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Numerical Semigroups with Monotone Apéry Set and Fixed Multiplicity and Ratio

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Abstract

We characterise the numerical semigroups with a monotone Apéry set (MANS-semigroups for short). Moreover, we describe the families of MANS-semigroups when we fix the multiplicity and the ratio.

Keywords Numerical semigroup · Multiplicity · Ratio · Frobenius number · Monotone Apéry set · Suitably monotone element

Mathematics Subject Classification 20M14 · 11D07

1 Introduction

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{N} = \{z \in \mathbb{Z} \mid z \geq 0\}$ be the sets of integer numbers and non-negative integers, respectively. A *numerical semigroup* is a subset S of $\mathbb{N}$ such that it is closed under addition, $0 \in S$, and $\mathbb{N} \backslash S = \{x \in \mathbb{N} \mid x \notin S\}$ is finite.

If A is a non-empty subset of $\mathbb{N}$, then we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \left\{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\}.$$

From Lemma 2.1 of Rosales and García-Sánchez (2009), we have that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

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If S is a numerical semigroup and $S = \langle A \rangle$, then we say that A is a *system of generators of* S. Moreover, if $S \neq \langle B \rangle$ for any subset $B \subsetneq A$, then A is a *minimal system of generators for* S. From Theorem 2.7 of Rosales and García-Sánchez (2009), each numerical semigroup admits a unique minimal system of generators and that such a system is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$, denoted by $\mathrm{e}(S)$, is the *embedding dimension of* S.

If S is a numerical semigroup, from the finiteness of $\mathbb{N} \backslash S$, we can define two invariants of S. Namely, the *Frobenius number of* S is the greatest integer that does not belong to S, denoted by $\mathrm{F}(S)$, and the *genus of* S is the cardinality of $\mathbb{N} \backslash S$, denoted by $\mathrm{g}(S)$.

The (extended) Frobenius problem (see Ramírez Alfonsín 2005) for a numerical semigroup S consists of finding formulas to compute $\mathrm{F}(S)$ and $\mathrm{g}(S)$ in terms of $\mathrm{msg}(S)$. Its solution is well known for numerical semigroups with embedding dimension two (see Sylvester 1883). However, the problem is open for $\mathrm{e}(S) \geq 3$. In fact, in Curtis (1990), it is proved that, in general, there is not possible to find polynomial formulas when $\mathrm{e}(S) \geq 3$. The difficulty of the Frobenius problem can be further argued. Indeed, it is known (see Ramírez-Alfonsín 1996) that finding $\mathrm{F}(S)$ is NP-hard in general, implying that it is unlikely to find an explicit formula for $\mathrm{F}(S)$ (only depending on the generators of S and not necessarily a polynomial) unless P=NP, which is widely believed to be false.

Let S be a numerical semigroup and $n \in S \backslash \{0\}$. The *Apéry set of* n *in* S (named so after Apéry 1946) is the set $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. In Lemma 2.4 of Rosales and García-Sánchez (2009), it is shown that $\mathrm{Ap}(S, n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n.

Recall that if S is a numerical semigroup, the least element of $S \backslash \{0\}$ (equivalently, the minimum of $\mathrm{msg}(S)$) is called the *multiplicity of* S, denoted by $\mathrm{m}(S)$. Now, following the notation introduced in Rosales et al. (2005), S is a *numerical semigroup with a monotone Apéry set* (*MANS-semigroup* for short) if S is a numerical semigroup fulfilling $w(1) < w(2) < \cdots < w(\mathrm{m}(S) - 1)$, where $w(i)$ is the least element of S congruent with i modulo $\mathrm{m}(S)$.

In Rosales et al. (2005), the authors study some families of numerical semigroups with monotone Apéry sets and fixed multiplicity. This work aims to characterise the family of MANS-semigroups.

Firstly, in Sect. 2, we discuss a necessary condition that, in particular, is sufficient in the case of embedding dimension two.

Then, in Sect. 3, we analyse MANS-semigroups with embedding dimension three. Thus, in Sect. 3.1, we characterise the triplet (n_1, n_2, n_3) such that $\langle \{n_1, n_2, n_3\} \rangle$ is a MANS-semigroup. Furthermore, we solve the (extended) Frobenius problem for those semigroups in Sects. 3.2 and, in Sects. 3.3 and 3.4, we study pseudo-Frobenius numbers and MANS-semigroups, with embedding dimension three, that are irreducible (recall that a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups containing it properly).

Finally, in Sect. 4, we characterise MANS-semigroups with arbitrary embedding dimension. Moreover, in Sect. 4.2, we describe the tree associated with the family of numerical semigroups with fixed multiplicity and ratio; in Sect. 4.3, we show how to construct irreducible MANS-semigroups in general; and, in Sect. 4.4, we study when numerical semigroups associated with arithmetic and almost arithmetic sequences are MANS-semigroups.

2 Case of Embedding Dimension Two

If S is a numerical semigroup and $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e\}$, then $\mathrm{m}(S) = n_1$, $\mathrm{r}(S) = n_2$, and $\mathrm{M}(S) = n_e$ are the multiplicity, the *ratio* and the *maximum minimal generator of* S, respectively. In particular, if S is a numerical semigroup with $\mathrm{e}(S) = 2$, then $\mathrm{msg}(S) = \{\mathrm{m}(S) < \mathrm{r}(S)\}$ and $\mathrm{r}(S) = \mathrm{M}(S)$.

We present two necessary conditions for MANS-semigroups. The first condition is a consequence of a well-known fact: $\{n_2, \ldots, n_e\} \subseteq \mathrm{Ap}(S, n_1)$.

As usual, for any $a, b \in \mathbb{N}$, $a \bmod b$ is the remainder of the division of a by b.

Lemma 2.1 *Let $S = \langle n_1 < n_2 < \cdots < n_e \rangle$ be a numerical semigroup with $n_1 \geq 2$. If S is a MANS-semigroup, then $n_2 \bmod n_1 < n_3 \bmod n_1 < \cdots < n_e \bmod n_1$.*

Lemma 2.2 *Let S be a numerical semigroup with $\mathrm{m}(S) \geq 2$. If S is a MANS-semigroup, then there exists $a \in \mathbb{N}\backslash\{0\}$ such that $\mathrm{r}(S) = a\mathrm{m}(S) + 1$.*

Proof From the definitions, we have that $\mathrm{r}(S) = \min(\mathrm{Ap}(S, \mathrm{m}(S))\backslash\{0\})$, and since S is a MANS-semigroup, we deduce there exists $a \in \mathbb{N}\backslash\{0\}$ such that $\mathrm{r}(S) = a\mathrm{m}(S) + 1$. □

In the case of embedding dimension two, the necessary condition of the above lemma is also sufficient.

Proposition 2.3 *Let $S = \langle n_1, n_2 \rangle$ be a numerical semigroup with $2 \leq n_1 < n_2$. Then S is a MANS-semigroup if and only if there exists $a \in \mathbb{N}\backslash\{0\}$ such that $n_2 = an_1 + 1$.*

Proof The necessity is just Lemma 2.2. For sufficiency, it is easy to check that $\mathrm{Ap}(S, n_1) = \{0 < n_2 < 2n_2 < \cdots < (n_1 - 1)n_2\}$ and $in_2 \equiv i \pmod{n_1}$ for all $i \in \{1, \ldots, n_1 - 1\}$. □

3 Case of Embedding Dimension Three

Observe that if S is a numerical semigroup with embedding dimension three, then $\mathrm{msg}(S) = \{\mathrm{m}(S) < \mathrm{r}(S) < \mathrm{M}(S)\}$.

3.1 Minimal Generators

In the following result, we show a necessary condition for MANS-semigroups with embedding dimension equal to three.

Proposition 3.1 *If S is a MANS-semigroup and $\mathrm{e}(S) = 3$, then there exists $\{m, a, b, t\} \subseteq \mathbb{N}$ such that $m \geq 3$, $a \geq 1$, $t \in \{2, \ldots, m - 1\}$, $(t - 1)(am + 1) < bm + t < t(am + 1)$, and $(\mathrm{m}(S), \mathrm{r}(S), \mathrm{M}(S)) = (m, am + 1, bm + t)$.*

Proof From Proposition 2.10 of Rosales and García-Sánchez (2009), we know that $\mathrm{e}(S) \leq \mathrm{m}(S)$ and, therefore, $\mathrm{m}(S) \geq 3$.

By applying Lemma 2.2, there exists $a \in \mathbb{N}\backslash\{0\}$ such that $\mathrm{r}(S) = a\mathrm{m}(S) + 1$. Moreover, we have that $\mathrm{M}(S) = b\mathrm{m}(S) + t$ with $b \in \mathbb{N}\backslash\{0\}$ and $t \in \{2, \ldots, m-1\}$.

If $\mathrm{Ap}(S, \mathrm{m}(S)) = \{w(0) = 0, w(1), \ldots, w(\mathrm{m}(S)-1)\}$, then clearly $w(t) = \mathrm{M}(S)$ and, since S is a MANS-semigroup, we have that $w(0) = 0 < w(1) = a\mathrm{m}(S) + 1 < w(2) = 2(a\mathrm{m}(S) + 1) < \cdots < w(t-1) = (t-1)(a\mathrm{m}(S) + 1) < w(t) = \mathrm{M}(S) = b\mathrm{m}(S) + t$. Finally, since $b\mathrm{m}(S) + t \notin \langle \mathrm{m}(S), a\mathrm{m}(S) + 1\rangle$, we deduce that $b\mathrm{m}(S) + t < t(a\mathrm{m}(S) + 1)$. □

Our next aim is to show that the condition given in Proposition 3.1 is also sufficient. From here, m, a, b, and t are positive integers such that $m \geq 3, a \geq 1, t \in \{2, \ldots, m-1\}$, $(t-1)(am+1) < bm+t < t(am+1)$; moreover, we consider $S = \langle m, am+1, bm+t\rangle$. Let us see that S is a numerical semigroup with embedding dimension equal to three, and then let us describe the Apéry set $\mathrm{Ap}(S, m)$.

Lemma 3.2 *S is a numerical semigroup with* $\mathrm{e}(S) = 3$.

Proof Since $\gcd\{m, am+1\} = 1$, we have that S is a numerical semigroup. Moreover, since $(t-1)(am+1) < bm+t$, then $am+1 < bm+t$. Finally, to prove that $\mathrm{e}(S) = 3$, it suffices to see that $mb+t \notin \langle a, am+1\rangle$, which is true because $bm+t < t(am+1)$. □

Three previous results, in which we determine the possible elements, are necessary to give the Apéry set $\mathrm{Ap}(S, m)$ explicitly.

Lemma 3.3 *Let $\lambda \in \mathbb{N}\backslash\{0\}$ such that $(\lambda-1)t < m \leq \lambda t$. Then $\lambda(bm+t) \notin \mathrm{Ap}(S, m)$.*

Proof It follows directly from the hypothesis that $\lambda t - m \in \{0, \ldots, t-1\}$. Moreover, we have that $\lambda(bm+t) \equiv (\lambda t-m)(am+1) \pmod{m}$. Then, since $(\lambda t-m)(am+1) \leq (t-1)(am+1) < bm+t \leq \lambda(bm-t)$, we deduce that $\lambda(bm+t) = (\lambda t-m)(am+1) + \mu m$ for some $\mu \in \mathbb{N}\backslash\{0\}$. Therefore, $\lambda(bm+t) \notin \mathrm{Ap}(S, m)$. □

An immediate consequence of the above lemma is the following one.

Lemma 3.4 *If $\lambda \in \mathbb{N}$ and $\lambda t \geq m$, then $\lambda(bm+t) \notin \mathrm{Ap}(S, m)$.*

Lemma 3.5 *If $\mu \in \mathbb{N}$ and $\mu \geq t$, then $\mu(am+1) \notin \mathrm{Ap}(S, m)$.*

Proof It is clear that $\mu(am+1) \equiv (bm+t) + (\mu-t)(am+1) \pmod{m}$. Moreover, $(bm+t) + (\mu-t)(am+1) \in S$ and $(bm+t) + (\mu-t)(am+1) < \mu(am+1)$ (because $bm+t < t(am+1)$). Therefore, there exists $\lambda \in \mathbb{N}\backslash\{0\}$ such that $\mu(am+1) = (bm+t) + (\mu-t)(am+1) + \lambda m$. In consequence, $\mu(am+1) \notin \mathrm{Ap}(S, m)$. □

We are ready to show $\mathrm{Ap}(S, m)$. As usual, $\lfloor x \rfloor = \max\{n \in \mathbb{N} \mid n \leq x\}$ for every $x \in \mathbb{R}$.

Proposition 3.6 *If* $\mathrm{Ap}(S, m) = \{w(0), w(1), \ldots, w(m-1)\}$, *then*

$$w(i) = \left\lfloor \frac{i}{t} \right\rfloor (bm+t) + (i \bmod t)(am+1), \; i \in \{0, 1, \ldots, m-1\}.$$

Proof By Lemmas 3.4 and 3.5, we deduce that $w(i) = \lambda(bm + t) + \mu(am + 1)$ for some $\{\lambda, \mu\} \subseteq \mathbb{N}$ such that $\lambda t < m$ and $\mu < t$. Then, since $\lambda t + \mu \equiv i \pmod{m}$ and $i = \left\lfloor \frac{i}{t} \right\rfloor t + (i \bmod t)$, we can conclude that $\lambda = \left\lfloor \frac{i}{t} \right\rfloor$ and $\mu = i \bmod t$. □

We end this section with the characterisation of MANS-semigroups with embedding dimension equal to three.

Theorem 3.7 *The following conditions are equivalent.*

1. *S is a MANS-semigroup with* $\mathrm{e}(S) = 3$.
2. $S = \langle m, am+1, bm+t \rangle$*, where* $\{m, a, b, t\} \subseteq \mathbb{N}$*,* $m \geq 3$*,* $a \geq 1$*,* $t \in \{2, \ldots, m-1\}$*, and* $(t-1)(am+1) < bm + t < t(am+1)$.

Proof (1. ⇒ 2.) This is Proposition 3.1.

(2. ⇒ 1.) From Lemma 3.2, we know that S is a numerical semigroup with $\mathrm{e}(S) = 3$. To finish the proof, it will be enough to see that if $\mathrm{Ap}(S, m) = \{w(0), w(1), \ldots, w(m-1)\}$, then $w(i) < w(i+1)$ for all $i \in \{0, \ldots, m-2\}$. On the one side, if $(i+1) \bmod t > i \bmod t$, then we deduce that $w(i) < w(i+1)$ by applying Proposition 3.6. On the other side, if $(i+1) \bmod t \leq i \bmod t$, then $(i+1) \bmod t = 0$ and, thereby, $\left\lfloor \frac{i+1}{t} \right\rfloor = \left\lfloor \frac{i}{t} \right\rfloor - 1$. By applying again Proposition 3.6, we have that $w(i) < w(i+1)$. □

3.2 Frobenius Problem

The following result is the first part of Proposition 2.12 in Rosales and García-Sánchez (2009).

Proposition 3.8 *If S is a numerical semigroup and $n \in S \backslash \{0\}$, then* $\mathrm{F}(S) = \max(\mathrm{Ap}(S, n)) - n$.

An immediate consequence of Propositions 3.6 and 3.8 is the following.

Proposition 3.9 *If* $S = \langle m, am+1, bm+t \rangle$ *is a MANS-semigroup with embedding dimension three, then* $\mathrm{F}(S) = r(am+1) + q(bm+t) - m$*, where* $q = \left\lfloor \frac{m-1}{t} \right\rfloor$ *and* $r = (m-1) \bmod t$.

Remark 3.10 By applying the second case of the first theorem in Byrnes (1974), we recover Proposition 3.9.

Let us see an example of the above proposition.

Example 3.11 Let $S = \langle 5, 6, 13 \rangle = \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\}$ (where the symbol $\rightarrow$ indicates that all integers greater than 15 belong to S). Then $\mathrm{Ap}(S, 5) = \{w(0) = 0, w(1) = 6, w(2) = 12, w(3) = 13, w(4) = 19\}$ and, therefore, S is a MANS-semigroup with $\mathrm{e}(S) = 3$. Moreover, since $m = 5$, $am + 1 = 6$, $bm + t = 13$, and $t = 3$, then $q = 1$, $r = 1$, and consequently $\mathrm{F}(S) = 6 + 13 - 5 = 14$.

The following result is the second statement of Proposition 2.12 in Rosales and García-Sánchez (2009).

Proposition 3.12 *If* S *is a numerical semigroup,* $n \in S\backslash\{0\}$*, and* $\mathrm{Ap}(S,n) = \{w(0), w(1), \ldots, w(n-1)\}$*, then* $\mathrm{g}(S) = \frac{w(0)+w(1)+\cdots+w(n-1)}{n} - \frac{n-1}{2}$.

We can now show a formula for the genus of a MANS-semigroup with embedding dimension three.

Proposition 3.13 *If* $S = \langle m, am+1, bm+t\rangle$ *is a MANS-semigroup with embedding dimension three, then*

$$\mathrm{g}(S) = \frac{qt(t-1)+r(r+1)}{2m}(am+1) + \frac{qt(q-1)+2q(r+1)}{2m}(bm+t) - \frac{m-1}{2},$$

where $q = \left\lfloor \frac{m-1}{t} \right\rfloor$ *and* $r = (m-1) \bmod t$.

Proof As a consequence of Proposition 3.6, we have that

$$\begin{aligned}\mathrm{Ap}(S,m) = \{&0, (am+1), \ldots, (t-1)(am+1),\\ &(bm+t), (am+1)+(bm+t), \ldots, (t-1)(am+1)+(bm+t), \ldots,\\ &(q-1)(bm+t), (am+1)+(q-1)(bm+t), \ldots,\\ &(t-1)(am+1)+(q-1)(bm+t),\\ &r(bm+t), (am+1)+q(bm+t), \ldots, r(am+1)+q(bm+t)\}\end{aligned}$$

Then, by applying Proposition 3.12, we get the result. □

Remark 3.14 Since $qt(t-1)+r(r+1)+(qt(q-1)+2q(r+1))t = m(m-1)$, we can rewrite the formula of the previous proposition as

$$\mathrm{g}(S) = \frac{qt(t-1)+r(r+1)}{2}a + \frac{qt(q-1)+2q(r+1)}{2}b.$$

Let us see an example of the content of the above proposition.

Example 3.15 Let $S = \langle 5, 6, 13\rangle$ the numerical semigroup of Example 3.11. Then $a = 1$, $b = 2$, $m = 5$, $t = 3$, $q = 1$, and $r = 1$. By applying Proposition 3.13, we have that $\mathrm{g}(S) = \frac{(3\times2+1\times2)\times6+(3\times0+2\times1\times2)\times13}{2\times5} - \frac{4}{2} = 8$.

3.3 Pseudo-Frobenius Numbers

Let S be a numerical semigroup. Following the terminology in Rosales and Branco (2002), a *pseudo-Frobenius number of* S is an element $x \in \mathbb{Z}\backslash S$ such that $x+s \in S$ for all $s \in S\backslash\{0\}$. We denote by $\mathrm{PF}(S) = \{x \mid x \text{ is a pseudo-Frobenius number of S}\}$. The cardinality of $\mathrm{PF}(S)$ is called the *type of* S, denoted by $\mathrm{t}(S)$. From Fröberg et al. (1987), we have that if S is a numerical semigroup with $\mathrm{e}(S) = 3$, then $\mathrm{t}(S) \in \{1, 2\}$.

Let S be a numerical semigroup. We define over $\mathbb{Z}$ the following binary relation: $a \le_S b$ if $b - a \in S$. It is clear that $\le_S$ is a non-strict partial order relation (that is, it is reflexive, transitive, and anti-symmetric).

The following result is Proposition 2.20 of Rosales and García-Sánchez (2009) (see also Proposition 7 of Fröberg et al. 1987) and characterises the pseudo-Frobenius numbers in terms of the maximal elements of $\mathrm{Ap}(S, n)$ with respect to the relation $\leq_S$.

Proposition 3.16 *Let S be a numerical semigroup and $n \in S\backslash\{0\}$. Then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, n))\}.$$

Before continuing, let us see one example.

Example 3.17 Let $S = \langle 5, 6, 13\rangle$ as in Example 3.11. Then $\mathrm{Ap}(S, 5) = \{0, 6, 12, 13, 19\}$ and, thereby, $\mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, 5)) = \{12, 19\}$. By applying Proposition 3.16, we have that $\mathrm{PF}(S) = \{7, 14\}$.

The next result follows from the proof of the Proposition 3.13.

Lemma 3.18 *Let $S = \langle m, am + 1, bm + t\rangle$ be a MANS-semigroup with $\mathrm{e}(S) = 3$, $q = \left\lfloor \frac{m-1}{t} \right\rfloor$, and $r = (m - 1) \bmod t$. Then $\{q(bm + t) + r(am + 1)\} \subseteq \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, m)) \subseteq \{(q-1)(bm+t)+(t-1)(am+1), q(bm+t)+r(am+1)\}$ and, consequently, $\{q(bm + t) + r(am + 1) - m\} \subseteq \mathrm{PF}(S) \subseteq \{(q - 1)(bm + t) + (t - 1)(am + 1) - m, q(bm + t) + r(am + 1) - m\}$.*

Let us characterise when a MANS-semigroup with embedding dimension three has type equals one or two.

Proposition 3.19 *Let $S = \langle m, am + 1, bm + t\rangle$ be a MANS-semigroup with $\mathrm{e}(S) = 3$. Then $\mathrm{t}(S) = 1$ if and only if $t \mid m$ (that is, t divides m).*

Proof Let us take $q = \left\lfloor \frac{m-1}{t} \right\rfloor$ and $r = (m - 1) \bmod t$.

(Necessity.) From Proposition 3.16 and Lemma 3.18, we deduce that if $\mathrm{t}(S) = 1$, then $q(bm+t)+r(am+1)-((q-1)(bm+t)+(t-1)(am+1)) \in S$ and, therefore, $bm + t + (r - t + 1)(am + 1) \in S$. By applying that $bm + t \in \mathrm{msg}(S)$, we have that $r - t + 1 \geq 0$ and, consequently, $r = t - 1$. Thus, $m - 1 = qt + t - 1$ and, thereby, $t \mid m$.

(Sufficiency.) If $t \mid m$, then there exists $k \in \mathbb{N}$ such that $m = kt$ and, therefore, $m - 1 = (k - 1)t + t - 1$. Thus, $r = t - 1$ and $q(bm + t) + r(am + 1) - ((q - 1)(bm + t) + (t - 1)(am + 1)) = bm + t \in S$. Now, by applying Proposition 3.16 and Lemma 3.18, we conclude that $\mathrm{t}(S) = 1$. □

We deduce the following result from Propositions 3.16 and 3.19 and Lemma 3.18. We denote by $t \nmid m$ that t does not divide m.

Proposition 3.20 *Let $S = \langle m, am + 1, bm + t\rangle$ be a MANS-semigroup with $\mathrm{e}(S) = 3$, $q = \left\lfloor \frac{m-1}{t} \right\rfloor$, and $r = (m - 1) \bmod t$.*

1. *If $t \mid m$, then $\mathrm{PF}(S) = \{q(bm + t) + r(am + 1) - m\}$.*
2. *If $t \nmid m$, then $\mathrm{PF}(S) = \{(q - 1)(bm + t) + (t - 1)(am + 1) - m, q(bm + t) + r(am + 1) - m\}$.*

Let us see an example of the last two propositions.

Example 3.21 Let $S = \langle 6, 7, 15\rangle$. Then $\mathrm{Ap}(S, 6) = \{w(0) = 0, w(1) = 7, w(2) = 14, w(3) = 15, w(4) = 22, w(5) = 29\}$. Therefore, S is a MANS-semigroup with $\mathrm{e}(S) = 3$. Since $m = 6$ and $t = 3$, from Proposition 3.19 we can assert that $\mathrm{t}(S) = 1$; indeed, $\mathrm{PF}(S) = \{23\}$.

3.4 Irreducibility

Recall that a numerical semigroup S is *irreducible* if it is not expressible as the intersection of two numerical semigroups properly containing S. This concept was introduced in Rosales and Branco (2003), where it is shown that a numerical semigroup S is irreducible if and only if it is maximal (with respect to the inclusion order) in the set formed by all numerical semigroups with Frobenius number $\mathrm{F}(S)$. From Barucci et al. (1997) and Fröberg et al. (1987), it follows that the family of irreducible numerical semigroups is the union of two well-known families, the symmetric numerical semigroups and the pseudo-symmetric numerical semigroups (see Rosales and Branco 2003). Furthermore, a numerical semigroup is symmetric (pseudo-symmetric, respectively) if it is irreducible and has an odd Frobenius number (even Frobenius number, respectively).

The following result is consequence of Corollaries 4.5, 4.11 and 4.16 in Rosales and García-Sánchez (2009).

Proposition 3.22 *Let S be a numerical semigroup.*

1. *S is symmetric if and only if* $\mathrm{t}(S) = 1$ *(equivalently,* $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$*).*
2. *S is symmetric if and only if* $\mathrm{F}(S) = 2\mathrm{g}(S) - 1$.
3. *S is pseudo-symmetric if and only if* $\mathrm{PF}(S) = \left\{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\right\}$.
4. *S is pseudo-symmetric if and only if* $\mathrm{F}(S) = 2\mathrm{g}(S) - 2$.

Let us observe that Proposition 3.19 characterises MANS-semigroups with embedding dimension three that are symmetric. Note also that, from Example 3.17 and Proposition 3.22, we know that $S = \langle 5, 6, 13\rangle$ is a MANS-semigroup with embedding dimension three that is pseudo-symmetric. We now propose to characterise this class of semigroups.

Proposition 3.23 *Let $S = \langle m, am + 1, bm + t\rangle$ be a MANS-semigroup with* $\mathrm{e}(S) = 3$. *Then S is pseudo-symmetric if and only if $t = \frac{m+1}{2}$ and $t = \frac{b+1}{a}$.*

Proof From Propositions 3.9, 3.13, and 3.22, S is pseudo-symmetric if and only if $r(am + 1) + q(bm + t)$ is equal to

$$\frac{qt(t-1) + r(r+1)}{m}(am+1) + \frac{qt(q-1) + 2q(r+1)}{m}(bm+t) - 1,$$

where $q = \left\lfloor \frac{m-1}{t} \right\rfloor$ and $r = (m - 1) \bmod t$. Since

$$\begin{aligned}&(mr-qt(t-1)-r(r+1))(am+1)+(qm-qt(q-1)-2q(r+1))(bm+t)\\&\quad=((m-r-1)r-qt(t-1))(am+1)+q(m-t(q-1)-2(r+1))(bm+t)\\&\quad=qt(r-t+1)(am+1)+q(t-1-r)(bm+t)=q(t-1-r)(ta-b)(-m),\end{aligned}$$

we deduce that S is pseudo-symmetric if and only if $q(t-1-r)(ta-b)=1$. Finally, since $q, t-1-r, ta-b \in \mathbb{N}$ (recall that $bm+t < t(am+1)$), we deduce that $q=t-1-r=ta-b=1$ and we get the result by observing that

- $t=\frac{m+1}{2}$ if and only if $q=1$ and $r=t-2$;
- $t=\frac{b+1}{a}$ if and only if $ta-b=1$.

□

Remark 3.24 With similar reasoning as in the above proof, we can recover Proposition 3.19 using condition 2 of Proposition 3.22.

Let us see three examples relate to the above proposition. In particular, from the first two, we conclude that conditions $t=\frac{b+1}{a}$ and $t=\frac{m+1}{2}$ are independent.

Example 3.25 Let $S=\langle 5,6,19\rangle$. Then $m=5, a=1, b=3$, and $t=4$. Thus, $t=\frac{b+1}{a}$ and $t\neq\frac{m+1}{2}$. Note that S is a MANS-semigroup and, since $\mathrm{PF}(S)=\{13,14\}$, it is not pseudo-symmetric.

Example 3.26 Let $S=\langle 5,11,23\rangle$. Then $m=5, a=2, b=4$, and $t=3$. Therefore, $t\neq\frac{b+1}{a}$ and $t=\frac{m+1}{2}$. We have that S is a MANS-semigroup and, since $\mathrm{PF}(S)=\{17,29\}$, it is not pseudo-symmetric.

Example 3.27 If $m=3$, then symmetric MANS-semigroups are of the form $S=\langle 3,3a+1\rangle$ and pseudo-symmetric MANS-semigroups are of the form $S=\langle 3,3a+1,6a-1\rangle$, with $a\in\mathbb{N}\backslash\{0\}$ in both cases. Note that, for each $a\in\mathbb{N}\backslash\{0\}$, $6a-1$ is the Frobenius number of $\langle 3,3a+1\rangle$.

4 Case of Arbitrary Embedding Dimension

In this section, we analyse the general case of MANS-semigroups, that is, we consider numerical semigroups of arbitrary embedding dimension.

As stated in Sect. 2, if S is a numerical semigroup with $\mathrm{msg}(S)=\{n_1<n_2<\cdots<n_e\}$, then $\mathrm{m}(S)=n_1$, $\mathrm{r}(S)=n_2$, and $\mathrm{M}(S)=n_e$ are the multiplicity, the ratio, and the greatest minimal generator of S, respectively. Moreover, if S is a MANS-semigroup, then $\mathrm{r}(S)=a\mathrm{m}(S)+1$ for some $a\in\mathbb{N}\backslash\{0\}$.

Let us first give a characterisation of MANS-semigroups. We start by detecting how we can add to a MANS-semigroup S a new minimal generator (greater than $\mathrm{M}(S)$) to obtain a new MANS-semigroup (with a higher embedding dimension).

Lemma 4.1 *Let S be a MANS-semigroup with $\mathrm{msg}(S)=\{n_1<n_2<\cdots<n_e\}$ ($2\le e\le n_1-1$) and $\mathrm{Ap}(S,n_1)=\{w(0),w(1),\ldots,w(n_1-1)\}$. If $n_{e+1}\in\mathbb{N}$ fulfil that $n_e<n_{e+1}$, $n_e \bmod n_1 < n_{e+1} \bmod n_1$, and $w(n_{e+1} \bmod n_1 - 1) < n_{e+1} < w(n_{e+1} \bmod n_1)$, then $S'=\langle n_1,\ldots,n_e,n_{e+1}\rangle$ is a MANS-semigroup with $\mathrm{e}(S')=\mathrm{e}(S)+1$.*

Proof From condition $n_{e+1} < w(n_{e+1} \bmod n_1)$, we deduce that $n_{e+1} \notin S$ and, since $n_e < n_{e+1}$, then $\mathrm{msg}(S') = \{n_1 < n_2 < \cdots < n_e < n_{e+1}\}$. Therefore, $\mathrm{e}(S') = \mathrm{e}(S) + 1$.

Let $\mathrm{Ap}(S', n_1) = \{w'(0), w'(1), \ldots, w'(n_1 - 1)\}$. By observing the construction of S', it is clear that $w'(i) \leq w(i)$ for all $i \in \{0, 1, \ldots, n_1 - 1\}$. In order to prove that S' is a MANS-semigroup, we analyse what happens between two consecutive elements of $\mathrm{Ap}(S', n_1)$. We will consider two cases (in which we take $i \geq 1$).

1. If $w'(i) \in S$, then $w'(i) = w(i)$ and, therefore, $w'(i-1) \leq w(i-1) < w(i) = w'(i)$.
2. If $w'(i) \notin S$, then $w'(i) = kn_{e+1} + w(j)$ for some $k \in \mathbb{N}\backslash\{0\}$ and some $j \in \{0, \ldots, n_1 - 1\}$. Once again, we distinguish two cases.
   (a) If $j \neq 0$, then $w'(i-1) \leq kn_{e+1} + w(j-1) < kn_{e+1} + w(j) = w'(i)$.
   (b) If $j = 0$, then $w'(i) = kn_{e+1}$ and, consequently, $w'(i-1) \leq (k-1)n_{e+1} + w(n_{e+1} \bmod n_1 - 1) < kn_{e+1} = w'(i)$.

□

Let us now see that if we remove the greatest minimal generator of a MANS-semigroup S, we get a new MANS-semigroup (with a smaller embedding dimension).

Lemma 4.2 *Let S be a MANS-semigroup with $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e < n_{e+1}\}$ ($e \geq 2$). Then $S' = \langle n_1, n_2, \ldots, n_e \rangle$ is a MANS-semigroup with $\mathrm{e}(S') = \mathrm{e}(S) - 1$.*

Proof The equality $\mathrm{e}(S') = \mathrm{e}(S) - 1$ is trivial by the construction of S'.

From Lemma 2.2, we know that if $e = 2$, then $S' = \langle n_1, n_2 \rangle = \langle n_1, an_1 + 1 \rangle$ for some $a \in \mathbb{N}\backslash\{0\}$. By Proposition 2.3, S' is a MANS-semigroup.

Let us now suppose that $e \geq 3$, $\mathrm{Ap}(S, n_1) = \{w(0), w(1), \ldots, w(n_1 - 1)\}$, and $\mathrm{Ap}(S', n_1) = \{w'(0), w'(1), \ldots, w'(n_1 - 1)\}$. To prove that S' is a MANS-semigroup, we analyse what happens for two consecutive elements of $\mathrm{Ap}(S', n_1)$.

Firstly, observe that $w'(i) = w(i)$ for all $i \in \{0, 1, \ldots, n_{e+1} \bmod n_1 - 1\}$. (This fact will be used again in Sect. 4.1.)

If $i \geq 2$, then there exists $k \in \{1, 2, \ldots, e\}$ such that $w'(i) = n_k + w'(j)$, where $j = (n_k - i) \bmod n_1$. Thus, $n_k \bmod n_1 - 1 + j \equiv (i-1) \pmod{n_1}$ and, therefore, $w'(i-1) \leq w'(n_k \bmod n_1 - 1) + w'(j)$. Now, since S is MANS-semigroup, then $w'(n_k \bmod n_1 - 1) < w'(n_k \bmod n_1) = n_k$. In conclusion, $w'(i-1) \leq w'(j) + w'(n_k \bmod n_1 - 1) < w'(j) + n_k = w'(i)$. □

The next result follows immediately from the above lemma.

Corollary 4.3 *Let S be a MANS-semigroup with $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e < n_{e+1}\}$ ($e \geq 2$). Then $S' = \langle n_1, n_2, \ldots, n_i \rangle$ is a MANS-semigroup with $\mathrm{e}(S') = i$ for all $i \in \{2, \ldots, e\}$.*

We can already state the characterisation of MANS-semigroups.

Theorem 4.4 *Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e < n_{e+1}\}$ ($e \geq 2$) and let $S' = \langle n_1 < n_2 < \cdots < n_e \rangle$ with $\mathrm{Ap}(S', n_1) = \{w'(0), w'(1), \ldots, w'(n_1 - 1)\}$. Then S is a MANS-semigroup if and only if*

1. *S' is a MANS-semigroup,*
2. $n_e \bmod n_1 < n_{e+1} \bmod n_1$,
3. *and* $w'(n_{e+1} \bmod n_1 - 1) < n_{e+1} < w'(n_{e+1} \bmod n_1)$.

Proof *(Necessity.)* By Lemma 4.2, we know that S' is a MANS-semigroup.

Since S is a MANS-semigroup and $n_e, n_{e+1} \in \mathrm{Ap}(S, n_1)$, we can state that $n_e \bmod n_1 < n_{e+1} \bmod n_1$.

If $\mathrm{Ap}(S, n_1) = \{w(0), w(1), \ldots, w(n_1 - 1)\}$, since S and S' are MANS-semigroups, we have that $w'(i) = w(i)$ for all $i \in \{0, 1, \ldots, n_{e+1} \bmod n_1 - 1\}$ and, consequently, $w'(n_{e+1} \bmod n_1 - 1) < n_{e+1}$.

Finally, since $e_{n+1} \in \mathrm{msg}(S)$, we have that $e_{n+1} \notin S'$, from which it follows that $n_{e+1} < w'(n_{e+1} \bmod n_1)$.

(Sufficiency.) Follows by Lemma 4.1. □

4.1 Apéry Sets

Let S and S' be MANS-semigroups with $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e\}$ and $\mathrm{msg}(S') = \{n_1 < n_2 < \cdots < n_e < n_{e+1}\}$. We now aim to construct $\mathrm{Ap}(S', n_1)$ from $\mathrm{Ap}(S, n_1)$.

Remark 4.5 Under the conditions of Lemma 4.1, let $t_{e+1} = n_{e+1} \bmod n_1$. Note that if we take $k = \left\lfloor \frac{n_1-1}{t_{e+1}} \right\rfloor + 1$, then $k \geq 2$ and $kn_{e+1} \bmod n_1 \leq n_{e+1} \bmod n_1$. Since $S' = \langle n_1 < n_2 < \cdots < n_e < n_{e+1} \rangle$ is a MANS-semigroup, we deduce that kn_{e+1} cannot appear as a summand in the elements of $\mathrm{Ap}(S', n_1)$.

We take $K = \left\lfloor \frac{n_1-1}{t_{e+1}} \right\rfloor$, $A_h = \{w(0), w(1), \ldots, w(n_1 - 1 - ht_{e+1})\}$, for $h \in \{0, 1, \ldots, K\}$, and $\mathrm{Ap}(S, n_1) = \{w(0), w(1), \ldots, w(n_1 - 1)\}$. From the proof of Lemma 4.1 and Remark 4.5, we have that $\mathrm{Ap}(S', n_1)$ is a subset of $B = A_0 \cup (n_{e+1} + A_1) \cup \cdots \cup (Kn_{e+1} + A_K)$ (where, as usual, if $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then $x + A = \{x + a \mid a \in A\}$). Thus if $\mathrm{Ap}(S', n_1) = \{w'(0), w'(1), \ldots, w'(n_1 - 1)\}$, then it is satisfied that

- $w'(i) = w(i)$ if $0 \leq i \leq t_{e+1} - 1$,
- $w'(i) = \min\{w(i), n_{e+1} + w(i - t_{e+1})\}$ if $t_{e+1} \leq i \leq 2t_{e+1} - 1$,
- $w'(i) = \min\{w(i), n_{e+1} + w(i - t_{e+1}), 2n_{e+1} + w(i - 2t_{e+1})\}$ if $2t_{e+1} \leq i \leq 3t_{e+1} - 1$,
- …
- $w'(i) = \min\{w(i), n_{e+1} + w(i - t_{e+1}), 2n_{e+1} + w(i - 2t_{e+1}), \ldots, (K-1)n_{e+1} + w(i - (K-1)t_{e+1})\}$ if $(K-1)t_{e+1} \leq i \leq Kt_{e+1} - 1$,
- $w'(i) = \min\{w(i), n_{e+1} + w(i - t_{e+1}), 2n_{e+1} + w(i - 2t_{e+1}), \ldots, (K-1)n_{e+1} + w(i - (K-1)t_{e+1}), Kn_{e+1} + w(i - Kt_{e+1})\}$ if $Kt_{e+1} \leq i \leq n_1 - 1$.

Let us see an example of this construction.

Example 4.6 Let $S_3 = \langle 13, 27, 55 \rangle$. Then

$$\mathrm{Ap}(S_3, 13) = \{0, 27, 54, 55, 82, 109, 110, 137, 164, 165, 192, 219, 220\}.$$

Therefore, S_3 is a MANS-semigroup.

If we take $n_4 = 96 = 7 \times 13 + 5$, then $82 < 96 < 109$ and, by Lemma 4.1, $S_4 = \langle 13, 27, 55, 96 \rangle$ is a MANS-semigroup. Moreover, $K = \left\lfloor \frac{13-1}{5} \right\rfloor = 2$.

To construct $\mathrm{Ap}(S_4, 13)$, we consider the following table.

$w(i)$	$96 + w(i)$	$2 \cdot 96 + w(i)$
0	–	–
27	–	–
54	–	–
55	–	–
82	–	–
109	96 + 0 = 96	–
110	96 + 27 = 123	–
137	96 + 54 = 150	–
164	96 + 55 = 151	–
165	96 + 82 = 178	–
192	96 + 109 = 205	192 + 0 = 192
219	96 + 110 = 206	192 + 27 = 219
220	96 + 137 = 233	192 + 54 = 246

Taking the minimum in each line, we conclude that

$$\mathrm{Ap}(S_4, 13) = \{0, 27, 54, 55, 82, 96, 110, 137, 151, 165, 192, 206, 220\}.$$

4.2 The Tree of MANS-Semigroups with Multiplicity and Ratio Fixed

Note that if we fix the multiplicity value ($m > 1$), then there are infinite MANS-semigroups S with $\mathrm{m}(S) = m$. In fact, by Proposition 2.3, we have that $S = \langle m, am + 1 \rangle$ is a MANS-semigroup for any $a \in \mathbb{N} \setminus \{0\}$. Incidentally, $\mathbb{N}$ is the unique MANS-semigroup with multiplicity $m = 1$.

However, if we fix the multiplicity ($m > 1$) and the ratio ($r > 2$), then the set $\mathcal{MA}(m, r) = \{S \mid S \text{ is a MANS-semigroup}, \mathrm{m}(S) = m, \text{ and } \mathrm{r}(S) = r\}$ is finite. Indeed, by Corollary 4.3, it is clear that every element of $\mathcal{MA}(m, r)$ must contain the numerical semigroup $\langle m, r \rangle$. Now, since $\mathbb{N} \setminus \langle m, r \rangle$ is finite, we conclude that $\mathcal{MA}(m, r)$ has finitely many elements.

Since we now want to find all the elements of $\mathcal{MA}(m, r)$, we will endow $\mathcal{MA}(m, r)$ with a tree structure.

Recall that a *directed graph* G is a pair (V, E) where V is a non-empty set and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of V and E are called *vertices* and *edges*, respectively. A *path*, of length n, connecting the vertices $u, v \in G$ is a sequence of distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = u$ and $v_n = v$.

A directed graph G is a *tree* if there exists a vertex v_r (known as the *root* of G) such that, for any other vertex $v \in G$, there exists a unique path connecting v and v_r. Moreover, if (u, v) is an edge of the tree, then u is a *child* of v.

To define the tree $G(\mathcal{MA}(m,r))$, we take $\mathcal{MA}(m,r)$ as the set of vertices and say that $(T, S) \in \mathcal{MA}(m,r) \times \mathcal{MA}(m,r)$ is an edge of $G(\mathcal{MA}(m,r))$ if and only if $\mathrm{msg}(S) = \mathrm{msg}(T)\backslash\{\mathrm{M}(T)\}$.

Given $S \in \mathcal{MA}(m,r)$, we define the following sequence: $S_0 = S$ and

$$S_{n+1} = \begin{cases} \langle \mathrm{msg}(S_n)\backslash\{\mathrm{M}(S_n)\}\rangle \text{ if } S_n \neq \langle m, r\rangle, \\ \langle m, r\rangle \text{ otherwise.} \end{cases}$$

From Lemma 4.2, we deduce the following result.

Proposition 4.7 *If $S \in \mathcal{MA}(m,r)$ and $\{S_n \mid n \in \mathbb{N}\}$ is the sequence defined above, then $S_n \in \mathcal{MA}(m,r)$ for all $n \in \mathbb{N}$. Moreover, $S_k = \langle m, r\rangle$ for all $k \geq \mathrm{e}(S) - 2$.*

As a consequence of Proposition 4.7 and the first two comments of this subsection, we obtain the following result.

Proposition 4.8 *$G(\mathcal{MA}(m,r))$ is a finite tree with root $\langle m, r\rangle$.*

Observe that, starting from its root, we can recurrently build a tree by connecting each vertex to its children through the corresponding edges. Thus, if we know the children of any vertex in $G(\mathcal{MA}(m,r))$, then we can build the tree and find all the elements of $\mathcal{MA}(m,r)$.

Let $S = \langle n_1 < n_2 < \cdots < n_e\rangle$ be a MANS-semigroup with $e \geq 2$. Moreover, let $\mathrm{Ap}(S, n_1) = \{w(0), w(1), \ldots, w(n_1 - 1)\}$. We will say that $n \in \mathbb{N}$ is a *suitably monotone element for* S if it fulfils the following three conditions:

1. $n_e < n$.
2. $n_e \bmod n_1 < n \bmod n_1$.
3. $w(n \bmod n_1 - 1) < n < w(n \bmod n_1)$.

From Lemma 4.1 and Proposition 4.8, we deduce the result that allows us to recurrently build $G(\mathcal{MA}(m,r))$.

Theorem 4.9 *Let $S = \langle n_1 < n_2 < \cdots < n_e\rangle \in \mathcal{MA}(m,r)$ be any vertex of $G(\mathcal{MA}(m,r))$. Then the children of S are the numerical semigroups $T_n = \langle n_1, n_2, \ldots, n_e, n\rangle$, where n is a suitably monotone element for S.*

Let us illustrate the above theorem with two examples. In both of them, the number above the arrows corresponds to the modulo, with respect to the multiplicity, of the new minimal generator.

Example 4.10 Let $m = 5$ and $r = 6$. Then the tree $G(\mathcal{MA}(5,6))$ is given by Fig. 1.

Example 4.11 Let $m = 5$ and $r = 11$. Then the tree $G(\mathcal{MA}(5,6))$ is given by Fig. 2.

From the two above examples, we observe that if S is a MANS-semigroup such that $\mathrm{M}(S) \bmod n_1 < n_1 - 1$, then $\mathrm{M}(S) + 1$ is a suitably monotone element for S. As we show in the following result, this fact is not casual.

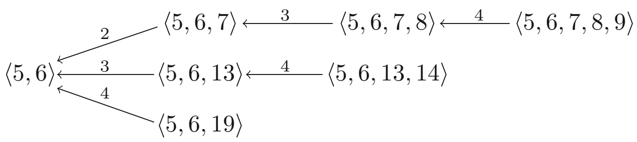

Fig. 1 $G(\mathcal{MA}(5,6))$

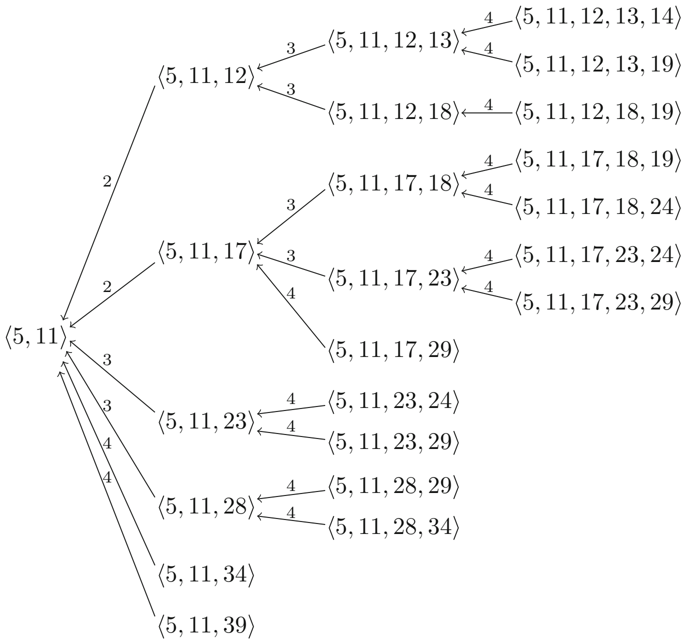

Fig. 2 $G(\mathcal{MA}(5,11))$

Proposition 4.12 *Let $S = \langle n_1 < n_2 < \cdots < n_e \rangle$ with $n_1 \geq 3$ and $e \geq 2$. If S is a MANS-semigroup such that $n_e \bmod n_1 < n_1 - 1$, then $n_e + 1$ is a suitably monotone element for S.*

Proof Firstly, we observe that $n_e < n_e + 1$ and, since $n_e \bmod n_1 < n_1 - 1$, then $n_e \bmod n_1 < n_e \bmod n_1 + 1 = (n_e + 1) \bmod n_1$.

Secondly, since $n_e \in \mathrm{msg}(S)$, if $\mathrm{Ap}(S, n_1) = \{w(0), w(1), \ldots, w(n_1 - 1)\}$, then $n_e = w(n_e \bmod n_1)$ and, in consequence, $w((n_e+1) \bmod n_1 - 1) = w(n_e \bmod n_1) = n_e < n_e + 1$.

At this moment, to prove that $n_e + 1$ is a suitably monotone element for S, it remains to be seen that $n_e + 1 < w((n_e + 1) \bmod n_1)$. For this purpose, since S is a MANS-semigroup, we note that $w(n_e \bmod n_1) < w((n_e + 1) \bmod n_1)$, that is, $n_e < w((n_e + 1) \bmod n_1)$. Therefore, $n_e + 1 \leq w((n_e + 1) \bmod n_1)$.

Suppose now that $n_e + 1 = w((n_e + 1) \bmod n_1)$. Then, we have that $n_e + 1 \in S$. Moreover, since $n_e + 1 \notin \mathrm{msg}(S)$, we can assert that there exist $i, j \in \{1, \ldots, e\}$ such that $i < j$ and $n_e + 1 = w(i) + w(j)$. Thus $n_e = w(i) + w(j - 1)$ or $n_e = w(i-1) + w(j)$ (recall again that $n_e = w(n_e \bmod n_1)$). However, since $n_e \in \mathrm{msg}(S)$, it is not possible that $n_e = w(i)+w(j-1)$. Furthermore, $n_e = w(i-1)+w(j)$ only if $i-1 = 0$ and $j = e$. In such a case, $n_e+1 = w(i)+w(j) = w(1)+w(e) = w(1)+n_e$, which is a contradiction because $w(1) = n_2 > n_1 \geq 3$. Thus, we conclude that $n_e + 1 < w((n_e + 1) \bmod n_1)$. □

An immediate consequence of Proposition 4.12 is the following result.

Corollary 4.13 *If $m \in \mathbb{N}\backslash\{0, 1, 2\}$ and $a \in \mathbb{N}\backslash\{0\}$, then the numerical semigroups $S_1 = \langle m, am + 1\rangle$, $S_2 = \langle m, am + 1, am + 2\rangle$, ..., $S_{m-1} = \langle m, am + 1, \ldots, am + (m - 1)\rangle$ belong to the tree $G(\mathcal{MA}(m, am + 1))$. Moreover, S_{i+1} is a child of S_i for $i \in \{1, 2, \ldots, m - 2\}$.*

Let G be a tree with root v_r. The *depth* of a vertex x of G is the number of edges in the unique path connecting x and v_r. The *n-level* of G is the set of all vertices with depth n. Moreover, the *height* of G is the maximum depth of any vertex of G. Lastly, a *leaf* is a vertex that has no children (see Rosen 2000, 9.1.2).

It is not difficult to see that the next proposition follows from the results and comments in this subsection.

Proposition 4.14 *Let G be the tree $G(\mathcal{MA}(m, r))$.*

1. *The height of G is equal to $m - 2$.*
2. *If $S \in G$, then S belongs to the n-level of G if and only if $\mathrm{e}(S) = n + 2$.*
3. *If $S \in G$, then S is a leaf if and only if $\mathrm{M}(S) \equiv m - 1 \pmod{m}$.*

To finish this subsection, we will see that it is possible to compute the number of children of a MANS-semigroup $S \in \mathcal{MA}(m, r)$ if we know its Frobenius number $\mathrm{F}(S)$. Indeed, if $\mathrm{Ap}(S, m) = \{w(0), w(1), \ldots, w(m - 1)$, then $n \in \mathbb{N}$ could be a suitably monotone element for S whenever $n \bmod m > \mathrm{M}(S) \bmod m$. Therefore, we will only find suitably monotone elements in intervals $(w(i), w(i + 1))$ such that $\mathrm{M}(S) \leq i \leq m-2$ and $w(i+1) - w(i) > m$. Moreover, there will be $\frac{w(i+1)-w(i)-1}{m}$ suitably monotone elements in the interval $(w(i), w(i + 1))$ (precisely, the numbers congruent to $w(i + 1)$ modulo m). From here, the number of children will be given by the expression $\sum_{i=\mathrm{M}(S)}^{m-2} \frac{w(i+1)-w(i)-1}{m}$. From a simple computation, it follows the next result.

Proposition 4.15 *A numerical semigroup $S \in \mathcal{MA}(m, r)$ has $\left\lfloor \frac{\mathrm{F}(S)-\mathrm{M}(S)}{m} \right\rfloor + 1$ children in the tree $G(\mathcal{MA}(m, r))$.*

Let us see an illustrative example of the above proposition.

Example 4.16 Let S be the numerical semigroup given by $\langle 7, 15, 16\rangle$. Then $\mathrm{Ap}(S, 7) = \{w(0) = 0, w(1) = 15, w(2) = 16, w(3) = 31, w(4) = 32, w(5) = 47, w(6) = 48\}$ (that is, S is a MANS-semigroup) and $\mathrm{F}(S) = 41$. Thus, S has $\left\lfloor \frac{41-16}{7} \right\rfloor + 1 = 4$ children in $\mathcal{MA}(7, 15)$. Indeed, S has

- two children in $(w(2), w(3)) = (16, 31)$ (for $n = 17$ and $n = 24$),
- and two children in $(w(4), w(5)) = (32, 47)$ (for $n = 33$ and $n = 40$).

4.3 Irreducibility

By Propositions 3.19, 3.22 and 3.23, we characterise MANS-semigroups that are symmetric or pseudo-symmetric in the case of embedding dimension three. In the general case, we are far from being able to give similar results. Nevertheless, from Propositions 3.5, 4.10, and 4.15 in Rosales and García-Sánchez (2009), we can say a few words on this question. Let us recall those results.

Proposition 4.17 *Let* $C = \{w(0) = 0, w(1), \ldots, w(n-1)\} \subseteq \mathbb{N}$ *be such that* $w(i) \equiv i \pmod{n}$ *for all* $i \in \{0, \ldots, n-1\}$ *and* $n \in \mathbb{N}\backslash\{0\}$. *If* $S = \langle \{n\} \cup C \rangle$, *then* $\mathrm{Ap}(S, n) = C$ *if and only if* $w(i) + w(j) \geq w((i+j) \bmod m)$ *for all* $i, j \in \{1, \ldots, n-1\}$.

Proposition 4.18 *Let* S *be a numerical semigroup and let* $n \in S\backslash\{0\}$. *If* $\mathrm{Ap}(S, n) = \{a_0 < a_1 < \cdots < a_{n-1}\}$, *then* S *is symmetric if and only if* $a_i + a_{n-1-i} = a_{n-1}$ *for all* $i \in \{0, \ldots, n-1\}$.

Proposition 4.19 *Let* S *be a numerical semigroup with even Frobenius number* $\mathrm{F}(S)$ *and let* $n \in S\backslash\{0\}$. *Then* S *is pseudo-symmetric if and only if*

$$\mathrm{Ap}(S, n) = \{a_0 < a_1 \cdots < a_{n-2} = \mathrm{F}(S) + n\} \cup \left\{\frac{\mathrm{F}(S)}{2} + n\right\}$$

and $a_i + a_{n-1-i} = a_{n-1}$ *for all* $i \in \{0, \ldots, n-2\}$.

Our purpose is to construct symmetric (and pseudo-symmetric) MANS-semigroups by fixing the multiplicity, m, and the Frobenius number, F. Observe that, since we want a MANS-semigroup, we need that $F \equiv m-1 \pmod{m}$. Indeed, if S is a MANS-semigroup with multiplicity m and Frobenius number F, then $w(m-1) = F + m \in \mathrm{Ap}(S, m)$.

We suppose $m \geq 4$ to have an embedding dimension greater than three.

Proposition 4.20 *Let* m, F *be positive integers such that* $m \geq 4$ *and* $F \equiv m-1 \pmod{m}$. *Let*

$$C_1 = \left\{w(1) < \cdots < w\left(\left\lfloor \frac{m-1}{2} \right\rfloor\right)\right\} \subseteq \left\{m+1, m+2, \ldots, \left\lfloor \frac{F+m}{2} \right\rfloor\right\}$$

be such that $w(i) \equiv i \pmod{m}$ *for all* $i \in \left\{1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor\right\}$. *Moreover, let* $C = \{w(0) = 0\} \cup C_1 \cup C_2 \cup \{w(m-1) = F + m\}$, *where*

$$C_2 = \left\{w(m-1-i) = w(m-1) - w(i) \;\middle|\; i \in \left\{1, 2, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor\right\}\right\}.$$

Then $S = \langle \{m\} \cup C \rangle$ *is a symmetric MANS-semigroup if and only if* $w(i) + w(j) \geq w(i+j)$ *for all* $i, j \in \left\{1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor\right\}$.

Proof By construction, $C = \{w(0) = 0 < w(1) < \cdots < w(m-1) = F + m\}$ and, therefore, the necessity follows from Proposition 4.17.

For sufficiency, let us see that $w(i) + w(j) \geq w((i + j) \bmod m)$ for all $i, j \in \{1, 2, \ldots, m-1\}$. For this, we distinguish two cases.

- If $m \leq i + j \leq 2m - 2$, then $(i + j) \bmod m < \min\{i, j\}$ and the conclusion follows from the monotonicity of C.
- If $2 \leq i + j \leq m - 1$, then $\min\{i, j\} \leq \left\lfloor \frac{m-1}{2} \right\rfloor$. Taking $i \leq j$, we have two subcases.
  - If $j \leq \left\lfloor \frac{m-1}{2} \right\rfloor$, we get the result from the hypothesis.
  - If $j \geq \left\lfloor \frac{m+1}{2} \right\rfloor$, then we have $w(j) = w(m-1) - w(m-1-j)$ and $w(i+j) = w(m-1) - w(m-1-i-j)$ with $m-1-j, m-1-i-j \in \left\{1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor\right\}$. Moreover, $w(i) + w(j) \geq w(i + j)$ if and only if $w(i) + w(m-1-i-j) \geq w(m-1-j)$ and, since $i + (m-1-i-j) = m-1-j$, the result again follows from the hypothesis.

Finally, the numerical semigroup $S = \langle \{m\} \cup C \rangle$ is symmetric from the construction of C. □

Reasoning as in Proposition 4.20, we have the following result.

Proposition 4.21 *Let m, F be positive integers such that $m \geq 4$ and $F \equiv m - 1 \pmod{m}$. In addition, let F be an even integer (or, equivalently, let m an odd integer). Let*

$$C_1 = \left\{ w(1) < \cdots < w\left(\frac{m-1}{2}\right) = \frac{F}{2} + m \right\} \subseteq \left\{ m+1, m+2, \ldots, \frac{F}{2} + m \right\}$$

be such that $w(i) \equiv i \pmod{m}$ for all $i \in \left\{1, 2, \ldots, \frac{m-1}{2}\right\}$. Moreover, let $C = \{w(0) = 0\} \cup C_1 \cup C_2 \cup \{w(m-1) = F + m\}$, where

$$C_2 = \left\{ w(m-1-i) = w(m-1) - w(i) \,\middle|\, i \in \left\{1, \ldots, \frac{m-3}{2}\right\} \right\}.$$

Then $S = \langle \{m\} \cup C \rangle$ is a pseudo-symmetric MANS-semigroup if and only if $w(i) + w(j) \geq w(i + j)$ for all $i, j \in \left\{1, 2, \ldots, \frac{m-1}{2}\right\}$.

4.4 Arithmetic and Almost Arithmetic Semigroups

It would be interesting to study the relationships between MANS-semigroups and other families of numerical semigroups. In this subsection, we apply the results of the above subsections to analyse families associated with arithmetic and almost arithmetic sequences.

Following Lewin (1975), an *almost arithmetic numerical semigroup* is a numerical semigroup such that all but one of the generators correspond to an arithmetic sequence. That is, such a numerical semigroup is of the form

$$AA(a, d, k, p) = \langle a, a + d, \ldots, a + kd, p \rangle,$$

where a, d, k, p are positive integers such that $\gcd(a, d, p) = 1$, $a \geq 2$, $p \geq 2$, and $p \notin \{a - d, a + (k + 1)d\}$.

Following Ritter (1997) (or Selmer 1977), a *generalised arithmetic numerical semigroup* is a numerical semigroup of the form

$$GA(m, h, d, k) = \langle m, hm + d, \dots, hm + kd \rangle,$$

where m, h, d, k are positive integers such that $\gcd(m, d) = 1$, and $m \geq 2$. Furthermore, to have minimal generators, we assume that $k \leq m - 1$.

Let us observe that

- $A(m, d, k) = GA(m, 1, d, k) = \langle m, m + d, \dots, m + kd \rangle$ is an *arithmetic numerical semigroup*;
- $GA(m, h, d, k)$ is an almost arithmetic semigroup when $h \geq 2$ (in fact, $GA(m, h, d, k) = AA(hm + d, d, k - 1, m)$).

We are interested in determining the generalised arithmetic semigroups that are MANS-semigroups. For this, it is necessary that $d \equiv 1 \pmod{m}$ and, thus, we can suppose that $d = cm+1$ with $c \in \mathbb{N}$. Consequently, a generalised arithmetic semigroup is a MANS-semigroup if it is of the form

$$GM(m, h, c, k) = \langle m, (h + c)m + 1, \dots, (h + kc)m + k \rangle,$$

where m, h, k are positive integers such that $2 \leq m$ and $k \leq m - 1$, and c is a non-negative integer.

The following result is Corollary 3.6 of Rosales and García-Sánchez (2009) and is useful for our purpose.

Lemma 4.22 *Let S be a numerical semigroup with multiplicity m and assume that $\mathrm{Ap}(S, m) = \{w(0) = 0, w(1), \dots, w(m - 1)\}$ with $w(i) \equiv i \pmod{m}$ for all $i \in \{1, \dots, m-1\}$. Then S has maximal embedding dimension if and only if $w(i)+w(j) > w((i + j) \bmod m)$ for all $i, j \in \{1, \dots, m - 1\}$.*

Let us suppose that $w(i) = (h+ia)m+i$ for all $i \in \{1, \dots, m-1\}$. Then, by a direct calculation, we have that $w(i)+w(j) > w((i+j) \bmod m)$ for all $i, j \in \{1, \dots, m-1\}$ and, by Lemma 4.22, we get that $GM(m, h, c, m - 1)$ is a numerical semigroup with maximal embedding dimension. Moreover, by construction, $GM(m, h, c, m - 1)$ is a MANS-semigroup. Therefore, from Corollary 4.3, we deduce the following two results.

Proposition 4.23 *For any $k \leq m - 1$, $GA(m, h, d, k)$ is a MANS-semigroup if and only if $d \equiv 1 \pmod{m}$.*

Corollary 4.24 *For any $k \leq m - 1$, $A(m, d, k)$ is a MANS-semigroup if and only if $d \equiv 1 \pmod{m}$.*

By applying Theorem 4.4 and Corollary 4.24, we can build more examples of almost numerical semigroup that are MANS-semigroups.

Corollary 4.25 *Let $A(m, d, k)$ be a MANS-semigroup. If p is a suitably monotone element for $A(m, d, k)$ and $p \neq m + (k + 1)d$, then $AA(m, d, k, p)$ is a MANS-semigroup.*

To finish this subsection, we must study the almost numerical semigroups $AA(m, d, k, p)$ with $p \equiv 1 \pmod{m}$. From now on, we assume that $p = bm + 1$, for some $b \in \mathbb{N}\backslash\{0\}$, and that $m + d = am + (i + 1)$, for some $a \in \mathbb{N}\backslash\{0\}$ and $i \in \{1, 2, \dots, m - 2\}$.

The purpose is to analyse when the almost numerical semigroup $AA(m, (a-1)m + (i + 1), k, bm + 1)$ is a MANS-semigroup.

Let $S^* = \langle m, p\rangle = \langle m, bm + 1\rangle$. Then S^* is a MANS-semigroup with

$$\mathrm{Ap}(S^*, m) = \{0, bm + 1, 2bm + 2, \dots, (m - 1)bm + (m - 1)\}.$$

Since we want $m + d = am + (i + 1)$ to be a suitable monotone element for S^*, we need that $ibm + i + 1 \leq am + (i + 1) < (i + 1)bm + (i + 1)$ and, consequently, that $ib \leq a < (i + 1)b$. Now, to show what happens, we distinguish two cases: $a = ib$ and $ib + 1 \leq a \leq (i + 1)b - 1$.

- For $a = ib$, let $S_{1,ib} = \langle m, bm + 1, ibm + (i + 1)\rangle$. By Theorem 4.4, we have that $S_{1,ib}$ is a MANS-semigroup. Moreover, from Sect. 4.1,

$$\begin{aligned}\mathrm{Ap}(S_{1,ib}, m) = \{&0, bm + 1, \dots, ibm + i,\\ &ibm + (i + 1), (i + 1)bm + (i + 2), \dots, 2ibm + (2i + 1),\\ &2ibm + 2(i + 1), \dots\}.\end{aligned}$$

  (Observe that, to get $2ibm + 2(i + 1) \in \mathrm{Ap}(S_{1,ib}, m)$, we are implicitly assuming that $i \leq \frac{m-3}{2}$.)
  Since $2ibm + (2i + 1), 2ibm + 2(i + 1) \in \mathrm{Ap}(S_{1,ib}, m)$, we have that $(2ib - 1)m + 2(i + 1)$ is not a suitable monotone element for $S_{1,ib}$ and, therefore, we conclude that $S_{2,ib} = AA(m, (ib - 1)m + (i + 1), 2, bm + 1)$ is not a MANS-semigroup.
- For $ib + 1 \leq a \leq (i + 1)b - 1$, let $S_{k,a} = AA(m, (a - 1)m + (i + 1), k, bm + 1)$ with $k(i + 1) \leq m - 1$. Since $ibm < (a - 1)m + 1$, we have that

$$\begin{aligned}\mathrm{Ap}(S_{k,a}, m) = \{&0, bm + 1, \dots, ibm + i,\\ &am + (i + 1), (am + b) + (i + 2), \dots, (a + ib)m + (2i + 1),\\ &(2a - 1)m + 2(i + 1), (2a + b - 1)m + (2i + 3), \dots,\\ &(2a + ib - 1)m + (3i + 2), \dots\}.\end{aligned}$$

  Therefore, $S_{k,a}$ is a MANS-semigroup.

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