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ONTO-SEMIOTIC ANALYSIS OF THE EMERGENCE AND EVOLUTION OF FUNCTIONAL REASONING

ANÁLISIS ONTOSEMIÓTICO DE LA EMERGENCIA Y EVOLUCIÓN DEL RAZONAMIENTO FUNCIONAL

ANÁLISE ONTOSEMIÓTICA DA EMERGÊNCIA E EVOLUÇÃO DO RACIOCÍNIO FUNCIONAL

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ABSTRACT

Developing students' adequate functional reasoning requires paying attention to the design and planning of teaching from the first educational levels. This implies considering and progressively articulating the diversity of meanings of the function, attending to the generality and formalization levels that emerged in its historical evolution. In this paper, we review historical and epistemological studies on function using theoretical tools of the Onto-semiotic Approach to characterize different levels of functional reasoning. We interpret meaning in terms of systems operative and discursive practices related to solving types of problems. In line with previous research, we identify partial meanings of function (operative-tabular, operative-graphic, algebraic-geometric, analytic, arbitrary correspondence between numerical sets, and mapping between arbitrary sets) that should be part of the overall reference meaning in the planning and management of function teaching and learning processes. This study provides a complementary view of the multiple

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investigations that describe the phylogenesis of the concept of function in mathematics with a historical and epistemological approach.

Keywords: epistemology; history; function; mathematics education; onto-semiotic approach.

Resumen

Desarrollar un adecuado razonamiento funcional en los estudiantes requiere prestar atención al diseño y planificación de la enseñanza de las funciones desde los primeros niveles educativos. Esto supone considerar la diversidad de significados de la función y articularlos de manera progresiva, atendiendo a los niveles de generalidad y formalización emergentes en las etapas de su evolución histórica. En este trabajo revisamos estudios históricos y epistemológicos sobre la función utilizando herramientas teóricas del Enfoque Ontosemiótico para caracterizar distintos niveles de razonamiento funcional. En particular, aplicamos la interpretación del significado en términos de sistemas de prácticas operativas y discursivas relativas a la resolución de tipos de problemas. De acuerdo con investigaciones previas, identificamos significados parciales de la función (operatorio-tabular, operatorio-gráfico, algebraico-geométrico, analítico, correspondencia arbitraria entre conjuntos numéricos y conjuntista) que pueden ser considerados como parte del significado de referencia global en la planificación y gestión de los procesos de enseñanza y aprendizaje de las funciones. Este estudio aporta una visión complementaria de las múltiples investigaciones que describen la filogénesis del concepto de función en matemáticas con un enfoque histórico y epistemológico.

Palabras clave: epistemología; historia; función; educación matemática; enfoque ontosemiótico.

Resumo

O desenvolvimento de um raciocínio funcional adequado nos alunos requer atenção à conceção e planificação do ensino das funções desde os primeiros níveis de ensino. Isto implica considerar a diversidade de significados de função e articulá-los de forma progressiva, atendendo aos níveis de generalidade e formalização emergentes nas etapas da sua evolução histórica. Neste artigo, fazemos uma revisão dos estudos históricos e epistemológicos sobre função, utilizando ferramentas teóricas da Abordagem Ontossemiótica para caraterizar diferentes níveis de raciocínio funcional. Em particular, aplicamos a interpretação do significado em termos de sistemas de práticas operacionais e discursivas relacionadas com a resolução de tipos de problemas. Na linha de investigações anteriores, identificamos significados parciais de função (operatório-tabular, operatório-gráfico, algébrico-geométrico, analítico, correspondência arbitrária entre conjuntos numéricos e conjuntista) que podem ser considerados como parte do significado global de referência na planificação e gestão dos processos de ensino e aprendizagem de funções. Este estudo fornece uma visão complementar das múltiplas investigações que descrevem a filogénese do conceito de função em matemática com uma abordagem histórica e epistemológica.

Palavras-chave: epistemologia; história; função; educação matemática; abordagem ontosemiótica; educação matemática.

INTRODUCTION

The concept of function is fundamental not only in analysis, but also in the other mathematical areas.

> Functions are all around in mathematics and its applications, albeit labelled in various ways: mapping, transformation, permutation, operation, process, functional, operator, sequence, morphism, functor, automaton, machine, which are used according to needs and opportunities:
> *Function* is preferred if the set of values is numerical, *mappings* and *transformations* come from geometry but serve as well, with certain attributes added in algebraic structures such as *morphisms*, prefixed with certain prepositions or adjectives, *functors*, acting on morphisms; *permutation* is the term for a one-to-one mapping on itself, in particular, if studied in a group theory context; *operation* or *process* is the term used with certain simple standard functions (addition, root extraction). [1] (p. 496)

Several historical and epistemological research works [2-7] clarified the nature and emergence of functions. Other mathematics education studies tried to describe and explain students' difficulties in understanding the concept of function [8-12], analyze curricular orientations [13], and teacher training [14]. Educational research linking historical-epistemological aspects with psychological and instructional issues is scarcer. This research emphasized the different definitions that have characterized the historical development of the concept of function or the students' difficulties understanding these definitions. In other words, a conceptualist view of mathematics has predominated, forgetting the mathematical problems and practices that motivate the emergence and evolution of functional reasoning. This approach has also relegated the characterization of levels of its development from a historical-cultural perspective.

[1] unveils an overwhelming phenomenological variety of the meaning of functions in mathematics and draws attention to the need to inquire into the use, the "what for" of functions:

> Is function a name I can attach to all that fulfills certain requirements or rather a signal how to act in certain contexts? Does one call a thing a function in order to do something with it, and if so, what? [1] (p. 511)

In this paper, we apply the assumptions and theoretical tools of the Onto-semiotic Approach to Mathematical Knowledge and Instruction (OSA) [15-17] to analyze the emergence of the concept of function and to characterize the various partial meanings attributed to it. The pragmatist view of meanings and broadening the mathematical concept understood as definition towards its view as onto-semiotic configuration help to understand the complexity of mathematical knowledge, to explain learning difficulties, and to support instructional decisions. Likewise, the algebraization levels of mathematical activity elaborated within the OSA framework [18] can help identify levels of functional reasoning associated with the evolution stages of the concept.

What is done with functions and what are they for are central questions in reconstructing the function meanings proposed from the OSA since problem situations are the raison d'être, the reason for mathematical activity [19]. However, the operative and discursive practices carried out to solve problems of identification of dependencies between variables (covariation, correspondence), analysis, and prediction of behaviors involve various types of objects (linguistic, concepts, propositions, procedures, arguments) and processes (representation, translation, definition, enunciation, syntactic and analytical calculation) that must be considered.

We include in the construct of functional reasoning this broad view of mathematical activity, its motivation, and the use of the objects and processes involved. Our historical-cultural (epistemological) approach leads us to use the expression functional reasoning instead of functional thinking, usually referring to the subject's cognitive abilities and processes.

To follow, we describe the research problem, theoretical framework, and method. Secondly, we describe the partial meanings of the concept of function: operative-tabular, operative-graphic, geometric-algebraic, analytic, arbitrary correspondence between numerical sets, and mapping between arbitrary sets. We then include the articulation in a global vision or holistic meaning and highlight the implications for mathematics education of the onto-semiotic model of functional reasoning.

PROBLEM, THEORETICAL FRAMEWORK AND METHOD

PROBLEM

The questions on functional reasoning we address in this paper are: 1) How has functional reasoning evolved in the different historical stages? 2) What meanings have been attributed to the concept of function? 3) How are such meanings distinguished according to the level of generality and formalization? 4) What educational implications are derived from our global vision of function and functional reasoning?

To answer these questions, we adopt a theoretical framework that provides tools to analyze mathematical activity and the various types of objects and processes involved in it. It must assume the plurality of meanings for mathematical constructs and provide criteria for identifying different generality and formalization levels of mathematical activity. As explained in the following section, the OSA offers the assumptions and tools necessary for this type of analysis.

The present study complements other work based on the OSA framework that characterize institutional meanings [20-23], and algebraization levels of mathematical activity [24,18].

THEORETICAL FRAMEWORK

To answer essential educational mathematics questions, such as what knowledge is or how learning occurs, the OSA introduces the constructs: mathematical practices, mathematical objects and processes, and contextual attributes of practices and objects [15]. These theoretical elements are articulated in the onto-semiotic configuration of practices, objects, and processes (Figure 1) through which mathematical activity can be analyzed, distinguishing different levels for such activity and different meanings for the mathematical objects involved. The articulation of mathematical knowledge's epistemic and cognitive facets is achieved in OSA by attributing a double character, personal (idiosyncratic of an individual) or institutional (shared within a community) to mathematical practices.

Mathematics as an activity

People's activity when solving problems in an ecological context (physical, biological, and social) is the central element in constructing of mathematical knowledge. The problems, which are the origin or motive of mathematical activity, can be extra-mathematical, thus involving material things, objects, and facts, or intra-mathematical, involving non-material or ideal objects.

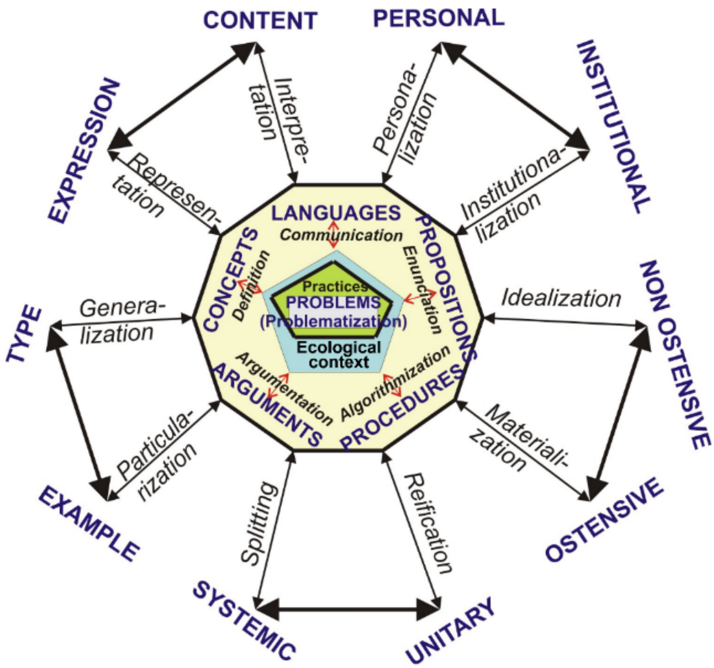


Figure 1. Onto-semiotic configuration of practices, objects, and processes [15]

Mathematics as a system of objects and processes

Mathematics cannot be understood simply as an activity of people but also as a system of culturally shared objects emerging from this activity. In OSA, mathematical practices, that is, the actions performed by people to solve certain types of problem situations, are the origin and raison d'être of mathematical abstractions, ideas, or objects [25]. The constitution of linguistic objects, problems, definitions, propositions, procedures, and arguments (Figure 1) occurs through the primary mathematical processes of communication, problematization, definition, enunciation, elaboration of procedures (algorithmization, routinization), and argumentation.

Mathematics as a system of signs

The different objects are not isolated entities but are placed in relation to each other. For example, between the symbol 2 and the concept of number 2, and between the concept of natural number and the system of operative and discursive practices from which this mathematical object emerges, a relationship is established that OSA calls a semiotic function. The semiotic function is the correspondence between an antecedent object (expression/signifier) and a consequent object (content/meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. We reflect the semiotic function in Figure 1 through the expression-content duality, which accounts for any use given to meaning: meaning is the content of a semiotic function [19].

Idealization, reification, and generalization according to OSA. Contextual dualities

The OSA introduces three pairs of contextual attributes to analyze the idealization, reification, and generalization processes, from which practices and primary objects can be considered: ostensive-non ostensive (material, immaterial), unitary-systemic, and extensive-intensive (particular-general). These contextual dualities make it possible to describe the types of abstraction (empirical and formal) at play in mathematical activity and the objects that intervene and emerge in these processes.

In OSA, ostensive is any object that is public and can be shown directly to another. Symbols, notations, gestures, graphic representations, and material artifacts have this character; they are real or concrete objects. Concepts, propositions, procedures, and arguments are constructs, creations of the human mind, non-ostensive objects; they depend on subjects, their actions, and artifacts for their existence. This duality allows us to account for the dual processes of idealization and materialization in mathematical activity.

In some circumstances, mathematical objects participate as unitary entities (assumed to be previously known), while others intervene as systems that must be unpacked for their analysis. Both onto-semiotic configurations (in their double socio-epistemic or cognitive version) and the primary objects that compose them can be considered from unitary or systemic perspectives, depending on the language game [26] in which they participate. In the first case, processes of reification (synthesis) occur, and in the second, a system breaks down into its components (analysis).

A characteristic feature of mathematical activity is the attempt to generalize the types of problems addressed, the solution procedures, definitions, propositions, and justifications. Solutions are organized and justified in progressively more general structures. However, in the instructional processes, one begins to study models of these general structures. The analysis of mathematical activity requires, therefore, to consider both processes, particularization, and generalization, and the objects involved in these processes, which are called in OSA extensive (particular) and intensive (general) objects. The generalization process consists of finding a pattern from similar cases, while particularization consists of generating or showing individual exemplars that follow a pattern.

Processes of abstraction and abstract objects in OSA

In a first approximation, the ostensive-non ostensive duality and the associated processes of materialization and idealization explain the concrete (ostensive) and abstract (ideal) objects usually considered in everyday language. However, the analysis of mathematical activity, from both a professional and an educational point of view, requires a deeper understanding of the nature of the abstraction process, the emerging abstract objects, and the inverse interpretation process. For this reason, OSA complements the ostensive-non ostensive duality with the unitary-systemic and example-type dualities: a mathematical abstract object is not only an ideal (non-ostensive) entity but also a generality, considered as a unitary whole or as a system, depending on the situation.

In the epistemic analysis of the concept of function, it is necessary to identify, in addition to the definitions used, the various elements indicated in Figure 1, which are mobilized to respond to the problems in which the object function participates in a determinant way. However, it may be implicitly in the first stages of its emergence. Each of these configurations is a partial pragmatic meaning of the function object that synthesizes the mathematical activity to solve specific problems in certain contexts or historical periods. The evolution of the concept implies a sequence of configurations through which definitions, procedures, properties, and arguments are generalized, passing from the use of ordinary, tabular, and graphic language to alphanumeric expression and from arithmetic to algebraic and analytical calculus. In this way, we analyze the evolution of the mathematical activity that today we call functional reasoning, which is undoubtedly a fundamental piece of mathematics architecture.

ALGEBRAIZATION LEVELS OF MATHEMATICAL PRACTICES

Within the OSA framework, we proposed a model to characterize the elementary algebraic reasoning (EAR) involved in Primary and Secondary mathematics, with six levels of algebraization [18]. Such levels consider the types of representations used, the degree of generality of the intensive objects involved, and the analytical computation done with such objects, which indicates the onto-semiotic complexity at stake.

Natural numbers are intensive objects (general, abstract entities) that emerge from collections of perceptual objects and the actions performed with them [18]. Therefore, they are assigned degree 1 of intension or generality, with degree 0 corresponding to material or ostensive objects. A new layer or generality degree 2 occurs when considering collections or sets of intensive objects of degree 1, and so on. In this way, the universe of mathematical objects is structured in increasing degrees of intension.

In this paper, we interpret and adapt the EAR levels model to analyze functional reasoning in different historical stages. In the EAR model, the first two levels are considered proto-algebraic since the language used to express unknowns or equations must be alphanumeric or operate with the intensive objects represented. In the third level, the mathematical activity with the unknowns is represented symbolically. In the case of functional reasoning (FR), we distinguish two first levels of proto-functional reasoning (Levels I and II), which include problems relating two or more variables to make forecasts or calculations. Still, the function object is implicit and represented with natural, numerical, or graphical language. In Level III, algebraic language begins to express relationships, although the function is restricted to the geometric (study of curves) or kinematic magnitudes.

The fourth and fifth levels in the EAR model describe the more general and abstract mathematical activity when parameters are used to indicate families of functions (Level 4) or when operating with parameters (Level 5). In FR, we use these two levels to describe two periods of mathematical activity in which functions are treated explicitly and represented analytically (Level IV), or the function is defined as an arbitrary correspondence between numerical sets (Level V). The sixth level in the EAR model studies abstract algebraic structures such as vector spaces, groups, etc. This level VI is appropriate for functional reasoning concerning the set definition of functions, applications, or correspondences between arbitrary sets.

METHOD

This research is a documentary study based on the selection of texts describing the emergence of the concept of function in different historical stages and epistemological analyses. The selected sources (Table 1) were compiled with a systematic search in databases (Web of Science, Scopus, Google Scholar) to identify the elements that characterize the partial meanings of function and the levels of functional reasoning.

Table 1. *Selected documentary sources*

Theme	Sources
History	[27]; [28]; [29]; [30]; [31]; [32]; [33]; [34]; [35]; [36]; [37]
Epistemology	[1]; [3]; [4]; [5]; [6]; [7]; [10]; [38]; [39]; [40]; [41]; [42]

Source: elaborated by the authors

Our analysis of the historical trajectory of function is inscribed in the anthropological [43] and ecological [44-46] styles of epistemological reasoning. It is based on OSA ontology and semiotics and thus assumes the "practice turn" [47] in philosophy and the history of science.

PARTIAL MEANINGS OF THE FUNCTION CONCEPT. GENERALITY LEVELS OF FUNCTIONAL REASONING

We distinguish six stages in the historical evolution of functional reasoning by considering the types of problems addressed and the mathematical activity performed. Likewise, we identify six levels of generality of functional reasoning when considering the algebraic activity in each stage[4].

STAGE I (ANTIQUITY). OPERATIVE-TABULAR MEANING

[4] We recognize that the six-stage, six-level model for functional reasoning elaborated here can be developed, either by distinguishing sublevels, or additional stages beyond the set definition of function. The aims of the epistemological or cognitive analysis addressed may justify this enlargement.

Solving problems of predicting unknown quantities by tabulating known data appears in the earliest historical records of mathematical activity in Babylonia and Egypt (2000 BC). Babylonian mathematicians widely used in their calculations sexagesimal tables of reciprocals, squares and square roots, cubes, and cube roots. Babylonian astronomers employed different types of tables to calculate ephemeris of the sun, moon, and planets.

> Despite the large gaps in their exponential tables, Babylonian mathematicians did not hesitate to interpolate by proportional parts to approximate intermediate values. Linear interpolation seems to have been a commonplace procedure in ancient Mesopotamia, and the positional notation lent itself conveniently to the rule of three. A clear instance of the practical use of interpolation within exponential tables is seen in a problem text that asks how long it will take money to double at 20 percent annually. [33] (pp. 27-28)

The Greeks did not limit themselves to using tables to express relationships between variable quantities. In ancient Greece, functions introduced in connection with mathematical and astronomical problems were subjected to studies like those carried out in the mathematical analysis of modern times. Depending on the objective pursued, tabulated functions were interpolated (linear interpolation) and, in the simplest cases, the limits of quotients of two infinitely small quantities were found as, for example, the limit of $\frac{sen\,(x)}{x}$ when $x \rightarrow 0$. Problems about extreme values and tangents were solved by procedures equivalent to the differential method; areas, volumes, lengths, and centers of gravity were calculated by equivalent methods to the calculation of integrals [7] (p. 41).

There are features in the mathematical work carried out by the Babylonians, Egyptians, and Greeks indicating the implicit handling of general rules. They did not reduce themselves to a simple tabulation of empirical data but also made interpolations and extrapolations suggesting the recognition of intensive objects with a certain degree of generality. In short, we do not find the function as we now use it, but there are characteristic elements of functional reasoning.

As a synthesis, in this Stage I (Antiquity), problems of calculating quantities of magnitudes from the relation of dependence with other magnitudes were deal with. These applications were mainly extra-mathematical (astronomy or land measurement) and intra-mathematical (tables for calculating squares, cubes, square roots). Procedures were elaborated, and properties were identified with a first degree of generality, so we qualify this activity as proto-functional of level I.

STAGE II (MIDDLE AGES). OPERATIVE-GRAPHIC MEANING

In the 14th century, mathematicians of the Oxford (Heytesbury; Swineshead) and Paris (Oresme) schools made progress in solving geometrical and kinematic problems with diverse procedures, particularly graphical, involving dependence relations between variables. For Oresme, qualities or forms are phenomena such as heat, light, color, density, distance, velocity, etc., which may possess varying degrees of intensity and which, in general, change continuously within given limits. These mathematicians considered intensities of forms such as, for example, the amount of matter, time, etc., concerning their extensions. During such considerations, several concepts were introduced, e.g., instantaneous or point velocity, acceleration, and variable quantity conceived as a degree or flow of quality. "In all this, a dominant role was played by a synthesis of kinematic and mathematical thought” [7] (p. 45).

Oresme studied the phenomenon of uniformly accelerated motion and worked out a solution considered the first graphical representation of physical laws. This representation seems to indicate that Oresme "have grasped the essential principle that a function of one unknown can be represented as a curve, but he was unable to make any effective use of this observation except in the case of the linear function” [33] (p. 240).

By way of summary, in Stage II (up to the 14th and 15th centuries), the study of concrete cases of dependencies between two magnitudes continues. Ordinary, numerical, and tabular language complements graphical language. Abstract concepts (intensive objects) are introduced, such as instantaneous velocity, acceleration, and variable quantity, conceived as a degree or flow of quality (empirical abstractions). We qualify this activity as proto-functional of level II.

STAGE III (MODERN PERIOD). GEOMETRICAL-ALGEBRAIC MEANING

In the 17th century, further progress in mathematics took place with a high impact on the development of functional reasoning, in particular, the creation of symbolic algebra together with the extension of the concept of number, which by the end of the 16th century encompassed not only the whole field of real numbers but also complex numbers [7] (p 50). These advances were necessary for introducing the concept of function as a relation between sets of numbers instead of "quantities" and for the analytical representation of functions using formulas. In the first half of the 17th century, there were relevant changes in mathematical activity, such as the invention of analytic geometry (Descartes, Fermat), which meant associating an algebraic equation with curves. In this way, the analytical aspects of curves were given priority over geometrical ones. Likewise, the study of motion (Kepler, Galileo) led to the enunciation of physical laws expressed as dependence between variable quantities.

Despite these advances, the calculus developed by Newton and Leibniz was not a calculus of functions. The main objects of study in 17th-century calculus were (geometric) curves. 17th-century analysis originated as a collection of methods for solving problems over curves (such as finding tangents, areas, lengths under curves, and velocities of points moving along curves). Since the problems that gave rise to calculus were geometric and kinematic, it would take time and thought to reformulate calculus in algebraic form [3].

The emergence of analytic geometry was a necessary preliminary step for the emergence of the construct function as a mathematical object. A procedure was available to create an infinity of curves, as Fermat did in the early 17th century with the infinite family of parabolas and hyperbolas ($y = kx^n$; $k > 0$, $n > 0$). However, equations between variables do not assume the use of functions unless there is explicit identification of the independent and dependent variables [5] (p. 127). Considering the types of problems addressed and the use of algebraic resources we qualify this mathematical activity as level III functional reasoning. Although the construct function has not been formulated, there are substantial differences concerning the proto-functional levels of the two previous stages.

Summarizing, in Stage III (Modern Period, XVI and XVII centuries), algebraic expressions began to prevail to express the relationships between geometric and kinematic quantities. Although the main focus of the works of Descartes, Newton, and Leibniz, among others, is the study of curves, the construct function begins its explicit emergence, which is why we assign a level III of functional reasoning to this stage.

Stage IV (18th century). Analytical meaning

During the eighteenth century´s first decades, calculus was gradually detached from its geometrical origin. The algebraic apparatus developed by Newton and Leibniz was augmented and exploited by their successors to solve problems not directly related to the geometry of curves. The formulas relating variables and their differentials began to take on their own life, independent of the geometrical objects they represented.

Leibniz and Johann Bernoulli searched for a concept to express this new reality and, finally, came up with the idea of function, a concept that had not been necessary in the previous stages. Although Leibniz first used the term function, it was J. Bernoulli who formulated an explicit statement of the concept in 1718:

> One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants. [37] (p. 72).

It took several decades for calculus to merge in algebraic terms with the concept of function as a centerpiece, thanks mainly to Euler and his influential textbooks of the mid-18th century. Euler turned the 17th-century calculus of variables and equations into a calculus of functions. Euler proposed this definition in 1748:

> A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities. [37] (p. 72)

For Euler, an "analytical expression" was an algebraic formula generated from algebraic and transcendental functions (i.e., polynomials, trigonometric, inverse trigonometric, exponential, and logarithmic functions) using the four algebraic operations plus the composition of functions and the taking of n-th roots.

Once the function object was created and linked to a specific form of representation, namely analytical expressions, new problems arose related to specific forms of these expressions and the study of the properties of the various emerging functions. Power series became a fundamental tool of calculus in the 18th century to the point that Euler stated that every function can be developed into a power series. Another way of producing functions and studying their properties took place through their expression in the form of integrals, for example, the logarithmic function $\int_1^x \frac{1}{t} dt$. Likewise, from any continuous function $f(x)$, one can define another function $F(x) = \int_1^x f(t)dt$.

Considering the function as an object allows us to introduce it as an unknown in equations or an argument in new functions. In a certain way, one operates with functions. In particular, differential equations involving an unknown function and one or more of its derivatives determine mathematical models of various physical phenomena. The development of methods to solve this type of equation constitutes an extensive and rich branch of mathematical analysis. The solution of equations in partial derivatives, such as the cases of the wave function and heat, motivated the extension of the concept of function, initially given by Euler as an analytical expression, i.e., a unique algebraic formula. As a result of the debate, the concept of function enlarged to include expressions given by various formulas and freely drawn curves. Euler himself modified his initial definition of function linked to analytical expressions in the following terms:

> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions. [37] (p. 72-73)

With Euler, the object function, which had previously intervened in mathematical activity in an operative or implicit way assumed as a new entity with different types and properties. Functions can be algebraic or transcendent, univalued or multivalued, implicit, or explicit, continuous, or discontinuous. The developments in power series constitute an essential mode of expression and treatment in the study of the formal properties of the new construct. Euler enunciated a controversial proposition that would be the starting point of later reflections: "We can develop any function as a power series".

Thus, in the first decades of the 18th century (Stage IV), the algebraic apparatus developed by Newton and Leibniz was augmented and exploited by their successors to solve problems not directly related to the geometry of curves. The concept of function, linked to its analytical expressions with Euler, was consolidated as the core construct of the pure or formal mathematical activity that characterizes mathematical analysis. Problems of classification of the types of functions and their properties (continuity, derivability, etc.) arose.

The type of mathematical activity performed in this historical stage, in which the explicitly defined construct operated to produce new functions, intervenes centrally, leads us to assign level IV to the functional reasoning of this stage.

STAGE V (19TH CENTURY). ARBITRARY CORRESPONDENCE BETWEEN NUMERICAL SETS MEANING

Fourier's work on another physical problem, heat conduction, led to emphasize the role of analytical expressions in the conceptualization of the function by formulating the following theorem:

Any function *f(x)* defined over (-*l, l*) is representable over this interval by a series of sines and cosines,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \operatorname{sen}\left(\frac{n\pi x}{l}\right)] \quad (1)$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\left(\frac{n\pi t}{l}\right) dt;\ b_n = \frac{1}{l}\int_{-l}^{l} f(t)\operatorname{sen}\left(\frac{n\pi t}{l}\right) dt \quad (2)$$

In the statement of his theorem, Fourier considered that the function f(x) represents a succession of values or ordinates, each of which is arbitrary:

In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa *x*, there are an equal number of ordinates $f(x)$. All have actual numerical values, either positive, or negative, or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity. [37] (p. 73)

This generality of the theorem is incorrect, as several later mathematicians showed. For a correct formulation and proof of Fourier's theorem, it was necessary to introduce clear notions of continuity, convergence, and definite integral, as well as a clear understanding of the concept of function [5]. This goal was achieved by Dirichlet, relying on works by Gauss, Abel, and Cauchy:

Theorem 1: *If a function f has only a finite number of discontinuities and a finite number of maxima and minima at (-l; l), then f can be represented by its Fourier series at (-l, l). The Fourier series converges pointwise to f where f is continuous and to [f(x+) + f(x-)]/2 at every point x where f is discontinuous.*

Dirichlet is credited with the separation of the concept of function from analytical expressions and its consideration as an arbitrary correspondence between real numbers:

Let us suppose that *a* and *b* are two definite values and *x* is a variable quantity which is to assume, gradually, all values located between *a*

> and *b*. Now, if to each *x* there corresponds a unique, finite *y* in such a way that, as *x* continuously passes through the interval from *a* to *b*, *y = f(x)* varies likewise gradually, then *y* is called a continuous function of *x* for this interval. It is, moreover, not at all necessary, that *y* depends on *x* in this whole interval according to the same law; indeed, it is not necessary to think of only relations that can be expressed by mathematical operations. Geometrically represented, i.e. *x* and *y* imagined as abscissa and ordinate, a continuous function appears as a connected curve, for which only one point corresponds to each abscissa between *a* and *b*. [37] (p. 74)

This attribution is also possibly due to the use and the example of function he proposed, the so-called Dirichlet function: D(*x*), which takes the value *c* if *x* is rational and the value *d* if *x* is irrational. The Dirichlet function, introduced in connection with the representativeness of a function by the Fourier series, reflects a new way of understanding the function object as an arbitrary correspondence between numerical sets. It is not a function given by an analytical expression, nor can it be represented by a curve. It is a new type of function, the first of many "pathological functions". The domain and range in Dirichlet's definition of function are sets of real numbers.

As a synthesis, we note that in the middle of the 18th century, the interpretation of functions as analytical expressions proved inadequate. During the same period a new general definition of function was introduced, which later became universally accepted in mathematical analysis: The function as an arbitrary correspondence between elements of numerical sets (Dirichlet). A new generalization of the concept of function took place, which we interpret as level V of functional reasoning.

STAGE VI. CORRESPONDENCE BETWEEN ARBITRARY SETS MEANING

Although Dirichlet's broad conception of a function as an arbitrary correspondence (between elements of numerical sets) prevailed for much of the 19th century, signs of dissatisfaction began to appear towards the end of that century [4]. In the early 20th century, the intuitionist and formalist schools of mathematical philosophy debated questions of the existence of mathematical objects. Applied to functions, for example, let *f(x)* be defined by *f(x)*=1 if *x* is a positive integer and there are *x* successive zeros in the decimal expansion of π; otherwise, *f(x)*=0. Does *f(x)* exist? Is it well-defined? While formalists would answer in the affirmative, intuitionists would take the opposite view. For them, *f(x)* is not a bona fide function since we cannot determine its values for all *x*-values in the domain. For example, what is f(99)? We do not know whether f(99)=1 or *f(x)*=0 because we do not know, and may never know, whether there are 99 consecutive zeros in the decimal expansion of π. Therefore, for intuitionists, this function is meaningless [4] (p. 205).

The notion of function as a correspondence between arbitrary sets gradually took hold in twentieth-century mathematics. Algebra impacted this development by placing the function in the general framework of application from one set to another. As early as 1887, Dedekind gave a very modern definition of the term "mapping" (application):

> By a mapping of a system S a law is understood, in accordance with which to each determinate element *s* of S there is associated a determinate object, which is called the image of *s* and is denoted by *φ(s)*; we say, too, that *φ(s)* corresponds to the element *s*, that *φ(s)* is caused or generated by the mapping *φ* out of *s*, that *s* is transformed by the mapping *φ* into *φ(s)*. [37] (p. 75)

The meaning of the function as an arbitrary correspondence can refer both to the character of the functional relation and to the values of the variables that span the domain and codomain, which can be numbers, tuples of numbers, points, curves, functions, permutations, elements of arbitrary sets.

> The functions of analysis, the geometric transformations, the permutations of finite sets and the mappings of arbitrary ones flow together, in order to generate the general function concept. This concept is used to comprise a great variety of things: algebraic operations, functionals, operators, even sequences, coordinates, logical predicates. [1] (p. 528)

This extension of the concept of function domain and range to other arbitrary sets took place gradually in the nineteenth century, although, implicitly, it was previously present in various applications and contexts: maps of the earth are functions of the sphere in the Euclidean plane; the derivative, as an operator, is a function with domain the set of differentiable functions and range the set of all functions; truth tables are functions with domain a set of statements and range the set {T, F}. Functions appear as transformations in geometry, as homomorphisms in algebra, and as operators in analysis, with domains given by Euclidean spaces, groups or rings, and sequences or function spaces, respectively.

The standard definition of function today is firmly based on set theory:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element *x* of E and a variable element *y* of F is

> called a functional relation in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x. We give the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given relation. Two equivalent functional relations determine the same function. [37] (p. 77)

At this stage, the set-based definition of function was introduced, and the function became the backbone concept of the architecture of mathematics. This level VI of functional reasoning is therefore characterized using abstract algebraic structures and functional and topological spaces.

OTHER MEANINGS OF THE FUNCTION CONCEPT

A pragmatic understanding of a concept meaning, as systems of operative and discursive practices involved in the solution of types of problems, it helps identify new meanings linked to subtypes of problems. Moreover, bearing in mind that functions are pervasive in mathematics, named in various ways [1] (p. 496), we need to characterize the specific meanings of the constructs referred to by terms such as application, transformation, operation, functional, operator, morphism, functor, etc. The various educational contexts are sources for identifying varieties of meanings of functions. As Freudenthal states,

> In mathematical instruction functions have moved downwards from Calculus, via graphs and supported by equations to the primary school, even to its lower grades, where they are concretised by imaginary machines and expressed and symbolised in table and arrow language. [1] (p. 528)

[3] (p. 16), following [48], considers as a development of the concept of function the use made in functional analysis of Hilbert spaces L2, the set of square integrable-Lebesgue functions. Two functions in L2 are identical if they coincide everywhere except possibly on a zero Lebesgue measure set. Thus, in function theory, one can always work with representatives of an equivalence class rather than individual functions.

> It is interesting to observe that this modern development really involves a further evolution of the concept of function. For an element in L_2 is not a function, either in Euler's sense of an analytic expression,

> or in Dirichlet's sense of a rule or mapping associating one set of numbers with another. It is function-like in the sense that it can be subjected to certain operations normally applied to functions (adding, multiplying, integrating). But since it is regarded as unchanged if its values are altered on an arbitrary set of measure zero, it is certainly not just a rule assigning values at each point in its domain. [48] (p. 293)

According to [3], a new meaning of function arose in the context of category theory. This theory emerged in the late 1940s to express formally certain aspects of homology theory; in it, the concept of function assumes a fundamental role. The concept is described as an "association" of one "object" A with another "object" B. Objects A and B do not need to have elements (i.e., they do not need to be usual sets). A category consists of arrows (or "maps"), understood as undefined (primitive) concepts that satisfy certain relations or axioms [49].

FUNCTION HOLISTIC MEANING

The diagram in Figure 2 summarizes the evolution of the concept of function and the levels of functional reasoning. We remark that with the appearance of explicit definitions of function (J. Bernoulli, Euler), the ontological nature of the concept and the type of activity it engaged in changed substantially. Just as in ontogenetic development, as proposed by theories of cognitive development (Piaget, Dubinski, Sfard), there was a transition from the operational, procedural stage to the objectual stage, where the concept becomes part of cognitive schemes that enable the individual to comprehend, make decisions, and act in similar situations.

In phylogenetic terms, the function became part of the body of mathematical objects, such as numbers, geometric figures, and equations. Various types of functions were invented to model various phenomena, and their specific properties were studied (continuity, differentiability, etc.), which allowed the definition of new functions and played a role in a new ecological niche characterized by formalization, generalization, and rigor.

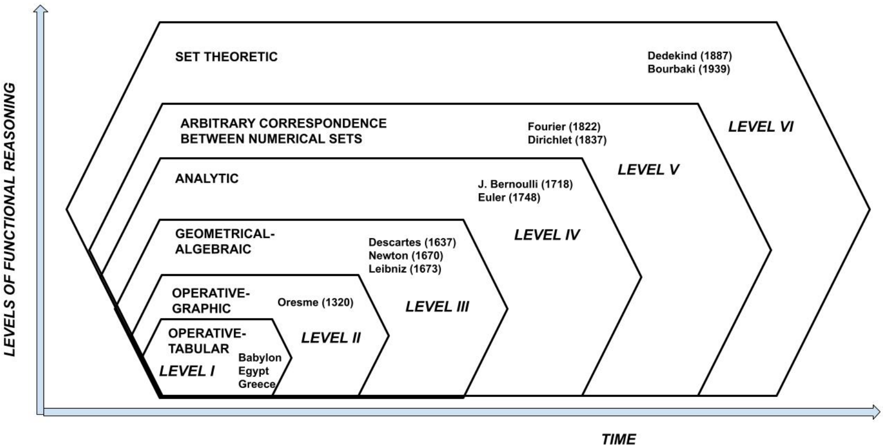


Figure 2. Evolution of the function concept. Levels of functional reasoning
Source: elaborated by the authors

This historical evolution of the function concept reflects the tendency or attitude of mathematicians to generalize concepts and procedures to solve increasingly complex and general problems. This happened because of the "practical necessity for unifying by means of underlying general principles those aspects of numerous theories that promise to be more than transitory interest" [29] (p. 470). Thus, the formulation of the function in terms of correspondence between the elements of sets according to arbitrary criteria, not necessarily using analytical expressions, responds to the need to account for work with functions that could not be drawn or expressed algebraically, such as the Dirichlet function. Another qualitative leap is the use of structural algebraic language in the study of functions, which fundamentally deals with conserving the structures resulting from applying morphisms (structure-preserving functions).

As [1] showed, there is a varied phenomenology involving the object function, which, together with diverse forms of expression, procedures, propositions, and arguments, characterizes functional reasoning. Is it possible to identify some common feature that justifies using the same term function to name such a variety of meanings? The ideas of dependence, covariation, and prediction are the nexus that connects the first three meanings or uses of functions (Figure 2). We can express such dependence tabularly, graphically, or analytically, but in any case, variable elements of numerical sets are related to other numbers. The ideas of variability and dependence do not appear in the set-based meaning, which is more general and abstract than the previous ones; however, the connection or correspondence between objects based on some rule or criterion persists.

In Figure 3, we show an example of the progressive generalization of the function, indicating that different species of the intensive object function are involved in the progressive sequence of representations and the mathematical activity involved in its use.

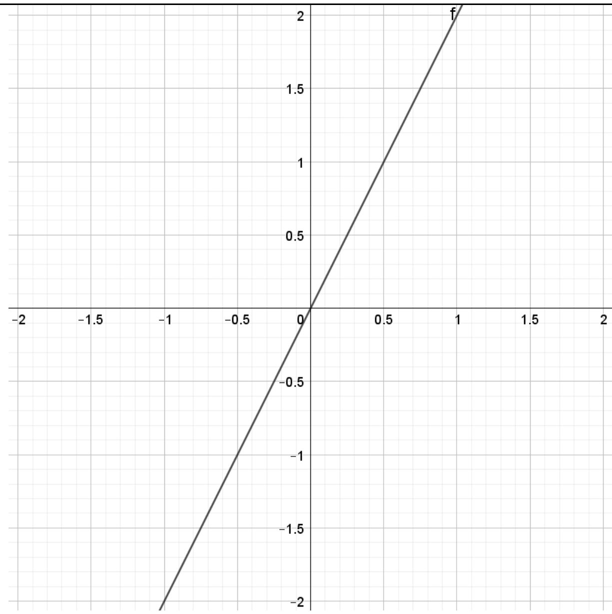
<table>
<tr><th>Level</th><th>Representation</th><th>Intensive</th></tr>
<tr><td>I</td><td><table><tr><td>Variable 1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>Variable 2</td><td>0</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr></table></td><td>Of 1st especie</td></tr>
<tr><td>II</td><td>
</td><td>Of 2nd specie</td></tr>
<tr><td>III</td><td>$y = 2x, x \in (-\infty, \infty)$</td><td>Of 2nd specie</td></tr>
<tr><td>IV</td><td>$y = ax, a \in \mathbf{R}, x \in D \subseteq (-\infty, \infty)$</td><td>Of 3rd specie</td></tr>
<tr><td>V</td><td>$y = f(x), x \in C, f(x) \in C'$,
with C y C' numerical sets</td><td>Of 4th specie</td></tr>
<tr><td>VI</td><td>$(y \sim f(x), A, B), x \in A, f(x) \in B$,
with A y B arbitrary sets</td><td>Of 5th specie</td></tr>
</table>

Figure 3. Levels of functional reasoning and species of intensives
Source: elaborated by the authors

At level I, the tabular representation indicates the use of finite collections of particular natural numbers, so the tabular representation of the function is an intensive of the 1st species. At level II, the continuous graph indicates the presence of intervals of real numbers in which the functional relation can be interpolated and extrapolated to any numbers, which is why we interpret it as a 2nd species intensive of the function concept.

At level III, the symbolic representation expresses the same level of generality and is therefore also classified as 2nd species intensive. At level IV, the presence of the parameter refers to a family of functions, implying the increase in the degree of generality and, therefore, in the function specie. At level V the increase in the species of the intensive derives from the change in the generality of the domain and range of the function, which become any numerical sets, and the expression is not necessarily analytic. The 5th specie of the intensive of level VI comes from considering a new generality in the type of relation and the nature of the correspondence domain and range.

IMPLICATIONS FOR MATHEMATICS EDUCATION

The analysis of the holistic meaning of function is epistemological and has revealed the diversity of senses or partial meanings it has taken on in different contexts and historical moments. In terms of the ecology of meanings [50, 16], we have tried to identify the ecological niche and the role that the function, in its different varieties or species, has been playing in mathematics, understood as a human activity and as a system of historical and cultural objects. We aimed to identify the reason or motive for the evolution of these species and the common characteristics that lead to speaking of the genus function.

In the OSA framework, this type of study must be previous to posing specific mathematics education problems. We cannot decontextualize the global analysis of mathematical instruction processes because they are consubstantial to the institution and time [51]. It is necessary to describe a global meaning [52] that allows addressing issues:

- Relative to the transformations and adaptations that mathematical knowledge needs at the various educational levels.
- On students' learning, in particular, their difficulties and levels of knowledge and understanding.

- On designing instructional processes with the maximum didactic suitability for the different educational contexts.

In OSA we understand learning as the progressive appropriation of the implemented institutional meanings by the students. The implemented meanings should be based on previous planning, which implies the informed selection of specific aspects of the proposed content. This process requires the prior reconstruction of a local reference meaning, that is to say, adapted to the context. The realization of this process of ecological adaptation requires the educational agents (curriculum, teachers, authors of didactic materials) to start from a global or holistic meaning of the teaching content so that a representative and well-founded selection of the planned and implemented knowledge can take place, as well as adequate evaluation of learning.

The holistic meaning model will help to relativize understanding and be aware of the complexity of practices, objects, and processes to consider in the progressive development of functional reasoning. History informs us when, why, and in what form the concept of function arose in mathematics and the reasons for its progressive generalization and formalization. In particular, the set-based definition and application in abstract algebra, topology, and other fields only addresses questions of pure mathematics that have nothing to do with the mathematics of change and covariation.

> In conclusion, we note that the definition of function as an expression or formula representing a relation between variables is for calculus or a pre-calculus course; is a rule of correspondence between reals for analysis; and a set theoretic definition with domain and range is required in the study of topology. [38] (p. 492)

Teaching the function concept should consider the different meanings, identifying criteria for selecting those suitable for the different educational levels and their progressive articulation. The stages or phases in the students' construction of the function concept proposed by various cognitive theories, such as the APOS model (action, process, object, and scheme) [53] or the operational and structural conception [6], should be applied to each of the species of the function that make up each meaning.

Closing remarks

The analysis of function we have undertaken leads us to conclude that it is inadequate to speak of the "function object" in the singular; at least we should recognize that such an object has a complex internal structure. Each possible definitions involves an onto-semiotic configuration, interconnected to form a conglomerate of practices, objects, and processes configurations. The different object-functions indeed share some features or family resemblances that lead to speaking of the concept of function. But when we are interested in teaching and learning processes, we cannot start the house from the roof, that is, from the most general and abstract object-function. Achieving this higher level of functional reasoning will not be possible if the preceding ones have not been previously worked on.

We have seen that the mathematical construct function is the fruit of social or communal activity. Consequently, the individual or mental learning activity, i.e., the ontogenesis of the object function, takes place in the ecological niche of phylogenesis. Hence, before addressing ontogenetic issues, i.e., mental processes of understanding and learning, a theoretical framework for understanding phylogenesis must be created, as we have advanced in this paper.

Recognizing the diverse meanings of functions is part of the epistemic facet of the teacher's didactic-mathematical knowledge required for the suitable teaching of this content [54]. As [52] state:

> The performance as teachers can be seriously impaired if it is not complemented by a deepening of specific epistemological training on the plurality of meanings of mathematical objects and the configurations of objects and processes in which such meanings crystallize. [52] (p. 581)

Moreover, in pre-university education, the function progresses from simple tabular representations, often associated with problems of direct proportionality with natural numbers, to graphical and symbolic representations in algebraic or transcendent relations between numerical sets in progressive degrees of generalization (natural numbers, positive fractional, positive decimal, whole, rational, real, complex). This evolution is not a mere accumulation of linear knowledge but a true epistemological challenge that has defied the great mathematicians throughout history.

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