facts occur according to Theorem 6. □

Corollary 4. *The variety* T *is generated by the class* $\mathsf{O} \cup \mathsf{W}$.

In order to complete our study of the defined varieties, we give a new result about T and the lattice of its subvarieties.

Corollary 5. W *is maximal in the lattice of subvarieties of* T.

Proof. Let X be a subvariety of T such that $\mathsf{W} \subset \mathsf{X}$, and let $\mathbf{A} \in \mathsf{X} \setminus \mathsf{W}$. Since $\mathbf{A} \notin \mathsf{W}$, then $F_0 \neq A$, and so $\mathbf{A_o}$ is on one hand non-trivial and on the other an out-of-joints algebra, as follows from Corollary 3. Let $D \in TspM(\mathbf{A_o})$. It is clear that $\mathbf{E_0} = \mathbf{A_o}/D$ is in the variety X. Since $\mathbf{E_0}$ generates O, then $\mathsf{O} \subseteq \mathsf{X}$ and so $\mathsf{O} \cup \mathsf{W} \subseteq \mathsf{X}$. We conclude by Theorem 7 that $\mathsf{X} = \mathsf{T}$. □

4. A Characterization of Simple Temporal Algebras

Our aim in this section is to prove that in any simple temporal algebra, an element different from 1 can be diminished progressively to 0 by means of consecutive applications of the operator L. Furthermore, this is not possible except in a simple algebra. Of course, in this statement L can be changed by M, whenever the roles of 1 and 0 are interchanged.

Given a temporal algebra $\mathbf{A}$, for all $a \in A$ we set by definition $L^0 a = a$ (resp. $M^0 a = a$) and $L^{n+1} a = L(L^n a)$ (resp. $M^{n+1} a = M(M^n a)$). The following lemma describes the elements of $D_t(X)$.

Lemma 8. *Let* $\mathbf{A}$ *be a temporal algebra,* $X \subseteq A$*, and* $H(X)$ *the set defined by the equality:*

$$H(X) = \{a \in A\colon \text{ there exists } Y \in \mathcal{P}_\omega(X)^* \text{ and } n \in \omega \text{ such that } L^n(\wedge Y) \leq a\}$$

Hence $D_t(X) = H(X)$.

Proof. It is easy to check that $1 \in H(X)$ and that $H(X)$ is a closed order filter for L; thus, $H(X)$ is a temporal filter. Moreover, if we take $a, b \in H(X)$ then $n_a, n_b \in \omega$, and $Y_a, Y_b \in \mathcal{P}_\omega(X)^*$ exist such that $L^{n_a}(\wedge Y_a) \leq a$ and $L^{n_b}(\wedge Y_b) \leq b$. The natural number $m = \max n_a, n_b$ verifies $L^m(\wedge Y_a) \leq L^{n_a}(\wedge Y_a)$ and $L^m(\wedge Y_b) \leq L^{n_b}(\wedge Y_b)$. Therefore,

$$L^m(\wedge(Y_a \cup Y_b)) \leq a \wedge b$$

and so $a \wedge b \in H(X)$. Since $H(X)$ is a filter and $X \subseteq H(X)$, we have that $D_t(X) \subseteq H(X)$. For the reciprocal inclusion, let $a \in H(X)$; then, $Y_a \in \mathcal{P}_\omega(X)^*$ and $n_a \in \omega$ must exist such that $L^{n_a}(\wedge Y_a) \leq a$. However, $X \subseteq D_t(X)$ and $D_t(X)$ are closed for operations $\wedge$ and L; therefore, $a \in D_t(X)$. □

Remark 2. *In the sequel, the symbol* $\rightarrow$ *will denote the binary operation in the universe of any temporal algebra defined by* $a \rightarrow b = \neg a \vee b$.

In the case that $X = D \cup \{a\}$, $D \in Tsp(\mathbf{A})$ and $a \in A$, $D_t(X)$ has a special description. The following result, actually an algebraic formulation of the "deduction theorem", gives this description.

Theorem 8 (Deduction theorem)**.** *Let* $\mathbf{A}$ *be a temporal algebra. For all* $D \in Tsp(\mathbf{A})$ *and* $a \in A$*, the equality:*

$$D_t(D,a) = \{x \in A : \text{ there exists } n \in \omega \text{ such that } L^n a \to x \in D\}$$

holds.

Proof. For convenience in the proof, let F be the right hand member of the conjunctive equality of the statement. The assertion is that F is a temporal filter of $\mathbf{A}$ containing $D \cup \{a\}$ and that it is also the smallest among those verifying this property.

It is immediate to verify that F is a filter of order and that $D \cup \{a\} \subseteq F$. Moreover, $L^0 a \to 1 = 1 \in D$, thus $1 \in F$. If we assume that $x, y \in F$, then two natural numbers n and m (e.g. $n \leq m$) exist such that $L^n a \to x$, $L^m a \to y \in D$. It follows that $L^m a \to x$, $L^m a \to y \in D$, and hence $L^m a \to (x \wedge y) \in D$. We thus conclude that $F \in Sp(A)$. Moreover, F is temporal because if $m \in \omega$ exists such that $L^m a \to x$ and D is temporal, then $L(L^m a \to x) \in D$ and so $L^{m+1} a \to Lx \in D$; hence, the temporality of F follows.

Finally, suppose that D' is an element of $Tsp(\mathbf{A})$ containing $D \cup \{a\}$ and let $x \in A$ and $m \in \omega$ such that $L^m a \to x \in D$. In such a case, $L^m a, L^m a \to x \in D'$, which implies that $x \in D'$. This concludes the proof. □

Remark 3. *Let* $\mathbf{A}$ *a temporal algebra. We consider that the precedence of the operator* L *is greater than that of the operator* $\to$ *(for all* $a, b \in A$, $a \to b = \neg a \vee b$*). So,* $La \to b$ *means* $L(a) \to b$*. For all nonempty finite subset* $X = \{x_0, \ldots, x_n\}$ *of* A, $\wedge X$ *is by definition the element* $x_0 \wedge \cdots \wedge x_n$*.*

Corollary 6. *Let* $D \in Tsp(\mathbf{A})$ *and* $X = \{a_1, \ldots, a_n\}$ *a finite subset of* A*. The temporal filter generated by* $D \cup X$ *is the set of* $x \in A$ *such that there is* $m \in \omega$ *verifying* $L^m(a_1 \wedge \cdots \wedge a_n) \to x \in D$*.*

Lemma 9. *Let* $\mathbf{A}$ *be a simple temporal algebra. If* $X \in \mathcal{P}(A) \setminus \{\emptyset, \{1\}\}$ *and* $a \in A$*, then there is* $Y_a \in \mathcal{P}_\omega(X)^*$ *and* $n_a \in \omega$ *such that* $L^{n_a}(\wedge Y_a) \leq a$*.*

Proof. If $X \notin \{\emptyset, \{1\}\}$, then $\{1\} \subset D_t(X)$. As $\mathbf{A}$ is simple, we have $D_t(X) = A$ and so $a \in D_t(X)$. The lemma follows at once from the definition of $D_t(X)$. □

Lemma 10. *Let* $\mathbf{A}$ *be a temporal algebra. Then, the statements:*

1. *For all* $X \in \mathcal{P}(A) \setminus \{\emptyset, \{1\}\}$ *and* $a \in A$*, there is* $Y_a \in \mathcal{P}_\omega(X)^*$ *and* $n_a \in \omega$ *such that* $L^{n_a}(\wedge Y_a) \leq a$*.*
2. *For all* $x \in A \setminus \{1\}$*, there is* $n_x \in \omega$ *such that* $L^{n_x} x = 0$*.*

are equivalent.

Proof. Let us assume the first statement and let $x \in A \setminus \{1\}$. For $X = \{x\}$ and $a = 0$ there is $n_x \in \omega$ such that $L^{n_x} x \leq 0$. Conversely, if the second statement is true, $X \in \mathcal{P}(A) \setminus \{\emptyset, \{1\}\}$, $a \in A$, and $x \in X \setminus \{1\}$, then $n_x \in \omega$ verifies $L^{n_x} x = 0$, and so $L^{n_x} x \leq a$. If we select $Y_a = \{x\}$ and $n_a = n_x$, the first statement is established. □

Corollary 7. *Let* $\mathbf{A}$ *be a simple temporal algebra. For every* $a \in A \setminus \{1\}$, $n_a \in \omega$ *exists such that* $L^{n_a} a = 0$*.*

Theorem 9 is the main result of the section and is very satisfactory because it characterises simple temporal algebras by means of an arithmetical criterion. The criterion consists in being able to reduce to 0 any element of the algebra, except 1, by iterated applications of the operator L. Of course, the dual criterion states that any non-zero element may be raised to 1 by iterated applications of the operator M.

Theorem 9. *Let* **A** *be a temporal algebra. Then, the statements:*

1. **A** *is simple.*
2. *For every* $a \in A \setminus \{1\}$, *there is* $n_a \in \omega$ *such that* $L^{n_a} a = 0$.
3. *For every* $a \in A \setminus \{0\}$, *there is* $m_a \in \omega$ *such that* $M^{m_a} a = 1$.
4. *For all non-trivial temporal algebra* **B**, *if* $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a morphism of temporal algebras, then* ϕ *is a monomorphism.*

are equivalent.

Proof. The first statement implies the second, as assured by Lemma 9 and Lemma 10. It is obvious that the second implies the third. Let us assume the third statement and let ϕ be a morphism from **A** to a non-trivial algebra **B**. If ϕ is non-injective, then there is $a \in Ker(\phi) \setminus \{1\}$ and $m \in \omega$ such that $M^n \neg a = 1$. We have:

$$\begin{aligned} 1 &= \neg M^m 0 = \neg M^m \phi(\neg a) \\ &= \phi(\neg M^m \neg a) = \phi(0) \\ &= 0 \end{aligned}$$

Therefore, **B** should be trivial, and this is impossible by hypothesis. So, the fourth statement is established. The first follows from the fourth since every temporal filter of **A** is the kernel of a morphism of temporal algebras. □

5. The Skew Product

In Definition 6, we introduce the modified product of temporal algebras, which serves our objective. The term "skew-product" comes from the fact that the definition of temporal operations is not componentwise, but it is in all modified components—except at least in one, the "not–skew" (i.e., the temporal operations of the usual product of Universal Algebra)—based on a specific condition of the adjacent components in the argument tuple. As we will see later on, our reasoning is about formulas α such that $p\pi(\alpha) \neq 0$ or $f\pi(\alpha) \neq 0$; hence, we consider, for example in Definition 6, the condition $b \leq pa$ because if $pa \neq 0$ in a finite temporal algebra, then an atom b must exist that is less than or equal to pa. The choice of the "skew factor" **B** will be made on purpose, as we will see, in order to cause effects or to catch them. In this section, we will make implicit use of Lemma 1, Lemma 2, and Lemma 3 almost everywhere; the applicability of these lemmas will be evident from the situation.

Definition 6. *Let* **A** *and* **B** *be finite temporal algebras. Let us assume that* $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $k \in \omega^*$, *and* $b \leq pa$. *We define in* $A^k \times B$ *the unary operations* g *and* h *as follows. For all* $\langle y_0, \ldots, y_k \rangle \in A^k \times B$:

$$\pi_i(h\langle y_0, \ldots, y_k \rangle) = \begin{cases} hy_i, & \text{if } i = 0 \text{ or } (i > 0 \text{ and } a \leq y_{i-1}), \\ hy_i \wedge \neg b, & \text{if } 1 \leq i \leq k-1 \text{ and } a \nleq y_{i-1}, \\ hy_i \wedge \neg c, & \text{if } i = k \text{ and } a \nleq y_{k-1}. \end{cases}$$

$$\pi_i(g\langle y_0, \ldots, y_k \rangle) = \begin{cases} gy_i, & \text{if } i = k \text{ or } (i = k-1 \text{ and } c \leq y_k) \\ & \text{or } (i < k-1 \text{ and } b \leq y_{i+1}), \\ gy_i \wedge \neg a, & \text{if } (i < k-1 \text{ and } b \nleq y_{i+1}) \\ & \text{or } (i = k-1 \text{ and } c \nleq y_k). \end{cases}$$

The proof of the following Lemma 11 boils down to a simple routine check.

Lemma 11. *Let* **A** *and* **B** *be two finite temporal algebras. Suppose that* $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $k \in \omega^*$, *and* $b \leq pa$. *The algebra* $\mathbf{A}^k_{a,b} \times \mathbf{B}_c = \langle A^k \times B, \wedge, \vee, \neg, g, h, 1 \rangle$ *where* $\wedge$, $\vee$, $\neg$, *and* 1 *are the operations componentwise on the product* $\mathbf{A}^k \times \mathbf{B}$ *and the operations g y h are as in Definition 6, is a temporal algebra. In fact, for this algebra the operations f and p are as follows:*

$$\pi_i(p\langle y_0,\ldots,y_k\rangle) = \begin{cases} py_i, & \text{if } i=0 \text{ or } (1 \le i \le k-1 \text{ and } a \not\le y_{i-1}) \\ & \text{or } (i=k \text{ and } a \not\le y_{k-1}), \\ py_i \vee b, & \text{if } 1 \le i \le k-1 \text{ and } a \le y_{i-1}, \\ py_i \vee c, & \text{if } i=k \text{ and } a \le y_{k-1} \end{cases}$$

$$\pi_i(f\langle y_0,\ldots,y_k\rangle) = \begin{cases} fy_i, & \text{if } (0 \le i < k-1 \text{ and } b \not\le y_{i+1}) \\ & \text{or } (i=k-1 \text{ and } c \not\le y_k) \text{ or } i=k, \\ fy_i \vee a, & \text{if } (0 \le i < k-1 \text{ and } b \le y_{i+1}) \\ & \text{or } (i=k-1 \text{ and } c \le y_k). \end{cases}$$

Remark 4. *In the hypotheses of Lemma 11 and according to its content we know that* $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ *is a temporal algebra; well, we suggest giving it the name skew product of A and B regarding to the atoms a, b, and c.*

To shorten in this paper we will implicitly assume that $\mathbf{A}$ and $\mathbf{B}$ are both finite temporal algebras. Moreover, when we write $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ we presupose that: $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $b \le pa$, $k \in \omega^*$, and, finally, that the temporal operations are according to Definition 6. Sometimes, we will represent the universe of $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ by $A^k_{a,b} \times B_c$, though this universe is in fact the set $A^k \times B$.

Nevertheless, the skew value of $g0$ matches with its componentwise value. In effect, we have the following simple but important lemma.

Lemma 12. *In* $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ *the equality* $g0 = \langle g0, \overset{k+1)}{\ldots}, g0\rangle$ *holds.*

Proof. We have the equality $g0 = \langle \neg a \wedge g0, \overset{k)}{\ldots}, \neg a \wedge g0, g0\rangle$. Since $b \le pa$, we have that $a \le \neg g0$, or equivalently, $g0 \le \neg a$; hence, $\neg a \wedge g0 = g0$. □

Lemma 13. *Let* $k \in \omega$ *such that* $2 \le k$ *and* $\langle y_0,\ldots,y_k\rangle \in A^k_{a,b} \times B_c$. *If there is* $m \in \omega^*$ *such that* $m \le k-1$ *and* $y_i = y_0$, *for all* $0 \le i \le m$, *then for all* $0 \le i \le m$, $\pi_i(h\langle y_0,\ldots,y_k\rangle) = \pi_0(h\langle y_0,\ldots,y_k\rangle)$ *and for all* $0 \le i \le m-1$, $\pi_i(g\langle y_0,\ldots,y_k\rangle) = \pi_0(g\langle y_0,\ldots,y_k\rangle)$.

Proof. From the hypotheses of the lemma we have $m \le k-1$. Suppose that $j \le k-1$ and that $a \not\le y_j$ or, equivalently, that $y_j \le \neg a$. Since $b \le pa$, it follows that $hy_j \le \neg b$, that is, $hy_j \wedge \neg b = hy_j$. Hence, if $0 \le i \le m$, then $\pi_i(h\langle y_0,\ldots,y_k\rangle) = hy_0$, from which the first statement follows. Moreover, according to the definition of g in $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ we have, for all $0 \le i \le m$,

$$\pi_i(g\langle y_0,\ldots,y_k\rangle) = \begin{cases} gy_0, & \text{if } (i=k-1 \text{ and } c \le y_k) \\ & \text{or } (i<k-1 \text{ and } b \le y_{i+1}), \\ gy_0 \wedge \neg a, & \text{if } (i=k-1 \text{ and } c \not\le y_k) \\ & \text{or } (i<k-1 \text{ and } b \not\le y_{i+1}). \end{cases}$$

From this, it follows that each $0 \le i \le m-1$ satisfies the equality $\pi_i(g\langle y_0,\ldots,y_k\rangle) = gy_0$ or $\pi_i(g\langle y_0,\ldots,y_k\rangle) = gy_0 \wedge \neg a$, depending on whether $b \le y_0$ holds or not. □

The following lemma indicates what can we expect about conmutativity between M and π_i. Its proof follows easily by induction on m given that $py_i \le \pi_i(p\langle y_0,\ldots,y_k\rangle)$, $fy_i \le \pi_i(f\langle y_0,\ldots,y_k\rangle)$ and that M is an increasing function.

Lemma 14. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite temporal algebras. For all* $m \in \omega$ *and* $0 \leq i \leq k$, *in* $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ *the inequality* $M^m \pi_i(\langle y_0, \ldots, y_k \rangle) \leq \pi_i(M^m \langle y_0, \ldots, y_k \rangle)$ *holds.*

Now, we introduce the hypothesis of simplicity in the factors of the new product. The simplicity of $\mathbf{A}$ and $\mathbf{B}$ implies that $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ is simple and conversely. Nevertheless, we will prove a weaker result.

Theorem 10. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two finite simple temporal algebras and* $y \in A \setminus \{0\}$. *If* $\langle y, \overset{k-1)}{\ldots}, y, z \rangle \in A^k_{a,b} \times B_c$, *then* $m \in \omega^*$ *exists such that* $M^m \langle y, \overset{k-1)}{\ldots}, y, z \rangle = \langle 1, \overset{k-1)}{\ldots}, 1, z' \rangle$, *for some* $z' \in B$ *satisfying* $M^m z \leq z'$.

Proof. Since $\mathbf{A}$ is simple and $y \neq 0$, by Theorem 9, $m \in \omega^*$ exists such that $M^m y = 1$. This and Lemma 14 imply that there is $z' \in B$ such that $M^m \langle y, \overset{k-1)}{\ldots}, y, z \rangle = \langle 1, \overset{k-1)}{\ldots}, 1, z' \rangle$. It is clear from Lemma 14 that $M^m z \leq z'$. □

Remark 5. *If* $\mathbf{B} = \mathbf{A}$ *and* $c = b$, *the symbol* $\mathbf{A}^{k+1}_{a,b}$ *(resp.* $\mathbf{A}$*) will represent the algebra* $\mathbf{A}^k_{a,b} \times \mathbf{A}_b$ *(resp.* $\mathbf{A}^1_{a,b}$*).*

Corollary 8. *Let a* $\mathbf{A}$ *be a finite simple temporal algebra and* $y \in A \setminus \{0\}$. *If* $\langle y, \ldots, y \rangle \in A^k_{a,b}$ *then there exists* $m \in \omega^*$ *such that* $M^m \langle y, \ldots, y \rangle = \langle 1, \ldots, 1 \rangle$ *and* $M^m y = 1$.

In the arguments that we will give later, finding bounds requires a more careful approach in the case $g0 \neq 0$. In Definition 7, we define the necessary parameters, except the notion of degree (see Definition 9), to build the appropriate algebra $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ to cover what we need. Moreover, in Definition 8 we give a specific temporal algebra that we are interested in using as algebra $\mathbf{B}$ in a specific skew-product $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$.

Definition 7. *Let* $\mathbf{A}$ *be a finite simple temporal algebra for which the condition* $g0 \neq 0$ *holds and there are* $a, b \in Atm(\mathbf{A})$ *such that* $b \leq pa$. *From Lemma 12 and Corollary 8, the set of all* $j \in \omega$ *such that* $M^j g0 = 1$ *at the same time in* $\mathbf{A}^k_{a,b}$ *and* $\mathbf{A}$ *is non-empty; so, it is possible to take the minimum* s *of this set. Let*

$$r = \min\{j \in \omega \colon \langle 0, \overset{k-1)}{\ldots}, 0, a \rangle \leq M^j g0\} \tag{1}$$

and

$$l = \max\{s - r, 1\}. \tag{2}$$

We define the value $t(\mathbf{A}, a)$ *by the equality*

$$t(\mathbf{A}, a) = \min\{2j \colon j \in \omega \text{ and } l + 1 \leq 2j\}. \tag{3}$$

Finally, let us define $\sigma \colon B \longrightarrow B$ *by* $\sigma(y) = ghy$.

Definition 8. *Given* $q \in \omega^*$, *if* B *represents the set* $\{0, 1\}$ *then we define the functions*

$$g, h \colon B^{2q} \longrightarrow B^{2q}$$

as follows ($0 \leq i \leq 2q - 1$)*:*

$$\pi_i(h\langle y_0, \ldots, y_{2q-1} \rangle) = \begin{cases} y_0 \wedge y_1, & \text{if } i = 0, \\ 0, & \text{if } i \text{ is odd}, \\ y_{i-1} \wedge y_i \wedge y_{i+1}, & \text{otherwise}. \end{cases}$$

$$\pi_i(g\langle y_0,\ldots,y_{2q-1}\rangle)=\begin{cases}y_i, & \textit{if } i \textit{ is even},\\ y_{2q-2}, & \textit{if } i=2q-1,\\ y_{i-1}\wedge y_{i+1}, & \textit{otherwise.}\end{cases}$$

Remark 6. *Let $q\in\omega^*$. It is clear that the algebra $\mathbf{B}^{2q}=\langle B^{2q},\wedge,\vee,\neg,g,h,1\rangle$ is a temporal algebra. For $\mathbf{B}^{2q}$, the operations f and p are as follows:*

$$\pi_i(f\langle y_0,\ldots,y_{2q-1}\rangle)=\begin{cases}y_i, & \textit{if } i \textit{ is even},\\ y_{2q-2}, & \textit{if } i=2q-1,\\ y_{i-1}\vee y_{i+1}, & \textit{otherwise.}\end{cases}$$

$$\pi_i(p\langle y_0,\ldots,y_{2q-1}\rangle)=\begin{cases}y_0\vee y_1, & \textit{if } i=0,\\ 0, & \textit{if } i \textit{ is odd},\\ y_{i-1}\vee y_i\vee y_{i+1}, & \textit{otherwise.}\end{cases}$$

Moreover, in $\mathbf{B}^{2q}$ the equality $g0=0$ holds.

Lemma 15 is a necessary tool to show Theorem 15. In order to grasp its meaning, understand that in increasing j, M^jg0 "saturates" the algebra $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$. Having $0\le j\le r$, the construction is designed so that the skew-factor $\mathbf{B}^{2q}_c$ does not have any effect on the first k components of M^jg0; the component $k+1$ remains at 0; and when j equals r, the value of the kth component of M^jg0 exceeds or equals a. Nevertheless, when the value of j exceeds r, the $k+1$ component of M^jg0 (i.e., the component of M^jg0 in the skew-factor $\mathbf{B}^{2q}_c$) leaves the value 0, and its evolution is helpful to measure the progress of j by means of a progressive spread of the value 1 from the first component to the last. Regarding "well chosen" q, the replacement of the value 0 in the $(k+1)$th component is not complete. Actually, the reason for our definitions is to enunciate and demonstrate Lemma 15.

Lemma 15. *Let $\mathbf{A}$ be a finite simple temporal algebra such that $g0\neq 0$, and let $a,b\in Atm(\mathbf{A})$ be such that $b\le pa$. Let s be the least $j\in\omega$ satifying $M^jg0=1$ at the same time in $\mathbf{A}^k_{a,b}$ and $\mathbf{A}$. If r is the value given by Equation (1), $c=\langle 1,0,\ldots,0\rangle$, and $q\in\omega^*$ then the following properties hold:*

1. *For all $j\le r$ and $0\le i\le k-1$, the value of $\pi_i(M^jg0)$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$ coincides with its value in $\mathbf{A}^k_{a,b}$.*
2. *For all $j\le r$, $\pi_k(M^jg0)=0$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$.*
3. *$a\le\pi_{k-1}(M^rg0)$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$.*
4. *If $r<s$ and $s-r<2q$, then for all $r<j\le s$, $\pi_k(M^jg0)=\langle 1,\overset{j-r)}{\ldots},1,0,\ldots,0\rangle$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$.*
5. *If $2q$ is the number $t(\mathbf{A},a)$, defined in (3), and $\langle y_0,\ldots,y_{2q-1}\rangle$ is $\pi_k(M^sg0)$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$, then $y_{2q-1}=0$.*

Proof. We will prove the first two statements at the same time by induction on j. Actually, in the two algebras the values of $g0$ are $\langle g0,\overset{k)}{\ldots},g0,0\rangle$ and $\langle g0,\overset{k)}{\ldots},g0\rangle$, respectively. So, the properties follow in the case $j=0$. Let us assume that the properties hold for $j<r$. It is easy to verify the first one in the cases $0\le i<k-1$. When $i=k-1$, since $\pi_k(M^jg0)=0$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$ then the value of $\pi_{k-1}(fM^jg0)$ is $f\pi_{k-1}(M^jg0)$. In the case of $\mathbf{A}^k_{a,b}$, the value of $\pi_{k-1}(fM^jg0)$ is $f\pi_{k-1}(M^jg0)$, but, by the inductive hypothesis, $\pi_{k-1}(M^jg0)$ has the same value in the two algebras under consideration, so the result holds for $k-1$. Since r is the least natural i such that $\langle 0,\overset{k-1)}{\ldots},0,a\rangle\le M^ig0$, it follows that $a\not\le\pi_{k-1}(M^jg0)$ in $\mathbf{A}^k_{a,b}\times\mathbf{B}^{2q}_c$, and so $\pi_k(pM^jg0)=p0$ and $\pi_k(fM^jg0)=f0$. This implies that $\pi_k(M^{j+1}g0)=0$. The third property is obvious from the first one. The fourth statement also follows by induction.

For $j = r+1$, the result holds. Actually, according to property 2 and property 3 we have $\pi_k(M^r g0) = 0$ and $a \le \pi_{k-1}(M^r g0)$; then, $\pi_k(pM^r g0) = \langle 1, 0, \ldots, 0\rangle$ and, furthermore, $\pi_k(M^{r+1} g0) = \langle 1, 0, \ldots, 0\rangle$. Let us assume that $1 \le i$, $r+i+1 \le s$ and that the result holds for $r+i$. If $\langle y_0, \ldots, y_{2q-1}\rangle$ stands for $\pi_k(M^{r+i} g0)$, the inductive hypothesis means that $y_0 = \cdots = y_{i-1} = 1$ and $y_i = \cdots = y_{2q-1} = 0$. It is clear that

$$\pi_k(M^{r+i+1} g0) = p\langle y_0, \ldots, y_{2q-1}\rangle \vee \langle y_0, \ldots, y_{2q-1}\rangle \vee f\langle y_0, \ldots, y_{2q-1}\rangle$$

Hence, all we need is to examine the right-hand side of this equality. Represent by $\langle z_0, \ldots, z_{2q-1}\rangle$ (resp. $\langle u_0, \ldots, u_{2q-1}\rangle$) the value $p\langle y_0, \ldots, y_{2q-1}\rangle$ (resp. $f\langle y_0, \ldots, y_{2q-1}\rangle$). Since $M^{r+1} g0 \le M^{r+i} g0$, then $y_0 = 1$ and so $z_0 = u_0 = 1$. On the other hand, $y_{2q-2} = 0$; hence, $z_{2q-1} = u_{2q-1} = 0$. When $j \notin \{0, 2q-1\}$, the values of z_j and u_j are as follows:

1. $y_j = 1$; in this case, the result is obvious,
2. $y_{j-1} = 1$ and $y_j = 0$; if j is even (resp. odd), then $z_j = 1$ (resp. $u_j = 1$),
3. $y_{j-1} = 0$, $y_j = 0$; therefore, $y_{j+1} = 0$, and so, if j is either even or odd, $z_j = 0$ and $u_j = 0$.

So, the fourth property is established. The fifth follows from the fourth and the given definitions since the equality

$$\langle y_0, \ldots, y_{2q-1}\rangle = \begin{cases} \langle 0, 0\rangle, & \text{if } s = r, \\ \langle 1, \ldots, 1, 0\rangle, & \text{if } s \neq r \text{ and } l \text{ is odd}, \\ \langle 1, \ldots, 1, 0, 0\rangle, & \text{if } s \neq r \text{ and } l \text{ is even}, \end{cases}$$

holds. □

Remark 7. *In the sequel, we adopt the following notational use. On the one hand, for all $0 \le i \le 2q-1$ let z_i be the element of $\mathbf{B}^{2q}$ satisfying for all $0 \le j \le 2q-1$ the condition:*

$$\pi_j(z_i) = \begin{cases} 1, & \textit{if } i = j, \\ 0, & \textit{otherwise}. \end{cases}$$

On the other hand, if $y = \langle y_0, \ldots, y_k\rangle$ is in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$, then for every $0 \le i \le 2q-1$, $\vartheta_i(y)$ will be the abbreviation of $\pi_i(y_k)$.

Lemma 16. *Let $\mathbf{A}$ be a finite simple temporal algebra such that $g0 \neq 0$ and $a, b \in Atm(\mathbf{A})$ such that $b \le pa$. Let $q \in \omega^*$, and let c be the atom z_0 of $\mathbf{B}^{2q}$. If $\langle 0, \ldots, 0, z_{2q-1}\rangle \nleq M^s g0$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$ for some $s \in \omega$, then $\langle 0, \ldots, a, 0\rangle \nleq \sigma^q(M^s g0)$.*

Proof. Let us assume that $\langle 0, \ldots, 0, z_{2q-1}\rangle \nleq M^s g0$, and use induction to show that for all $0 \le i \le q-1$, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \nleq \sigma^i(M^s g0)$. In the case $i = 0$, the result follows directly from the hypotheses. Suppose that $0 \le i < q-1$, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \nleq \sigma^i(M^s g0)$; nevertheless, $\langle 0, \ldots, 0, z_{2(q-(i+1))-1}\rangle \le \sigma^{i+1}(M^s g0)$. Hence, $\langle 0, \ldots, 0, z_{2(q-i)-3}\rangle \le gh\sigma^i(M^s g0)$. This implies that $\vartheta_{2(q-i)-3}(gh\sigma^i(M^s g0)) = 1$. According to the definition of g, since $2(q-i)-3$ is odd and different from $2q-1$, we conclude that $\vartheta_{2(q-i)-2}(h\sigma^i(M^s g0)) = 1$. Since $2(q-i)-2$ is even, $i < q-1$; therefore, by the definition of h, the equality $\vartheta_{2(q-i)-1}(\sigma^i(M^s g0)) = 1$ holds or, equivalently, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \le \sigma^i(M^s g0)$, which is contradictory with the inductive hypothesis. In particular, we have $\langle 0, \ldots, 0, z_1\rangle \nleq \sigma^{q-1}(M^s g0)$. By contradiction, let us suppose now that $\langle 0, \ldots, a, 0\rangle \le gh\sigma^{q-1}(M^s g0)$, that is, $a \le \pi_{k-1}(gh\sigma^{q-1}(M^s g0))$. Since $a \neq 0$ and $z_0 = c$, we have that $z_0 \le \pi_k(h\sigma^{q-1}(M^s g0))$; hence, $z_0 \le h\pi_k(\sigma^{q-1}(M^s g0))$ and, consequently, $z_1 \le \pi_k(\sigma^{q-1}(M^s g0))$, which is contradictory. Therefore, $\langle 0, \ldots, a, 0\rangle \nleq \sigma^q(M^s g0)$. □

6. The Bound $g0 \wedge h0$

It is intuitively clear that for formulas $\pi(\alpha)$, with $\alpha \in F(X)$, such that $p\pi(\alpha) \neq 0$ or $f\pi(\alpha) \neq 0$ it will be possible to find in $\mathbf{F}_t(X)$ non-zero lower bounds. To fix ideas, here we show the reasoning in the case of $p\pi(\alpha) \neq 0$, and we leave the case $f\pi(\alpha) \neq 0$, which deserves similar treatment in an obvious way. In order to determine which formulas are atoms, we will show that it is sufficient to study these cases.

Given $\alpha \in F(X)$, we need an algebra $\mathbf{A}_\alpha$ with appropriate properties, including simpleness, in order to build a skew-product $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ that is useful for our purposes. The existence of this algebra, as we shall see, is warranted by the results of [9,20] and is here where we bring to our argument—and this is essential—the concept of filtration studied in [9]. Later, we will set the temporal algebra $\mathbf{B}$, which in this case will be very simple: the Boolean algebra with two elements: g a constant function, and $h = g$; having done this, the choice of the atom c is univocal.

Theorem 11. *Let $\alpha \in F(X)$ such that $p\pi(\alpha) \neq 0$. A finite simple temporal algebra $\mathbf{A}_\alpha$ and a morphism $\tilde{v}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ exist such that $\tilde{v}(\pi(\alpha)) \neq 0$ and $\tilde{v}(p\pi(\alpha)) \neq 0$.*

Proof. Since $\neg p\pi(\alpha) \neq 1$, a finite temporal algebra $\mathbf{A}$ and a morphism $\tilde{u}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}$ exist (see [9]) such that $\tilde{u}(\neg p\pi(\alpha)) \neq 1$; hence, $\tilde{u}(p\pi(\alpha)) \neq 0$. Since $\mathbf{A}$ is finite, it is isomorphic to a product of finite simple temporal algebras (see [20]). Composing $\tilde{u}$ with the canonical projection over the convenient simple factor of $\mathbf{A}$, $\mathbf{A}_\alpha$, we obtain the mapping $\tilde{v}$ of the statement. It is clear that $\tilde{v}(p\pi(\alpha)) \neq 0$. Therefore, the equality $\tilde{v}(\pi(\alpha)) = 0$ is impossible. □

In the following, we will consider that $\pi(\alpha) \in F_t(X)$ satisfies $p\pi(\alpha) \neq 0$. So, $\mathbf{A}_\alpha$ and $\tilde{v}$ will represent the algebra and the morphism of Theorem 11. We will also assume that $a, b \in Atm(\mathbf{A}_\alpha)$ are such that $a \leq \tilde{v}(\pi(\alpha))$ and $b \leq \tilde{v}(p\pi(\alpha))$. Moreover, if $\mathbf{B}$ is a temporal algebra, we define the map $\tau\colon B \longrightarrow B$ by the equality $\tau y = fpy$. The reason for involving $\mathbf{E}_0$ here (see Example 1) is exactly to be able to count on Equation (4) as a valid result; it is the key to Theorem 14 obtained through Corollary 9.

Theorem 12. *Let $\mathbf{A}$ be a finite temporal algebra. In the algebra $\mathbf{A}^{m+1}_{a,b} \times (\mathbf{E}_0)_c$, where $m \in \omega^*$ and $c = 1$, the equality:*

$$\tau^i(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) = \langle 0, \overset{m-i)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^i(a), 0\rangle \tag{4}$$

holds for all $0 \leq i \leq m$.

Proof. The proof is by induction on i. For $i = 0$, the statement is obviously true. Let us assume that the property holds for j, and let us take $i = j + 1$. It is straightforward to show that $a \leq \tau a$. Since τ is an increasing funtion, $\{\tau^j a\}_{j\in\omega}$ increases with j; hence, for all $j \in \omega$, $a \leq \tau^j a$ and so $b \leq p\tau^j(a)$. By definition of f and p we have:

$$\begin{aligned}
\tau^i(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) &= fp\langle 0, \overset{m-j)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^j(a), 0\rangle \\
&= f\langle 0, \overset{m-j)}{\ldots}, 0, pa, p\tau(a), p\tau^2(a), \ldots, p\tau^j(a), 1\rangle \\
&= \langle 0, \overset{m-j-1)}{\ldots}, 0, a, a \vee \tau(a), a \vee \tau^2(a), \ldots, a \vee \tau^{j+1}(a), 0\rangle \\
&= \langle 0, \overset{m-i)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^i(a), 0\rangle
\end{aligned}$$

which is just what we wanted to show. □

Equation (4) is valid when $i = m$, even if $m = 0$; therefore, the following corollary holds.

Corollary 9. *Let* $\mathbf{A}$ *be a finite temporal algebra. In the algebra* $\mathbf{A}_{a,b}^{m+1} \times (\mathbf{E}_0)_c$*, where* $m \in \omega$ *and* $c = 1$*, the equality* $\tau^m(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) = \langle a, \tau(a), \tau^2(a), \ldots, \tau^m(a), 0\rangle$ *holds.*

In order to apply Corollary 9 within the proof of Theorem 14, we need to assign an appropriate value for m; this value will be selected by means of a concept of degree for formulas α derived from their unique writing. In exchange for involve in our reasoning, a notion of degree conceptually simpler to the others used in the literature, we need to define two functions. In Definition 9, the function *deg* is given, which should actually be called $\deg_g$; similarly, we should give and use a function $\deg_h$ in the case $f\pi(\alpha) \neq 0$. Our notion of degree for a formula α, $\deg_g(\alpha)$, counts the maximum number of nested g-symbols in the formula α.

Remark 8. *In the following, we will assume that* X *is the finite set* $\{x_1, \ldots, x_n\}$*, where* $n \in \omega^*$*.*

Definition 9. *Let* $\alpha \in F(X)$*; the degree of* α*,* $\deg(\alpha)$*, is defined as follows:*

$$\deg(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ \deg(\beta), & \text{if } \alpha = \neg\beta, \\ \max\{\deg(\beta), \deg(\gamma)\}, & \text{if } \alpha = \beta \wedge \gamma \text{ or } \alpha = \beta \vee \gamma, \\ \deg(\beta), & \text{if } \alpha = h\beta, \\ \deg(\beta) + 1, & \text{if } \alpha = g\beta. \end{cases}$$

The proof of the following theorem follows by induction over the complexity of the formula α. It is straightforward after the definition of degree and Lemma 13. Indeed, validity of Theorem 13 is the reason underlying our definition of degree.

Theorem 13. *Let* $v : X \longrightarrow \mathbf{A}$ *be a temporal valuation and* $\bar{v}$ *its extension as a morphism to* $\mathbf{F}(X)$*. Let us consider the temporal valuation* $w \colon X \longrightarrow A_{a,b}^k \times B_c$ *defined, for all* $1 \leq i \leq n$*, by* $w(x_i) = \langle v(x_i), v(x_i), \ldots, v(x_i), 0\rangle$*. For all* $\alpha \in F(X)$ *and* $0 \leq j \leq k - \big(\deg(\alpha) + 1\big)$*,* $\pi_j\big(\bar{w}(\alpha)\big) = \bar{v}(\alpha)$ *(or equivalently,* $\pi_j\big(\tilde{w}(\pi(\alpha))\big) = \tilde{v}\big(\pi(\alpha)\big)$*), whenever* $\deg(\alpha) < k$*.*

Now, we give the referred lower bounds. Everything was based on $\alpha \in F(X)$ such that $p\pi(\alpha) \neq 0$ and some simple algebra $\mathbf{A}_\alpha$ selected from $\alpha \in F(X)$ in Theorem 11. The case study is suggested by the different behavior of the skew-product depending on whether the equality $g0 = 0$ occurs in $\mathbf{A}_\alpha$. In the case $g0 = 0$, the bound is given in Theorem 14; moreover, in the case of that $g0 \neq 0$, then our bound is given in Theorem 16.

Theorem 14. *Let* $\alpha \in F(X)$ *be such that* $p\pi(\alpha) \neq 0$*;* $\mathbf{A}_\alpha$ *a finite simple temporal algebra;* $\tilde{v} \colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ *such that* $\tilde{v}\big(p\pi(\alpha)\big) \neq 0$*; and consider the element* $\tau^m(fg0) \in F_t(X)$*, where* $m = \deg(\alpha)$*. If in* $\mathbf{A}_\alpha$ *the equality* $g0 = 0$ *holds, then* $0 < \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$*.*

Proof. We have $\tilde{v}\big(\tau^m(fg0)\big) = \tau^m\big(f\tilde{v}(g0)\big) = \tau^m(f0) = 0$. Since $\tilde{v}\big(\pi(\alpha)\big) \neq 0$, the inequality $\pi(\alpha) \leq \tau^m(fg0)$ is not possible; therefore, $0 \leq \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$. Now, all we need is to show that $0 < \pi(\alpha) \wedge \tau^m(fg0)$. For this, take the algebra $\mathbf{A}_{a,b}^k \times (\mathbf{E}_0)_c$, where $c = 1$ and $k = \deg(\alpha) + 1$, and define the temporal valuation $w \colon \{x_1, \ldots, x_n\} \longrightarrow A_{a,b}^k \times B_c$ by the equality $w(x_i) = \langle v(x_i), \overset{k)}{\ldots}, v(x_i), 0\rangle$. According to Theorem 13, $\pi_0\big(\tilde{w}(\pi(\alpha))\big) = \tilde{v}\big(\pi(\alpha)\big)$; hence, $\langle a, 0, \overset{k)}{\ldots}, 0\rangle \leq \tilde{w}\big(\pi(\alpha)\big)$. Since $g0 = 0$ holds in $\mathbf{A}_\alpha$, the value of $fg0$ in $\mathbf{A}_{a,b}^k \times (\mathbf{E}_0)_c$ is $\langle 0, \overset{k-1)}{\ldots}, 0, a, 0\rangle$, where $\mathbf{A}$ is the algebra $\mathbf{A}_\alpha$. According to Corollary 9, we have $\langle a, 0, \overset{k)}{\ldots}, 0\rangle \leq \tilde{w}\big(\tau^m(fg0)\big)$. Since $\tilde{w}\big(\pi(\alpha)\big) \wedge \tilde{w}\big(\tau^m(fg0)\big) = \tilde{w}\big(\pi(\alpha) \wedge \tau^m(fg0)\big)$, it follows that $\pi(\alpha) \wedge \tau^m(fg0) \neq 0$, and so $0 < \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$. □

Theorem 15. *Let $\alpha \in F(X)$ be such that $p\pi(\alpha) \neq 0$ and both $\mathbf{A}_\alpha$ and $\bar{v}$ the algebra and the morphism whose existence ensures Theorem 11. Let us assume that condition $g0 \neq 0$ holds in $\mathbf{A}_\alpha$ and take the atoms $c = \langle 1, 0, \ldots, 0 \rangle$ and $d = \langle 0, \ldots, 0, 1 \rangle$ of $\mathbf{B}^{t(\mathbf{A}_\alpha, a)}$. If $\deg(\alpha) = m$, $k = m + 1$, $\mathbf{A}$ is $\mathbf{A}_\alpha$, and $\bar{w}$ is the extension to $\mathbf{F}_t(X)$ of the temporal valuation:*

$$w\colon X \longrightarrow \mathbf{A}^k_{a,b} \times \mathbf{B}^{t(\mathbf{A}_\alpha, a)}_c$$

defined by $w(x_i) = \langle v(x_i), \overset{k)}{\ldots}, v(x_i), 0 \rangle$, then the properties:

1. $\bar{v}\big(\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big) = 1$.
2. $\langle 0, \overset{k)}{\ldots}, 0, d \rangle \not\leq \bar{w}(M^s g0)$.
3. $\langle a, 0, \overset{k)}{\ldots}, 0 \rangle \not\leq \bar{w}\big(\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big)$.

hold.

Proof. According to the choice of s (see Definition 7), the equality $M^s g0 = 1$ holds in $\mathbf{A}$. Since for all $j \in \omega$ we have that $\sigma^j(1) = 1$, it follows that $\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0) = 1$ and so $\bar{v}\big(\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big) = 1$. The second property is an immediate consequence of part 5 in Lemma 15. For the third property, we will show that, under the hypotheses of the theorem, if q represents to $t(\mathbf{A}_\alpha, a)/2 (\geq 1)$ and, for all $0 \leq i \leq k - 1$, u_i is the element of $A^k \times B^{2q-1}$ satisfying for $0 \leq j \leq k$:

$$\pi_j(u_i) = \begin{cases} a, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

then in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$ the condition $u_{m-i} \not\leq \sigma^{i+q}(M^s g0)$ holds for all $0 \leq i \leq m$. If $i = 0$, the result is true by Lemma 16. Let us suppose that $0 \leq i < m$ and that the result holds for i, i.e., $u_{m-i} \not\leq \sigma^{i+q}(M^s g0)$. If $u_{m-i-1} \leq gh\sigma^{i+q}(M^s g0)$, then $a \leq \pi_{m-i-1}\big(gh\sigma^{i+q}(M^s g0)\big)$ and, so long as $m - i - 1 < k - 1$, we have

$$b \leq \pi_{m-i}\big(h\sigma^{i+q}(M^s g0)\big) \tag{5}$$

Since $m - i \leq k - 1$, we deduce from (5) that $b \leq h\pi_{m-i}\big(\sigma^{i+q}(M^s g0)\big)$; but, $b \leq pa$, hence $u_{m-i} \leq \sigma^{i+q}(M^s g0)$, which contradicts the inductive hypothesis. In particular, we have $u_0 \not\leq \sigma^{m+q}(M^s g0)$; that is to say, $\langle a, 0, \overset{k)}{\ldots}, 0 \rangle \not\leq \sigma^{i+q}(M^s g0)$, which proves part 3. □

Theorem 16. *Let $\alpha \in F(X)$ such that $p\pi(\alpha) \neq 0$, and let us assume that in $\mathbf{A}_\alpha$ the condition $g0 \neq 0$ holds. Let $m = \deg(\alpha)$ and both s and $t(\mathbf{A}_\alpha, a)$ the values defined in Definition 7. Then,*

$$0 < \pi(\alpha) \wedge \sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0) < \pi(\alpha)$$

Proof. The morphism $\bar{v}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ satisfies that $\bar{v}\big(\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big) = 1$ and that $\bar{v}\big(\pi(\alpha)\big) \neq 0$. So, $0 < \bar{v}\big(\pi(\alpha) \wedge \sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big)$. For $\bar{w}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}^k_{a,b} \times \mathbf{B}^{t(\mathbf{A}_\alpha, a)}$, where $\mathbf{A}$ is here the algebra $\mathbf{A}_\alpha$ and a is the selected atom satisfying $a \leq \bar{v}\big(\pi(\alpha)\big)$, the condition $\langle a, 0, \overset{k)}{\ldots}, 0 \rangle \not\leq \bar{w}\big(\sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)\big)$ holds (see Theorem 15). Nevertheles, since $k = m + 1$, Theorem 13 ensures that $\pi_0\big(\bar{w}(\pi(\alpha))\big) = \bar{v}\big(\pi(\alpha)\big)$; therefore, $\langle a, 0, \overset{k)}{\ldots}, 0 \rangle \leq \bar{w}\big(\pi(\alpha)\big)$. It follows that

$$\pi(\alpha) \not\leq \sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0)$$

and so $0 < \pi(\alpha) \wedge \sigma^{m + \frac{t(\mathbf{A}_\alpha, a)}{2}}(M^s g0) < \pi(\alpha)$. □

Gathering the information provided by Theorem 14 and Theorem 16, we obtain in Corollary 10 an upper bound of the elements in $Atm\big(F_t(X)\big)$.

Corollary 10. *Let* $\alpha \in F(X)$ *be such that* $\pi(\alpha) \in Atm\big(F_t(X)\big)$*. Then,* $\pi(\alpha) \leq g0 \wedge h0$*.*

Proof. If $\pi(\alpha) \nleq g0 \wedge h0$, then $p\pi(\alpha) \neq 0$ or $f\pi(\alpha) \neq 0$. If $p\pi(\alpha) \neq 0$, then take β equal to $\tau^m fg0$ or $\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)$ as needed (see Theorem 14 and Theorem 16). It follows that

$$0 < \pi(\alpha) \wedge \beta < \pi(\alpha),$$

hence the result. In case that $f\pi(\alpha) \neq 0$, it is feasible to give a dual reason. □

7. Atoms of Free Temporal Algebras

In this section, we show an application of the construction presented above and its properties, namely, the exposition and counting of the atoms of any finitely generated free temporal algebra. This result is well known (see [7]), although our proof is different from the one given there. In fact, Corollary 10 provides a necessary condition for "being atom"; therefore, it is appropriate to select the atoms between the formulas that verify this condition. The technique is to give a particular bijection between the set of atoms under investigation and other that is well known. Schematically, the main idea of this last section is to split up the temporal algebra $\mathbf{F}_t(X)$ into two pieces, one of which provides no "atoms" and the other —a "trivialization" of $\mathbf{F}_t(X)$ that makes it practically a free Boolean algebra finitely generated— that provides all the atoms.

Our interest is now focused on free temporal algebras finitely generated. Let us denote the free temporal algebra of O (resp. W) over the non-empty set X by $\mathbf{F}_o(X)$ (resp. $\mathbf{F}_w(X)$). From the results obtained in Section 3, we can easily deduce these others. We have $\mathbf{F}_t(X) \cong \mathbf{F}_o(X) \times \mathbf{F}_w(X)$. It is straightforward to verify that $\mathbf{F}_o(X)$ (resp. $\mathbf{F}_w(X)$) is isomorphic to $\mathbf{F}_t(X)/F_0$ (resp. $\mathbf{F}_t(X)/F_1$). Moreover, $\mathbf{F}_t(X)/F_0$ is freely generated by $X/F_0 = \{x/F_0 \colon x \in X\}$. The temporal algebra $\mathbf{F}_o(X)$ is isomorphic to $\langle B(X), \wedge, \vee, \neg, k, k, 1\rangle$, where $\mathbf{B}(X) = \langle B(X), \wedge, \vee, \neg, 1\rangle$ is the free Boolean algebra freely generated by X, and $k\colon B(X) \to B(X)$ is the map defined by $k(a) = 1$ for all $a \in B(X)$. Let $n \in \omega^*, \sigma \in \{-1,1\}^n$, and $X = \{x_1, \ldots, x_n\}$; define $\xi_\sigma \in F_t(X)/F_0$ by the equality $\xi_\sigma = \bigwedge\{y_i^{\sigma_i} : 1 \leq i \leq n\}$, where $y_i = x_i/F_0$ and

$$\alpha^r = \begin{cases} \alpha, & \text{if } r = 1, \\ \neg\alpha, & \text{if } r = -1. \end{cases}$$

Then, $\mathbf{F}_o(X)$ has 2^n atoms and $Atm(\mathbf{F}_o(X)) = \{\xi_\sigma \colon \sigma \in \{-1,1\}^n\}$.

It is straightforward to show that if $\pi(\alpha)/F_0 \in Atm(\mathbf{F}_t(X)/F_0)$ then $g0 \wedge h0 \wedge \pi(\alpha) \in Atm\big(\mathbf{F}_t(X)\big)$. Moreover, if $\pi(\alpha)/F_0, \pi(\beta)/F_0 \in F_t(X)/F_0$ then it is clear that $\pi(\alpha)/F_0 \neq \pi(\beta)/F_0$ if and only if $h0 \wedge g0 \wedge \pi(\alpha) \neq h0 \wedge g0 \wedge \pi(\beta)$. Therefore, the mapping

$$\Psi\colon Atm\big(\mathbf{F}_t(X)/F_0\big) \longrightarrow Atm\big(\mathbf{F}_t(X)\big)$$

given by $\Psi(\alpha/F_0) = h0 \wedge g0 \wedge \pi(\alpha)$, for all $\pi(\alpha)/F_0 \in Atm\big(\mathbf{F}_t(X)/F_0\big)$, is well defined. Now, our goal is to show that Ψ is in fact a bijective map; for this, we will find an inverse.

Lemma 17. *Let* α *be an element of* $F(X)$*. If* $\pi(\alpha) \in Atm(\mathbf{F}_t(X))$*, then* $\pi(\alpha)/F_0 \in Atm(\mathbf{F}_t(X)/F_0)$*.*

Proof. Let us assume that $\pi(\alpha) \in Atm(\mathbf{F}_t(X))$. It follows from Corollary 10 that $\pi(\alpha) \wedge g0 \wedge h0 = \pi(\alpha)$. Since $\pi(\alpha)$ is an atom, it is different to 0; therefore, $\pi(\alpha) \wedge g0 \wedge h0 \neq 0$, so $\pi(\alpha)/F_0 \neq 0/F_0$. Then, suppose that $\pi(\beta)/F_0 \leq \pi(\alpha)/F_0$. It follows that $h0 \wedge g0 \wedge \pi(\beta) \leq \pi(\alpha)$, and this implies that either $h0 \wedge g0 \wedge \pi(\beta) = 0$ or $h0 \wedge g0 \wedge \pi(\beta) = \pi(\alpha)$. The first equality is equivalent to $\pi(\beta)/F_0 = 0/F_0$ and the second one to $\pi(\beta)/F_0 = \pi(\alpha)/F_0$. □

Theorem 17 and Corollary 11 (neither stated nor proved before) are the main results of this section. In Theorem 17, we take advantage of a well-known algebra to establish a bijection between its atoms and those of the finitely generated temporal algebra; this allows us to count its atoms and to know their form.

Theorem 17. *Let $n \in \omega^*$ and $X = \{x_1, \dots, x_n\}$ be a finite set with cardinality n. $\mathbf{F}_t(X)$ has 2^n atoms and $Atm\big(\mathbf{F}_t(X)\big) = \{\xi_\sigma \wedge h0 \wedge g0 \colon \sigma \in \{-1, 1\}^n\}$.*

Proof. Let us consider the mapping $\Upsilon\colon Atm\big(\mathbf{F}_t(X)\big) \to Atm\big(\mathbf{F}_t(X)/F_0\big)$ given by $\Upsilon\big(\pi(\alpha)\big) = \pi(\alpha)/F_0$, which is also well-defined, as indicated by Lemma 17. On the one hand,

$$\begin{aligned}(\Upsilon \circ \Psi)(\pi(\alpha)/F_0) &= \Upsilon\big(h0 \wedge g0 \wedge \pi(\alpha)\big) = \big(h0 \wedge g0 \wedge \pi(\alpha)\big)/F_0 \\ &= (h0 \wedge g0)/F_0 \wedge \pi(\alpha)/F_0 = 1/F_0 \wedge \pi(\alpha)/F_0 \\ &= \pi(\alpha)/F_0\end{aligned}$$

and, on the other hand, $(\Psi \circ \Upsilon)(\pi(\alpha)) = \Psi(\pi(\alpha)/F_0) = h0 \wedge g0 \wedge \pi(\alpha)$. Since $0 \leq h0 \wedge g0 \wedge \pi(\alpha) \leq \pi(\alpha)$ and $\pi(\alpha) \in Atm\big(\mathbf{F}_t(X)\big)$, then either $h0 \wedge g0 \wedge \pi(\alpha) = 0$ or $h0 \wedge g0 \wedge \pi(\alpha) = \pi(\alpha)$. The first case is impossible, since then we would have $\pi(\alpha)/F_0 = 0/F_0$ and so $\pi(\alpha)/F_0$ would not be in $Atm\big(\mathbf{F}_t(X)/F_0\big)$, in contradiction with Lemma 17. Therefore, $h0 \wedge g0 \wedge \pi(\alpha) = \pi(\alpha)$ and so $(\Psi \circ \Upsilon)(\pi(\alpha)) = \pi(\alpha)$, for all $\pi(\alpha) \in Atm\big(\mathbf{F}_t(X)\big)$. It follows that Ψ and Υ are mutually inverse and bijective mappings. Hence, the cardinality of $Atm\big(\mathbf{F}_t(X)\big)$ coincides with the cardinality of $Atm\big(\mathbf{F}_t(X)/F_0\big)$. From the definition of Ψ, it follows that $Atm\big(\mathbf{F}_t(X)\big) = \{\xi_\sigma \wedge h0 \wedge g0 \colon \sigma \in \{-1, 1\}^n\}$. □

The proof of Corollary 11 is a straightforward consequence of Lemma 17 and Theorem 17. It establishes the non-existence of atoms in the free time algebra when it is generated by a non-finite set and announces the non-existence of atoms in the free algebra of the variety W. Finally, it establishes that the free time algebra generated by a finite set is not atomic even if it has atoms.

Corollary 11. *Let X be a non-empty set. The following statements hold:*

1. *If X is infinite, then $\mathbf{F}_t(X)$ is atomless.*
2. *$\mathbf{F}_w(X)$ is atomless.*
3. *If X is finite, then $\mathbf{F}_t(X)$ is not atomic.*

Proof. For the first statement, let us assume that $Atm(F_t(X)) \neq \emptyset$; then, by Lemma 17, $Atm(\mathbf{F}_t(X)/F_0) \neq \emptyset$. But, $\mathbf{F}_t(X)/F_0$ is atomless whenever X is infinite. So, if X is infinite then $Atm(F_t(X)) = \emptyset$. For the second, we apply $\mathbf{F}_t(X) \cong \mathbf{F}_t(X)/F_0 \times \mathbf{F}_t(X)/F_1$ (see Theorem 7) and $F_w(X) \cong \mathbf{F}_t(X)/F_1$. If X is infinite, we apply that $\mathbf{F}_t(X)$ is atomless (the first statement) to obtain that $Atm\big(\mathbf{F}_w(X)\big) = \emptyset$; moreover, if X is finite then, from Theorem 17, we have $|Atm(\mathbf{F}_t(X))| = |Atm(\mathbf{F}_o(X))|$; hence, $Atm\big(\mathbf{F}_w(X)\big) = \emptyset$. For the third statement, the second makes impossible to be $\mathbf{F}_o(X) \times \mathbf{F}_w(X)$ atomic; therefore, $\mathbf{F}_t(X)$ is not atomic. □

8. Conclusions

In this paper, we have investigated how the element $g0 \wedge h0$ intervenes in the order of the temporal algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1\rangle$. We have studied the varieties of temporal algebras for which $[0, g0 \wedge h0]$ has extreme values, i.e., $\{0\}$ and A; this has provided the varieties W and O, respectively. This has allowed us to obtain that every temporal algebra is isomorphic to one, which is a product of a certain element of O and another in W; this is the content of Theorem 7. Subsequently, this result has been particularized to the free temporal algebra generated by X, and this has allowed one, in the application of the 7 section, to count atoms the contributed by the O factor (all) and those contributed by the W factor (nothing). We consider this approach to be one of the essential aspects of the original methodology we propose in our article.

As a second contribution, we have been able to obtain an original characterization of simple temporal algebras. The characterization we provide makes use of the operators L and M. The characterizing fact (see Theorem 9) is to be able, for each element of A other than 1 (resp. 0), to take it to 0 (resp. 1) by a (finite) iteration of the application of L (resp.

M). The study of simplicity in algebraic structures is a classical topic in universal algebra; in this case, we have been able to obtain a very satisfactory criterion due to the clarity of the information it provides and being arithmetical in nature.

On our way to investigate the role played by the element $g0 \wedge h0$, we have devised an unconventional product of temporal algebras that we have provisionally called skew-product (see Definition 6). It is based on finite temporal algebras and their previously selected atoms. The definition of the temporal operations in this "product" algebra is not componentwise, and in applications it must be tailored to the purpose; in our case, it was the study of the role of $g0 \wedge h0$.

Section 5 is very technical, but we highlight Lemma 15 in it. Also of interest are Corollary 8 and Lemma 16, where we establish the behavior of the construction with respect to the simplicity of the skew–factors.

Theorem 14 and Theorem 16 show the use of the product construction introduced above. They show how given a formula of the free temporal algebra satisfying certain conditions compatible with nonatoms, it is possible to take advantage of the properties of some conveniently constructed skew-product algebra in order to find a nonzero lower bound of it. This allows one to prove easily, as we have done, that every atom of the free temporal algebra is bounded by $g0 \wedge h0$. It is here that we see something reminiscent of the classical lifting technique.

Finally, as an application, we give another proof of Theorem 1.1 of [7] (p. 61) about the number of atoms of $\mathbf{F}_t(X)$, whenever X is of cardinality n (see Theorem 17). Specifically, the technique of proof is to give a particular bijection between the set of atoms under investigation and another well known finite set that we specify. As an original contribution, we add the information on atomicity provided by Corollary 11. Along the way, we have particularized the structure theorem we gave based on O and W to the case of the free temporal algebra, noting the key role it plays in the study of atomicity. As far as we know, the statement of Theorem 1.1 is the only known thing that has been included in this work; however, the proof that we give as an application of the previous constructions and the results are totally novel; in fact, what we present here is a technique.

We believe that we have introduced a tool that could be useful as part of a demonstration technique. In the future, we will investigate the role of this construction in the field of polymodal algebras (see [11,14,17,19]). However, in the short term, contributing to the study of order and atomicity in temporal varieties along the lines of [7] will be a high priority for us. In [22], we presented an approach to the study of independently introduced $\mathcal{D}_n$ varieties in [8]. There, we sensed the need to organize the ideas contained in this article before delving into varieties that contemplated the stabilization of the powers of L starting from a given one; in fact, when studying [8] we see the profuse use it makes of $g0 \wedge h0$, which has been deeply studied in this work. Note that an important point of connection between this work and [8] is in Theorem 8. We strongly believe that our general results could be used to study the finitely generated free algebra on the variety $\mathcal{D}_n$. On the other hand, Tomasz Kowalski has left his study on $\mathcal{D}_2$ so we could make progress in the subscript. In fact, as far as we know the current state of the problem is that left by those papers.

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Abbreviations

The following abbreviations are used in this manuscript:

iff	if and only if
def.	definition
induc hyp.	induction hypothesis
resp.	respectively
p.	page
pp.	pages

References

1. Goble, L. (Ed.). *Philosophical Logic*; Blackwell Publishers Ltd.: Chichester, UK, 2017.
2. Reichenback, H. *Elements of Symbolic Logic*; Macmillan: New York, NY, USA, 1947.
3. Mani, I.; Pustejovsky, J.; Gaizauskas, R. *The Language of Time: A Reader*; Oxford University Press: Oxford, UK, 2005.
4. Fisher, M.; Gabbay, D.; Vila, L. *Handbook of Temporal Reasoning in Artificial Intelligence*; Elsevier: Amsterdam, The Netherlands, 2005.
5. Fisher, M. Chapter 12 Temporal Representation and Reasoning. In *Handbook of Knowledge Representation*; Van Harmelen, F., Lifschitz, V., Porter, B., Eds.; Springer: Berlin/Heidelberg, Germany, 2008.
6. Bolc, L.; Szalas, P. *Time and Logic: A Computational Approach*; UCL Press: London, UK, 1995.
7. Bellissima, F. Atoms of tense algebras. *Algebra Universalis* **1991**, *28*, 52–78. [CrossRef]
8. Kowalski, T. Varieties of Tense Algebras. *Rep. Math. Log.* **1998**, *32*, 53–95.
9. García Olmedo, F.M.; Rodríguez Salas, A.J. A Structure Theorem for Free Temporal Algebras. *Math. Log. Quartely* **1995**, *41*, 249–256. [CrossRef]
10. Kracht, M. Even more about the lattice of tense logics. *Arch. Math. Log.* **1992**, *31*, 243–257. [CrossRef]
11. Wolter, F. A note on atoms in polymodal algebras. *Algebra Universalis* **1997**, *37*, 334–341. [CrossRef]
12. Bellissima, F. Atoms in Modal Algebras. *Zeitschr. Math. Log. Grund. Math.* **1984**, *30*, 303–312. [CrossRef]
13. Gardies, J.L. *La Logique du Temps*; Presses Universitaires de France: Paris, France, 1975.
14. Pakhomov, F.N. Linear GLP-algebras and their elementary theories. *Izv. Math.* **2016**, *80*, 1159–1199. [CrossRef]
15. Van Benthem, J. *The Logic of Time*, 2nd ed.; Springer-Science+Business Media, B.V.: Berlin/Heidelberg, Germany, 1991.
16. Burgess, J.P. Basic Tense Logic. In *Handbook of Philosophical Logic*; Gabbay, D.M., Guenthner, F., Eds.; Reidel Publishing Company: Dordrecht, The Netherlands, 1984; Volume 2, pp. 89–133.
17. Goldblatt, R. Algebraic polymodal logic: A survey. *Log. J. IGPL* **2000**, *8*, 393–450. [CrossRef]
18. Japaridze, G. The Polymodal Logic of Provability. In *Intensional Logics and Logical Structure of Theories*; 1988; pp. 16–48.
19. Bazhenov, N. Categoricity Spectra for Polymodal Algebras. *Stud. Log.* **2016**, *104*, 1083–1097. [CrossRef]
20. García Olmedo, F.M.; Rodríguez Salas, A.J. Temporal Algebras, Pretemporal Algebras, and Modal Algebras. A Relation between Time and Necessity. *Math. Log. Quartely* **1995**, *41*, 24–38. [CrossRef]
21. Burris, S.; Sankappanavar, H.P. *A Course in Universal Algebra*; Springer: Berlin/Heidelberg, Germany, 1981.
22. García Olmedo, F. Un Estudio Algebraico de las LóGicas Temporales. Ph.D. Thesis, University of Granada, Granada, Spain, 1994.