Universidad de Granada

Departamento de Ciencias de la Computación
e Inteligencia Artificial

Programa de Doctorado en Tecnologías de la Información y la Comunicación

Consensus Evolution Networks and Adjustment Cost for Group Decision Making: Large Scale, Trust Social Networks and Group Recommendation Approaches

Tesis Doctoral

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Granada, Septiembre de 2023

Universidad de Granada

Consensus Evolution Networks and Adjustment Cost for Group Decision Making: Large Scale, Trust Social Networks and Group Recommendation Approaches

MEMORIA PRESENTADA POR

Tong Wu

Para optar al grado de doctor por la Universidad de Granada dentro del

Programa de Doctorado Tecnologías de la Información y la Comunicación

Septiembre de 2023

DIRECTORES

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Departamento de Ciencias de la Computación
e Inteligencia Artificial

Editor: Universidad de Granada. Tesis Doctorales
Autor: Tong Wu
ISBN: 978-84-1195-213-2
URI: https://hdl.handle.net/10481/89870

La memoria titulada "*Consensus Evolution Networks and Adjustment Cost for Group Decision Making: Large Scale, Trust Social Networks and Group Recommendation Approaches*", que presenta D. Tong Wu para optar al grado de doctor, ha sido realizada dentro del Programa de Doctorado en "*Tecnologías de la Información y la Comunicación*", en el Departamento de Ciencias de la Computación e Inteligencia Artificial de la Universidad de Granada bajo la dirección de los doctores D. Francisco Herrera Triguero y D. Xinwang Liu.

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Granada, Septiembre de 2022

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This doctoral thesis has been developed with the financial support from China Scholarship Council, which provides a scholarship selected through a rigid academia evaluation process.

Acknowledgements

I would like to express my sincere gratitude to all those who helped me during my study at the University of Granada. Living and studying in Granada opened my eyes and was one of the most colorful moments.

First of all, I would like to thank my supervisor Francisco Herrera, who provided me with a valuable opportunity to study at the University of Granada. I was deeply impressed by his patience, enthusiasm, dedication, and superb professional ability. He would provide critical opinions and suggestions for each journal paper we cooperated, which significantly improved my scientific research level and our paper's quality. In addition to the study, he also took special care of us in daily life, including accommodation, registration of school roll, going back to China for vacation, and other procedures, which made us feel warm when we were in a foreign country. In a word, I am honor to be his student.

I would also like to thank my Ph.D. supervisor from Southeast University, Xinwang Liu. During the past seven years (from master to doctor), he guided me carefully and led me to scientific research. He taught me to look at academic research with a profound perspective and look at life's gains and losses with a healthy heart. During my study in Spain, he also maintained a weekly one-to-one online academic discussion with me. I am very grateful for his great support and thoughtful kindness in these years.

At the same time, I would also like to thank Qin Jindong, who is in the same major but some years ahead and is currently an associate professor at the Wuhan University of Technology. I benefited a lot from his academic guidance during the doctoral period to the present scientific research cooperation. I made many friends in Granada, Cristina, Xunjie Gou, Xia Liu, Ruxi Ding, Weike Zhang, Wanying Xie, and many other close friends. Thank them for their selfless help in my life and study in Granada. Besides, I am also deeply indebted to the China Scholarship Council for supporting me to experience the beautiful life and study at the University of Granada.

Finally, I would like to express my gratitude to my parents for their support and encouragement to me over the years. I am lucky to have you as my parents. Thank my brother, Dr. Heng Wu, for helping me solve many technical problems in my thesis. Thank my husband, Mr. Changyan Cao. His optimism is my unconditional support and encouragement. Thank my daughter and son, who keeps me motivated and hopeful every day!

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PhD dissertation

1 Introduction

With the development of the Internet and Web 2.0, social networking services appear, and people increasingly communicate through social media [1]. For example, the appearance of Facebook, Twitter, Wechat, etc., provides a luxurious carrier for users to discuss topics, participate in group decision-making (GDM) problems, exchange shopping experience, and so on. The feature that social media facilitates interpersonal interaction promotes the emergence of social network group decision making (SNGDM) [2, 3]. Furthermore, social media also breaks the limitations of interaction such as space, distance, and time, making the community scale larger, which accelerates the formation of large-scale group decision-making (LSGDM) [4].

SNGDM and LSGDM are both new decision-making methods based on traditional GDM. SNGDM mainly focus on the consensus behavior of experts decision makers (DMs) under social communications. LSGDM pays more attention to the scale of DMs, attributes, and alternatives to meet the decision-making needs in the age of social media and big data [5]. For example, e-government [6] provides an opportunity for large number of citizens to participate in the decision-making process, responding to the real demand for scientific, fair, and transparent decision-making. Social media makes citizens interact more frequently and influence each other more widely. Thus, LSGDM contains many complicated factors related to group behavior, like communication, conflict, and collaboration. Besides, decision-making members may differ in culture and knowledge background, cognitive ability, information expression form and judgment level, etc., which leads to high complexity and uncertainty of LSGDM [4]. Since traditional models and methods are gradually unable to deal with complex problems under social and big data environments, SNGDM and LSGDM are becoming new research hotspots in decision-making [7].

In addition to the traditional family or colleague group tour, home theater, etc., individuals on social networks can often form different communities spontaneously by exchanging their interests and hobbies, such as film forums, music forums, game communities, etc. Therefore, group recommendation has become a new challenge in the recommendation field [8]. Group recommendation not only needs to pay attention to individual preferences but also needs to comprehensively consider group preferences and provide a list of group satisfaction [9]. In commodity transactions, the group recommendation is from items to users, while the process of purchasing decision is from users to items, which is two homogeneous problems in opposite directions, both of which aim to make a group of users buy satisfactory commodities. Therefore, LSGDM, focusing on the study of decision-making attitude and behavior, has a certain reference for group recommendations [10].

Besides, users can exchange shopping experience with more users through online communication, and such interaction will affect their shopping behavior. In the past few years, group recommendation has become popular and developed rapidly with the increasing frequency of web-based social activities [11]. Thus, group recommendation also needs the support of SNGDM theory and method.

Traditional GDM is the foundation of SNGDM and LSGDM. Although the research on traditional GDM has become increasingly mature, there are still some drawbacks in group consensus research, like the evolution of consensus is rarely considered in essence. In terms of SNGDM, most of the study still focus on the role of social relationships in promoting consensus, ignoring the influence of relationship conflict on decision-making [12]. For LSGDM, most studies mainly concentrate on preference information expression[13], clustering analysis process (CAP) [14], and consensus reaching process (CRP) [15], there is also a small focus on non-cooperative behavior [16], minority opinions [17], and social network LSGDM [12]. Because of the increase of decision-making scale and consideration of social relationships, the consensus research of LSGDM and SNGDM will be more complicated and interesting than traditional GDM. In the application domain, although the group recommendation system has recently considered group consensus [18] and social relationships [19], there are few types of research focusing on the group recommendation model based on LSGDM methods.

Based on the above analysis, this thesis starts with an in-depth study of the CRP in traditional GDM, then comprehensively studies the CRP of SNGDM with social network analysis tools, then focus on the CRP of LSGDM from the point of clustering analysis, and finally applies a proposed LSGDM model which considers user behavior to a restaurant recommendation. Specifically, we mainly achieve the above objective from the following four aspects.

(1) To explore group consensus's composition and evolution, we introduce a new tool, consensus evolution networks (CENs). Inspired by social network analysis, we define CENs based on the similarity of DMs' preferences, analyze the structure of CENs with different consensus thresholds, determine the suitable agreed consensus thresholds based on the sensitive CEN, introduce a new consensus index according to network structure, and design a pairwise feedback adjustment method for improving consensus.

(2) To comprehensively study the CRP of SNGDM, we mainly focus on the influence of trust relationships from the following two aspects. (i) We propose several minimum cost consensus models based on implicit trust, which is obtained from opinions similarity. In this study, the moderator is regarded as a trustworthy coordinator to persuade individuals to reach a consensus that he/she expects to pay the lowest cost. The implicit trust of individuals to the moderator is computed based on the opinions' similarity. Two minimum cost consensus models and the dual models are proposed based on the implicit trust. (ii) We explore the CRP of SNGDM based on multiplex network structures. Firstly, we construct the trust consensus evolution multiplex network combining trust relationships and consensus relations. We then compute DMs' influence based on their comprehensive importance in the layer of trust networks and CENs using PageRank centrality. With DMs' influence, the consensus evolution and trust development are both investigated during interactions, in which, the consensus evolves under both the positive and negative effects of trust.

(3) To study the CRP of LSGDM, we study the clustering analysis process (CAP) from some interesting perspectives. (i) Firstly, we design a dynamic CAP based on CENs with different consensus thresholds according to a community detection method. We then evaluate the clustering validity based on intra-cluster and inter-cluster consensus levels. Finally, we give a feedback adjustment algorithm based on the clustering analysis. This study balances the

dynamic clustering and CRP based on CENs with higher consensus thresholds. (ii) Secondly, we regard the preference information and adjustment cost as dual attributes in the CAP. The former plays a significant role, and the latter represents a supporting role. We then compute the distance between individuals based on dual attributes. The adjustment cost is attached to the clustering analysis with a parameter determined by balancing the conflict between the intra-cluster total adjustment costs and intra-cluster consensus levels. We also define the initial clustering centers based on this parameter, combining the consensus levels and adjustment cost to obtain stable clustering results.

(4) To deal with the large-scale group recommendation characteristics, we propose an LSGDM method considering users' behavior and providing a corresponding group recommendation model for the online catering platform. Firstly, we compute the distance between users based on the weighted OWA (WOWA) operator to deal with the sparse evaluation information. Then, we utilize the Louvain method, which considers the primary community partition with the concept of modularity, to find users with similar shopping preferences and behaviors. Moreover, we discuss the polarization effects to obtain collective preferences of clusters and manage minority opinions with the importance induced ordered weighted averaging (I-IOWA) operator. An LSGDM method is proposed based on the above techniques and used to develop a group recommendation model for Dianping.com.

This thesis mainly consists of two parts: the first one illustrates the existing problems, the basic concepts, methods, and models that are used to deal with problems. The second part is a compilation of the main publications that are associated with this thesis. To improve the readability of subsequent content, we explain three essential abbreviations (GDM, SNGDM, and LSGDM) and group recommendation in Fig.1.

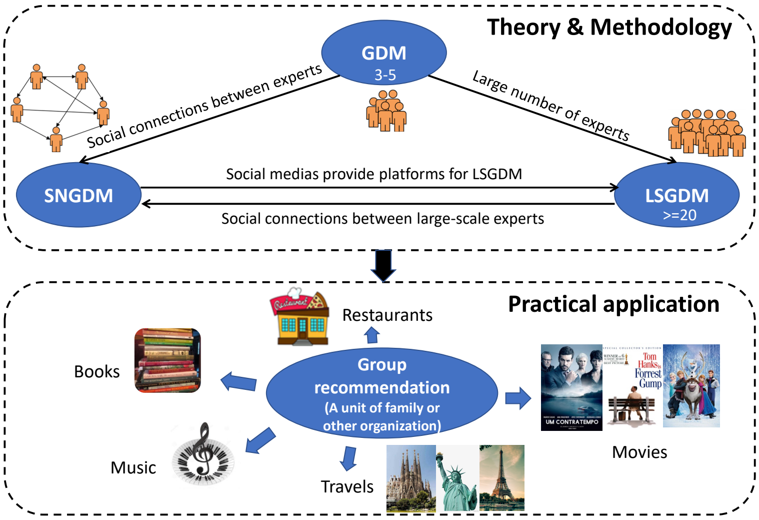


Fig.1 The connection between GDM, SNGDM, LSGDM and group recommendation

(1) GDM refers to the decision-making activity carried out by 3-5 people, including alternative evaluation, CRP, and solution selection process.

(2) SNGDM considers the social relationships among DMs based on GDM, including alternative evaluation, social relationships analysis, CRP, and solution selection.

(3) LSGDM refers to the decision-making activities carried out by a group of at least 20 people. The decision process is implemented as a dynamic process aimed to select the final

decision from multiple alternatives under several criteria/attributes. LSGDM often includes alternative evaluation, clustering analysis, CRP, and solution selection process.

(4) The group recommendation is a group-oriented technique of e-commerce development. It is an advanced business intelligence platform based on mass data mining, which provides information services and decision support to group users.

The rest of the thesis is organized as follows: Section 2 introduces some related preliminaries. Section 3 justifies the development of the thesis through discussing the basic ideas and challenges of current researches. Section 4 presents the objectives of the thesis. Section 5 introduces the methodologies used in the thesis. Section 6 discusses the results of the proposals in the thesis. Section 7 presents a discussion of the results obtained in the thesis. Section 8 gives conclusions of the thesis. Finally, some future works are discussed in Section 9.

Introducción

Con el desarrollo de Internet y Web 2.0, los servicios de redes sociales aparecen y las personas se comunican con más frecuencia mediente las redes sociales [AY12]. Por ejemplo, la aparición de Facebook, Twitter, Wechat, etc., ofrece un operador lujoso para que los usuarios adquieran experiencia de compra e influyan en su comportamiento de compra. Esta característica de la red social brinda nuevas oportunidades para la comercialización, lo que hace que el modelo tradicional de comercio electrónico evolucione a un nuevo modelo de comercio social [LT11]. La recomendación del grupo de comercio social (SCGR) es una estrategia recomendada basada en las relaciones sociales de los usuarios. Es el elemento central del comercio social y una parte esencial de la era inteligente [LWL13]. Las redes sociales rompen los límites de la interacción interpersonal, como el espacio, la distancia y el tiempo, haciendo que la escala de la comunidad se amplíe. Por eso, SCGR tiene las características de una gran escala.

La SCGR a menudo necesita considerar la preferencia del grupo y dar el artículo recomendado con alta satisfacción general. Por eso, SCGR es esencialmente un problema de toma de decisiones grupales a gran escala (LSGDM) [ZLHF19]. LSGDM es un nuevo método de toma de decisiones basado en la toma de decisiones grupal tradicional (GDM) para satisfacer las necesidades de toma de decisiones en la era de las redes sociales y big data [TL19]. Por ejemplo, la aparición del gobierno electrónico [Cha08] proporciona una oportunidad para más y más personas a participar en el proceso de toma de decisiones, respondiendo a la demanda real de toma de decisiones científica, justa, abierta y transparente. El aumento y el crecimiento explosivo de las redes sociales hacen que las que tomen decisiones interactúen más frecuentemente e influyan entre sí de manera más amplia. Por eso, en el proceso de LSGDM, hay muchos factores complejos relacionados con el comportamiento del grupo, como la comunicación, el conflicto y la colaboración entre los miembros del grupo. Además, los diferentes miembros que toman decisiones difieren en el contexto cultural y de conocimiento, la capacidad cognitiva, la forma de expresión de información y el nivel de juicio, etc., lo que conduce a la alta complejidad e incertidumbre de LSGDM [DPW+20]. Debido a la complejidad de interacción, la escala del LSGDM está limitada a no más de 50 en la investigación actual [ZDHV18, DWS+19, PMH14]. Sin embargo, el modelo tradicional de GDM y el método con expertos 3-5 todavía son incapaces de tratar con el problema de LSGDM. Por eso, LSGDM se está convirtiendo en un nuevo punto caliente de investigación en el campo de toma de decisiones [Pal18].

En la actualidad, la mayoría de los estudios de LSGDM se centran en la expresión de información de preferencia [ZDHV18], el análisis de agrupamiento [DWS+19], el proceso de llegar al consenso (CRP) [PMH14], también hay un pequeño enfoque en el comportamiento no cooperativo [QPM15] y la opinión minoritaria [XDC15] A medida que los investigadores se dan cuenta del papel significativo y activo de las redes sociales en GDM, la toma de decisiones de grupos en redes sociales (SNGDM) aparece gradualmente y desarrolla rápidamente [UKD+19, DZZ+18], las relaciones sociales también se consideran en el campo LSGDM [DWSH19]. Aunque la investigación sobre GDM se ha vuelto cada vez más madura, todavía hay algunos inconvenientes en la investigación de consenso del grupo. Por ejemplo, la mayoría de la investigación se centra en la expresión de preferencias y el ajuste de retroalimentación de preferencias, y la evolución del consenso rara vez se considera en esencia. Debido a la complejidad del LSGDM, su investigación de consenso es más complicada que la GDM tradicional. Además, en los últimos años, la recomendación grupal se ha vuelto popular y se ha desarrollado rápidamente con la frecuencia creciente de actividades sociales basadas en la web [FBST18]. Recientemente el sistema de recomendación grupal ha considerado el consenso de grupo [BAC06] y las relaciones sociales [CSA16]. Aunque la esencia de SCGR puede considerarse como el problema de LSGDM, existen pocos tipos de investigación centrados en el modelo de recomendación grupal basado en el método de LSGDM.

Basado en el análisis de arriba, esta tesis comenzará con el GDM tradicional, tomará la SCGR como el objeto de investigación y hará del análisis de redes sociales una herramienta para enfocarse en el análisis de agrupamiento, CRP y otros problemas en LSGDM basados en el comportamiento del usuario. Específicamente, logramos principalmente el objetivo anterior desde los siguientes cuatro aspectos.

(1) Para explorar la composición y evolución del consenso de grupo, presentamos una nueva herramienta, las redes de evolución del consenso (CEN). Los CEN estudian la CRP basados en técnicas de análisis de redes sociales. Definimos los CENs, determinamos los umbrales de consenso convenidos aptos basado en el umbral de consenso sensible, introducimos un nuevo índice de consenso y diseñamos un método de ajuste de retroalimentación por pares para mejorar el consenso.

(2) Para explorar los efectos de la confianza en el CRP de SNGDM, nos centramos principalmente en los siguientes dos aspectos. (a) Proponemos varios modelos de consenso de costo mínimo basados en la confianza implícita, que es obtenido de la similitud de opiniones. En este estudio, se considera que el moderador es un coordinador confiable para persuadir a las personas de llegar a un consenso que espera a pagar el costo más bajo. La confianza implícita de las personas en el moderador se calcula basado en la similitud de las opiniones. Se proponen dos modelos de consenso de costo mínimo y los modelos duales basados en la confianza implícita. (b) Exploramos el consenso de grupo basado en estructuras de red multicine. En primer lugar, construimos la red multicine de evolución de consenso de confianza combinando relaciones de confianza y relaciones de consenso. El consenso y la evolución de la confianza ambos se investigan en el ajuste del consenso. Luego calculamos la influencia de los expertosbasao en su importancia integral en la capa de redes de confianza y CEN utilizando la centralidad de PageRank. Además, exploramos la evolución del consenso bajo los efectos tanto positivos como negativos de la confianza basados en la influencia de los expertos. Al final, consideramos el desarrollo de relaciones de confianza basadas en la negociación entre los expertos.

(3) Para estudiar la PCR en LSGDM, equilibramos el análisis de agrupamiento y la PCR y consideramos la influencia de los factores de agrupación en la PCR. (a) En primer lugar, diseñamos un proceso de análisis de agrupamiento dinámico basado en CEN con diferentes umbrales de consenso basados en el método de detección de la comunidad. Luego evaluamos la validez de la agrupación basado en los niveles de consenso dentro del grupo y entre grupos. Al final, damos un algoritmo de ajuste de retroalimentación basado en el análisis de agrupamiento. Este estudio equilibra la agrupación dinámica y la PCR basada en CEN con umbrales de consenso más altos. (b) En primer lugar, consideramos la información de preferencia y el costo de ajuste como atributos duales del análisis de agrupamiento. El primero juega un papel importante, y el segundo representa un papel de soporte. Luego calculamos la distancia entre los individuos basado en los atributos duales. El costo de ajuste se adjunta al análisis de agrupación con un parámetro, que se determina equilibrando el conflicto entre los costos de ajuste total dentro del grupo y los niveles de consenso dentro del grupo. Basado en este parámetro, también definimos los centros de agrupación iniciales que combinan los niveles de consenso y el costo de ajuste.

(4) Para tratar las características a gran escala de SCGR, proponemos un modelo LSGDM que considera el comportamiento de los usuarios y brindamos un modelo SCGR correspondiente para una plataforma de compras en línea. En primer lugar, computamos la distancia entre

usuarios basado en el operador ponderado OWA (WOWA) para tratar la escasez de datos de la información de evaluación. Luego, utilizamos el método de Louvain, que considera la partición previa de la comunidad con el concepto de modularidad, para encontrar usuarios con preferencias y comportamientos de compra similares. Además, discutimos los efectos de polarización para obtener preferencias colectivas de grupos y manejar opiniones minoritarias con la importancia del operador de promediado ponderado ordenado (I-IOWA). Se propone un método LSGDM basado en las técnicas anteriores. Finalmente, desarrollamos un modelo SCGR basado en el método LSGDM e ilustramos su efectividad en Dianping.com.

Esta tesis principalmente contiene dos partes: la primera ilustra los problemas existentes, los conceptos básicos y los modelos, y los resultados obtenidos de los modelos propuestos. La segunda parte es una recopilación de las principales publicaciones asociadas con esta tesis. Para mejorar la legibilidad del contenido subsecuente, explicamos las tres abreviaturas esenciales que a menudo aparecen en esta tesis.

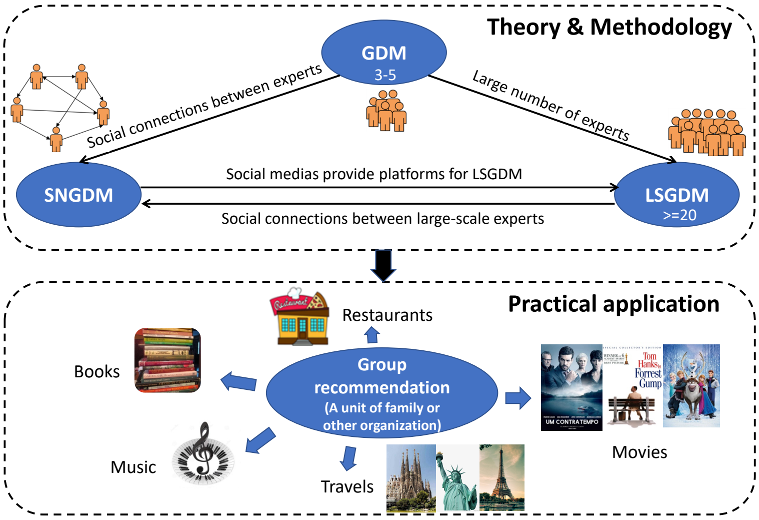


Fig. 1 The connection between GDM, SNGDM, LSGDM, and group recommendations

(1) GDM se refiere a las actividades de toma de decisiones realizadas por 3-5 personas, incluyendo la evaluación alternativa, CRP y el proceso de selección de soluciones.

(2) SNGDM considera las relaciones sociales entre los DMs basadas en GDM, incluyendo la evaluación alternativa, el análisis de las relaciones sociales, CRP y la selección de soluciones.

(3) LSGDM se refiere a las actividades de toma de decisiones realizadas por un grupo de al menos 20 personas. El proceso de decisión se implementa como un proceso dinámico para seleccionar la decisión final de múltiples alternativas bajo varios criterios / atributos. LSGDM a menudo incluye evaluación alternativa, análisis de agrupamiento, CRP y proceso de selección de soluciones.

(4) SCGR es una técnica de desarrollo de redes sociales y comercio electrónico. Es una plataforma de inteligencia empresarial avanzada basada en la minería de datos de masa, que brinda servicios de información y soporte de decisiones a los usuarios del grupo.

El resto de la tesis es organizada de la siguiente forma: la Sección 2 introduce algunos preliminares relacionados. La Sección 3 justifica el desarrollo de la tesis mediante la discusión de las ideas básicas y los desafíos de las investigaciones actuales. La Sección 4 presenta los objetivos de la tesis. La Sección 5 presenta las metodologías usadas en la tesis. La Sección 6 discute los resultados de las propuestas en la tesis. La Sección 7 presenta una discusión de los resultados obtenidos en la tesis. La Sección 8 da conclusiones de la tesis. Finalmente, algunos trabajos futuros son discutidos en la Sección 9.

2 Preliminaries

2.1 Group decision making

The process of GDM is generally composed of preference information expression process, CRP, and alternative selection process. The fuzzy preference relations (FPRs) are commonly used to represent DMs' preferences. The distance and similarity method based on FPRs is significant for consensus measurement. Alternative selection process is the result of the aggregation of the latest opinions after the group has reached consensus or communication has ceased, and the ordered weighted averaging (OWA) operator is a primary aggregation operator.

2.1.1 Preference representation

Definition 1. *[Tan88] An FPR F is a fuzzy set on the alternative set $X \times X$, which is characterized by a membership function $\mu_F : X \times X \to [0,1]$, where $\mu_F(x_i, x_j) = f_{ij}$ is interpreted as the preference degree of alternative x_i over x_j $(i, j = 1, 2, ..., n)$: $f_{ij} = 0.5$ indicates indifference between x_i and x_j. $f_{ij} > 0.5$ indicates that x_i is preferred to x_j, $f_{ij} < 0.5$ indicates that x_i is inferior to x_j, and fulfilling $f_{ij} + f_{ji} = 1$. Generally, the FPR of DM d_k $(k = 1, 2, ..., m)$ alternative x_i over x_j can be represented as $F_k = \left(f_{ij}^k\right)_{n\times n}$, and $f_{ij}^k + f_{ji}^k = 1$.*

2.1.2 Consensus reaching process

According to different purposes, the consensus level calculation can be divided into two types: the consensus level between DMs and the consensus level between DMs and groups. Since consensus level is often measured based on distance functions [CTGdMHV13], the commonly used distance functions are also divided into two types, namely the distance between the pairwise DMs and the distance between DMs and the group [DZZ+18].

The consensus level can be computed based on the distance between DMs' preferences.

Definition 2. *[DZHV16] A similarity matrix $SM_{kl} = \left(sm_{ij}^{kl}\right)_{m\times m}$ between DM d_k and d_l on the preference of alternative x_i over x_j is defined as:*

$$SM_{kl} = \begin{pmatrix} - & ... & sm_{1i}^{kl} & ... & sm_{1m}^{kl} \\ ... & - & ... & ... & ... \\ sm_{i1}^{kl} & ... & - & ... & sm_{im}^{kl} \\ ... & ... & ... & - & ... \\ sm_{m1}^{kl} & ... & sm_{mi}^{kl} & ... & - \end{pmatrix}_{m\times m} \tag{I.1}$$

where sm_{ij}^{kl} is computed based on FPRs by means of a similarity function introduced in [DZHV16]: $sm_{ij}^{kl} = 1 - \left|f_{ij}^k - f_{ij}^l\right|$, $i, j = 1, 2, ..., n; i \neq j$, and $k, l = 1, 2, ..., m$; $k \neq l$.

A consensus matrix $CM = (cm_{ij})_{n\times n}$ is computed by aggregating similarity matrices with experts' weights, $w_{kl} \in [0,1]$ associated with each pair of experts (d_k, d_l), $k < l$. Each element $cm_{ij} \in [0,1]$, $i \neq j$, is computed as the weighted average of similarity degrees:

$$cm_{ij} = \frac{\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} w_{kl} sm_{ij}^{kl}}{\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} w_{kl}} \tag{I.2}$$

Once the consensus matrix is computed, consensus degree is computed at three different levels:

(1) Level of pair of alternatives cp_{ij}, obtained from CM: $cp_{ij} = cm_{ij}$, $i, j = 1, 2, ..., n$, $i \neq j$.

(2) Level of alternatives ca_i: The level of agreement on each alternative $x_l \in X$ is computed as

$$ca_i = \frac{\sum_{i=1, i\neq j}^{n} cp_{ij}}{n-1} \tag{I.3}$$

(3) Level of preference relation (overall consensus degree, OCL)

$$OCL = \frac{\sum_{i=1}^{n} ca_i}{n} \tag{I.4}$$

The consensus level between individual DMs and the group is defined as below.

Definition 3. *[CMM⁺08] The consensus level associated with the DM d_k and the group is defined based on his/her individual preference $F_k = \left(f_{ij}^k\right)_{n\times n}$ and the group FPR $\bar{F} = \left(\bar{f}_{ij}\right)_{n\times n}$ as:*

$$CL_k = 1 - \sum_{i,j=1; i\neq j}^{n} \frac{\left|f_{ij}^k - \bar{f}_{ij}\right|}{n\left(n-1\right)} \tag{I.5}$$

where $CL_k \in [0, 1]$.

Based on the consensus level CL_k, the overall consensus level OCL associated with the group FPR can be computed as [LRM⁺19]:

$$OCL = \frac{1}{m}\sum_{k=1}^{m} CL_k \tag{I.6}$$

where $OCL \in [0, 1]$.

The preference of group can be obtained by optimizing individual preferences. For a GDM problem, let $c_k \in [0, 1]$ represents the unit adjustment cost of the DM d_k. The traditional minimum adjustment cost consensus model was given based on utility preference [BAE07, GZF⁺15, ZKP19]:

$$\min\ \phi\left(o\right) = \sum_{k=1}^{m} c_k \left|\bar{o} - o_k\right| \tag{I.7}$$

$$s.t.\quad o \in O$$

where o_k represents the opinion of the DM d_k and $\bar{o}$ denotes the consensus opinion.

2.1.3 Selection process

The selection process is used to obtain the ranking of alternatives from a group of prefernce relations. The collective preference used to rank alternatives is usually obtained using aggregation operators, such as OWA and its family operators.

Definition 4. *[Yag88] A mapping* Φ *from* $R^n \to R\,(R = [0,1])$ *is called an OWA operator of dimension* n *if associated with* Φ *is a weighting vector* $w = (w_1, w_2, ..., w_n)^T$ *such that* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1$ *, and it is defined to aggregate the set of arguments* $(a_1, a_2, ..., a_n)$ *according to the following expression:*

$$\Phi_{OWA}\,(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)} \tag{I.8}$$

where $\{\sigma(1), ..., \sigma(n)\}$ *is a permutation of* $\{1, ..., n\}$ *with* $a_{\sigma(j-1)} \geq a_{\sigma(j)}$ *for all* $i = 2, ..., n$ *and* $a_{\sigma(j)}$ *is the jth largest element in the collection* $(a_1, ..., a_n)$.

A number of methods have been proposed for determining the weighting vector since it is a key point in the OWA operator to obtain the aggregating results with its associated weights [Yag88, Yag96, Xu05, Yag12]. Yager [Yag88] proposed the regular increasing monotone (RIM) quantifier for obtaining the OWA weighting vectors via linguistic quantifiers, which is guided by verbally expressed concepts.

Definition 5. *[Yag88] Given a RIM quantifier* Q*, the OWA weighting vectors can be obtained using* $w_j = Q\,(j/n) - Q\,(j-1/n)$ *.The degree of orness associated with the OWA operator* Φ *is defined as*

$$orness\,(Q) = \frac{1}{n-1} \sum_{j=1}^{n} ((n-j) * w_j) \tag{I.9}$$

Let $n \to \infty$*, a degree of orness can be associated with this quantifier as*

$$\begin{aligned} orness\,(Q) &= \lim_{n\to\infty} \tfrac{1}{n-1} \sum_{j=1}^{n-1} Q\left(\tfrac{j}{n}\right) \\ &= \int_0^1 Q\,(y)\,dy \end{aligned} \tag{I.10}$$

Definition 6. *[Yag96] Yager define the parameterized family of RIM quantifiers*

$$Q\,(y) = y^\alpha \alpha \geq 0 \tag{I.11}$$

then $orness\,(Q) = \int_0^1 y^\alpha dy = \frac{1}{\alpha+1} y^\alpha|_0^1 = \frac{1}{\alpha+1}$

2.2 Social network analysis

In this section, we introduce some basic definitions of social networks and social network analysis techniques.

2.2.1 Some basic definitions

The definitions of general social networks, trust networks and some structured indicators are introduced as follows.

Definition 7. *[AW03] For a simple weighted social network* $G = \{D, E, W\}$ *,* $D = \{d_1, d_2, ..., d_m\}$ *denotes the non-empty set of* n *DMs,*$E = \{e_{kl}\}\,(k, l = 1, 2, ..., m, k \neq l)$ *represents the finite set of social connections between DMs, and* $W = \{w_{kl}\}\,(k, l = 1, 2, ..., m, k \neq l)$ *denotes the weights of these connections,i.e.,*e_{kl} *denotes the connection between DM* d_k *and* d_l *with the weight* w_{kl}.

When the connections between DMs in Definition 7 have a specific meaning, like trust relationships, the general social networks become trust networks $G = \{D, E, T\}$, where $E = \{e_{kl}\}\,(k, l = 1, 2, ..., m, k \neq l)$ represents the finite set of trust relationships between DMs, and $T = \{t_{kl}\}\,(k, l = 1, 2, ..., m, k \neq l)$ denotes trust levels, i.e., DM d_k trust d_l with level t_{kl}.

Definition 8. *[WF94] For a simple unweighted network $G = (D, E)$,its density $d\,(G)$ can be used to describe the density of edges between nodes:*

$$d\,(G) = \frac{2\,|E|}{m\,(m-1)} \tag{I.12}$$

where $|E|$ and m are the number of edges and nodes in network G, respectively.

Social networks exhibit different forms of network structure due to different distribution of node relationships, such as small-world networks and complex networks. The clustering coefficient can be used to determine whether a graph is a small-world network and identify the connections between nodes in complex networks [WS98].

Definition 9. *[WS98] For a simple weighted network $G = \{D, E\}$, let $N_k = \{d_l : e_{kl} \in E\}$ denote the direct neighbor nodes of the DM d_k, the local clustering coefficient LCC_k of d_k in G is determined as:*

$$LCC_k = \frac{2\,|\{e_{lh} : d_l, d_h \in N_k, e_{lh} \in E\}|}{N\,(d_k)\,[N\,(d_k) - 1]} \tag{I.13}$$

where d_l and d_h are the element of set N_k, i.e., they are neighbor nodes of the DM d_k, $N\,(v_k)$ denotes the number of neighbors of the DM d_k, $|\{e_{lh} : d_l, d_h \in N_k, e_{lh} \in E\}|$ represents the number of edges between the neighbors of the DM d_k, $N\,(d_k)\,[N\,(d_k) - 1]/2$ represents the number of edges between d_k and its neighbors.

The total clustering coefficient CC of the network G can be obtained as:

$$CC = \frac{1}{N} \sum_{k=1}^{n} LCC_k \tag{I.14}$$

where $CC \in [0, 1]$, G is a complete network when $CC = 1$.

2.2.2 Social network analysis techniques

Social network analysis is a research method to study the relationship between a group of actors. Centrality is a relatively common social network analysis index to determine the weights of DMs in social networks, and the community detection algorithm is commonly used to analyze complex networks. There are many kinds of centrality indicators and community detection algorithms. Next, we mainly introduce the methods used in this thesis.

PageRank centrality is a well-known tool used to sort web pages by Google [BP98].

Definition 10. *[BP98] For a simple weighted network $G = \{V, E, W\}$ with a constant weight of 1, the PageRank centrality p_k of the DM d_k can be determined as:*

$$p_k = \alpha \sum_{l} W_{kl} \frac{p_l}{g_l} + (1 - \alpha)\,\frac{1}{m} \tag{I.15}$$

where W equals to the adjacency matrix of G, $g_l = \max\left(1, \sum_h W_{hl}\right)$, $u = 1, 2, ..., m$, the DM d_k randomly walks to one of its neighbors with probability α and walks to other neighbors with probability $1 - \alpha$,$\alpha > 0$ is called the damping factor.

In community detection, the modularity is a key technique for measuring link density within and between communities [New04]. The closer the modularity is to 1, the more stable the community structure divided by the network and the better the community discovery quality. Therefore, Blondel et al. [BGLL08] proposed a commonly used Louvain method to detect communities for large networks based on modularity gain ΔQ.

Definition 11. *[BGLL08] Supposing there is a large network LG that is classified into t subgroups $LG = \{SG_1, SG_2, ..., SG_t\}$, d_r^k $(r = 1, 2, ..., t)$ denotes the kth DM belongs to the subgroup SG_r, then the gain in modularity ΔQ is computed by:*

$$\Delta Q = \left[\frac{\sum_{in} + 2W_{r,in}}{M_{rs}} - \left(\frac{\sum_{tot} + W_r}{M_{rs}}\right)^2\right] - \left[\frac{\sum_{in}}{M_{rs}} - \left(\frac{\sum_{tot}}{M_{rs}}\right)^2 - \left(\frac{W_r}{M_{rs}}\right)^2\right] \tag{I.16}$$

where $\sum_{in}$ denotes the sum of the edge weights in the subgroup SG_s, $\sum_{tot}$ is the sum of the edge weights incident to DMs in SG_s, $W_{r,in}$ represents the sum of the edge weights from the DM d_r^k to DMs in SG_s, $W_r = \sum_{r=1}^{t} c_{rs}$ is the sum of the edge weights incident to DM d_r^k, $M_{rs} = \sum_{r,s=1,r\neq s}^{t} c_{rs}$ means the sum of the weights of all DMs in LG, where c_{rs} means the weights between SG_r and SG_s, and it is obtained based on the sum of the edge weights of DMs belong to SG_r and SG_s.

If $\Delta Q \geq 0$, then remove d_r^k from SG_r to SG_s with $\max \Delta Q$, otherwise, the final community detection result is obtained.

2.3 Group recommendation system

In this section, we introduce some basic definitions of social networks and social network analysis techniques.

The group recommendation system is one of the most challenging and important studies in the field of recommendation system because it needs to produce a list to the target group with a satisfactory consensus level [38]. Group recommendation generally has two forms [39]. One is that aggregates individual preferences to obtain group preferences to generate the group recommendation list. The other is that aggregates personalized recommendations to obtain the group recommendation list. Both forms need to consider the consensus among group members to avoid causing recommendation conflicts. The process of group recommendation considering CRP based on the above two forms is shown in Fig.2.

(1) In the preference aggregation process, group preferences are evaluated by integrating individual preferences.

(2) In recommendation aggregation, the personalized recommendation list of each user is integrated to generate the group recommendation list. Personalized recommendations can be obtained according to multiple recommendation methods, such as collaborative filtering.

(3) In the CRP, unreasonable recommendations in the initial group recommendation list are removed according to group preferences, and the final group recommendation list is produced when group members reach an agreed consensus level.

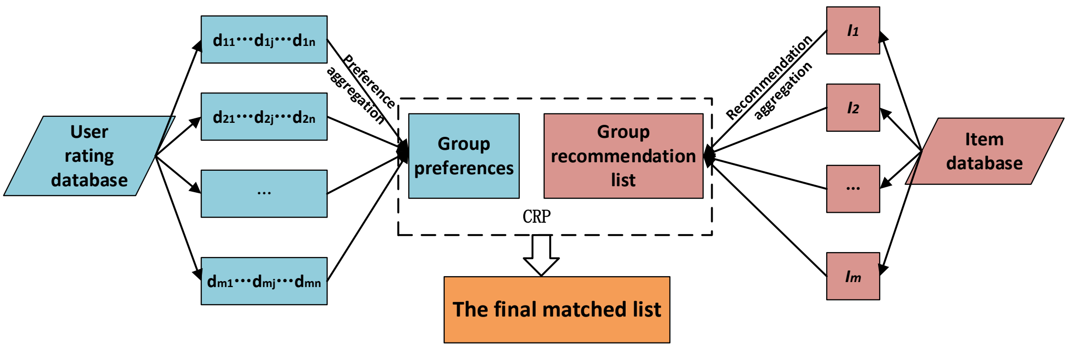


Fig.2 The process of the group recommendation

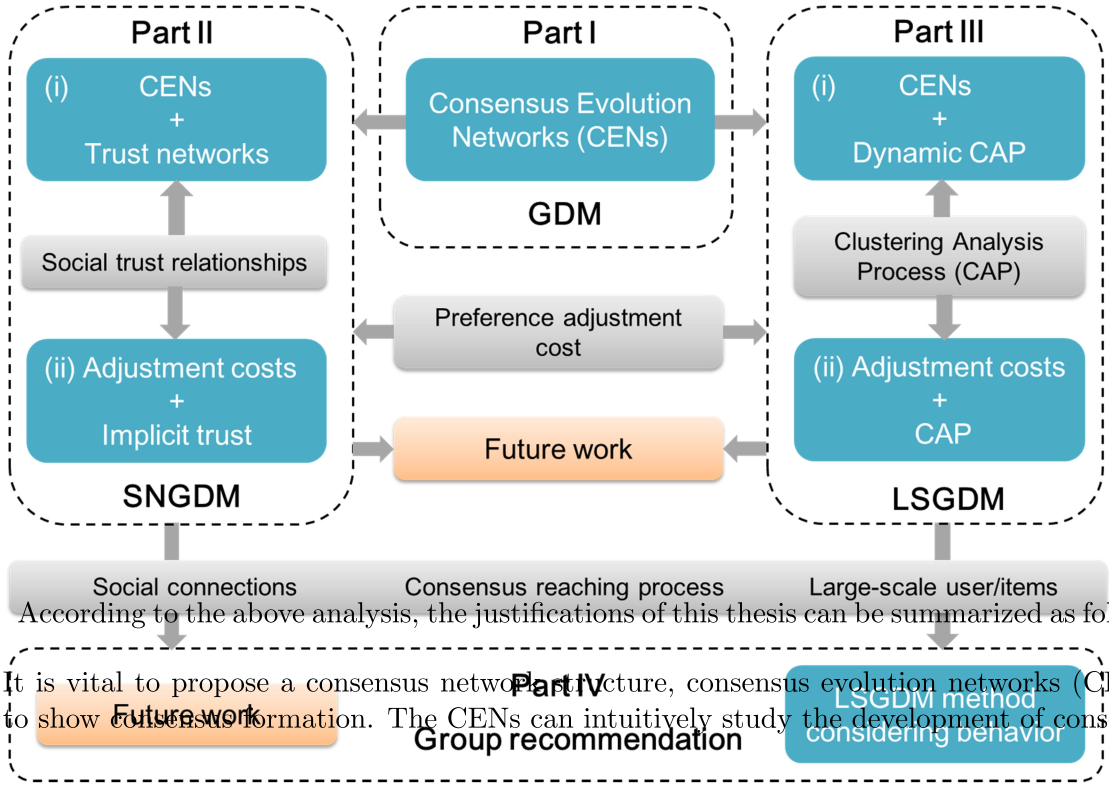
3 Justification

As we mentioned above, SNGDM and LSGDM are new forms developed from traditional GDM under the social context and big data environment, and they are also the decision-making basis for group recommendation. According to the connection between GDM, SNGDM and LSGDM shown in Fig.1, we can find that the social media accelerates the formation of LSGDM and the theory/method of SNGDM is essential to handle LSGDM under social context; the theory/method of LSGDM also has some implications for solving the SNGDM problem in complex networks; the theory/method of SNGDM and LSGDM are the basis to solve the group recommendation problems with both social and large-scale characteristics.

The basis, tools and, framework that are used to improve the accuracy and satisfaction of group recommendations based on SNGDM and LSGDM are shown in Fig.3. The main basis that is used to propose SNGDM, LSGDM, and group recommendation approaches are CENs, trust relationships, and the preference adjustment cost. Social network analysis provides main tools for this thesis, like the density, centrality, and community detection methods, which can be used to compute DMs influence, measure the consensus levels and classify large number of DMs into communities.

CENs is proposed for studying consensus of traditional GDM in our first essential work namely as Part I. In Part II, we further analyze the CRP of SNGDM from two aspects: (i) we show the effect of trust relationships on consensus. (ii) we combine CENs and trust networks to survey their interaction effect. In Part III, we investigate the CRP of LSGDM, starting with the point of clustering analysis from two aspects: (i) we balance the CRP and dynamic CAP based on CENs. (ii) we consider the effect of the preference adjustment cost in the CAP. In Part IV, we provide a group recommendation approach based on a proposed LSGDM method that considers user behavior and minority opinions.

In addition to Part I-Part IV, Fig.3 also marks the challenges between SNGDM and LSGDM, and the application of SNGDM in group recommendation.



According to the above analysis, the justifications of this thesis can be summarized as follows:

Part I. It is vital to propose a consensus network structure, consensus evolution networks (CENs), to show consensus formation. The CENs can intuitively study the development of consensus relations. Based on this structure, we can deeply explore consensus measurement and CRP based on social network analysis tools.

Part II (i). In current SNGDM studies, the role of the trust relationship is relatively essential and widely studied. However, the promotion effect of trust on consensus is still at the hypothesis stage without in-depth analysis. To examine the effectiveness of trust on group consensus, some minimum cost consensus models should represent the relationship between trust and the preference adjustment cost. Besides, the rationality of the adjustment cost perceived by DMs should be checked according to their weights.

Part II (ii). A perception of interpersonal incompatibility among DMs can be caused in decision scenarios by the conflict between internal preferences and external relationships. Such incompatibility can be detrimental to decision quality because it may provoke DMs' negative decision behavior. In other words, the role of trust in promoting consensus may not be exact in all cases. Besides, trust relationships can be built or interrupted during negotiations. Therefore, the evolution of trust networks and CENs should be explored deeply.

Part III (i). The CRP of LSGDM is more complicated than the traditional GDM and the CAP is often implemented in LSGDM to reduce the complication. But a conflict may arise between the CAP and CRP since the CAP is carried out when there are differences among DMs, while CRP is to minimize such differences. To deal with this conflict, we study the dynamic characteristics of the CAP based on CENs. Besides, the clustering validity should be checked to facilitate the following CRP. The conflict between the dynamic CAP and CRP is solved based on CENs with higher consensus thresholds.

Part III (ii). The clustering elements will affect the clustering results and then affect the CRP. When the clustering analysis is carried out mainly based on preference information, DMs with different preference adjustment costs might be classified into the same clusters, which will significantly increase the adjustment difficulty. To reduce the complexity of consensus negotiation, we consider the preference adjustment cost as a secondary influence factor in the CAP in addition to the primary factor of preferences.

Part IV. As mentioned above, LSGDM is more complicated than traditional GDM. In addition to clustering analysis and CRP, we also need to pay attention to the incompleteness of preference information, decision behavior, and minority opinions in LSGDM. To deal with the large-scale characteristics of the group users and improve the satisfaction of group members, we apply LSGDM models in group recommendation.

4 Objectives

Since consensus plays a vital role in GDM, this thesis aims to take consensus as the mainline to study the theories and methods of traditional GDM, SNGDM, and LSGDM, and finally apply LSGDM models to group recommendations. For traditional GDM, we mainly focus on consensus research based on CENs. For SNGDM, we primarily use social network analysis tools to study the social influence on group consensus. For LSGDM, we study the problem of large-scale consensus from the perspective of clustering analysis, and also consider other issues, such as the incompleteness of preference information, decision behavior, and minority opinions. Finally, we try to apply the LSGDM methods to group recommendations to promote the transformation of scientific research achievement. Specifically, the objectives of this thesis are summarized as follows:

Part I. **To explore the evolution of consensus in GDM deeply.** Construct CENs based on the preference similarity among DMs. According to the clustering coefficient of CENs, the suitable agreed consensus threshold is determined based on the sensitive consensus threshold. Besides, a new consensus index is designed based on the structure of CENs, and a pairwise feedback adjustment method is introduced for improving consensus.

Part II (i). **To show the effectiveness of trust on group consensus.** Propose several improved minimum cost consensus models considering the implicit trust, which is obtained based on the similarity of opinions. In the enhanced minimum cost consensus model, the moderator is considered to be a trustworthy coordinator to persuade DMs to reach an agreed consensus level with the lowest cost. Besides, a minimum cost consensus model is proposed to modify the unit preference adjustment cost's irrationality based on the weights of DMs, which is determined based on implicit trust.

Part II (ii). **To study the interaction between CENs and trust networks.** Combine trust networks and CENs based on multiplex network structures. In the trust consensus evolution multiplex network, the consensus evolution is uncovered under the impact of trust, but the trust development is also investigated during the consensus adjustment. The influence of DMs is computed using PageRank centrality in the multiplex network. Under such influence, the consensus evolution is explored under the positive and negative effects of trust. Trust development is also considered based on the complete negotiation between DMs.

Part III (i). **To balance the dynamic CAP and CRP in LSGDM.** Design the dynamic CAP based on CENs according to a community detection method. The clustering dynamic mainly reflects in the multi-structure of CENs with different consensus thresholds and the consensus evolution. To select a suitable result for the following decision process, we evaluate the clustering validity based on intra-cluster and inter-cluster consensus levels. We enhance the consensus thresholds of CENs to effectively proceed with the clustering analysis when the consensus level is improved.

Part III (ii). **To facilitate the CRP based on the CAP.** We consider the adjustment cost as a clustering factor of K-means except for the preference information. A parameter is attached to the adjustment cost to show its supporting role. The distance between DMs is determined based on the combined effect of preferences and the adjustment cost. To obtain stable clustering results to facilitate the following CRP, we determine the adjustment cost parameter after several rounds of iterations, and the initial clustering centers of K-means are defined in advance.

Part IV. To propose a group recommendation approach based on LSGDM models. An LSGDM model is proposed considering users' behavior and managing minority opinions based on OWA and its family operators. In the LSGDM model, users' similarity is computed using the WOWA operator to deal with the incomplete evaluation information. The orness of OWA is used to measure the polarization effects of intra-cluster users. The I-IOWA operator is utilized to manage minority opinions. Finally, we use the LSGDM model to provide a group-buying list for merchants in a live service website, Dianping.com.

In short, our purpose is to deepen and improve the consensus research of traditional GDM, SNGDM, and LSGDM to promote the development of group recommendation.

5 Methodology

Based on the above aims, this thesis's main idea is to investigate further the theory and method of traditional GDM, SNGDM, LSGDM, and the application in group recommendation. The principal used method is an interdisciplinary research method, which refers to the method of comprehensive research on a subject by using multi-disciplinary theories, methods, and achievements. This thesis involves the integration of decision science, operational research, computer science, mathematics, and social psychology. Other related main research methods are introduced as follows:

(1) **Literature research method.** It is a method to get information by investigating literature according to specific research purposes or topics to understand and master the research problems comprehensively and correctly. Through literature research, we find the limitations of current GDM, SNGDM, and LSGDM research, and the practical needs of SCGR. According to these problems, the research objectives and main contents of this thesis are determined.

(2) **Mathematical methods.** It uses numerical tools to deal with a series of quantities of the research objects to make precise explanations and judgment and get the numerical results. This thesis uses the optimization models, distance functions, similarity functions, consensus measurement, weight determination, aggregation operators, etc. The optimization model is fundamental to determine the optimal consensus opinion and clustering results. Other methods are the critical basis of preference expression and processing, consensus calculation, and alternative ranking.

(3) **Statistical analysis.** It refers to the study and research on the quantitative relationship of the scale, scope, and degree of the research object, to understand and reveal the mutual relationship, change rule and development trend of objects. This thesis mainly uses clustering analysis and community detection algorithm to find a similar relationship between large-scale group DMs and aggregate them into small groups, to reduce the complexity of LSGDM.

(4) **Quantitative analysis.** In scientific research, people's understanding of the research object can be further accurate through quantitative analysis, to reveal the law more scientifically, grasp the essence, clarify the relationship, and predict the development trend of things. For example, this paper needs to quantify partial qualitative information such as evaluation information, social relationship strength, consensus relations, etc.

(5) **Systematic and scientific method.** In addition to the logical and mathematical methods, the decision-making also adopts the scientific system methods, including system method, information method, and control feedback method, which makes the decision-making more scientific and accurate. For example, the feedback adjustment method is vital to improve the consensus level in CRP.

(6) **Simulation analysis.** It is to use simulation tools to simulate and analyze the validity and rationality of the proposed methods. For example, we use the simulation analysis to show the community structure consists of similar DMs, simulate the improvement of consensus under the influence of social relationships after the feedback adjustment process, and reflect the ranking results of alternatives.

(7) **Comparative study.** It can reflect innovation and highlight the advantages of the proposed methods and models in SNGDM and LSGDM. For instance, we can compare the modified consensus measurement method based on CENs, the evolution of consensus in SNGDM, and the clustering analysis considering other clustering factors in LSGDM with existing methods.

6 Summary

In this section, we summarize the main proposals in this thesis and explain the main contents and the obtained results associated with the journal publications. The published and submitted works are listed as follows:

(1) T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Consensus evolution networks: A consensus reaching tool for managing consensus thresholds in group decision making. Information Fusion, 52 (2019) 375-388.

(2) T. Wu, X.W. Liu, Z.W. Gong, H.H. Zhang, F. Herrera, The minimum cost consensus model considering the implicit trust of opinions similarities in social network group decision-making. International Journal of Intelligent Systems, 35(2020) 470-493.

(3) T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Trust-consensus multiplex networks by combining trust social network analysis and consensus evolution methods in group decision-making. IEEE Transactions on Fuzzy Systems, early access, 2022. Doi: 10.1109/TFUZZ.2022.3158432.

(4) T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Balance dynamic clustering analysis and consensus reaching process with consensus evolution networks in large-scale group decision making. IEEE Transactions on Fuzzy Systems, 29(2021) 357-371.

(5) T. Wu, X.W. Liu, J.D. Qin, F. Herrera, A new clustering algorithm with preference adjustment cost to reduce the cooperation complexity in large scale group decision making. IEEE Transactions on Systems, Man, and Cybernetics: Systems, early access, 2021. Doi: 10.1109/TSMC.2021.3120809.

(6) T. Wu, C. Zuheros, X.W. Liu, F. Herrera, Managing minority opinions in large-scale group decision-making based on community detection and group polarization. Submitted to Computers & Industrial Engineering, (2022).

Four aspects organize the rest of this section according to the primary research objectives (Section 4). Subsection 6.1 introduces the consensus research of traditional GDM based on CENs. Subsection 6.2 studies the consensus research of SNGDM based on trust networks and CENs. Subsection 6.3 investigates the consensus research of LSGDM from the perspective of clustering. Subsection 6.4 proposes a new LSGDM model and illustrates its application in group recommendation.

6.1 The consensus research of traditional GDM based on CENs

The consensus research of traditional GDM based on CENs mainly covers the construction of CENs, a new consensus index based on CENs, and a feedback adjustment algorithm based on CENs.

(1) The construction of CENs

The CENs is the basis for intuitively studying consensus relations, which can be determined based on the similarity of FPRs among DMs:

$$sm_{ij}^{kl} = 1 - \left| f_{ij}^{k} - f_{ij}^{l} \right| \tag{I.17}$$

where $i, j = 1, 2, ..., n,\ k, l = 1, 2, ..., m$. The similarity matrix between the pairwise DMs can be obtained based on the similarity sm_{ij}^{kl}:

$$SM_{kl} = \begin{pmatrix} - & ... & sm_{1i}^{kl} & ... & sm_{1m}^{kl} \\ ... & - & ... & ... & ... \\ sm_{i1}^{kl} & ... & - & ... & sm_{im}^{kl} \\ ... & ... & ... & - & ... \\ sm_{m1}^{kl} & ... & sm_{mi}^{kl} & ... & - \end{pmatrix}_{m\times m} \tag{I.18}$$

Based on the similarity matrix SM_{kl}, the consensus matrix CM can be constructed as:

$$CM = \begin{pmatrix} - & ... & cm_{1k} & ... & cm_{1n} \\ ... & - & ... & ... & ... \\ cm_{k1} & ... & - & ... & cm_{kn} \\ ... & ... & ... & - & ... \\ cm_{n1} & ... & cm_{nk} & ... & - \end{pmatrix}_{n\times n} \tag{I.19}$$

where,$cm_{kl} = \frac{1}{n(n-1)/2}\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} sm_{ij}^{kl}$, $cm_{kl} \in [0,1]$.

According to the consensus matrix CM, the consensus threshold ε can be determined to construct CENs. When $cm_{kl} \geq \varepsilon$, then the consensus level between d_k and d_l can be accepted, i.e., there is consensus among DM d_k and d_l and the consensus level is cm_{kl}. Otherwise, there is consensus among DM d_k and d_l.

Definition 12. *For a consensus evolution network $G = (D, E, C)$, D denotes the set of n DMs $D = \{d_1, d_2, ..., d_n\}$,$E = \{e_{kl}\}\,(k, l = 1, 2, ..., n, k \neq l)$ denotes the set of consensus relations among DMs, levels of these consensus relations are larger than or equal to the consensus threshold ε, which is denoted by $C = \{c_{kl} = cm_{kl} | k, l = 1, 2, ..., n, k \neq l, cm_{kl} \geq \varepsilon\}$ .CENs can show different forms according to different consensus thresholds ε.*

When the consensus thresholds ε is taken as the maximum and minimum elements of the consensus matrix CM, two extreme forms of consensus networks can be obtained: the empty CEN G_E and the complete CEN G_C. Other cases are incomplete CEN G_I. $\min\{cm_{kl}\}$ denotes the consensus threshold boundary between G_C and G_I. $\max\{cm_{kl}\}$ denotes the consensus threshold boundary between G_I and G_E.

Besides, DMs are considered to be sensitive to ε_r when the gap of the clustering coefficient between two CENs reaches maximum:

$$\max\left(CC_{r-1} - CC_r\right) \tag{I.20}$$

where CC_{r-1} and CC_r are the clustering coefficients of relevant CEN G_{r-1} and G_r obtained according to Definition 8. G_r is regarded to be the sensitive CEN.

The smaller the CC_r, the less stable the triangle relationship is among DM. Thus, the sensitive consensus threshold ε_r suggests that the CEN G_r becomes vulnerable and unstable. It is also suggests that the consensus relation values between most pairs of DM do not reach ε_r. Therefore, the value of ε_r can be the reference for decision managers to set the agreed consensus threshold $\bar{\varepsilon}$ for GDM. The more $\bar{\varepsilon}$ is larger than ε_r, the higher the cost of the feedback adjustment in CRP.

(2) A new consensus index based on CENs

A new consensus index is proposed structurally and numerically in this section to explore the formation of consensus in depth.

The level of the numerical index can be computed based on the weighted aggregating consensus of all DMs:

$$CR_N = \sum_{k=1}^{m} \omega_N^k u_N^k \tag{I.21}$$

where u_N^k is the unit consensus of d_k , $u_N^k = \frac{\sum_{l=1}^{m} c_{kl}}{\deg(d_k)}$ $(k \neq l)$, ω_N^k is the consensus based weight of d_k , $\omega_N^k = \sum_{l=1}^{m} c_{kl} \Big/ \sum_{k=1}^{m} \sum_{l=1}^{m} c_{kl}$ $(k \neq l)$.

The level of the structured index CR_S can be computed by the weighted averaging operator:

$$CR_S = \sum_{k=1}^{m} \omega_S^k u_S^k \tag{I.22}$$

where ω_S^k is the structure based weight of d_k , $\omega_S^k = \deg(d_k) \Big/ \sum_{k=1}^{m} \deg(d_k)$, u_S^k is the unit degree of d_k , $u_S^k = \deg(d_k)/(m-1)$.

The comprehensive consensus index is determined with the combination of CR_N and CR_S :

$$CR = CR_N \times CR_S \tag{I.23}$$

where $CR_N \in [0,1], CR_S \in [0,1] \Rightarrow CR \in [0,1]$, $CR = 1$ means all DM have reached complete consensus.

(3) The feedback adjustment algorithm based on CENs

The complementary CENs are the supplementary form of general CENs, which shows the consensus situation that most DM have not achieved. Contrary to the general CENs, the complementary CENs become tight with the increasing ε . It is easy to distinguish those DMs that contribute less to consensus from the complementary CENs. When the complementary CEN is too compact, the adjustment cost increases, while when the complementary CEN is too sparse, the low consensus connections are hard to find. Hence, the complementary CENs change from a compact to sparse, especially from the sensitive consensus threshold. A feedback adjustment algorithm is introduced based on complementary sensitive CEN.

Definition 13. *Based on the universal CEN $G_C = (D_C, E_C, C_C)$, the complementary sensitive CEN consists of $\bar{G}_r = \left(D, \bar{E}_r, \bar{C}_r\right)$ with m DMs $D = \{d_1, d_2, ..., d_m\}$, consensus relations $\bar{E}_r = \{e_{kl} | e_{kl} \notin E_r\}$ and consensus relation values $\bar{C}_r = \left\{\bar{c}_{kl} | \bar{c}_{kl} = c_{kl}^C < \varepsilon_r\right\}$, $k, l = 1, 2, ..., m$, $k \neq l$. If $\bar{c}_{kl} = c_{kl}^C < \varepsilon_r$, then there is an edge e_{kl} in $\bar{G}_r$ o connect d_k and d_l together with the consensus relation value $\bar{c}_{kl}$, $\bar{E}_r \cup E_r = E_C$, $\bar{c}_{kl}$ called weight of the edge e_{kl} and $\bar{C}_r \cup C_r = C_C$.*

According to the identification and direction rule of CRP, suppose the FPR of d_k is identified to be adjusted. To make the consensus similarity between d_k and d_l as similar as possible, a consensus improving model to obtain the adjusted FPR $F_{k'} = \left(f_{ij}^{k'}\right)_{n \times n}$ is represented as:

$$f_{ij}^{k'} = \begin{cases} f_{ij}^k + \left| f_{ij}^k - f_{ij}^{kl} \right| \Big/ 2, \ f_{ij}^k \leq f_{ij}^{kl} \\ f_{ij}^k - \left| f_{ij}^k - f_{ij}^{kl} \right| \Big/ 2, \ f_{ij}^k > f_{ij}^{kl} \end{cases} \tag{I.24}$$

where $i \le j$,f_{ij}^{kl} is the averaging preference of d_k and d_l , and $f_{ij}^{kl} = \left(f_{ij}^{k} + f_{ij}^{l}\right)\Big/2$, when $i > j$, $f_{ji}^{k'} = 1 - f_{ij}^{k'}$.

The journal paper concerning this part is:

T. Wu, X.W Liu, J.D. Qin, F. Herrera, Consensus evolution networks: A consensus reaching tool for managing consensus thresholds in group decision making. Information Fusion, 52 (2019) 375-388.

6.2 The consensus research of SNGDM based on trust networks and CENs

According to the advantages of social network analysis, we further study the consensus research of SNGDM from two aspects, one is to show the influence of social relationships to the adjustment cost in the CRP; the other one is to investigate the interaction between trust relationships and consensus relations.

6.2.1 The consensus research based on the implicit trust in SNGDM

The consensus research based on the implicit trust in SNGDM mainly includes the definition of the implicit trust, the minimum cost consensus models based on implicit trust and the modified adjustment cost.

(1) The definition of the implicit trust

Suppose there is an SNGDM problem consisting of n individuals $\{d_1, d_2, ..., d_n\}$ and a moderator M. Let $o_i \in R$ represents the opinion of individual d_i, $o_M \in R$ be the expect consensus opinion of M, c_i be the unit cost of d_i for making concession, o_d be the consensus opinion actively formed by all individuals.

The similarity between the individual d_i and d_j is defined by the similarity function $s_{ij}(o)$:

$$s_{ij}(o) = 1 - \frac{|o_i - o_j|}{\max\{o_i\}} \tag{I.25}$$

where $s_{ij}(o) \in [0, 1]$.

The more similar between the opinion o_i and o_j, the more implicit trust between the individual d_i and d_j. Thus, the definition of the implicit trust is given based on the similarity function as follows.

Definition 14. *Let the implicit trust function $t_{ij}(o) = s_{ij}(o)$, then the implicit trust of d_i to d_j equals to the implicit trust of d_j to d_i:*

$$t_{ij}(o) = t_{ji}(o) = 1 - \frac{|o_i - o_j|}{\max\{o_i\}} \tag{I.26}$$

since $o_i, o_j \in R$, and $t_{ij}(o) \in [0, 1]$.

According to the expected consensus opinion o_M of the moderator M and the opinions of individuals, the implicit trust of the individuals to the moderator M can be determined:

$$t_{iM} = 1 - \frac{|o_i - o_M|}{\max\{o_i\}} \tag{I.27}$$

where $t_{iM} \in [0,1]$ since $o_i, o_M \in R$.

Similarly, the more similarity between the opinion o_i and the consensus opinion o_d, the more consensus willingness the individual d_i has to reach such consensus. Thus, the definition of the consensus willingness is given as follows.

Definition 15. *According to the similarity function $s_{ij}(o)$, the consensus willingness of the individual d_i to reach to the consensus opinion o_d is defined as:*

$$w_{id} = 1 - \frac{|o_i - o_d|}{\max\{o_i\}} \tag{I.28}$$

where $w_{id} \in [0,1]$ since $o_i, o_M \in R$.

(2) The minimum cost consensus model based on implicit trust

In terms of the consensus reaching, implicit trust is benefit attribute, while the adjustment costs are cost attribute. Thus, the implicit trust needs to be transformed into cost attribute t'_{iM} as:

$$t'_{iM} = 1 - t_{iM} \tag{I.29}$$

Based on the implicit trust, a minimum cost model $NLP(c,t)$ is proposed as below:

$$\begin{aligned} NLP(c,t): \quad & \min\ \phi(o_M) = \sum_{i=1}^{n} c_i {t'}_{iM} f_i(o_M) \\ & \qquad = \tfrac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left({o_M}^2 + {o_i}^2 - 2o_i o_M\right) \\ & s.t. \quad o_M \geq 0 \end{aligned} \tag{I.30}$$

To discuss the further economic significance about Model, its the dual problem is constructed based on the Lagrange multiplier method:

$$\begin{aligned} DNLP(c,t): \quad & \max\ \psi(o_d) = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(-{o_d}^2 + {o_i}^2\right) \\ & s.t. \quad o_d = \sum_{i=1}^{n} c_i o_i \Big/ \sum_{i=1}^{n} c_i \end{aligned} \tag{I.31}$$

where o_d represents the weighted average opinion of all individuals since $\sum_{i=1}^{n} c_i = 1$.

(3) The minimum cost consensus model based on the modified adjustment cost The adjustment costs are subjectively given by DM. Sometimes, the subjective costs may not be so reasonable, which will cause unfair decision results. Thus, we provide DMs the adjustment suggestions based on their importance obtained from trust levels.

The in-degree trust index t_j of DM d_j is computed based on. With the in-degree trust index of individuals, the weights ω_j of d_j can be computed:

$$\omega_j = t_j \Big/ \sum_{j=1}^{n} t_j \tag{I.32}$$

where $\sum_{j=1}^{n} \omega_j = 1$.

Let o'_s denotes the expected consensus opinion of the moderator M when the weights of individuals are considered in the process of negotiation. With the weights ω , the modified optimal model is given as:

$$\begin{aligned} NLP\left(c,t,w\right): \quad & \min\ \phi\left(o'_M\right) = \tfrac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i\omega_i \left({o'_M}^2 + {o_i}^2 - 2o_i o'_M\right) \\ & s.t. \quad o'_M \geq 0 \end{aligned} \tag{I.33}$$

Similarly, the dual problem of Model can be determined through the introduction of the Lagrange multiplier λ':

$$\begin{aligned} DNLP\left(c,t,w\right): \quad & \max\ \psi\left(o'_d\right) = \tfrac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i\omega_i \left(-{o'_d}^2 + {o_i}^2\right) \\ & s.t. \quad o'_d = \sum_{i=1}^{n} c_i\omega_i o_i \Big/ \sum_{i=1}^{n} c_i\omega_i \end{aligned} \tag{I.34}$$

The journal article associated to this part is:

T. Wu, X.W. Liu, Z.W. Gong, H.H. Zhang, F. Herrera, The minimum cost consensus model considering the implicit trust of opinions similarities in social network group decision-making. International Journal of Intelligent Systems, 35(2020) 470-493.

6.2.2 The interaction between trust networks and CENs in SNGDM

The consensus research in SNGDM combining trust networks and CENs mainly includes the construction of trust consensus evolution multiplex networks, CENs and trust networks' evolution based on the PageRank centrality of DMs.

(1) The construction of trust consensus evolution multiplex networks

Suppose the trust network among n DMs $D = \{d_1, \ldots, d_n\}$ is given to be $G_A = (D, E_A, T)$, let E_A denote the set of trust relationships and $T = (T_{ij})_{n\times n}$ denote the corresponding set of trust degrees, i.e. the weighted adjacency matrix of G_A.

Definition 16. *A trust consensus evolution multiplex network $MG = (G_A, G_B, E_{AB}, W_{AB}, ME_{BA})$ consists of a trust network $G_A = (D, E_A, T)$ and a CEN $G_B = (D, E_B, T)$, where the set of DMs are the same in layer G_A and G_B, E_A and E_B denote the relationship between DMs in layer G_A and G_B, respectively, E_{AB} denotes the direct impact of layer G_A on layer G_B and W_{AB} represents the related values of the impact relations, and ME_{BA} reflects the indirect influence of layer G_B on layer G_A since the adjustment occur in layer G_B.*

An example of trust consensus evolution multiplex network MG is shown in Fig.2, where the solid lines in layer G_A and G_B means the in-layer connections, the solid lines from layer G_A to G_B denotes the direct impact of trust relationships on consensus, and the dotted lines from G_B to G_A mean the indirect impact of layer G_B on layer G_A.

(2) The evolution of CENs

According to the concept of PageRank centrality shown in Definition 9, the comprehensive influence $Y \in [0, 1]$ of DMs in the multiplex network MG can be obtained.

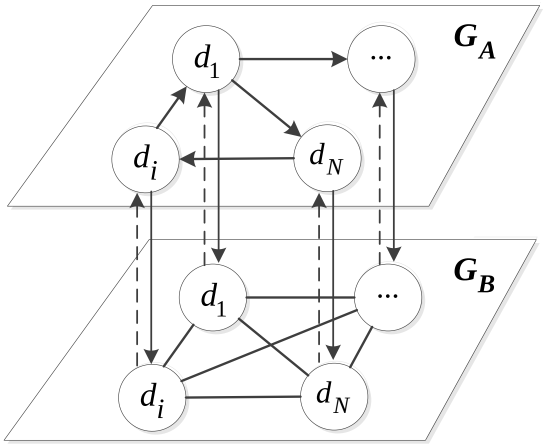


Fig.4 An example of a multiplex network

Since the influence Y can affect DMs' decisions, it can be considered to be the impact factor of trust relationships on consensus relations, which suggests that the consensus level between a pair of DMs will be improved with value Y when one expert completely trusts another one. Thus, the influence Y can be regarded as the weights W_{AB} of the impact relations E_{AB}. The consensus relations in G_B will change according to the impact of trust relationships E_{AB} with the weights Y.

The effects of trust on consensus is codetermined by the trust degrees T, the consensus levels C and the influence Y. We propose a function $Q(C,T,Y)$ to evaluate the modified consensus matrix $C' = Q(C,T,Y)$ for CENs G_B under the influence of the trust networkG_A:

$$q(C_{ij}, T_{ij}, Y_j) = \frac{C_{ij} + Y_j T_{ij}}{\max(C_{ij} + Y_j, 1)} \tag{I.35}$$

where C_{ij} is the initial consensus level between the expert d_i and d_j, Y_j denotes the influence of the expert d_j over d_i, the modified consensus level between the expert d_i and d_j is $C'_{ij} = q(C_{ij}, T_{ij}, Y_j)$.

The consensus matrix C is symmetric, while the trust matrix T is asymmetric. Thus, there may be some deviation of the consensus between some pairs of DMs whose consensus levels are adjusted based on the directional trust relationships, i.e. $q(C_{ij}, T_{ij}, Y_j) \neq q(C_{ji}, T_{ji}, Y_j)$. To deal with the inconsistent consensus relations C'_{ij} between DMs in the evolved consensus matrix $C' = (C'_{ij})_{n\times n}$, a symmetric consensus matrix $MC = (MC_{ij})_{n\times n}$ is obtained as:

$$MC_{ij} = \frac{q(C_{ij}, T_{ij}, Y_j) + q(C_{ji}, T_{ji}, Y_j)}{2} \tag{I.36}$$

(3) The evolution of trust networks

Propagation is one of the most crucial trust properties in complex trust networks. It is challenging to consider all the propagation paths. Due to the diminishing information, the longer the propagation path, the weaker the trust degree is transferred. Thus, we compute the transitive trust for DMs in trust networks based on their shortest propagation paths.

Suppose $d_i \to d_j \to d_k$ is the shortest propagation path between d_i and d_k except the direct trust degree T_{ik}, the transitive value is denoted by PT_{ik} and it is commonly computed based on the algebraic t-norm operator:

$$PT_{ik} = T_{ij} \times T_{jk} \tag{I.37}$$

Referring to assumptions, the trust network G_A is changing positively during adjustment. The transitive trust degree is considered as the gains of the directed trust value between the paired DMs. The influence of trusters can affect their transitive degree to trustees. The larger the influence of the truster, the more the trustee trusts the truster. Thus, the varying degree of d_i trusts d_k in the round of adjustment can be computed based (I.36):

$$T_{ik}^{(r)} = T_{ik}^{(r-1)} + \frac{1}{N\left(p_{ik}^{(r-1)}\right)} \sum_{p_{ik}^{(r-1)} \in P_{ik}^{(r-1)}} \min\left(Y_{ik}^{(r-1)}\right) PT_{ik}^{(r-1)} \tag{I.38}$$

where $PT_{ik}^{(r-1)}$ is the transitive trust value from d_i to d_k in the shortest path $p_{ik}^{(r-1)}$ after rounds of adjustment, $N\left(p_{ik}^{(r-1)}\right)$ is the number of the shortest paths, $\min\left(Y_{ik}^{(r-1)}\right)$ is the minimum influence of the trusters in the trustees on $p_{ik}^{(r-1)}$, i.e. $\min(Y_{ik})$ is determined based on the influence of experts on p_{ik} except d_i.

The journal article associated to this part is:

T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Trust-Consensus Multiplex Networks by Combining Trust Social Network Analysis and Consensus Evolution Methods in Group Decision-Making. IEEE Transactions on Fuzzy Systems, Early Access, 2022. Doi: 10.1109/TFUZZ.2022.3158432.

6.3 The consensus research of LSGDM from the perspective of clustering analysis

According to the characteristic of LSGDM, we started the consensus research with the perspective of clustering analysis from two aspects: one is to balance the dynamic CAP and CRP based on CENs, the other one is to consider the role of the adjustment cost in the CAP to reduce the complexity of the CRP.

6.3.1 Balance the dynamic CAP and CRP based on CENs in LSGDM

To balance dynamic clustering analysis and CRP based on CENs in LSGDM, we mainly focus on the dynamic clustering analysis method based on CENs, the clustering validity test based on the local and global CENs, and the feedback adjustment method based on the clustering analysis.

(1) The dynamic clustering analysis method based on CENs

The main reasons for the dynamic clustering results are: on the one hand, the CENs present different network structures with varying thresholds of consensus; on the other hand, CENs evolve with the progress of group communication. According to the Louvain method, the dynamic clustering method is proposed based on CEN from two phases:

The first phase: For individuals

Step 1: For the CEN LG, assign a different subgroup to each individual. For m experts, there is m subgroups $LG^{(1)} = \left\{SG_1^{(1)}, ..., SG_m^{(1)}\right\}$ in the initial partition.

Step 2: For each DM d_k, consider its gains in modularity $\Delta Q > 0$ with its neighbors $d_h\,(h = 1, ..., m, h \neq k)$, and remove d_k from its subgroup $SG_k^{(1)}$ to $SG_h^{(1)}$ with $\max \Delta Q$ based on

(15). Repeat this step until $\Delta Q \leq 0$ and no node can be moved, then go to the second phase.

The second phase: For the independent subgroups

Step 1: Assuming that $x\,(x < m)$ subgroups are determined after p rounds in the first phase, and $LG^{(p)} = \left\{SG_1^{(p)}, ..., SG_x^{(p)}\right\}$. For each subgroup $SG_r^{(p)}\,(r = 1, ..., x)$, consider the gains in modularity $\Delta Q'$ with its neighboring subgroup $SG_s^{(p)}\,(s = 1, ..., x, r \neq s)$, and remove $SG_r^{(p)}$ to $SG_s^{(p)}$ with $\max \Delta Q'$ based on (I.15)

Step 2: Repeat Step 1 of the second phase until there are no more changes. Finally, the CEN LG is classified into t independent subgroups $LG = \{SG_1, ..., SG_t\}$, $t \leq x < m$.

(2) The test of the clustering validity based on the local and global CENs

The clustering validity is usually verified based on intra-cluster compactness and inter-cluster sparsity. The larger the intra-cluster compactness and the inter-cluster sparsity, the better the clustering is. Similarly, the cluster validity can be extended into LSGDM based on the intra-cluster consensus levels and the inter-cluster consensus level. The local consensus level (LCL) reflects the intra-cluster consensus levels of local CENs, and the global consensus level (GCL) represents the inter-cluster consensus level among local CENs. The higher the LCL and the lower the GCL, the better the clustering will be. Moreover, the GCL should be smaller than any of the LCLs. If not, the subgroup whose LCL is smaller than the GCL should be integrated into other subgroups with the closest consensus level. We propose an evaluation algorithm of clustering validity based on the following three rules:

Rule 1: The number of isolated experts in each subgroup should not be higher than 2.

Rule 2: The GCL should not be larger than any of the LCLs.

Rule 3: The clustering result with a minimum ratio between the GCL and the LCL should be generally determined as a suitable result.

In dynamic clustering analysis, more and more isolated experts appear with the increasing consensus threshold. Rule 1 is proposed to remove the invalid clustering results that include more isolated experts. Rule 2 is proposed to further remove unqualified results from the remaining clustering results after Rule 1. After Rules 1 and 2, Rule 3 is used to select a suitable clustering result from the final remaining dynamic results.

(3) The adjustment method based on the clustering analysis

According to the identification rule, the local CEN $SG_r^{(p)}$ with the largest weight $\max\left(w_r^{(p)}\right)$ is identified. According to the direction rule, experts in other local CENs are advised to modify their FPRs according to the collective FPR of the local CEN $SG_r^{(p)}$. As such, the adjustment cost may be reduced in the process of consensus reaching regardless of whether the small weights of other local CENs are caused by the small number of members or the lower compactness. Besides, to improve the adjustment effect and reduce the adjustment costs as much as possible, the LCL of $SG_r^{(p)}$ is checked first if $CL_{loc}^r \geq \overline{CL}$. If so, go on to the adjustment process of other subgroups based on the collective preferences of $SG_r^{(p)}$; otherwise, improve the LCL of $SG_r^{(p)}$ first.

The journal article associated to this part is:

T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Balance Dynamic Clustering Analysis and Consensus Reaching Process with Consensus Evolution Networks in Large-scale Group Decision Making. IEEE Transactions on Fuzzy Systems, 29(2021) 357-371.

6.3.2 A new clustering algorithm with preference adjustment cost in LSGDM

The new clustering algorithm with preference adjustment cost in LSGDM is developed from three main aspects: the distance computation based on preference adjustment cost, the determination of the coefficient of the adjustment cost in the clustering method, and the determination of initial clustering centers.

(1) The distance computation based on preference adjustment cost

We hold that FRPs and the preference adjustment costs are dual attributes of individuals in clustering analysis. Let $D : \{F, C\}$ denotes the dual attributes, where $F = (F_h)_{1\times M}$, $C = (C_h)_{1\times M}$.

Suppose the adjustment cost is attached to the proposed clustering algorithm with the coefficient $\alpha \in [0, 1]$. To consider the clustering analysis's dual attributes, we determine the combined distance $D\ \ (d_h, d_l)$ based on the preference-based and cost-based pairwise distanceα using the Euclidean distance as:

$$\begin{aligned} D\ \ (d_h, d_l) &= \left(dis^2 (F_h, F_l) + \alpha \times dis^2 (C_h, C_l)\right)^{\frac{1}{2}} \\ &= \left(\left(\sum_{i,j\in N;i\neq j} \frac{\left|f_{ij}^h - f_{ij}^l\right|}{n(n-1)}\right)^2 + \alpha \times |C_h - C_l|^2\right)^{\frac{1}{2}} \end{aligned} \tag{I.39}$$

where $D\ \ (d_h, d_l) \in [0, 1]$ since $dis (F_h, F_l) = \sum\limits_{i,j\in N;i\neq j} \frac{\left|f_{ij}^h - f_{ij}^l\right|}{n(n-1)}$, $dis (C_h, C_l) = |C_h - C_l|$.

(2) The determination of the coefficient of the adjustment cost

According Eq. (5), obtain the optimal FPR $F^* = \left(f_{ij}^*\right)_{n\times n}$. Then, the mean error of intra-cluster consensus levels ME_{consen}^{α} can be determined as:

$$ME_{consen}^{\alpha} = \frac{1}{K} \sum_{r\in K} \left(\frac{\sum\limits_{d_h,d_l\in G_r} \eta_\alpha (d_h, d_l)}{N(G_r)(N(G_r) - 1)}\right) \tag{I.40}$$

where K is the number of clusters and $N (G_r)$ is the number of individuals in G_r, $\eta_\alpha (d_h, d_l) = \left|CL_r^h - CL_r^l\right|$, CL_r^h and CL_r^l can be computed with F_h, and the group FPR F^* based on (1).

Similarly, the mean error of the intra-cluster total adjustment cost ME_{cost}^{α}can be determined as:

$$ME_{cost}^{\alpha} = \frac{1}{K} \sum_{r\in K} \left(\frac{\sum\limits_{d_h,d_l\in G_r} \xi_\alpha (d_h, d_l)}{N(G_r)(N(G_r) - 1)}\right) \tag{I.41}$$

where K is the number of clusters and $N (G_r)$ is the number of individuals in G_r, $\xi_\alpha (d_h, d_l) = |TC_h - TC_l|$, in which, $TC_h = \sum\limits_{i,j\in N;i\neq j} C_h \frac{\left|f_{ij}^h - f_{ij}^*\right|}{n(n-1)}$.

According to the clustering principle, the closer the intra-cluster individuals are and the more sparse the clusters are, the better the clustering effect is. Similarly, the smaller both ME_{consen}^{α} and ME_{cost}^{α}, the better the clustering effect is. To determine the value of α, we propose a weighted method as:

$$\min\left(\omega \times ME_{consen}^{\alpha} + \mu \times ME_{cost}^{\alpha}\right) \tag{I.42}$$

where ω and μ are the weights of ME_{consen}^{α} and ME_{cost}^{α}, respectively.

Suppose $\exists \max ME_{consen}^{\alpha} \neq \min ME_{consen}^{\alpha}$ and $\max ME_{cost}^{\alpha} \neq \min ME_{cost}^{\alpha}$, then let $a^* = \max ME_{consen}^{\alpha}$, $a_* = \min ME_{consen}^{\alpha}$, $b^* = \max ME_{cost}^{\alpha}$, and $b_* = \min ME_{cost}^{\alpha}$, the value of the weight ω and μ can be determined as:

$$\omega = 1 - \frac{a^* - a_*}{a^* - a_* + b^* - b_*} \tag{I.43}$$

$$\mu = 1 - \frac{b^* - b_*}{a^* - a_* + b^* - b_*} \tag{I.44}$$

where $\omega + \mu = 1$, which is consistent with the above analysis that the weight ω and μ checks and balances.

(3) The determination of initial clustering centers

To obtain stable clustering results, we define K initial clustering centers for the proposed algorithm based on the dual attributes with the value of α.

First, find the individual d_h who has the highest level of consensus and most similar unit adjustment cost with others as the first clustering center:

$$\max\left(\sum_{h,l\in M,h\neq l} CL_{h,l} - \alpha \times \sum_{h,l\in M,h\neq l} dis\left(C_h, C_l\right)\right) \tag{I.45}$$

Then, find the individual $d_x\,(x \in M, x \neq h)$ who has the lowest level of consensus and the least similar unit adjustment cost with d_h as the second clustering center:

$$\min\left(CL_{x,h} - \alpha \times dis\left(C_x, C_h\right)\right) \tag{I.46}$$

Next, find the individual d_y who has the as lowest level of consensus and the least similar unit adjustment cost both with d_h and d_x as the next clustering center:

$$\min\left(\frac{1}{N\left(P\right)}\left(CL_{y,x} + CL_{y,h}\right) - \frac{\alpha}{N\left(P\right)} \times \left(dis\left(C_y, C_x\right) + dis\left(C_y, C_h\right)\right)\right) \tag{I.47}$$

where $N\left(P\right)$ denotes the number of clustering centers that has been determined, i.e. $N\left(P\right) = 2$, $x, y, h \in M$, $y \neq x$, and $y \neq h$.

Repeat (46) until all $K\,(if\ K > 3)$ initial clustering centers are defined.

The journal article associated to this part is:

T. Wu, X.W. Liu, J.D. Qin, F. Herrera, A New Clustering Algorithm with Preference Adjustment Cost to Reduce the Cooperation Complexity in Large Scale Group Decision Making. IEEE Transactions on Systems, Man, and Cybernetics: Systems, early access, 2021. Doi: 10.1109/TSMC.2021.3120809.

6.4 The application of LSGDM methods in group recommendations

To study the application of LSGDM methods in SCGR, we mainly focus on the incomplete preference information, the polarization behaviors of experts in subgroups, the management of minority opinions with the I-IOWA operator, and the application of LSGDM model in practical social commerce platform.

(1) The community detection among users with flexible similarity thresholds

A graph is a meaningful way to represent data structures, and it is widespread to model data items as a graph in many hierarchical clustering algorithms. Since the relationship between objects is generally sparse, sparse graphs can be constructed based on similarity thresholds obtained using the OWA operator flexibly. The adjacency matrix $A = (A_{ik})_{m\times m}$ of the sparse graph G is constructed as

$$A_{ik} = \begin{cases} 1 & S_{ik} \geq \theta \\ 0 & otherwise \end{cases} \tag{I.48}$$

where S_{ik} represents the similarity between user d_i and d_k obtained based on the imcomplete preference information using the WOWA operate, θ denotes the similarity threshold and can be determined using the OWA operator (6):

$$\theta = \Phi_{OWA}\left(S_{ik}\right) = \sum_{g=1}^{m'} w_g S_{ik}^{\sigma(g)} \tag{I.49}$$

where $m' = m\,(m-1)/2$ denotes the number of pairwise users , $S_{ik}^{\sigma(g)}$ $(i < k)$ is the permutation of similarity among users $\left(S_{ik}, ..., \ S_{m(m-1)}\right)$, the weighting vector w_g can be determined using the RIM quantifier with suitable parameter according to the purpose of the decision, $\sum\limits_{g=1}^{m'} w_g = 1$.

(2) The polarization behaviors of experts in subgroups

Group polarization effect provides a reasonable explanation for reaching consensus inside clusters and shows that a group can make decisions that are more extreme than the average of individuals' preferences. The reference point K in group polarization model can be seen as a quantitative representation of group pressure, and it can be determined using the OWA operator:

$$K = \Phi_{OWA}\left(v_{iz}^k\right) = \sum_{k=1}^{m} w_k \, v_{iz}^k \tag{I.50}$$

where v_{iz}^k is the preference of the DM d_k on the alternative x_i concerning the criteria f_z, the weighting vector w_k can be determined using the RIM quantifier Q based on (9) with suitable parameter α'_{OWA} referring to the purpose that to emphasize the preferences of the majority, $\sum\limits_{k=1}^{m} w_k = 1$.

The shift parameter ϕ is determined based on the difference between the orness measure of the majority and the neutralizing attitude:

$$\begin{aligned} \phi &= \left|orness\,(Q) - 0.5\right| \\ &= \left|\tfrac{1}{1+\alpha'_{OWA}} - 0.5\right| \end{aligned} \tag{I.51}$$

where ϕ is constrained to be nonnegative and $\phi \in [0, 1]$.

According to assumptions, individuals in the same cluster are regarded as equally important, i.e., $\lambda_i = 1/|G_r|$ $(d_i \in G_r)$. Therefore, the subgroup preference U_{lj}^r of the cluster G_r on the lth item concerning the jth criteria can be evaluated as:

$$\begin{aligned} U_{iz}^r &= \bar{u}_{iz}^r + \phi\left(\bar{u}_{iz}^r - K\right) \\ &= \tfrac{1}{|G_r|} \sum_{d_k \in G_r} v_{iz}^k + \left| \tfrac{1}{1+\alpha'_{OWA}} - 0.5 \right| \sum_{d_k \in G_r} \left(\left(\tfrac{1}{|G_r|} - w_k \right) v_{iz}^i \right) \end{aligned} \tag{I.52}$$

where $|G_r|$ denotes the number of individuals in the cluster G_r and $d_k \in G_r$ represents the user d_k belongs to G_r.

(3) Managing minority opinions with the I-IOWA operator

This study tries to use the importance induced ordered weighted averaging (I-IOWA) operator to deal with the minority opinion with flexible weighting vectors while protecting the minority's rights.

Firstly, the clusters with minority opinions should be identified according to two conditions: (a) the cluster has the farthest opinion or lowest consensus from all the other clusters. (b) let $[n/t]$ be the threshold to judge whether clusters hold the minority opinion in view of the cluster's size, where $[n/t]$ is the bracket function of the value of n divided by t.

According to the IOWA operator, the minority opinion is omitted (considered) when most experts are optimistic (pessimistic) about its impact on the outcome of decision-making. However, the minority opinion is considered more than the majority opinion when experts are pessimistic. Based on subgroup preferences U_{iz}^r, the overall preference U_{iz} can be computed by the I-IOWA operator with the weighting vector w_r:

$$\begin{aligned} U_{iz} &= \Phi_{I-IOWA}\left(\left\langle I_1, U_{iz}^1 \right\rangle, \left\langle I_2, U_{iz}^2 \right\rangle, \ldots, \left\langle I_t, U_{iz}^t \right\rangle\right) \\ &= \sum_{r=1}^{t} w_r U_{iz}^{\sigma(r)} \end{aligned} \tag{I.53}$$

where the weighting vector w_r is determined using the RIM quantifier Q based on (9) with a suitable parameter α_{I-IOWA}, $\left\langle I_{\sigma(r)}, U_{iz}^{\sigma(r)} \right\rangle$ is the 2-tuple with $I_{\sigma(r)}$ the rth largest order inducing value, w_r is determined considering the associated weights μ_r of clusters G_r:

$$w_r = Q\left(\sum_{s \le r} \mu_{\sigma(r)}\right) - Q\left(\sum_{s < r} \mu_{\sigma(r)}\right) \tag{I.54}$$

Finally, the LSGDM model is proposed based on the above three main points and used in the SCGR of Jingdong.com.

The journal article associated to this part is:

T. Wu, C. Zuheros, X.W. Liu, F. Herrera, Managing Minority Opinions in Large-Scale Group Decision-Making Based on Community Detection and Groups Polarization, Computers & Industrial Engineering (Submitted)

7 Discussion of results

This section mainly makes several discussions about the results obtained in all the mentioned stages of this thesis.

7.1 Consensus evolution analysis of traditional GDM

A new tool for CRP is proposed based on the CENs o explore the composition and evolution of consensus in GDM. With the help of the CENs, the consensus measure and feedback adjustment in CRP are processed with a substantial advantage, managing the consensus thresholds and its evolution. Four important CRP research points in traditional GDM are obtained in this study: 1) Different kinds of CENs are built with different consensus thresholds based on the consensus matrix. 2) The SCEN is distinguished by the sensitive consensus threshold, which can act as a reference for determining the agreed consensus threshold. 3) According to CENs, a new index for measuring the overall consensus degree is proposed structurally and numerically. 4) A pairwise feedback adjustment method is proposed based on complementary sensitive CEN.

According to the above analysis, this study mainly has the following four important advantages:

1) This paper studies consensus from a new perspective with the help of network analysis tools. The structure of CENs can show the formation and evolution of consensus in GDM more clearly. It is also a useful tool for LSGDM to reduce the interaction and negotiation complexity among experts.

2) The agreed consensus threshold is determined based on the sensitive consensus threshold obtained from the sparsity of the CENs, which will promote a balance between adjustment costs and the agreed consensus threshold. Besides, this method is more scientific than the traditional method based on experience to determine the consensus threshold.

3) In this study, the overall consensus degree is calculated based on the structured and numerical index based on CENs. The new consensus index is approximately equivalent to traditional methods when experts are considered undifferentiated. It is more effective than conventional methods in reflecting changes in consensus structures.

4) The pairwise adjustment strategy, which is proposed based on the complementary sensitive CEN, can pair up two experts that differ significantly in their consensus values. Such a combination can reduce the negotiation complexity and improve the overall consensus degree as much as possible.

7.2 Consensus adjustment of SNGDM considering the implicit trust

To explore the effects of trust on consensus reaching, we propose several minimum cost consensus models considering the implicit trust. The moderator is considered to be a trustworthy coordinator to persuade individuals to reach a consensus that he expects to pay the lowest cost. Compared with the traditional models, individuals are more willing to compromise to the moderator considering the implicit trust. At the same time, they are also easier to agree on the compensation with their consensus willingness. Three important points of consensus research based on implicit trust in SNGDM are obtained in this study: 1) The implicit trust of individuals to the moderator is determined based on the similarity of opinions. 2) A minimum cost consensus model and the dual model are proposed based on implicit trust. 3) Another minimum cost consensus model is developed based on the improved unit adjustment costs of individuals.

This study provides a new perspective for SNGDM to measure the effectiveness of social relationships in CRP. According to the economic significance of the primal-dual models, the proposed models show the offset role of the implicit trust to the adjustment costs in CRP and reveal the regulation role of the implicit trust modifying the adjustment costs of large deviation.

7.3 The interaction between trust relationships and consensus evolution in SNGDM

Since multiplex networks can uncover the interaction among multiple relationships in a complicated system, a consensus model for SNGDM was proposed based on trust consensus evolution multiplex networks combining trust relationships and consensus evolution methods. Four important consensus research points based on multiplex networks in SNGDM are obtained in this study: 1) The trust consensus evolution multiplex network, which shows the complicated connections between trust relationships and consensus relations, are constructed. 2) The comprehensive influence of experts in the multiplex networks is determined using the PageRank centrality, considering both the connections among experts in both layers. 3) The evolution of CENs is considered based on the direct impact of trust networks. 4) The evolution of trust networks is evaluated based on the indirect impact of CENs.

According to the above analysis, this study mainly has the following four critical advantages:

1) This study provides a new perspective to deal with the complicated consensus process combining multiple relationships in SNGDM based on the concept of multiplex networks. Based on the constructed trust consensus evolution multiplex networks, experts' influence, and the evolution of consensus and trust can be investigated intuitively.

2) The influence of experts in trust consensus evolution multiplex networks is determined based on their comprehensive importance in the layer of trust networks and the layer of CENs using PageRank centrality. The acquisition of experts' influence provided the basis for quantifying the degree of the interaction between trust networks and CENs.

3) The change of consensus is evaluated under both the positive and negative effects of trust based on experts' influence, which flexibly uncovers the evolution of consensus under the positive and negative effects of trust relationships with the influence of experts.

4) The variation of trust relationships is measured during the negotiation process based on trust propagation, reflecting the dynamic changes of trust and is closer to the decision facts.

7.4 Balance the dynamic clustering analysis and CRP in LSGDM

A dynamic clustering analysis process is designed based on CENs managing consensus thresholds to deal with the complex LSGDM. The clustering analysis is reconsidered after each round of feedback adjustment in CRP to balance the contradiction between the dynamic clustering analysis and CRP in LSGDM. This study has four important points: 1) According to the famous community detection method, we design the dynamic clustering analysis process based on CENs for LSGDM. 2) We define the clustering validity indicator based on the intra-cluster and inter-cluster consensus levels. 3) We compute the weights of subgroups based on the size of intra-cluster members and the compactness of local CENs. 4) We balance the conflict between the clustering analysis and CRP with higher consensus thresholds in CENs.

To highlight this study's advantages, we give a comparative analysis with traditional and social network LSGDM models. The advantages of this study can be presented as follows:

1) Individuals are classified dynamically based on their consensus relations using the community detection method. The dynamic clustering results in LSGDM can be adapted to different decision situations by managing consensus thresholds.

2) Meanwhile, the validity of dynamic clustering is verified based on the intra-cluster and inter-cluster consensus levels. This method enables us to select the best result from the dynamic clustering results and provides a train of thought for the validity determination of many multi-result clustering algorithms.

3) The weights of individuals inside of clusters are considered as equally important. In contrast, the weights of subgroups are determined with the combination of experts' size and the compactness of local CENs. This method avoids the unfairness caused by the majority principle.

4) We reconsider the clustering analysis during the CRP and balance the dynamic clustering analysis and CRP with higher consensus thresholds, which conform to the nature of LSGDM.

7.5 A new clustering algorithm with preference adjustment cost in LSGDM

To reduce the adjustment complexity in LSGDM, we propose a new clustering algorithm based on K-means considering the preference adjustment cost. We regard the adjustment cost as additional information to the preferences with a parameter in the proposed clustering algorithm. This study's three critical points are obtained: 1) The parameter's value is determined to balance the conflict between the consensus levels and the total adjustment cost among intra-cluster individuals. 2) The adjustment cost coefficient, which reflects the importance degree of the adjustment cost in the preference-based clustering method, is computed. 3) The initial clustering centers are determined in advance based on consensus levels among experts.

The advantages of this study are concluded as follows:

1) The adjustment cost is considered to be an impact factor of the proposed clustering algorithm. We regard the preference information and adjustment cost as dual attributes of individuals in the clustering analysis. The former plays a significant role, and the latter represents a supporting role.

2) The distance between individuals is computed based on the dual attributes, where the adjustment cost is attached to the clustering analysis with a coefficient. After multiple random clustering processes, the impact factor's coefficient is determined by balancing the conflict between the intra-cluster total adjustment costs and the intra-cluster consensus levels.

3) The initial clustering centers are defined by combining the consensus levels and adjustment cost using the determined coefficient of the impact factor, which is conducive to obtaining stable clustering results convenient for the following consensus analysis.

7.6 The application of LSGDM methods in S-commerce group recommendation

This thesis mainly deals with the incomplete preference information, the polarization of group behavior, minority opinions in LSGDM, and preliminary explore LSGDM models in S-commerce group recommendation (SCGR) which was developed from traditional E-commerce to promote products through users' social relationships.

The advantages of this study are summarized as follows:

1) The similarity among users is computed by the WOWA operator dealing with the incomplete preference information, which is a common practical application situation. Users are classified

based on a similarity graph, drawn with a threshold that is determined using OWA. The OWA and WOWA operator can flexibly adjust the importance of incomplete preference information under different alternatives and attributes.

2) The reaching of consensus inside clusters is explained by the group polarization effect, which shows that a group can make more extreme decisions than the average of individuals' preferences. Besides, the reference point and the shift parameter of the group preference polarization model are determined based on the orness measure of the OWA operator.

3) This study uses the I-IOWA operator to manage the minority opinion considering the decision manager's attitude with flexible weighting vectors, which satisfies the majority's requirements while protecting the minority's rights and avoids time-consuming for experts in negotiating to determine whether to consider minority opinions.

Besides, the application in SCGR shows the application significance of LSGDM models and has preliminarily achieved the transformation of scientific research results.

8 Concluding remarks

In this section, we present the results obtained from the research carried out during this Ph.D. dissertation. This thesis expands and deepens the study of traditional GDM, promotes the development of SNGDM and LSGDM. In terms of the decision-making requirements under the complex social network background, LSGDM theory is applied to SCGR practice, which enriches the theory of decision-making and provides an effective method for practical applications. Thus, this study has important theoretical significance and application value. The research results obtained in this thesis are described in detail from the following four main points.

(1) Inspired by social network analysis, we defined CENs to intuitively study the consensus evolution in GDM. Based on CENs, we calculated the consensus levels and experts' weights by analyzing the network structure. The CENs provide a new tool for deeply investigating consensus problems of traditional GDM and also promote the application of community detection methods in LSGDM, which can effectively reduce the complexity of LSGDM.

(2) Based on the particularity of trust relationships, we analyzed the consensus situation in SNGDM deeply. At the beginning, we proposed minimum cost consensus models based on experts' implicit trust to study its effectiveness on consensus from the model analysis. The proposed models not only consider the offset effect of trust on consensus adjustment cost but also judge and modify the subjective irrationality of experts' adjustment cost. Furthermore, based on the multiplex network structure, we investigated the consensus evolution and trust development with experts' comprehensive influence obtained using the PageRank centrality. The above-related researches have conducted an in-depth study on group consensus under the influence of social relations.

(3) We mainly studied the consensus problem in LSGDM from the perspective of clustering analysis. On the one hand, a dynamic clustering analysis method is proposed based on CENs for LSGDM to balance the clustering analysis and CRP with managing consensus thresholds. In this method, the clustering analysis and CRP are in a dynamic cycle, which isis closer to the actual LSGDM situation. On the other hand, the preference adjustment cost is considered a new element in the clustering analysis, which is commonly dominated by preference information in current LSGDM. This clustering method can classify experts with both the similar preferences and unit adjustment costs, effectively reducing the negotiation cost and decision time. The above researches can promote the progress of large-scale group consensus research and provide a broader perspective for future investigations.

(4) We preliminarily applied the LSGDM model to SCGR. Aiming at the large scale characteristics of users and the low consensus level in SCGR, an LSGDM model is proposed considering the incomplete preference information, group polarization effects, and minority opinions. Then, the above LSGDM model is utilized to provide group purchase commodity lists for users in Dianping.com. This study provides an LSGDM model considering more group behaviors from a theoretical perspective and highlights the practical application significance of the LSGDM model in SCGR.

The current research started from the traditional group consensus problem, and then analyzed the social network group consensus issue, and consequently extended to the research on the large-scale group consensus situation, and finally tried to apply the LSGDM model to SCGR, which lays a solid theoretical foundation for our subsequent application research.

Conclusiones

En esta sección, presentamos los resultados obtenidos de la investigación realizada en esta disertación de doctor. Esta tesis amplía y profundiza el estudio de GDM tradicional, promueve el desarrollo de SNGDM y LSGDM. En términos de los requisitos de toma de decisiones en el contexto complejo de redes sociales, la teoría LSGDM se aplica a la práctica de SCGR, que enriquece la teoría de la toma de decisiones y ofrece un método eficaz para aplicaciones prácticas. Por eso, este estudio tiene una importancia teórica importante y un valor de aplicación. Los resultados de la investigación obtenidos en esta tesis se describen en detalle desde los puntos principales a continuación.

(1) Inspirados por análisis de redes sociales, definimos CEN para la investigación intuitiva de la evolución de consenso de GDM. Basado en CENs, calculamos los niveles de consenso y los pesos de expertos analizando la estructura de la red. Los CEN proporcionan una nueva herramienta para investigar a fondo los problemas de consenso de la DMG tradicional y también promueven la aplicación de métodos de detección comunitaria en LSGDM, que pueden reducir efectivamente la complejidad de la LSGDM.

(2) Basado en la particularidad de las relaciones de confianza, analizamos la situación de consenso en SNGDM profundamente. Al principio, propusimos modelos de consenso de costo mínimo basados en la confianza implícita de los expertos para estudiar su efectividad en el consenso desde el análisis del modelo. Los modelos propuestos no solo consideran el efecto de compensa de la confianza en el costo de ajuste por consenso, sino también juzgan y modifican la irracionalidad subjetiva del costo de ajuste de los expertos. Además, basado en la estructura de red multicine, investigamos la evolución del consenso y el desarrollo de la confianza con la influencia integral de los expertos obtenida utilizando la centralidad de PageRank. Las investigaciones relacionadas arriba han realizado un estudio en profundidad sobre el consenso de grupo bajo la influencia de las relaciones sociales.

(3) Estudiamos principalmente el problema del consenso en LSGDM desde la perspectiva del análisis de agrupamiento. Por un lado, se propone un método de análisis de agrupamiento dinámico basado en CENs para LSGDM para equilibrar el análisis de agrupamiento y CRP con la gestión de los umbrales de consenso. En este método, el análisis de agrupamiento y la PCR están en un ciclo dinámico, que está cerca de la situación real de LSGDM. Por otro lado, el costo de ajuste de preferencia se considera un nuevo elemento en el análisis de agrupamiento, que comúnmente está dominado por la información de preferencia en LSGDM actual. Este método de agrupamiento puede clasificar tanto los DM con las preferencias similares como los costos de ajuste de la unidad, reduciendo efectivamente el costo de negociación y el tiempo de decisión. Las investigaciones anteriores pueden promover el progreso de la investigación de consenso grupal a gran escala y brindar una perspectiva más amplia para futuras investigaciones.

(4) Aplicamos preliminarmente el modelo LSGDM a SCGR. Apuntando a las características a gran escala de los usuarios y el bajo nivel de consenso en SCGR, se propone un modelo LSGDM considerando la información de preferencia incompleta, los efectos de polarización de grupo y las opiniones minoritarias. Luego, el modelo LSGDM anterior es utilizado para ofrecer listas de productos de compra de grupo para los usuarios en Dianping.com. Este estudio brinda un modelo LSGDM que considera más comportamientos grupales desde una perspectiva teórica y destaca la importancia práctica de la aplicación del modelo LSGDM en SCGR.

La investigación actual empezó con el problema tradicional de consenso de grupo, luego analizó el problema de consenso de grupo de las redes sociales, y en consecuencia se extendió a la investigación de la situación de consenso de grupo a gran escala, y al final trató de aplicar el modelo LSGDM a SCGR, que establece un fundamento teórico sólido para nuestra investigación subsecuente de aplicaciones.

9 Future works

Although we have done some research in this thesis, there are still some new challenges and interesting research topics in dealing with LSGDM. For example, the opinion dynamics among large-scale experts on the complex network structure, the non-cooperative behavior among experts, the language comments process, and emotion recognition in the practical application. Based on these exsiting topics, we introduce our future work in detail as follows:

9.1 Overlapping community detection and influence propagation in LSGDM

The main objective is to dig deeper into the influence of social relations on LSGDM by complex network theory and technology. The structural hole is a phenomenon in which some individuals in a social network have direct contact with others, but disconnection with others, which seems to be a cave in the network structure. We will analyze the role of critical nodes occupying structural holes in community communication and cooperation.

People in a social network are naturally characterized by multiple community memberships. For example, a person usually has connections to several social groups, and a researcher may be active in several areas. Therefore, we intend to discover the overlapping community structures in LSGDM where overlapping nodes occupy the structural holes between communities.

Based on the above analysis of influence propagation among communities, we can investigate the opinions propagation and evolution to predict the consensus situation of the LSGDM. We hope that this study can provide suggestions for decision managers to improve consensus with suitable information dissemination strategies and control strategies.

9.2 Irrational and non-cooperative behavior in LSGDM

Non-cooperative behavior is caused by different views, specialties, and interests among experts, and is also the product of cooperative evolution. The theory of network games is the basis for analyzing the collaborative evolution of human activities. The main objective is to study the non-cooperative behavior in LSGDM based on the theory of network games.

Dynamic analyses of general network structures are usually very complicated. The mature direction of research in network games is commonly used to model social interactions by letting two-player games be simultaneously played by connected players. Thus, we can classify large-scale experts into two communities according to cooperative willingness. One is willing to accept the plan, and the other is unwilling to accept it. A game arises when communities seek to maximize their interests. Thus, we can analyze the non-cooperative behavior based on network game theory.

Besides, the evolution of social relationships between experts can also be investigated to avoid non-cooperative behavior, like breaking the inverse relationship between non-partners and promoting the establishment of the cooperative association, enabling participants to trust each other more and achieve win-win results in decision-making.

9.3 Applications of LSGDM methods in reality

The primary purpose is to promote the practical application of the LSGDM model in SCGR. The practical application of social commerce still faces many problems, such as processing language comments, mining and analyzing social relationships between experts, etc. To deal with these problems, we need to do semantic analysis and emotional analysis on the evaluation data extracted from the real e-commerce platform, especially the linguistic comments, and conduct quantitative processing.

In previous studies, we focused more on the elements of people (like user preferences and social relationships), but ignored other influencing factors of online shopping, such as commodity attributes, platform subsidies and stimulating consumption policies, which affect users' shopping behaviors. It is of great help for us to give a more accurate recommendation to discover the shopping behavior patterns and influencing factors of users through the review data.

The LSGDM methods can be used to segment customer market and rank recommendation lists based on user evaluation information for the SCGR application. However, the selection of recommendation algorithms is also outstanding, such as combining the recommendation model based on collaborative filtering and association rules to improve the surprising degree of group recommendation and reduce data sparseness and cold startup problems.

Chapter II

Publications: Published Papers

1 Consensus evolution networks: A consensus reaching tool for managing consensus thresholds in group decision making

- T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Consensus evolution networks: A consensus reaching tool for managing consensus thresholds in group decision making. **Information Fusion.** 52 (2019) 375-388.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 12.975
  - Subject Category: Computer Science, Artificial Intelligence, Ranking 3/139 (Q1).
  - Subject Category: Computer Science, Theory and Methods, Ranking 1/110 (Q1).

Consensus Evolution Networks: A Consensus Reaching tool for Managing Consensus Thresholds in Group Decision Making

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Abstract: The consensus reaching process (CRP) is a critical part of group decision making (GDM). In order to explore the evolution of consensus, a new CRP tool is proposed based on consensus evolution networks (CENs). The CENs are built based on the consensus degrees among decision makers (DMs) and allow us to manage the consensus thresholds and its evolution. A new consensus index is introduced based on the structured and numerical aspects of the CENs. The new consensus index can not only deeply analyze the constitution of consensus, but also determine the weights of DMs. According to the clustering coefficient, the sensitive consensus threshold is identified and the sensitive consensus evolution network (SCEN) is built. Based on the complementary SCEN, a pairwise feedback adjustment method is proposed to improve consensus. Besides, the sparsity of the CENs can act as a reference to determine the agreed consensus thresholds, which is considered an important issue in traditional models. A numerical example is used to verify the usefulness of the proposed CRP tool. The numerical results show that the evolution of consensus can be clearly found based on CENs and the pairwise method can improve consensus in only four rounds.

Keywords: Consensus reaching process; consensus evolution networks; group decision making

1 Introduction

Group decision making (GDM) is regarded as a useful technique to be able to make an optimal decision when multiple options are offered by a couple of stakeholders [1]. These decision-makers (DMs) may have different knowledge and experience, even different goals. To make a decision that satisfies most DMs, the consensus reaching process (CRP) is a crucial tool that promotes the formation of a consensus view [2]. As it is difficult to reach a complete consensus under the theory of "hard" consensus [3, 4], "soft" consensus is proposed [5-7] and developed rapidly [8-12]. In the "soft" CRP, the iterative and dynamic process is carried out until an agreed consensus threshold is achieved.

The fuzzy preference relation (FPR) is commonly used to represent DMs' preference information [13]. Many researchers have applied the FPR to describe DMs' pairwise comparison information under GDM environment [14-18]. Especially, many CRPs are mainly

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proposed based on FPRs [19-24]. For example, Pérez et al. [19] proposed a new consensus model for GDM with FPRs for non-homogeneous DMs. Dong et al. [20] measure consensus based on FPRs with the dynamic weights of DMs. Xu et al. [21] proposed a local adjustment strategy to reach consensus with FPRs, and also introduced a new consensus model based on the revised hesitant FPRs [22]. Furthermore, Xu et al. [23] and Liu et al. [24] explored CRPs in the large-scale group decision making based on FPRs. With the advantages of information representation and consensus measure, the FPRs are used to express DMs' opinions in this study.

In recent years, a variety of achievements have been made concerning consensus reaching process models (CRPs) based on soft consensus [2, 11, 14-16, 19, 20, 25-42]. The consensus development in this study is introduced by the following three key points: the consensus measurement [16, 18, 20, 31-35], the feedback adjustment [2, 9-11, 16, 19, 20, 32, 35-39], the consensus analysis based on social network analysis [2, 34, 35, 43-47].

(1) The consensus measurement. The consensus measurement is handled based on the similarity degree among DMs [48]. And most of the similarity is mainly computed based on the distance function from two aspects [41]. One is consensus measure based on distances to the collective preference [10, 18, 31, 32], in which, the collective preference is represented by the group opinion. The other is consensus measure based on pairwise distances between DMs [16, 20, 33-35].

(2) The feedback adjustment. The nature of the feedback adjustment is that DMs contributing less to consensus are encouraged to modify their opinions so that they are more similar to the collective one [9, 33]. Researchers have paid more attention to the feedback adjustment methods, such as the extension of the traditional methods [11, 19, 36], the optimization-based consensus rules [9, 10, 32, 35, 49], and the non-cooperative behavior-based consensus rules [2, 16, 20, 37-39].

(3) The consensus analysis based on social network analysis. The social network analysis (SNA) is a useful tool for consensus reaching, which promotes the generation of the social network group decision making (SNGDM) [2, 34, 35, 43-47]. Since consensus analysis based on SNA is currently one of the hottest research points of CRPs, the developments of the SNGDM have been reviewed in Ref. [40-42]. Herrera-Viedma et al. [40] have introduced and offered a broader perspective on some CRPs based on SNA. Ureña et al. [42] discussed the function of trust, reputation and influence for fostering decision making processes and recommendation mechanisms in social networks scenarios. Dong et al. [41] gave a detailed introduction of these SNA-based CRPs, especially those based on opinion dynamics [44-47] and trust relations [2, 34,

35, 43]. It has been verified that the propagation of opinions and trust relations is beneficial for consensus reaching. Based on the DeGroot model [44, 45], Dong et al. [47] developed a CRP model in opinion dynamics based on the concept of leadership. Ding et al. [46] proposed an opinion control rule to support consensus reaching. Wu et al. [34, 35, 43] continuously studied trust based consensus models with SNA. Zhang et al. [2] introduced a novel consensus framework based on social trust networks to deal with non-cooperative behaviors. In short, not only can the effectiveness of CRPs be improved with opinion dynamics and trust propagation, but the CRPs can also be conveniently computed with SNA techniques.

Although many CRP models in existing research are important for helping group members to reach a consensus, they still need to be further improved to figure out the evolution of the consensus in depth. The discussion is based on the following limited problems and issues.

(1) Many CRPs paid close attention to consensus measure and feedback adjustment, especially with the help of SNA. In SNGDM, researchers used to assume that DMs were linked together because of some kinds of relations and that the group consensus was improved using the transmission of relations or influence of DMs. The existing SNGDM CRPs focus on the consensus with the evolution of relations among DMs but ignore the consensus change based on preference, which can more effectively reveal the essence that consensus forms.

(2) The over consensus degree is mainly aggregated with individual DMs' consensus degrees by the simple averaging operator or other operators, rather than being analyzed from the structure of consensus. Besides, the weights of DMs are given subjectively or are difficult to determine, some even assume that there is no difference between DMs. However, such an assumption may be improbable in some scenarios.

(3) The iterative and dynamic feedback adjustment process to reach a consensus which can satisfy all DMs has a high cost. Although opinion dynamic and trust propagation can reduce the cost in some ways, sometimes opinion and trust may be too subjective to be susceptibly manipulated by malicious information. If opinion and trust are mismanaged, the GDM will be lead to fragmentation and polarization.

(4) The determination of the agreed consensus threshold is still an open problem in CRP. The agreed consensus thresholds in many studies are mainly set subjectively based on decision experience and goals according to the requirements of the particular problem. Very few studies have focused on judging the reasonableness of agreed consensus threshold. As mentioned before, pursuing a high consensus is not only costly, but the significance of the GDM can also be easily lost with too much assimilation.

To deal with these limitations, we introduce a new tool for CRP in GDM based on the consensus evolution networks (CENs) to manage the consensus thresholds and its evolution, which is based on the following hypotheses:

(1) Suppose DMs reach higher consensus with more DMs, have greater knowledge about the GDM problem and have higher weights in decision making.

(2) Suppose DMs connected with more DMs in CENs have more influence over others and have higher weights in decision making.

(3) Suppose each DM has an equal willingness to reach consensus, that is, to change their opinion to reach consensus.

According to the above hypotheses, the main purpose of this study is to propose a new tool for CRP based on CENs, including exploring the evolution of consensus for the determination of the agreed consensus thresholds, introducing a new consensus index and designing a pairwise feedback adjustment method for improving consensus. In order to deal with the limited problems in GDM mentioned above, we will explore the new CRP tool in GDM based on the graph theory with following aspects:

(1) For the first issue, the effective consensus relations are distinguished managing consensus thresholds, and the complete, incomplete and empty CENs among DMs are built with different consensus thresholds. We analyze the composition and evolution of consensus based on different CENs. In addition, we distinguish the sensitive consensus threshold and determine the sensitive CEN (SCEN) from the dynamic CENs.

(2) For the second issue, we propose a new index for consensus measure based on CENs, taken mainly from consensus levels and network structures, to analyze the formation of consensus in depth. The weights of DMs' are determined in the calculation of this index. Also, the effectiveness of the new consensus index and traditional consensus measure methods are compared so as to explain the rationality of the new consensus index.

(3) For the third issue, we introduce the pairwise feedback adjustment method based on the complementary sensitive CEN (CSCEN). According to CSCEN, DMs contribute to the consensus less or more are easy to be identified. DMs with low consensus adjust their preference according to the DMs with the high consensus in their neighbors, with such process, the gap between DMs with low consensus and other DMs can also be narrowed in some degree.

(4) For the fourth issue, the agreed consensus threshold can be set within a reasonable range based on the structures of CENs, especially by the reference of the sensitive consensus threshold. The reference of the sensitive consensus threshold can avoid too many deviations between the agreed consensus threshold and the actual consensus situation among DMs, so as to avoid the excessive adjustment costs in CRP.

This study has something in common with the previous consensus approaches, yet it has an important advantage. Regarding to the common approaches: a) the CENs are built based on the preference similarity, like in the traditional GDM, b) the structured weights are determined based on degree centrality like the SNA in SNGDM. Regarding the advantages, we highlight the determination of the agreed consensus threshold, it is an important problem both in traditional GDM and SNGDM, while the sparsity of the CENs in this study can act as a reference for the determination of the agreed consensus thresholds.

The proposed CRP tool based on CENs is examined by a numerical example. In the example, the sensitive consensus threshold and its corresponding SCEN are identified, and the overall consensus degree is computed by the new consensus index structurally and numerically. The consensus is achieved to the agreed value in only four rounds using the proposed feedback adjustment method. After the adjustment, the sensitive consensus threshold increases and the consensus relation values between all the pairwise DMs increase evenly, which shows the efficiency and usefulness of the proposed CRP tool. At the end of this paper, after the current proposal has been explained in depth, we analyze the main difference between this study and another two well know approaches: the classical GDM based on consensus degree at three levels and the GDM based on social relations and SNA.

The rest of this study is organized as follows. In section 2, the preliminaries of FPRs and graph theory are introduced. In section 3, some definitions of CENs are given. In section 4, a new consensus index is proposed based on CENs. In section 5, the feedback adjustment method is presented based on CENs. In section 6, the CRP tool based on CENs is described. In section 7, a numerical example is used to illustrate the feasibility of the proposed CRP tool. In section 8, the comparison and analysis are provided. In section 9, the conclusion and ideas for further studies are given.

2 Preliminaries

Before introducing the new tool for CRP based on CENs, some basic knowledge of CRP and graph theory needs to be reviewed briefly. The definition of the FERs and the similarity matrix based on FERs are given in Section 2.1. And some definitions and measurable indicators about graph theory are described in Section 2.2.

2.1 The fuzzy preference relations

Consensus degrees in GDM are often computed based on the preferences of DMs with respect to alternatives. The FRRs is critical for uncertain GDM, and the definition of FPRs is given as below.

Definition 1. [13] An FPR F is a fuzzy set on the alternative set $X \times X$, which is characterized by a membership function $\mu_F: X \times X \rightarrow [0,1]$, where $\mu_F(x_i, x_j)$ is interpreted as the preference degree of the alternative x_i over x_j, and fulfilling $\mu_F(x_i, x_j) + \mu_F(x_i, x_j) = 1$.

Generally, let $D = \{d_1, d_2, ..., d_m\}$ be the set of DMs involved in the GDM problem and $F_k = \left(f_{ij}^k\right)_{n \times n}$ be the FPR on the alternative set $X = \{x_1, x_2, ..., x_n\}$ of DM d_k, $i, j = 1, ..., n$, $k = 1, ..., m$. The FPR matrix of DM d_k can be represented as:

$$F_k = \begin{pmatrix} 0.5 & \cdots & f_{1i}^k & \cdots & f_{1n}^k \\ \cdots & 0.5 & \cdots & \cdots & \cdots \\ f_{i1}^k & \cdots & 0.5 & \cdots & f_{in}^k \\ \cdots & \cdots & \cdots & 0.5 & \cdots \\ f_{n1}^k & \cdots & f_{ni}^k & \cdots & 0.5 \end{pmatrix}_{n \times n} \tag{1}$$

where $f_{ij}^k = \mu_{F_k}(x_i, x_j)$ and $f_{ij}^k + f_{ji}^k = 1$.

Based on the FPRs, Palomares et al. [16] defined the similarity matrix to determine the consensus matrix for CRP. The definition of the similarity matrix is shown as below.

Definition 2. [16] A similarity matrix $SM_{kl} = \left(sm_{ij}^{kl}\right)_{n \times n}$ between DM d_k and d_l on the preference of alternative x_i over x_j is defined as:

$$SM_{kl} = \begin{pmatrix} - & \cdots & sm_{1i}^{kl} & \cdots & sm_{1n}^{kl} \\ \cdots & - & \cdots & \cdots & \cdots \\ sm_{i1}^{kl} & \cdots & - & \cdots & sm_{in}^{kl} \\ \cdots & \cdots & \cdots & - & \cdots \\ sm_{n1}^{kl} & \cdots & sm_{ni}^{kl} & \cdots & - \end{pmatrix}_{n \times n} \tag{2}$$

where sm_{ij}^{kl} is computed by means of a similarity function introduced in [33]: $sm_{ij}^{kl} = 1 - \left| f_{ij}^k - f_{ij}^l \right|$, $i, j = 1, 2, ..., n; i \neq j$, and $k, l = 1, 2, ..., m; k \neq l$.

2.2 Some definitions of graph theory

In graph theory, the set of vertices in a classical graph is denoted as $V = \{v_k\}$. Due to the fact that DMs are our main research objects, the set of vertices in graph are denoted as

$D=\{d_k\}$ in this paper.

Definition 3. [50, 51] A simple weighted graph $G=\{D,E,W\}$ consists of a non-empty finite set $D=\{d_1,d_2,...,d_m\}$ of m vertices and a finite set $E=\{e_{kl}\}(k,l=1,2,...,m,k\neq l)$ of edges with a finite set $W=\{w_{kl}\}(k,l=1,2,...,m,k\neq l)$ of weights, in which, an edge e_{ij} indicates the connection between d_k and d_l with weight w_{kl}.

Let $N(E)$ be the number of edges in G, if G is a complete graph, then:

$$N(E)=\frac{m(m-1)}{2} \tag{3}$$

Definition 4. [50, 51] The degree of a vertex d_k is the number of edges incident with d_l, and is written as $\deg(d_k)$:

$$\deg(d_k)=N(e_{kl}) \quad (k,l=1,2,...,m,k\neq l) \tag{4}$$

where $N(e_{kl})$ is the number of adjacency edges of node d_k.

According to the relation of vertices-degree in a simple graph [50, 51], the sum of all the degrees is equal to the double numbers of edges:

$$\sum_{k=1}^{m}\deg(d_k)=2N(E)=\sum_{k=1}^{m}\sum_{l=1}^{m}N(e_{kl}) \tag{5}$$

In terms of specific relations among vertices, general graphs can be called as specific networks, such as the small-world networks and complex networks. Watts and Strogatz [52] introduced the definition of a clustering coefficient to determine whether or not a graph is a small-world network. The clustering coefficient is widely used to verify the degree of connectivity between points in complex networks.

Definition 5. [52] In a simple undirected graph $G=\{D,E\}$, in which, $D=\{d_1,d_2,...,d_m\}$ and $E=\{e_{kl}\}(k,l=1,2,...,m,k\neq l)$, the neighborhood N_k for a vertex d_k is defined as its immediately connected neighbors such as $N_k=\{d_l:e_{kl}\in E\}$. The local clustering coefficient for undirected graphs G is defined as:

$$LCC_k=\frac{2\left|\{e_{la}:d_l,d_a\in N_k,e_{la}\in E\}\right|}{N(d_k)\left[N(d_k)-1\right]} \tag{6}$$

where d_l and d_a are neighbors of vertex d_k in N_k, $N(d_k)$ is the number of neighbors of d_k, $\left|\left\{e_{la}: d_l, d_a \in N_k, e_{la} \in E\right\}\right|$ is the number of edges among the neighbors of d_k, and there are $N\left(d_k\right)\left[N\left(d_k\right)-1\right]/2$ edges existing among d_k within the neighborhood.

The overall level of clustering coefficient CC is computed as the average of the local clustering coefficients of all the vertices in G:

$$CC = \frac{1}{m}\sum_{k=1}^{m} LCC_k \tag{7}$$

where $CC = 1$ means G is a complete network.

3 The consensus evolution networks

In this section, the CENs are built according to some definitions of graph theory managing consensus thresholds. And the SCEN is identified based on the clustering coefficient of complex networks. The consensus evolution of the GDM can be conveniently found based on CENs with different consensus thresholds. In Section 3.1, the consensus matrix is constructed. In Section 3.2, some definitions of CENs are given. In Section 3.3, the SCEN is determined.

3.1 The construction of consensus matrix

Firstly, based on Eq. (2), the similarity degrees between the pairwise DMs are computed, and the similarity matrix $SM_{kl} = \left(sm_{ij}^{kl}\right)_{n\times n}$ is constructed. And then, the consensus matrix $CM = \left(cm_{kl}\right)_{m\times m}$ among DMs is determined as:

$$CM = \begin{pmatrix} 0 & \cdots & cm_{1k} & \cdots & cm_{1m} \\ \cdots & 0 & \cdots & \cdots & \cdots \\ cm_{k1} & \cdots & 0 & \cdots & cm_{km} \\ \cdots & \cdots & \cdots & 0 & \cdots \\ cm_{m1} & \cdots & cm_{mk} & \cdots & 0 \end{pmatrix}_{m\times m} \tag{8}$$

where $cm_{kl}\left(k \neq l\right)$ is computed as:

$$cm_{kl} = \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} sm_{ij}^{kl}}{n\left(n-1\right)/2} \tag{9}$$

where $n\left(n-1\right)/2$ represents the number of different pairs of alternatives $\left\{x_i, x_j\right\}$, which means that alternatives are considered to be indistinguishable from one another, $k \neq l$, and $i \neq j$, so the diagonal elements of CM are set to 0.

The consensus matrix CM is a symmetric matrix, that is $cm_{kl} = cm_{lk}$, which means the consensus relation between a pair of DMs is unique and unidirectional. Besides, it is easy to see that $cm_{kl} \in [0,1]$.

A simple example (such as Example 1 which is taken from [53]) is used to show the computation of the consensus matrix.

Example 1: Assuming there are four FPRs given by four DMs on four alternatives as follows:

$$F_1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0.5 & 0.7 & 0.9 & 0.5 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.5 & 0.3 & 0.2 & 0.5 \end{pmatrix} F_3 = \begin{pmatrix} 0.5 & 0.3 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.1 & 0.3 \\ 0.5 & 0.9 & 0.5 & 0.25 \\ 0.3 & 0.7 & 0.75 & 0.5 \end{pmatrix} F_4 = \begin{pmatrix} 0.5 & 0.25 & 0.15 & 0.65 \\ 0.73 & 0.5 & 0.6 & 0.8 \\ 0.85 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}$$

Based on Eq. (2) and (9), the consensus matrix $CM = (cm_{kl})_{4\times4}$ is computed as:

$$CM = \begin{bmatrix} 0 & 0.716 & 0.708 & 0.774 \\ 0.716 & 0 & 0.591 & 0.708 \\ 0.708 & 0.591 & 0 & 0.766 \\ 0.774 & 0.708 & 0.766 & 0 \end{bmatrix}_{4\times4}$$

3.2 Some consensus evolution network definitions

Based on the consensus degrees between the pairwise DMs shown in $CM = (cm_{kl})_{m\times m}$, CENs can be built managing consensus thresholds ε. If $cm_{kl} \geq \varepsilon$, then the consensus degree between d_k and d_l is acceptable, which means there is consensus relation e_{kl} between d_k and d_l, otherwise, e_{kl} does not exist. Based on the consensus relations, the CENs can be built. And the satisfied consensus degree cm_{kl} is called as the consensus relation value between d_k and d_l. The consensus threshold ε can be determined according to the consensus degrees in $CM = (cm_{kl})_{m\times m}$. According to the definition of general graphs, the definition of general CEN is given as below.

Definition 6. A consensus evolution network (CEN) consists of $G = (D, E, C)$ with m DMs $D = \{d_1, d_2, ..., d_m\}$, consensus relations $E = \{e_{kl}\}(k, l = 1, 2, ..., m, k \neq l)$ and consensus relation values $C = \{c_{kl} = cm_{kl} \mid k, l = 1, 2, ..., m, k \neq l, cm_{kl} \geq \varepsilon\}$, where ε fixes the consensus thresholds. If $c_{kl} = cm_{kl} \geq \varepsilon$, then there is an edge e_{kl} in G to connect d_k and d_l together with the consensus relation value c_{kl}, and c_{kl} called the weight of the edge e_{kl}, otherwise, when $c_{kl} < \varepsilon$, there is no edge between d_k and d_l.

The existence of edges in the CENs indicates that a certain degree of consensus has been reached between a pair of DMs. The CEN built is simple, weighted and undirected. That is, neither loops happened to any DMs, nor was there more than one edge between a pair of DMs in the CEN. The ‘undirected’ feature suggests that the consensus relation between the pairwise DMs is mutual and unique. According to the layouts of CENs, the consensus situation can be measured structurally and numerically.

The CEN G can be shown in different structures with different ε. It is worth noting that the different layouts of CENs are caused by the changing edges and their corresponding weights, not including the variation of DMs. In many types of CENs, there are two extremes: the complete and empty CENs, and they are denoted as G_C and G_E. Automatically, other types are considered incomplete CEN denoted as G_I. Based on Definition 6, the definitions of G_C, G_E, and G_I are respectively given as below.

Definition 7. The complete consensus evolution network (CCEN) consists of $G_C = \left(D, E_C, C_C\right)$ with m DMs, let D be defined in G, consensus relations $E_C = \left\{e_{kl}^C \mid k,l = 1,2,...,m, k \neq l\right\}$, consensus relation values $C_C = \left\{c_{kl}^C = cm_{kl} \mid k,l = 1,2,...,m, k \neq l, cm_{kl} \geq \varepsilon_C\right\}$, in which, $\varepsilon_C = \min\left\{cm_{kl}\right\}$. There is always an edge e_{kl}^C in G_C to connect d_k and d_l together with weight c_{kl}^C. If $\exists\, cm_{kl} = 0$, then the edge e_{kl}^C does not exist, and the CCEN G_C is non-existent.

In G_C, all DMs are interconnected which suggests that the consensus among DMs has been reached structurally, and all DMs have full consensus structurally and numerically when $\min\left\{cm_{kl}\right\} = 1$. Under a given agreed consensus threshold $\overline{\varepsilon}$, the larger $\min\left\{cm_{kl}\right\}$, the greater the possibility for building a CCEN, and vice versa. If $\overline{\varepsilon} \leq \min\left\{cm_{kl}\right\}$, then the CRP is finished, otherwise, the feedback adjustment needs to be carried out. The larger $\min\left\{cm_{kl}\right\}$, the smaller the cost of CRP.

Definition 8. The empty consensus evolution network (ECEN) consists of $G_E = \left(D, E_E, C_E\right)$ with m DMs, let D be defined in G, consensus relations $E_E = \left\{e_{kl}^E \mid k,l = 1,2,...,m, k \neq l\right\}$, and consensus relation values $C_E = \left\{c_{kl}^E = 0 \mid k,l = 1,2,...,m, k \neq l, \max\left\{cm_{kl}\right\} < \varepsilon_E\right\}$.

In G_E, all DMs are disconnected which shows that the consensus among DMs has not been reached structurally at all, and there is no consensus structurally and numerically at all when

$\max\{cm_{kl}\}=0$. Under a given agreed consensus threshold $\overline{\varepsilon}$, the smaller the $\max\{cm_{kl}\}$, the greater the possibility for forming an ECEN G_E, and vice versa. If $\overline{\varepsilon} \geq \max\{cm_{kl}\}$, the feedback adjustment needs to be carried out, and the smaller the $\max\{cm_{kl}\}$, the larger the cost of CRP, which indicates that it is inefficient to set the threshold too high.

Definition 9. The incomplete consensus evolution network (ICEN) consists of $G_I=(D,E_I,C_I)$ with m DMs, let D be defined in G, consensus relations $E_I=\{e_{kl}^I \mid k,l=1,2,...,m,k\neq l\}$, consensus relation values $C_I=\{c_{kl}^I=cm_{kl} \mid k,l=1,2,...,m,k\neq l,cm_{kl}\geq \varepsilon_I\}$, in which, $\min\{cm_{kl}\}<\varepsilon_I\leq \max\{cm_{kl}\}$. There is an edge e_{kl}^I in G_I to connect d_k and d_l together with weight c_{kl}^I.

Obviously, $\min\{cm_{kl}\}$ is the boundary between G_C and G_I, and $\max\{cm_{kl}\}$ is the boundary between G_I and G_E. So the number of edges of G_I is located between G_C and G_E: $N(E_E)<N(E_I)<N(E_C)$. It differs to CCEN and ECEN in that there are different layouts for ICENs with different ε_I. The sensitive consensus threshold can be identified mainly based on the variation of G_I.

3.3 The determination of the sensitive consensus evolution network

According to the definition of complex networks [52, 54], some of the main characteristics of complex networks are also present in CENs, such as network evolution, connection diversity and dynamic complexity. The clustering coefficient in complex networks reflects the degree of network collectivization, that is, the cohesive tendency of the network, or the degree of the small world effect. Since the CENs have similar characteristics to the complex networks, the clustering coefficient is used to distinguish the sensitive consensus threshold from CENs.

To distinguish the sensitive consensus threshold from $\varepsilon=\{\varepsilon_C,\varepsilon_I,\varepsilon_E\}$, compute the clustering coefficient CC_r for the consensus evolution network G_r based on Eq. (7). Suppose the number of ε_I is t, $r=0,...,t+1$. The greater the gap of the clustering coefficient between neighboring CENs, the more sensitive DMs are to the higher consensus between neighboring values. Therefore, based on the clustering coefficient $CC_r\,(r=0,...,t+1)$, DMs are considered to be sensitive to $\varepsilon_r\,(r=0,...,t+1)$ when the gap of the clustering coefficient between two CENs reaches maximum:

$$\max\left(CC_{r-1} - CC_r\right) \tag{10}$$

where CC_{r-1} and CC_r are the clustering coefficients of relevant CEN G_{r-1} and G_r. With ε_r, corresponding CEN is referred to as the sensitive consensus evolution network (SCEN) .

The smaller the CC_r, the less stable the triangle relationship is among DMs. Obviously, the sensitive consensus threshold suggests that the CENs become vulnerable and unstable due to ε_r. It is also suggests that the consensus relation values between most pairs of DMs do not reach ε_r. Therefore, the value of ε_r can be a point of reference for people to set the agreed consensus threshold $\overline{\varepsilon}$ for CRP. The more $\overline{\varepsilon}$ is larger than ε_r, the higher the cost of the feedback adjustment in CRP.

Example 2. Let $CM = (cm_{kl})_{4\times 4}$ be as in **Example 1**:

Based on $CM = (cm_{kl})_{4\times 4}$, $\varepsilon_C = \min\{cm_{kl}\} = 0.591$, $\varepsilon_E > \max\{cm_{kl}\} = 0.774$, $\varepsilon_I = \{\varepsilon_I^1, \varepsilon_I^2, \varepsilon_I^3, \varepsilon_I^4\} = \{0.708, 0.716, 0.766, 0.774\}$. According to the Definition 7-9, the CCEN G_C, ECEN G_E and ICEN $G_I = \{G_I^1, G_I^2, G_I^3, G_I^4\}$ are built in Fig. 1.

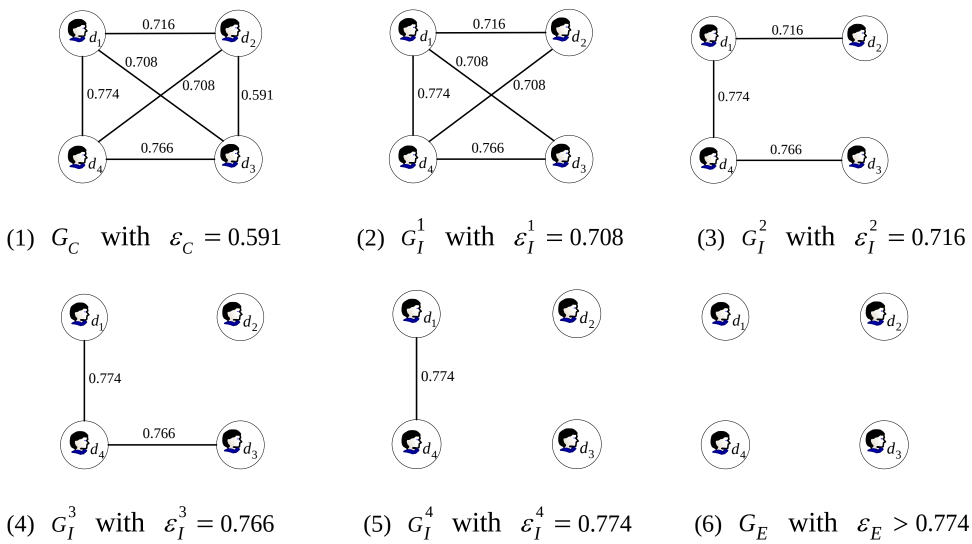


Fig.1 The layouts of all consensus evolution networks with four DMs

In addition, the clustering coefficients of all the built CENs are computed based on Eq. (7) as: $CC_0 = 1$, $CC_1 = 0.83$, $CC_2 = CC_3 = CC_4 = CC_5 = 0$. Based on Eq. (10), it is known that DMs are sensitive for $\varepsilon_I^2 = 0.716$, and G_I^2 can be considered to be a SCEN. According to Fig.1, we can also see that the CENs become apparently sparse and weak when $\varepsilon_I^2 = 0.716$.

4 A new consensus index based on consensus evolution networks

The consensus measure is important for CRP. To explore the formation of consensus in depth, a new consensus index is proposed in this section structurally and numerically. Based on CENs, the importance of DMs can also be determined from a structured and numerical perspective. In Section 4.1 and 4.2, the numerical and structured consensus index is introduced. In Section 4.3, the comprehensive index is computed and the comparative analysis with traditional methods is given.

4.1 The numerical consensus index

According to the layouts of CENs, the overall consensus degree can be determined structurally and numerically. The numerical index and the structured index are denoted as CR_N and CR_S, respectively. The numerical index measures the overall consensus degree based on the consensus levels among DMs, while the structured index measures the overall consensus degree based on the consensus relations among DMs. Similarly, the importance of DMs is also reflected as the numerical weight ω_N^k and the structured weight ω_S^k. CR_N and CR_S are both determined at three levels: (i) Level of weights determination (ii) Level of unit consensus or connection strength (iii) Level of the collective consensus or connection strength.

The numerical consensus index CR_N is computed as:

i) The level of weights determination (ω_N^k): the weight ω_N^k is computed based on the consensus ratio of d_k and all DMs in G:

$$\omega_N^k = \frac{\sum_{l=1}^{m} c_{kl}}{\sum_{k=1}^{m}\sum_{l=1}^{m} c_{kl}} \left(i \neq j\right) \tag{11}$$

where $\sum_{l=1}^{m} c_{kl}$ denotes the sum of consensus between d_k and its neighboring DMs, $\sum_{k=1}^{m}\sum_{l=1}^{m} c_{kl}$ represents the sum of consensus between each DM and its neighboring DMs, $\omega_N^k \in \left[0,1\right]$, $\sum_{l=1}^{m} \omega_N^k = 1$, and it changes with the preference of DMs. When all DMs are considered to be equally important, $\omega_N^k = 1/m$.

ii) The level of unit consensus (u_N^k): the unit consensus u_N^k of DM d_k is determined as:

$$u_N^k = \frac{\sum_{l=1}^{m} c_{kl}}{\deg(d_k)} \quad (k \neq l) \tag{12}$$

where $\deg(d_k)$ is the degree of DM d_k and indicates the number of edges $e_{kl} \in E$ between d_k and other DMs, $c_{kl} \in C$ are the weights of the corresponding edges and represent the consensus relation value between d_k and other connected DMs.

iii) The level of the collective consensus degree (the numerical index CR_N): CR_N is computed based on the weighted aggregating consensus of all DMs:

$$CR_N = \sum_{k=1}^{m} \omega_N^k u_N^k \tag{13}$$

where u_N^k is the unit consensus of d_k, ω_N^k is the consensus based weight of d_k.

According to the properties of consensus, all DMs reach complete consensus when the overall consensus degree equals to 1. Hence, a desired property of the numerical index CR_N is given as below.

Property 1. The consensus based index CR_N varies from zero to one based on Eq. (13), i.e., $CR_N \in [0,1]$.

Proof. Since $c_{kl} \in [0,1]$, we have $\sum_{l=1}^{m} c_{kl} \in [0, m-1] \ (k \neq l)$. When $\sum_{l=1}^{m} c_{kl} = 0$, then based on Eq. (12): $\sum_{l=1}^{m} c_{kl} = 0 \Rightarrow u_N^k = 0$. When $\sum_{l=1}^{m} c_{kl} = m-1$, which means that there are $m-1$ edges between d_k and others, and the weights of all edges are equal to 1, and obviously $\deg(d_k) = m-1$, then based on Eq. (12): $\sum_{l=1}^{m} c_{kl} = m-1 \Rightarrow u_N^k = 1$. When $0 < \sum_{l=1}^{m} c_{kl} < m-1$, it always $\exists\, c_{kl} < 1$, then based on Eq. (12): $\sum_{l=1}^{m} c_{kl} < \deg(d_k) \Rightarrow 0 < u_N^k < 1$. Hence, $u_N^k \in [0,1]$. It is easy to see that $\omega_N^k \in [0,1]$ and $\sum_{k=1}^{m} \omega_N^k = 1$ from Eq. (11). Based on Eq. (13), we can conclude that $CR_N \in [0,1]$.

4.2 The structured consensus index

Similarly, the structured index CR_S can also be determined at three levels:

i) The level of weights determination (ω_S^k): the weight ω_S^k is computed based on the connection strengths among DMs:

$$\omega_S^k = \frac{\deg\left(d_k\right)}{\sum_{k=1}^{m} \deg\left(d_k\right)} \tag{14}$$

where $\deg\left(d_k\right)$ is the degree of DM d_k, $\omega_S^k \in [0,1]$, $\sum_{k=1}^{m} \omega_S^k = 1$, and it changes with the consensus structure of the consensus evolution networks. When all DMs are considered to be equally important, $\omega_N^k = 1/m$.

ii) The level of unit connection strength (u_S^k): the unit connection strength u_S^k of DM d_k is determined:

$$u_S^k = \frac{\deg\left(d_k\right)}{m-1} \tag{15}$$

where $m-1$ is the maximum degree of nodes in G.

iii) The level of the collective consensus degree (the structured index CR_S): CR_S is computed by the weighted averaging operator:

$$CR_S = \sum_{k=1}^{m} \omega_S^k u_S^k \tag{16}$$

where ω_S^k is the structure based weight of d_k, u_S^k is the unit degree of d_k.

As with the numerical index CR_N, a desired property of CR_S is given as below.

Property 2. The structure based index CR_S varies from zero to one based on Eq. (16), i.e., $CR_S \in [0,1]$.

Proof. Since the maximum degree of individual DM is $m-1$, thus $\deg\left(d_k\right) \in [0, m-1]$, then $u_S^k \in [0,1]$. It is easy to see that $\omega_S^k \in [0,1]$ and $\sum_{k=1}^{m} \omega_S^k = 1$. Based on Eq. (16), we can obtain $CR_S \in [0,1]$.

4.3 The comprehensive consensus index

If and only if $CR_N = 1$ and $CR_S = 1$, then all DMs reach complete consensus numerically and structurally, that is all DMs have achieved consensus comprehensively. Thus, the comprehensive consensus index CR is determined with the combination of CR_N and CR_S:

$$CR = CR_N \times CR_S \tag{17}$$

where $CR_N \in [0,1], CR_S \in [0,1] \Rightarrow CR \in [0,1]$, $CR = 1$ means all DMs have reached complete consensus.

Many studies have been carried out to measure consensus based on distances between DMs while considering weights of DMs [16, 20, 33-35]. The weights of DMs are mainly given or computed unvaryingly [33-35]. Dong et al. [20] and Palomares et al. [16] provided dynamic weights for DMs for coping with the continuously changing CRP. In this paper, we measure the overall consensus degree from a numerical and structured perspective. The detailed comparative information is shown in Table 1.

Table 1 The comparisons of consensus measure methods

References	Consensus measure	Determination/Status of DMs' weights
Herrera-Viedma et al. [33]	Based on distances between DMs at three levels	Aggregation operators/ Stationary
Wu et al. [34, 35]		Based on trust degrees/ Stationary
Dong et al. [20]		Multiple attribute mutual evaluations/ Dynamic
Palomares et al. [16]		Based on non-cooperative behavior/ Dynamic
The new consensus index based on consensus evolution networks	Based on structured and numerical consensus at three levels	Based on consensus relation values and connection strength/ Dynamic

To illustrate the application of the new consensus index, the overall consensus degrees of the CENs built in Example 2 are computed in **Example 3**.

Example 3. Let CENs be as in **Example 2**:

Based on Eq. (11)-(17), the overall consensus degree G_C, G_I^1, G_I^2, G_I^3, G_I^4, and G_E are computed and compared with the traditional consensus measure in Table 2. Comparisons are given to verify the availability for the new consensus index. Since the weights of DMs are obtained in different ways in multiple consensus measure methods, the comparisons are mainly carried out based on the assumption that all DMs are considered to be equally important.

Table 2 The comparisons of the overall consensus degrees obtained between different consensus measure methods when all DMs are considered to be equally important

Consensus networks	G_C	G_I^1	G_I^2	G_I^3	G_I^4	G_E
The traditional method [16, 33]	0.710	0.612	0.376	0.256	0.129	0
The new consensus index	0.710	0.610	0.374	0.192	0.064	0

According to Table 2, it is obvious that the overall consensus degree of the new consensus index is exactly the same as that of the traditional method both for G_C and G_E. For the ICEN

G_I^1, G_I^2, G_I^3, and G_I^4, although the overall consensus degrees obtained using different methods are different to each other, the same decreasing trend indicates the rationality of the new consensus index. It is worth noting that there is not much difference between the overall consensus degrees of G_I^1 and G_I^2, while the difference between the overall consensus degrees of G_I^3 and G_I^4 is obvious. From the network structure of G_I^3 and G_I^4 shown in Fig.1, we can see that more and more DMs become isolated nodes in the network, which means that there is no consensus between them and any other DMs. Thus, the sharp drop of the overall consensus degrees of G_I^3 and G_I^4 corresponds exactly to their sparse structures.

Therefore, the new consensus index is approximately equivalent to traditional methods when DMs are considered to be undifferentiated, and it is more effective than traditional methods in reflecting changes of consensus structures.

5 The feedback adjustment based on consensus evolution networks

An adjustment strategy based on the CENs is introduced to improve the consensus in GDM. To reflect the fairness of decision making, the adjustment aims at improving the averaging consensus level of the whole group. To find a balance between the consensus connections and adjustment cost, the feedback adjustment is carried out based on the complementary sensitive consensus evolution network (CSCENs). The CSCEN is defined in Section 5.1. In Section 5.2, a pairwise adjustment strategy is proposed. In Section 5.3, the adjustment process based on CENs is introduced.

5.1 The complementary sensitive consensus evolution networks

The complementary CENs are the supplementary form of general CENs. Contrary to the general CENs, the complementary CENs become tight with the increasing ε. The general CENs show the consensus situation that most DMs have reached, while the complementary CENs show the consensus situation that most DMs have not achieved. Therefore it is easy to distinguish those DMs that contribute less to consensus from the complementary CENs. When the complementary CEN is too compact, the adjustment cost increases, while when the complementary CEN is too sparse, the low consensus connections are hard to find. Obviously, the complementary CENs change from a compact to sparse, especially from the sensitive consensus threshold. Hence, the complementary CEN is introduced based on the sensitive CEN and is referred to as the CSCENs.

As with the definition of universal set, the CCEN G_C can be regarded as the universal CEN of all DMs. Based on the universal CEN G_C, the CSCEN of the sensitive CEN

$G_r = (D, E_r, C_r)$ can be denoted as $\overline{G}_r = (D, \overline{E}_r, \overline{C}_r)$. To be more precise, the sets of edges and weights are complementary in $\overline{G}_r$ and G_r, except for the set of DMs. The definition of the CSCEN is given as below.

Definition 10. Based on the universal CEN $G_C = (D_C, E_C, C_C)$, the CSCEN consists of $\overline{G}_r = (D, \overline{E}_r, \overline{C}_r)$ with m DMs $D = \{d_1, d_2, ..., d_m\}$, consensus relations $\overline{E}_r = \{e_{kl} \mid e_{kl} \notin E_r\}$ and consensus relation values $\overline{C}_r = \{\overline{c}_{kl} \mid \overline{c}_{kl} = c_{kl}^C < \varepsilon_r\}$, $k, l = 1, 2, ..., m, k \neq l$. If $\overline{c}_{kl} = c_{kl}^C < \varepsilon_r$, then there is an edge e_{kl} in $\overline{G}_r$ to connect d_k and d_l together with the consensus relation value $\overline{c}_{kl}$, $\overline{E}_r \cup E_r = E_C$, $\overline{c}_{kl}$ called weight of the edge e_{kl} and $\overline{C}_r \cup C_r = C_C$.

If $CR_0 < \overline{\varepsilon}$, according to the identification rule [55], DMs who have low consensus levels need to be identified to adjust their FPRs. Based on Eq. (13), compute the consensus level for DMs in the universal CEN G_C:

$$cl_k = \frac{\sum_{l=1}^{m} \omega_N^k c_{kl}}{\deg(d_k)} (k \neq l) \tag{18}$$

where c_{kl} is the consensus relation value between d_k and d_l, ω_N^k is the numerical weight of d_k, $\deg(d_k)$ denotes the degree of d_k.

5.2 The pairwise adjustment strategy

To improve the overall consensus level of GDM, many feedback adjustment methods have been designed [9, 17, 20]. The aim of feedback adjustment is to narrow the gap between the preferences of individual DMs that have a low consensus level and the collective preference. Here, a pairwise adjustment strategy is proposed based on the distance to the collective preference to improve similarities in the pairwise DMs.

Since most of the existing adjustment strategies are proposed based on the majority opinion, we put forward the adjustment strategy based on the pairwise DMs (d_k, d_l), in which, d_k has the lower consensus level and d_l has the largest consensus level in the neighboring DMs of d_k. Based on the direction rule, let d_k adjust preferences according to d_l. We refer to d_k as the adjustment DM and d_l as the reference DM. Because d_l has highest consensus level out of the neighboring DMs of d_k, the adjustment of d_k according to d_l can also improve the consensus level between d_k and other neighboring DMs. Thus, the pairwise adjustment strategy allows each adjustment to maximize the overall consensus.

Suppose $cl_k = \min\{cl_k, cl_l\}$, according to the direction rule [55], the FPR of d_k needs to be adjusted. To make the consensus similarity between d_k and d_l as similar as possible, a consensus improving model to obtain the adjusted FPR $F_{k'} = \left(f_{ij}^{k'}\right)_{n\times n}$ is represented as:

$$f_{ij}^{k'} = \begin{cases} f_{ij}^{k} + \left|f_{ij}^{k} - f_{ij}^{kl}\right| \big/ 2,\ f_{ij}^{k} \le f_{ij}^{kl} \\ f_{ij}^{k} - \left|f_{ij}^{k} - f_{ij}^{kl}\right| \big/ 2,\ f_{i}^{k} > f_{ij}^{kl} \end{cases} \tag{19}$$

where $i \le j$, f_{ij}^{kl} is the averaging preference of d_k and d_l, and $f_{ij}^{kl} = \left(f_{ij}^{k} + f_{ij}^{l}\right)\big/2$, when $i > j$, $f_{ji}^{k'} = 1 - f_{ij}^{k'}$.

According to the adjustment rule proposed by Dong et al. [20], $f_{ij}^{k'}$ should satisfy $f_{ij}^{k'} \in \left[\min\left(f_{ij}^{k}, f_{ij}^{kl}\right), \max\left(f_{ij}^{k}, f_{ij}^{kl}\right)\right]$, so the property 3 is proposed and proofed based on Eq. (19) as below.

Property 3. For the initial FPRs $F_k = \left(f_{ij}^{k}\right)_{n\times n}$, the modified FPRs $F_{k'} = \left(f_{ij}^{k'}\right)_{n\times n}$ obtained using Eq. (19) satisfy $f_{ij}^{k'} \in \left[\min\left(f_{ij}^{k}, f_{ij}^{kl}\right), \max\left(f_{ij}^{k}, f_{ij}^{kl}\right)\right]$.

Proof. When $f_{ij}^{k} \le f_{ij}^{kl}$, $\min\left(f_{ij}^{k}, f_{ij}^{kl}\right) = f_{ij}^{k}$, $\max\left(f_{ij}^{k}, f_{ij}^{kl}\right) = f_{ij}^{kl}$, $f_{ij}^{k'} = f_{ij}^{k} + \left(f_{ij}^{kl} - f_{ij}^{k}\right)\big/2$, $f_{ij}^{kl} - f_{ij}^{k} \ge 0 \Rightarrow f_{ij}^{k} \le f_{ij}^{k'}$; $f_{ij}^{k'} = f_{ij}^{k} + \left(f_{ij}^{kl} - f_{ij}^{k}\right)\big/2 = f_{ij}^{k}\big/2 + f_{ij}^{kl}\big/2$, $f_{ij}^{k} \le f_{ij}^{kl} \Rightarrow f_{ij}^{k'} \le f_{ij}^{kl}\big/2 + f_{ij}^{kl}\big/2 = f_{ij}^{kl}$, so $f_{ij}^{k'} \in \left[\min\left(f_{ij}^{k}, f_{ij}^{kl}\right), \max\left(f_{ij}^{k}, f_{ij}^{kl}\right)\right]$. When $f_{ij}^{k} > f_{ij}^{kl}$, $\min\left(f_{ij}^{k}, f_{ij}^{kl}\right) = f_{ij}^{kl}$, $\max\left(f_{ij}^{k}, f_{ij}^{kl}\right) = f_{ij}^{k}$, $f_{ij}^{k'} = \left(f_{ij}^{kl} + f_{ij}^{k}\right)\big/2 < \left(f_{ij}^{k} + f_{ij}^{k}\right)\big/2 = f_{ij}^{k}$ and $f_{ij}^{k'} = \left(f_{ij}^{kl} + f_{ij}^{k}\right)\big/2 > \left(f_{ij}^{kl} + f_{ij}^{kl}\right)\big/2 = f_{ij}^{kl}$, so $f_{ij}^{k'} \in \left[\min\left(f_{ij}^{k}, f_{ij}^{kl}\right), \max\left(f_{ij}^{k}, f_{ij}^{kl}\right)\right]$.

Based on $f_{ij}^{kl} = \left(f_{ij}^{k} + f_{ij}^{l}\right)\big/2$, Eq. (19) can be further simplified as:

$$f_{ij}^{k'} = \frac{3 f_{ij}^{k} + f_{ij}^{l}}{4} (i \le j) \tag{20}$$

where $f_{ji}^{k'}=1-f_{ij}^{k'}\,(i>j)$.

5.3 The adjustment process based on consensus evolution networks

In order to reduce the cost of negotiations, we design the adjustment strategy based on pairs of DMs rather than having all DMs interact with each other in CSCENs. The main adjustment strategy is to pair up two DMs that differ greatly in their consensus values. Such combination can improve the overall consensus degree as much as possible.

The flowchart of the adjustment in CRP based on CSCENs is given in Fig. 2. The main steps of the adjustment process are also introduced as below.

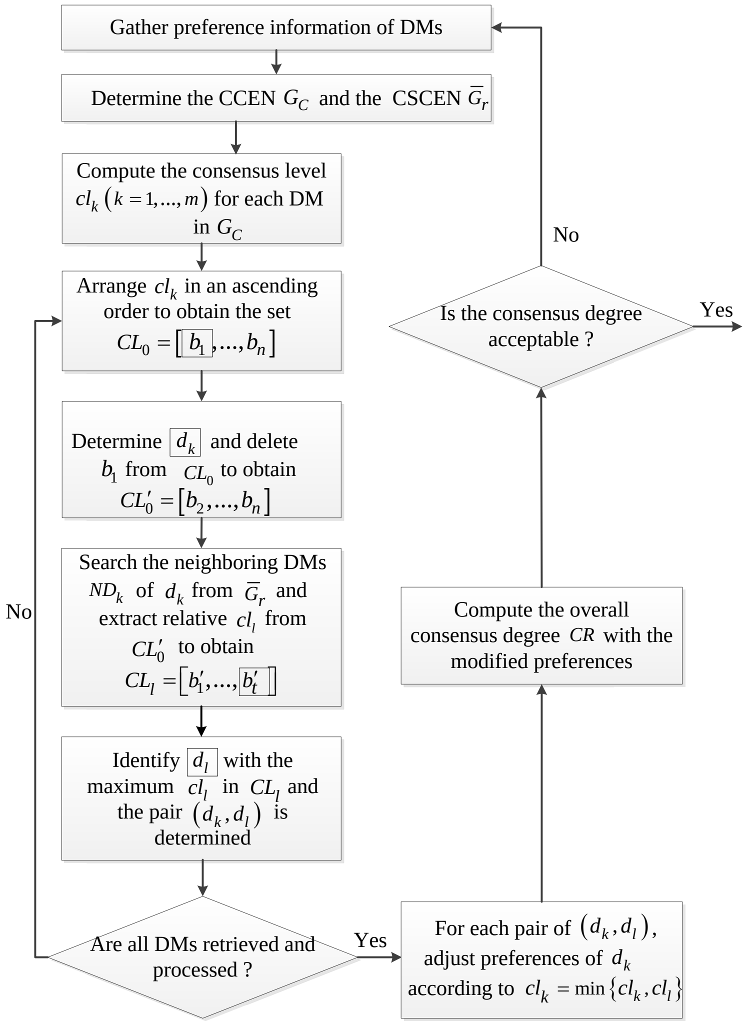


Fig.2 The flowchart of adjustment process in CRP based on CSCENs

Step 1: Compute the consensus levels $cl_k\,(k=1,...,m)$ of each DM based on Eq. (18) and arrange cl_k in an ascending order as the set $CL_0=\{b_1,...,b_m\}$, in which, $b_1=\min\{cl_k\}(k=1,...,m)$, $b_n=\max\{cl_k\}(k=1,...,m)$. Determine the adjustment DM d_k

from G_0 with the minimum consensus level b_1 and delete b_1 from CL_0 to obtain $CL_0' = \{b_2, ..., b_n\}$.

Step 2: Search the neighboring DMs of d_k from $\overline{G}_r$ as the set $ND_k = \{d_l | e_{kl} \in \overline{E}_r, l = 1, ..., m, k \neq l\}$. Extract the consensus levels of the neighboring DMs from CL_0' and arrange them in an ascending order $CL_l = [b_1', ..., b_t']$, and then determine the reference DM d_l with biggest consensus level $c_t = \max\{cl_l\}$ in ND_k. Thus, the pair of DMs (d_k, d_l) is determined. Delete the corresponding consensus level d_l from CL_0'.

Step 3: Repeat Step 1 and 2 until all pair of DMs in $\overline{G}_r$ are identified. Assign the isolated DM to the next round if necessary when the number of DMs in $\overline{G}_r$ is odd. As for the identified pairs of DMs, according to the identification rule: $cl_k = \min\{cl_k, cl_l\}$, adjust preferences of d_k from the pair (d_k, d_l) to obtain the modified FPR $F_{k'} = \left(f_{ij}^{k'}\right)_{n \times n}$ based on Eq. (20). So far, the first round of feedback adjustment is finished.

Step 4: Build the new consensus evolution networks with the modified FPRs of DMs, compute the overall consensus degree $CR_0^{(1)}$ obtained in the first round based on the proposed consensus index using Eq. (17). If $CR_0^{(1)} \geq \overline{\varepsilon}$, the feedback adjustment is stopped, otherwise, go to the next round of adjustment.

Example 4. Let G_C be as in **Example 2**:

Let $\overline{\varepsilon} = 0.80$. According to the computation in **Example 3**, the overall consensus degree of G_C is computed as $CR_0 = 0.710$. Obviously, $CR_0 < \overline{\varepsilon}$, so the FPRs of some DMs need to be adjusted. As in **Example 2**, G_I^2 is the sensitive CEN, so build the CSCEN $\overline{G}_I^2$ in Fig. 3.

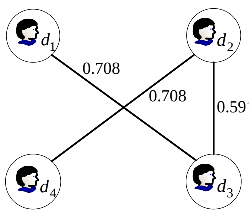


Fig.3 The complementary consensus evolution network of four DMs when $\varepsilon_2 = 0.716$

Based on Eq. (18), obtain the set $CL_0 = \{0.158, 0.166, 0.188, 0.197\}$ with increasing order of consensus of d_2, d_3, d_1, and d_4. Firstly, d_2 is identified with the minimum consensus level. Search the neighboring DMs of d_2 from $\overline{G}_I^2$ as the set $ND_2 = \{d_3, d_4\}$, since d_4 has the biggest consensus level in the set of ND_2, the pair of DMs (d_2, d_4) is determined. Similarly, (d_3, d_1) is determined. Since $cl_2 = \min\{cl_2, cl_4\}$, $cl_3 = \min\{cl_3, cl_1\}$, adjust the preferences of d_2 and d_3 based on Eq. (20) and obtain the modified FPRs of d_2 and d_3 as:

$$F_2^{(1)} = \begin{pmatrix} 0.5 & 0.475 & 0.525 & 0.575 \\ 0.515 & 0.5 & 0.6 & 0.75 \\ 0.475 & 0.4 & 0.5 & 0.65 \\ 0.425 & 0.25 & 0.35 & 0.5 \end{pmatrix} \text{ and } F_3^{(1)} = \begin{pmatrix} 0.5 & 0.25 & 0.55 & 0.55 \\ 0.75 & 0.5 & 0.5 & 0.5 \\ 0.45 & 0.5 & 0.5 & 0.275 \\ 0.45 & 0.5 & 0.725 & 0.5 \end{pmatrix}$$

With the modified FPRs: F_1, $F_2^{(1)}$, $F_3^{(1)}$ and F_4, compute the overall consensus degree using Eq.(17): $CR_0^{(1)} = 0.820$. Therefore, $CR_0^{(1)} \geq \overline{\varepsilon}$, so the CRP is finished.

6 The consensus reaching process in GDM based on consensus evolution networks

Based on the construction of CENs, the new consensus index and the feedback adjustment method, the main framework of CRP in GDM based on CENs is given in this section. The flowchart of the framework is shown in Fig. 4, and the main steps of are described as below.

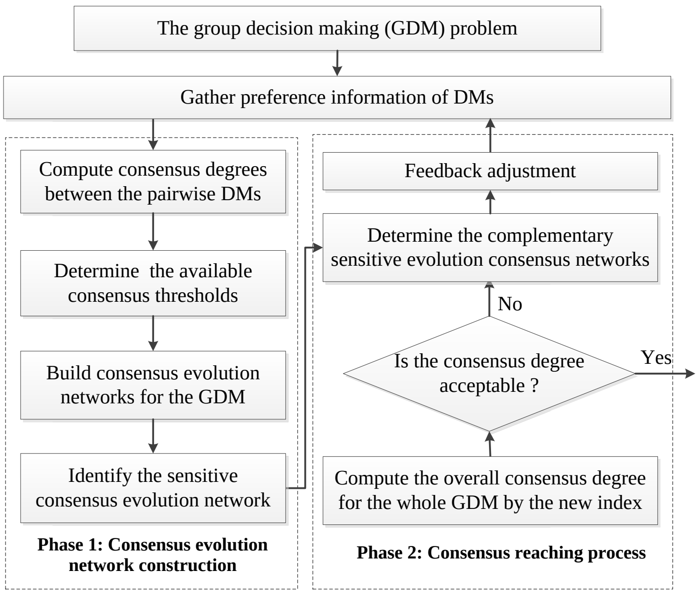


Fig.4 The framework of the proposed CRP model for GDM

(I) Phase 1: Consensus evolution network construction

Step 1: Gather preferences of DMs $D = \{d_1, d_2, ..., d_m\}$ with respect to the pairwise alternatives $X = \{x_1, x_2, ..., x_n\}$, and construct FPR matrices $F_k = \left(f_{ij}^k\right)_{n\times n}$ using Eq. (1).

Step 2: Construct the consensus relation matrix among DMs. Firstly, calculate the similarity $sm_{ij}^{kl}\left(i, j = 1, 2, ..., n, i \neq j, k, l = 1, 2, ..., m;\ k \neq l\right)$ between each pair of DMs and construct the similarity matrix $SM_{kl} = \left(sm_{ij}^{kl}\right)_{n\times n}$ using Eq. (2). And then, construct the consensus matrix $CM = \left(cm_{kl}\right)_{m\times m}$ using Eq. (8) and (9).

Step 3: Build the CCEN G_C, ECEN G_E and ICENs G_I with different consensus thresholds ε. Identify the sensitive consensus threshold ε_r and the corresponding SCEN G_r with the maximum difference of clustering coefficient based on Eq. (10).

(II) Phase 2: Consensus reaching process based on consensus evolution networks

Step 1: Compute the overall consensus degree CR_0 of G_C using Eq. (17). If $CR_0 < \overline{\varepsilon}$, use the proposed feedback adjustment to improve the consensus. Build the CSCEN $\overline{G}_r$ based on the SCEN G_r.

Step 2: Identify the DM d_k with the minimum consensus level in $cl_k\left(k = 1, ..., m\right)$. Search the neighboring DMs $ND_k = \left\{d_l \middle| e_{kl} \in \overline{E}_r, l = 1, ..., m, k \neq l\right\}$ of d_k, and determine all pairs of DMs $\left(d_k, d_l\right)$ from $\overline{G}_r$ until all DMs are identified and distributed.

Step 3: Adjust preferences of d_k in $\left(d_k, d_l\right)$ to obtain the modified FPR $F_{k'} = \left(f_{ij}^{k'}\right)_{n\times n}$ based on Eq. (20). Build the modified CEN G_C' and compute the overall consensus degree CR_0' using Eq. (17). If $CR_0' < \overline{\varepsilon}$, repeat the feedback adjustment until $CR_0' \geq \overline{\varepsilon}$, otherwise, go to Phase 3.

7 A numerical example and analysis

To demonstrate our proposal, consider the example which is used by Dong et al. [20]. In the example, eight DMs $D = \left\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\right\}$ provide their preferences over a set of six alternatives $X = \left\{x_1, x_2, x_3, x_4, x_5, x_6\right\}$. The FPRs $F_k\left(k = 1, ..., 8\right)$ are shown below.

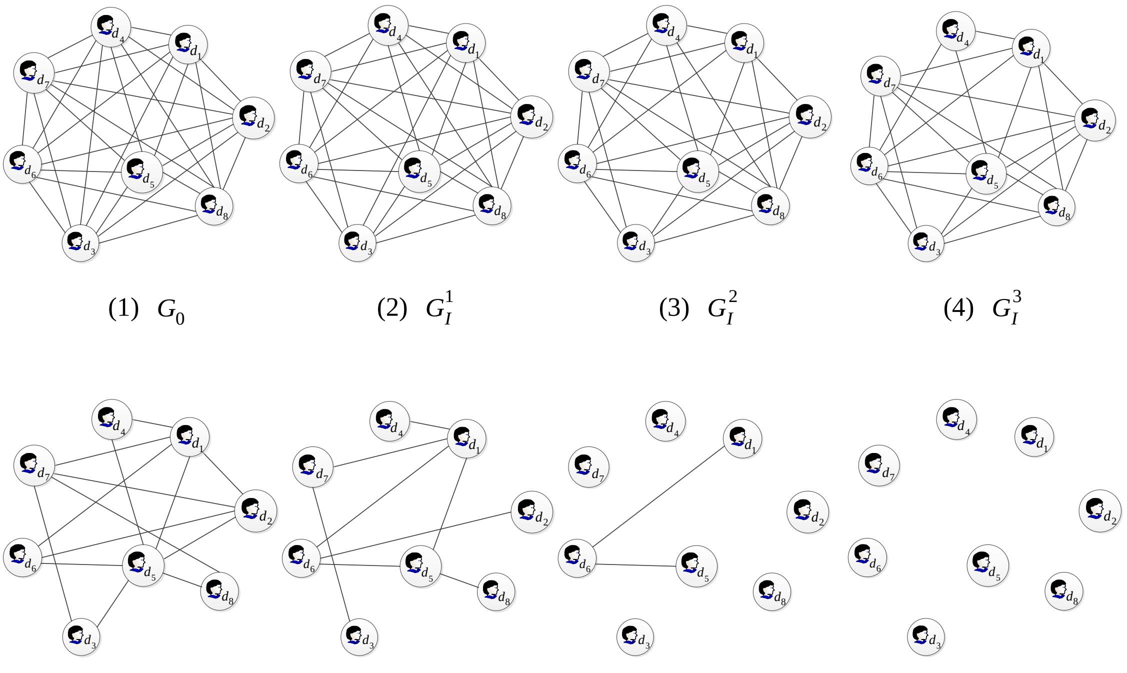
$$F_1 = \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.9 & 0.7 & 0.8 \\ 0.6 & 0.5 & 0.7 & 1 & 0.8 & 0.9 \\ 0.4 & 0.3 & 0.5 & 0.8 & 0.6 & 0.7 \\ 0.1 & 0 & 0.2 & 0.5 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.4 & 0.7 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0.3 & 0.6 & 0.4 & 0.5 \end{bmatrix} F_2 = \begin{bmatrix} 0.5 & 0.7 & 0.8 & 0.6 & 1 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.8 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.3 & 0.7 & 0.6 \\ 0.4 & 0.6 & 0.7 & 0.5 & 0.9 & 0.8 \\ 0 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 & 0.2 & 0.6 & 0.5 \end{bmatrix} F_3 = \begin{bmatrix} 0.5 & 0.69 & 0.12 & 0.2 & 0.36 & 0.9 \\ 0.31 & 0.5 & 0.06 & 0.1 & 0.2 & 0.8 \\ 0.88 & 0.94 & 0.5 & 0.64 & 0.8 & 0.98 \\ 0.8 & 0.9 & 0.36 & 0.5 & 0.69 & 0.97 \\ 0.64 & 0.8 & 0.2 & 0.31 & 0.5 & 0.94 \\ 0.1 & 0.2 & 0.02 & 0.03 & 0.06 & 0.5 \end{bmatrix} F_4 = \begin{bmatrix} 0.5 & 0.1 & 0.36 & 0.69 & 0.16 & 0.26 \\ 0.9 & 0.5 & 0.84 & 0.95 & 0.62 & 0.76 \\ 0.64 & 0.16 & 0.5 & 0.8 & 0.25 & 0.39 \\ 0.31 & 0.05 & 0.2 & 0.5 & 0.08 & 0.14 \\ 0.84 & 0.38 & 0.75 & 0.92 & 0.5 & 0.66 \\ 0.74 & 0.24 & 0.61 & 0.86 & 0.34 & 0.5 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} 0.5 & 0.55 & 0.45 & 0.25 & 0.7 & 0.3 \\ 0.45 & 0.5 & 0.7 & 0.85 & 0.4 & 0.8 \\ 0.55 & 0.3 & 0.5 & 0.65 & 0.7 & 0.6 \\ 0.75 & 0.15 & 0.35 & 0.5 & 0.95 & 0.6 \\ 0.3 & 0.6 & 0.3 & 0.05 & 0.5 & 0.85 \\ 0.7 & 0.2 & 0.4 & 0.4 & 0.15 & 0.5 \end{bmatrix} F_6 = \begin{bmatrix} 0.5 & 0.7 & 0.75 & 0.95 & 0.6 & 0.85 \\ 0.3 & 0.5 & 0.55 & 0.8 & 0.4 & 0.65 \\ 0.25 & 0.45 & 0.5 & 0.7 & 0.6 & 0.45 \\ 0.05 & 0.2 & 0.3 & 0.5 & 0.85 & 0.4 \\ 0.4 & 0.6 & 0.4 & 0.15 & 0.5 & 0.75 \\ 0.15 & 0.35 & 0.55 & 0.6 & 0.25 & 0.5 \end{bmatrix} F_7 = \begin{bmatrix} 0.5 & 0.34 & 0.25 & 0.82 & 0.75 & 0.87 \\ 0.66 & 0.5 & 0.25 & 0.18 & 0.82 & 0.91 \\ 0.75 & 0.75 & 0.5 & 0.94 & 0.91 & 1 \\ 0.18 & 0.82 & 0.06 & 0.5 & 0.34 & 0.75 \\ 0.25 & 0.18 & 0.09 & 0.66 & 0.5 & 0.82 \\ 0.13 & 0.09 & 0 & 0.25 & 0.18 & 0.5 \end{bmatrix} F_8 = \begin{bmatrix} 0.5 & 0.13 & 0.18 & 0.34 & 0.75 & 0.09 \\ 0.87 & 0.5 & 0.66 & 0.82 & 0.91 & 0.25 \\ 0.82 & 0.34 & 0.25 & 0.75 & 0.87 & 0.82 \\ 0.66 & 0.18 & 0.25 & 0.5 & 0.75 & 0.91 \\ 0.25 & 0.09 & 0.13 & 0.25 & 0.5 & 0.97 \\ 0.91 & 0.75 & 0.18 & 0.09 & 0.03 & 0.5 \end{bmatrix}$$

Firstly, construct the consensus matrix $CM = (cm_{kl})_{8\times8}$ based on Eq. (2) and (9) as:

$$CM = \begin{bmatrix} 0 & 0.733 & 0.594 & 0.763 & 0.763 & 0.820 & 0.781 & 0.682 \\ 0.733 & 0 & 0.666 & 0.564 & 0.743 & 0.786 & 0.700 & 0.651 \\ 0.594 & 0.666 & 0 & 0.515 & 0.716 & 0.671 & 0.757 & 0.664 \\ 0.763 & 0.564 & 0.515 & 0 & 0.707 & 0.688 & 0.611 & 0.647 \\ 0.763 & 0.743 & 0.716 & 0.707 & 0 & 0.810 & 0.670 & 0.780 \\ 0.820 & 0.786 & 0.671 & 0.688 & 0.810 & 0 & 0.680 & 0.651 \\ 0.781 & 0.700 & 0.757 & 0.611 & 0.670 & 0.680 & 0 & 0.702 \\ 0.682 & 0.651 & 0.664 & 0.647 & 0.780 & 0.651 & 0.702 & 0 \end{bmatrix}_{8\times8}$$

From the consensus matrix $CM = (cm_{kl})_{8\times8}$, $\varepsilon_C = \min\{cm_{kl}\} = 0.515$, $\varepsilon_E > \max\{cm_{kl}\} = 0.820$, other ε should be $\{0.564, 0.594, ..., 0.810\}$. However, other values of ε is distributed too densely to highlight the structural difference of consensus evolution networks. So let $\{\varepsilon_1, ..., \varepsilon_6\}$ be $\{0.55, 0.60, 0.65, 0.70, 0.75, 0.80\}$. Then the CCEN G_C, ECEN G_E, and the ICEN G_I^1, G_I^2, G_I^3, G_I^4, G_I^5, and G_I^6 are built in Fig. 5.



(1) G_0 (2) G_I^1 (3) G_I^2 (4) G_I^3

(5) G_I^4 (6) G_I^5 (7) G_I^6 (8) G_7

Fig. 5 The structure of the complete consensus evolution networks

Based on Eq. (7), (13), (16), and (17), the overall consensus degrees and the clustering coefficients of all built CENs are computed. The relative CR_C, CR_S, CR and CC of these consensus evolution networks are given in Table 3.

Table 3 The relative ε, CR_C, CR_S, CR and CC of all CENs

CENs	G_C	G_I^1	G_I^2	G_I^3	G_I^4	G_I^5	G_I^6	G_E
ε	0.515	0.55	0.6	0.65	0.7	0.75	0.8	0
CR_C	0.698	0.694	0.714	0.721	0.754	0.782	0.815	0
CR_S	1	0.984	0.908	0.857	0.581	0.375	0.214	0
CR	0.698	0.683	0.649	0.618	0.439	0.293	0.174	0
CC	1	0.964	0.911	0.890	0.450	0.208	0	0

According to Table 1, the variation trends of CR_C, CR_S, CR, and CC with ε of all CENs are shown in Fig. 6.

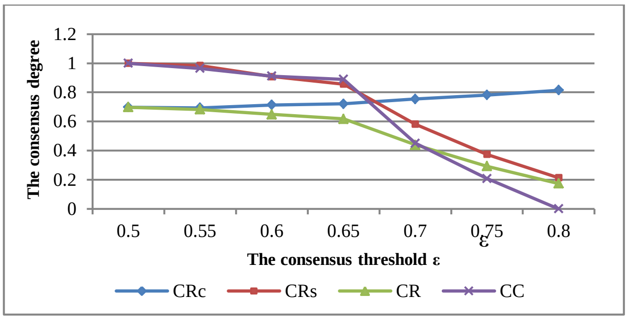


Fig. 6 The variation trends of CR_C, CR_S, CR, and CC with ε

According to Fig. 6, we can give the following findings:

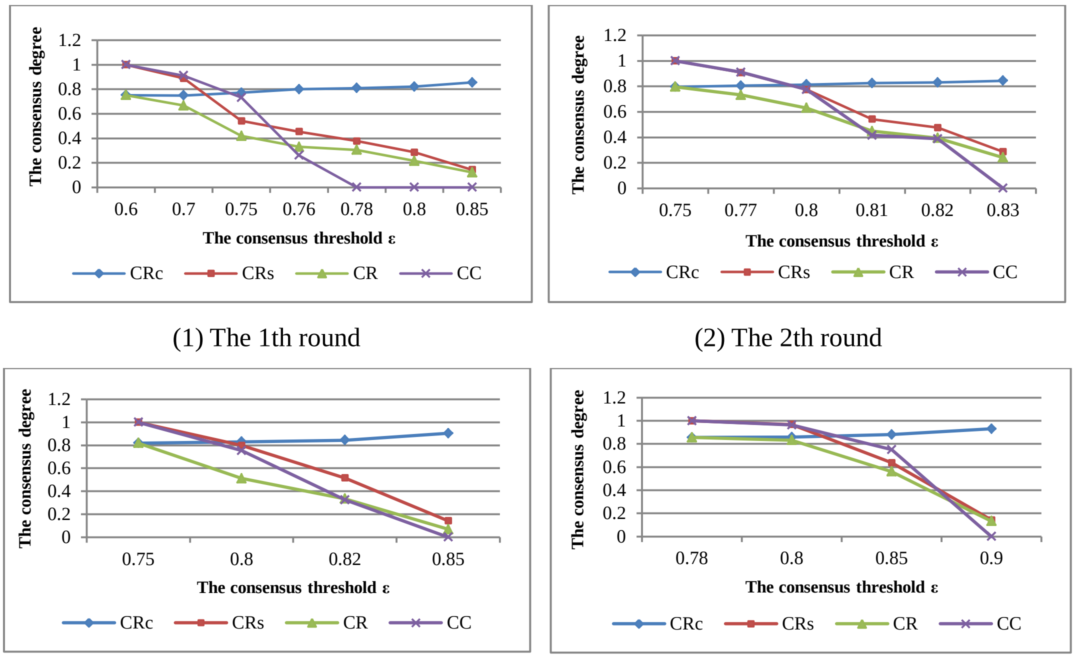
(a) CR_S and CC decreases with the increasing ε, which also illustrates that the structures of consensus evolution networks become sparse with the increasing ε.

(b) CR_C increases due to most of the lower consensus connections being discarded with the increasing ε. CR_S falls much faster than CR_C rises, which causes the decline of CR. This finding suggests the importance of building consensus links between DMs.

(c) There are obvious changes for CR_C, CR_S, CR and CC from $\varepsilon_3 = 0.65$, which implies that consensus relation values between most pairs of DMs do not reach 0.7 and the consensus evolution networks among DMs become weak from $\varepsilon_4 = 0.7$. Based on Eq. (10), $\max\left(CC_3 - CC_4\right) = 0.440$, so the sensitive consensus evolution network G_I^4 can be identified easily.

(d) The trend line of CR and CC intersect at the point of the sensitive consensus threshold $\varepsilon_4 = 0.7$, which suggests that there is also a significant drop for CR in $\varepsilon_4 = 0.7$. Thus, it is also easy to identify the sensitive point from the trend lines of CR and CC. The sensitive consensus threshold means that the consensus of most DMs does not reach 0.7, so the agreed consensus threshold should be set too high than 0.7. Otherwise, the adjustment cost will be high.

Let the agreed consensus threshold $\overline{\varepsilon} = 0.85$, so $CR_0 = 0.698 < \overline{\varepsilon}$, the proposed feedback adjustment is used to improve the consensus. The agreed consensus threshold is achieved after four rounds of adjustment, that is $CR_0^{(4)} = 0.856 > \overline{\varepsilon}$. The variation trends of CR_C, CR_S, CR, and CC in each round are shown in Fig. 7.



(1) The 1th round (2) The 2th round

(3) The 3th round (4) The 4th round

Fig. 7 The variation trends of CR_C, CR_S, CR, and CC with ε shown in four rounds

From Fig. 7, we can see that both the minimum and maximum consensus relation value increases when reaching a high level of consensus. To be specific, the boundary of the CCEN and ICEN is improved from $\min\left\{cm_{kl}\right\} = 0.515$ to $\min\left\{cm_{kl}\right\} = 0.789$, and the boundary of the ICEN and ECEN is improved from $\max\left\{cm_{kl}\right\} = 0.820$ to $\max\left\{cm_{kl}\right\} = 0.936$. In addition, the points at which the trend line of CR and CC intersects in Fig. 7 correspond to the sensitive consensus thresholds, respectively. The sensitive consensus threshold also increases along with the minimum and maximum consensus relation value.

After four rounds of adjustment, the final consensus matrix is determined as:

$$\hat{CM} = \begin{bmatrix} 0 & 0.847 & 0.881 & 0.897 & 0.863 & 0.826 & 0.924 & 0.864 \\ 0.847 & 0 & 0.842 & 0.828 & 0.845 & 0.806 & 0.847 & 0.884 \\ 0.881 & 0.842 & 0 & 0.863 & 0.854 & 0.936 & 0.844 & 0.835 \\ 0.897 & 0.828 & 0.863 & 0 & 0.862 & 0.833 & 0.856 & 0.889 \\ 0.863 & 0.845 & 0.854 & 0.862 & 0 & 0.828 & 0.852 & 0.899 \\ 0.826 & 0.806 & 0.936 & 0.833 & 0.828 & 0 & 0.789 & 0.814 \\ 0.924 & 0.847 & 0.844 & 0.856 & 0.852 & 0.780 & 0 & 0.866 \\ 0.864 & 0.884 & 0.835 & 0.889 & 0.899 & 0.814 & 0.866 & 0 \end{bmatrix}_{8\times 8}$$

According to the final determined consensus matrix, we can see that the consensus levels between pairwise DMs basically reach a relatively balanced state. This phenomenon shows that the feedback adjustment is mainly used to improve the consensus of DMs who contribute less to the CRP. The adjustment strategy makes sure that most DMs have similar decision making weights, thus ensuring the fairness of decision making.

8 Comparison and analysis

To show the advantages of this study, we first give the numerical comparison analysis between this study and other method from microcosmic point of view. Next, we also give the comparison analysis between this study and other GDM models from macroscopic perspective.

8.1 The numerical comparison between the CENs based consensus with other method

In Ref. [20], the DMs' weights are dynamically derived from the multi-attribute mutual evaluation matrices (MMEMs). The original overall consensus degree is computed as 0.6973, the agreed consensus threshold 0.85 is satisfied after two rounds adjustment, and the final overall consensus degree is 0.8837. In this study, the DMs' weights are codetermined with the combination of consensus degree and CENs structures. The original overall consensus degree is computed as 0.696, the agreed consensus threshold 0.85 is satisfied after four rounds adjustment, and the final overall consensus degree is 0.856. Since the comparison about consensus measure has given in Example 3, we will compare this study with Ref. [20] from the aspect of weights determination and consensus adjustment.

(1) Weights determination

In this study, the DM's weights are codetermined by numerical and structured weights. Due to the difference of adjustment strategy, the DM's weights between Ref. [20] and this study in the middle adjustment process are hard to be compared. Thus, the comparison of DMs' weights is mainly given based on the original and final round. Take the CCEN as an example, the comparison between the numerical weights and weights of Ref. [20] is shown as Fig.8. Similarly, take the ICEN with $\varepsilon = 0.75$ as an example, the comparison between the structured weights and weights of Ref. [20] are shown as Fig.9.

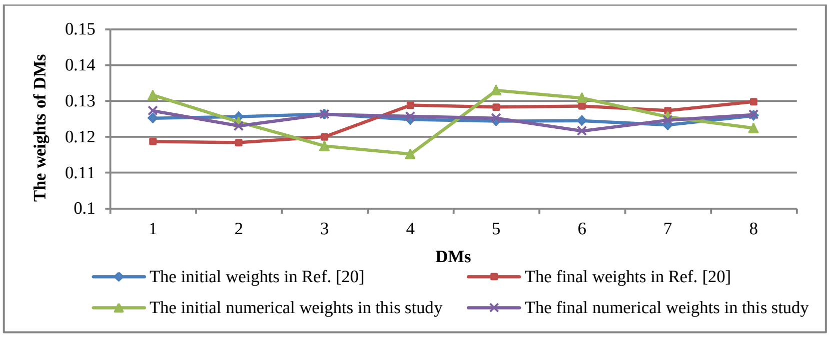


Fig.8. The comparison between DM's weights in Ref. [20] and the numerical weights in this study

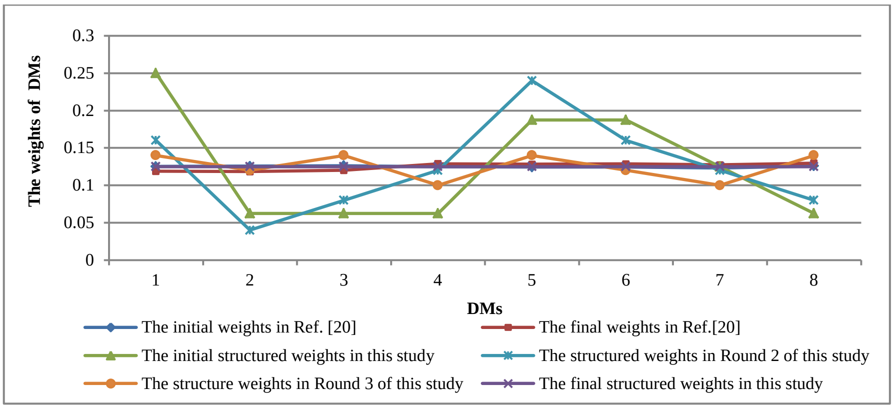


Fig.9. The comparison between DM's weights in Ref. [20] and the structured weights in this study

Regarding Fig.8 and Fig.9, for Ref. [20], the DMs' weights in the original round basically show an average trend, and show difference in the final round. However, for this study, whatever the numerical weights or the structured weights both show the opposite trend with DM's weights in Ref. [20]. Because of the DMs' weights in Ref. [20] are changing with the their performance like the cooperative behavior, the weights change from no-difference to difference in the CRP. However, the numerical and structured weights in this study are computed based on the consensus contribution and the closeness of consensus relations, they both change from difference to no-difference with the consensus improvement.

Although the main objective of Ref. [20] and this study is different, there is little difference in the final weights and the consensus measure. With the weights determination of this study, the difference among DMs is easy to be found, and such difference becomes more and more evident with the increasing consensus threshold. It is benefit for analyzing the contribution of DMs on consensus reaching.

(2) Consensus reaching

In each round of Ref. [20], each DM needs to modify their preference according the collective one. However, the DMs who contribute more to the consensus may be less willing to make an equivalent concession with DMs who contribute less to the consensus. In this study, according to the commonly used identification and direction rule of consensus, we propose a pairwise adjustment strategy based on the CSCEN. In the 1th round, the pairs of DMs are (d_4, d_6), (d_3, d_1), (d_8, d_2), (d_7, d_5), in the 2th round, the pairs of DMs are (d_4, d_5), (d_8, d_1), (d_2, d_7), (d_3, d_6), in the 3th round, the pairs of DMs are (d_7, d_6), (d_2, d_1), (d_4, d_3), in the 4th round, the pairs of DMs are (d_5, d_7), (d_1, d_3), (d_4, d_2), (d_8, d_6). From the four rounds, it is evident that d_4 and d_8 contribute less to the consensus, while d_1, d_5 and d_6 contribute more to the consensus, and other DMs contribute more or less to the consensus. In the first three rounds, the preference of DMs who contribute less is mainly modified refer to the preference of DMs who contribute more to the consensus. Inversely, in the final round, the DMs who contribute more to consensus also need to modify their preference to improve the whole consensus.

In this study, we try to find a balance between the consensus improvement and the adjustment cost controlling with the pairwise adjustment strategy. The DMs are easy to be distinguished in the CSCEN based on their contribution to the consensus. In the previous rounds, the DMs who contribute less to the consensus should compromise first, and in the later rounds, the DMs who contribute more to the consensus also need to make some compromises to reflect the fairness in some degree.

8.2 CENs based consensus versus traditional and social network based consensus models

It is known that there are abundant CRPs in GDM that have been proposed to improve group consensus. According to the literature review, the existing studies of GDMs involving consensus research can be divided into two broad categories: the traditional GDM and the SNGDM. As this study is structurally and ideologically distinct from the existing studies, the comparative analysis between this study and others is given as below.

The main difference between this study and another two kinds of studies are shown in detail in Table 4.

Table 4 The difference between this study and others

References	Differences			
	Main objective	Consensus measure	Feedback adjustment	The agreed consensus threshold
Traditional GDMs ([10, 11, 16, 32, 33, 35, 36])	Improve consensus with considering behavior factors or adjustment cost	Based on consensus degrees at three levels	Based on optimization models or modification of traditional models	Given directly based on the decision experience of DMs or the decision requirements of the GDM
SNGDMs ([2, 34, 35, 43-47])	Study the CRP based on social relations among DMs	Based on the social relations at three levels	Based on the transmission of social relations	

The CENs based CRP in this study	Explore the evolution of consensus with networks	Based on the structures of consensus evolution networks	Based on the complementary sensitive consensus evolution networks	Based on the sparsity of the CENs, especially the sensitive consensus threshold

(1) The main objective. The main idea of the traditional GDM is to improve consensus by considering noncooperative behaviors in CRP and using an optimization model to make adjustment costs as low as possible. The SNGDM mainly depends on the social relations among DMs, and its main idea is to study the CRP based on the propagation of social relations. This study is meant to build the consensus relations for DMs based on the similar preference, and construct CENs managing the consensus threshold to explore the formation and evolution of consensus.

(2) Consensus measure. In traditional GDM, the overall consensus degree is mainly computed with individual DMs' consensus degree at the three levels. In SNGDM, the overall consensus degree is mainly computed based on social relations using the three levels method in traditional GDM. In this study, the overall consensus degree is calculated based on the structured and numerical index based on CENs. In traditional GDM and SNGDM, the weights of DMs are mainly given subjectively or computed based on social relations. In this study, the weights of DMs are computed structurally and numerically.

(3) Feedback adjustment. In traditional GDM, the adjustment strategy is mainly proposed based on the optimization models or the modification of traditional models. In SNGDM, the adjustment suggestion is mainly provided based on the transmission of social relations, such as opinion dynamics, trust propagation, and influence diffusion. In this study, a pairwise adjustment strategy is proposed based on the CSCEN, which makes the final consensus among the DMs more balanced.

(4) The agreed consensus threshold. In traditional GDM and SNGDM, most of the agreed consensus thresholds are given directly based on the decision experience of DMs or the decision requirements of the GDM. In this study, we provide a reference for the determination of the agreed consensus threshold based on the sparsity of the CENs. The sensitive consensus threshold obtained from the sparsity of the CENs can act as a numerical reference for the determination of the agreed consensus threshold, which will promote a balance between adjustment costs and the agreed consensus threshold.

This study also has something in common with the other two kinds of studies. Such as, the CENs are built based on the preference similarity like in the traditional GDM, the structured weights are determined based on degree centrality like the SNA in SNGDM.

9 Conclusions

To explore the composition and evolution of consensus in GDM, a new tool for CRP is proposed based on the CENs. With the help of the CENs, the consensus measure and feedback adjustment in CRP are processed with an important advantage, managing the consensus thresholds and its evolution.

In this study, we build different kinds of CENs with different consensus thresholds based on the consensus matrix, including the CCEN, ICEN, ECEN, SCEN, and CSCEN. The CCEN, ICEN, and ECEN are general forms of CENs that can be shown directly with different consensus thresholds. The SCEN is distinguished by the sensitive consensus threshold which can act as a reference for the determination of the agreed consensus threshold. According to CENs, a new index for measuring the overall consensus degree is proposed structurally and numerically. Compared with the traditional methods, the new consensus index consisting of the structured and numerical index can show the consensus evolution more clearly. The CSCEN is constructed as the complementary form of the SCEN. A pairwise feedback adjustment method is proposed based on the CSCEN. The usefulness of this new CRP tool is shown by a numerical example. The numerical results suggest that the feedback adjustment improves the consensus regularly through limited rounds. The advantage of this study is highlighted by the numerical and the theoretical comparison, respectively.

In short, the CRP tool based on CENs allows us to study the evolution of consensus from a more visible perspective. In terms of the common social relations among DMs, we will try to consider the comprehensive impact of the social relations and preference on the evolution of consensus in future work. Furthermore, the large-scale group decision making (LSGDM) problems are becoming more popular [56-60]. Apparently, the CRP in the LSGDM is more complex than the GDM. In future research, we will try to explore the evolution of consensus in LSGDM based on CENs.

Acknowledgements

This work was supported by the National Science Foundation of China (NSFC) (71371049, 71701158); the Scientific Research Foundation of Graduate School of Southeast University (YBTJ1831); the MOE (Ministry of Education in China) Project of Humanities and Social Sciences (17YJC630114); the Fundamental Research Funds for the Central Universities (2018IVB036 and 2019VI030).

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2 The minimum cost consensus model considering the implicit trust of opinions similarities in social network group decision-making

- T. Wu, X.W. Liu, Z.W. Gong, H.H. Zhang, F. Herrera, The minimum cost consensus model considering the implicit trust of opinions similarities in social network group decision-making. **International Journal of Intelligent Systems.** 35(2020) 470-493.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 8.709
  - Subject Category: Computer Science, Artificial Intelligence, Ranking 11/175 (Q1).

The minimum cost consensus model considering the implicit trust of opinions similarities in social network group decision making

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Abstract: The social network group decision making is popular due to the advantages of social relationships in consensus reaching process, especially the trust relationships. To explore the effects of trust on consensus, some minimum cost consensus models are proposed based on the implicit trust between individuals and the moderator. The implicit trust is computed based on the similarity of opinion and it is implied into the traditional minimum cost consensus model to obtain a new quadratic programming problem and the related dual problem. The weights of individuals can be determined based on the implicit trust and can be used to modify the possible deviations among individuals' adjustment cost. A numerical example and the comparative analysis are given to analyze the effectiveness of the proposed models, which suggests that individuals are willing to give up some benefit to reach consensus due to their implicit trust to the moderator and make minor revisions to their adjustment cost due to their implicit trust to each others.

Keywords: Social network group decision making; trust relationships; consensus opinion; the minimum cost consensus model

1 Introduction

Consensus reaching process (CRP) is an important part in GDM. Many studies were focus on improving consensus and proposed many consensus models in recent years. Due to the advantages of social network analysis, the social network group decision making (SNGDM)[1-4] is becoming one of the hottest research points in nowadays. In SNGDM, the CRP is improved based on many kinds of social relationships, especially the trust relationships[5].

The CRP in GDM is mainly composed by two parts: consensus judgement and feedback

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adjustment[6, 7]. The consensus degrees are usually computed to judge the consensus of the whole group[8-12]. If the overall consensus degree is unsatisfactory, implement the feedback adjustment[13-15]. The feedback adjustment based on optimization models[10, 16-18] and social relationships[9, 10, 19-21] are popular in CRP. In CRP, a moderator who presents the collective interest is important to help the group reach to a consensus[22, 23]. The moderator is predetermined and possesses an effective leadership and strong negotiation skills to convince most of the individuals reach a final consensus by spending cost[16, 17, 24].

In CRP, the improvement of consensus may lead to the adjustment cost increase while the cost controlling may cause low consensus. The optimal consensus models are usually used to solve such contradiction from the aspect of minimum adjustment cost[16-18, 25] and maximum utility under limited budget[26, 27], respectively. On one hand, Gong et al.[16, 17] introduced consensus models to obtain the minimum cost for the moderator and the maximum return for individuals. Zhang et al.[18] proposed a minimum-cost consensus model under aggregation operators. Furthermore, Zhang et al.[25] considered the degree of consensus in the minimum cost consensus model by defining a consensus level function and a generalized aggregation operator. On the other hand, Gong et al.[26, 27] maximized the GDM utility under limited cost and nonlinear utility constraints.

Social network is another useful tool to solve the contradiction between consensus reaching and adjustment cost[9, 10, 21, 28]. Trust is a persuasive relationship to promote consensus reaching and it is widely used in SNGDM[2, 9, 10, 21, 28]. The studies of CRPs based on trust relationships are summarized in Ref.[2, 3]. Wu et al.[9] proposed a trust based consensus model under an incomplete linguistic information context. Moreover, Wu et al.[21] studied the CRP using a trust based recommendation mechanism. Zhang et al.[28] introduced a consensus framework based on social trust networks to deal with non-cooperative behaviors. Wu et al.[10] proposed a consensus model based on a minimum adjustment cost feedback mechanism with distributed linguistic trust.

Depending on the source of trust, it can be divided into explicit and implicit trust[29]. The explicit trust can be acquired through social interaction and influence, and the implicit trust can be inferred based on users' similarity characteristics[30]. In fact, it is difficult to gather the explicit trust since it is the expression of people's subjective will. Besides, the subjective given trust of has the disadvantage of being less objective. It has been proven that there is a positive correlation between trust and user similarity in online communities[31]. Ureña et al.[32] also measured the confidence level between agents in GDM based on the similarity of their opinions. Thus, the implicit trust relationships between the individuals in SNGDM can be determined based on the similarity of their opinions.

Moreover, many studies assigned the weights to individuals based on trust degrees since trust can reflect the importance of individuals[21, 33-35]. Wu et al.[21] computed the weights for individuals based on trust scores in the trust based recommendation mechanism. Wu et al.[35] applied the indirect trust relationships via trusted third partners as a reliable resource to determine experts' weights. Besides, the consensus degree which is obtained based on the similarity of opinion is also regarded as the critical index to determine the importance for individuals[12, 36]. Liu et al.[33] assigned weights to individuals taking both trust and consensus on assessment into consideration. Since the implicit trust in this study is defined based on the similarity of opinions which is consistent with the definition of the consensus degree, we will assign the importance to individuals based on the implicit trust levels.

According to the previous reviews, trust plays an important role in CRP and the optimal model is a useful tool to balance the budget and consensus level in CRP. However, there are still some limitations in current SNGDM researches:

(1) The moderator is always regarded as an independent subject and the implicit trust of individuals on the moderator is rarely considered in SNGDM.

(2) The promotion of trust on consensus reaching is rarely analyzed from the aspect of cost offsetting using optimal models.

(3) The adjustment cost of individuals are given subjectively, but the reasonability of these adjustment cost is rarely discussed and adjusted.

To solve the above limitations, some minimum cost consensus models are proposed based on the implicit trust. As the basis to these models, three assumptions are given as follows.

Assumption 1: The moderator has an expected consensus opinion for seeking the lowest negotiation cost. Individuals may trust the moderator due to their similar opinion and they also have willingness to build a consensus that they can obtain the expected return.

Assumption 2: Based on the implicit trust of individuals, the moderator can save cost when persuading individuals to reach the consensus that he/she expected. Referring to the consensus willingness of individuals, they can give up some benefits to reach a consensus voluntarily.

Assumption 3: The adjustment cost given by individuals subjectively may be unreasonable which will cause injustice in CRP.

Based on the above assumptions, we try to solve the limitations of current SNGDM studies using some minimum cost consensus models considering implicit trust.

(1) Define the implicit trust of individuals to the moderator based on the similarity of the individuals' opinion and the expected consensus opinion of the moderator. Define the consensus willingness for individuals to reach a consensus that they actively want to achieve.

(2) Construct the primal and dual minimum cost consensus models based on the implicit trust and the consensus willingness of individuals. Solve the proposed models and analyze the effects of implicit trust in CRP, and explain the economic significance of the duality models.

(3) Compute the weights for individuals based on how much they are trusted by others, and modify their adjustment cost based on the weights if they have a large deviation from others' cost. The primal and dual models considering the modified adjustment cost are built and solved to analyze the significance of justice.

The proposed models are examined by a numerical example. In the example, the optimal consensus opinion and the minimum costs are determined considering the implicit trust and the modified adjustment cost, respectively. Through the comparative analysis, we find that individuals are willing to reach consensus with low return due to their trust to the moderator. They are also willing to modify their adjustment cost in some degree for free based on their implicit trust to others.

The rest of paper is organized as follows: the basic knowledge of this study is introduced in Section 2. The minimum cost consensus model and its dual model are proposed in Section 3. The unreasonable adjustment costs are modified in Section 4. The application of the proposed models and the comparative analysis are given in Section 5. The conclusion is given in Section 6.

2 Preliminaries

The basic knowledge of trust, the quadratic programming problem, and the traditional minimum cost consensus model are described as below.

2.1 Some basic knowledge of trust

Trust has been used in different disciplines to model different type of relations[37], such as trust between individuals in social networks, trust between consumers and commodities in social commerce, and trust between electors and candidates in campaign. According to the different ways of obtaining trust, trust can be divided into explicit trust and implicit trust[29]. The implicit trust is usually inferred based on the similarity of opinions.

Degree centrality is widely used to measure the importance of vertices in networks[38]. The in-degree index of individuals in the complete directed trust networks is defined as follows.

Definition 1.[39] For a complete directed trust network $G=\left(D,E,T\right)$, $D=\left\{d_1,\ldots,d_n\right\}$ be the set of individuals, $E=\left\{e_{12},\ldots,e_{n,n-1}\right\}$ be the set of directed trust arcs, the number of the directed arcs is $n\left(n-1\right)$, and $T=\left\{t_{12},\ldots,t_{n,n-1}\right\}$ is the attached trust value of the directed trust arcs, then the in-degree trust index t_j of the individual d_j is determined as:

$$t_j = \frac{1}{n-1}\sum_{i=1}^{n} t_{ij} \tag{1}$$

where $i, j = 1,...,n$ and $i \neq j$.

2.2 The quadratic programming problem and its dual problem

The quadratic programming is a special form of convex optimization and it plays important rule in the operational research[40]. According to different forms of the constraint condition, the quadratic programming can be divided into the unconstrained optimization problem and constrained optimization problem. In the constrained optimization problem, the quadratic programming problems are commonly used[41].

Definition 2.[41] A quadratic program is a problem of seeking the minimum of a quadratic function of n variables subject to a finite number of constraints in the form of linear equations and linear inequalities. A typical minimum quadratic program is generally described as:

$$\min \phi(\boldsymbol{X}) = \frac{1}{2}\boldsymbol{X}^T H\boldsymbol{X} + \boldsymbol{g}^T\boldsymbol{X} + c \tag{2}$$

$$s.t. \begin{cases} \boldsymbol{a}_i^T\boldsymbol{X} - b_i = 0, \ i \in E = \{1,2,...,l\} \\ \boldsymbol{a}_i^T\boldsymbol{X} - b_i \geq 0, \ i \in I = \{l+1,...,m\} \end{cases}$$

where ϕ is a quadratic function in R^n, H is a symmetric matrix of order n, $H^T = H$, $H \in R^{n\times n}$, $\boldsymbol{a}_i, \boldsymbol{g}, \boldsymbol{X} \in R^n$, $b_i \in R^m$, c is a scalar. If the quadratic coefficient matrix H is a positive definite, then Eq. (2) is a strict convex quadratic programming problem. For the strict convex quadratic program, the local optimal solution is equal to the global optimal solution.

The quadratic program can be solved by using the Lagrange multiplier method which aims to find out the K-T (Kuhn-Tucker) point from feasible region[42]. Firstly, the Lagrange function of Model (2) can be denoted as:

$$L(\boldsymbol{\lambda}, \boldsymbol{X}) = \frac{1}{2}\boldsymbol{X}^T H\boldsymbol{X} + \boldsymbol{g}^T\boldsymbol{X} + c - \boldsymbol{\lambda}^T(A\boldsymbol{X} - \boldsymbol{b}) \tag{3}$$

where $\boldsymbol{\lambda}$ is the Lagrange multiplier and $\boldsymbol{\lambda} \in R^m$.

For the strict convex quadratic program, the K-T point must be the global minimum point, at this point, solve the Eq. (2) equals to solve the following model:

$$\begin{cases} \boldsymbol{g} + H\boldsymbol{X} = A\boldsymbol{\lambda} & \\ a_i^T \boldsymbol{X} = b_i & i \in E \\ a_i^T \boldsymbol{X} \geq b_i & i \in I \\ \lambda_i \left[a_i^T \boldsymbol{X} - b_i \right] = 0 & i \in I \\ \lambda_i \geq 0 & i \in I \end{cases} \tag{4}$$

where $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$, $A = [a_1, ..., a_m]$.

Based on the first equation in Model (4), the Lagrange multiplier $\boldsymbol{\lambda}$ can be represented by $\boldsymbol{X}$:

$$\boldsymbol{\lambda} = A^{-1}(\boldsymbol{g} + H\boldsymbol{X}) \tag{5}$$

And then, a new function $\psi(\boldsymbol{X})$ can be defined by replacing $\boldsymbol{\lambda}$ by $\boldsymbol{X}$ based on (5):

$$\begin{aligned} \psi(\boldsymbol{X}) &= L(\boldsymbol{\lambda}, \boldsymbol{X}) \\ &= -\frac{1}{2}\boldsymbol{X}^T H^{-1} \boldsymbol{X} + \boldsymbol{b}\boldsymbol{\lambda}^T + c \end{aligned} \tag{6}$$

$$s.t. \begin{cases} \boldsymbol{g} + H\boldsymbol{X} = A\boldsymbol{\lambda} \\ \boldsymbol{\lambda} \geq 0 \end{cases}$$

If $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, then $\boldsymbol{X} = \boldsymbol{X}^*$, and hence

$$\psi(\boldsymbol{X}^*) = L(\boldsymbol{\lambda}^*, \boldsymbol{X}^*) = \phi(\boldsymbol{X}^*)$$

Thus, $\psi(\boldsymbol{X})$ is the dual program of $\phi(\boldsymbol{X})$. To distinguish the primal and dual program, we use a new variable $\boldsymbol{Y}$ to replace $\boldsymbol{X}$ in $\psi(\boldsymbol{X})$:

$$\max \psi(\boldsymbol{Y}) = -\frac{1}{2}\boldsymbol{Y}^T H^{-1} \boldsymbol{Y} + \boldsymbol{b}\boldsymbol{\lambda}^T + c \tag{7}$$

$$s.t. \begin{cases} \boldsymbol{g} + H\boldsymbol{Y} = A\boldsymbol{\lambda} \\ \boldsymbol{\lambda} \geq 0 \end{cases}$$

According to the K-T point, the primal and dual program can be solved. In addition, the primal and dual quadratic program problems satisfy the strong and weak duality theorem:

Theorem 1. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ is a feasible solution of $\phi(\boldsymbol{X})$ and $\psi(\boldsymbol{Y})$, respectively, then $\phi(\boldsymbol{X}) \geq \psi(\boldsymbol{Y})$, which is called as the weak duality theorem.

Theorem 2. If Model $\phi(\boldsymbol{X})$ and $\psi(\boldsymbol{Y})$ both have an optimal solution $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$, respectively, then $\min \phi(\boldsymbol{X}^*) = \max \psi(\boldsymbol{Y}^*)$, which is called as the strong duality theorem..

2.3 The traditional minimum cost consensus model

Actually, the consensus opinion o has been solved by a traditional minimum cost model without considering trust[17]. Let $f_i(o)=|o-o_i|$ be the deviation between the opinion o_i of the individual $d_i\,(i\in N)$ and the consensus opinion o. Then, $c_i f_i(o)$ denotes the cost that paid by the moderator M to persuade the individual $d_i\,(i\in N)$ whose unit cost is c_i. Thus, the total cost of all individuals for reaching consensus can be described as $\sum_{i=1}^{n} c_i f_i(o)$. The smaller the total cost is, the greater the consensus will be.

Thus, a nonlinear optimization model $NLP(c)$ under the assumption that the minimum total cost can be determined with a consensus opinion:

$$NLP(c):\quad \min \phi(o)=\sum_{i=1}^{n} c_i |o-o_i| \tag{8}$$

$$s.t.\quad o'\in O$$

Referring to the primal-dual theory of linear programming, the dual problem of the Model (8) is presented as:

$$DLP(c):\quad \max \psi(y)=\sum_{i=1}^{n} y_i\left(o_i-o^*\right) \tag{9}$$

$$s.t.\quad \begin{cases} \sum_{i=1}^{n} y_i = 0 \\ |y_i|\le c_i,\ i\in N \end{cases}$$

where $|y_i|$ indicates the unit return that individual DM d_i expects to obtain for changing his/her opinion to improve the consensus, so Model (9) reflects the total return that is expected by all individuals for changing their opinions toward the consensus.

There are two theorems given in[17] to present the relationship between the unit return y_i and the unit cost c_i:

Theorem 3. Suppose that the individuals opinions satisfy $o_1\le\ldots\le o_i\le\ldots\le o_n$. If o^* is the optimal solution to the primal problem $NLP(c)$, then there must exist a $t_0\in N$ such that $o_{t_0}\le o_i\le o_{t_0+1}$, and

$$\sum_{i=1}^{t_0} c_i = \sum_{j=t_0+1}^{n} c_j \tag{10}$$

if and only if $DLP(c)$ has optimal solutions, and one of the optimal solution is $\left(-y_1,...,-y_{t_0},y_{t_0+1},...,y_n\right)^T$.

Theorem 4. The statement $y_i=-c_i$ holds when $o^*>o_i$ holds; and $y_i=c_i$ holds when $o^*<o_i$. This denotes $\left|y_i\right|=c_i$ holds when $o^*\neq o_i$ holds; $-c_i\le y_i\le c_i$ holds when $o^*=o_i$ holds.

3 The minimum cost consensus model and its dual model based on the implicit trust

It is difficult to reach a high consensus since some individuals are reluctant to change their opinion or need a lot of payoff to change their opinions. Based on the advantage of trust in consensus reaching, the offsetting effects of trust on the adjustment cost are considered. According to similar opinion between individuals and the moderator, the implicit trust of individuals to the moderator can be constructed. Next, the implicit trust is considered in the minimum cost consensus model and the corresponding dual model is built. Finally, the proposed models are solved and the effects of implicit trust are analyzed based on the optimal solutions. The critical techniques of the proposed model are described as follows.

3.1 Model description

Suppose there is a SNGDM problem consisting of n individuals $\{d_1,d_2,...,d_n\}$ and a moderator M. Let $o_i\in R$ represents the opinion of individual d_i. In the SNGDM, the purpose of the moderator M is to persuade individuals to reach an expected consensus based on the implicit trust.

According to Assumption 1, the moderator M has an expected consensus opinion for seeking the minimum persuasion cost. Let $o_M\in R$ be the expect consensus of M, c_i be the unit cost of d_i for making concession, then $c_i\left|o_M-o_i\right|$ denotes the total cost of d_i for changing his/her opinion. The greater value $c_i\left|o_M-o_i\right|$ is, the more cost the moderator should to pay. Regarding Assumption 1, individuals may trust the moderator M based on the similarity between individuals' opinions and o_M. Let t_{iM} be the implicit trust of d_i to M, the aim of this study is to analyze the effectiveness of the implicit trust t_{iM} on the total cost $c_i\left|o_M-o_i\right|$.

Referring to Assumption 1, the individuals also have willingness to reach a consensus under the acceptable compensation. Let y_i be the unit return expected by the individual d_i, o_d be the consensus opinion actively formed by all individuals, then $y_i\left|o_d-o_i\right|$ denotes the total return of d_i for changing his/her opinion. The greater the value $y_i\left|o_d-o_i\right|$ is, the more the total return they expect. Suppose all individuals will voluntarily form a consensus o_d to obtain the maximum return from the moderator M under their consensus willingness w_{id}.

3.2 The definitions of the implicit trust and the consensus willingness

According to Assumption 1, we suppose that there are two different kinds of consensus, one is the consensus expected by the moderator based on the minimum cost and the other is the consensus that individuals can reach voluntarily. The implicit trust and the consensus willingness of individuals are distinguished and defined based on the similarity between their opinion and the two kinds of consensus opinion in this section.

Firstly, the similarity function is given based on the opinions. Without loss of generality, let $o_i,o_j\in R$. The similarity between the individual d_i and d_j is defined by the similarity function $s_{ij}\left(o\right)$:

$$s_{ij}\left(o\right)=1-\frac{\left|o_i-o_j\right|}{\max\left\{o_i\right\}} \tag{11}$$

where $s_{ij}\left(o\right)\in\left[0,1\right]$.

The more similar between the opinion o_i and o_j, the more implicit trust between the individual d_i and d_j. Thus, the definition of the implicit trust is given based on the similarity function as follows.

Definition 3. Let the implicit trust function $t_{ij}\left(o\right)=s_{ij}\left(o\right)$, then the implicit trust of d_i to d_j equals to the implicit trust of d_j to d_i:

$$t_{ij}\left(o\right)=t_{ji}\left(o\right)=1-\frac{\left|o_i-o_j\right|}{\max\left\{o_i\right\}} \tag{12}$$

since $o_i,o_j\in R$, and $t_{ij}\left(o\right)\in\left[0,1\right]$. Apparently, the implicit trust changes with the change of individuals' opinions, which is consistent with the context characteristic of trust [25]. The more similar the opinions two subjects have, the higher the implicit trust between them is.

According to the expected consensus opinion o_M of the moderator M and opinion of individuals, the implicit trust of the individuals to the moderator M can be determined. The structure of the implicit trust of individuals to the moderator is shown in Fig.1. For example, the implicit trust of the individual d_i to the moderator M can be computed based on (12):

$$t_{iM} = 1 - \frac{|o_i - o_M|}{\max\{o_i\}} \tag{13}$$

where $t_{iM} \in [0,1]$ since $o_i, o_M \in R$. t_{iM} equals to 0 as long as $|o_i - o_M| = \max\{o_i\}$, which means that the individual d_i has no implicit trust on the moderator M at all. When $o_i = o_M$, we can obtain $t_{iM} = 1$, which means that the individual d_i fully trusts on the moderator M.

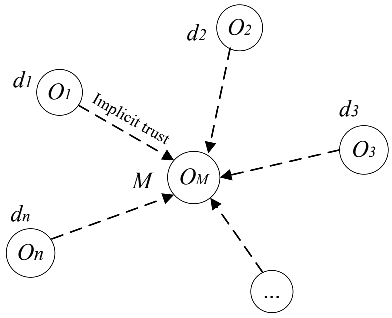

Fig.1. The implicit trust of individuals to the moderator

Similarly, the more similarity between the opinion o_i and the consensus opinion o_d, the more consensus willingness the individual d_i has to reach such consensus. Thus, the definition of the consensus willingness is given as follows.

Definition 4. Let the consensus willingness function $w_{id}(o) = s_{ij}(o)$, then the consensus willingness of the individual d_i equals to the implicit trust of d_j to d_i:

$$w_{id} = 1 - \frac{|o_i - o_d|}{\max\{o_i\}} \tag{14}$$

where $w_{id} \in [0,1]$ since $o_i, o_M \in R$. w_{id} equals to 0 as long as $|o_i - o_d| = \max\{o_i\}$, which means that the individual d_i has no willingness to make any concessions to the consensus opinion o_d. When $o_i = o_d$, we can obtain $w_{id} = 1$, which means that the individual d_i has full willingness to make concessions to consensus opinion o_d.

The structure of the consensus willingness of individuals based on the consensus o_d is shown in Fig.2.

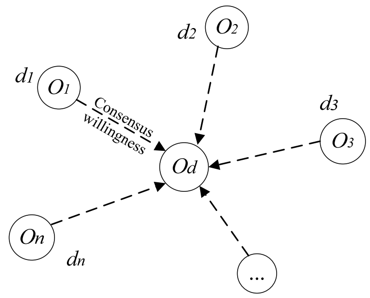

Fig.2. The consensus willingness of individuals

3.3 The minimum cost consensus model based on the implicit trust

In terms of the consensus reaching, the implicit trust is benefit attribute, while the adjustment cost is cost attribute. Thus, the implicit trust needs to be transformed into cost attribute t_{iM} as:

$$t'_{iM} = 1 - t_{iM} \tag{15}$$

where $t'_{iM} \in [0,1]$, when $t_{iM} = 0 \Rightarrow t'_{iM} = 1$, the implicit trust totally becomes into the cost attribute; when $t_{iM} = 1 \Rightarrow t'_{iM} = 0$, the implicit trust has no offsetting rule on the consensus cost at all.

Obviously, the higher the implicit trust t_{iM} between individuals and the moderator is, the lower t'_{iM} is, so as the lower the unit adjustment cost the moderator pays. Based on the traditional minimum cost consensus model shown in Eq. (8), a implicit trust based minimum cost model $NLP(c,t)$ is proposed as below:

$$\begin{aligned} NLP(c,t): \quad \min \phi(o_M) &= \sum_{i=1}^{n} c_i t'_{iM} f_i(o_M) \\ &= \sum_{i=1}^{n} c_i t'_{iM} \left| o_M - o_i \right| \\ s.t.\quad & o_M \geq 0 \end{aligned} \tag{16}$$

Based on Eq. (13) and (15), the above model is reorganized:

$$\begin{aligned} NLP(c,t): \quad \min \phi(o_M) &= \sum_{i=1}^{n} \left(c_i \frac{\left| o_M - o_i \right|}{\max\{o_i\}} \right) \left| o_M - o_i \right| \\ &= \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left({o_M}^2 + o_i^2 - 2 o_i o_M \right) \\ s.t.\quad & o_M \geq 0 \end{aligned} \tag{17}$$

We can find that the Eq. (17) is a quadratic programming model. Since $c_i \in [0,1]$, it is easy to verify that the quadratic coefficient matrix is a positive definite matrix, that is, Eq. (17) is a strict convex quadratic programming problem. For a strict convex quadratic programming problem, the global minimum is unique, and the global optimal solution is equal to the local optimal solution. According to the model analyzer, such as Lingo, it is easy to determine the local optimal solution $o_M{}^*$ for the above model.

When $|o_i - o_M| = \max\{o_i\}$, $t_{iM} = 0 \Rightarrow t'_{iM} = 1$, there is no change in the cost paid by the moderator M to the individual d_i since $t_{iM} = 0$. On the contrary, when $o_i = o_M$, $t_{iM} = 1 \Rightarrow t'_{iM} = 0$, the cost paid by the moderator M to the individual d_i is completely offset. Thus, the individual d_i is willing to adjust his/her opinion for free since he/she fully trusts on the moderator M according to Assumption 2.

3.4 The dual problem of the minimum cost consensus model

To discuss the further economic significance about Eq. (17) based on the dual theory of quadratic programming, the dual problem of Eq. (17) is constructed based on the Lagrange multiplier method[41]. The Lagrange function $L(\lambda, o_M)$ of $\min \phi(o_M)$ is constructed using a vector Lagrange multiplier $\lambda \in R$ as:

$$L(\lambda, o_M) = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_M{}^2 + o_i^2 - 2o_i o_M\right) - \lambda o_M \tag{18}$$

Let the partial derivatives of $L(\lambda, o_M)$ with respect to the component of o_M equal to zero:

$$\frac{\partial L}{\partial o_M} = \frac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_M - o_i\right) - \lambda = 0, \quad \lambda \ge 0 \tag{19}$$

We define a new function $\psi(o_M)$ to replace λ by $\lambda = \frac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_M - o_i\right)$ in $L(\lambda, o_M)$:

$$\begin{aligned} \psi(o_M) &= L\left(o_M, \frac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_M - o_i\right)\right) \\ &= \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(-o_M{}^2 + o_i^2\right) \end{aligned} \tag{20}$$

$$s.t.\quad \begin{cases} \lambda = \dfrac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_M - o_i\right) \\ \lambda \geq 0 \end{cases}$$

If $\lambda = \lambda^*$, then $o_M = o_M{}^*$, and hence

$$\psi\left(o_M{}^*\right) = L\left(\lambda^*, o_M{}^*\right) = \phi\left(o_M{}^*\right)$$

Thus, $\psi\left(o_M\right)$ is the dual program of $\phi\left(o_M\right)$. To distinguish the quadratic programming composed with $\phi\left(o_M\right)$ from the dual quadratic programming composed with $\psi\left(o_M\right)$, we use another independent variable o_d to replace o_M in $\psi\left(o_M\right)$:

$$DNLP\left(c,t\right):\quad \max \psi\left(o_d\right) = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(-o_d{}^2 + o_i^2\right) \tag{21}$$

$$s.t.\quad \begin{cases} \lambda = \dfrac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_d - o_i\right) \\ \lambda \geq 0 \end{cases}$$

Since $\lambda \geq 0$, $\lambda = \dfrac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_d - o_i\right) \geq 0$. For Model (21), the optimal solution is obtained when and only when $\lambda = \dfrac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(o_d - o_i\right) = 0$. Thus, the dual quadratic program shown in Eq. (21) can be redefined:

$$DNLP\left(c,t\right):\quad \max \psi\left(o_d\right) = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} c_i \left(-o_d{}^2 + o_i^2\right) \tag{22}$$

$$s.t.\quad o_d = \sum_{i=1}^{n} c_i o_i \bigg/ \sum_{i=1}^{n} c_i$$

where o_d represents the weighted average opinion of all individuals since $\sum_{i=1}^{n} c_i = 1$. That is, o_d is the consensus opinion formed by all individuals for seeking the highest return. Actually, $o_d{}^* = \sum_{i=1}^{n} c_i o_i \bigg/ \sum_{i=1}^{n} c_i$ is the optimal solution of Model (22). The optimal solution $o_d{}^*$ means the optimal consensus opinion formed by all individuals for seeking the highest return. The $\max \psi\left(o_d\right)$ denotes the total return that is expected by all individuals for changing their opinions under their trust relationships to the moderator M .

To analyze the economic significance of Model (22), the objective function $\psi\left(o_d\right)$ can be reorganized:

$$\begin{aligned}\psi\left(o_d\right) &= \frac{1}{\max\left\{o_i\right\}}\sum_{i=1}^{n} c_i\left(-o_d^{\ 2}+o_i^2\right) \\ &= \frac{1}{\max\left\{o_i\right\}}\sum_{i=1}^{n} c_i\left(o_d^{\ 2}+o_i^2-2o_d o_i\right)+\frac{1}{\max\left\{o_i\right\}}\sum_{i=1}^{n} c_i\left(2o_d o_i-2o_d^{\ 2}\right) \\ &= \sum_{i=1}^{n} c_i\frac{\left|o_d-o_i\right|}{\max\left\{o_i\right\}}\left|o_d-o_i\right|+\frac{1}{\max\left\{o_i\right\}}\sum_{i=1}^{n} c_i\left(2o_d o_i-2o_d^{\ 2}\right)\end{aligned} \tag{23}$$

We can easy to obtain that $\frac{1}{\max\left\{o_i\right\}}\sum_{i=1}^{n} c_i\left(2o_d o_i-2o_d^{\ 2}\right)=0$ since $o_d=\sum_{i=1}^{n} c_i o_i \Big/ \sum_{i=1}^{n} c_i$.

Thus,

$$\begin{aligned}\psi\left(o_d\right) &= \sum_{i=1}^{n} c_i\frac{\left|o_d-o_i\right|}{\max\left\{o_i\right\}}\left|o_d-o_i\right| \\ &= \sum_{i=1}^{n}\left(y_i\left(1-s_i\right)\right)\left(o_i-o_d\right)\end{aligned} \tag{24}$$

where y_i denotes the unit return of the individual d_i, $\left|y_i\right|\le c_i$, $\sum_{i=1}^{n}\left|y_i\right|=1$. The similarity $s_i=1-\frac{\left|o_d-o_i\right|}{\max\left\{o_i\right\}}$ between o_i and o_d represents the consensus willingness of the individual d_i to reach the consensus o_d for seeking the highest return. The larger the similarity between o_i and o_d, the stronger the consensus willingness of d_i to reach the consensus o_d. According to Eq. (13), we can find that when $o_M^{\ *}$ and $\left(\lambda^*, o_d^{\ *}\right)$ exist, $o_M^{\ *}=o_d^{\ *}$, then the consensus willingness $s_i=1-\frac{\left|o_d^{\ *}-o_i\right|}{\max\left\{o_i\right\}}$ of d_i equals to the implicit trust $t_{iM}=1-\frac{\left|o_M^{\ *}-o_i\right|}{\max\left\{o_i\right\}}$ of d_i to the moderator M. When $o_d^{\ *}\ne o_i$ and $\left|y_i\right|=c_i$, $o_d^{\ *}$ exists.

According to **Theorem 3** and **4**, we can given two theorems based on (24):

Theorem 5. Suppose that the individuals opinions satisfy $o_1\le\ldots\le o_i\le\ldots\le o_n$, and the unit cost satisfy $\sum_{i=1}^{n} c_i=1$. If $o_M^{\ *}$ is the optimal solution to the primal problem $NLP\left(c,t\right)$, then there must exist a $t_0 \in N$ such that $o_{t_0} \le o_i \le o_{t_0+1}$, and

$$\sum_{i=1}^{t_0} c_i = \sum_{j=t_0+1}^{n} c_j = 0.5 \tag{25}$$

if and only if $DNLP(c,t)$ has optimal solution $\left(0, o_d^*\right)$, and the unit return of individuals is $\left(-y_1, \ldots, -y_{t_0}, y_{t_0+1}, \ldots, y_n\right)^T$ and satisfy $\sum_{i=1}^{n} |y_i| = 1$.

Theorem 6. When $DNLP(c,t)$ has optimal solution $\left(0, o_d^*\right)$, $o_d^* = \sum_{i=1}^{n} c_i o_i \Big/ \sum_{i=1}^{n} c_i$ holds, then $o_d^* \ne o_i$ holds, and then the statement $y_i = -c_i$ holds when $o_d^* > o_i$ holds; and $y_i = c_i$ holds when $o_d^* < o_i$. This denotes $|y_i| = c_i$ holds when $o_d^* \ne o_i$ holds.

3.5 The economic explanation of the proposed primal and dual models

In the primal Model $NLP(c,t)$, the variable o_M represents the expected consensus opinion of the moderator M for seeking the lowest cost. In the dual Model $DNLP(c,t)$, the variable o_d represents the consensus opinion formed by all individuals for seeking the highest return. According to the strong and weak duality theorem of quadratic programming, we can give the following two corollaries for the proposed primal and dual model.

Corollary 1 According to the weak duality theorem of quadratic programming, if o_M and o_d is the feasible solution of Model $NLP(c,t)$ and $DNLP(c,t)$, respectively, then $\phi(o_M) \ge \psi(o_d)$, that is the maximum value of $\psi(o_d)$ is the greatest lower bound of $\phi(o_M)$.

The economic explanation: Under the premise of consensus reaching, the total compensation that all individuals received for making concessions is less than or equal to the total cost that the moderator is willing to pay.

Corollary 2 According to the strong duality theorem, if Model $NLP(c,t)$ and $DNLP(c,t)$ both have an optimal solution o_M^* and o_d^*, respectively, then $\min \phi\left(o_M^*\right) = \max \psi\left(o_d^*\right)$.

The economic explanation: If all individuals agree to achieve a consensus o_d^* under the maximum compensation given by the moderator M, the moderator M pay the minimum cost under the expected consensus opinion o_M^*, and $\min \phi\left(o_M^*\right) = \max \psi\left(o_d^*\right)$.

Corollary 3 According to the complementary slackness property, when Model $NLP(c,t)$ has an unique optimal solution o_M^*, then the necessary condition for the optimal solution (λ^*, o_d^*) of the dual problem $DNLP(c,t)$ is: $\lambda^* o_M^* = 0$. If $\lambda^* = 0$ holds in Model $DNLP(c,t)$ shown in (21), then $o_d^* = \sum_{i=1}^{n} c_i o_i \Big/ \sum_{i=1}^{n} c_i$ and $o_M^* > 0$ hold, and $o_M^* = o_d^*$.

The economic explanation: If all individuals agree to achieve the consensus o_M^* which is expected by the moderator M under seeking for the minimum cost $\min \phi(o_M^*)$, then $\lambda^* = 0$, which means that the unit return of the group will not change whatever the consensus opinion o_M changes. That is, the consensus o_M^* equals to the consensus opinion o_d^* formed by all individuals for seeking the highest return, that is $o_d^* = \sum_{i=1}^{n} c_i o_i \Big/ \sum_{i=1}^{n} c_i = o_M^*$. In another word, all the individuals and the moderator M can find out the maximum return and the minimum cost, respectively, when they reach a consensus $o_M^* = o_d^*$.

Corollary 4 If Model $NLP(c,t)$ and $DNLP(c,t)$ both have an optimal solution o_M^* and o_d^*, respectively, then $s_i = t_{iM}$.

The economic explanation: If the consensus o_d^* formed by all individuals equals to the consensus o_M^* expected by the moderator M, then the implicit trust of individuals to the moderator M equals to their consensus will to reach to o_d^*.

Besides, the Lagrange multiplier λ represents the shadow price in economics. Similarly, it has practical economic significance in the dual Model (21). For $\lambda = \frac{2}{\max\{o_i\}} \sum_{i=1}^{n} c_i (o_d - o_i)$, $\lambda^* = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} y_i (o_d^* - o_i)$ when $DNLP(c,t)$ has optimal solution (λ^*, o_d^*). According to **Theorem 6**, $\lambda^* = \frac{1}{\max\{o_i\}} \sum_{i=1}^{n} y_i (o_d^* - o_i)$ represents the unit compensation of the whole group when the consensus opinion o_d changes by one unit.

In summary, in the primal Model $NLP(c,t)$, individuals are willing to reduce the unit cost to reach the consensus o_M that the moderator M expect based on their trust relationships to

M. In the dual Model $DNLP(c,t)$, individuals are willing to give up some benefits to reach the consensus o_d based on their consensus willingness which can be determined based on the similarity between their opinions and o_d. The moderator M can obtain the minimum cost and the group $\{d_1, d_2, ..., d_n\}$ can obtain the highest return when and only when the consensus opinion o_d volunteered by all individuals equals to the consensus opinion o_M expected by the moderator M.

3.6 The relation between the proposed models and the traditional ones

When the implicit trust is not considered in the coordination process, the traditional primal and dual model $NLP(c)$ and $DLP(c)$ are special forms of the proposed primal and dual models $NLP(c,t)$ and $DNLP(c,t)$, respectively. Besides, **Theorem 5** and **6** of the proposed models are consistent with **Theorem 3** and **4** of the traditional ones obtained in [13].

When we consider the implicit trust in the minimum cost consensus model, the optimal solution to $NLP(c,t)$ is not equal to any of the individuals' opinions, then the moderator has to pay more effort and cost to persuade individuals to change their opinions. The optimal consensus opinion o_d^* of $DNLP(c,t)$ exists only when the individuals' expected unit returns $|y_i|$ attains the upper limit value of the unit cost, i.e., $|y_i| = c_i$. In the traditional model $NLP(c)$, the optimal solution is always equal to one of the individuals' opinions, and there is always an individual's unit return cannot attain the upper limit value of the unit cost, i.e., $\forall |y_i| < w_i$.

Compared with the traditional primal model, the individuals are more willing to compromise to the moderator in $NLP(c,t)$ with considering the implicit trust, while they are also easier to agree on the compensation in $DNLP(c,t)$ with their consensus willingness.

4 The optimal models based on the modified adjustment cost

The adjustment cost is usually given by individuals in a subjective way. Sometimes, the subjective cost may be not so reasonable, especially when some individuals lack relevant experience or knowledge. We need to adjust the unit cost for individuals since the unreasonable cost will cause the unfair decision results. Similar with the adjustment of individuals' preference, they may be reluctant to make free concessions to modify their cost. To persuade individuals changing their unit cost voluntarily, we are meant to provide them the adjustment suggestions based on their importance which comes from the trust levels given by others.

4.1 The weights determination for the individuals

In practical, people who are highly trusted have more influence on others. It shows that trust reflects the importance of a person to some extent. Thus, we try to compute the weights for individuals based on how much others trust them.

Suppose that we can obtain all the trust relationships between the pairwise individuals based on the trust function Eq. (11). The implicit trust structure based on opinions is constructed in Fig.3, it is evident that the implicit trust relationship among individuals is mutual.

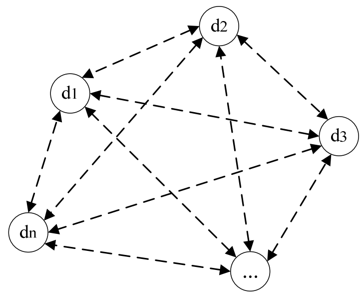

Fig.3. The implicit trust structure between individuals

According to Fig.3, the trust relationship matrix $T = \left[t_{ij}\right]_{n\times n}$ is constructed:

$$T = \begin{bmatrix} - & t_{12} & t_{13} & \cdots & t_{1n} \\ t_{21} & - & t_{23} & \cdots & t_{2n} \\ t_{31} & t_{32} & - & \cdots & t_{3n} \\ \cdots & \cdots & \cdots & - & \cdots \\ t_{n1} & t_{n2} & t_{n3} & \cdots & - \end{bmatrix} \tag{26}$$

where $t_{ij} = t_{ji}$, $i, j = 1,\ldots,n, i \neq j$.

The in-degree trust index t_j of d_j is computed based on Eq. (1). With the in-degree trust index of individuals, the weights ω_j of individuals d_j can be computed:

$$\omega_j = t_j \Big/ \sum_{j=1}^{n} t_j \tag{27}$$

where $\sum_{j=1}^{n} \omega_j = 1$.

The weight ω_j reflects how much the individual d_j is trusted by others. The higher the in-degree trust value is, the larger his/her weight is. Next, we will adjust the unit cost for individuals with their weights $\boldsymbol{\omega}$.

4.2 The modified optimal models based on the revision of the adjustment cost

The adjustment cost is commonly evaluated by the individuals themselves based on their expectation. However, the subjective adjustment cost with large deviation may cause injustice in CRP. For example, people may earn trust based on their high prestige or professional level, and they may also have high confidence in the decision-making process. Because of this, they are less likely to change their views in consensus negotiations, so it is possible that their unit cost is higher than the ordinary individuals. However, it may not be reasonable if the unit cost of the ordinary individuals is higher than that of highly trusted individuals. Therefore, we propose the modified optimal model to narrow the possible unreasonable gaps.

Let o'_S denotes the expected consensus opinion of the moderator M when the weights of individuals are considered in the process of negotiation. With the weights $\boldsymbol{\omega}$, the modified optimal model is given based on Model (17):

$$NLP(c,t,w): \quad \min \phi\left(o'_M\right) = \frac{1}{\max\left\{o_i\right\}} \sum_{i=1}^{n} c_i \omega_i \left(o_M'^{\,2} + o_i^2 - 2 o_i o'_M\right) \tag{28}$$

$$s.t. \quad o'_M \geq 0$$

In Model (28), if $c_i \geq c_j \left(c_i \leq c_j\right)$ and $\omega_i \geq \omega_j$, the importance of cost c_i and c_j will be emphasized and weakened, respectively, and the gap between them will be widened (narrowed). If $c_i \geq c_j \left(c_i \leq c_j\right)$ and $\omega_i \leq \omega_j$, the importance of cost c_i and c_j will be weakened and emphasized, respectively, and the gap between them will be narrowed (widened).

Similarly, the dual problem of Model (28) can be determined according to (18)-(21) through the introduction of the Lagrange multiplier λ':

$$DNLP(c,t,w): \quad \max \psi\left(o'_d\right) = \frac{1}{\max\left\{o_i\right\}} \sum_{i=1}^{n} c_i \omega_i \left(-o_d'^{\,2} + o_i^2\right) \tag{29}$$

$$s.t. \quad o'_d = \sum_{i=1}^{n} c_i \omega_i o_i \Big/ \sum_{i=1}^{n} c_i \omega_i$$

Based on Model (28), the Lagrange multiplier λ' is determined as $\lambda' = \frac{2}{\max\left\{o_i\right\}} \sum_{i=1}^{n} c_i \omega_i \left(o_d - o_i\right)$ referring (18)-(21), thus, the unit return of individuals is also adjusted with their importance: $y'_i \leq c_i \omega_i$.

It is evident that Model (28) and Model (29) are strict convex quadratic programming problems, and the local optimal solution $o_M'^{\,*}$ and $o_d'^{\,*}$ are easily to be determined.

Compared with the optimization models of the dual problem (17) and (22), the optimal solutions of the dual problem (28) and (29) are changing with the modified unit cost. However, it has the similar economic explanation with the dual problem (28) and (29) given in sub-section 3.5.

5 Numerical example and comparative analysis

In this section, a numerical example is proposed to show the application of the proposed models, and the comparative analysis is given to verify the effectiveness of the proposed models.

5.1 The applications of the proposed models

Suppose there are eight individuals $\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\}$ participate in a decision making problem, in which there is a moderator who can coordinate with individuals to reach a consensus. The corresponding opinion of these individuals is $O=\{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8\}=\{0, 2, 3, 4, 6, 7, 8, 10\}$, the unit cost they would like to pay to reach a consensus is $C=\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}=\{0.1, 0.2, 0.3, 0.1, 0.2, 0.4, 0.3, 0.5\}$, and the expected consensus opinion of the moderator M for seeking the minimum cost is o_M.

(1) The application of the optimal models based on the implicit trust

Besides, without loss of generality, the unit cost of individuals needs to be normalized:

$$\overline{c}_1 = c_1 \Big/ \sum_{i=1}^{8} c_i = 0.048 \tag{30}$$

Similarly, the normalized unit cost of others can be computed in the same way: $\overline{C} = \{0.048, 0.095, 0.143, 0.048, 0.095, 0.190, 0.143, 0.238\}$.

Based on Eq. (13), the implicit trust values of individuals to the moderator M are computed as: $t_{1M} = 1 - o_M/10$, $t_{2M} = 1 - |o_M - 2|/10$, $t_{3M} = 1 - |o_M - 3|/10$, $t_{4M} = 1 - |o_M - 4|/10$, $t_{5M} = 1 - |o_M - 6|/10$, $t_{6M} = 1 - |o_M - 7|/10$, $t_{7M} = 1 - |o_M - 8|/10$, and $t_{8M} = 1 - |o_M - 10|/10$. Then, the implicit trust of individuals with respect of cost attribute can be computed based on Eq. (15). According to the proposed optimal consensus model (17), the minimum cost model $NLP(c,t)$ based on the implicit trust in this problem is constructed as below:

$$\min \phi(o_M) = \frac{1}{10}\begin{bmatrix} 0.048{o_M}^2 + 0.095(o_M - 2)^2 + 0.143(o_M - 3)^2 + 0.048(o_M - 4)^2 \\ +0.095(o_M - 6)^2 + 0.190(o_M - 7)^2 + 0.143(o_M - 8)^2 + 0.238(o_M - 10)^2 \end{bmatrix} \tag{31}$$
$$s.t. \quad o_M \geq 0$$

The local optimal solution and the optimal value of model (31) are solved as ${o_M}^* = 6.238$

and $\min\phi\left(o_M^*\right)=0.923$, respectively. Based on the local optimal solution, the deviation between eight individuals' opinions and the consensus opinion are $\left|o_M^*-o_1\right|=6.238$, $\left|o_M^*-o_2\right|=4.238$, $\left|o_M^*-o_3\right|=3.238$, $\left|o_M^*-o_4\right|=2.238$, $\left|o_M^*-o_5\right|=0.238$, $\left|o_M^*-o_6\right|=0.762$, $\left|o_M^*-o_7\right|=1.762$, and $\left|o_M^*-o_8\right|=3.762$, respectively. The implicit trust values of individuals to the moderator M are $t_{1M}=0.3762$, $t_{2M}=0.5762$, $t_{3M}=0.6762$, $t_{4M}=0.7762$, $t_{5M}=0.9762$, $t_{6M}=0.9238$, $t_{7M}=0.8238$, and $t_{8M}=0.6238$.

Based on Model (22), the dual problem $DNLP(c,t)$ of Model (31) is built as below:

$$\max\psi\left(o_d\right)=\frac{1}{10}\left(-o_d^{\ 2}+48.143\right) \tag{32}$$
$$s.t.\quad o_d=6.238$$

The unique solution and the optimal value of model (32) are solved as $o_d^*=6.238$ and $\max\psi\left(o_d^*\right)=0.923$, respectively.

According to the strong and weak duality theorem of quadratic program, we can deduce that individuals get the maximum compensation $\max\psi\left(o_d^*\right)=0.923$ and the moderator pays the minimum cost $\min\phi\left(o_M^*\right)=0.923$ when the consensus $o_M^*=o_d^*=6.238$ is reached. In the coordination process, individuals are easy to compromise to reach the consensus since they trust the moderator M.

(2) The application of the modified optimal models based on the adjustment of unit cost

According to the modified optimal model, we need to compute the weights of individuals based on the most prefer trust levels that given by others. Firstly, construct the trust relationship matrix $T=\left(t_{ij}\right)_{8\times8}$:

$$\mathrm{T}=\begin{bmatrix} - & 0.8 & 0.7 & 0.6 & 0.4 & 0.3 & 0.2 & 0.0 \\ 0.8 & - & 0.9 & 0.8 & 0.6 & 0.5 & 0.4 & 0.2 \\ 0.7 & 0.9 & - & 0.9 & 0.7 & 0.6 & 0.5 & 0.3 \\ 0.6 & 0.8 & 0.9 & - & 0.8 & 0.7 & 0.6 & 0.4 \\ 0.4 & 0.6 & 0.7 & 0.8 & - & 0.9 & 0.8 & 0.6 \\ 0.3 & 0.5 & 0.6 & 0.7 & 0.9 & - & 0.9 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.6 & 0.8 & 0.9 & - & 0.8 \\ 0.0 & 0.2 & 0.3 & 0.4 & 0.6 & 0.7 & 0.8 & - \end{bmatrix}_{8\times8} \tag{33}$$

According to the trust matrix $T=\left(t_{ij}\right)_{8\times 8}$, the in-degree trust value of individuals can be determined based on Eq. (1). For example, the in-degree trust index t_1 of d_1 is computed:

$$\begin{aligned} t_1 &= \frac{1}{7}\sum_{i=1}^{8} t_{i1} \\ &= \left(0.8+0.7+0.6+0.4+0.3+0.2+0\right)/7 = 0.429 \end{aligned} \tag{34}$$

Similarly, the in-degree trust values of others can be computed in the same manner. $t_2=0.6$, $t_3=0.657$, $t_4=0.686$, $t_5=0.686$, $t_6=0.657$, $t_7=0.6$, and $t_8=0.429$. With these in-degree trust values, the weights of individuals are calculated based on Eq. (27). For example, the weight of d_1 is computed as:

$$\omega_1 = \frac{0.429}{0.429+0.6+0.657+0.686+0.657+0.6+0.429} = 0.09 \tag{35}$$

Similarly, the weights of other individuals can be computed in the same manner. $\omega_2=0.127$, $\omega_3=0.139$, $\omega_4=0.145$, $\omega_5=0.145$, $\omega_6=0.139$, $\omega_7=0.127$, and $\omega_8=0.09$.

According to Model (28), the optimal consensus model $NLP(c,t,w)$ based on implicit trust with the adjustment of unit cost is built as below:

$$\begin{gathered} \min \phi\left(o'_M\right) = \frac{1}{10}\left[\begin{array}{l} 0.004{o'_M}^2 + 0.012\left(o'_M-2\right)^2 + 0.020\left(o'_M-3\right)^2 + 0.007\left(o'_M-4\right)^2 \\ +0.014\left(o'_M-6\right)^2 + 0.026\left(o'_M-7\right)^2 + 0.018\left(o'_M-8\right)^2 + 0.021\left(o'_M-10\right) \end{array}\right] \\ s.t.\ \ o'_M \geq 0 \end{gathered} \tag{36}$$

The local optimal solution and the optimal value of model (36) are solved as ${o'_M}^* = 6.008$ and $\min \phi\left({o'_M}^*\right) = 0.099$, respectively. Based on the local optimal solution, the deviation between eight individuals' opinions and the consensus opinion are $\left|{o'_M}^* - o_1\right| = 6.008$, $\left|{o'_M}^* - o_2\right| = 4.008$, $\left|{o'_M}^* - o_3\right| = 3.008$, $\left|{o'_M}^* - o_4\right| = 2.008$, $\left|{o'_M}^* - o_5\right| = 0.008$, $\left|{o'_M}^* - o_6\right| = 0.992$, $\left|{o'_M}^* - o_7\right| = 1.992$, and $\left|{o'_M}^* - o_8\right| = 3.992$, respectively.

Similarly, the dual problem $DNLP(c,t,w)$ of Model (36) is shown as:

$$\max \psi\left(o'_d\right) = \frac{1}{10}\left[-0.123{o'_d}^2 + 5.436\right] \tag{37}$$

$$s.t.\ \ o'_d = \frac{0.739}{0.123}$$

The unique solution and the optimal value of model (32) are solved as ${o'_d}^* = 6.008$ an $\max \psi\left({o'_d}^*\right) = 0.099$, respectively.

Similarly, we can deduce that individuals get the maximum compensation $\max \psi\left({o'_d}^*\right) = 0.099$ and the moderator pays the minimum cost $\min \phi\left({o'_M}^*\right) = 0.099$ when the consensus ${o'_M}^* = {o'_d}^* = 6.008$ is reached. In the coordination process, individuals are willing to modify their unit cost and give up some benefits to reach the consensus since they trust the moderator M.

5.2 The comparative analysis

In order to highlight the effectiveness of the proposed models, the comparative analysis is given in this section. Firstly, the numerical example is solved using the traditional model. Secondly, compare the solution obtained based on the traditional model with the proposed Model (31) and (32) which are based on the implicit trust relationships. Finally, compare the solution obtained based on Model (31) and (32) with Model (36) and (37) where the cost of individuals is modified.

(1) The solution of the numerical example based on the traditional model

In order to highlight the effectiveness of the proposed models, the numerical example is solved by the traditional minimum cost consensus Model (8):

$$\begin{gathered}\min \phi(o) = 0.048o + 0.095|o-2| + 0.143|o-3| + 0.048|o-4| \\ + 0.095|o-6| + 0.190|o-7| + 0.143|o-8| + 0.238|o-10| \\ s.t. \quad o \in O\end{gathered} \tag{38}$$

The unique solution and the optimal value of model (38) are solved as $o^* = 7$ and $\min \phi\left(o^*\right) = 2.476$, respectively. Based on the unique solution, the deviation between 4 individuals' opinions and the consensus opinion are $\left|o^* - o_1\right| = 7$, $\left|o^* - o_2\right| = 5$, $\left|o^* - o_3\right| = 4$, $\left|o^* - o_4\right| = 3$, $\left|o^* - o_5\right| = 1$, $\left|o^* - o_6\right| = 0$, $\left|o^* - o_7\right| = 1$, and $\left|o^* - o_8\right| = 3$, respectively.

Similarly, the dual problem of Model (38) can be given based on Model (9):

$$\begin{gathered}\max \psi(y) = -7y_1 - 5y_2 - 4y_3 - 3y_4 - y_5 + y_7 + 3y_8 \\ s.t. \quad \begin{cases} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 0 \\ |y_1| \le 0.048; |y_2| \le 0.095; |y_3| \le 0.143; |y_4| \le 0.048 \\ |y_5| \le 0.095; |y_6| \le 0.190; |y_7| \le 0.143; |y_8| \le 0.238 \end{cases}\end{gathered} \tag{39}$$

The unique solution and the optimal value of Model (39) are solved as

$Y = \{-0.048, -0.095, -0.143, -0.048, -0.095, 0.048, 0.143, 0.238\}$ and $\max \psi\left(y^*\right) = 2.476$. The solution in Y means the unit reward the individuals can obtain from the moderator and they can obtain 2.476 at most.

Individuals obtain the highest return $\max \psi\left(y^*\right) = 2.476$ when the consensus opinion $o^* = 7$ is reached and the moderator pay the lowest cost $\min \phi\left(o^*\right) = 2.476$.

(2) The comparison between Model (31), (32) and Model (38), (39)

When the consensus opinion $o^* = 7$, we can also compute the total cost that the moderator M needs to pay to all individuals with considering the implicit trust relationships. Let $o_M = o^* = 7$ in Model (31), the implicit trust of individuals to the moderator is denoted as $t_{iM_{o^*=7}}$ when $o_M = o^* = 7$. The comparison between the traditional and the implicit trust-based minimum cost consensus model is given in Table 1.

Table 1 The comparison results when $o^* = 7$

Individuals	$\left\|o^* - o_i\right\|$	$\overline{c_i}$	$t_{iM_{o^*=7}}$	$c_i t'_{iM_{o^*=7}}$	$\phi\left(o^*\right)$	$\phi\left(o_M\right)$
d1	7	0.048	0.3	0.038	0.333	0.233
d2	5	0.095	0.5	0.048	0.476	0.238
d3	4	0.143	0.6	0.057	0.571	0.229
d4	3	0.048	0.7	0.014	0.143	0.043
d5	1	0.095	0.9	0.010	0.095	0.010
d6	0	0.190	1	0	0	0
d7	1	0.143	0.9	0.014	0.143	0.014
d8	3	0.238	0.7	0.071	0.714	0.214
Total cost	-	-	-	-	2.476	0.981

From Table 1, we can find that the unit cost of individuals is offset by their trust to the moderator M. The higher the trust level is, the more cost is offset. Especially, the cost of individual is completely offset when he/she fully trusts M, i.e., $t_{iM} = 1$, while there is no change in cost when he/she fully distrusts M, i.e., $t_{iM} = 0$. It is evident that the total cost of the group decreases with the offsetting of trust information.

Regarding Model (31) and Model (38), we can find that the consensus opinion decreases from 7 to 6.238 with considering the implicit trust, that is the overall trust level of individuals to the moderator M reaches to the maximize. Let $t_{iM_{o^*=6.238}}$ denotes the trust value of d_i to the moderator M when $o_M{}^* = 6.238$. The trust comparison between $o^* = 7$ and $o_M{}^* = 6.238$ is shown in Table 2.

Table 2 The trust comparison between $o^* = 7$ and $o_M{}^* = 6.238$

Individuals	$o^* = 7$	$o_M{}^* = 6.238$	$t_{iM_{o^*=6.238}} - t_{iM_{o^*=7}}$
	$t_{iM_{o^*=7}}$	$t_{iM_{o^*=6.238}}$	
d1	0.3	0.3762	0.0762
d2	0.5	0.5762	0.0762
d3	0.6	0.6762	0.0762
d4	0.7	0.7762	0.0762
d5	0.9	0.9762	0.0762
d6	1	0.9238	-0.0762
d7	0.9	08238	-0.0762
d8	0.7	0.6238	-0.0762
Total trust	5.6	5.7524	0.1524

From Table 2, we can find that the total trust value of the group increases due to the trust value of most individuals increases when the consensus opinion decreases from 7 to 6.238. Therefore, the optimal consensus opinion is found when the consensus opinion equals to 6 with considering the implicit trust.

Similarly, the consensus willingness of individuals also makes them would give up some benefits to reach a consensus in the dual model (32). When the consensus opinion decreases from 7 to 6.238, the consensus willingness and the compensation both are at their maximum.

(3) The comparison between Model (31), (32) and Model (36), (37)

Regarding Model (31) and Model (36), the unreasonable unit cost of individuals is adjusted in Model (36), the optimal consensus correspondingly changes from $o_M{}^* = 6.238$ to $o'_M{}^* = 6.008$. The comparison results between $o_M{}^* = 6.238$ and $o'_M{}^* = 6.008$ is shown in Table 3.

Table 3 The comparison results when $o_M{}^* = 6.238$ and $o'_M{}^* = 6.008$

Individuals	$o_M{}^* = 6.238$	$o'_M{}^* = 6.008$	
	$\overline{c_i}$	ω_i	$c_i * \omega_i$
d1	0.048	0.09	0.004
d2	0.095	0.127	0.012
d3	0.143	0.139	0.020
d4	0.048	0.145	0.007
d5	0.095	0.145	0.014
d6	0.190	0.139	0.026
d7	0.143	0.127	0.018
d8	0.238	0.09	0.021

Referring the weights of individuals in Table 5, we can distinguish that there are unreasonable situations in the unit cost, i.e., $c_8 > c_7$, while $\omega_7 > \omega_8$. With the weights of individuals, the unreasonable gap between the modified cost is narrowed largely than that between the initial cost, i.e., the gap between c_7 and c_8 which equals to $|0.238-0.143|$ is reduced to $|0.021-0.018|$. Similarly, the consensus opinion formed by all individuals in the dual Model (37) will change since their unit return will also be adjusted according to their importance in the group.

In summary, the above two comparisons suggest the importance of trust in consensus reaching. In the first comparison, individuals are more likely to be persuaded to adjust their opinions to reach consensus if they trust the moderator. In the second comparison, individuals agree to make minor adjustment to the unit cost if they trust the moderator. Thus, we can obtain more reasonable consensus opinion with the adjusted unit cost.

6 Conclusions and future research

To explore the effects of trust on consensus reaching, we propose minimum cost consensus models considering implicit trust which is obtained based on the similarity of opinion.

In this paper, the moderator is considered to be a trustworthy coordinator to persuade individuals to reach a consensus which he/she expects to pay the lowest cost. The implicit trust of individuals to the moderator is determined based on the similarity of opinions. The minimum cost consensus model and its dual model are proposed based on the implicit trust. According to the economic significance of the primal and dual models, individuals who have implicit trust on the moderator are willing to give up some benefits to reach the consensus as that the moderator expected. Besides, the individuals are also willing to make minor revision to their unit adjustment cost according to their weights obtained based on the implicit trust among individuals. A numerical example and the comparative analysis are given to analyze the application of the proposed models.

In summary, this study provides a new perspective for SNGDM to measure the effectiveness of social relationships in CRP. The proposed models not only show the offset role of the implicit trust to the adjustment cost in CRP but also reveal the regulation role of the implicit trust modifying the adjustment cost of large deviation. Besides, the offset and regulation role of the implicit trust are analyzed according to the economic significance of the primal-dual models.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (NSFC) (71771051), the 2019 Jiangsu Province Policy Guidance Program (Soft Science Research) (BR2019064), the Scientific Research Foundation of Graduate School of Southeast University (YBJJ1831) and the China Scholarship Council (201806090164).

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3 Trust-consensus multiplex networks by combining trust social network analysis and consensus evolution methods in group decision-making

- T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Trust-Consensus Multiplex Networks by Combining Trust Social Network Analysis and Consensus Evolution Methods in Group Decision-Making. **IEEE Transactions on Fuzzy Systems.** 30 (2022) 4741-4753.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 12.029
  - Subject Category: Engineering, Electrical & Electronic, Ranking 6/273 (Q1).
  - Subject Category: Computer Science, Artificial Intelligence, Ranking 8/175 (Q1).

A New Multiplex Social Network Group Decision-Making Model for Consensus Reaching Process Combining Trust Relationships and Consensus Evolution Method

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***Abstract*—Recently, the consensus research considering trust relationships is popular in social network group decision-making and the consensus evolution networks (CENs) were developed to explore the evolved consensus relations. To study consensus under the impact of trust relationships, we propose a consensus model based on trust consensus evolution multiplex networks by combing trust relationships and consensus evolution methods. According to the PageRank centrality, experts' influence is computed based on their comprehensive importance in the layer of trust networks and CENs. With experts' influence, we consider the interactive impacts between the layer of trust networks and CENs. Besides, we compute the overall consensus level based on the connection density and strength of trust consensus evolution multiplex networks. The proposed consensus model is illustrated by an example to show the positive and negative effects of trust on consensus, and its flexibility for studying the consensus evolution under the influence of trust is analyzed by a comparative analysis.**

***Index Terms*—Social network group decision making, consensus reaching process, trust networks, consensus evolution networks, multiplex networks**

I. Introduction

CONSENSUS is essential in the areas where collaboration is required, such as group decision-making (GDM) [1-3] and multiagent systems [4-6]. In traditional GDM, consensus research is mainly carried on based on experts' opinions [7-9]. Due to the development of information technology and the strengthening of inter-organizational cooperation, the social network group decision-making (SNGDM) appears and develops rapidly [10-12]. The popularity of SNGDM is due to its ability to visualize the variation of experts' relationships [13-15] and deal with complicated situations with various social network analysis tools [16].

Regarding the consensus research in GDM, abundant studies focused on consensus measuring [17, 18] and consensus modeling [19, 20]. The consensus is usually measured mathematically via similarity functions, which are commonly determined based on distance functions [17]. Chiclana et al. [18] found that different kinds of distance functions have various manifestations in consensus measurements. Besides, many consensus models were proposed to deal with the unsatisfying consensus situation [20-22]. In current research, the optimization consensus modeling is popular when providing compensations for experts who make concessions [2, 7, 20]. However, experts in SNGDM may automatically reach a consensus without benefits based on trust relationships [22-25] and opinion dynamics [21, 26, 27].

In some cases, social relationships among people can be specific to be trust relationships, which are usually judged by humans' perception of others' reputation based on their prior knowledge or experience [28]. Because of the specificity of definitions and clarity of relationships, trust is widespread in SNGDM as a vital role in decision support systems [22-24]. Ureña et al. [12] discussed the function of trust, reputation, and influence for fostering decision-making processes in social networks. Wu et al. [13, 22, 23] designed various consensus models based on trust relationships for different decision-making scenarios. Zhang et al. [24] introduced a novel consensus framework based on social trust networks to deal with non-cooperative behaviors. Besides, trust is also critical for processing large-scale GDM [14, 25, 29]. However, most previous studies focus on trust effect on preferences changing rather than the direct influence of trust in consensus evolution, neither the impact of consensus evolution on trust.

Inspired by the advantages of social network analysis in the study of SNGDM, we defined the consensus evolution networks (CENs) with consensus relations which obtained based on preference similarity [30]. SNGDM considers the

This work was supported in part by the National Natural Science Foundation of China under Grant 71771051 and 71701158, in part of the Ministry of Education in China Project of Humanities and Social Sciences under Grant 17YJC630114, and in part by the China Scholarship Council under Grant 201806090164. (Corresponding authors: Xinwang Liu and Jindong Qin)

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external social relationships among experts, while CENs concern the internal preference relations. Since hybrid information are commonly helpful for solving problems comprehensively in the decision-making area [8, 9, 31], we intend to explore consensus evolution by combining trust networks and CENs.

But, two kinds of conflict may appear in decision scenarios between the internal preferences and external relationships: task conflict and relationship conflict [32-34]. Task conflict is the disagreement among experts' preferences, and relationship conflict is the interpersonal incompatibility among experts [32]. Task conflict may produce effective decisions due to the diversity of views, while relationship conflict is detrimental to decision quality due to it may provoke negative decision behavior [35]. Most of the existing studies show the positive effects of trust relationships in the decision process [22-24, 29]. Liu et al. [25] and Ding et al. [36] realized the relationship conflict but rarely analyzed the direct negative effects of trust on consensus.

According to the above discussion, we summarized the limitations of current SNGDM in the following three points.

(1) The current consensus research and experts' importance determination is mainly based on external trust relationships or internal preference relations rather than their combination.

(2) Trust relationships' effects, especially the negative ones, are not directly considered in the consensus relations but preference adjustments.

(4) The evolution of trust relationships is rarely investigated, while the relation network structure may change during the interactive adjustment.

The complex interaction between trust networks and CENs needs to be solved to deal with the above problems. Multiplex network is a useful tool to analyze the complex systems considering multiple types of interactions [37-39]. The multiplex network analysis is not based on the simple aggregation procedure of all single interacting ways, which might result in information loss [40]. For example, in multiplex network centrality analysis, the importance of individual nodes on each layer is considered. Still, their influence on other layers is also considered, i.e., the overall efficiency of multiplex networks is obtained across layers [41]. Besides, the same objects in multiplex networks can be co-evolving as the various kinds of connections affect one another over time [39]. In terms of the multiple links among experts in SNGDM, we are mainly focused on the trust relationships and consensus relations. In addition to the inner connections among experts in trust networks or CENs, there are interactions between these two forms of networks with different kinds of connections. Therefore, the multiplex network is useful to uncover the evolved nature of the complicated consensus by combining trust relationships and consensus relations.

In this study, we intend to explore the interaction between trust and consensus in SNGDM based on multiplex network structures by combining trust networks and CENs. We firstly give three assumptions that can help to address the above limitations.

Assumption 1. In multiplex networks, experts' influence is reflected not only in neighbors of the same layer but also in neighbors of other layers.

Assumption 2. Experts are willing to make concessions to reach a consensus under trust influence, but a conflict may arise when they have a low trust degree and high consensus level.

Assumption 3. All experts have a desire to promote consensus, and their trust relationships can become tighter and tighter in full and benign interaction adjustment.

Based on these assumptions, our main contributions are to deal with the above limitations from the following three points.

(1) Construct trust consensus evolution multiplex networks and compute experts' comprehensive influence. By combining trust relationships and consensus relations, we can build multiplex networks for experts, where the consensus evolution and trust development is mutually affected during the consensus adjustment. Based on Assumption 1, experts' overall influence can be determined in the multiplex network using PageRank centrality.

(3) Explore the evolution of consensus relations. According to Assumption 2, experts will cooperate with others when they have trust relationships. Trust may affect consensus negatively when a conflict arises. Thus, the consensus evolution is evaluated under both the positive and negative effects of trust based on experts' influence.

(4) Consider the development of trust relationships. Depending on Assumption 3, the trust relationships among experts may change when each round of adjustment is finished in the consensus reaching process. According to experts' overall influence, the variation of trust relationships can be measured based on its propagation characteristic.

According to the above contributions, a new consensus model can be proposed based on trust consensus evolution multiplex networks. The consensus relations are first adjusted based on the initial trust relationships, and then the trust relationships may change after the negotiation. If the evolved consensus still does not reach the agreed level, the updated trust relationships can continually promote the consensus evolution. This iterative process continues until the agreed consensus is achieved.

We use a numerical example to examine the proposed consensus model to show its flexibility for studying the consensus evolution and trust development during three rounds of adjustment. In each round, experts' overall influence is also updated using the PageRank centrality. Correspondingly, the regenerative trust consensus evolution multiplex network is output in the final round. According to the final consensus level, the whole positive effects of the trust on consensus outweigh its negative effects.

The remainder of this article is organized as follows: Section II introduces the related concepts of this study. Section III proposes the consensus model based on trust consensus evolution multiplex networks. Section IV examines the proposed consensus model using an illustrative example. Section V gives comparative analysis to show the advantages of the proposed consensus model. Section VI concludes this article.

II. PRELIMINARIES

In this section, the concepts of trust networks, multiplex networks, and some social analysis tools are described in section *A*. The concept of CENs is introduced based on fuzzy preference relations in section *B*.

A. Trust networks and multiplex networks

Social network analysis plays an essential role in network research [16, 42]. The network structure and node properties of trust networks can be measured via social network analysis tools. The definition of a general network is given first.

Definition 1. [43] An undirected and weighted network $\boldsymbol{G}=(\boldsymbol{D},\boldsymbol{E},\boldsymbol{W})$ consists of the set of nodes $\boldsymbol{D}=\{d_1,d_2,...,d_N\}$, the set of edges $\boldsymbol{E}=(E_{ij})$, and the set of edge weights $\boldsymbol{W}=(W_{ij})$.

A trust network $\boldsymbol{G}_A=(\boldsymbol{D},\boldsymbol{E}_A,\boldsymbol{T})$ can be defined when nodes are connected with trust relationships $\boldsymbol{E}_A=(E_{ij}^A)$ and trust degree $\boldsymbol{T}=(T_{ij})$, where E_{ij}^A denoting the expert d_i trusts d_j with the value T_{ij}.

The system in which the same nodes interact in multiple relationships is typically defined as a multiplex network [41].

Definition 2. [41] Let $\boldsymbol{MG}=(\boldsymbol{G}_1,...,\boldsymbol{G}_\lambda,...,\boldsymbol{G}_L,\boldsymbol{ME},\boldsymbol{MW})$ be a multiplex network, where $\lambda=1,2,...,L$ represent the number of layers in $\boldsymbol{MG}$, $\boldsymbol{G}_\lambda=(\boldsymbol{D}_\lambda,\boldsymbol{E}_\lambda,\boldsymbol{W}_\lambda)$ denoting the λth layer network consisting of the set of nodes $\boldsymbol{D}_\lambda$, edges $\boldsymbol{E}_\lambda$ and weights $\boldsymbol{W}_\lambda$, $\boldsymbol{ME}=(ME_{ij})(d_i\in\boldsymbol{G}_\lambda, d_j\notin\boldsymbol{G}_\lambda)$ denoting the relationships between nodes who belong to different layers, $\boldsymbol{MW}=(MW_{ij})$ denoting the corresponding weights of the cross-layer edges $\boldsymbol{ME}$.

Density is commonly used to measure the compactness of edge connections in the network [16].

Definition 3. [16] For an undirected and unweighted network $\boldsymbol{G}=(\boldsymbol{D},\boldsymbol{E})$, its density $d(\boldsymbol{G})$ is used to describe the intensity of edge connections between nodes:

$$d(\boldsymbol{G})=\frac{2|E|}{N(N-1)} \tag{1}$$

where $|E|$ and N denote the number of edges and nodes in the network $\boldsymbol{G}$, respectively.

Centrality is the most direct measure of node importance in social network analysis. The PageRank centrality is a famous tool used by Google for ranking websites [44]. It was originally proposed for directed networks, describing a random walker d_j jumps to one of d_j's out-neighbors with probability α, and to any other site at random with probability $1-\alpha$.

Definition 4. [44] For a directed and unweighted network $\boldsymbol{G}=(\boldsymbol{D},\boldsymbol{E},\boldsymbol{W})$, the PageRank v_i of a node d_i in $\boldsymbol{G}$ with N nodes is defined as:

$$v_i=\alpha\sum_j W_{ij}\frac{v_j}{g_j}+(1-\alpha)\frac{1}{N} \tag{2}$$

where W_{ij} are the adjacency matrix elements that are equal to 1 if node d_j connect to node d_i and 0 otherwise; $\alpha>0$ is the damping factor; $g_j=\max\left(1,\sum_u W_{uj}\right)$, $u=1,...,N$.

B. Consensus evolution networks based on fuzzy preference relations

The CENs is proposed to explore the consensus relations among experts [30]. The consensus relations are computed based on the similarity of experts' preferences, which are widely represented with the fuzzy preference relations (FPRs).

Definition 5. [3] An FPR F characterized by a membership function μ_F is a fuzzy set on the alternative set $X\times X\to[0,1]$, where $\mu_F(x_h,x_l)=f_{hl}$ describes the preference degree of alternative x_h over x_l $(h,l=1,2,...,M)$: $f_{hl}=0.5$ indicates indifference between x_h and x_l. $f_{hl}>0.5$ indicates that x_h is preferred to x_l, $f_{hl}<0.5$ indicates that x_h is inferior to x_l, and satisfying $f_{hl}+f_{lh}=1$.

Definition 6. [45] A similarity matrix $S_{ij}=(S_{hl}^{ij})_{N\times N}$ between expert d_i and d_j is defined as:

$$S_{hl}^{ij}=1-\left|f_{hl}^i-f_{hl}^j\right| \tag{3}$$

where f_{hl}^i denoting the preference of d_i with respect to x_h over x_l, $i,j=1,2,...,N$, $i\neq j$, $h,l,=1,2,...,M$, and $h\neq l$.

Definition 7. [30] Based on the similarity matrices $S_{ij}\ (i\neq j)$, the consensus relation $C_{ij}\ (i\neq j)$ between the expert d_i and d_j is computed:

$$C_{ij}=\frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}S_{hl}^{ij}}{N(N-1)/2} \tag{4}$$

where $N(N-1)/2$ represents the number of pairs of experts.

Therefore, the consensus matrix $\boldsymbol{C}=(C_{ij})_{N\times N}$ can be constructed among experts:

$$\boldsymbol{C}=\begin{bmatrix} 0 & \cdots & C_{1N} \\ \cdots & 0 & \cdots \\ C_{N1} & \cdots & 0 \end{bmatrix}_{N\times N} \tag{5}$$

Definition 8. [30] An CEN $\boldsymbol{G}_B^{\varepsilon}=\left(\boldsymbol{D},\boldsymbol{E}_B,\boldsymbol{C}\right)$ consists of N experts $\boldsymbol{D}=\left\{d_1,d_2,...,d_N\right\}$, the consensus relations $\boldsymbol{E}_B=\left\{E_{ij}^B\right\}$ with levels $\boldsymbol{C}=\left\{C_{ij}\geq\varepsilon\right\}$, $i,j=1,2,\ldots,N$ $i\neq j$. If the consensus level C_{ij} is higher than a consensus threshold $\varepsilon\in[0,1]$, then the edge E_{ij}^B exists between d_i and d_j with C_{ij}. Otherwise, E_{ij}^B does not exist.

The CENs is dynamic with different consensus threshold ε. The value of ε depends on the characteristics of decision problem and the attitudes of DMs.

III. A Multiplex Network Consensus Research Model Combining Trust Relationships and Consensus Relations

In this section, we propose a consensus model, based on multiplex networks by combining trust relationships and consensus relations, and considering experts' influence using the PageRank centrality, with the following tasks:

- Build trust consensus evolution multiplex networks in *section A*
- Compute experts' influence using the PageRank centrality in *section B*
- Investigate the consensus evolution under the effects of trust relationships in *section C*
- Consider the trust development during the consensus adjustment in *section D*
- Measure the overall consensus level based on the density in *section E*
- Introduce the main procedures of the proposed consensus model in *section F*

A. Construct trust consensus evolution multiplex networks

Let $\boldsymbol{G}_A=\left(\boldsymbol{D},\boldsymbol{E}_A,\boldsymbol{T}\right)$ be the trust network with the adjacency matrix $\boldsymbol{T}=\left(T_{ij}\right)_{N\times N}$ and $\boldsymbol{G}_B=\left(\boldsymbol{D},\boldsymbol{E}_B,\boldsymbol{C}\right)$ be the CEN with weighted adjacency matrix $\boldsymbol{C}=\left(C_{ij}\right)_{N\times N}$ when ε takes on a particular value.

According to Assumption 1, a trust consensus evolution multiplex network $\boldsymbol{MG}$ can be built based on $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$. Except the intra-layer relationships both in $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$, the trust network $\boldsymbol{G}_A$ can affect the CEN $\boldsymbol{G}_B$ when experts negotiate with each other to reach a consensus. This effect can be considered as a direct relationship between layers. However, the trust network $\boldsymbol{G}_A$ will also change after the consensus adjustment in the CEN $\boldsymbol{G}_B$. This effect can be considered as an indirect relationship between layers since it is not that the consensus relations but the negotiation process directly affects the trust relationships.

Definition 9. The trust consensus evolution multiplex network $\boldsymbol{MG}=\left(\boldsymbol{G}_A,\boldsymbol{G}_B,\boldsymbol{E}_{AB},\boldsymbol{W}_{AB},\boldsymbol{ME}_{BA}\right)$ is defined based on the trust network $\boldsymbol{G}_A=\left(\boldsymbol{D},\boldsymbol{E}_A,\boldsymbol{T}\right)$ and the CEN $\boldsymbol{G}_B=\left(\boldsymbol{D},\boldsymbol{E}_B,\boldsymbol{C}\right)$ with the same set of experts $\boldsymbol{D}$ in both layers, $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ denotes the relationship between experts in the layer $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$, respectively, $\boldsymbol{E}_{AB}$ denotes the direct impact of the layer $\boldsymbol{G}_A$ on the layer $\boldsymbol{G}_B$ with the degree $\boldsymbol{W}_{AB}$, and $\boldsymbol{ME}_{BA}$ reflects the indirect influence of the layer $\boldsymbol{G}_B$ on the layer $\boldsymbol{G}_A$ since the adjustment occur in $\boldsymbol{G}_B$.

Fig.1 shows a simple example of $\boldsymbol{MG}$, where the solid lines in the layer $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$ means the in-layer connections, the solid lines from layer $\boldsymbol{G}_A$ to $\boldsymbol{G}_B$ denotes the direct impact of trust on consensus, and the dotted lines from $\boldsymbol{G}_B$ to $\boldsymbol{G}_A$ mean the indirect impact of consensus on trust.

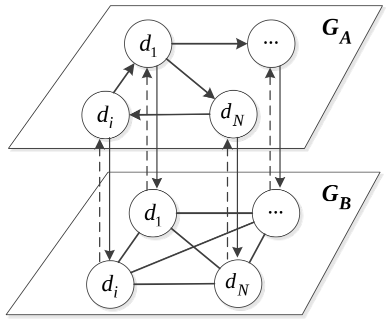

Fig. 1. An example of a multiplex network

The trust consensus evolution multiplex network $\boldsymbol{MG}$ is dynamic since the trust network $\boldsymbol{G}_A$ changes with the trust relationships development and the CEN $\boldsymbol{G}_B$ varies with the consensus evolution. The complicated relation among experts can be analyzed with the structure of trust consensus evolution multiplex networks.

B. Compute experts' influence using the PageRank centrality

Experts' influence plays an important role in their decision behavior. According to Assumption 2, experts' influence in trust consensus evolution multiplex networks can be determined with the combination of social influence and professional influence using the PageRank centrality.

Considering the effect of trust on consensus, the PageRank centrality of experts in the trust network $\boldsymbol{G}_A$ should be determined first. According to $\boldsymbol{T}=\left(T_{ij}\right)_{N\times N}$, the initial trust value $T_i^{(0)}\left(i\in N\right)$ of the expert d_i obtained from the adjacent experts d_k is determined:

$$T_i^{(0)}=\frac{1}{N\left(d_k\right)}\sum_{k\in N}T_{ki} \tag{6}$$

where $N(d_k)$ denotes the number of neighbors the expert d_i have except the neighbor d_j. Based on the initial trust value $T_i^{(0)}$, the initial centrality $v_i^{(0)}$ of d_i can be computed:

$$v_i^{(0)} = T_i^{(0)} \Big/ \sum_{i \in N} T_i^{(0)} \tag{7}$$

where $\sum_{i \in N} v_i^{(0)} = 1$.

According to the initial centrality $v_i^{(0)}$, the PageRank centrality $v_i^{(t)}$ of experts d_i in the layer of $\boldsymbol{G}_A$ at time t is computed based on (2):

$$v_i^{(t)} = \alpha_A \sum_j T_{ji} \frac{v_j^{(t)}}{g_j} + \left(1 - \alpha_A\right) v_i^{(0)} \tag{8}$$

where T_{ji} are elements of the weighted adjacency matrix $\boldsymbol{T}$, meaning that d_j trusts d_i; $g_j = \sum_u T_{ju} + \delta\left(0, \sum_u T_{ju}\right)$, $\delta\left(0, \sum_u T_{ju}\right)$ is the Kronecker delta; $\alpha_A > 0$ is the damping factor, which means that if the expert d_i is no longer trusted by the neighbor d_j, he/she may be trusted by other neighbors d_r; $\sum_u T_{ju}$ means the sum of trust values that d_j trusts other neighbors; $v_i^{(0)}$ is the initial centrality of the expert d_i, $v_j^{(t)}$ is the PageRank centrality of d_j at time t. The iteration ends when for some small η: $\sum_{i \in N} \left| v_i^{(t)} - v_i^{(t-1)} \right| < \eta$.

Based on the PageRank centrality of experts in the trust network $\boldsymbol{G}_A$, their PageRank centrality in the CEN $\boldsymbol{G}_B$ can be determined [41]. The combined PageRank reflects that the more importance an expert is in $\boldsymbol{G}_B$, the more influence the expert can extract from the connections received from important neighbors in $\boldsymbol{G}_A$, such influence is reflected not only in the importance change of experts over time but also in their initial centrality. According to the consensus matrix $\boldsymbol{C} = \left(C_{ij}\right)_{N \times N}$, the initial consensus level $C_i^{(0)} \left(i \in N\right)$ of the expert d_i between neighbor experts d_k is determined:

$$C_i^{(0)} = \frac{1}{N\left(d_k\right)} \sum_{k \in N} C_{ik} \tag{9}$$

where $N\left(d_k\right)$ denoting the number of neighbors d_k of the expert d_i except the neighbor d_j. Then, the normalized initial consensus level $\overline{C}_i^{(0)}$ can be computed:

$$\overline{C}_i^{(0)} = C_i^{(0)} \Big/ \sum_{i \in N} C_i^{(0)} \tag{10}$$

where $\sum_{i \in N} \overline{C}_i^{(0)} = 1$.

According to Assumption 2, the importance of experts in $\boldsymbol{G}_B$ is also affected by experts' importance in $\boldsymbol{G}_A$. Thus, the initial centrality $y_i^{(0)} \left(i \in N\right)$ of the expert d_i at $t = 0$ can be computed based on the combination of the normalized initial consensus level $\overline{C}_i^{(0)}$ and the PageRank centrality $v_i^{(t)}$ obtained in $\boldsymbol{G}_A$:

$$y_i^{(0)} = v_i^{(t)} \overline{C}_i^{(0)} \Big/ \left\langle v^{(t)} \right\rangle \tag{11}$$

where $\left\langle v^{(t)} \right\rangle = \frac{1}{N} \sum_{i \in N} v_i^{(t)} = \frac{1}{N}$ is the average PageRank centrality of experts in $\boldsymbol{G}_A$, and $\sum_{i \in N} y_i^{(0)} = 1$.

Finally, the PageRank centrality $y_i^{(t)} \left(i \in N\right)$ of the expert d_i in $\boldsymbol{G}_B$ is determined:

$$y_i^{(t)} = \alpha_A \sum_j v_i^{(t)} C_{ij} \frac{y_j^{(t)}}{g_j} + \left(1 - \alpha_A\right) y_i^{(0)} \tag{12}$$

where $v_i^{(t)}$ is the PageRank centrality of the expert d_i obtained in network $\boldsymbol{G}_A$, C_{ij} are the elements of the weighted adjacency matrix $\boldsymbol{C}$, $g_j = \sum_r C_{rj} v_r^{(t)} + \delta\left(0, \sum_r C_{rj} v_r^{(t)}\right)$, $\sum_r C_{jr}$ means the whole consensus levels between d_j and neighbors, $y_j^{(t)}$ is the PageRank centrality of d_j at time t, $\alpha_A > 0$ is the damping factor which suggests that if the expert d_i has no longer consensus with the neighbor d_j, he/she may construct the consensus with other neighbors d_r, $y_i^{(0)}$ is the initial centrality of the expert d_i at $t = 0$. The iteration ends when for some small η: $\sum_{i \in N} \left| y_i^{(t)} - y_i^{(t-1)} \right| < \eta$.

According to the PageRank centrality, the comprehensive influence $\boldsymbol{Y} = \left(y_1, y_2, ..., y_N\right)$ of all experts in the trust consensus evolution multiplex network can be computed:

$$y_i = y_i^{(t)} \Big/ \sum_{i \in N} y_i^{(t)} \tag{13}$$

where $\sum_{i \in N} y_i = 1$.

To determine the comprehensive influence for experts using the PageRank centrality in the trust consensus evolution multiplex network $\boldsymbol{MG}$ clearly, Algorithm 1 is given as follows.

Algorithm 1-Experts' influence determination

Inputs: The trust consensus evolution multiplex network $\boldsymbol{MG} = \left(\boldsymbol{G}_A, \boldsymbol{G}_B\right)$, the value of α_A, α_B and η

Phase 1: *Determine the PageRank centrality of experts in* $\boldsymbol{G}_A$

Step 1: Compute the initial centrality $v_i^{(0)}\left(i \in N\right)$ of experts based on (6) and (7).

Step 2: Determine the PageRank centrality $v_i^{(t)}$ of experts based on (8) until $\sum_{i\in N}\left|v_i^{(t)} - v_i^{(t-1)}\right| < \eta$.

Phase 2: *Determine the PageRank centrality of experts in* $\boldsymbol{G}_B$

Step 1: Compute the initial centrality $y_i^{(0)}$ of experts based on (9), (10), and (11) with the impact of the PageRank centrality $v_i^{(t)}$ in layer $\boldsymbol{G}_A$.

Step 2: Calculate the PageRank centrality $y_i^{(t)}$ of experts in layer $\boldsymbol{G}_B$ based on (12) till $\sum_{i\in N}\left|y_i^{(t)} - y_i^{(t-1)}\right| < \eta$.

Step 3: Determine experts comprehensive influence Y_i in $\boldsymbol{MG}$ based on (13).

Outputs: The comprehensive influence $\boldsymbol{Y} = \left(Y_1, Y_2, ..., Y_N\right)$

Since the influence of experts obtained using Algorithm 1 considers the PageRank centrality of experts in two layers, it will play a critical role to measure the consensus evolution and trust relationships development in $\boldsymbol{MG}$.

C. The consensus reaching process in trust consensus evolution multiplex networks

According to Assumption 3, trust relationships can promote the consensus reaching of groups. But, there may be a conflict between experts when they have a low trust degree and a high consensus level. Thus, trust relationships sometimes may be obstacles for consensus reaching. In this section, the consensus evolution in trust consensus evolution multiplex networks is discussed under trust's positive and negative effects. These effects are considered to be the direct impacts of the layer $\boldsymbol{G}_A$ on the layer $\boldsymbol{G}_B$.

The effects of trust on consensus can be quantified via an impact factor. Since experts' influence obtained in $\boldsymbol{MG}$ can affect others' decisions, it can be considered to be the impact factor. For the trust relationship between experts, the more influence any expert has, the larger the impact of their trust on consensus, vice versa. Let $\boldsymbol{Y} \in [0,1]$ be the impact factor, which suggests that the consensus level between experts will be improved with value $\boldsymbol{Y}$ when one expert completely trusts another one. Thus, the influence $\boldsymbol{Y}$ can be regarded as the weights $\boldsymbol{W}_{AB}$ of the direct relations $\boldsymbol{E}_{AB}$. The consensus relations in $\boldsymbol{G}_B$ will change according to the impact of trust relationships $\boldsymbol{E}_{AB}$ with the weights $\boldsymbol{Y}$.

The effects of trust on consensus is codetermined by the trust degrees $\boldsymbol{T}$, the consensus levels $\boldsymbol{C}$ and the influence $\boldsymbol{Y}$. We propose a function $Q\left(\boldsymbol{C}, \boldsymbol{T}, \boldsymbol{Y}\right)$ to evaluate the modified consensus matrix $\boldsymbol{C}' = Q\left(\boldsymbol{C}, \boldsymbol{T}, \boldsymbol{Y}\right)$ for the CEN $\boldsymbol{G}_B$ under the influence of the trust network $\boldsymbol{G}_A$:

$$q\left(C_{ij}, T_{ij}, Y_j\right) = \frac{C_{ij} + Y_j T_{ij}}{\max\left(C_{ij} + Y_j, 1\right)} \tag{14}$$

where C_{ij} is the initial consensus level between the expert d_i and d_j, Y_j denotes the influence of the expert d_j over d_i, the modified consensus level is $C'_{ij} = q\left(C_{ij}, T_{ij}, Y_j\right)$.

Since $C_{ij} \in [0,1]$, $T_{ij} \in [0,1]$, and $Y_j \in [0,1]$, $C_{ij} + Y_j T_{ij} \le C_{ij} + Y_j$. $0 \le C'_{ij} = C_{ij} + Y_j T_{ij} \le 1$ when $C_{ij} + Y_j \le 1$. $0 \le C'_{ij} = \left(C_{ij} + Y_j T_{ij}\right) \big/ \left(C_{ij} + Y_j\right) \le 1$ when $C_{ij} + Y_j > 1$. Thus, $C'_{ij} \in [0,1]$.

When the impact factor $Y_j = 0$, there is no effect of trust on consensus. To discuss the effects of trust networks, we mainly consider the situation of $Y_j \ne 0$ in the rest of this part. It is obvious that (14) satisfies the following four properties when C_{ij} and T_{ij} are equals to 0 or 1.

Property 1. $C'_{ij} = q\left(0, 1, Y_j\right) = Y_j$, which shows the positive effect of trust on consensus.

Property 2. $C'_{ij} = q\left(1, 1, Y_j\right) = 1$, which shows the non-negative effect of trust on consensus.

Property 3. $C'_{ij} = q\left(0, 0, Y_j\right) = 0$, which shows the non-positive effect of trust on consensus.

Property 4. $C'_{ij} = q\left(1, 0, Y_j\right) = 1 \big/ \left(1 + Y_j\right)$, which shows the negative effect of trust on consensus.

According to the above properties, we can find that there is no conflict between a pair of experts when the consensus level is low enough, and trust will promote the consensus no matter what the trust degree is. When the consensus level is high enough, the high degree of trust can promote or even consolidate consensus. Whereas, the low trust degree may harm the high consensus level. We conclude three rules to judge the positive and negative effect of trust as follows.

Rule 1. Trust has a positive effect on consensus when $C_{ij} + Y_j < 1$ and $T_{ij} > 0$ or when $C_{ij} + Y_j > 1$ and the trust value T_{ij} satisfies:

$$T_{ij} > C_{ij}\left(C_{ij} + Y_j - 1\right) \big/ Y_j \tag{15}$$

Rule 2. Trust has a negative effect on consensus when $C_{ij} + Y_j > 1$ and the trust value T_{ij} satisfies:

$$T_{ij} < C_{ij}\left(C_{ij} + Y_j - 1\right) \big/ Y_j \tag{16}$$

Rule 3. Trust has no effect on consensus when the trust value T_{ij} satisfies:

$$T_{ij} = C_{ij}\left(C_{ij} + Y_j - 1\right) \big/ Y_j \tag{17}$$

When trust negatively affects consensus, then we can deduce that there is a conflict between experts. Hence, **Rule 2** is also a conflict judgment condition.

The evolved consensus matrix $\boldsymbol{C}' = \left(C'_{ij}\right)_{N\times N}$ might be asymmetric since the initial consensus matrix $\boldsymbol{C}$ is symmetric and the trust matrix $\boldsymbol{T}$ is often asymmetric. Thus, there may be some deviation between experts' updated consensus levels, i.e. $q\left(C_{ij}, T_{ij}, Y_j\right) \neq q\left(C_{ji}, T_{ji}, Y_j\right)$. To deal with this situation, a symmetric consensus matrix $\boldsymbol{MC} = \left(MC_{ij}\right)_{N\times N}$ is obtained as:

$$MC_{ij} = \frac{q\left(C_{ij}, T_{ij}, Y_j\right) + q\left(C_{ji}, T_{ji}, Y_j\right)}{2} \tag{18}$$

Algorithm 2 is given to show the direct impact of trust to consensus with experts' influence clearly.

Algorithm 2-The consensus evolution

Inputs: The initial trust consensus evolution multiplex network $\boldsymbol{MG}^{(0)} = \left(\boldsymbol{G}_A^{(0)}, \boldsymbol{G}_B^{(0)}\right)$ and the initial influence $\boldsymbol{Y}^{(0)}$

Step 1: According to the initial trust matrix $\boldsymbol{T}^{(0)}$, the initial consensus matrix $\boldsymbol{C}^{(0)}$, and the influence $\boldsymbol{Y}^{(0)}$ in $\boldsymbol{MG}^{(0)}$, determine the evolved consensus matrix $\boldsymbol{C}'$ based on (14).

Step 2: Determine the normalized evolved consensus matrix $\boldsymbol{MC}^{(1)}$ based on $\boldsymbol{C}'$ by (18). Then obtain the evolved CEN $\boldsymbol{G}_B^{(1)}$ based on $\boldsymbol{MC}^{(1)}$.

Outputs: The evolved CEN $\boldsymbol{G}_B^{(1)}$ after the first round of negotiation

D. Evaluating the evolution of trust relationships in trust consensus evolution multiplex networks

In general, an agreement may be reached through several rounds of discussion. According to Assumption 3, trust relationships can develop during interactions among experts. This development is caused by the indirect influence of the CEN layer on the trust layer.

Propagation is one of the most important trust properties and can be multipath in complex trust networks. The change of trust relationships is mainly caused by the propagation path other than direct paths. However, it is challenging to consider all the propagation paths in complicated trust networks. Besides, the longer the path, the weaker the trust degree is transferred due to information diminishing [46]. Thus, we intend to compute the transitive trust for experts based on the shortest propagation path.

Suppose we would like to obtain the degree that d_i trusts d_k, then d_i is named a trustee and d_k is a truster. Let $P_{ik} = \left(p_{ik}\right)$ be the path set from d_i to d_k with the length set $L_{ik} = \left(l_{ik}\right)$. p_{ik} denotes the direct path from d_i to d_k when $l_{ik} = 1$ and indirect path when $l_{ik} > 1$. A condition for identifying the shortest indirect path from d_i and d_k is given.

Condition 1. Determine the shortest trust path p_{ik} from d_i to d_k with its length $l_{ik} = \min\left(L_{ik}\right)\left(L_{ik} > 1\right)$.

Suppose $d_i \to d_j \to d_k$ is one of the shortest indirect path between d_i and d_k, the transitive trust PT_{ik} can be computed based on the algebraic t-norm operator [47]:

$$PT_{ik} = T_{ij} \times T_{jk} \tag{19}$$

The transitive trust PT_{ik} is considered as the gains of the directed trust value T_{ik}. Trusters' influence can also affect the transitive trust degree to trustees. The larger the influence of the truster, the more the trustee trusts the truster. Thus, the varying degree of d_i trusts d_k in the rth round of adjustment can be computed as:

$$T_{ik}^{(r)} = T_{ik}^{(r-1)} + \frac{1}{N\left(p_{ik}^{(r-1)}\right)} \sum_{p_{ik}^{(r-1)} \in P_{ik}^{(r-1)}} \min\left(Y_{ik}^{(r-1)}\right) PT_{ik}^{(r-1)} \tag{20}$$

where $PT_{ik}^{(r-1)}$ is the transitive trust value in the shortest path $p_{ik}^{(r-1)}$ after $r-1$ rounds of adjustment, $N\left(p_{ik}^{(r-1)}\right)$ denotes the number of the shortest paths, $\min\left(Y_{ik}^{(r-1)}\right)$ is the minimum influence of trusters to trustees in the path $p_{ik}^{(r-1)}$.

We propose Algorithm 3 to introduce the CEN layer's indirect impact to the trust layer based on trust propagation during the consensus adjustment.

Algorithm 3-The trust relationships development

Inputs: The initial trust network $\boldsymbol{G}_A^{(0)} = \left(\boldsymbol{D}, \boldsymbol{E}_A^{(0)}, \boldsymbol{T}^{(0)}\right)$ and the influence $\boldsymbol{Y}^{(0)}$

Step 1: Randomly select two experts $\left(d_i, d_j\right)$ from $\boldsymbol{G}_A^{(0)}$ as the source and the sink nodes. Search all the paths from d_i to d_j and select the shortest ones according to Condition 1.

Step 2: Compute the transitive value $PT_{ik}^{(0)}$ using (19) to obtain $T_{ik}^{(1)} = PT_{ik}^{(0)}$.

Step 3: Take $PT_{ik}^{(0)}$ as the increase that d_i trusts d_j and compute the total value $T_{ik}^{(1)}$ that d_i trusts d_j using (20) with the initial influence $\boldsymbol{Y}^{(0)}$ of experts.

Step 4: Repeat Step1 to Step 3 till the trust degrees in $\boldsymbol{G}_A^{(0)}$ are stable and obtain $\boldsymbol{G}_A^{(1)} = \left(\boldsymbol{D}, \boldsymbol{E}_A^{(1)}, \boldsymbol{T}^{(1)}\right)$.

Outputs: The evolved trust network $\boldsymbol{G}_A^{(1)}$ after the first round of adjustment

E. The overall consensus measure in trust consensus evolution multiplex networks

Consensus measurement is fundamental to judge the consensus situation of the group. The edges in the CEN denote the consensus levels between two experts. We can then measure consensus based on the density and intensity of connections in the CEN. We extend the density in (1) into trust consensus evolution multiplex networks to compute the overall consensus level (OCL) based on the consensus matrix $\boldsymbol{MC}$:

$$OCL = d\left(\boldsymbol{MG}\right) = \frac{1}{N\left(N-1\right)} \sum_{i,j\in N, i\neq j} MC_{ij} \tag{21}$$

which can be transformed based on (18):

$$OCL = \frac{\sum_{i,j\in N, i\neq j} \left[q\left(C_{ij}, T_{ij}, Y_j\right) + q\left(C_{ji}, T_{ji}, Y_j\right)\right]}{2N\left(N-1\right)} \tag{22}$$

According to the effects of trust on consensus, the OCL in several special situations is introduced.

(1) When $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(1,1,1\right)$, $i, j \in N$, all the network relationships in $\boldsymbol{MG}$ have the highest density and intensity, then all experts reach a complete agreement, i.e., $OCL = 1$.

(2) When $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(0,1,1\right)$, $i, j \in N$, all the trust relationships in $\boldsymbol{G}_A$ have the highest density and intensity, while all the consensus relations in $\boldsymbol{G}_B$ have the lowest density and intensity, then the trust relationships positively affect the consensus relations, i.e., $OCL = \frac{1}{N\left(N-1\right)} \sum_{i,j\in N, i\neq j} Y_j$.

(3) When $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(0,1,0\right)$ or $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(0,0,1\right)$, $i, j \in N$, half of the trust relationships in $\boldsymbol{G}_A$ have the highest density and intensity, while all the consensus relationships in $\boldsymbol{G}_B$ have the lowest density and intensity, then such half trust relationships positively affect the consensus relations, i.e., $OCL = \frac{1}{2N\left(N-1\right)} \sum_{i,j\in N, i\neq j} Y_j$.

(4) When $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(1,1,0\right)$ or $\left(C_{ij}, T_{ij}, T_{ji}\right) = \left(1,0,1\right)$, $i, j \in N$, all the consensus relationships in $\boldsymbol{G}_B$ have the lowest density and intensity, while half of the trust relationships in $\boldsymbol{G}_A$ have the highest density and intensity, then such half trust relationships negatively affect on consensus, i.e.,
$OCL = \frac{1}{2N\left(N-1\right)} \sum_{i,j\in N, i\neq j} \left(\frac{C_{ij}}{C_{ij}+Y_j} + 1\right)$.

Especially, when the trust effect is not considered, i.e. $\boldsymbol{T} = \boldsymbol{Y} = 0$, then (22) degenerates into

$$OCL = \frac{1}{N\left(N-1\right)} \sum_{i,j\in N, i\neq j} C_{ij} \tag{23}$$

where $OCL = d\left(\boldsymbol{G}_B\right)$ denotes the intensity of $\boldsymbol{G}_B$, which is consistent with the conventional consensus measurement.

F. The computational process of consensus reaching in trust consensus evolution multiplex networks

The proposed consensus model's critical techniques, including the construction of trust consensus evolution multiplex networks, experts' influence, the consensus evolution, the trust development, and the overall consensus measurement, have been introduced. Next, the framework of the proposed model is shown in Fig.2.

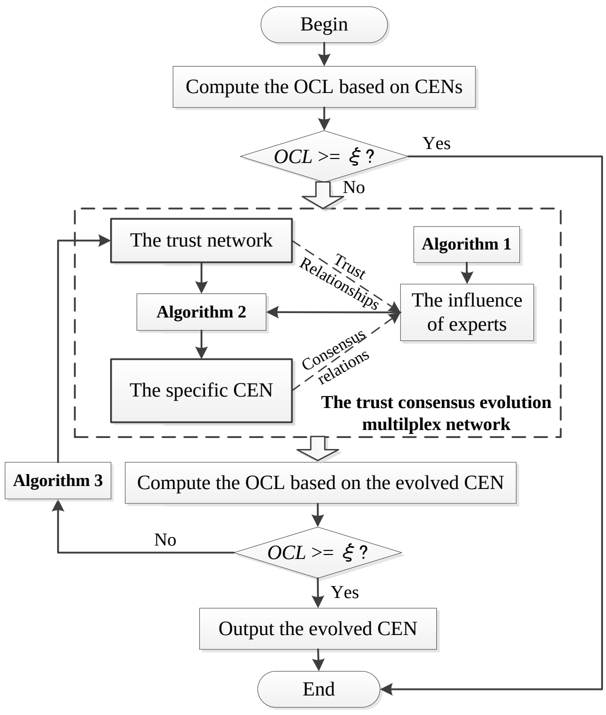

Fig. 2. The framework of the consensus model based on trust consensus evolution multiplex networks

According to Fig.2, the main procedures of the proposed model are given as follows.

Step 1. For an GDM problem consists of N experts $\boldsymbol{D} = \left\{d_1, d_2, ..., d_N\right\}$ and M alternatives $\boldsymbol{X} = \left\{x_1, x_2, ..., x_M\right\}$, gather the preference relations of experts and determine an initial CEN $\boldsymbol{G}_B^{(0)}$ based on the consensus matrix $\boldsymbol{C}^{(0)}$ with a specific consensus threshold ε, where $\boldsymbol{C}^{(0)}$ is computed using (3) and (4). Besides, build the initial trust network $\boldsymbol{G}_A^{(0)}$ for experts according to their trust relationships.

Step 2. We measure the initial $OCL^{(0)}$ for the GDM based on the initial CEN $\boldsymbol{G}_B^{(0)}$ using (23). Then, compare $OCL^{(0)}$ with an agreed consensus threshold ξ, if $OCL^{(0)} \geq \xi$, then the group has reached to the agreed consensus and the consensus reaching process is end. Otherwise, go to the next step.

Step 3. Construct the initial trust consensus evolution multiplex network $\boldsymbol{MG}^{(0)}$ based on the initial trust network $\boldsymbol{G}_A^{(0)}$ with the trust matrix $\boldsymbol{T}^{(0)}$ and the initial CEN $\boldsymbol{G}_B^{(0)}$ with the consensus matrix $\boldsymbol{C}^{(0)}$. According to Algorithm 1, compute the initial influence $\boldsymbol{Y}^{(0)}$ of experts in $\boldsymbol{MG}^{(0)}$.

Step 4. During the first round of adjustment, considering the effects of trust, obtain the modified CEN $\boldsymbol{G}_B^{(1)}$ by determining the normalized consensus matrix $\boldsymbol{MC}^1$ based on $\boldsymbol{T}^{(0)}$ and $\boldsymbol{Y}^{(0)}$ using Algorithm 2. Compute $OCL^{(1)}$ based on the evolved CEN $\boldsymbol{G}_B^{(1)}$ using (21). If $OCL^{(1)} \geq \xi$, stop the consensus reaching process and output the evolved CEN $\boldsymbol{G}_B^{(1)}$. Otherwise, go to the next step.

Step 5. Since the trust relationships may change during the interactions among experts, obtain the modified trust network $\boldsymbol{G}_A^{(1)}$ by computing the trust network matrix $\boldsymbol{T}^{(1)}$ using Algorithm 3.

Step 6. Repeat from Step 3 to Step 5 until the agreed consensus is reached. Supposing that the agreed consensus is achieved after R rounds of adjustment, the modified consensus matrix in the iteration process is obtained:

$$\boldsymbol{C}'^{(R)} = Q\left(\boldsymbol{MC}^{(R-1)}, \boldsymbol{T}^{(R-1)}, \boldsymbol{Y}\right) \tag{24}$$

where $\boldsymbol{MC}^{(0)} = \boldsymbol{C}^{(0)}$ and the normalized consensus matrix $\boldsymbol{MC}^{(R)}$ is obtained using (18).

IV. Numerical Example

A numerical example given in [17] is used to examine the proposed consensus model. In the original example, eight experts $\boldsymbol{D} = \{d_1, d_2, ..., d_8\}$ evaluated six alternatives $\boldsymbol{X} = \{x_1, x_2, ..., x_6\}$ with FPRs $F_i\,(i = 1, ..., 8)$.

$$F_1 = \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.9 & 0.7 & 0.8 \\ 0.6 & 0.5 & 0.7 & 1 & 0.8 & 0.9 \\ 0.4 & 0.3 & 0.5 & 0.8 & 0.6 & 0.7 \\ 0.1 & 0 & 0.2 & 0.5 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.4 & 0.7 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0.3 & 0.6 & 0.4 & 0.5 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0.5 & 0.7 & 0.8 & 0.6 & 1 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.8 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.3 & 0.7 & 0.6 \\ 0.4 & 0.6 & 0.7 & 0.5 & 0.9 & 0.8 \\ 0 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 & 0.2 & 0.6 & 0.5 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0.5 & 0.69 & 0.12 & 0.2 & 0.36 & 0.9 \\ 0.31 & 0.5 & 0.06 & 0.1 & 0.2 & 0.8 \\ 0.88 & 0.94 & 0.5 & 0.64 & 0.8 & 0.98 \\ 0.8 & 0.9 & 0.36 & 0.5 & 0.69 & 0.97 \\ 0.64 & 0.8 & 0.2 & 0.31 & 0.5 & 0.94 \\ 0.1 & 0.2 & 0.02 & 0.03 & 0.06 & 0.5 \end{bmatrix} \quad F_4 = \begin{bmatrix} 0.5 & 0.1 & 0.36 & 0.69 & 0.16 & 0.26 \\ 0.9 & 0.5 & 0.84 & 0.95 & 0.62 & 0.76 \\ 0.64 & 0.16 & 0.5 & 0.8 & 0.25 & 0.39 \\ 0.31 & 0.05 & 0.2 & 0.5 & 0.08 & 0.14 \\ 0.84 & 0.38 & 0.75 & 0.92 & 0.5 & 0.66 \\ 0.74 & 0.24 & 0.61 & 0.86 & 0.34 & 0.5 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} 0.5 & 0.55 & 0.45 & 0.25 & 0.7 & 0.3 \\ 0.45 & 0.5 & 0.7 & 0.85 & 0.4 & 0.8 \\ 0.55 & 0.3 & 0.5 & 0.65 & 0.7 & 0.6 \\ 0.75 & 0.15 & 0.35 & 0.5 & 0.95 & 0.6 \\ 0.3 & 0.6 & 0.3 & 0.05 & 0.5 & 0.85 \\ 0.7 & 0.2 & 0.4 & 0.4 & 0.15 & 0.5 \end{bmatrix} \quad F_6 = \begin{bmatrix} 0.5 & 0.7 & 0.75 & 0.95 & 0.6 & 0.85 \\ 0.3 & 0.5 & 0.55 & 0.8 & 0.4 & 0.65 \\ 0.25 & 0.45 & 0.5 & 0.7 & 0.6 & 0.45 \\ 0.05 & 0.2 & 0.3 & 0.5 & 0.85 & 0.4 \\ 0.4 & 0.6 & 0.4 & 0.15 & 0.5 & 0.75 \\ 0.15 & 0.35 & 0.55 & 0.6 & 0.25 & 0.5 \end{bmatrix}$$

$$F_7 = \begin{bmatrix} 0.5 & 0.34 & 0.25 & 0.82 & 0.75 & 0.87 \\ 0.66 & 0.5 & 0.25 & 0.18 & 0.82 & 0.91 \\ 0.75 & 0.75 & 0.5 & 0.94 & 0.91 & 1 \\ 0.18 & 0.82 & 0.06 & 0.5 & 0.34 & 0.75 \\ 0.25 & 0.18 & 0.09 & 0.66 & 0.5 & 0.82 \\ 0.13 & 0.09 & 0 & 0.25 & 0.18 & 0.5 \end{bmatrix} \quad F_8 = \begin{bmatrix} 0.5 & 0.13 & 0.18 & 0.34 & 0.75 & 0.09 \\ 0.87 & 0.5 & 0.66 & 0.82 & 0.91 & 0.25 \\ 0.82 & 0.34 & 0.25 & 0.75 & 0.87 & 0.82 \\ 0.66 & 0.18 & 0.25 & 0.5 & 0.75 & 0.91 \\ 0.25 & 0.09 & 0.13 & 0.25 & 0.5 & 0.97 \\ 0.91 & 0.75 & 0.18 & 0.09 & 0.03 & 0.5 \end{bmatrix}$$

Suppose there are trust relationships between experts and the initial trust network $\boldsymbol{G}_A^{(0)}$ is shown in Fig.3 with the initial weighted adjacency matrix $\boldsymbol{T}^{(0)} = \left(T_{ij}^{(0)}\right)_{8\times 8}$ of $\boldsymbol{G}_A^{(0)}$.

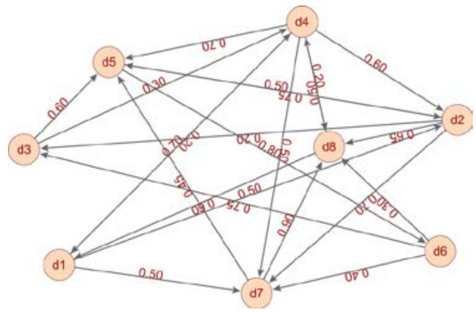

Fig.3. The structure of the trust network $\boldsymbol{G}_A^{(0)}$

$$\boldsymbol{T}^{(0)} = \begin{bmatrix} 0 & 0.5 & 0 & 0.7 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 & 0.75 & 0 & 0.7 & 0.65 \\ 0 & 0 & 0 & 0.3 & 0.6 & 0 & 0 & 0 \\ 0.2 & 0.6 & 0 & 0 & 0.7 & 0 & 0.5 & 0.2 \\ 0 & 0.5 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.9 \\ 0.6 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}_{8\times 8}$$

Step 1. With the FPRs, construct the initial consensus matrix $\boldsymbol{C}^{(0)}$ based on (3) and (4):

$$\boldsymbol{C}^{(0)} = \begin{bmatrix} 0 & 0.733 & 0.594 & 0.763 & 0.763 & 0.820 & 0.781 & 0.682 \\ 0.733 & 0 & 0.666 & 0.564 & 0.743 & 0.786 & 0.700 & 0.651 \\ 0.594 & 0.666 & 0 & 0.515 & 0.716 & 0.671 & 0.757 & 0.664 \\ 0.763 & 0.564 & 0.515 & 0 & 0.707 & 0.688 & 0.611 & 0.647 \\ 0.763 & 0.743 & 0.716 & 0.707 & 0 & 0.810 & 0.670 & 0.780 \\ 0.820 & 0.786 & 0.671 & 0.688 & 0.810 & 0 & 0.680 & 0.651 \\ 0.781 & 0.700 & 0.757 & 0.611 & 0.670 & 0.680 & 0 & 0.702 \\ 0.682 & 0.651 & 0.664 & 0.647 & 0.780 & 0.651 & 0.702 & 0 \end{bmatrix}_{8\times 8}$$

According to **Definition 8**, the CENs can be built in different formats with the consensus thresholds ε. We take the complete CEN $\boldsymbol{G}_B = \left(\boldsymbol{G}_B^{\varepsilon=0.515}\right)$ which is shown in Fig.4 as an example to illustrate the proposed model. Besides, let $\boldsymbol{C}^{(0)}$ denote the weighted adjacency matrix of $\boldsymbol{G}_B^{(0)}$.

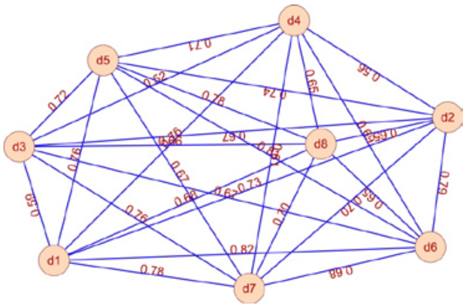

Fig. 4. The structure of the complete CEN $\boldsymbol{G}_B^{(0)}$

Step 2. Compute the initial $OCL^{(0)} = 0.697$ using (23). Experts should negotiate with each other if we assume that the consensus among groups is at least 0.75.

Step 3. Referring to **Definition 8**, the initial trust consensus evolution multiplex network $\boldsymbol{MG}^{(0)} = \left(\boldsymbol{G}_A^{(0)}, \boldsymbol{G}_B^{(0)}\right)$ is built based on $\boldsymbol{G}_A^{(0)}$ and $\boldsymbol{G}_B^{(0)}$ in Fig.5, where $\boldsymbol{G}_A^{0}$ has direct impact on $\boldsymbol{G}_B^{(0)}$ and $\boldsymbol{G}_B^{(0)}$ has indirect impact on $\boldsymbol{G}_A^{(0)}$.

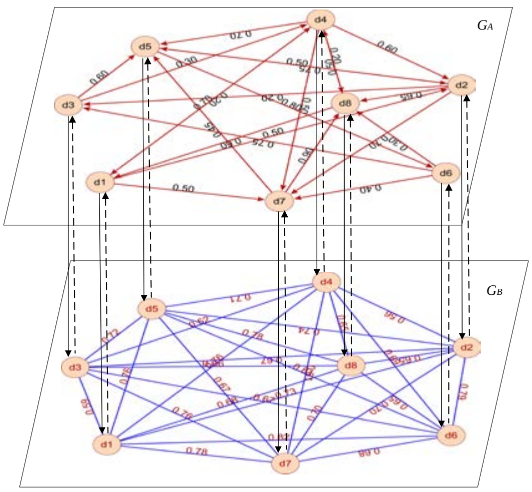

Fig.5. The trust consensus evolution multiplex network $\boldsymbol{MG}^{(0)}$

Let the damping factor α_A and α_B in Algorithm 1 be equal to 0.85 and $\eta = 10^{-6}$. The initial influence $\boldsymbol{Y}^{(0)} = (0.127, 0.124, 0.111, 0.116, 0.140, 0.130, 0.126, 0.126)$ of experts in $\boldsymbol{MG}^{(0)}$ is determined based on the PageRank centrality in $\boldsymbol{G}_A^{(0)}$ and $\boldsymbol{G}_B^{(0)}$ respectively obtained after 76 and 34 rounds.

Step 4. Depending on Algorithm 2, determine the modified consensus matrix $\boldsymbol{MG}^{(1)}$ based on $\boldsymbol{T}^{(0)}$ and $\boldsymbol{Y}^{(0)}$ during the first round of adjustment, and obtain the modified $OCL^{(1)} = 0.725$.

$$\boldsymbol{MC}^{(1)} = \begin{bmatrix} 0 & 0.764 & 0.594 & 0.817 & 0.763 & 0.820 & 0.813 & 0.720 \\ 0.764 & 0 & 0.677 & 0.601 & 0.826 & 0.786 & 0.744 & 0.692 \\ 0.594 & 0.677 & 0 & 0.533 & 0.757 & 0.713 & 0.757 & 0.664 \\ 0.817 & 0.601 & 0.533 & 0 & 0.755 & 0.688 & 0.643 & 0.689 \\ 0.763 & 0.826 & 0.757 & 0.755 & 0 & 0.862 & 0.701 & 0.780 \\ 0.820 & 0.786 & 0.713 & 0.688 & 0.862 & 0 & 0.705 & 0.670 \\ 0.813 & 0.744 & 0.757 & 0.643 & 0.701 & 0.705 & 0 & 0.759 \\ 0.720 & 0.692 & 0.664 & 0.689 & 0.780 & 0.670 & 0.759 & 0 \end{bmatrix}_{8\times 8}$$

Step 5. Since $OCL^{(1)} \le 0.75$, the negotiation process should be carried on. Referring to Algorithm 3, determine the modified trust matrix $\boldsymbol{T}^{(1)}$ after the first round of adjustment.

$$\boldsymbol{T}^{(1)} = \begin{bmatrix} - & 0.549 & 0.011 & 0.703 & 0.028 & 0.023 & 0.543 & 0.016 \\ 0.001 & - & 0.250 & 0.007 & 0.763 & 0.078 & 0.703 & 0.651 \\ 0.007 & 0.021 & - & 0.308 & 0.624 & 0.062 & 0.017 & 0.007 \\ 0.215 & 0.643 & 0.013 & - & 0.728 & 0.072 & 0.552 & 0.248 \\ 0.018 & 0.508 & 0.011 & 0.003 & - & 0.800 & 0.043 & 0.030 \\ 0.023 & 0.01 & 0.752 & 0.017 & 0.023 & - & 0.409 & 0.345 \\ 0.068 & 0.028 & 0.005 & 0.052 & 0.487 & 0.047 & - & 0.914 \\ 0.600 & 0.035 & 0.007 & 0.549 & 0.041 & 0.033 & 0.029 & - \end{bmatrix}_{8\times 8}$$

Step 6. Repeat Step 3 to Step 5 till $OCL^{(3)} = 0.786$ after three iterations. The final revised trust consensus evolution multiplex network $\boldsymbol{MG}^{(3)} = \left(\boldsymbol{G}_A^{(3)}, \boldsymbol{G}_B^{(3)}\right)$ with the consensus matrix $\boldsymbol{MG}^{(3)}$ and the trust matrix $\boldsymbol{T}^{(3)}$ is shown in Fig.6.

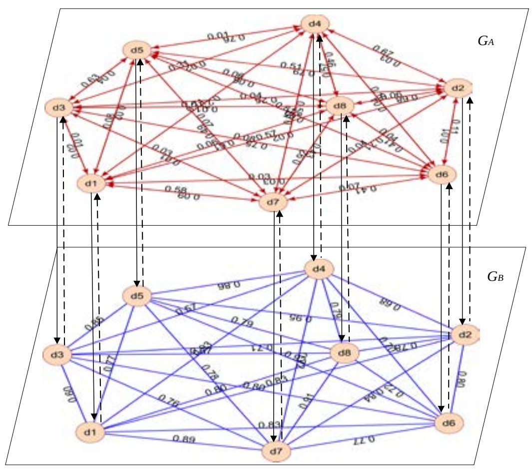

Fig.6. The trust consensus evolution multiplex network $\boldsymbol{MC}^{(3)}$

$$\boldsymbol{MC}^{(3)} = \begin{bmatrix} 0 & 0.834 & 0.596 & 0.926 & 0.771 & 0.826 & 0.893 & 0.798 \\ 0.834 & 0 & 0.708 & 0.684 & 0.955 & 0.798 & 0.839 & 0.782 \\ 0.596 & 0.708 & 0 & 0.570 & 0.847 & 0.803 & 0.760 & 0.666 \\ 0.926 & 0.684 & 0.570 & 0 & 0.858 & 0.701 & 0.721 & 0.792 \\ 0.771 & 0.955 & 0.847 & 0.858 & 0 & \mathbf{0.924} & 0.776 & 0.791 \\ 0.826 & 0.798 & 0.803 & 0.701 & \mathbf{0.924} & 0 & 0.765 & 0.720 \\ 0.893 & 0.839 & 0.760 & 0.721 & 0.776 & 0.765 & 0 & 0.906 \\ 0.798 & 0.782 & 0.666 & 0.792 & 0.791 & 0.720 & 0.906 & 0 \end{bmatrix}_{8\times 8}$$

$$\boldsymbol{T}^{(3)}=\begin{bmatrix} 0 & 0.568 & 0.017 & 0.705 & 0.078 & 0.030 & 0.575 & 0.061 \\ 0.020 & 0 & 0.253 & 0.026 & 0.785 & 0.107 & 0.713 & 0.681 \\ 0.010 & 0.042 & 0 & 0.309 & 0.634 & 0.084 & 0.027 & 0.014 \\ 0.225 & 0.665 & 0.022 & 0 & 0.762 & 0.101 & 0.583 & 0.457 \\ 0.020 & 0.509 & 0.037 & 0.005 & 0 & 0.802 & 0.073 & 0.058 \\ 0.034 & 0.014 & 0.752 & 0.035 & 0.050 & 0 & 0.415 & 0.406 \\ 0.092 & 0.043 & 0.007 & 0.074 & 0.492 & 0.065 & 0 & 0.917 \\ 0.605 & 0.064 & 0.009 & 0.566 & 0.064 & 0.038 & 0.432 & 0 \end{bmatrix}_{8\times 8}$$

The comprehensive influence of experts in $\boldsymbol{MC}^{(3)}$ is obtained to be $\boldsymbol{Y}^{(3)}=\left(0.123, 0.124, 0.103, 0.121, 0.136, 0.121, 0.133, 0.139\right)$.

V. Comparative Analysis

The comparative analysis is given to highlight the advantages of this study from the following three points: the determination of experts' influence, the trust effect on consensus, and the consensus measure.

(1) The determination of experts' influence

In conventional consensus models, the experts' influence is mainly determined based on the consensus levels or trust degrees. In this article, we compute experts' influence by the interactions between the consensus evolution and trust development based on multiplex networks' structure.

The related indicators used to compute experts' influence in the initial trust consensus evolution multiplex network $\boldsymbol{MG}^{(0)}$ are shown in Table 1, including the initial and final PageRank centrality $v_i^{(0)}$ and $v_i^{(76)}$ in $\boldsymbol{G}_A^{(0)}$, the normalized consensus levels $\overline{C}_i^{(0)}$ in $\boldsymbol{G}_B^{(0)}$, the initial and final PageRank centrality $y_i^{(0)}$ and $y_i^{(34)}$ in $\boldsymbol{G}_B^{(0)}$, $i=1,\ldots,8$.

Table 1 The related indicators of experts' influence in $\boldsymbol{MG}^{(0)}$

Experts	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$v_i^{(0)}$	**0.065**	0.130	**0.077**	0.122	0.203	0.065	0.171	0.167
$v_i^{(76)}$	**0.107**	0.149	**0.086**	0.152	0.205	0.145	0.154	0.186
$\overline{C}_i^{(0)}$	**0.13**	**0.12**	0.12	0.12	0.13	0.13	0.13	0.12
$y_i^{(0)}$	**0.096**	**0.125**	0.068	0.118	0.184	0.128	0.131	0.153
$y_i^{(34)}$	**0.127**	**0.124**	0.111	0.116	0.140	0.131	0.126	0.127

According to the traditional method, the trust importance of d_i in $\boldsymbol{G}_A^{(0)}$ and the consensus importance of d_i in $\boldsymbol{G}_B^{(0)}$ can be computed be the initial PageRank centrality $v_i^{(0)}$ and $y_i^{(0)}$, respectively. In this study, we regard the final PageRank centrality $v_i^{(76)}$ and $y_i^{(34)}$ considering the trust development and consensus evolution as the trust and consensus importance of d_i, respectively. For example, $v_1^{(0)}<v_3^{(0)}$ while $v_1^{(76)}>v_3^{(76)}$ because the expert d_1 gains more trust than d_3 during the trust development. $y_1^{(0)}<y_2^{(0)}$ while $y_1^{(34)}>y_2^{(34)}$ because d_1 has more similar preference with other neighbors than d_2 during the consensus evolution. Besides, the normalized consensus levels $\overline{C}_i^{(0)}$ in $\boldsymbol{G}_B^{(0)}$, which is used to compute the initial PageRank centrality $y_i^{(0)}$, is obtained considering the final PageRank centrality $x_i^{(76)}$ in $\boldsymbol{G}_A^{(0)}$. For instance, $\overline{C}_1^{(0)}>\overline{C}_2^{(0)}$ while $y_1^{(0)}<y_2^{(0)}$ because d_1 is more important than d_2 in $\boldsymbol{G}_A^{(0)}$, i.e., $x_1^{(76)}<x_2^{(76)}$.

(2) The trust effect on consensus

The positive effect of trust on consensus has been discussed in previous studies. However, its negative effect is rarely considered when the relationship conflict exists. In this article, we investigate both the positive and negative effects of trust on the consensus evolution with experts' influence obtained using the PageRank centrality.

In the second round of adjustment, the symmetrical consensus matrix $\boldsymbol{MC}^{(2)}$, the trust network $\boldsymbol{T}^{(2)}$, and experts' influence $\boldsymbol{Y}^{(2)}=\left(0.123, 0.126, 0.107, 0.116, 0.141, 0.126, 0.132, 0.129\right)$ are obtained. Based on $\boldsymbol{MC}^{(2)}$, $\boldsymbol{T}^{(2)}$, and $\boldsymbol{Y}^{(2)}$, the consensus matrix $\boldsymbol{C}^{(3)}$ is determined using Algorithm 2.

$$\boldsymbol{MC}^{(2)}=\begin{bmatrix} 0 & 0.798 & 0.595 & 0.871 & 0.766 & 0.823 & 0.851 & 0.759 \\ 0.798 & 0 & 0.692 & 0.641 & 0.910 & 0.792 & 0.790 & 0.736 \\ 0.595 & 0.692 & 0 & 0.551 & 0.801 & 0.758 & 0.758 & 0.665 \\ 0.871 & 0.641 & 0.551 & 0 & 0.806 & 0.694 & 0.680 & 0.737 \\ 0.766 & 0.910 & 0.801 & 0.806 & 0 & \mathbf{0.915} & 0.737 & 0.785 \\ 0.823 & 0.792 & 0.758 & 0.694 & \mathbf{0.915} & 0 & 0.734 & 0.694 \\ 0.851 & 0.790 & 0.758 & 0.680 & 0.737 & 0.734 & 0 & 0.819 \\ 0.759 & 0.736 & 0.665 & 0.737 & 0.785 & 0.694 & 0.819 & 0 \end{bmatrix}_{8\times 8}$$

$$\boldsymbol{T}^{(2)}=\begin{bmatrix} 0 & 0.558 & 0.014 & 0.704 & 0.052 & 0.026 & 0.559 & 0.038 \\ 0.010 & 0 & 0.251 & 0.016 & 0.774 & 0.092 & 0.705 & 0.666 \\ 0.009 & 0.031 & 0 & 0.308 & 0.629 & 0.073 & 0.022 & 0.010 \\ 0.219 & 0.654 & 0.017 & 0 & 0.745 & 0.086 & 0.565 & 0.350 \\ 0.019 & 0.508 & 0.024 & 0.004 & 0 & \mathbf{0.801} & 0.058 & 0.044 \\ 0.028 & 0.011 & 0.752 & 0.026 & \mathbf{0.037} & 0 & 0.410 & 0.375 \\ 0.080 & 0.035 & 0.006 & 0.063 & 0.489 & 0.056 & 0 & 0.915 \\ 0.602 & 0.049 & 0.008 & 0.557 & 0.050 & 0.035 & 0.424 & 0 \end{bmatrix}_{8\times 8}$$

$$\boldsymbol{C}^{(3)}=\begin{bmatrix} 0 & 0.868 & 0.597 & 0.953 & 0.773 & 0.826 & 0.925 & 0.763 \\ 0.799 & 0 & 0.719 & 0.643 & 0.969 & 0.803 & 0.883 & 0.822 \\ 0.596 & 0.696 & 0 & 0.587 & 0.890 & 0.768 & 0.761 & 0.666 \\ 0.898 & 0.724 & 0.553 & 0 & 0.911 & 0.705 & 0.755 & 0.782 \\ 0.768 & 0.940 & 0.804 & 0.806 & 0 & \mathbf{0.976} & 0.745 & 0.790 \\ 0.826 & 0.793 & 0.839 & 0.697 & \mathbf{0.871} & 0 & 0.788 & 0.742 \\ 0.861 & 0.795 & 0.759 & 0.688 & 0.806 & 0.741 & 0 & 0.937 \\ 0.833 & 0.742 & 0.666 & 0.801 & 0.792 & 0.698 & 0.875 & 0 \end{bmatrix}_{8\times 8}$$

Most of the trust relationships in $\boldsymbol{T}^{(2)}$ have positive effect on consensus in $\boldsymbol{MC}^{(2)}$ except for the trust degree between d_5 and d_6. $T_{56}^{(2)}=0.801$ has a positive effect on consensus level $MC_{56}^{(2)}=0.915$ to obtain $C_{56}^{(3)}=0.976$ while $T_{65}^{(2)}=0.037$ has a negative effect on consensus level $MC_{65}^{(2)}=0.915$ to obtain $C_{65}^{(3)}=0.871$. However, the consensus level between d_5 and d_6 is modified to be $C_{56}^{(3)}=C_{65}^{(3)}=0.924$ according to the obtained $\boldsymbol{MC}^{(3)}$. Thus, the positive effect of $T_{56}^{(2)}=0.801$ on $MC_{56}^{(2)}=0.915$ outweighs the negative effect of $T_{65}^{(2)}=0.037$ on $MC_{65}^{(2)}=0.915$.

(3) The consensus measure

In traditional consensus models, the consensus level is mainly measured based on FRPs through three levels process. In this article, we measure consensus based on the density and intensity of trust consensus evolution networks.

The initial $OCL^{(0)}=0.697$ computed based on (23) without considering the effect of trust relations is consistent with the result obtained using the traditional method. In this study, the density of trust networks and the CEN is not aggregated using simple aggregators, but considering the positive and negative effects of trust in consensus. Such as, the OCL of the trust consensus evolution multiplex network $\boldsymbol{MC}^{(3)}$ is computed directly based on the modified consensus matrix $\boldsymbol{MC}^{(3)}$ as $OCL^{(3)}=0.786$. It is more intuitive to compute the evolved consensus from density and intensity than the conventional method.

VI. Conclusions

Since multiplex networks can uncover the interaction among multiple relationships in a complicated system, a consensus model for SNGDM was proposed based on trust consensus evolution multiplex networks considering the consensus evolution and the trust development.

According to trust consensus evolution multiplex networks, the complicated interactions between trust relationships and consensus relations are expressed more clearly. Experts' influence, which plays a vital role in the consensus evolution and trust development, is determined comprehensively using the PageRank centrality considering both the connections among experts in both layers. Based on assumptions, the consensus evolution and trust development are in a dynamic virtuous cycle until a satisfactory consensus level is reached. Especially, both the positive and negative effects of the trust on consensus are considered. Besides, the overall consensus level was measured more intuitively based on the density and intensity of trust consensus evolution multiplex networks. The proposed model was analyzed using an illustrative example and corresponding comparative analysis.

In summary, this study provides a new perspective to deal with the complicated consensus process combining multiple relationships in SNGDM based on the structure of multiplex networks. In response to the need of large-scale GDM, we will focus on the consensus evolution among a large number of experts and consider the application of other features of multiplex networks to this complicated situation, such as the community detection methods and the clustering coefficient.

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4 Balance dynamic clustering analysis and consensus reaching process with consensus evolution networks in large-scale group decision making

- T. Wu, X.W. Liu, J.D. Qin, F. Herrera, Balance Dynamic Clustering Analysis and Consensus Reaching Process with Consensus Evolution Networks in Large-scale Group Decision Making. **IEEE Transactions on Fuzzy Systems.** 29 (2021) 357-371.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 9.518
  - Subject Category: Computer Science, Computational Theory and Mathematics, Ranking 3/113 (Q1).
  - Subject Category: Computer Science, Artificial Intelligence, Ranking 7/189 (Q1).

Balance Dynamic Clustering Analysis and Consensus Reaching Process with Consensus Evolution Networks in Large-Scale Group Decision Making

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***Abstract*—Large-scale group decision making solution is usually based on the clustering analysis process (CAP) and consensus reaching process (CRP). However, CAP and CRP can be contradictory since CAP is performed based on the differences between potentially small groups and CRP is conducted to improve the overall similarity of a large group. To balance CAP and CRP, a dynamic clustering analysis process (DCAP) based on consensus evolution networks is proposed. A clustering algorithm proposed based on community detection method can be used to handle the diverse network structures with dynamic consensus thresholds. The clustering validity based on the intra-cluster consensus levels in subgroups and the inter-cluster consensus level among subgroups is evaluated. Then, the DCAP after each feedback adjustment round in CRP is reanalyzed. In such a way, effective clustering can also be found after a satisfying consensus is reached. Finally, a case study shows the availability of this approach and comparative analyses are provided to highlight the advantages from both theoretical and numerical perspectives.**

***Index Terms*—Consensus reaching process, Consensus evolution networks, Community detection, Dynamic clustering analysis, Large-scale group decision making**

I. Introduction

LARGE-scale group decision making (LSGDM) is a new branch in group decision making (GDM) research area with a multiple numbers of decision makers (DMs) involved [1].

This work was supported in part by the National Natural Science Foundation of China under Grant 71771051, 71701158, and 71371049, in part by the Scientific Research Foundation of Graduate School of Southeast University under Grant YBJJ1831, in part by the Ministry of Education in China Project of Humanities and Social Sciences under Grant 17YJC630114, and in part by the Scholarship from China Scholarship Council under Grant 201806090164.
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Driven by current societal and technological developments, LSGDM has become more and more popular [2-6]. The expertise and experience of a large number of DMs may create low consensus while providing diverse perspectives for decision making [7-11]. The consensus reaching process (CPR) plays an important part in helping large-scale DMs to reach consensus [11]. However, the interaction among DMs in CRP is much more complicated for LSGDM [12-15].

Preference relation is commonly used to express DMs' opinions in LSGDM [16]. Preference relation has diverse forms in current studies with different research priorities [3, 7, 8, 17]. To deal with the subjectivity of DM representation, Li et al. [7] proposed a consensus model for LSGDM with linguistic preference relation. Liu et al. [3] and Gou et al. [8] introduced CRP models for LSGDM with hesitant fuzzy preference relation. However, traditional fuzzy preference relations (FPRs) is the most commonly used form in LSGDM [11, 17, 18] since it is hard for DMs to maintain their expression consistency with the linguistic or hesitant FPRs throughout the complex interactions in CRP. Therefore, we will use the traditional FPRs to represent DM opinions in this study.

Based on various forms of preference relations, LSGDM models are mainly proposed from a clustering analysis process (CAP) perspective [19-22]. CAP is crucial when dealing with the complexity of LSGDM since it can classify individual DMs into several different small subgroups. The interactions among DMs in small groups are smoother because of their similar preferences, which is convenient for the decision manager when negotiating with intra-cluster DMs using similar strategies. Liu et al. [19] proposed an interval-valued intuitionistic fuzzy principal component analysis model to classify DMs. Xu et al. [20] designed a double clustering method based on distance and direction of preferences. Wu et al. [21] studied an interval type-2 fuzzy dynamic clustering analysis for LSGDM under uncertainty. Liu et al. [22] introduced a partial binary tree DEA-discriminant analysis cyclic classification model for the complex LSGDM. However, previous studies rarely focused on the consensus reaching of LSGDM.

More and more LSGDM models are being proposed from both a CAP and CRP perspective to study consensus reaching based on clustering analysis [3, 5, 8, 12-15]. Liu et al. [3] proposed a reliability index-based consensus reaching process based on an alternative ranking-based clustering method. Shi et al. [5] presented a novel CRP model based on a behavior classification model that can classify three kinds of modification behaviors. Xu et al. [12] proposed a two-stage consensus model for LSGDM, in which the DMs are classified into small subgroups using self-organizing maps. Palomares et al. [13] and Rodriguez et al. [14] both studied consensus models based on fuzzy c-means. Wu and Xu [15] introduced a consensus model for LSGDM with changeable clusters based on k-means. Most of the clustering models are performed based on DMs' preference. Previous studies, however, rarely consider the CAP during CRP since the preferences of DMs might change if they compromise with others to reach a consensus.

Recently, social network has become popular in LSGDM due to the development of information and network technology [6, 20, 23, 24]. Liu et al. [6] proposed a trust relationship-based conflict detection and elimination decision making model that can be applied to LSGDM problems in social network contexts. Xu et al. [20] introduced a method based on a trust model for LSGDM with incomplete preference information. Wu et al. [23, 24] studied the CAP in LSGDM by utilizing community detection methods. To take advantage of network analysis and to explore the evolution of consensus based on DMs' preference information, Wu et al. [25] proposed consensus evolution networks (CENs) based model for GDM. Currently, almost no one has simultaneously studied CAP and CRP in LSGDM based on social network models.

Despite the fact that diverse LSGDM models have already been proposed, they still suffer from several limitations:

(1) In most LSGDM models, CAP and CRP are usually considered to be independent parts. CAP is mainly performed with preference similarity using traditional clustering methods and the dynamic clustering analysis is seldom considered after the feedback adjustment of CRP.

(2) Clustering validity is rarely checked in most LSGDM models since it is an important indicator when evaluating the clustering effect. An unreasonable clustering result in LSGDM may increase the complexity of consensus building and lead to wrong decisions being made.

(3) The conflict between CAP and CRP in LSGDM is rarely considered since CAP can be effectively implemented when there are differences among DMs, while the purpose of CRP is to minimize the amount of differences among DMs.

Our interest mainly focuses on dealing with the above limitations by exploring the dynamic CAP and CRP based on the CENs of LSGDM. Several assumptions in our proposal need to be explained in advance:

(1) Suppose DMs in the same subgroups hold equal importance and have equal willingness to change their opinions to reach consensus.

(2) Suppose the subgroups that have more DMs have higher weights in decision making, i.e. the aggregation of subgroups follows the majority principle.

(3) Suppose the subgroups that have higher consensus levels have higher weights in decision making, i.e. subgroups with stronger cohesion have a greater discourse competence.

Based on the above assumptions, the main contributions of this study are given as follows:

(1) Design the dynamic clustering analysis process (DCAP). We construct CENs for LSGDM by managing consensus thresholds and propose a clustering method based on a community detection method. The DCAP appears when the proposed clustering method is used to classify diversified CENs and reused after CRP. The role of the DCAP in CRP is similar to the identification rule that identifies DMs that have similar consensus levels and classify them into the same subgroups.

(2) Evaluate clustering validity. We define the overall consensus levels in subgroups as intra-cluster consensus levels and that among subgroups as inter-cluster consensus level. Depending on the clustering principle, an evaluation algorithm is proposed based on intra-cluster and inter-cluster consensus levels. We evaluate the clustering validity after each round of DCAP and select a suitable result for the following decision process.

(3) Balance the DCAP and CRP. Depending on the identification and direction rule, we give a feedback adjustment algorithm based on the intra-cluster consensus levels and the inter-cluster consensus level. The DCAP is reanalyzed during CRP and after the satisfying consensus is achieved. The conflict between the DCAP and CRP may appear after some rounds of iterations. Thus, we balance the DCAP and CRP by classifying the modified CENs with higher consensus thresholds.

The proposed LSGDM model is examined using a case study which shows its flexibility when dealing with LSGDM based on the DCAP and CRP. In the DCAP, the suitable clustering result is determined based on the clustering validity algorithm. The consensus is achieved to an agreed value in only two rounds of iteration using the clustering-based feedback adjustment algorithm. Since the contradiction between the DCAP and CRP becomes more evident after two rounds of iterations, we balance the contradiction using a higher consensus threshold with which an effective clustering result is obtained.

The rest of this paper is organized as follows: the basic concepts of this study are introduced in Section II. The DACP is designed based on CENs, the clustering validity is verified, and CRP is studied based on the DCAP in Section III. The whole framework of this study and a comparison from a theoretical perspective are provided in Section IV. A case study is applied to illustrate the proposed model and a related comparison is given to show its advantages in Section V. Finally, the conclusion and discussion are given in Section VI.

II. Preliminaries

In this section, the basic concepts of traditional CRP, the selection process, the definition of CENs and the community detection based on modularity are given.

A. Basic concepts of traditional consensus reaching process

The CRP mainly consists of preference representation, consensus measure, and feedback adjustment. The related concepts of CRP are introduced from these perspectives.

(1) Fuzzy preference relations

Definition 1. [26, 27] An FPR F is a fuzzy set on the alternative set $X \times X$, which is characterized by a membership function $\mu_F: X \times X \to [0,1]$, where $\mu_F(x_i, x_j) = f_{ij}$ is interpreted as the preference degree of alternative x_i over x_j $(i, j = 1,...,n)$: $f_{ij} = 0.5$ indicates indifference between x_i and x_j. $f_{ij} > 0.5$ indicates that x_i is preferred to x_j, $f_{ij} < 0.5$ indicates that x_i is inferior to x_j, and fulfilling $f_{ij} + f_{ji} = 1$.

Generally, the FPR of DM d_k alternative x_i over x_j can be represented as $F_k = \left(f_{ij}^k\right)_{n \times n}$, and $f_{ij}^k + f_{ji}^k = 1$. Notably, the FPR F_k can be denoted as $F_k(b)$ when all its elements f_{ij}^k are equal to $b \in [0,1]$.

(2) Consensus measure

The consensus is usually measured via similarity functions that are commonly determined based on distance functions. Chiclana et al. [28] found that Manhattan distance is sensitive to the number of DMs when measuring consensus, which helps the consensus process converge faster. These characteristics are convenient when producing clustering results and promote consensus convergence in LSGDM [13, 18]. A similarity method based on Manhattan distance is given as follows.

Definition 2. [29] A similarity matrix $S_{hk} = \left(s_{ij}^{hk}\right)_{n \times n}$ between DM d_h and d_k on the preference of alternative x_i over x_j is defined as:

$$s_{ij}^{hk} = 1 - \left| f_{ij}^h - f_{ij}^k \right| \quad (1)$$

where $i, j = 1,2,...,n; i \neq j$, and $h, k, = 1,2,...,m;\ h \neq k$.

Different consensus models have been proposed over recent decades [13, 30, 31]. Generally, the consensus is usually measured based on the consensus matrix.

Definition 3. [13] A function $cm: [0,1]^n \to [0,1]$ is defined as being a consensus measure among all DMs with respect to a pair of alternatives (x_i, x_j) based on the weighted average of the similarity matrices s_{ij}^{hk}:

$$cm_{ij} = \frac{\sum_{h=1}^{m-1} \sum_{k=h+1}^{m} \omega_{hk} s_{ij}^{hk}}{\sum_{h=1}^{m-1} \sum_{k=h+1}^{m} \omega_{hk}} \quad (2)$$

where ω_{hk} are the weights associated with each pair of DMs (d_h, d_k), and $\omega_{hk} = \min\{\omega_h, \omega_k\}$ [13, 29], ω_h and ω_k are weights of DM d_h and d_k, respectively.

Based on the consensus measure cm, the consensus matrix $CM = \left(cm_{ij}\right)_{n \times n}$ can be constructed as:

$$CM = \begin{array}{c} \\ x_1 \\ \ldots \\ x_n \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & \ldots & x_n \end{array} \\ \begin{bmatrix} 0 & \ldots & cm_{1n} \\ \ldots & 0 & \ldots \\ cm_{n1} & \ldots & 0 \end{bmatrix}_{n \times n} \end{array} \quad (3)$$

Definition 4. [13, 29, 32] A function $CL_a: [0,1]^m \to [0,1]$ is defined as being an overall consensus measure based on the consensus matrix $CM = \left(cm_{ij}\right)_{n \times n}$:

$$CL_a = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} cm_{ij}}{n(n-1)/2} \quad (4)$$

Beliakov et al. [30] summarized seven consensus properties to define a reasonable consensus measure: **C1** (unanimity), **C2** (Minimum consensus for $m = 2$), **C3** (Symmetry), **C4** (Maximum dissension), **C5** (Reciprocity), **C6** (Replication invariance), **C7** (Monotonicity with respect to the majority). It is easy to verify that the overall consensus function CL_a satisfies the seven properties.

(3) Feedback adjustment

In the feedback adjustment of CRP, there are two consensus rules that are usually used [33]: (1) Identification rule. It identifies the DMs that contribute less to reach high consensus. (2) Direction rule. It finds out the direction required to be taken to change the DMs' preferences.

Suppose the d_k is identified based on the identification rule, then d_h is determined as the referenced FPR based on the direction rule. The adjustment strategy that is used to obtain the adjusted $F_{k'} = \left(f_{ij}^{k'}\right)_{n \times n}$ for d_k is represented as [25]:

$$f_{ij}^{k'} = \begin{cases} f_{ij}^k + \left| f_{ij}^k - f_{ij}^{hk} \right| / 2, & f_{ij}^k \leq f_{ij}^{hk} \\ f_{ij}^k - \left| f_{ij}^k - f_{ij}^{hk} \right| / 2, & f_{ij}^k > f_{ij}^{hk} \end{cases} \quad (5)$$

where $i \leq j$, f_{ij}^{hk} is the averaging preference of d_k and d_h, and $f_{ij}^{hk} = \left(f_{ij}^h + f_{ij}^k\right)/2$, when $i > j$, $f_{ji}^{k'} = 1 - f_{ij}^{k'}$.

B. The selection process

Since the ordered weighted averaging (OWA) operator is simply built up of the pairwise association coefficients and the weights are defined by linguistic quantifiers, it is commonly used to obtain the overall preferences in the selection process of GDM [32-34].

Definition 5. [35] The OWA is defined as

$$OWA(f_1, f_2, ..., f_n) = \sum_{i=1}^{n} \pi_i c_i \quad (6)$$

where c_i is the ith largest value in $\{f_1, f_2, ..., f_n\}$, and $\pi = (\pi_1, \pi_2, ..., \pi_n)^T$ is an associated weight vector such that $\pi_i \in [0,1]$ and $\sum_{i=1}^{n} \pi_i = 1$. π can be computed with linguistic quantifiers [36], $\pi_i = Q(i/n) - Q((i-1)/n)$. Some commonly used examples of linguistic quantifiers are "at least half," "most," "as many as possible," defined by the interval values (a,b), (0, 0.5), (0.3, 0.8) and (0.5, 1), respectively, using the following expression:

$$Q(\kappa) = \begin{cases} 0, & \text{if } \kappa \le a \\ \dfrac{\kappa - a}{b - a}, & \text{if } a < \kappa < b \\ 1, & \text{if } \kappa \ge b \end{cases} \tag{7}$$

where $a, b, \kappa \in [0,1]$.

Suppose the collective preference matrix $F = (f_{ij})_{n \times n}$ shows the preference of the whole group with respect to n alternatives. Based on the OWA operator with a corresponding fuzzy quantifier, the overall collective preference value $f_i \ (i = 1, ..., n)$ of the alternative x_i can be computed based on the ith elements of F [32, 33]:

$$f_i = OWA(f_{i1}, f_{i2}, ..., f_{in}) \tag{8}$$

C. Consensus evolution networks

Similarly to social relationships that formed by common connections, the consensus relations can be built among DMs based on their similar preferences. The definition of consensus relation is extracted from [25].

Definition 6. [25] The consensus relation between the DM d_h and d_k exists if their consensus level c_{hk} is higher than a consensus threshold $\varepsilon \in [0,1]$.

Different to the consensus matrix $CM = (cm_{ij})_{n \times n}$ obtained by (2), Wu et al. [25] reconstructed the consensus matrix $C = (c_{hk})_{m \times m}$ using the similarity degree between each pair of DMs (d_h, d_k) to construct consensus relations among DMs:

Definition 7. [25] A function $c : [0,1]^m \to [0,1]$ is defined as being a consensus measure between a pair of DMs (d_h, d_k) with respect to all alternatives based on similarity matrices s_{ij}^{hk}:

$$c_{hk} = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{ij}^{hk}}{n(n-1)/2} \tag{9}$$

where $n(n-1)/2$ denotes the number of paired alternatives.

Based on the consensus measure c, a modified consensus matrix $C = (c_{hk})_{m \times m}$ can be constructed as:

$$C = \begin{array}{c} \\ d_1 \\ ... \\ d_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & ... & d_m \end{array} \\ \begin{bmatrix} 0 & ... & c_{1m} \\ ... & 0 & ... \\ c_{m1} & ... & 0 \end{bmatrix}_{m \times m} \end{array} \tag{10}$$

According to Definition 6, the definition of CENs is given based on the consensus matrix $C = (c_{hk})_{m \times m}$ as follows.

Definition 8. [25] An CEN consists of $G = (D, E, C)$ with m DMs $D = \{d_1, d_2, ..., d_m\}$, consensus relations $E = \{e_{hk} \mid h, k = 1, 2, ..., m, h \ne k\}$ and consensus levels $C = \{c_{hk} \ge \varepsilon, h, k = 1, 2, ..., m, h \ne k\}$. If $c_{hk} \ge \varepsilon$, then there is an edge e_{hk} in G to connect d_h and d_k together with consensus level c_{hk}, and c_{hk} called the weight of the edge e_{hk}. Otherwise, there is no edge between d_h and d_k.

The determination of the consensus threshold depends on the characteristics of the decision making problem and the attitudes of DMs, i.e. it is context-based.

D. Community detection based on modularity

Community detection is a useful tool when studying large networks [37]. In community detection, modularity is a crucial technique to measure the density of links in the communities as compared to links between communities [38]. The value of modularity mainly depends on the community allocation of nodes in networks, i.e. modularity can be used to measure the quality of community allocation. The closer the value of modularity is to 1, the stronger the community structure divided by the network is, the better the quality of the partition is. Therefore, Blondel et al. [39] proposed a commonly used Louvain method to detect communities for large networks based on the gain in modularity ΔQ:

Definition 9. [39] Supposing there is a large network LG that is classified into t subgroups $LG = \{SG_1, ..., SG_t\}$, $r, s = 1, ..., t$, d_r^k denotes the DM that belongs to the subgroup SG_r, then the gain in modularity ΔQ obtained by moving the DM d_r^k from SG_r to SG_s is computed by

$$\Delta Q = \left[\frac{\sum_{in} + 2W_{r,in}}{M_{rs}} - \left(\frac{\sum_{tot} + W_r}{M_{rs}} \right)^2 \right] - \left[\frac{\sum_{in}}{M_{rs}} - \left(\frac{\sum_{tot}}{M_{rs}} \right)^2 - \left(\frac{W_r}{M_{rs}} \right)^2 \right] \tag{11}$$

where $\sum_{in}$ denotes the sum of the edge weights in the SG_s, $\sum_{tot}$ is the sum of the edge weights incident to DMs in SG_s,

$W_{r,in}$ represents the sum of the edge weights from d_r^k to DMs in SG_s, $W_r = \sum_{r=1}^{t} c_{rs}$ is the sum of the edge weights incident to DM d_r^k from DMs in SG_s, $M_{rs} = \sum_{r,s=1,r\neq s}^{t} c_{rs}$ means the sum of the weights of all DMs in G, in which c_{rs} means the weights between SG_r and SG_s and is obtained based on the sum of the edge weights of DMs belong to SG_r and SG_s.

If $\Delta Q \geq 0$, then remove d_r^k from SG_r to SG_s with $\max \Delta Q$, otherwise, the final community detection result is obtained.

III. The Dynamic Clustering Analysis and Consensus Reaching Process in LSGDM Based on Consensus Evolution Networks

In this study, the idea is to balance high consensus and effective clustering analysis by considering the DCAP during CRP. The DCAP is derived using a proposed clustering analysis algorithm in LSCENs with different CENs. Actually, the clustering nature of the proposed DCAP is to identify DMs that have compact consensus relations and allot them into same subgroups, so the role of the DCAP is similar to the identification rule in CRP. It is worth noting that the clustering validity is verified based on the intra-cluster consensus levels of subgroups and inter-cluster consensus level among subgroups. A feedback adjustment algorithm based on the direction rule is proposed depending on the clustering results. Additionally, the clustering is reanalyzed in the DCAP after each round of CRP. In short, the main idea of this study is depicted in Fig.1.

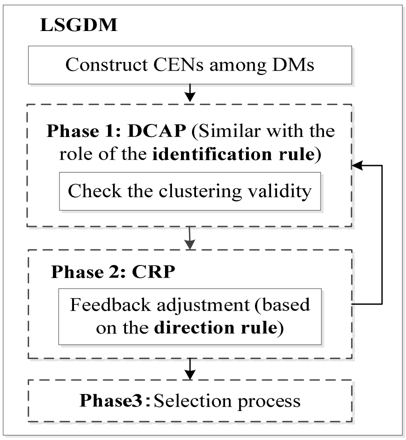


Fig. 1. The main structure of the LSGDM model based on CENs

The four main contributions will be shown in Section III (A-F). Firstly, we introduce a modified consensus measure for LSGDM in Section III-A. Next, we propose a clustering algorithm in CENs to carry out the DCAP in Section III-B. Afterwards, we define the local and global CENs and verify the clustering validity based on the local and global OCL in Section III-C and III-D, respectively. Finally, we propose a feedback adjustment method in Section III-E and reanalyze the clustering in Section III-F.

A. A modified consensus measure method

With the FPRs $F = \{F_1, \ldots, F_m\}$ of m DMs, the large-scale similarity matrix $LS_{hk} = \left(ls_{ij}^{hk}\right)_{n\times n}$ and consensus matrix $LC = \left(lc_{hk}\right)_{m\times m}$ are computed using (1), (9), and (10), respectively.

Definition 10. A function $CL: [0,1]^m \to [0,1]$ is defined as being a modified overall consensus measure based on the weighted average of consensus matrix $LC = \left(lc_{hk}\right)_{m\times m}$:

$$CL = \frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk} lc_{hk}}{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}} \tag{12}$$

where ω_h and ω_k are weights of DM d_h and d_k, respectively, and $\omega_{hk} = \min\{\omega_h, \omega_k\}$.

If all DMs are given equal weights, i.e. $\omega_{hk} = \omega_h = \omega_k = 1/m$, then (12) is transformed to:

$$CL = \frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m} lc_{hk}}{m(m-1)/2} \tag{13}$$

According to (1), (9) and (12), the FPRs $F_k = \left(f_{ij}^k\right)_{n\times n}$, $\forall i, j = 1, \ldots, n$, $k = 1, \ldots, m$, of DMs is the main variable of the overall consensus function CL. It is obvious that (1) $ls_{ij}^{hk} = 1 \Rightarrow lc_{hk} = 1$ when $f_{ij}^h = f_{ij}^k$; (2) $ls_{ij}^{hk} = 0 \Rightarrow lc_{hk} = 0$ when $F_h(0), F_k(1)$ or $F_h(1), F_k(0)$; (3) $ls_{ij}^{hk} = ls_{ij}^{kh} \Rightarrow lc_{hk} = lc_{kh}$. (4) $ls_{ji}^{hk} = ls_{ij}^{hk}$ since $f_{ji}^h = 1 - f_{ij}^h$ and $f_{ji}^k = 1 - f_{ij}^k$. (5) If $\left|f_{ij}^l - f_{ij}^h\right| \leq \left|f_{ij}^l - f_{ij}^k\right|$, then $ls_{ij}^{hl} \geq ls_{ij}^{kl} \Rightarrow lc_{hl} \geq lc_{kl}$.

Considering above discussion, it is easy to prove that the extended overall consensus function CL satisfies the seven consensus properties summarized in [30].

C1. CL is complete unanimity for all $F_k = \left(f_{ij}^k\right)_{n\times n}$, $f_{ij}^k = b \in [0,1]$, we have $CL(b, \ldots, b) = 1$.

C2. CL is the minimum consensus for $m = 2$ when for the special case of two inputs F_h and F_k, we have $CL(F_h(0), F_k(1)) = CL(F_h(1), F_k(0)) = 0$.

C3. CL is symmetrical when for all permutations $\pi(k)$ on $\{1, \ldots, m\}$, we have $CL(F_1, \ldots, F_m) = CL\left(F_{\pi(1)}, \ldots, F_{\pi(m)}\right)$.

C4. CL satisfies the property of maximum dissension when for $m = 2q$, if the q of the FPRs are equal to 0 and the q of the

FPRs are equal to 1, then $CL\left(F_1(0),...,F_q(0),F_{q+1}(1),...,F_{2q}(1)\right)=0$ for all input vector permutations.

C5. CL is reciprocal if for a strong negation N, i.e. $N\left(f_{ij}^h\right)=1-f_{ij}^h$, we have $CL(F_1,...,F_m)=CL(N(F_1),...,N(F_n))$.

C6. CL is a replication invariant when for any FPRs $F=\{F_1,...,F_m\}$, duplicating the inputs does not alter the level of consensus, i.e. $CL(F)=CL(F,F)=CL(F,F,F)$ and so on.

C7. For $n=2q$, let half of the FPRs have the same equal and thus denoted by $f_{ij}^l=(b,...,b)$. CL is monotone with respect to the majority if when $\left|b-f_{ij}^h\right|\le\left|b-f_{ij}^k\right|$ for all $h,k,l=1,...,m$, then $CL\left(F_l,F_h,...,F_{h+q}\right)\ge CL\left(F_l,F_k,...,F_{k+q}\right)$.

The above proofs show that the FPRs are the main variables both for the overall consensus measure CL_a and CL, and they both satisfy the seven consensus properties, the relationship between CL_a and CL is described in the following theorem.

Theorem 1: The overall consensus measure CL_a obtained with the consensus matrix $CM=\left(cm_{ij}\right)_{n\times n}$ using the traditional consensus measure method is equal to the overall consensus measure CL obtained with the consensus matrix $LC=\left(lc_{hk}\right)_{m\times m}$ using the modified consensus measure method.

Correspondingly, the proof of **Theorem 1** is given as:

Proof: Based on (1), (2) and (4), CL_a is denoted as:

$$CL_a=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}s_{ij}^{hk}}{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}}\right).$$

Based on (1), (9) and (12), CL can be represented as:

$$CL=\frac{2}{n(n-1)}\frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}ls_{ij}^{hk}\right)}{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}}.$$

Since

$$s_{ij}^{hk}=ls_{ij}^{hk}\Rightarrow\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}s_{ij}^{hk}}{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}}\right)=\frac{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}ls_{ij}^{hk}\right)}{\sum_{h=1}^{m-1}\sum_{k=h+1}^{m}\omega_{hk}},$$

the second part of the deduction represents the sum of the similarity degrees among all DMs with respect to all the pairs of alternatives, therefore, $CL_a=CL$. QED.

The theorem suggests the feasibility of the extended overall consensus measure CL and implies the practical significance of the extended consensus matrix $LC=\left(lc_{hk}\right)_{m\times m}$. The main difference is the computation of the consensus matrix, i.e. $CM=\left(cm_{ij}\right)_{n\times n}$ considers consensus among all DMs on pairwise alternatives and $LC=\left(lc_{hk}\right)_{m\times m}$ considers consensus between pairwise DMs on all alternatives. We can find consensus relation and construct CENs for DMs based on LC. Compared to CL_a, the computing process of CL reflects the consensus relation among DMs more directly.

According to Definition 8, the CEN $LG=(LD,LE,LC)$ in LSGDM with m DMs $LD=\{d_1,d_2,...,d_m\}$, consensus relations $LE=\left\{le_{hk}\mid h,k=1,2,...,m,h\ne k\right\}$ and consensus levels $LC=\left\{lc_{hk}\ge\varepsilon\mid h,k=1,2,...,m,h\ne k\right\}$ can be constructed based on $LC=\left(lc_{hk}\right)_{m\times m}$. Similarly, the diverse CENs can also be identified with the consensus threshold ε. Based on the diverse CENs, the DACP can be carried out by a proposed clustering analysis method in the next section.

B. The dynamic clustering of large-scale consensus evolution networks based on community detection

Depending on social network analysis, the frequently used Louvain method [39, 40] is used to detect communities from CENs. The Louvain method is an unsupervised and iterative two-phase algorithm used to extract community structures from large networks. In the first phase, all DMs are classified into subgroups until there are no gains in modularity. In the second phase, the independent subgroups are combined to become a bigger subgroup until there is no subgroup that can be moved. The clustering analysis for the CEN using the Louvain method is given as Algorithm 1.

Algorithm 1-The clustering analysis algorithm

The first phase: For individuals

Step 1: For the CEN LG, assign a different subgroup to each individual. Thus, there is m subgroups $LG^{(1)}=\left\{SG_1^{(1)},...,SG_m^{(1)}\right\}$ in the initial partition.

Step 2: For each DM d_k, consider its gains in modularity $\Delta Q>0$ with its neighbors $d_h(h=1,...,m,h\ne k)$, and remove d_k from its subgroup $SG_k^{(1)}$ to $SG_h^{(1)}$ with $\max\Delta Q$ based on (11). Repeat this step until $\Delta Q\le 0$ and no node can be moved, then go to the second phase.

The second phase: For the independent subgroups

Step 1: Assuming that $x(x<m)$ subgroups are determined after p rounds in the first phase, and $LG^{(p)}=\left\{SG_1^{(p)},...,SG_x^{(p)}\right\}$. For each subgroup $SG_r^{(p)}(r=1,...,x)$, consider the gains in modularity $\Delta Q'$ with its neighboring subgroup $SG_s^{(p)}(s=1,...,x,r\ne s)$, and remove $SG_r^{(p)}$ to $SG_s^{(p)}$ with $\max\Delta Q'$ based on (11)

Step 2: Repeat *Step 1* of the second phase until there are no more changes. Finally, the CEN LG is classified into t independent subgroups $LG=\{SG_1,...,SG_t\}$, $t\le x<m$.

The different order of node access will lead to different results in Algorithm 1, the experiment found that this order only affects the calculation time to some extent, but does not affect the result. The time complexity of the iteration in the first phase is $O(M)$, where M is the number of edges of the CEN LG. The time complexity in the second phase is $O(M+N(SG))$, where $N(SG)$ denotes the number of subgroups.

For the LSGDM, dynamic clustering results can be obtained using Algorithm 1 based on different CENs LG_{ε} with the consensus threshold ε. Suppose there are p dynamic clustering results of LG, and that they are shown as $LG=\left(LG_{\varepsilon}^{(1)},...,LG_{\varepsilon}^{(p)}\right)$. That is, individuals may belong to different subgroups with different consensus levels. Only partial DMs who have higher consensus levels with others can be classified into subgroups with the increasing ε. Thus, the clustering results become less and less effective with the increasing ε.

C. The local and global consensus evolution networks

According to the clustering analysis, the LSGDM composes of several subgroups. Similarly, the CEN can also be seen as a combination of several small CENs. These small CENs consist of DMs in the same subgroups that are referred to as local CENs. Depending on the clustering purpose, DMs that achieve greater group consensus with each others are classified into the same local CENs. Therefore, each local CEN can be regarded as an individual and collective FPRs of local CENs can be computed using the averaging operator. Then, a global CEN can be built with local CENs and the consensus relations among them. Assuming there are q kinds of dynamic clustering results, the definition of the rth$\left(r=1,...,t,t<m\right)$ local and global CEN are given in the case of the pth$\left(p=1,...,q\right)$ clustering result, as shown below.

Definition 11. The local CEN $SG_r^{(p)}=\left(D_r^{(p)},E_r^{(p)},C_r^{(p)}\right)$ $(1<r\leq t<m)$ consists of $N\left(SG_r^{(p)}\right)$ DMs $D_r^{(p)}=\left\{d_r^k \mid d_r^k\in SG_r^{(p)}\right\}$, $E_r^{(p)}=\left\{e_{hk}^r \mid d_r^k,d_r^h\in SG_r^{(p)}\right\}$, $C_r^{(p)}=\left\{c_{hk}^r \mid d_r^k,d_r^h\in SG_r^{(p)},c_{hk}^r=lc_{hk}\right\}$, where $h,k=1,2,...,m$, $h\neq k$. If $c_{hk}^r\geq 0$, then there is an edge e_{hk}^r in $SG_r^{(p)}$ to connect d_r^k and d_r^h together with consensus relation value e_{hk}^r, and c_{hk}^r denotes the weight of the edge e_{hk}^r.

Accordingly, the OCL of local CENs are referred to as the local consensus levels (LCLs) and they are represented by $\left\{CL_{loc(1)}^{(p)},...,CL_{loc(t)}^{(p)}\right\}$. Since DMs in the same local CEN have similar preferences, they are considered to be equally important in the determination of LCLs, i.e. LCLs are computed by (13).

Definition 12. The global CEN $\tilde{G}^{(p)}=\left(SG^{(p)},\tilde{E}^{(p)},\tilde{C}^{(p)}\right)$ consists of t subgroups $\left\{SG_1^{(p)},...,SG_r^{(p)},...,SG_t^{(p)}\right\}$, $\tilde{E}^{(p)}=\left\{\tilde{e}_{rs}^{(p)}\right\}$, and $\tilde{C}^{(p)}=\left\{\tilde{c}_{rs}^{(p)} \mid SG_r^{(p)},SG_s^{(p)}\in\tilde{G}^{(p)}\right\}$, $r,s=1,...,t,r\neq s$. The edge $\tilde{e}_{rs}^{(p)}$ connects $SG_r^{(p)}$ and $SG_s^{(p)}$ together with the consensus relation value $\tilde{c}_{rs}^{(p)}$ and is referred to as the edge weight of $\tilde{e}_{rs}^{(p)}$.

In Definition 12, the consensus relation $\tilde{c}_{rs}^{(p)}$ can be computed with the collective FPRs of local CENs. Firstly, the collective FPRs are computed under the assumption that all DMs in local CEN $SG_r^{(p)}$ are considered to be equally important:

$$f_{ij}^{SG_r}=\frac{1}{N\left(SG_r^{(p)}\right)}\sum_{d_r^k\in SG_r^{(p)}}f_{ij}^k \tag{14}$$

where $N\left(SG_r^{(p)}\right)$ is the number of DMs d_r^k $(k=1,...,m)$ in $SG_r^{(p)}$, $r=1,...,t$.

Then, the similarity relation $SM_{rs}^{(p)}=\left(sm_{ij}^{rs}\right)_{t\times t}$ between the local CENs $SG_r^{(p)}$ and $SG_s^{(p)}$ is calculated based on (1). Finally, the consensus relation $\tilde{c}_{rs}^{(p)}$ is determined based on (9).

Similarly, the consensus level of the global CEN is referred to as the global consensus level (GCL) and denoted as $CL_{glo(r)}^{(p)}$. In several previous studies, the weights of subgroups are determined based on the majority principle, that is, the subgroup consists of more individuals that may be more important in the decision making process. The majority principle may work most of time. However, it is easy to cause unfairness when only one factor is considered. Sometimes, more solidarity also means more decision-making power. The LCL reflects the compactness or unity among individuals in the whole local CEN. In this study, we adjust the majority principle based weights for local CENs with their LCLs. The majority principle based importance is denoted as $\mu_{M(r)}^{(p)}$:

$$\mu_{M(r)}^{(p)}=\frac{N\left(SG_r^{(p)}\right)}{\sum_{r=1}^{t}N\left(SG_r^{(p)}\right)} \tag{15}$$

where $N\left(SG_r^{(p)}\right)$ is the number of DMs in $SG_r^{(p)}$.

The local compactness based importance is denoted as $\mu_{C(r)}^{(p)}$:

$$\mu_{C(r)}^{(p)}=\frac{CL_{loc(r)}^{(p)}}{\sum_{r=1}^{t}CL_{loc(r)}^{(p)}} \tag{16}$$

where $CL_{loc(r)}^{(p)}$ is the LCL of $SG_r^{(p)}$.

Rodriguez et al. [14] computed subgroups' weights based on size and cohesion with a parameter $\beta>0$ to increase/decrease the impact of cohesion. Depending on their proposal, the weights $w_r^{(p)}\left(r=1,...,t\right)$ of local CENs are codetermined based on the majority principle and compactness:

$$w_r^{(p)}=\left(1+\mu_{M(r)}^{(p)}\right)^{\beta\mu_{C(r)}^{(p)}} \tag{17}$$

where $\beta>0$ is a parameter to increase/decrease the impact of compactness in the determination of $SG_r^{(p)}$. Rodriguez et al. [14] suggested that $\beta=0.3$ based on several experiments.

With the weights $w_r^{(p)}\left(r=1,...,t\right)$ of local CENs, the GCL $CL_{glo(r)}^{(p)}$ can be computed using (12).

D. The clustering validity based on the local and global consensus evolution networks

In classical clustering analysis, the clustering validity is usually verified based on intra-cluster compactness and inter-cluster sparsity. The larger the intra-cluster compactness and the inter-cluster sparsity, the better the clustering is. Similarly, the cluster validity can be extended into LSGDM based on the intra-cluster consensus levels and the inter-cluster consensus level. The LCL reflects the intra-cluster consensus levels of local CENs, and the GCL represents the inter-cluster consensus level among local CENs. The higher the LCL and the lower the GCL, the better the clustering will be. Moreover, the GCL should be smaller than any of the LCLs. If not, the subgroup whose LCL is smaller than the GCL should be integrated into other subgroups with the closest consensus level. In this study, we propose an evaluation algorithm based the following three rules to evaluate the clustering validity to determine a suitable clustering result from the dynamic clustering analysis.

Rule 1: The number of isolated DMs in each subgroup should not be higher than 2.

Rule 2: The GCL should not be larger than any of the LCLs.

Rule 3: The clustering result with a minimum ratio between the GCL and the LCL should be generally determined as the suitable result.

In dynamic clustering analysis, more and more isolated DMs appear with the increasing consensus threshold ε. Rule 1 is proposed to remove the invalid clustering results that include more isolated DMs. Rule 2 is proposed to further remove unqualified results from the remaining clustering results after Rule 1. After Rules 1 and 2, Rule 3 is used to select a suitable clustering result from the final remaining dynamic results.

Algorithm 2 is given based on the above three rules to identify valid clustering results and determine a suitable clustering result from the dynamic results.

In the CRP, the feedback adjustment needs to be carried out if the agreed consensus level $\overline{CL}$ is not satisfied. In the following sections, we propose a feedback adjustment method based on the clustering analysis and reanalyze the new clustering analysis after the agreed consensus is reached.

Algorithm 2-The clustering validity test algorithm

Step 1: Based on ***Rule 1***, the clustering results that include isolated DMs and less than 2 subgroups are eliminated.

Step 2: Compute the LCLs $\left\{CL_{loc(1)}^{(p)},...,CL_{loc(t)}^{(p)}\right\}$ for all local CENs using (13) with the equal weights of DMs. And then, compute the overall intra-cluster consensus level $CL_{loc}^{(p)}$ of all local CENs:

$$CL_{loc}^{(p)}=\sum_{r=1}^{t}w_r^{(p)}CL_{loc(r)}^{(p)} \tag{18}$$

where w_r is the weight of local CENs obtained using (17).

Step 3: Based on (14) and (17), compute the GCL $CL_{glo}^{(p)}$ using (12).

Step 4: According to ***Rule 2***, compare the GCL $CL_{glo}^{(p)}$ with all the LCLs $\left\{CL_{loc(1)}^{(p)},...,CL_{loc(t)}^{(p)}\right\}$. If $\min\left(CL_{loc(r)}^{(p)}\right)\geq CL_{glo}^{(p)}$, then the clustering is valid. Otherwise, the clustering is invalid.

Step 5: From the remaining valid clustering results, select a suitable one to carry out the following decision making process based on ***Rule 3*** with $\min\left(CL_{glo}^{(p)}\big/CL_{loc}^{(p)}\right)$.

E. The adjustment method based on the clustering analysis

After the clustering analysis, the complexity of consensus reaching in LSGDM is greatly reduced. Depending on the determined weights $w_r^{(p)}\left(r=1,...,t\right)$ of local CENs shown in (17), the local CEN with more DMs and higher compactness should have a greater influence on decision making than a subgroup with less DMs and lower compactness. In terms of the importance of local CENs, we propose the adjustment algorithm based on the clustering analysis.

According to the identification rule, the local CEN $SG_r^{(p)}$ with the largest weight $\max\left(w_r^{(p)}\right)$ is identified. Based on the direction rule, the DMs in other local CENs are advised to modify their FPR according to the collective FPR of the local CEN $SG_r^{(p)}$. As such, the adjustment cost may be reduced in the process of consensus reaching regardless of whether the small weights of other local CENs are caused by the small number of members or the lower compactness. Besides, in order to improve the adjustment effect and reduce the adjustment costs as much as possible, the LCL of $SG_r^{(p)}$ is checked first if $CL_{loc}^{r}\geq\overline{CL}$. If so, go on to the adjustment process of other subgroups based on the collective preferences of $SG_r^{(p)}$; otherwise, improve the LCL of $SG_r^{(p)}$ first.

The detailed feedback adjustment algorithm based on the clustering analysis is given as Algorithm 3.

Algorithm 3-The feedback adjustment algorithm

Step 1: Compute the weights $w_r^{(p)}\left(r=1,\ldots,t\right)$ for all local CENs based on (17).

Step 2: Identify the local CEN $SG_r^{(p)}$ with the largest weight $\max\left(w_r^{(p)}\right)$ and other local CENs denoted as $SG_s^{(p)}$, $r,s=1,\ldots,t,r\neq s$.

Step 3: If $CL_{loc(r)}^{(p)}\geq\overline{CL}$, then go to the next step. If $CL_{loc(r)}^{(p)}<\overline{CL}$, then improve the internal consensus of $SG_r^{(p)}$ first. For example, for DM d_r^k in $SG_r^{(p)}$, compute the adjusted FPR $F_k'=\left(f_{ij}'^{k}\right)_{n\times n}$ based on the collective FPR $f_{ij}^{SG_r^{(p)}}$ using (5). Repeat this step until $CL_{loc(r)}^{(p)}\geq\overline{CL}$, and then determine the final local collective FPR $F_c'=F'_{SG_r^{(p)}}=\left(f_{ij}'^{SG_r^{(p)}}\right)_{n\times n}$ for $SG_s^{(p)}$ based on the modified FPRs.

Step 4: Take the collective FPR F_c' as a reference to suggest DMs d_s^h in $SG_s^{(p)}$ **to** adjust their FPRs $F_h'=\left(f_{ij}'^{h}\right)_{n\times n}$ based on (5).

After the feedback adjustment, compute CL_{ori} with the modified FPRs using (13). If $CL_{ori}\geq\overline{CL}$, then stop the adjustment and continue the decision making process; otherwise, repeat the feedback adjustment until $CL_{ori}\geq\overline{CL}$.

F. Reanalyze the clustering after the feedback adjustment

The structures of CENs and the relative clustering results might be changed when the FRPs are modified. The modified CENs becomes more compact and harder to classify when the consensus is improved. After each round of feedback adjustment, the clustering analysis of the modified CENs might not be available until the consensus threshold ε is large enough. Thus, the contradiction between the DCAP and CRP might be solved with a larger consensus threshold ε.

For the new clustering analysis obtained after each round, the clustering validity should be checked again and the suitable clustering result also needs to be reselected using Algorithm 2. If the agreed consensus $\overline{CL}$ is achieved, then the subsequent decision making process continues. Otherwise, a new round of adjustment is carried out based on the new clustering result.

IV. The Solution for LSGDM Balancing the Dynamic Clustering Analysis and Consensus Research Process

Since the critical techniques for settling LSGDM based on the DCAP and CRP have already been introduced above, the integrative framework for dealing with LSGDM based on CENs is concluded in Section IV-A. To differentiate this study from previous studies, a comparative analysis is given from a theoretical perspective in Section IV-B.

A. The main steps of the LSGDM solution

To make the processes of the LSGDM solution more legible, see the flowchart in Fig.2. The main steps of the solution in Fig.2 are introduced as below.

Phase 1: The dynamic clustering process

Step 1: The fuzzy preference relations of large-scale DMs

Identify a LSGDM problem, and gather preferences of DMs $LD=\{d_1,d_2,\ldots,d_m\}$ with respect to the pairwise alternatives $X=\{x_1,x_2,\ldots,x_n\}$ with FPRs $F_k=\left(f_{ij}^k\right)_{n\times n}\left(k=1,\ldots,m\right)$.

Step 2: Compute the consensus levels among large-scale DMs

First, calculate the similarity s_{ij}^{hk}, $i,j=1,2,\ldots,n,i\neq j$ $h,k=1,2,\ldots,m;\ h\neq k$, between each pair of DMs and construct the similarity matrix $LS_{hk}=\left(ls_{ij}^{hk}\right)_{n\times n}$ using (1). Then, construct the consensus matrix $LC=\left(lc_{hk}\right)_{m\times m}$ using (1) and (9).

Step 3: Build CENs with different consensus thresholds

Categorize the unique elements in LC as values of consensus thresholds ε. According to Definition 8, build different CENs with different ε.

Step 4: Dynamically classify large-scale DMs into subgroups

The different CENs can be dynamically classified using Algorithm 1. Suppose that q kinds of dynamic clustering results are obtained.

Step 5: Define the local and global CENs

Based on the clustering analysis, define the local and global CENs for the LSGDM according to Definition 11 and 12. For the pth$\left(p=1,\ldots,q\right)$ clustering result, the local CENs are denoted as $\left\{SG_1^{(p)},\ldots,SG_t^{(p)}\right\}$, where t is the number of subgroups, and the global CEN is denoted as $\tilde{G}^{(p)}$.

Step 6: Verify the clustering validity and determine a suitable clustering result

According to Algorithm 2, compute the LCLs $\left\{CL_{loc(1)}^{(p)},\ldots,CL_{loc(t)}^{(p)}\right\}$ and GCL $CL_{glo(r)}^{(p)}$ for local and global CENs, respectively. Then, verify the clustering validity and determine a suitable clustering result based on the LCLs and GCL.

Phase 2: The analysis of CRP

Step 1: Compute the OCL for the LSGDM

Compute the OCL for the LSGDM using (13). If $CL_{ori}\geq\overline{CL}$, then go to the selection process in **Phase 3**. Otherwise, go to the next step.

Step 2: Perform the feedback adjustment using Algorithm 3

If the agreed consensus $\overline{CL}$ is not satisfied, then carry out the feedback adjustment using Algorithm 3. In Algorithm 3, the weights $w_r^{(p)}\left(r=1,\ldots,t\right)$ of local CENs are computed to identify the local CENs in which individuals need to modify their FPRs from $F_k=\left(f_{ij}^k\right)_{n\times n}$ to $F_k'=\left(f_{ij}'^{k}\right)_{n\times n}$.

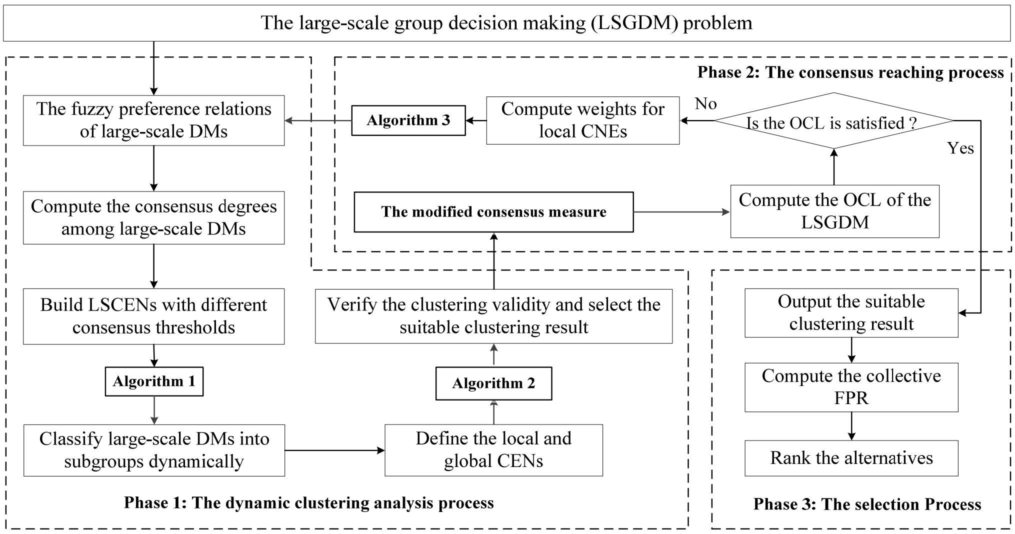


Fig. 2. The framework of dealing with LSGDM based on CENs

Step 3: Repeat the dynamic clustering analysis process

According to the revised FPRs, repeat the dynamic clustering analysis process shown in **Phase 1**. And then, recheck the OCL CL'_{ori} for the LSGDM with the modified FPRs. If $CL'_{ori} \geq \overline{CL}$, output the determined clustering result and go to the subsequent decision making process. Otherwise, repeat the feedback adjustment until $\overline{CL}$ is satisfied.

Phase 3: The selection process

Step 1: Output the suitable clustering result

Suppose $\overline{CL}$ is satisfying based on the modified FPRs $\{F''_1,...,F''_m\}$ after two adjustment rounds, output the suitable clustering result $LG = \{SG_1,...,SG_{t''}\}$ and the updated weights $w_{r''}(r''=1,...,t'')$.

Step 2: Compute the collective FPR

First, compute the collective FPRs $\left\{F''_{SG_1},...,F''_{SG_t}\right\}$ for subgroups with the equal weights of individuals using the averaging operator:

$$f''^{SG_r}_{ij} = \frac{\sum_{d^k_r \in SG_{r''}} f''^k_{ij}}{N(SG_{r''})} \tag{19}$$

Similarly, compute the collective FPR F'' for the LSGDM with the weights $w_{r''}$ using the weighted averaging operator:

$$f''_{ij} = \sum\nolimits_{r''=1}^{t''} w_{r''} * f''^{SG_r}_{ij} \tag{20}$$

Step 3: Rank the alternatives

Based on the collective FPR $F'' = \left(f''_{ij}\right)_{n\times n}$, determine the collective preference value $f_i\,(i=1,...,n)$ using the OWA operator based on (8), in which, the weights $\pi = (\pi_1,\pi_2,...,\pi_n)^T$ for OWA are computed with the fuzzy quantifier "most" based on (7).

B. *The comparison from a theoretical perspective*

To clearly differentiate between this study and others, we mark the traditional LSGDM as T-LSGDM, and the LSGDM based on social networks as SN-LSGDM, and this study as CEN-LSGDM.

First, we differentiate the three kinds of LSGDM models according to their main ideas. (1) T-LSGDM aims to classify individuals based on their preference similarity, and to improve consensus by considering the consensus behavior of individuals. (2) SN-LSGDM aims to classify individuals based on their social relations using community detection methods, and to study CRP based on the propagation of such relations. (3) This study aims to classify individuals based on CENs using a community detection method, and promote consensus reaching based on clustering analysis.

Secondly, we categorize the three kinds of LSGDM using the following details: (a) clustering analysis, (b) weights determination, and (c) the CRP and the following clustering reanalysis. The detailed comparison analysis is shown in Table I.

(a) Clustering analysis

Regarding the clustering analysis, (1) In T-LSGDM, the clustering analysis is mainly performed based on the preference similarity [12-15] using traditional clustering method, i.e. k-means [15], fuzzy c-means [13, 14] and self-organizing maps [12]. (2) In SN-LSGDM, individuals are often classified based on their social connections without considering consensus reaching [20, 23, 24]. (3) In this study, individuals are classified dynamically based on their consensus relations using the community detection method. Meanwhile, the validity of dynamic clustering is verified.

(b) Weights determination

Regarding the determination of the weights, (1) In T-LSGDM, individuals are usually considered as equally important [14-16, 20] and the weights of subgroups are mainly computed based on the majority principle [15, 16, 20, 23, 24]. (2) In SN-LSGDM, the weights of individuals and subgroups are mainly computed based on some centrality indices [20, 23, 24]. (3) In this study, the weights of individuals are also considered as equally important, while the weights of subgroups are computed with the combination of the size and compactness of the local CENs.

(c) The CRP and the following clustering reanalysis

Regarding the CRP and the following clustering reanalysis, (1) In most of T-LSGDM, the CRPs are usually studied while the clustering is not reconsidered after each round of adjustment. (2) In SN-LSGDM, the CRP is seldom considered, still less the clustering reanalysis. (3) In this study, we reconsider the clustering analysis during CRP. Besides, we balance DCAP and CRP with higher consensus thresholds.

TABLE I

THE COMPARISON ANALYSIS AMONG MULTIPLE LSGDM MODELS

LSGDM	References	Clustering analysis	CRP	The weights determination Individuals	Subgroups	Reanalyze the clustering analysis after CRP
T-LSGDM	Shi et al. [5];	√	√	Based on consensus behavior	—	×
	Wu and Xu [15]	√	√	Set as equal	Majority principle	√
	Zhang et al. [16]	√	√	Set as equal	Majority principle	×
	Xu et al. [12]	√	√	Based on contribution to consensus		×
	Palomares et al. [13]	√	√	Based on contribution to consensus		×
	Rodriguez et al. [14]	√	√	Set as equal	Based on size and cohesion	×
SN-LSGDM	Wu et al. [23, 24]	√	×	Based on the cohesion	Majority principle	—
	Xu et al. [20]	√	×	Set as equal	Majority principle	—
	Liu et al. [6]	×	√	Based on conflict level	—	—
CEN-LSGDM	This study	√	√	Set as equal	Based on size and compactness	√

V. CASE STUDY

This section presents a case study to show the application of the proposed models. This case study is designed in [18] to illustrate the large group emergency decision making due to the occurrence of a 7.0 magnitude earthquake in Ya'an City, Sichuan Province, in China on April 20, 2013. Since it was an emergency and there were insufficient rescue staff and medical facilities due to the paralyzed traffic, the rescue team, which consisted of 20 DMs $LD=\{d_1,...,d_{20}\}$ needed to determine an optimal alternative from the four rescue plans $X=\{x_1,x_2,x_3,x_4\}$ in order to minimize the damage.

To do so, CENs are built first. Next, the dynamic clustering analysis of CENs is performed using Algorithm 1 and is verified by Algorithm 2. The consensus is reached using Algorithm 3 and the subsequent clustering is reanalyzed. Finally, the alternatives are ranked in the selection process.

A. The construction of large-scale consensus evolution networks

The FRPs $F_k=\left(f_{ij}^k\right)_{n\times n}$ $\left(k=1,...,20,\ i,j=1,2,3,4,i\neq j\right)$ on four alternatives are given by 20 DMs as follows:

$$F_1=\begin{pmatrix}0.5&0.9&0.9&0.8\\0.1&0.5&0.7&0.8\\0.1&0.3&0.5&0.4\\0.2&0.2&0.6&0.5\end{pmatrix}\quad F_2=\begin{pmatrix}0.5&0.3&0.7&0.8\\0.7&0.5&0.3&0.6\\0.3&0.7&0.5&0.3\\0.2&0.4&0.7&0.5\end{pmatrix}\quad F_3=\begin{pmatrix}0.5&0.1&0.6&0.4\\0.9&0.5&0.6&0.4\\0.4&0.4&0.5&0.3\\0.4&0.6&0.7&0.5\end{pmatrix}$$

$$F_4=\begin{pmatrix}0.5&0.1&0.8&0.4\\0.9&0.5&0.6&0.6\\0.2&0.4&0.5&0.8\\0.6&0.4&0.2&0.5\end{pmatrix}\quad F_5=\begin{pmatrix}0.5&0.4&0.4&0.4\\0.6&0.5&0.1&0.5\\0.6&0.9&0.5&0.4\\0.6&0.5&0.6&0.5\end{pmatrix}\quad F_6=\begin{pmatrix}0.5&0.3&0.6&0.7\\0.7&0.5&0.8&0.8\\0.4&0.2&0.5&0.9\\0.3&0.2&0.1&0.5\end{pmatrix}$$

$$F_7=\begin{pmatrix}0.5&0.2&0.6&0.6\\0.8&0.5&0.8&0.8\\0.4&0.2&0.5&0.6\\0.4&0.2&0.4&0.5\end{pmatrix}\quad F_8=\begin{pmatrix}0.5&0.4&0.7&0.6\\0.6&0.5&0.4&0.8\\0.3&0.6&0.5&0.7\\0.4&0.2&0.3&0.5\end{pmatrix}\quad F_9=\begin{pmatrix}0.5&0.6&0.6&0.7\\0.4&0.5&0.9&0.9\\0.4&0.1&0.5&0.9\\0.3&0.1&0.1&0.5\end{pmatrix}$$

$$F_{10}=\begin{pmatrix}0.5&0.7&0.6&0.8\\0.3&0.5&0.6&0.7\\0.4&0.4&0.5&0.9\\0.2&0.3&0.1&0.5\end{pmatrix}\quad F_{11}=\begin{pmatrix}0.5&0.4&0.4&0.6\\0.6&0.5&0.3&0.4\\0.6&0.7&0.5&0.7\\0.4&0.6&0.3&0.5\end{pmatrix}\quad F_{12}=\begin{pmatrix}0.5&0.3&0.6&0.4\\0.7&0.5&0.6&0.6\\0.4&0.4&0.5&0.6\\0.6&0.4&0.4&0.5\end{pmatrix}$$

$$F_{13}=\begin{pmatrix}0.5&0.6&0.2&0.3\\0.4&0.5&0.4&0.3\\0.8&0.6&0.5&0.4\\0.7&0.7&0.6&0.5\end{pmatrix}\quad F_{14}=\begin{pmatrix}0.5&0.9&0.7&0.8\\0.1&0.5&0.8&0.7\\0.3&0.2&0.5&0.1\\0.2&0.3&0.9&0.5\end{pmatrix}\quad F_{15}=\begin{pmatrix}0.5&0.7&0.4&0.5\\0.3&0.5&0.1&0.2\\0.6&0.9&0.5&0.4\\0.5&0.8&0.6&0.5\end{pmatrix}$$

$$F_{16}=\begin{pmatrix}0.5&0.4&0.4&0.2\\0.6&0.5&0.1&0.2\\0.6&0.9&0.5&0.4\\0.8&0.8&0.6&0.5\end{pmatrix}\quad F_{16}=\begin{pmatrix}0.5&0.4&0.4&0.2\\0.6&0.5&0.1&0.2\\0.6&0.9&0.5&0.4\\0.8&0.8&0.6&0.5\end{pmatrix}\quad F_{17}=\begin{pmatrix}0.5&0.4&0.4&0.1\\0.6&0.5&0.5&0.4\\0.6&0.5&0.5&0.7\\0.9&0.6&0.3&0.5\end{pmatrix}$$

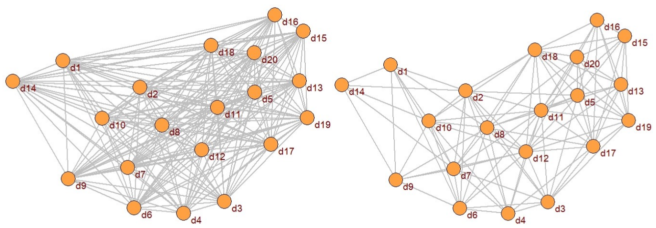
$$F_{18}=\begin{pmatrix}0.5&0.6&0.4&0.2\\0.4&0.5&0.3&0.7\\0.6&0.7&0.5&0.6\\0.8&0.3&0.4&0.5\end{pmatrix}\quad F_{19}=\begin{pmatrix}0.5&0.6&0.2&0.3\\0.4&0.5&0.4&0.3\\0.8&0.6&0.5&0.4\\0.7&0.7&0.6&0.5\end{pmatrix}\quad F_{20}=\begin{pmatrix}0.5&0.6&0.4&0.1\\0.4&0.5&0.3&0.4\\0.6&0.7&0.5&0.7\\0.9&0.6&0.3&0.5\end{pmatrix}$$

Based on (1), and (9), compute the consensus matrix $LC=\left(lc_{hk}\right)_{20\times 20}$ as:

$$LC=\begin{pmatrix}0&0.750&...&0.800&0.616&...&0.566\\0.750&0&...&0.750&0.799&...&0.683\\...&...&0&...&...&...&...\\0.800&0.750&...&0&0.750&...&0.700...\\0.616&0.799&...&0.750&0&...&0.883\\...&...&...&...&...&0&...\\0.566&0.683&...&0.700&0.883&...&0\end{pmatrix}_{20\times 20}$$

Suppose that all DMs are considered to be equally important, $\omega_k=0.05\,(k=1,2,...,20)$, the OCL of all the LSGDM is computed using (13) as: $CL=0.739$.

Based on $LC=\left(lc_{hk}\right)_{20\times 20}$, two examples of CENs are built in Fig.3.



(a) When $0<\varepsilon\leq 0.516$ (b) When $\varepsilon=0.733$

Fig. 3. The CENs with different consensus thresholds

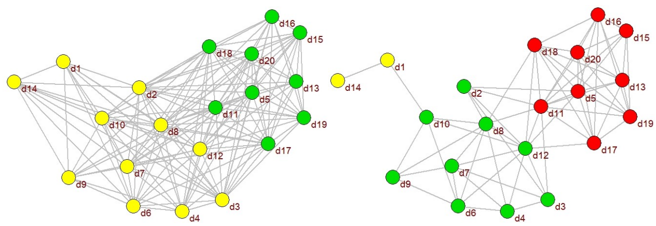
B. Dynamic clustering analysis of CENs

The CENs constructed with different consensus thresholds can be dynamically classified using Algorithm 1. The clustering numbers increase with the increasing consensus threshold ε. However, based on Rule 1, there are no effective clustering results when $\varepsilon>0.833$. The effective dynamic results are generated within $\varepsilon\in\left[0.516,0.833\right]$, they are $LG^{(1)}_{\varepsilon\in(0.516,0.783]}=\left\{SG^{(1)}_1,SG^{(1)}_2\right\}$ and $LG^{(2)}_{\varepsilon\in(0.783,0.833]}=\left\{SG^{(2)}_1,SG^{(2)}_2,SG^{(2)}_3\right\}$. The valid clustering results of the initial LSGDM based on Rule 1 are shown in Table II. For example, one of the clustering results of $LG^{(1)}_{\varepsilon=0.683}$ and $LG^{(2)}_{\varepsilon=0.799}$ can be shown in Fig.4. (a) and (b), respectively.

TABLE II

THE VALID CLUSTERING RESULTS OF THE INITIAL LSGDM BASED ON RULE 1

The consensus thresholds	The clustering results
$\varepsilon\in\left(0.516,0.783\right]$	$SG^{(1)}_1=\left\{d_1,d_2,d_3,d_4,d_6,d_7,d_8,d_9,d_{10},d_{12},d_{14}\right\}$
	$SG^{(1)}_2=\left\{d_5,d_{11},d_{13},d_{15},d_{16},d_{17},d_{18},d_{19},d_{20}\right\}$
$\varepsilon\in\left(0.783,0.833\right]$	$SG^{(2)}_1=\left\{d_1,d_{14}\right\}$
	$SG^{(2)}_2=\left\{d_2,d_3,d_4,d_6,d_7,d_8,d_9,d_{10},d_{12}\right\}$
	$SG^{(1)}_2=\left\{d_5,d_{11},d_{13},d_{15},d_{16},d_{17},d_{18},d_{19},d_{20}\right\}$



(a) $LG^{(1)}_{\varepsilon=0.683}$ (b) $LG^{(2)}_{\varepsilon=0.799}$

Fig. 4. The clustering results of $LG^{(1)}_{\varepsilon=0.683}$ and $LG^{(2)}_{\varepsilon=0.799}$

According to Algorithm 2, the related indicators that used to judge the validity of the remaining clustering and determine a suitable clustering result based on Rule 2 and 3 are shown in Table III.

In Table III, the weights of local CENs are $w^{(1)}_1=0.512$, , $w^{(2)}_1=0.278$ $w^{(2)}_2=0.357$, and $w^{(2)}_3=0.365$, respectively. The LCLs $CL^{(1)}_{loc(1)}=0.777$, $CL^{(1)}_{loc(2)}=0.845$, $CL^{(2)}_{loc(1)}=0.883$, $CL^{(2)}_{loc(2)}=0.799$, and $CL^{(2)}_{loc(3)}=0.845$, respectively. The overall intra-cluster consensus levels of $LG^{(1)}_{\varepsilon\in(0.516,0.783]}$ and $LG^{(2)}_{\varepsilon\in(0.783,0.883]}$ are $CL^{(1)}_{loc}=0.810$ and $CL^{(2)}_{loc}=0.839$, respectively. The GCL of $LG^{(1)}_{\varepsilon\in(0.516,0.783]}$ and $LG^{(2)}_{\varepsilon\in(0.783,0.883]}$ are $CL^{(1)}_{glo}=0.751$ and $CL^{(2)}_{glo}=0.697$, respectively. Obviously, $\min\left(CL^{(1)}_{loc(r)}\right)\geq CL^{(1)}_{glo}\left(r=1,2\right)$, and $\min\left(CL^{(2)}_{loc(r)}\right)\geq CL^{(2)}_{glo(r)}\left(r=1,2,3\right)$, so $LG^{(1)}_{\varepsilon\in(0.516,0.783]}$ and $LG^{(2)}_{\varepsilon\in(0.783,0.883]}$ are both valid clustering results. Besides, according to $\min\left(CL^{(p)}_{loc(r)}\middle/CL^{(p)}_{glo}\right)=0.830$ when $p=2$ and $r=1,2,3$, $LG^{(2)}_{\varepsilon\in(0.783,0.883]}$ is determined to be the suitable clustering result which can be used in the following decision making processes.

TABLE III

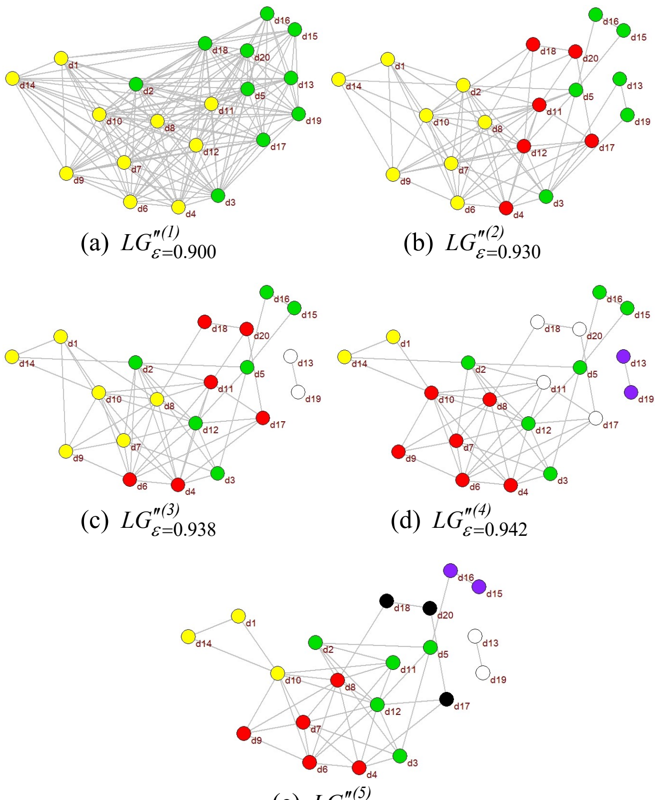
JUDGE THE CLUSTERING VALIDITY OF THE REMAINING DYNAMIC RESULTS

Dynamic clustering		w	LCL		GCL	GCL/LCL
$LG^{(1)}_{\varepsilon\in(0.516,0.783]}$	$SG^{(1)}_1$	0.512	0.777	0.810	0.751	0.926
	$SG^{(1)}_2$	0.488	0.845			
$LG^{(2)}_{\varepsilon\in(0.783,0.883]}$	$SG^{(2)}_1$	0.278	0.883	0.839	0.697	0.830
	$SG^{(2)}_2$	0.357	0.799			
	$SG^{(2)}_3$	0.365	0.845			

C. Consensus reaching process and the subsequent clustering analysis

Let the agreed consensus level $\overline{CL}=0.9$. According to $CL=0.739$, it is obvious that the OCL of the LSGDM does not reach 0.9, so the feedback adjustment method needs to be employed using Algorithm 3. In the first round, according to $w^{(2)}_1=0.278$ $w^{(2)}_2=0.357$, and $w^{(2)}_3=0.365$, DMs in $SG^{(2)}_1$ and $SG^{(2)}_2$ are advised to adjust their FPRs according to the collective FPR of $SG^{(2)}_3$. Then we obtain the revised $CL'=0.870$, so repeat the feedback adjustment. Finally, the revised $CL''=0.922$ is accepted after the second round iteration.

Based on Rule 1, the effective results generate when $\varepsilon\in\left[0.892,0.950\right]$, they are $LG''^{(1)}_{\varepsilon=0.900}=\left\{SG''^{(1)}_1,SG''^{(1)}_2\right\}$, $LG''^{(2)}_{\varepsilon=0.930}=\left\{SG''^{(2)}_1,SG''^{(2)}_2,SG''^{(2)}_3\right\}$, $LG''^{(3)}_{\varepsilon=0.938}=\left\{SG''^{(3)}_1,SG''^{(3)}_2,SG''^{(3)}_3,SG''^{(3)}_4\right\}$, $LG''^{(4)}_{\varepsilon=0.942}=\left\{SG''^{(4)}_1,SG''^{(4)}_2,SG''^{(4)}_3,SG''^{(4)}_4,SG''^{(4)}_5\right\}$ and $LG''^{(5)}_{\varepsilon=0.944}=\left\{SG''^{(5)}_1,SG''^{(5)}_2,SG''^{(5)}_3,SG''^{(5)}_4,SG''^{(5)}_5,SG''^{(5)}_6\right\}$, and they are shown in Fig.5 (a), (b), (c), (d) and (e), respectively.



(a) $LG''^{(1)}_{\varepsilon=0.900}$ (b) $LG''^{(2)}_{\varepsilon=0.930}$

(c) $LG''^{(3)}_{\varepsilon=0.938}$ (d) $LG''^{(4)}_{\varepsilon=0.942}$

(e) $LG''^{(5)}_{\varepsilon=0.944}$

Fig. 5. The dynamic clustering results of the final revised LSGDM

According to Algorithm 2, the related indicators that used to judge the validity of the remaining clustering and determine the suitable clustering result based on Rule 2 and 3 are shown in Table IV.

Based on Rule 2, the valid clustering should satisfy $\min\left(CL^{(p)}_{loc(r)}\right) \geq CL^{(p)}_{glo}\ (p=1,2,3,4,5)$, $LG''^{(2)}_{\varepsilon=0.930}$, $LG''^{(3)}_{\varepsilon=0.938}$, $LG''^{(4)}_{\varepsilon=0.942}$ and $LG''^{(5)}_{\varepsilon=0.944}$ are all satisfied except $LG''^{(1)}_{\varepsilon=0.900}$, so $LG''^{(1)}_{\varepsilon=0.900}$ is removed. Based on Rule 3, $G''^{(4)}_{\varepsilon=0.942}$ is selected as the suitable clustering result for the final determined LSGDM with $\min\left(CL^{(4)}_{loc} / CL^{(4)}_{glo}\right) = 0.962$.

D. Ranking alternatives

Since the agreed OCL is satisfied, output the clustering result $LG = LG''^{(4)}_{\varepsilon=0.942} = \left\{SG''^{(4)}_1, SG''^{(4)}_2, SG''^{(4)}_3, SG''^{(4)}_4, SG''^{(4)}_5\right\}$ and the relevant weights $\left\{0.190, 0.210, 0.209, 0.200, 0.191\right\}$. Based on (21) and (22), the collective FPR is computed as:

$$F'' = \begin{pmatrix} 0.5 & 0.529 & 0.391 & 0.346 \\ 0.471 & 0.5 & 0.337 & 0.422 \\ 0.609 & 0.630 & 0.5 & 0.518 \\ 0.652 & 0.576 & 0.481 & 0.5 \end{pmatrix}$$

Based on (7), compute weights for OWA as $\left\{0, 0.4, 0.5, 0.1\right\}$. Based on (8), compute $f_i\,(i=1,2,3,4)$ as $\left\{0.430, 0.433, 0.552, 0.528\right\}$, therefore, the alternatives are ranked as $x_3 \succ x_4 \succ x_2 \succ x_1$.

TABLE IV

EVALUATE THE CLUSTERING VALIDITY OF THE REMAINING DYNAMIC RESULTS

Dynamic clustering		w	LCL		GCL	GCL/LCL
$LG''^{(1)}_{\varepsilon=0.900}$	$SG''^{(1)}_1$	0.504	0.948	0.935	0.936	1.001
	$SG''^{(1)}_2$	0.496	0.923			
$LG''^{(2)}_{\varepsilon=0.930}$	$SG''^{(2)}_1$	0.351	0.950	0.942	0.934	0.991
	$SG''^{(2)}_2$	0.325	0.940			
	$SG''^{(2)}_3$	0.324	0.935			
$LG''^{(3)}_{\varepsilon=0.938}$	$SG''^{(3)}_1$	0.258	0.948	0.954	0.932	0.977
	$SG''^{(3)}_2$	0.257	0.937			
	$SG''^{(3)}_3$	0.257	0.935			
	$SG''^{(3)}_4$	0.228	1.0000			
$LG''^{(4)}_{\varepsilon=0.942}$	$SG''^{(4)}_1$	0.190	0.971	0.961	0.924	0.962
	$SG''^{(4)}_2$	0.210	0.958			
	$SG''^{(4)}_3$	0.209	0.937			
	$SG''^{(4)}_4$	0.200	0.942			
	$SG''^{(4)}_5$	0.191	1.000			
$LG''^{(5)}_{\varepsilon=0.944}$	$SG''^{(5)}_1$	0.166	0.956	0.958	0.922	0.963
	$SG''^{(5)}_2$	0.173	0.958			
	$SG''^{(5)}_3$	0.172	0.949			
	$SG''^{(5)}_4$	0.165	0.936			
	$SG''^{(5)}_5$	0.162	0.950			
	$SG''^{(5)}_6$	0.162	1.000			

E. The comparison from a numerical perspective

As summarized in Table I, we have proposed a new LSGDM model to handle the limitations of previous studies. However, it is difficult to make a complete comparison between this study and previous studies since they were proposed to deal with LSGDM from different angles. Therefore, we have tried to compare previous references numerically by considering aspects of the clustering analysis, the weights determination, and the following clustering analysis after CRP, respectively.

(1) Clustering analysis

The clustering analysis in LSGDM is mainly performed based on preference similarity [8, 20, 21]. For example, based on $LC = \left(lc_{hk}\right)_{20\times 20}$, the consensus similarity between d_1 and d_2, d_1 and d_{13} is $lc_{12} = 0.750$ and $lc_{1,13} = 0.800$, respectively. The possibility of d_1 and d_{13} belonging to the same subgroup should be greater than d_1 and d_2 because $lc_{1,13} > lc_{1,2}$. However, from Fig.4 (a), d_1 and d_2 are located in the same subgroup $LG^{(1)}_{\varepsilon=0.683}$, while d_1 and d_{13} are never located in the same subgroup, whatever the value of ε is.

That's because Algorithm 1 takes the consensus similarity of the neighbors into account, i.e. the consensus similarity of the common neighbors of d_1 and d_2 is larger than that of d_1 and d_{13}.

(2) Weights determination

According to Table I, the majority principle is usually used to measure the weights of subgroups [15, 16, 22, 23]. In this study, the weights of subgroups are computed based on the size index $\mu_{M(r)}^{(p)}$ and the compactness index $\mu_{C(r)}^{(p)}$, in which, size index is similar to the majority principle. For example, the weights $w_r^{(2)}$ of subgroups in the $LG^{(2)}$ of the original network LG are shown based on $\mu_{M(r)}^{(2)}$ and $\mu_{C(r)}^{(2)}$ in Table V.

TABLE V

THE WEIGHTS DETERMINATION OF SUBGROUPS IN $LG^{(2)}$

r	$\mu_{M(r)}^{(2)}$	$\mu_{C(r)}^{(2)}$	$w_r^{(2)}$
1	0.1	0.350	0.278
2	0.45	0.316	0.357
3	0.45	0.334	0.365

It is easy to cause unfairness if the importance of local CENs is only determined based on the majority principle or compactness. For example, the size index $\mu_{M(1)}^{(2)}$ and the compactness index $\mu_{C(1)}^{(2)}$ shows a completely different importance when $r=1$. Therefore, these two extreme situations need to be adjusted, i.e. the weight $w_1^{(2)}$ is determined based on the combination of $\mu_{M(1)}^{(2)}$ and $\mu_{C(1)}^{(2)}$. What's more, the parameter β in (17) can also be changed to increase or decrease the importance of the intra-cluster consensus compactness.

(3) The CRP and the following clustering analysis

The evolution of the OCL of the LSGDM between Ref [18] and this study is compared in Table VI. From Table VI, we can see that the evolution of consensus in [18] is not very obvious after two rounds of adjustment.

TABLE VI

THE EVOLUTION OF OCL

	Original	Round 1	Round 2
Ref. [39]	0.7358	0.7612	0.7943
This study	0.739	0.870	0.922

It is known that emergency LSGDM requires efficient decision making with limited time. Our study significantly reduces decision making time and yields high consensus results by finding a more convenient negotiation scheme during dynamic clustering. Moreover, one of the explicit features of this study is that we reanalyzed the clustering after the CRP. Therefore, the final clustering result of the LSGDM should be $LG_{\varepsilon=0.942}^{\prime\prime(4)}=\left\{SG_1^{\prime\prime(4)},SG_2^{\prime\prime(4)},SG_3^{\prime\prime(4)},SG_4^{\prime\prime(4)},SG_5^{\prime\prime(4)}\right\}$ rather than $LG_{\varepsilon\in(0.783,0.883]}^{(2)}=\left\{SG_1^{(2)},SG_2^{(2)},SG_3^{(2)}\right\}$.

VI. CONCLUSION

To deal with the complex LSGDM, a dynamic clustering analysis process is designed based on consensus evolution networks with managing consensus thresholds. The clustering analysis is reconsidered after each round of feedback adjustment in CRP to balance the contradiction between the DCAP and CRP in LSGDM.

The advantages of this proposal have been analyzed based on the theoretical and numerical comparison analysis. Differing from the remaining proposal, this study flexibly deals with CRP in LSGDM based on the DCAP. The DCAP is performed based on the consensus evolution relations and the clustering validity is examined based on the intra-cluster consensus levels and inter-cluster consensus level. The proposed model also has managerial significance in practical applications. The DCAP in LSGDM can be adapted to different decision situations by managing consensus thresholds. The reanalysis clustering after CRP is convenient for studying changes in the DMs' preferences and their consensus behavior when influenced by other DMs.

In complex LSGDM, it is important to control adjustment costs. However, in most of the LSGDM models, the adjustment costs are seldom considered despite many CRPs being given. In future work, we intend to determine the overall consensus opinion with the lowest adjustment cost using an optimal model with that uses clustering validity as a constraint condition.

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5 A new clustering algorithm with preference adjustment cost to reduce the cooperation complexity in large-scale group decision making

- T. Wu, X.W. Liu, J.D. Qin, F. Herrera, A New Clustering Algorithm with Preference Adjustment Cost to Reduce the Cooperation Complexity in Large Scale Group Decision Making. **IEEE Transactions on Systems, Man, and Cybernetics: Systems.** 52 (2022) 5271-5283.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 13.451
  - Subject Category: Automation & control Systems, Ranking 1/63 (Q1).
  - Subject Category: Computer Science, Cybernetics, Ranking 1/23 (Q1).

A New Clustering Algorithm with Preference Adjustment Cost to Reduce the Cooperation Complexity in Large-Scale Group Decision-Making

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***Abstract*—In large-scale group decision-making, appropriate clustering analysis is important to consensus reaching since it can reduce the interactive complexity among individuals. According to the traditional clustering method, a conflict may arise between the consensus reaching levels and total adjustment costs within clusters when individuals have different unit adjustment cost, which reflects their willingness to make concessions. Since this conflict may aggravate the consensus complexity, we propose a new K-means clustering method that considers both preferences and the preference adjustment cost. The preference adjustment cost is attached to preferences with a parameter that can be determined by balancing this conflict. Because of such conflict, the proposed clustering algorithm can improve the similarity of intra-cluster individuals on the preference adjustment cost through offsetting some acceptable consensus reaching levels within clusters. According to the proposed clustering algorithm, individuals who have both similar preferences and adjustment willingness are classified into the same clusters. In this way, the moderator can provide similar compensation strategies for intra-cluster individuals, which will decrease the adjustment complexity. A practical case study of team construction examines the application of the proposed algorithm, and the related comparative analysis shows that it is convenient for managers to persuade individuals to reach a consensus under the improved clustering results.**

This work was supported in part by the National Natural Science Foundation of China under Grant (72071045, 71771051, 72071151, 71971115), the Natural Science Foundation of Jiangsu Province (BK20210293), and the Natural Science Foundation of Hubei Province (2020CFB773). (Corresponding authors: Xinwang Liu and Jindong Qin)

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***Index Terms*—Large-scale group decision-making, clustering analysis, consensus reaching process, adjustment cost**

I. INTRODUCTION

THE large-scale group decision-making (LSGDM) is becoming popular when more and more people participate in group decision-making (GDM) due to the development of socialization and networking [1-3]. The complexity of LSGDM is not just caused by its size, but also the interaction among individuals [4-6]. Thus, LSGDM is more complex than traditional GDM, especially for the consensus reaching process [7-9]. Clustering analysis is usually used to classify individuals with similar preferences into subgroups to reduce the complexity of consensus building in LSGDM [10-12]. However, seldom studies improve the efficiency of consensus interaction from the aspect of the clustering analysis.

Variety formats of preference information have been used in existing GDM or LSGDM research, such as fuzzy preference relations (FPRs) [13-16], hesitant FPRs [17-19], linguistic preference relations [20-22], and other preference forms [23-25]. The FPRs is commonly used in GDM since it was introduced by Tanino [13]. Li et al. [14] studied consensus building based on the consistency control with FPRs. González-Arteaga et al. [15] defined a novel consensus measurement method based on FPRs. Chu et al. [16] proposed a social network community analysis-based LSGDM approach with incomplete FPRs. The FPRs may facilitate the decision-making process smoothly [13], and be helpful in the aggregation process of group preferences [26, 27]. Besides, the distance and similarity computations of FPRs have been widely utilized for measuring consensus [28]. Thus, we will classify individuals with opinions represented by FPRs.

To reduce the dimension of LSGDM, many studies mainly focused on the improvement of clustering algorithms [25, 29, 30]. Since consensus is the key to the success of LSGDM, more and more studies showed interest in the consensus reaching process based on clustering results [8-10]. Wu et al. [8] balanced the dynamic clustering analysis and consensus reaching process based on consensus evolution networks. Li et al. [9] conducted the consensus reaching process of LSGDM based on the fuzzy clustering results. Liao et al. [10] introduced an LSGDM model with probabilistic linguistic information to process local and global consensus. However, the critical factor of consensus reaching, i.e., the preference adjustment cost, is rarely considered in the existing LSGDM clustering models except for the preference information.

The preference adjustment cost usually refers to time, effort, or money that individuals will take to adjust their opinions [31-33]. Thus, much attention has been paid to motivate the cooperation behavior among individuals with minimum adjustment or limited cost through mathematical programming [34-40]. Dong and Xu [34] and Yu et al. [35] studied the consensus building in GDM with minimum adjustment. Gong et al. [37] and Zhang et al. [38] paid more attention to consensus reaching with minimum cost. Recently, Zhang et al. [39] analyzed the origin and basic research paradigm of the feedback mechanism with minimum adjustment or cost (FMMA/C), and reviewed FMMA/C in complex GDM contexts, including LSGDM. However, there is no direct correlation between experts' preference and adjustment cost. Thus, when the unit adjustment cost is large enough, the total adjustment cost is not necessarily small even though the preference adjustment reaches the minimum.

When the clustering analysis is carried out mainly based on preference information, individuals with different unit adjustment costs might be classified into the same cluster. For example, when a majority of company stakeholders deny launching a new product, product managers may have more to lose from changing their views than marketing managers although they all fall into the cluster that supports the proposal. Thus, the moderator needs to provide different compensation strategies for individuals within clusters, which will increase the difficulty of intra-cluster individuals in making decisions about whether to adjust preferences or how much to adjust. That is, the adjustment cost is an important influence factor in the clustering analysis of LSGDM, together with reducing the negotiation complexity, which is an important goal. In this paper, we tackle this goal considering the preference adjustment cost in an improved K-means clustering algorithm, which is popular in LSGDM [23, 41]. To achieve this goal, we need to solve the following three interlocking challenges:

(1) How to view the relationship between the preference information and the adjustment cost in the clustering analysis?

(2) How to measure the role of adjustment cost in the clustering analysis based on its relationship with preferences?

(3) How to determine the initial clustering centers to obtain stable k-means clustering results?

To investigate the influence of adjustment cost in the clustering analysis, we give three assumptions as follows.

Assumption 1. Suppose individuals with similar preferences may have different adjustment costs.

Assumption 2. Suppose the adjustment cost of individuals is independent of their preferences.

Assumption 3. Suppose intra-cluster individuals are equally important, i.e., their weights are the same as others.

Based on these assumptions, our contributions mainly focus on settling the above problems from the following three points.

(1) According to Assumption 1, the preference adjustment cost is considered to be an impact factor of the proposed clustering algorithm. We consider the preference information and adjustment cost as dual attributes of individuals in the clustering analysis, where the former plays a significant role, and the latter represents a supporting role.

(2) The distance between individuals is computed based on the dual attributes. According to Assumption 2, the adjustment cost is attached to the clustering analysis with a parameter. After multiple random clustering processes, the parameter of the impact factor is determined by balancing the conflict between the intra-cluster total adjustment costs and the intra-cluster consensus reaching levels.

(3) According to Assumption 3, the clustering centers can be aggregated based on the equal weights of intra-cluster individuals. The initial clustering centers are defined in advance, combining the consensus reaching levels and preference adjustment cost using the determined parameter of the impact factor. Then, we can obtain stable clustering results that are convenient for the following consensus analysis.

A practical case examines the proposed clustering algorithm in team construction. After 10^4 iterations, the parameter of the impact factor is determined to be 0.3258, which considers the conflict between the intra-cluster total adjustment costs and consensus reaching levels. With the determined parameter, the stable clustering results are obtained under the defined four initial clustering centers. Our proposal is compared with traditional K-means without considering the preference adjustment cost. The comparison results show that ignoring partially acceptable intra-cluster consensus with the tradeoff of cost similarity may reduce the negotiation complexity and save decision-making time.

The rest of this paper organizes as follows: Section II introduces the basic concepts of consensus measure and clustering analysis. Section III presents the critical techniques of the proposed clustering analysis considering the preference adjustment cost. Section IV examines the proposal using a case study of team construction and provides a comparative analysis to show its advantages. Section V gives a conclusion.

II. Preliminaries

This section introduces the basic knowledge of the proposed clustering algorithm, including the consensus measure in section *A*, the traditional minimum cost consensus model in section *B*, and K-means in section *C*.

A. Consensus measure

We give the definition of FPRs first since it is commonly used to show individuals' opinions and to measure consensus.

Definition 1. [42] An FPR F is a fuzzy set on the alternative set $X \times X$, which is characterized by a membership function $\mu_F: X \times X \to [0,1]$, where $\mu_F(x_i, x_j) = f_{ij}$ is interpreted as the preference degree of alternative x_i over x_j: x_i and x_j $(i, j = 1,2,...,N)$ are indifference when $f_{ij} = 0.5$. x_i is preferred to x_j when $f_{ij} > 0.5$, x_i is inferior to x_j when

$f_{ij} < 0.5$, and fulfilling $f_{ij} + f_{ji} = 1$. Generally, the FPR of $d_h\left(h = 1,2,...,M\right)$ with respect of the alternative x_i over x_j can be represented as $F_h = \left(f_{ij}^h\right)_{N\times N}$, where $f_{ij}^h + f_{ji}^h = 1$.

The consensus level is often measured based on distance functions [43]. The commonly used distance functions are the distance between individuals' preferences and the group preference, and the pairwise distances between individuals [28], respectively. The first one is useful when we need to measure consensus levels between individuals and the group [14].

Definition 2. [44] The consensus level associated with the individual d_h and the group is defined based on his/her individual preference $F_h = \left(f_{ij}^h\right)_{N\times N}$ and the group FPR $\overline{F} = \left(\overline{f}_{ij}\right)_{N\times N}$ as:

$$CL_h = 1 - \sum_{i,j\in N; i\neq j} \frac{\left|f_{ij}^h - \overline{f}_{ij}\right|}{N\left(N-1\right)} \tag{1}$$

where $CL_h \in \left[0,1\right]$.

Based on the consensus level CL_h, the overall consensus level OCL associated with all individuals in the group can be computed as [14]:

$$OCL = \frac{1}{M}\sum_{h\in M} CL_h \tag{2}$$

where $OCL \in \left[0,1\right]$.

The second one is useful when we need to measure consensus levels between individuals [32].

Definition 3. [32] The consensus level associated with the pair of individual d_h and d_l is defined based on their FPRs F_h and F_l as:

$$CL_{h,l} = 1 - \sum_{i,j\in N; i\neq j} \frac{\left|f_{ij}^h - f_{ij}^l\right|}{N\left(N-1\right)} \tag{3}$$

where $CL_{h,l} \in \left[0,1\right]$.

Based on the consensus level $CL_{h,l}$, the overall consensus level associated to all individuals can be computed as:

$$OCL = \frac{1}{M\left(M-1\right)}\sum_{h,l\in M; h\neq l} CL_{h,l} \tag{4}$$

where $OCL \in \left[0,1\right]$.

B. Minimum adjustment or cost consensus model through mathematical programming

In group decision-making contexts, the feedback mechanism with minimum adjustment or cost (FMMA/C) has been developed widely [39]. For an LSGDM problem, let $o_h \in \left[0,1\right]$ denote the initial preference of the individual d_h, $h = 1,2,...,M$, and $\overline{o}$ denotes the group consensus preference, $C_h \in \left[0,1\right]$ represents the unit adjustment cost of d_h to adjust per unit preference. The distance between d_h and the consensus preference $\overline{o}$ is determined to be $\left|o_h - \overline{o}\right|$. Ben-Arieh and Easton [45] defined the linear consensus cost to move d_h's preference from o_h to $\overline{o}$ as $C_h\left|o_h - \overline{o}\right|$. A nonlinear optimization model was constructed under the premise that a consensus preference can be obtained with the minimum total cost ϕ [36]:

$$\min \phi = \sum_{h\in M} C_h\left|o_h - \overline{o}\right| \qquad s.t.\ \overline{o} \in \overline{O} \tag{5}$$

where $\overline{O} = \left\{\overline{o} \in R \middle| \overline{o} > 0\right\}$ is the set of all possible consensus opinions.

C. K-means clustering analysis

K-means is a famous and commonly used clustering algorithm based on the iterative refinement technique [23, 41]. The goal of K-means is to assign objects into K clusters when the Euclidean distance $D\left(d_l, M_r\right)$ of each point from the clustering centers is minimal:

$$\min \varphi = \sum_{r\in K}\sum_{d_l\in G_r} D\left(d_l, M_r\right) \tag{6}$$

where K is the number of clusters, d_l belongs to the rth cluster G_r, and M_r denotes the center of the cluster G_r.

The classical K-means mainly consists of two steps:

Step 1. Select K initial clustering centers randomly from all objects and assign the remaining objects into K clusters according to the minimum distance between objects and initial clustering centers.

Step 2. Recalculate the new centers of clusters and redefine the distance of each object from the new centers. The process stops when the assignments do not change, or the mean error of the distance is smaller than a set threshold.

III. Clustering Algorithm Considering the Preference Adjustment Cost for LSGDM

Table 1 shows the main procedures of the proposed LSGDM clustering algorithm considering the combination of the preference information and adjustment cost.

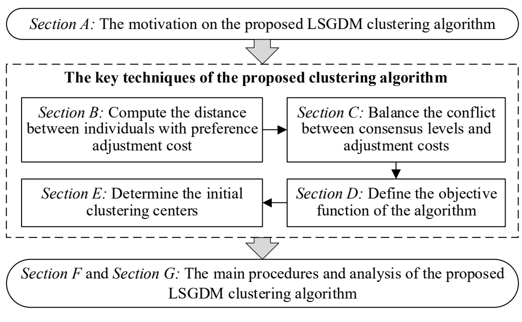


Fig.1 The main tasks of the proposed clustering algorithm

A. The motivation on the LSGDM clustering algorithm with preference adjustment cost

In LSGDM, preference conflict is widespread since individuals may have different knowledge and experiences. Persuading individuals to reach some kind of consensus is critical for decision effect and quality. However, according to Assumption 1, individuals are unwilling to adjust their preferences unless they gain corresponding compensation. The preference adjustment cost reflects individuals' willingness to make concessions indirectly. Thus, we believe that the preference adjustment cost plays an essential role in the consensus reaching process when individuals are not entirely altruistic.

In reality, large groups can divide into informal clusters due to similar preferences or interests spontaneously. Clustering analysis is a useful tool to discover these clusters. However, the traditional clustering analysis in LSGDM is mainly carried out based on preference information without considering the preference adjustment cost. The total adjustment cost of intra-cluster individuals can vary considerably when they have different unit adjustment costs, which may cause adjustment conflict within clusters. Thus, the moderator may need to cost more time and energy to persuade intra-cluster individuals to reach a consensus of adjustment strategies. Thus, we propose an LSGDM clustering analysis algorithm that considers the preference adjustment cost.

Preference, which reflects experts' professions directly, is the basis of scientific decision-making. Compared with preferences, the preference adjustment cost has a limited impact on the outcome of decisions. In some cases, we would like to invest more cost to pursue professional choices, rather than reduce decision quality for saving costs. Thus, we hold that preferences still play a decisive role in the clustering process and the preference adjustment cost plays a supporting role to modify the clustering results.

B. Compute the distance between individuals combining the preference information and adjustment cost

The clustering algorithm considering the preference adjustment cost is proposed based on K-means. In K-means, distance computation is the foundation for measuring cohesion and separation among individuals. Thus, it is vital to compute the distance among individuals considering the combination of the preference information and the preference adjustment cost.

We hold that preferences and the preference adjustment costs are dual attributes of individuals in clustering analysis. Let $D:\{F,C\}$ denotes the dual attributes, where $F=\left(F_h\right)_{1\times M}$ means preferences of individual d_h represented with FPRs and $C=\left(C_h\right)_{1\times M}$ means the unit adjustment cost of d_h , $C_h\in[0,1]$. We call the distance between the preference of d_h and d_l the preference-based pairwise distance $dis\left(F_h,F_l\right)$. According to (3), $dis\left(F_h,F_l\right)$ can be computed based on the FPRs $F_h=\left(f_{ij}^h\right)_{N\times N}$ and $F_l=\left(f_{ij}^l\right)_{N\times N}$:

$$dis\left(F_h,F_l\right)=\sum_{i,j\in N;i\neq j}\frac{\left|f_{ij}^h-f_{ij}^l\right|}{N\left(N-1\right)} \tag{7}$$

where $dis\left(F_h,F_l\right)\in[0,1]$.

We call the pairwise distance $dis\left(C_h,C_l\right)$ between d_h and d_l the cost-based pairwise distance. According to Manhattan distance, $dis\left(C_h,C_l\right)$ can be computed as:

$$dis\left(C_h,C_l\right)=\left|C_h-C_l\right| \tag{8}$$

where $dis\left(C_h,C_l\right)\in[0,1]$.

Let the preference adjustment cost be the additional attribute of the proposed clustering algorithm with a parameter $\alpha\in[0,1]$. To consider the dual attributes in the clustering analysis, we determine the combined distance $D\left(d_h,d_l\right)$ based on the preference-based pairwise distance and the cost-based pairwise distance using the Euclidean distance:

$$\begin{aligned}D\left(d_h,d_l\right)&=\left(dis^2\left(F_h,F_l\right)+\alpha\times dis^2\left(C_h,C_l\right)\right)^{\frac{1}{2}}\\&=\left(\left(\sum_{i,j\in N;i\neq j}\frac{\left|f_{ij}^h-f_{ij}^l\right|}{N\left(N-1\right)}\right)^2+\alpha\times\left|C_h-C_l\right|^2\right)^{\frac{1}{2}}\end{aligned} \tag{9}$$

where $D\left(d_h,d_l\right)\in[0,1]$ since $dis\left(F_h,F_l\right),dis\left(C_h,C_l\right)\in[0,1]$.

The combined distance $D\left(d_h,d_l\right)$ is used to measure the individuals' similarity in the proposed clustering algorithm. The function of the preference adjustment cost in the proposed clustering analysis reduces as α decreases and reaches to its minimum when $\alpha=0$, i.e., $D\left(d_h,d_l\right)=dis\left(F_h,F_l\right)$. In this context, individuals are only classified with FPRs. The function of the preference adjustment cost in the proposed clustering analysis improves as α increases and reaches to its maximum when $\alpha=1$, i.e., $D\left(d_h,d_l\right)=\sqrt{dis^2\left(F_h,F_l\right)+dis^2\left(C_h,C_l\right)}$. Correspondingly, the role of FPRs in clustering analysis reduces or improves with the increase or decrease of α . Thus, the impact of preference information and preference adjustment cost in clustering analysis is mutually inhibitory with the change of α . That is, when we pay more attention to preferences, there is a higher level of consensus among intra-cluster individuals, but there may be a large deviation between their unit adjustment costs.

When we pay more attention to the preference adjustment costs, the unit adjustment costs is similar among intra-cluster individuals, but their preference difference may be large, resulting in a large deviation of the total adjustment costs. Thus, the change of α may cause the conflict between consensus reaching levels and the total adjustment costs of each intra-cluster member.

C. Balance the conflict between the consensus reaching levels and the total adjustment cost

To balance the conflict between consensus reaching levels and the total adjustment costs, we analyze the value of α based on two kinds of gaps. One is the mean error of intra-cluster consensus reaching levels, and the other one is the mean error of intra-cluster total adjustment costs. For computing these gaps, the consensus preference of the large group should be determined first. Since the coordinator often gives compensation for individuals based on their adjustment costs, we define the consensus preference based on the minimum cost model.

Based on Model (5), the minimum total adjustment cost model can be extended with FPRs as:

$$\min \phi(F) = \sum_{h\in M} \sum_{i,j\in N; i\neq j} C_h \frac{\left|f_{ij}^h - \overline{f}_{ij}\right|}{N(N-1)} \quad (10)$$
$$s.t.\ \overline{f}_{ij} + \overline{f}_{ji} = 1;\ i,j = 1,2,...,N$$

where $\overline{F} = \left(\overline{f}_{ij}\right)_{N\times N}$ is the possible consensus FPR of the large group and $\overline{f}_{ij} > 0$.

According to the resolver tool, Lingo, it is easy to solve Model (10) to obtain the optimal FPR $F^* = \left(f_{ij}^*\right)_{N\times N}$. Then, the mean error of intra-cluster consensus reaching levels can be determined as follows.

(1) Suppose M individuals are classified into K clusters $G_r\left(r=1,2,...,K\right)$ with a certain value of α and the optimal consensus FPR F^* denotes the preference of the whole group, the consensus reaching level CL_r^h between each individual $d_h \in G_r$ and F^*, $h=1,2,...,M$, $r=1,2,...,K$, $K<M$, can be computed based on (1) as:

$$\begin{aligned} CL_r^h &= 1 - dis\left(F_h, F^*\right) \\ &= 1 - \sum_{i,j\in N; i\neq j} \frac{\left|f_{ij}^h - f_{ij}^*\right|}{N(N-1)} \end{aligned} \quad (11)$$

(2) Based on consensus reaching level CL_r^h, the error of intra-cluster consensus reaching levels $\eta_\alpha\left(d_h, d_l\right)$ between each pair of individuals $\left(d_h, d_l\right)$ can be calculated as:

$$\eta_\alpha\left(d_h, d_l\right) = \left|CL_r^h - CL_r^l\right| \quad (12)$$

where individual d_h and d_l belongs to the same cluster, i.e., $d_h, d_l \in G_r$.

(3) Based on the error of intra-cluster consensus reaching levels $\eta_\alpha\left(d_h, d_l\right)$, the mean error of intra-cluster consensus reaching levels ME_{consen}^α can be computed for each clustering result as:

$$ME_{consen}^\alpha = \frac{1}{K}\sum_{r\in K}\left(\frac{\sum_{d_h,d_l\in G_r} \eta_\alpha\left(d_h,d_l\right)}{N\left(G_r\right)\left(N\left(G_r\right)-1\right)}\right) \quad (13)$$

where K is the number of clusters and $N\left(G_r\right)$ is the number of individuals in G_r.

Similarly, the mean error of the intra-cluster total adjustment cost can be determined as follows.

(1) The total adjustment costs TC_h of individuals for adjusting FPRs toward to the optimal FPR F^* can be computed with the distance $dis\left(F_h, F^*\right)$ and the unit adjustment cost C_h as:

$$TC_h = \sum_{i,j\in N; i\neq j} C_h \frac{\left|f_{ij}^h - f_{ij}^*\right|}{N(N-1)} \quad (14)$$

(2) Based on TC_h, the error of intra-cluster total adjustment cost $\xi_\alpha\left(d_h, d_l\right)$ between each pair of individuals $\left(d_h, d_l\right)$ is computed as:

$$\xi_\alpha\left(d_h, d_l\right) = \left|TC_h - TC_l\right| \quad (15)$$

where $d_h, d_l \in G_r$.

(3) Based on $\xi_\alpha\left(d_h, d_l\right)$, compute the mean error of the intra-cluster total adjustment cost ME_{cost}^α for each clustering result when α takes different values as:

$$ME_{cost}^\alpha = \frac{1}{K}\sum_{r\in K}\left(\frac{\sum_{d_h,d_l\in G_r} \xi_\alpha\left(d_h,d_l\right)}{N\left(G_r\right)\left(N\left(G_r\right)-1\right)}\right) \quad (16)$$

where K is the number of clusters and $N\left(G_r\right)$ is the number of individuals in G_r.

Since the influence of the preference information in the clustering analysis decreases and the impact of the preference adjustment costs increases with the increasing value of α, the general trend of the mean error of the intra-cluster consensus reaching levels ME_{consen}^α increases and the general direction of the mean error of the intra-cluster total adjustment cost ME_{cost}^α decreases as the value of α increases.

According to the clustering principle, the closer the intra-cluster individuals are and the more sparse the clusters are, the better the clustering effect is. Similarly, the smaller both ME_{consen}^α and ME_{cost}^α, the better the clustering effect is. To determine the value of α, we propose a weighted method as:

$$\min\left(\omega \times ME_{consen}^\alpha + \mu \times ME_{cost}^\alpha\right) \quad (17)$$

where ω and μ are the weights of ME_{consen}^α and ME_{cost}^α, respectively.

Since the general trend of ME_{consen}^{α} and ME_{cost}^{α} is opposite as the value of α changes, when one of the weights ω and μ increases, the other one should accordingly decreases to obtain the minimum value of (17). Because k-means randomly selects the initial clustering centers in each iteration, the result of ME_{consen}^{α} and ME_{cost}^{α} may be different for the same value of α each time. Generally, we can select proper clustering results and determine the value of α based on the results of ME_{consen}^{α} and ME_{cost}^{α} through multiple iterations. Suppose $\exists \max ME_{consen}^{\alpha} \neq \min ME_{consen}^{\alpha}$ and $\max ME_{cost}^{\alpha} \neq \min ME_{cost}^{\alpha}$, let $a^{*} = \max ME_{consen}^{\alpha}$, $a_{*} = \min ME_{consen}^{\alpha}$, $b^{*} = \max ME_{cost}^{\alpha}$, and $b_{*} = \min ME_{cost}^{\alpha}$, then the value of the weight ω and μ can be determined as:

$$\omega = 1 - \frac{a^{*} - a_{*}}{a^{*} - a_{*} + b^{*} - b_{*}} \tag{18}$$

$$\mu = 1 - \frac{b^{*} - b_{*}}{a^{*} - a_{*} + b^{*} - b_{*}} \tag{19}$$

where $\omega + \mu = 1$, which is consistent with the above analysis that the weight ω and μ can check and balance.

D. Define the objective function of the proposed clustering algorithm

The objective function, that is used to determine the minimum distance between individuals and clustering centers, is the basis of K-means. Based on the combined distance $D(d_h, d_l)$ among individuals with respect of the dual attributes $D_h : \{F_h, C_h\}$ and the analysis of the parameter α, the objective function of the clustering algorithm considering adjustment cost can be constructed.

Let P_r, $r = 1, 2, ..., K$, denotes the clustering centers of K clusters G_r. Similar to individuals, the clustering center P_r of the corresponding cluster G_r can be considered as an virtual expert and he/she can be characterized with the dual attributes of the average FPRs PF_r and the average cost PC_r, i.e. $P_r : \{PF_r, PC_r\}$.

The average FPRs PF_r of the cluster G_r can be computed as:

$$pf_{ij}^{r} = \frac{1}{N(G_r)} \sum_{d_h \in G_r} f_{ij}^{h} \tag{20}$$

where $N(G_r)$ is the number of individuals belonging to the cluster G_r, and PF_r can be regarded as the group FPR of G_r.

Similarly, the average adjustment cost PC_r of the cluster G_r can be computed as:

$$PC_r = \frac{1}{N(G_r)} \sum_{d_h \in G_r} C_h \tag{21}$$

Based on the clustering centers $P_r : \{PF_r, PC_r\}$, the extended objective function $\varphi(F, C, \alpha)$ of the proposed clustering algorithm is constructed based on (6) as:

$$\begin{aligned} \min \varphi(F, C, \alpha) &= \sum_{r \in K} \sum_{d_l \in G_r} D(d_l, P_r) \\ &= \sum_{r \in K} \sum_{d_l \in G_r} \left(dis^2(F_l, PF_r) + \alpha \times dis^2(C_l, PC_r) \right)^{\frac{1}{2}} \end{aligned} \tag{22}$$

Based on (1), (8), (17), (18), and (19), the optimization function $\min \varphi(F, C, \alpha)$ can be constructed as:

$$\min \varphi(F, C, \alpha)$$

$$= \sum_{r \in K} \sum_{d_l \in G_r} \left(\left(\sum_{i,j \in N; i \neq j} \frac{\left| f_{ij}^{l} - pf_{ij}^{r} \right|}{N(N-1)} \right)^2 + \alpha \times \left| C_l - PC_r \right|^2 \right)^{\frac{1}{2}} \tag{23}$$

$$s.t. \begin{cases} \min \left(\omega \times ME_{consen}^{\alpha} + \mu \times ME_{cost}^{\alpha} \right) \\ \omega = 1 - \dfrac{a^{*} - a_{*}}{a^{*} - a_{*} + b^{*} - b_{*}} \\ \mu = 1 - \dfrac{b^{*} - b_{*}}{a^{*} - a_{*} + b^{*} - b_{*}} \end{cases}$$

According to the critical techniques introduced above, the general form of the proposed clustering algorithm is given as Algorithm 1. Much more importantly, the value of α can be determined using this algorithm.

Algorithm 1: The determination of the value of α

Inputs: M individuals with double attributes $D : \{F, C\}$, the number of clusters $K(K < M)$ and the number of iterations T.

Step 1: Randomly choose K different individuals $d_h : \{F_h, C_h\}$ as initial clustering centers $P_h : \{PF_h, PC_h\}$, $h = 1, 2, ..., M$.

Step 2: Assign individuals $d_l\ (l = 1, 2, ..., M, h \neq l)$ to the closest clusters with random α based on the optimal model (23), and produce the new clusters G_r, $r = 1, 2, ..., K$. If there is one individual in the cluster G_r, exit this algorithm directly and try a new iteration. Otherwise, continue to the next step.

Step 3: Reconstruct the clustering centers $P_r : \{PF_r, PC_r\}$ of each G_r using (20) and (21).

Step 4: Repeat from **Step 2** until there is no change in the clustering results.

Step 5: Repeat **Step 1-4** to determine the value of α until T iterations are finished.

Output: The value of α.

E. Determine the initial clustering centers for the proposed clustering algorithm

The randomly selected initial clustering centers tend to produce unstable clustering results that are not conducive to the following consensus reaching analysis. In this section, we define K stable initial clustering centers for the proposed algorithm based on the dual attributes of individuals with the value of α.

First, find the individual d_h who has the highest level of consensus and most similar unit adjustment costs with others based on (3) and (8) as the first clustering center:

$$\max\left(\sum_{l\in M,l\neq h} CL_{h,l} - \alpha\times\sum_{h,l\in M,h\neq l} dis\left(C_h,C_l\right)\right) \quad (24)$$

Then, find the individual $d_x\left(x\in M, x\neq h\right)$ who has the lowest level of consensus and the least similar unit adjustment cost with d_h as the second clustering center:

$$\min\left(CL_{x,h} - \alpha\times dis\left(C_x,C_h\right)\right) \quad (25)$$

Next, find the individual d_y who has the as lowest level of consensus and the least similar unit adjustment cost both with d_h and d_x as the next clustering center:

$$\min\begin{pmatrix}\left(CL_{y,x}+CL_{y,h}\right)/N\left(P\right)\\ \alpha\left(dis\left(C_y,C_x\right)+dis\left(C_y,C_h\right)\right)/N\left(P\right)\end{pmatrix} \quad (26)$$

where $N\left(P\right)$ denotes the number of clustering centers that has been determined, i.e. $N\left(P\right)=2$, $x,y,h\in M$, $y\neq x$, and $y\neq h$.

Finally, repeat (26) until all $K\left(\text{if } K>3\right)$ initial clustering centers are defined. Based on the defined initial clustering centers and the value of α, the specific form of the proposed clustering algorithm is given as Algorithm 2.

Algorithm 2: The clustering algorithm considering the preference adjustment costs

Inputs: M individuals with double attributes $D:\left\{F,C\right\}$, the number of clusters K, and the value of α.

Step 1: Define K $\left(\text{if } K>3\right)$ initial clustering centers $P_h:\left\{PF_h,PC_h\right\}$ of the clusters G_h based on (24), (25), and (26), $h=1,2,...,M$, $K<M$.

Step 2: Repeat **Step 2-4** of Algorithm 1 until there is no change in the clustering results.

Output: The clustering results G_r, $r=1,2,...,K$.

F. The main procedures of the proposed clustering algorithm

Based on the critical techniques introduced above, the main procedures of the proposed clustering algorithm are introduced in details.

Step 1. Solve the optimal FPR for the group

Identify an LSGDM that consists of M individuals $D=\left\{d_1,d_2,...,d_M\right\}$ and N alternatives $X=\left\{x_1,x_2,...,x_N\right\}$. Gather the preference information of all individuals with FPRs $F_h\left(h=1,2,...,M\right)$ and obtain the optimal FPR F^* for the group based on (10) using *Lingo 10*.

Step 2. Determine the parameter of the impact factor using the proposed clustering algorithm

Consider the FPRs F_h and the adjustment cost $C_h\left(h=1,2,...,M\right)$ as the dual attributes of individuals in the clustering algorithm. Set the clustering number K and the iterations T according to the context and demand of the LSGDM. Determine the appropriate value of α using Algorithm 1.

Step 3. Define the initial clustering centers

With the determined value of α, define the first three clustering centers based on (24), (25), and (26), respectively. Then, learn the rest of the clustering centers based on (26) until $K\left(\text{if } K>3\right)$ initial clustering centers are selected.

Step 4. Obtain the clustering results using the proposed clustering algorithms

With the determined value of α and K initial clustering centers, obtain the stable clustering results G_r using the specific Algorithm 2.

Step 5. Analyze the consensus situation after clustering

Compute the consensus reaching level CL_r^* between each cluster's FPR F_h and the optimal group FPR F^*, the overall consensus level OCL_r^* of clusters, the error of intra-cluster consensus reaching levels ME_{consen}^{α}, and the error of intra-cluster total adjustment cost ME_{cost}^{α} to analyze the consensus situation after clustering.

In the proposed clustering algorithm, the preference information and the adjustment costs are regarded as dual attributes of individuals with a parameter α. A constraint condition with the parameter α is determined under the objective function to obtain clustering results with high intra-cluster consensus reaching levels and low intra-cluster total adjustment costs. Based on the determined value of α and the initial clustering centers, a stable clustering result is obtained to analyze the following consensus reaching process.

G. The analysis of the proposed clustering algorithm

In the proposed clustering algorithm, the optimal preference obtained based on the minimum cost model provides a reference to measure the contradiction between intra-cluster consensus reaching levels and the total adjustment costs.

The main challenge of the proposed clustering algorithm is the determination of the parameter α, which reflects the supporting degree of the preference adjustment costs and the dominant role of preference information in the clustering analysis. The conflict between intra-cluster consensus reaching levels and the total adjustment costs may appear when the dual attributes are considered in the clustering algorithms. The condition that the parameter needs to satisfy is given as (17), (18), and (19). The parameter value generated under this condition is exactly where the two contradictory indicators can get the final tradeoff. Once the value of α is obtained, the supporting role of the preference adjustment cost is determined and the specific clustering Algorithm 2 can be used to get stable clustering results, that can be used to analyze the following consensus problem.

IV. A Practical Case Study of Team Construction

In this section, a practical case study of team construction shows the application of the proposed clustering algorithm. The main processes include introducing the background of team construction, implementing the proposed clustering algorithm, and discussing the comparative analysis.

A. Background of team construction

As enterprises realize the importance of team management more and more, team construction has become a necessary project for internal activities of the company. Most of the managers believe that team construction can enhance communications among colleagues, improve teamwork ability, establish a harmonious working relationship, share the company's core values, and release the pressure on employees.

Nowadays, team construction is mainly contracted by travel companies. Ctrip.com is a Chinese online travel company. It opened up a new market for customized online corporate travel to be the first professional platform built for a large number of corporate customers and suppliers. Ctrip.com reported its first annual "customized travel index report of Chinese enterprise team construction" in June 2018. The report analyzed the massive team construction order data completed by more than 34,000 enterprise customers and more than 1,200 customized suppliers on Ctrip's customization platform in 2018. In the first half of 2018, the number of orders for enterprise customization increases by 200% year on year. The team construction customization accounted for 15% of the total enterprise customization, in which internet companies have the highest percentage of team construction. There are three main types of team construction: one is relaxation and leisure, such as meditation, beautiful scenery, and food tour. The other is outdoor expansion activities, such as desert island survival, grassland hiking, and desert crossing. The last is overseas study and investigation, such as enterprise investigation and elite training.

Suppose a manager of team construction of an internet company with more than 300 employees has selected four alternatives from multiple plans given by Ctrip.com. Currently, the head of team construction needs to determine the final proposal from the following four options, (1) x_1: A trip to Saipan (2) x_2: Lingshan retreat for meditation (3) x_3: Inner Mongolia grassland hiking and desert crossing (4) x_4: World famous universities training. To save decision time and cost, the head of team construction gathered 25 group leaders to evaluate the four alternatives. Each of the group leader represents the interests of a small group of more than a dozen employees. The group leaders give their evaluations on four options with the FPRs. Meanwhile, all group leaders evaluate the degree of non-cooperation among members in their group from 0 to 1, the larger the estimated value, the more difficult the group is to coordinate. In other words, we regard such intra-group consistency value as the unit adjustment costs of group leaders.

The complexity of LSGDM reflects not only in the size of members but also in the complicated interactions between individuals. Thus, the group decision making process of team construction consists of 25 group leaders can be considered as an LSGDM problem. The group leaders are regarded as individuals $D=\{d_1,d_2,...,d_{25}\}$ in the LSGDM. The FPRs $F_h\ (h=1,2,...,25)$ evaluated by 25 group leaders concerning four alternatives $X=\{x_1,x_2,x_3,x_4\}$ and their corresponding unit adjustment costs $C=\{C_1,C_2,...,C_{25}\}$ are given as follows.

$$F_1=\begin{pmatrix}0.5&0.8&0.9&0.3\\0.2&0.5&0.7&0.4\\0.1&0.3&0.5&0.4\\0.7&0.6&0.6&0.5\end{pmatrix}\quad F_2=\begin{pmatrix}0.5&0.2&0.6&0.8\\0.8&0.5&0.6&0.7\\0.4&0.4&0.5&0.3\\0.2&0.3&0.7&0.5\end{pmatrix}\quad F_3=\begin{pmatrix}0.5&0.6&0.7&0.4\\0.4&0.5&0.6&0.4\\0.3&0.4&0.5&0.3\\0.6&0.6&0.7&0.5\end{pmatrix}$$

$$F_4=\begin{pmatrix}0.5&0.1&0.3&0.4\\0.9&0.5&0.6&0.6\\0.7&0.4&0.5&0.8\\0.6&0.4&0.2&0.5\end{pmatrix}\quad F_5=\begin{pmatrix}0.5&0.4&0.4&0.1\\0.6&0.5&0.1&0.5\\0.6&0.9&0.5&0.4\\0.9&0.5&0.6&0.5\end{pmatrix}\quad F_6=\begin{pmatrix}0.5&0.4&0.6&0.7\\0.6&0.5&0.9&0.8\\0.4&0.1&0.5&0.9\\0.3&0.2&0.1&0.5\end{pmatrix}$$

$$F_7=\begin{pmatrix}0.5&0.2&0.7&0.6\\0.8&0.5&0.9&0.8\\0.3&0.2&0.5&0.6\\0.4&0.1&0.4&0.5\end{pmatrix}\quad F_8=\begin{pmatrix}0.5&0.6&0.7&0.8\\0.4&0.5&0.4&0.8\\0.3&0.6&0.5&0.7\\0.2&0.2&0.3&0.5\end{pmatrix}\quad F_9=\begin{pmatrix}0.5&0.2&0.6&0.6\\0.8&0.5&0.9&0.8\\0.4&0.1&0.5&0.9\\0.4&0.2&0.1&0.5\end{pmatrix}$$

$$F_{10}=\begin{pmatrix}0.5&0.7&0.6&0.5\\0.3&0.5&0.6&0.7\\0.4&0.4&0.5&0.9\\0.5&0.3&0.1&0.5\end{pmatrix}\quad F_{11}=\begin{pmatrix}0.5&0.8&0.6&0.3\\0.2&0.5&0.7&0.5\\0.4&0.3&0.5&0.2\\0.7&0.5&0.8&0.5\end{pmatrix}\quad F_{12}=\begin{pmatrix}0.5&0.3&0.3&0.2\\0.7&0.5&0.6&0.6\\0.7&0.4&0.5&0.6\\0.8&0.4&0.4&0.5\end{pmatrix}$$

$$F_{13}=\begin{pmatrix}0.5&0.6&0.2&0.3\\0.4&0.5&0.4&0.3\\0.8&0.6&0.5&0.4\\0.7&0.7&0.6&0.5\end{pmatrix}\quad F_{14}=\begin{pmatrix}0.5&0.8&0.7&0.2\\0.2&0.5&0.8&0.3\\0.3&0.2&0.5&0.1\\0.8&0.7&0.9&0.5\end{pmatrix}\quad F_{15}=\begin{pmatrix}0.5&0.6&0.4&0.5\\0.4&0.5&0.2&0.2\\0.6&0.8&0.5&0.4\\0.5&0.8&0.6&0.5\end{pmatrix}$$

$$F_{16}=\begin{pmatrix}0.5&0.4&0.4&0.3\\0.6&0.5&0.1&0.2\\0.6&0.9&0.5&0.4\\0.7&0.8&0.6&0.5\end{pmatrix}\quad F_{17}=\begin{pmatrix}0.5&0.3&0.4&0.1\\0.7&0.5&0.5&0.4\\0.6&0.5&0.5&0.7\\0.9&0.6&0.3&0.5\end{pmatrix}\quad F_{18}=\begin{pmatrix}0.5&0.7&0.4&0.2\\0.3&0.5&0.3&0.7\\0.6&0.7&0.5&0.6\\0.8&0.3&0.4&0.5\end{pmatrix}$$

$$F_{19}=\begin{pmatrix}0.5&0.6&0.7&0.1\\0.4&0.5&0.9&0.3\\0.3&0.1&0.5&0.4\\0.9&0.7&0.6&0.5\end{pmatrix}\quad F_{20}=\begin{pmatrix}0.5&0.8&0.4&0.1\\0.2&0.5&0.3&0.4\\0.6&0.7&0.5&0.7\\0.9&0.6&0.3&0.5\end{pmatrix}\quad F_{21}=\begin{pmatrix}0.5&0.7&0.7&0.3\\0.3&0.5&0.8&0.4\\0.3&0.2&0.5&0.2\\0.7&0.6&0.8&0.5\end{pmatrix}$$

$$F_{22}=\begin{pmatrix}0.5&0.3&0.2&0.3\\0.7&0.5&0.6&0.6\\0.8&0.4&0.5&0.6\\0.7&0.4&0.4&0.5\end{pmatrix}\quad F_{23}=\begin{pmatrix}0.5&0.6&0.2&0.2\\0.4&0.5&0.4&0.3\\0.8&0.6&0.5&0.4\\0.8&0.7&0.6&0.5\end{pmatrix}\quad F_{24}=\begin{pmatrix}0.5&0.5&0.7&0.8\\0.5&0.5&0.2&0.7\\0.3&0.8&0.5&0.9\\0.2&0.3&0.1&0.5\end{pmatrix}$$

$$F_{25}=\begin{pmatrix}0.5&0.7&0.4&0.5\\0.3&0.5&0.1&0.2\\0.6&0.9&0.5&0.4\\0.5&0.8&0.6&0.5\end{pmatrix}\quad C=\left\{\begin{matrix}0.1,&0.3,&0.5,&0.7,&0.8,&0.1,&0.5,\\0.7,&0.4,&0.5,&0.5,&0.8,&0.4,&0.6,\\0.2,&0.8,&0.1,&0.4,&0.7,&0.3,&0.6,\\0.4,&0.9,&0.2,&0.7&&&\end{matrix}\right\}$$

B. The application of the proposed clustering algorithm on the case study

The implementation of the proposed clustering algorithm on the case study of team construction is analyzed from the following main procedures.

Step 1. Solve the optimal FPR for the group

Based on the FPRs of all individuals, the optimal FPR $F^*=\left(f_{ij}^*\right)_{4\times 4}$ is computed based on the minimum total adjustment cost Model (10).

$$F^*=\begin{pmatrix}0.5&0.6&0.4&0.3\\0.4&0.5&0.6&0.5\\0.6&0.4&0.5&0.4\\0.7&0.5&0.6&0.5\end{pmatrix}$$

Step 2. Determine the parameter of the impact factor using the proposed clustering algorithm

Set the number of iterations as $T=10^4$ and the cluster number $K=4$. Run Algorithm 1 and obtain $\alpha=0.3258$. Besides, the trend line of ME_{consen}^{α} and ME_{cost}^{α} obtained through 10^4 random iterations is simulated in Fig.2. We can find that the general trend of ME_{consen}^{α} increases with the increasing value of α and the general direction of ME_{cost}^{α} decreases.

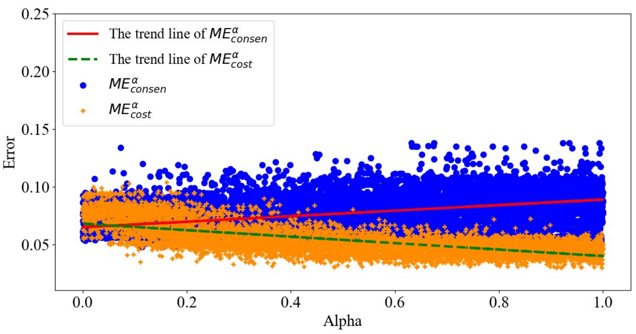


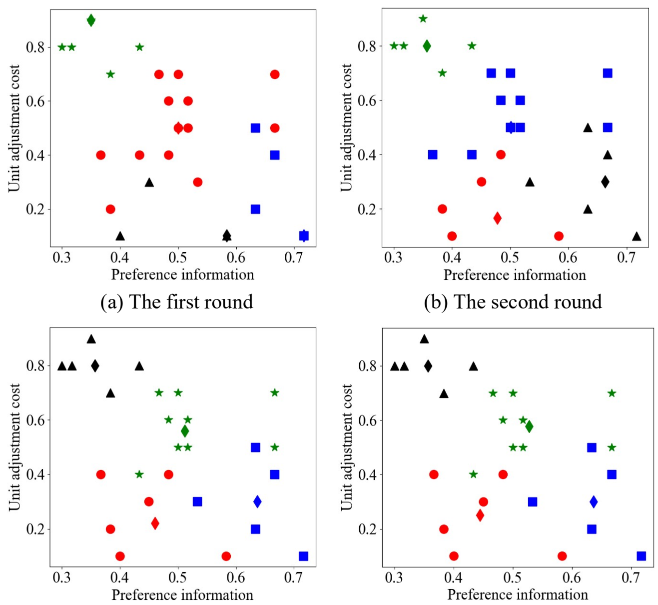
Fig.2. The distribution and trend of ME_{consen}^{α} and ME_{cost}^{α} through 10^4 random iterations

Step 3. Define the initial clustering centers

Referring to (24), (25), and (26), individuals d_3, d_6, and d_{23} are respectively identified as the first three initial clustering centers: P_1, P_2, and P_3. Based on P_1, P_2, and P_3, the individual d_1 is determined as the last clustering center P_4 using (26).

Step 4. Obtain the clustering results using the proposed clustering algorithm

Based on the defined initial clustering centers and $\alpha=0.3258$, obtain the stable clustering results after four rounds of Algorithm 2. Fig.3 shows the clustering results. In Fig.3, the thin diamond markers represent cluster centers, the vertical axis represents the unit adjustment costs of individuals, and the horizontal axis represents individuals' preference information, which is determined based on the mean of the upper trig elements in the FPR matrix. For example, for the first initial clustering center P_1, which is defined based on the individual d_3, its coordinate is determined to be $(0.5, 0.5)$.



(a) The first round (b) The second round

(c) The third round (d) The fourth round

Fig.3. The clustering results with $\alpha=0.3258$

According to the clustering results, 25 individuals are classified into four clusters, $G_1=\{d_1,d_{13},d_{15},d_{17},d_{18},d_{20}\}$, $G_2=\{d_2,d_6,d_7,d_9,d_{24}\}$, $G_3=\{d_3,d_4,d_8,d_{10},d_{11},d_{14},d_{19},d_{21},d_{22}\}$, and $G_4=\{d_5,d_{12},d_{16},d_{23},d_{25}\}$ with the set of unit adjustment costs {0.1, 0.4, 0.2, 0.1, 0.4, 0.3}, {0.3, 0.1, 0.5, 0.4, 0.2}, {0.5, 0.7, 0.7, 0.5, 0.5, 0.6, 0.7, 0.6, 0.4}, and {0.8, 0.8, 0.8, 0.9, 0.7}, respectively. We can find that there is little difference between the unit adjustment costs of intra-cluster individuals.

Let $P_r:\{PF_r, PC_r\}\left(r=1,2,3,4\right)$ represents four cluster centers with dual attributes. According to (20), the FPRs of cluster centers can be obtained based on the average preferences of intra-cluster individuals as:

$$PF_1=\begin{pmatrix}0.500&0.633&0.450&0.250\\0.367&0.500&0.400&0.400\\0.550&0.600&0.500&0.533\\0.750&0.600&0.467&0.500\end{pmatrix}\quad PF_2=\begin{pmatrix}0.500&0.300&0.640&0.700\\0.700&0.500&0.700&0.780\\0.360&0.300&0.500&0.720\\0.300&0.220&0.280&0.500\end{pmatrix}$$

$$PF_3=\begin{pmatrix}0.500&0.580&0.580&0.367\\0.420&0.500&0.667&0.511\\0.420&0.333&0.500&0.467\\0.633&0.489&0.533&0.500\end{pmatrix}\quad PF_4=\begin{pmatrix}0.500&0.480&0.340&0.260\\0.520&0.500&0.260&0.360\\0.660&0.740&0.500&0.440\\0.740&0.640&0.560&0.500\end{pmatrix}$$

Based on (21), the average unit adjustment costs of all clusters are obtained: $PC_1=0.25$, $PC_2=0.30$, $PC_3=0.58$, and $PC_4=0.80$. We can find that there are differences both in preferences and costs between clusters, which shows that the clustering effects obtained considering dual attributes.

Moreover, Fig.4 shows the tendencies of the distance $dis(F_h,F^*)$ and the total adjustment cost TC_h, the former indicator is computed using (7), and the later indicator is computed using (14). The tendency of $dis(F_h,F^*)$ reflects a similar situation of intra-cluster preferences. The smoother the tendency of $dis(F_h,F^*)$, the more similar the intra-cluster preferences. The tendency of TC_h is codetermined by the unit adjustment cost and $dis(F_h,F^*)$. The more similar the intra-cluster individuals' adjustment costs are, the tendency of TC_h is more consistent with $dis(F_h,F^*)$.

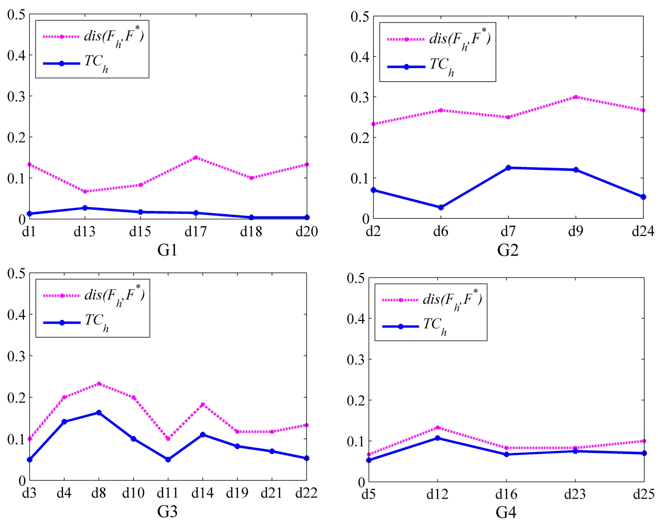


Fig.4. The tendencies of $dis\left(F_h,F^*\right)$ and TC_h obtained based on the proposed clustering analysis

From Fig.4, more than half of the trends of $dis\left(F_h,F^*\right)$ and TC_h are consistent since the proposed clustering analysis is carried out based on the dual attributes of preferences and preference adjustment costs. The opposite trends are mainly caused by the inconsistent changes between intra-cluster individuals in preferences and unit adjustment costs.

Step 5. Analyze the consensus situation after clustering

The consensus reaching level $CL_r^*\left(r=1,2,3,4\right)$ of each cluster and the optimal FPR F^* can be computed using (1) and (2). The overall consensus level OCL_r within each cluster G_r can be computed based on the consensus levels among individuals using (3) and (4). The mean error of intra-cluster consensus reaching levels $ME_{consen}^{\alpha=0.3258}$ and the total adjustment cost $ME_{cost}^{\alpha=0.3258}$ are calculated using (13) and (16), respectively. All these indicators are shown in Table 1.

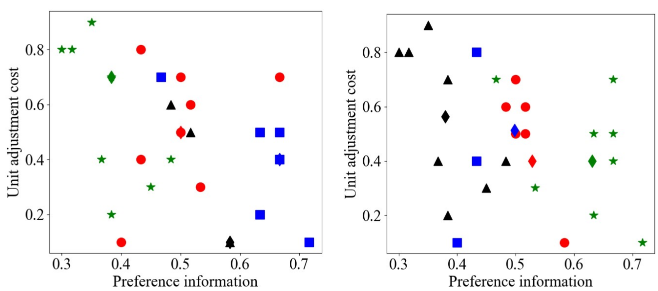
Table 1 The related consensus indicators after the proposed clustering analysis

	G1	*G2*	*G3*	*G4*	Mean value
CL_r^*	0.889	**0.737**	**0.846**	0.907	0.845
OCL_r	**0.8**	**0.82**	**0.769**	**0.827**	0.804
$ME_{consen}^{\alpha=0.3258}$	0.031	0.047	0.064	0.033	0.044
$ME_{cost}^{\alpha=0.3258}$	0.023	0.061	0.054	0.016	0.038

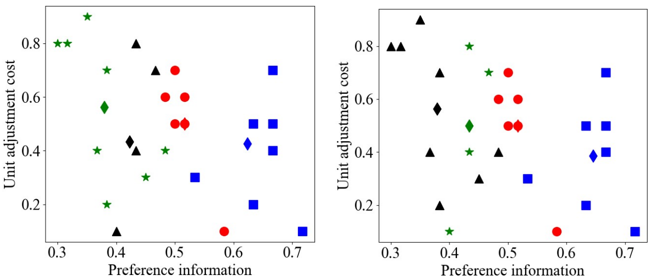
Let the optimal FPR F^* be the final collective FPR of the whole group. Suppose 0.85 is the acceptable consensus level within clusters, and 0.8 is the satisfactory consensus threshold between clusters and F^*, members in the cluster G_2 and G_3 should modify their preferences based on given adjustment strategies. In general, the adjustment strategies of the two clusters will be different. Still, each cluster has a unified adjustment strategy that intra-cluster members all agree. In this way, the moderator does not need to provide multiple adjustment strategies for intra-cluster members with similar preferences and large adjustment cost deviations which was usually obtained using traditional clustering methods. In other words, intra-cluster consensus coordination becomes easier.

C. *Comparative analysis*

In this section, we give a comparison with Refs [23, 41] that without considering adjustment costs to show the advantages of the proposed clustering algorithm. When $\alpha=0$, 25 group leaders are only classified based on preferences according to previous LSGDM K-means method. The initial clustering centers d_3, d_9, d_{25}, and d_1 can also be defined using (24), (25), and (26). The stable clustering results are obtained after four rounds of iterations and shown in Fig.5.



(a) The first round   (b) The second round



(c) The third round (d) The fourth round

Fig.5. The clustering process without considering the adjustment cost

The obtained four clusters are $\overline{G}_1=\{d_1,d_3,d_{11},d_{14},d_{19},d_{21}\}$, $\overline{G}_2=\{d_2,d_6,d_7,d_8,d_9,d_{10},d_{24}\}$, $\overline{G}_3=\{d_4,d_{12},d_{17},d_{22}\}$, and $\overline{G}_4=\{d_5,d_{13},d_{15},d_{16},d_{18},d_{20},d_{23},d_{25}\}$ with the set of unit adjustment costs {0.1, 0.5, 0.5, 0.6, 0.7, 0.6}, {0.3, 0.1, 0.5, 0.7, 0.4, 0.5, 0.2}, {0.3, 0.1, 0.5, 0.7, 0.4, 0.5, 0.2}, {0.7, 0.8, 0.1, 0.4}, and {0.8, 0.4, 0.2, 0.8, 0.4, 0.3, 0.9, 0.7}, respectively. Fig.6 shows the tendencies of $dis\left(F_h,F^*\right)$ and TC_h.

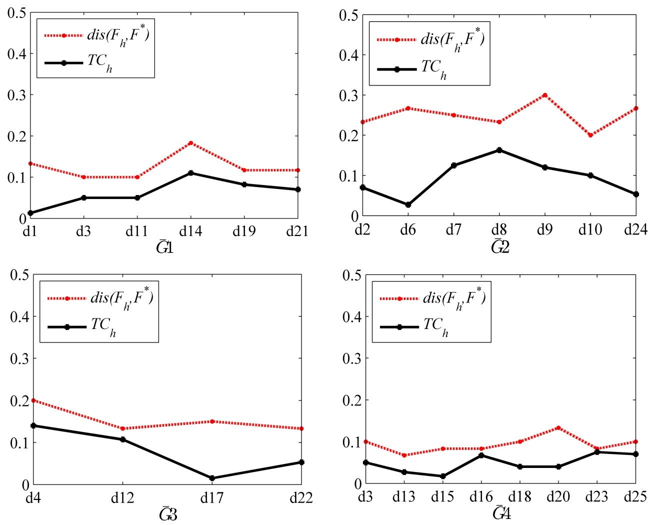


Fig.6. The tendencies of $dis\left(F_h,F^*\right)$ and TC_h obtained based on conventional clustering analysis

From Fig.6, except for the first cluster $\overline{G}_1$, the two indicators within the other clusters vary significantly in their tendencies, which is caused by the large gap of unit adjustment costs between intra-cluster members. For the similar trends in cluster $\overline{G}_1$, it is mainly because of that the unit adjustment costs of individuals d_3, d_{11}, d_{14}, d_{19}, and d_{21} are precisely the same. Compared with Fig.4, Fig.6 shows more stable overall trend lines of $dis\left(F_h,F^*\right)$, but displays more inconsistent trend lines of $dis\left(F_h,F^*\right)$ and TC_h.

According to (1)-(4), (13) and (16), the related indicators for measuring consensus are computed and shown in Table 2.

Table 2 The related consensus indicators after the cluster analysis without considering the adjustment cost

	$\overline{G}_1$	$\overline{G}_2$	$\overline{G}_3$	$\overline{G}_4$	Mean value
$\overline{CLr}^*$	0.875	**0.75**	**0.846**	0.91	0.845
$\overline{OCLr}$	0.884	**0.816**	0.894	0.850	0.861
$ME_{consen}^{\alpha=0}$	0.049	0.062	0.039	0.035	0.046
$ME_{cost}^{\alpha=0}$	0.054	0.066	0.071	0.050	0.060

Compared with Table 1, Table 2 has some changes except for the last data in the first row. Firstly, the average consensus reaching level of $CL_r^*\left(r=1,2,3,4\right)$ is the same no matter how individuals are divided into, which is caused by the stable members of the whole group. Then, most OCL_r in Table 2 is larger than that in Table 1, which is because that the consideration of adjustment costs in the clustering analysis sacrifices part of the intra-cluster consensus levels. The benefit of such loss is reflected in ME_{cost}^{α}, ME_{cost}^{α} in Table 2 is also greater than that in Table 1. The difference between ME_{cost}^{α} and ME_{consen}^{α} in Table 1 is smaller than that in Table 2. The bias about ME_{cost}^{α} and ME_{consen}^{α} between Table 1 and Table 2 is not large because we regard preference information as the main clustering factor and the preference adjustment costs as the support factor with a smaller parameter α. However, the existence of such bias is sufficient to illustrate the importance of considering the preference adjustment costs in the clustering process.

Under the same assumptions as of the case study's consensus analysis, members in cluster $\overline{G}_2$ and $\overline{G}_3$ should modify their preferences based on the given adjustment strategies. However, since members in most clusters have a different willingness for making concessions, it is difficult for intra-cluster members to agree on a fixed adjustment strategy. That is, not only the adjustment strategies of the two clusters are different, but also, their internal adjustment strategies are challenging to be unified. Under such circumstances, the existing classification might be destroyed by some members in pursuit of similar adjustment intentions, thus affects the smooth processes of consensus reaching and even extending the decision-making time.

According to the comparison analysis, we can find that the proposal may facilitate the subsequent consensus researching process since some acceptable intra-cluster consensus reaching levels can be ignored with the tradeoff of the similarity of intra-cluster adjustment costs. Although the calculation may not be faster, considering the adjustment costs in clustering can get more reasonable and realistic results than just focusing on the preference information, thus reducing the negotiation complexity and saving the decision time. The management implication of the comparison is that when considering consensus interactions, more practical factors that influencing decision makers' behavior needs to consider according to the decision context. That is, in addition to the possible deviation between the adjustment costs and similar preferences, experts' position, reputation, and concern for fairness can also have different degrees of correlation influence on the clustering analysis and consensus reaching process [46].

V. Conclusions

To reduce the consensus adjustment complexity in LSGDM, we propose a new clustering algorithm based on K-means considering the preference adjustment costs.

In the proposed LSGDM clustering algorithm, we regard the preference adjustment costs as additional information to the preferences with a parameter. The value of the parameter is determined by balancing the conflict between the consensus reaching levels and the total adjustment costs among intra-cluster individuals. A practical case study of team construction in a company illustrates the application of the proposed clustering algorithm. The clustering results, obtained with a determined parameter of the preference adjustment costs and the specified initial clustering centers, show the effectiveness of the preference adjustment costs on clustering analysis. A comparative study between the proposed clustering algorithm and the traditional LSGDM K-means, which only focuses on the preference information, shows the usefulness of considering the preference adjustment costs.

The clustering algorithm considering adjustment costs classifies individuals into small groups according to their similar preferences and adjustment willingness. The moderator can persuade all intra-cluster individuals to make concessions with a similar compensation strategy rather than provide multiple strategies according to traditional methods. In summary, when there is a significant gap in the adjustment intention between individuals, the proposed clustering algorithm can facilitate the consensus reaching process of LSGDM.

Acknowledgment

The authors would like to thank the editors and the anonymous reviewers for their insightful comments and constructive suggestions which significantly improved the quality of this article.

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6 Managing minority opinions in large-scale group decision-making based on community detection and group polarization

- T. Wu, C. Zuheros, X.W. Liu, F. Herrera, Managing minority opinions in large-scale group decision-making based on community detection and group polarization. **Computers & Industrial Engineering.** 170 (2022) 108337.
  - Status: **Published**.
  - Impact Factor (JCR 2020): 5.431
  - Subject Category: Computer Science, Interdisciplinary Applications, Ranking 21/111 (Q1).
  - Subject Category: Engineering, Industrial, Ranking 14/49 (Q1).

Managing minority opinions in large-scale group decision-making based on community detection and groups polarization

Abstract: As the decision environment becomes more and more complicated, the demand for large-scale group decision-making (LSGDM) is increasing. Because of the differences in decision makers' (DMs') personalities, knowledge, and experience, incomplete information, irrational decision-making behavior, and minority opinions frequently appear. An LSGDM method is proposed considering the above phenomenon. A similar network of DMs is first built based on incomplete preference information and the large number of DMs is divided into several clusters using the community detection method. The group polarization effects of individuals within communities are analyzed. Besides, minority opinions are identified and managed. In the above processes, the family aggregation operators of OWA are used to show the attitude of DMs in different decision scenarios. Given the large-scale decision-making characteristics of public participation in restaurant reviews, the proposed LSGDM method is used to determine the recommendation list of Dianping.com. The results of sensitivity analysis and comparative analysis show that the LSGDM method is flexible and applicable because of considering the attitude of DMs.
Keywords: Large-scale group decision-making, incomplete information, group polarization effect, minority opinions, aggregation operators

1 Introduction

Under the background of the rapid development of modern social economy and information technology, a highly interconnected social network is gradually forming, and large-scale group decision-making (LSGDM) problems with multi-domain experts and complex decision factors appear. LSGDM has a wide range of application scenarios, such as the democratic election (Klimek et al., 2012), the major emergency decision-making (Xu et al., 2020), and the recommendation of online shopping (Zhou et al., 2019). With the help of social media, the recommendation faces both opportunities and challenges of large group participation, diverse group preferences, and complicated decision-making behavior (Cao et al., 2019; Dara et al., 2020; Felfernig et al., 2018). Since the purchasing behavior of new users of online shopping mainly comes from the consensus reviews from a large group of experienced users, Ding et al. (2020) argued that LSGDM methods can provide support for improving the accuracy and satisfaction of group recommendations from the perspective of user decision behavior. This study proposes an LSGDM method that focuses on the group polarization phenomenon and minority opinions with incomplete information, and applies it to the determination of a group recommendation list.

LSGDM appears driven by societal and technological developments (Tang et al., 2020). The research on the LSGDM problem mainly focuses on preference information, clustering analysis, and consensus reaching process (Ding et al., 2019a; Tang et al., 2021).

- **Preference information processing.** Preferences are direct information that can predict or influence the outcome of a decision. Preference information processing mainly includes uncertain preference analysis (Ding et al., 2019b; Liu et al., 2019; Yu et al., 2022; Zhang et al., 2019; Zheng et al., 2021; Zheng et al., 2021; Zhou and Chen, 2021), heterogeneous preference processing (Chao et al., 2021; Li et al., 2021; Tang et al., 2021; Tang et al., 2019; Tian et al., 2021; Zhang et al., 2018; Zhong et al., 2022), and incomplete preference supplement (Chao et al., 2018; Du et al., 2021; Li and Wei, 2019; Lu et al., 2022; Song and Li, 2019; Tian et al., 2019; Xu et al., 2016; Zhou et al., 2022), et al.

- **Clustering analysis.** Clustering analysis is essential to discover the hidden subgroups in LSGDM, so as to mine the user needs, reduce the complexity and identify minority opinions. Depending on the decision information used for clustering, the commonly used clustering analysis algorithm can be divided into classical clustering algorithm (on the basis of preference information) (Li et al., 2022; Wu et al., 2021a; Yu et al., 2022; Zheng et al., 2021; Zhong et al., 2022) and community detection methods (on the basis of social relationships) (Du et al., 2020; Liao et al., 2021; Qin et al., 2022; Wu et al., 2021b). Meanwhile, minority opinions can be detected and managed based on the clustering results (Gou et al., 2021; Xiao et al., 2021; Zhou and Chen, 2021).
- **Consensus reaching process.** Consensus reaching process is a very important and complex process, its research topics can be divided into minimum adjustment (or minimum adjustment cost) (Lu et al., 2021; Qin et al., 2022; Rodriguez et al., 2021; Wang et al., 2021; Wu et al., 2021a; Zhang et al., 2021) and non-cooperative behavior (Chao et al., 2021; Li et al., 2021; Tian et al., 2021; Xu et al., 2019; Zhang et al., 2021; Zhou and Chen, 2021). Besides, some researchers focus on the local and global consensus of large groups based on clustering analysis (Liao et al., 2021; Wu et al., 2021b).

Incomplete information is common in real-life decision-making because of differences or limitations of DMs in knowledge, cognition, and experience (Li and Wei, 2019; Song and Li, 2019). To complete the missing information is a common incomplete information processing method at present. Social (trust) relationships are often used to supplement missing information (Lu et al., 2022; Tian et al., 2019; Xu et al., 2016). Besides, (Zhou et al., 2022) developed a statistical perspective to deduce several important parameters of the missing information. Du et al. (2021) handled incompleteness based on the knowledge structure of DMS. Chu et al. (2020) utilized community detection to restore incomplete fuzzy preference relations. In the field of recommendations, the problem of missing information is more common and has been studied earlier based on statistical methods (Ye et al., 2011), preference similarity (Ghazarian and Nematbakhsh, 2015), and optimization model (Ergu et al., 2016). However, supplementary information is imprecise and rapidly increases in complexity as the decision size increases. Different from the above studies, Li and Wei (2019) and Chao et al. (2018) calculated incomplete information similarity by improving the classical similarity method.

Clustering analysis in LSGDM mainly focuses on the improvement of traditional clustering algorithms and the design of community detection considering hybrid information of social relationships and preferences. Dynamic clustering is common in complex decision scenarios caused by the complexity and variability of DMs' preferences or behaviors. Gou et al. (2018) showed the dynamic clustering results using a hierarchical clustering method. Wu et al. (2021a) studied the conflict between dynamic clustering analysis and consensus reaching process. The clustering dynamics mentioned above are mainly reflected in preference adjustment and the change of clustering parameters, but the influence of preference information integrity on clustering results is not considered.

When minority groups are found in the clustering results, most LSGDM studies mainly follow the majority principle that the majority usually has a larger weight than the minority (Karanik et al., 2016; Zhang et al., 2018). In practice, the minority sometimes is beneficial to decision-making since it is not easy to free ride (Erb et al., 1998). In addition, according to the fairness rule of a group decision, it is important to protect the interests of the minority when meeting the needs of the majority. Recently, more and more research is focusing on the interests of minority opinions (Gou et al., 2021; Ren et al., 2020; Xiao et al., 2021; Zhou and Chen, 2021), where the weights of minority opinions are mainly determined by the majority through iterative interactions. This process is complex and highly dependent on majority opinions, which is difficult to avoid subjectivity.

The mechanism of consensus formation is complex and influenced by many factors such as decision-making behavior and environment. Most current researches on LSGDM assume that subgroups can reach a consensus due to their similar viewpoints and compute subgroup opinions using a simple averaging method (Wu et al., 2019; Wu and Xu, 2018; Zhang et al., 2018), while such assumption and solution still lack theoretical support. Many experiments identified that groups tend to exhibit group polarization phenomenon under group pressure, informational influence, and social comparison (Myers and Lamm, 1976). According to the group polarization phenomenon, members in a small group or community tend to make more extreme decisions than individuals, which can make the group members reach a consensus under a kind of polarization preference (Li et al., 2013). Dillenberger and Raymond (2016) regarded the group polarization consensus as a consensus effect that is equivalent to the strict quasi-convexity of preferences. Sieber and Ziegler (2019) argued that a group with initially distinct opinions may converge to a consensus because of the polarization effect. Thus, the group polarization phenomenon provides a theoretical explanation for the spontaneous formation of group consensus.

However, compared to intra-cluster consensus, the inter-cluster consensus is not easily formed under group pressures owing to the polarization effect makes subgroup members overconfident in their opinions and underestimates different points of view (Rao and Steckel, 1991). That is, even communities with minority opinions are less likely to compromise with the majority, which can also explain the emergence of large-scale group non-cooperative behavior and the focus on minority opinions.

According to the above analysis, there are still the following limitations in the study of LSGDM:

(1) Human decision-making information is influenced by knowledge, experience, behavior, and context. The rationality and accuracy of supplementary information are still questionable. The clustering analysis algorithms commonly used in existing research are more sensitive to missing information.

(2) Current researches usually assume that the intra-cluster consensus is reached based on the similarity of individual preferences and computes the subgroup opinion directly based on the opinions of group members. The above process ignores the essential reasons for the formation of intra-cluster consensus, such as individual psychology, group pressure, and social comparison.

(3) The influence of decision context and the applicability of decision scenarios are seldom considered. The importance of minority opinions is still determined by the majority, which is not objective enough. That is, the usefulness of minority opinions can sometimes be underrated or overrated.

To deal with the above limitations, we propose a new LSGDM method to obtain dynamic clustering results and detect and manage minority opinions based on incomplete information. In the above process, several family operators of the ordered weighted averaging (OWA) operator are used to reflect DMs' attitudes. The following assumptions are used to define the research scope and application scenarios of the proposed LSGDM method.

Assumption 1. Different DMs have different degrees of tolerance for incomplete information.

Assumption 2. Under group pressure, subgroup members will spontaneously change their strategies in response to the majority.

Assumption 3. Minority opinions can be flexibly reinforced or ignored depending on the decision context.

Depending on the above assumptions, our main contributions are given as follows:

(1) According to Assumption 1, we propose a similarity method for incomplete preference information, which can flexibly deal with the risk attitudes of DMs under different decision scenarios. Meanwhile, we construct a dynamic virtual network for a large number of DMs and classify them dynamically with flexible similarity thresholds.

(2) According to Assumption 2, we explore the group polarization behavior of DMs in subgroups to explain the formation of their consensus and compute the collective preference of subgroups with the reference point and the shift parameter in the group polarization model.

(3) According to Assumption 3, we improve the existing method of identifying the minority and managing minority opinions according to the criterion that protects the rights of the minority while satisfying the majority's requirements.

Based on the above techniques, we propose an LSGDM method and apply it to determine the group recommendation list on Dianping.com. The experimental data with five restaurants rated by 63 users concerning four criteria is extracted from raw data. With the incomplete preference information, 63 users are firstly divided into four and six clusters in the optimistic and pessimistic cases, respectively. Correspondingly, diverse sort orders are obtained in both cases by analyzing the group polarization effect and managing minority opinions.

The rest of the paper is organized as follows: Section 2 introduces the community detection method based on modularity, the group polarization effect, and the basic knowledge of aggregation operators; Section 3 proposes the LSGDM considering the incomplete preference information, group polarization behavior, and minority opinions; Section 4 displays the main steps of the proposed LSGDM method; Section 5 provides a group recommendation case of a life service website called Dianping.com; Section 6 gives conclusions.

2 Preliminaries

The basic knowledge of the community detection method and the aggregation operators for dealing with an LSGDM problem with incomplete information are introduced.

2.1 The community detection method based on modularity

Modularity is a crucial technique to measure the density of links inside communities as compared to links between communities (Blondel et al., 2008). In other words, modularity is an effective tool for detecting the effectiveness of community detection algorithms.

Definition 1. (Blondel et al., 2008) For a network LG with large number of DMs $D=\{d_1,d_2,...,d_n\}$, the gain in modularity of one of its temporary community SG_j is represented as:

$$\Delta M_j=\left[\frac{\sum_{in}+2LC_{i,in}}{LC_{ij}}-\left(\frac{\sum_{tot}+LC_i}{LC_{ij}}\right)^2\right]-\left[\frac{\sum_{in}}{LC_{ij}}-\left(\frac{\sum_{tot}}{LC_{ij}}\right)^2-\left(\frac{LC_i}{LC_{ij}}\right)^2\right] \tag{1}$$

where $\sum_{in}$ denotes the sum of edge weights inside of SG_j, $\sum_{tot}$ is the sum of edge weights incident to DMs in SG_j, $LC_{i,in}$ represents the sum of edge weights from d_i to DMs in SG_j, $LC_i=\sum_{i=1}^{n} lc_{ij}$ is the sum of edge weights incident to expert d_i, $LC_{ij}=\sum_{i,j=1,i\neq j}^{n} lc_{ij}$ means the sum of weights of all DM in LG, in which, lc_{ij} means the edge weights between SG_i and SG_j.

To determine the optimal community partition, Blondel et al. (2008) proposed the Louvain method based on modularity gain. Louvain method mainly consists of two phases:

(1) The first phase: for individual DMs

Step 1: For LG, assign each DM to be a subgroup, i.e., $LG^{(1)}=\{SG_1^{(1)},SG_2^{(1)},...,SG_n^{(1)}\}$ in the initial partition.

Step 2: For each DM d_i, consider the gains in modularity $\Delta M_j\left(\Delta M_j>0\right)$ with its neighbors

$d_j\left(j=1,2,...,n;i\neq j\right)$, and remove d_i from its subgroup $SG_i^{(1)}$ to $SG_j^{(1)}$ with $\max\Delta M_j\left(j=1,2,...,n,i\neq j\right)$ based on Eq. (1). Repeat this step until $\Delta M_j\leq 0$ and no node can be moved, then go to the second phase.

(2) The second phase: for independent subgroups

Step 1: Suppose that $x\left(x<n\right)$ subgroups are determined after p rounds in the first phase, and $LG^{(p)}=\left\{SG_1^{(p)},SG_2^{(p)},...,SG_x^{(p)}\right\}$. For each subgroup $SG_r^{(p)}\left(r=1,2,...,x\right)$, consider the gains in modularity $\Delta M_r\left(\Delta M_r>0\right)$ with its neighboring subgroup $SG_s^{(p)}$, $s=1,2,...,x$, $r\neq s$, and remove $SG_r^{(p)}$ to $SG_s^{(p)}$ with $\max\Delta M_r'$ based on Eq. (1).

Step 2: Repeat ***Step 1*** until there are no more changes. Finally, the network LG is classified into t independent communities $LG=\left\{SG_1,SG_2,...,SG_t\right\}$, $t\leq x<n$.

2.2 The family operators of OWA

OWA and its family operators which can consider DMs' attitudinal character are widely used in the decision-making area (Yager, 1988). Among all OWA family operators, the weighted ordered weighted averaging (WOWA) operator (Torra, 1997) which was found useful for decision-making under uncertainty and risk problems and the induced ordered weighted averaging (IOWA) operator which can sort argument values with induced factor (Yager, 2003) are relatively common. The basic knowledge of OWA family operators is introduced below.

Definition 2. (Yager, 1988) A mapping Φ from $R^n\rightarrow R\left(R=\left[0,1\right]\right)$ is called an OWA operator of dimension n if associated with Φ is a weighting vector $\boldsymbol{w}=\left(w_1,w_2,...,w_n\right)^T$ such that $w_j\in\left[0,1\right]$ and $\sum_{j=1}^{n}w_j=1$, and it is defined to aggregate the set of arguments $\left(a_1,a_2,...,a_n\right)$ according to the following expression:

$$\Phi_{OWA}\left(a_1,a_2,...,a_n\right)=\sum_{j=1}^{n}w_j a_{\sigma(j)} \tag{2}$$

where $\left\{\sigma(1),\sigma(2),...,\sigma(n)\right\}$ is a permutation of $\left\{1,2,...,n\right\}$ with $a_{\sigma(j-1)}\geq a_{\sigma(j)}$ for all $j=2,...,n$ and $a_{\sigma(j)}$ is the jth largest element in the collection $\left(a_1,a_2,...,a_n\right)$.

The regular increasing monotone (RIM) quantifier is commonly used to obtain the OWA weighting vector $\boldsymbol{w}$ via linguistic quantifiers. Yager (Yager, 1996) defined the parameterized family of RIM quantifiers as:

$$Q\left(y\right)=y^{\alpha}\quad \alpha\geq 0 \tag{3}$$

Yager (1996) proposed the *orness* measure for the OWA operator that can be used to represent the degree of DMs' optimism.

Definition 3. (Yager, 1996) Based on RIM quantifier Q, the degree of *orness* associated with the OWA operator Φ is defined as:

$$
\begin{aligned}
orness(Q) &= \lim_{n\to\infty}\frac{1}{n-1}\sum_{j=1}^{n-1} Q\left(\frac{j}{n}\right) \\
&= \int_0^1 Q(y)\,dy = \frac{1}{\alpha+1}
\end{aligned}
\tag{4}
$$

Definition 4. (Torra, 1997) A mapping Φ : $R^n \to R$ is a weighted ordered weighted averaging (WOWA) operator of dimension n associated with a weighting vector $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)$ and defined to aggregate the set of arguments $(a_1, a_2, ..., a_n)$ according to the following expression:

$$
\Phi_{WOWA}(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} \omega_j a_{\sigma(j)} \tag{5}
$$

where $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ is a permutation of $\{1,2,...,n\}$ with $a_{\sigma(j-1)} \geq a_{\sigma(j)}$, $\forall i = 2,...,n$, and $a_{\sigma(j)}$ is the jth largest element in the collection $(a_1, a_2, ..., a_n)$, and the weight ω_j is defined based on two weighting vectors $\boldsymbol{w} = (w_1, w_2, ..., w_n)$ and $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ such with $w_j, p_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and $\sum_{j=1}^{n} p_j = 1$ as

$$
\omega_j = w^*\left(\sum_{h\leq j} p_{\sigma(h)}\right) - w^*\left(\sum_{h<j} p_{\sigma(h)}\right) \tag{6}
$$

with w^* a monotone increasing function that interpolates the points $\left(j/n, \sum_{h\leq j} w_h\right)$ together with the point $(0,0)$.

Definition 5. (Yager, 2003) An IOWA operator of dimension n is a function $\Phi_{IOWA} : R^n \times R^n \to R$, in which a weighting vector is associated $\boldsymbol{w} = (w_1, w_2, \mathrm{K}, w_n)$, such that $w_j \in [0,1]$ and $\sum_{i=1}^{n} w_j = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \mathrm{K}, \langle u_n, a_n\rangle\}$ according to the following expression:

$$
\Phi_{IOWA}\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \mathrm{K}, \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} w_j a_{\sigma(j)} \tag{7}
$$

where $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ is a permutation of $\{1,2,...,n\}$ with $a_{\sigma(j-1)} \geq a_{\sigma(j)}$, $\forall i = 2,...,n$, i.e., $\langle u_{\sigma(j)}, a_{\sigma(j)}\rangle$ is the 2-tuple with $u_{\sigma(j)}$ the jth largest element in the collection $(u_1, u_2, ..., u_n)$.

When the importance $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)$ associated to the information source a_j is considered, the IOWA is named as the Importance IOWA (I-IOWA) (Chiclana et al., 2007):

$$w_j = Q\left(\frac{S(j)}{S(n)}\right) - Q\left(\frac{S(j-1)}{S(n)}\right) \tag{8}$$

where $S(j) = \sum_{k=1}^{j} u_{\sigma(k)}$.

3 A large-scale group decision-making method managing minority opinions based on community detection and group polarization

This section introduces an LSGDM method that considers the group polarization effect and manages minority opinions through flexible clustering analysis based on incomplete preference information. We measure the similarity between DMs with incomplete preference information in subsection 3.1; we detect the subgroups flexibly from the network of large-scale DMs in subsection 3.2; we consider the group polarization effects within subgroups in subsection 3.3; we identify and manage the minority opinions in subsection 3.4.

Suppose there is an LSGDM problem consists of n experts $D = \{d_1, d_2, ..., d_n\}$, m criteria $F = \{f_1, f_2, ..., f_m\}$ with the associated weights $\boldsymbol{p} = \{p_1, p_2, ..., p_m\}$, and z alternatives $X = \{x_1, x_2, ..., x_z\}$ with the associated weights $\boldsymbol{q} = \{q_1, q_2, ..., q_z\}$. Let $V^i = \left(v_{kl}^i\right)_{m \times z}$ $(i = 1, 2, ..., n)$ be the decision matrix given by the DM d_i , where $v_{kl}^i \in [0,1]$ represents the opinion of d_i towards to the alternative $x_l \in X$ concerning the criterion $f_j \in F$.

In subsection 3.1, we compute the similarity $SC_{ij}^k\,(k = 1, 2, ..., m)$ considering the kth criteria and the similarity $SA_{ij}^l\,(l = 1, 2, ..., z)$ considering the lth alternative based on the incomplete preference information, respectively. Obtain the similarity matrix $S = \left(S_{ij}\right)_{n \times n}\,(i, j = 1, 2, ..., n; i \neq j)$ of DMs combining SC_{ij}^k and SA_{ij}^l using the WOWA operator.

In subsection 3.2, we determine the similarity threshold θ for the similarity matrix S using the OWA operator associated with the parameter $\alpha_{OWA}^{similarity}$. Based on the similarity threshold θ , we construct a network G with the adjacency matrix $A = \left(A_{ij}\right)_{n \times n}$ and divide the large group of DMs into t subgroups $SG_r\,(r = 1, 2, ..., t; SG_r \in G)$ using Louvain method.

In subsection 3.3, according to the group polarization effects, we determine the collective preference $U_{kl}^r\,(k = 1, 2, ..., m; l = 1, 2, ..., z; \quad r = 1, 2, ..., t)$ of subgroups based on the deviation ϕ between the group preference and the average group preference, the reference point of the lth alternative concerning the kth criteria K_{kl} , and the average preference of subgroups $\overline{u}_{kl}^r$.

In subsection 3.4, we identify the minority opinions according to the comprehensive opinion identification index I_r and the threshold $\overline{I}$ that obtained based on the preference distance

$S_{rs}\left(r,s=1,2,...,t;SG_r,SG_s\in G\right)$. We manage the minority opinions using the I-IOWA operator and rank alternatives flexibly considering stakeholders' attitude with the weights μ_r.

3.1 Similarity measurement of incomplete preference information

Suppose DMs evaluated z alternatives concerning m criteria, the preference matrix of the DM $d_i\left(i=1,2,...,n\right)$ can be defined as:

$$V^i=\begin{bmatrix} v_{11}^i & v_{12}^i & \cdots & v_{1z}^i \\ v_{21}^i & v_{22}^i & \cdots & v_{2z}^i \\ \cdots & \cdots & \cdots & \cdots \\ v_{m1}^i & v_{m2}^i & \cdots & v_{mz}^i \end{bmatrix}$$

To express the incompleteness of $V^i\left(i=1,2,...,n\right)$ more clearly, V^i can be represented as a judgement matrix $B^i=\left(b_{kl}^i\right)_{m\times z}$:

$$b_{kl}^i=\begin{cases}1, & \exists v_{kl}^i\notin\varnothing \\ 0, & \exists v_{kl}^i\in\varnothing\end{cases} \tag{9}$$

Information incompleteness can affect the result of the similarity calculation. For instance, when the preference matrix $V^i\left(i=1,2,...,n\right)$ is incomplete or B^i contains at least one 0 element, the similarity calculated on the basis of the row element is not equal to the similarity calculated on the basis of the column element. Therefore, the similarity between DMs who may give incomplete preference information is computed from the aspect of alternative and criteria, respectively. Since this paper does not focus on the similarity method, we choose a commonly used Jaccard similarity (Chiclana et al., 2013) method to calculate the similarity of DMs.

At the level of criteria, the Jaccard similarity $SC_{ij}^k\left(k=1,2,...,m\right)$ between DM d_i and d_j is computed as (which is shown in **Table 1**):

$$SC_{ij}^k=\frac{\sum_{l=1}^{z}v_{kl}^i v_{kl}^j}{\sum_{l=1}^{z}\left(v_{kl}^i\right)^2+\sum_{l=1}^{z}\left(v_{kl}^j\right)^2-\sum_{l=1}^{z}v_{kl}^i v_{kl}^j} \tag{10}$$

where $SC_{ij}^k\in\left[0,1\right]$, $i,j=1,2,...,n$, and $i\neq j$. $SC_{ij}^k=0$ when $\left(b_{kl}^i\right)_{1\times z}\times\left[\left(b_{kl}^j\right)_{1\times z}\right]^T=0$, $l=1,2,...,z$.

At the level of alternative, the similarity $SA_{ij}^l\left(l=1,2,...,z\right)$ between expert d_i and d_j is computed as (which is shown in **Table 2**):

$$SA_{ij}^l=\frac{\sum_{k=1}^{m}v_{kl}^i v_{kl}^j}{\sum_{k=1}^{m}\left(v_{kl}^i\right)^2+\sum_{k=1}^{m}\left(v_{kl}^j\right)^2-\sum_{k=1}^{m}v_{kl}^i v_{kl}^j} \tag{11}$$

where $SA_{ij}^{l} \in [0,1]$, $i, j = 1,2,...,n$, and $i \neq j$. $SA_{ij}^{l} = 0$ when $\left[\left(b_{kl}^{i}\right)_{m\times 1}\right]^{T} \times \left(b_{kl}^{j}\right)_{m\times 1} = 0$, $k = 1,2,...,m$.

Table 1 Similarity computation at the level of criteria

	Alternatives							*Alternatives*					
d_i	x_1	x_2	$\cdots$	x_l	$\cdots$	x_z	d_j	x_1	x_2	$\cdots$	x_l	$\cdots$	x_z
f_1	v_{11}^{i}	v_{12}^{i}	$\cdots$	v_{1l}^{i}	$\cdots$	v_{1z}^{i}	f_1	v_{11}^{i}	v_{12}^{i}	$\cdots$	v_{1l}^{i}	$\cdots$	v_{1z}^{i}
f_2	v_{21}^{i}	v_{22}^{i}	$\cdots$	v_{2l}^{i}	$\cdots$	v_{2z}^{i}	f_2	v_{21}^{i}	v_{22}^{i}	$\cdots$	v_{2l}^{i}	$\cdots$	v_{2z}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_k	v_{k1}^{i}	v_{k2}^{i}	$\cdots$	v_{kl}^{i}	$\cdots$	v_{kz}^{i}	f_k	v_{k1}^{i}	v_{k2}^{i}	$\cdots$	v_{kl}^{i}	$\cdots$	v_{kz}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\cdots$
f_m	v_{m1}^{i}	v_{m2}^{i}	$\cdots$	v_{ml}^{i}	$\cdots$	v_{mz}^{i}	f_m	v_{m1}^{i}	v_{m2}^{i}	$\cdots$	v_{ml}^{i}	$\cdots$	v_{mz}^{i}

Criteria

Table 2 Similarity computation at the level of alternative

	Alternatives					
d_i	x_1	x_2	$\cdots$	x_l	$\cdots$	x_z
f_1	v_{11}^{i}	v_{12}^{i}	$\cdots$	v_{1l}^{i}	$\cdots$	v_{1z}^{i}
f_2	v_{21}^{i}	v_{22}^{i}	$\cdots$	v_{2l}^{i}	$\cdots$	v_{2z}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_k	v_{k1}^{i}	v_{k2}^{i}	$\cdots$	v_{kl}^{i}	$\cdots$	v_{kz}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\cdots$
f_m	v_{m1}^{i}	v_{m2}^{i}	$\cdots$	v_{ml}^{i}	$\cdots$	v_{mz}^{i}
d_j	x_1	x_2	$\cdots$	x_l	$\cdots$	x_z
f_1	v_{11}^{i}	v_{12}^{i}	$\cdots$	v_{1l}^{i}	$\cdots$	v_{1z}^{i}
f_2	v_{21}^{i}	v_{22}^{i}	$\cdots$	v_{2l}^{i}	$\cdots$	v_{2z}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_k	v_{k1}^{i}	v_{k2}^{i}	$\cdots$	v_{kl}^{i}	$\cdots$	v_{kz}^{i}
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\cdots$
f_m	v_{m1}^{i}	v_{m2}^{i}	$\cdots$	v_{ml}^{i}	$\cdots$	v_{mz}^{i}

Criteria *Criteria*

The similarity $S = \left(S_{ij}\right)_{n\times n}$ between DMs can be computed using the WOWA operator with different attitudes. The weights of aggregation operators can measure the importance of a distance value indifferent with the information source. Meanwhile, the importance $\boldsymbol{p} = \{p_1, p_2, ..., p_m\}$ associated with criterion F and the importance $\boldsymbol{q} = \{q_1, q_2, ..., q_z\}$ associated with alternatives is also crucial for the aggregated results since it measures the reliability of information source. Thus, the similarity S_{ij} between d_i and d_j can be computed using the similarity-based WOWA (S-WOWA) operator:

$$
\begin{aligned}
S_{ij} &= S - WOWA\left(SC_{ij}^{k}, SA_{ij}^{l}\right) = \frac{1}{2}\left(\Phi_{WOWA}\left(SC_{ij}^{k}\right) + \Phi_{WOWA}\left(SA_{ij}^{l}\right)\right) \\
&= \frac{1}{2}\left(\sum_{k=1}^{m} \omega_k SC_{ij}^{\sigma(k)} + \sum_{l=1}^{z} \omega_l' SA_{ij}^{\sigma'(l)}\right)
\end{aligned} \tag{12}
$$

where $S_{ij} \in [0,1]$, $\left(SC_{ij}^{\sigma(1)}, SC_{ij}^{\sigma(2)}, ..., SC_{ij}^{\sigma(m)}\right)$ and $\left(SA_{ij}^{\sigma'(1)}, SA_{ij}^{\sigma'(2)}, ..., SA_{ij}^{\sigma'(z)}\right)$ are permutations of $\left(SC_{ij}^{1}, SC_{ij}^{2}, ..., SC_{ij}^{m}\right)$ and $\left(SA_{ij}^{1}, SA_{ij}^{2}, ..., SA_{ij}^{z}\right)$, respectively. $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_m)$ and $\boldsymbol{\omega}' = (\omega_1', \omega_2', ..., \omega_z')$ are the associated weighting vectors of the WOWA function $\Phi_{WOWA}^{Q}\left(SC_{ij}^{k}\right)$ and $\Phi_{WOWA}^{Q'}\left(SA_{ij}^{l}\right)$, respectively, and they can be computed using the RIM quantifier Q with $\alpha_{WOWA}^{criteria}$ and $\alpha_{WOWA}^{alternative}$ based on Eqs. (3) and (6), respectively.

Some important properties of the S-WOWA operator are discussed as follows.

Property 1: S-WOWA can reduce to the weighted mean operator with $w_k = 1/m$ and $w_l = 1/z$, $\forall k = 1,2,...,m$, $\forall l = 1,2,...,z$. Then, $S_{ij} = WA\left(SC_{ij}^{k}, SA_{ij}^{l}\right) = \frac{1}{2}\left(\sum_{k=1}^{m} p_{\sigma(k)} SC_{ij}^{\sigma(k)} + \sum_{l=1}^{z} q_{\sigma'(l)} SA_{ij}^{\sigma'(l)}\right)$.

Property 2: S-WOWA can reduce to OWA with $p_k = 1/m$ and $q_l = 1/z$, $\forall k = 1,2,...,m$, $\forall l = 1,2,...,z$. Then, $S_{ij} = OWA\left(SC_{ij}^{k}, SA_{ij}^{l}\right) = \frac{1}{2}\left(\sum_{k=1}^{m} w_k SC_{ij}^{\sigma(k)} + \sum_{l=1}^{z} w_l' SA_{ij}^{\sigma'(l)}\right)$.

Property 3: $\left(S_{ij}\right)_* = \frac{1}{2}\left(\min_k\left[SC_{ij}^{\sigma(k)}\right] + \min_l\left[SA_{ij}^{\sigma'(l)}\right]\right) = \frac{1}{2}\left(SC_{ij}^{\sigma(m)} + SA_{ij}^{\sigma'(z)}\right)$ when $\boldsymbol{\omega} = \boldsymbol{\omega}' = (0,0,...,1)$ and $\left(S_{ij}\right)^* = \frac{1}{2}\left(\max_k\left[SC_{ij}^{\sigma(k)}\right] + \max_l\left[SA_{ij}^{\sigma'(l)}\right]\right) = \frac{1}{2}\left(SC_{ij}^{\sigma(1)} + SA_{ij}^{\sigma'(1)}\right)$ when $\boldsymbol{\omega} = \boldsymbol{\omega}' = (1,0,...,0)$. Then, $\frac{1}{2}\left(SC_{ij}^{\sigma(m)} + SA_{ij}^{\sigma'(z)}\right) \le S_{ij} \le \frac{1}{2}\left(SC_{ij}^{\sigma(1)} + SA_{ij}^{\sigma'(1)}\right)$.

The proof of the above properties is given in Appendix.

Based on the above analysis, the similarity matrix $S = \left(S_{ij}\right)_{n\times n}$ is constructed considering the incomplete preference information with respect to criteria and alternative in different ways using the WOWA operator.

3.2 Community detection based on the similarity network

A network is a meaningful way to visualize data structures. Given a matrix whose elements represent the similarity between data restaurants, a network structure between these restaurants can be constructed. Thus, the network of the large group is firstly built based on the similarity matrix among experts. Owing to dense networks occupying more data storage and the relationship between objects being generally sparse, sparse networks are often constructed based on similarity thresholds. The similarity threshold can be computed flexibly based on the similarity matrix $S = \left(S_{ij}\right)_{n\times n} (i, j = 1,2,...,n, i \ne j)$ using the OWA operator. Let $A = \left(A_{ij}\right)_{n\times n}$ denotes the adjacency matrix of the sparse network G:

$$A_{ij} = \begin{cases} 1 & S_{ij} \ge \theta \\ 0 & otherwise \end{cases} \tag{13}$$

where θ denotes the similarity threshold and it is computed using the OWA operator:

$$\theta = \Phi_{OWA}\left(S_{ij} \right) = \sum_{g=1}^{n'} w_g S_{ij}^{\sigma(g)} \tag{14}$$

with the number of pairs of DMs $n' = n(n-1)/2$, $S_{ij}^{\sigma(g)}(i<j)$ is the permutation of $\left(S_{ij},..., S_{n(n-1)}\right)$, the weighting vector w_g can be determined using the RIM quantifier Q based on (3) with a suitable parameter $\alpha_{OWA}^{similarity}$, $\sum_{g=1}^{n'} w_g = 1$.

Figure 1 shows the trend lines of the RIM quantifier $Q(y)$ when $y \in [0,1]$ and the degree of *orness* under the changes of the parameter α.

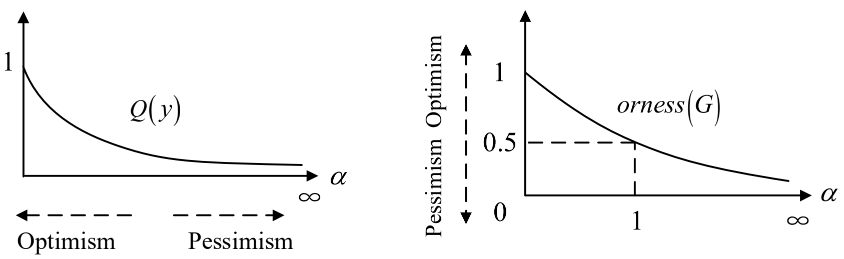


Figure 1 The trend lines of the RIM quantifier and *orness* with the parameter α

According to **Figure 1**, we can find that:

(1) The larger the parameter $\alpha_{OWA}^{similarity}$ is, the more pessimistic the decision stakeholders are towards the sparse similarity matrix S, i.e., the sparse matrix is considered not enough to represent the actual similarity between experts. When $\alpha_{OWA}^{similarity} \to +\infty$, we have $w_g = (0,0,...,1)$ and $orness(Q) = 0$, which means that the decision stakeholders are completely pessimistic, i.e., $\theta = \min S_{ij}$ and G reaches maximum density. In this case, the adjacency matrix A of G is the same as that of the traditional method, that is, the traditional expert network is a special case of the expert network constructed in this paper.

(2) The smaller the parameter $\alpha_{OWA}^{similarity}$ is, the more optimistic the decision stakeholders are towards the sparse similarity matrix S, i.e., they believe that the sparse matrix is enough to represent the actual similarity between experts. When $\alpha_{OWA}^{similarity} = 0$, we have $w_g = (1,0,...,0)$ and $orness(Q) = 1$, which means that the decision stakeholders are completely optimistic, i.e., $\theta = \max S_{ij}$ and G reaches minimum density.

(3) When $\alpha_{OWA}^{similarity} = 1$, we have $w_g = (1/n, 1/n, ..., 1/n)$ and $orness(Q) = 0.5$, which means that the decision stakeholders hold a neutral attitude towards the sparse similarity matrix S and do not pay attention to the impact of the sparsity of the matrix on their understanding of experts preferences.

Based on the sparse network G, experts can be classified using the Louvain method. The community detection process of network $G=\{SG_1,SG_2,...,SG_t\}$ is shown as **Algorithm 1**.

Algorithm 1 Community detection based on incomplete preference information

Input: *Incomplete preference information*

Step 1. Define the appropriate RIM quantifiers Q and Q' to compute the weights $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ of the WOWA operator.

Step 2. With weights $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, construct the similarity matrix $S=(S_{ij})_{n\times n}$ based on Eq. (12).

Step 3. Choose the similarity threshold θ from S and build a sparse network G using Eqs. (13) and (14) with $\boldsymbol{w}$ which is determined using RIM quantifier Q'' of OWA.

Step 4. Take θ as the clustering level and classify experts flexibly using the Louvain method. The optimal clustering result can be determined by maximizing the modularity.

Output: *The clustering result* $G=\{SG_1,SG_2,...,SG_t\}$

3.3 Group polarization model within communities

After the clustering analysis, we evaluate the collective preferences of subgroups with the group polarization effect since it can provide a reasonable explanation for the consensus formatting within subgroups. Rao and Steckel (1991) developed a model for describing the polarization phenomenon in the formation of the collective preference of a group concerning its individuals' attitudes:

$$U_g=\sum_{i=1}^{n}\lambda_i u_i+\phi(\bar{u}-K) \tag{15}$$

where U_g is the group preference, u_i is the individual preference of the DM d_i, λ_i is the weight associated with the DM d_i, $\bar{u}$ is the average of the preferences of the group members, K is the pivot point and ϕ is a shift parameter constrained to be nonnegative.

However, the group preference polarization model given in Ref. (Rao and Steckel, 1991) is tested based on experiments where the parameter ϕ and the reference point K were not provided. K in Eq. (15) is specific to a particular group and the decision context it faces. The meaning of the pivot point K is shown in **Figure 2**. Suppose the polarization is caused by risk attitude, K will be relatively low, and the shift will be upward if the group has a culture to avoid risk. In turn, K will be relatively high, and the shift will be down if the group has a culture to seek risk.

The reference point K can be seen as a quantitative representation of group pressure, and it can be obtained based on the aggregation of majority opinions within subgroups. Therefore, the reference point K_{kl} of the alternative x_l concerning the criteria f_k can be determined using the OWA operator:

$$K_{kl}=\Phi_{OWA}\left(v_{kl}^{i}\right)=\sum_{i=1}^{n}w_i\, v_{kl}^{i} \tag{16}$$

where v_{kl}^{i} is the preference of d_i on the alternative x_l concerning the criteria f_k, the weighting vector w_i can be determined using the RIM quantifier Q based on Eq. (3) with $\alpha_{OWA}^{reference}$ referring to the purpose that to emphasize majority opinions, $\sum_{i=1}^{n} w_i = 1$.

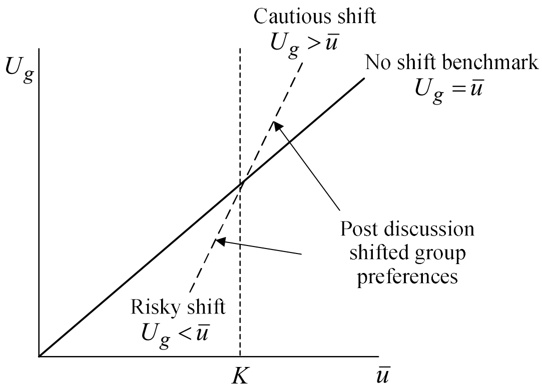


Figure 2 Graphical representation of the polarization hypothesis

The shift parameter ϕ reflects the degree of deviation of the group preference from the weighted mean group preference. Suppose that there is no difference in the degree of polarization between the two extreme values (two extreme attitude, i.e., $orness(Q)$ is equals to 0 or 1). Thus, ϕ is determined based on the difference between the majority's attitude and the neutralizing attitude:

$$\begin{aligned} \phi &= \left| orness(Q) - 0.5 \right| \\ &= \left| \frac{1}{1+\alpha_{OWA}^{reference}} - 0.5 \right| \end{aligned} \tag{17}$$

where ϕ is constrained to be nonnegative and $\phi \in [0, 0.5]$, $\phi\left(\alpha_{OWA}^{reference} = 1\right) = 0$ and $\phi\left(\alpha_{OWA}^{reference} = 0\right) = \phi\left(\alpha_{OWA}^{reference} \to +\infty\right) = 0.5$.

Suppose individuals in the same cluster are regarded as equally important, i.e., $\lambda_i = 1/\left|SG_r\right| \left(d_i \in SG_r\right)$ and $\left|SG_r\right|$ denotes the number of members in the community $SG_r \left(r = 1, 2, ..., t\right)$, we can determine the average preference $\bar{u}_{kl}^{r}$ of subgroups:

$$\bar{u}_{kl}^{r} = \frac{1}{\left|SG_r\right|} \sum_{d_i \in G_r} v_{kl}^{i} \tag{18}$$

Finally, under the influence of group polarization behavior, the subgroup preference U_{kl}^{r} of the cluster SG_r on the lth alternative concerning the kth criteria can be determined based on Eq. (15) as:

$$U_{kl}^{r} = \bar{u}_{kl}^{r} + \phi\left(\bar{u}_{kl}^{r} - K_{kl}\right)$$
$$= \frac{1}{|SG_r|}\sum_{d_i \in G_r} v_{kl}^{i} + \left|\frac{1}{1+\alpha_{OWA}^{reference}} - 0.5\right| \sum_{d_i \in G_r}\left(\left(\frac{1}{|SG_r|} - w_i\right) v_{kl}^{i}\right) \tag{19}$$

where $d_i \in SG_r$ represents the expert d_i belongs to SG_r.

According to Eq. (19), the average group opinions can become more extreme after discussions. The possible extreme situations are explained as follows:

(1) When most members in the cluster SG_r give higher evaluations, the more optimistic about risk they are, the larger the *orness* is, i.e. the larger ϕ is. K will be relatively high and the shift will be downward $U_{kl}^{r} < \bar{u}_{kl}^{r}$ since $\bar{u} < K$ and $\phi > 0$.

(2) When members in SG_r have opposing opinions, we consider the whole cluster has a neutral attitude with $orness\left(Q\right) = 0.5$ and $\phi = 0$. Thus, there is no polarization phenomenon in SG_r, i.e., $U_{kl}^{r} = \bar{u}_{kl}^{r}$ since $\bar{u} = K$ and $\phi = 0$.

(3) When most individuals in SG_r give lower evaluations, the more pessimistic about risk they are, the lower the *orness* is, i.e. the larger ϕ is. K will be relatively low and the shift will be upward $U_{kl}^{r} > \bar{u}_{kl}^{r}$ since $\bar{u} > K$ and $\phi > 0$.

3.4 Identify and manage minority opinions

Xu et al. (Xu et al., 2015) suggested that minority opinions can be identified with two conditions: (a) the cluster has an opinion farthest from the overall group opinion; (b) the cluster includes only one or a few individuals. In this method, condition (a) is first used to find a cluster farthest from the group opinion, and then condition (b) which concerns the size of the subgroup is used to judge whether it belongs to the minority opinion group. However, this method is easier to ignore true minority views, that is, minority groups whose preferences are somewhat different (not the furthest) from the group opinion and whose size is small. Besides, the overall group opinion is determined based on the preferences of all clusters, including the minority opinions, so condition (a) is much more easily influenced by the association of the overall group opinion. Therefore, we modify the first condition as: (a’) the cluster has a far opinion from all the other clusters.

(a’) The cluster has far opinion from all the other clusters. The distance between the subgroup SG_r and others $SG_s\left(s = 1,2,...,t; r \neq s\right)$ can be measured based on the similarity S_{rs} between their collective preferences U_{kl}^{r}, $r = 1,2,...,t$, $k = 1,2,...,m$, and $l = 1,2,...,z$, which can be determined based on Eqs. (10), (11) and (12) with the specific value of $\alpha_{WOWA}^{criteria}$ and $\alpha_{WOWA}^{alternative}$. Then, the averaging similarity $\bar{S}_r$ between SG_r and all the others can be computed as:

$$\overline{S}_r = \frac{1}{t-1}\sum_{s=1,r\neq s}^{t} S_{rs} \tag{20}$$

The larger the $\overline{S}_r$, the smaller the difference. Let $\frac{1}{t}\sum_{r=1}^{t}\overline{S}_r$ be the threshold to determine whether subgroups hold the minority opinion in view of the difference between others.

(b) Obtain groups of minority opinions. In general, let $[n/t]$ be the threshold to judge whether subgroups hold minority opinions in view of the subgroup's size (Xu et al., 2015), where $[n/t]$ is the bracket function of the value of n divided by t. Let $I_r = |SG_r| \times \overline{S}_r$ be the comprehensive opinion identification index of the subgroup SG_r, and $\overline{I}$ be the comprehensive threshold to identify whether subgroups have minority opinions concerning both conditions:

$$\overline{I} = [n/t] \times \frac{1}{t}\sum_{r=1}^{t}\overline{S}_r \tag{21}$$

when $I_r \leq \overline{I}$, the cluster SG_r represent the minority opinion.

Based on the comprehensive identification index I_r, the weights μ_r associated with SG_r can be computed as:

$$\mu_r = I_r \Big/ \sum_{r=1}^{t} I_r \tag{22}$$

where $\sum_{r=1}^{t}\mu_r = 1$.

According to the I-IOWA operator, minority opinions can be highly omitted or considered when the decision manager is optimistic or pessimistic about its impact on decision-making. Besides, the minority opinion is considered more than the majority opinion when experts are pessimistic. The I-IOWA with the weights μ_r, which is associated with the information source, is used to adjust the importance of minority opinions while protecting the rights of the minority.

The overall preference U_{kl} is obtained using the I-IOWA operator based on subgroup preferences $U_{kl}^r\,(r=1,2,...,t;k=1,2,...,m;l=1,2,...,z)$ with the weighting vector w_r:

$$\begin{aligned} U_{kl} &= \Phi_{I-IOWA}\left(\langle I_1, U_{kl}^1\rangle, \langle I_2, U_{kl}^2\rangle, \ldots, \langle I_t, U_{kl}^t\rangle\right) \\ &= \sum_{r=1}^{t} w_r U_{kl}^{\sigma(r)} \end{aligned} \tag{23}$$

where the weighting vector w_r is determined using the RIM quantifier Q based on Eq. (3) with a suitable parameter $\alpha_{I-IOWA}^{minority}$, $\langle I_{\sigma(r)}, U_{kl}^{\sigma(r)}\rangle$ is the 2-tuple with $I_{\sigma(r)}$ the rth largest order inducing value, w_r is determined considering the associated weights μ_r of clusters SG_r based on Eq. (8):

$$w_r = Q\left(\sum_{s\leq r}\mu_{\sigma(r)}\right) - Q\left(\sum_{s<r}\mu_{\sigma(r)}\right) \tag{24}$$

According to the RIM quantifier Q shown in Eq. (3), the management of minority opinions using I-IOWA is analyzed as follows.

(1) Suppose decision stakeholders holds that minority opinions are not worth considering, i.e., they are optimistic about the influence of minority opinions on the decision result, $orness(Q) > 0.5$ and $\alpha_{I-IOWA}^{minority} \in [0,1]$. The lower $\alpha_{I-IOWA}^{minority}$, the more critical majority opinions are in the overall preference.

(2) Suppose decision stakeholders is indifferent to the minority opinions, i.e., they never consider whether minority opinions will have an impact on the decision result, $orness(Q) = 0.5$ and $\alpha_{I-IOWA}^{minority} = 1$.

(3) Suppose decision stakeholders hold that the minority opinions are worth considering, i.e., they are pessimistic about the influence of minority opinions on the decision result, $orness(Q) < 0.5$ and $\alpha_{I-IOWA}^{minority} > 1$.

In the last case, the higher the parameter $\alpha_{I-IOWA}^{minority}$ is, the more important minority opinions are in the overall preference. However, the consideration of minority opinions in this paper is based on the premise of not harming the interests of the majority. Suppose the cluster $SG_{\sigma(s)}$ holds majority opinions with $\min w_s$ and the cluster $SG_{\sigma(r)}$ holds minority opinions with $\max w_r$, to avoid the overrating of the minority opinion to cause injustice to the majority, we let the parameter $\alpha_{I-IOWA}^{minority} > 1$ satisfy $w_r \le w_s$ when increasing the importance of the minority opinion, i.e.,

$$\left(\sum_{r'\le r}\mu_{\sigma(r)}\right)^{\alpha_{I-IOWA}^{minority}} - \left(\sum_{r'<r}\mu_{\sigma(r)}\right)^{\alpha_{I-IOWA}^{minority}} \le \left(\sum_{r'\le s}\mu_{\sigma(s)}\right)^{\alpha_{I-IOWA}^{minority}} - \left(\sum_{r'<s}\mu_{\sigma(s)}\right)^{\alpha_{I-IOWA}^{minority}} \tag{25}$$

Before $w_r = w_s$, the weight of minority opinions w_r increases with the increase of $\alpha_{I-IOWA}^{minority}$, while the weight of majority opinions w_s decreases. Thus, the maximum value of $\alpha_{I-IOWA}^{minority}$ in Eq. (25) can be obtained by simulating the curve and finding its intersection point.

4 An LSGDM method based on incomplete information and its application

Based on the above key techniques, we propose an LSGDM method based on incomplete information, including similarity measurement, clustering analysis, collective preference determination of subgroups, and minority opinions identification and management. The recommendation list is that the recommendation system will prioritize the solutions that meet the needs of the group and recommend them to users. The determination of a recommendation list-oriented to group demand needs to consider group preferences and attitudes. Considering the personalized needs of users, it is necessary to segment large-scale users, namely through clustering analysis. Therefore, we determine the group recommendation list based on the proposed LSGDM method which can pay attention to users' evaluation of items with relative criteria and avoid internal contradictions in group recommendation. **Figure 3** takes the restaurant group recommendation as an example to show the application of the proposed LSGDM method.

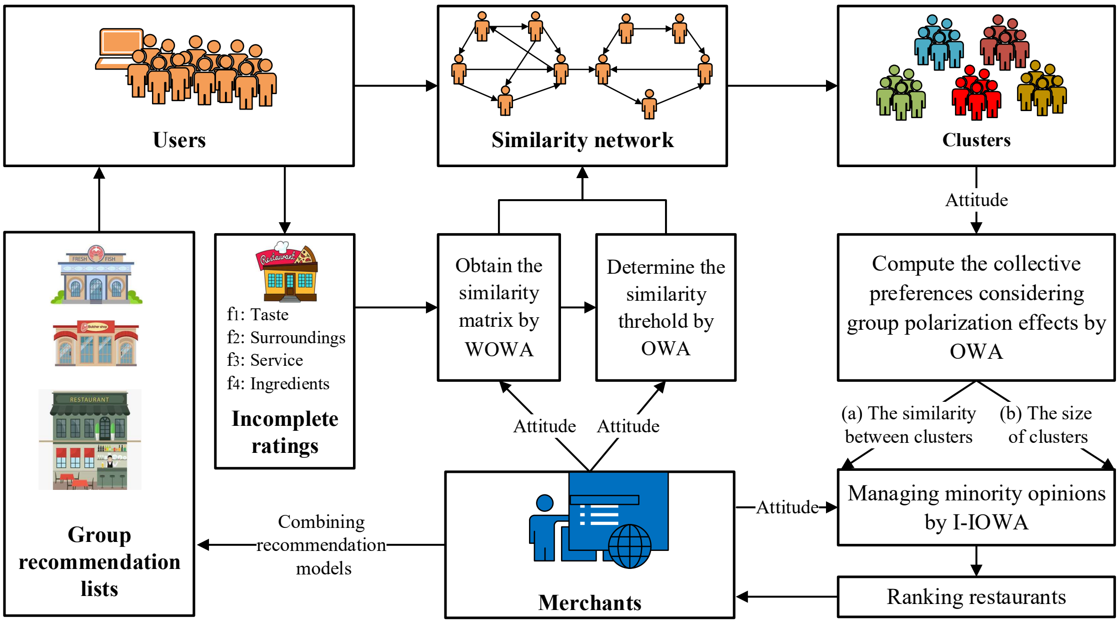


Figure 3 The framework of the proposed LSGDM method and its application in restaurant group recommendation

Suppose an online platform manager wants to recommend restaurants to a group pf users based on their existing reviews. The recommended programs are selected from all popular restaurants on the platform. The group recommendation problem is descripted as below: the platform manager intends to provide restaurant recommendations to n users $D=\{d_1,d_2,...,d_n\}$ from z selected restaurants $X=\{x_1,x_2,...,x_z\}$ with the associated weights $\boldsymbol{q}=\{q_1,q_2,...,q_z\}$, some of the above users may have commented on several target restaurants with respect of m criteria $F=\{f_1,f_2,...,f_m\}$ with the associated weights $\boldsymbol{p}=\{p_1,p_2,...,p_m\}$. Let $V^i=\left(v_{kl}^i\right)_{m\times z}$ $(i=1,2,...,n)$ be the preference matrix given by the user d_i, where $v_{kl}^i\in[0,1]$ represents the opinion of d_i towards to the restaurant $x_l\in X$ concerning the criterion $f_j\in F$.

According to **Figure 3**, the detailed procedures of determining the recommended restaurant lists are introduced as follows:

Step 1: Compute similarity matrix based on incomplete information

According to the incomplete preference information, compute the similarity among users using Eqs. (10) and (11). Based on the specific value of $\alpha_{WOWA}^{criteria}$ and $\alpha_{WOWA}^{alternative}$, determine the similarity matrix $S=\left(S_{ij}\right)_{n\times n}$ among users using Eq. (12).

Step 2: Construct similarity network with similarity thresholds

According to platform manager's attitude towards the influence of incomplete information on the similarity measurement among users, choosing a suitable value for $\alpha_{OWA}^{similarity}$ to determine the similarity threshold θ based on Eq. (14). Based on the similarity matrix S and the threshold θ, draw the initial similarity network G of users using Eq. (13).

Step 3: Classify users using the Louvain method

Based on the similarity network G, divide users into t subgroups $G=\{SG_1,SG_2,...,SG_t\}$ using the Louvain method. Step 1-3 can be implemented by the **Algorithm 1**.

Step 4: Obtain collective preferences of subgroups considering group polarization effects

Compute the average preference $\overline{u}_{kl}^{r}\left(r=1,2,...,t;k=1,2,...,m;l=1,2,...,z\right)$ of each subgroup SG_r. According to the improved group polarization model, determine each subgroup's collective preference U_{kl}^{r} using Eqs. (16)-(19) with the corresponding values of $\alpha_{OWA}^{reference}$.

Step 5: Identify clusters with minority opinions

According to the judgment condition (a') and (b), compute the averaging similarity $\overline{S}_r$ between the cluster SG_r and all the other clusters by Eq. (20) and determine $\left[n/t\right]$, respectively. Compute the comprehensive identification index I_r and identify clusters that have minority opinions based on Eq. (21).

Step 6: Manage minority opinions and rank restaurants

According to the attitude of platform managers towards minority opinions, evaluate the value of $\alpha_{I-IOWA}^{minority}$. Obtain the overall preference $U_{kl}\left(k=1,2,...,m;l=1,2,...,z\right)$ using the I-IOWA operator based on Eqs. (23)-(25) through managing minority opinions. Compute the comprehensive evaluation $U_l\left(l=1,2,...,z\right)$ of z restaurants with the weights $\boldsymbol{p}$ considering m criteria:

$$U_l=\sum_{k=1}^{m}p_kU_{kl} \tag{26}$$

Finally, sort z restaurants referring to the comprehensive evaluation U_l.

User behavior analysis is vital to improve the accuracy and satisfaction of recommendation systems. The proposed LSGDM method focuses on solving several common problems in the field of group recommendation, such as incomplete information, customer segmentation, group polarization behavior, minority opinions, etc. According to the proposed LSGDM method, target restaurants can be ranked considering behavioral factors. Furthermore, platform managers can continuously recommend new relative restaurants to users according to the recommendation lists based on association rules, or recommend restaurants to new users according to their similar preferences to existing users.

5 Case study

In recent years, online group buying has developed rapidly because of price advantage and the development of online shopping techniques. Group-buying platform managers usually recommend products or services based on the majority's preferences, while ignoring the needs of the minority. The key minority sometimes can also affect public sentiment and even guide public opinions, platforms should consider more consumption and preference levels of users to improve the overall satisfaction and transaction volume. In this section, we take Dianping.com, which is a life service (includes food, movies, travel, hotels, etc.) group purchase website in China, as an example to prepare the recommendation list for a large group of users based on the proposed LSGDM method.

In Dianping.com, users can give ratings and comments to restaurants concerning four attributes "Flavor" (f_1), "Environment" (f_2), "Service" (f_3), and "Ingredients" (f_4) (an example is given in **Figure 4**).

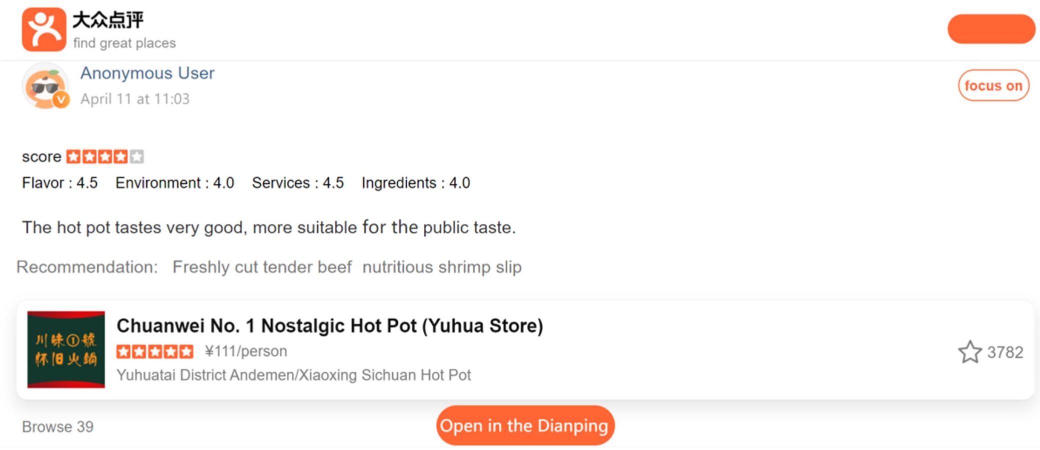


Figure 4 The evaluation interface of Dianping.com

5.1 The application of the proposed LSGDM method

We extract 100,000 comments from 5,500 restaurants and 12,000 users in Nanjing. Although the size of the data set is large, it has great sparsity, i.e., a group of users is less likely to have co-evaluated a group of restaurants. Thus, we process the raw data firstly to obtain experimental data that can satisfy an LSGDM situation with multiple attributes and alternatives: (1) we first select the set of users who have rated Top 100 restaurants; (2) then sort unique users by the frequency they appear in the above set to determine the set of 63 users; (3) finally identify Top 5 restaurants (x_1: 2120, x_2: 12251, x_3: 8143, x_4: 9030, x_5: 282) evaluated by the above users.

With the increasing attention to health, people are concerning on food safety more and more, so we assume that "ingredients" is weighted above other criteria. We describe an LSGDM problem consists of 63 users $D=\{d_1,d_2,...,d_{63}\}$, 5 alternatives $X=\{x_1,x_2,x_3,x_4,x_5\}$ with equal weights $\boldsymbol{q}=\{0.2,0.2,0.2,0.2,0.2\}$ and 4 attributes $F=\{f_1,f_2,f_3,f_4\}$ with associated weights $\boldsymbol{p}=\{0.25,0.25,0.2,0.3\}$. The incomplete ratings of partial users are given in **Table 3** and the overall average rating of five restaurants obtained from the experimental data is given in **Table 4**.

Table 3 The ratings of partial users

d_1	f_1	f_2	f_3	f_4
x_1	3	2	2	2
x_2	-	-	-	-
x_3	-	-	-	-
x_4	-	-	-	-
x_5	-	-	-	-

d_2	f_1	f_2	f_3	f_4
x_1	-	-	-	-
x_2	-	-	-	-
x_3	-	-	-	-
x_4	4	3	3	2
x_5	-	-	-	-

d_3	f_1	f_2	f_3	f_4
x_1	3	2	3	2
x_2	4.5	3	3	3
x_3	3	2	2.5	3
x_4	-	-	-	-
x_5	-	-	-	-

d_4	f_1	f_2	f_3	f_4
x_1	5	2	5	4
x_2	4	2	4.5	4
x_3	5	2.5	5	3
x_4	-	-	-	-
x_5	-	-	-	-

d_8	f_1	f_2	f_3	f_4
x_1	3.5	3	3	2
x_2	-	-	-	-
x_3	-	-	-	-
x_4	-	-	-	-
x_5	-	-	-	-

d_9	f_1	f_2	f_3	f_4
x_1	4	3	3	3
x_2	-	-	-	-
x_3	-	-	-	-
x_4	-	-	-	-
x_5	-	-	-	-

Table 4 The overall average rating of five alternatives

	f_1	f_2	f_3	f_4
x_1	3.527	2.543	2.833	2.704
x_2	3.973	2.892	3.462	2.978
x_3	3.601	2.627	2.954	2.673
x_4	3.923	3.103	3.482	3.087
x_5	3.458	2.354	2.931	2.389

In current data, four criteria is considered in the evaluation of restaurants by users. Still, there are few overlapping evaluations on restaurants among users, so the data is sparse. According to the proposed LSGDM method, the group buying list is sorted.

Step 1: Compute similarity matrix based on incomplete information

The similarity degree between the pairwise users is inversely proportional to the optimism of the platform manager. The more optimistic platform managers toward to incomplete information, i.e., they think that information integrity has little impact on their understanding of user behavior, and the less similar users are. Since the ratings of users on the criteria is more complete than that of restaurants, we let $\alpha_{WOWA}^{alternative} \le \alpha_{WOWA}^{criteria}$.

Suppose platform managers believe that data sparsity has less impact on their understanding of user behavior, let $\alpha_{WOWA}^{criteria} = 0.5$ and $\alpha_{WOWA}^{alternative} = 0.3$, and we can obtain the similarity matrix $S = \left(S_{ij}\right)_{63\times 63}$ based on Eq.(10)-(12).

Step 2: Construct the similarity network with similarity thresholds

Similarly, platform managers' attitude has influence on the determination of similarity thresholds. The more optimistic platform managers toward to similarity degree, i.e., they think that the similarity degree among users has little impact on their understanding of user behavior, and the more sparse the user similarity network is.

Suppose platform managers are optimistic about understanding user behavior, i.e., a highly segmented user market is not necessary, let $\alpha_{OWA}^{similarity} = 0.75$ to determine a relatively loose similarity threshold $\theta = 0.147$ based on Eq.(14). With $\theta = 0.147$ and the similarity matrix S, draw the similarity network G of users in **Figure 5** using Eq.(13).

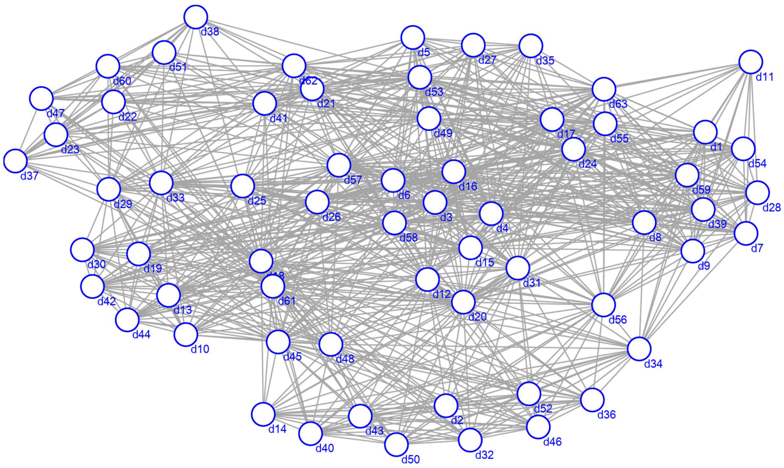


Figure 5 The similarity network G of users when the similarity threshold $\theta = 0.147$

Step 3: Classify users using the Louvain method

Based on the similarity network G, users are classified into four subgroups using the Louvain method: $SG_1 = \{d_1, d_7, d_8, d_9, d_{11}, d_{17}, d_{24}, d_{28}, d_{39}, d_{49}, d_{54}, d_{55}, d_{59}, d_{63}\}$, $SG_2 = \{d_2, d_{12}, d_{14}, d_{18}, d_{20}, d_{31}, d_{32}, d_{34}, d_{36}, d_{40}, d_{43}, d_{46}, d_{48}, d_{50}, d_{52}, d_{56}, d_{61}\}$, $SG_3 = \{d_3, d_4, d_6, d_{10}, d_{13}, d_{15}, d_{16}, d_{19}, d_{25}, d_{26}, d_{29}, d_{30}, d_{33}, d_{42}, d_{44}, d_{45}, d_{57}, d_{58}\}$, and $SG_4 = \{d_5, d_{21}, d_{22}, d_{23}, d_{27}, d_{35}, d_{37}, d_{38}, d_{41}, d_{47}, d_{51}, d_{53}, d_{60}, d_{62}\}$. **Figure 6** shows the above clustering effect.

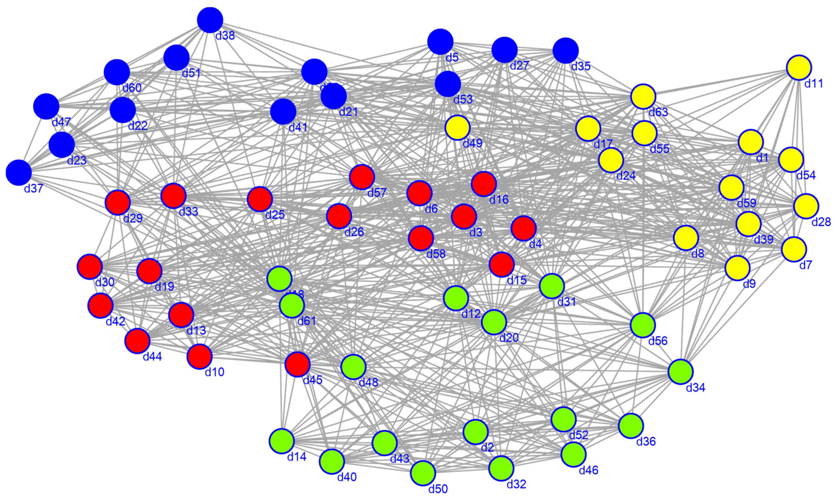


Figure 6 The clustering results of users when the similarity threshold $\theta = 0.147$

Step 4: Obtain collective preferences of subgroups considering group polarization effects

According to the clustering result, the average preference of each subgroup can be obtained in **Table 5**. From **Table 5**, we can find that all the four clusters give lower ratings for five restaurants than the overall average rating in **Table 4**.

Table 5 The average preference of clusters

SG_1	f_1	f_2	f_3	f_4
x_1	3.357	2.429	2.714	2.643
x_2	-	-	-	-
x_3	0.875	0.500	0.571	0.643
x_4	-	-	-	-
x_5	0.214	0.143	0.143	0.143

SG_2	f_1	f_2	f_3	f_4
x_1	0.588	0.412	0.412	0.471
x_2	0.412	0.412	0.529	0.294
x_3	0.353	0.353	0.294	0.294
x_4	3.765	2.765	3.176	2.765
x_5	0.176	0.118	0.118	0.118

SG_3	f_1	f_2	f_3	f_4
x_1	1.500	1.000	1.389	1.111
x_2	4.056	2.722	3.389	2.833
x_3	1.611	1.111	1.389	1.056
x_4	0.111	0.111	0.111	0.111
x_5	0.500	0.333	0.389	0.333

SG_4	f_1	f_2	f_3	f_4
x_1	0.214	0.143	0.214	0.143
x_2	-	-	-	-
x_3	1.714	1.500	1.571	1.357
x_4	-	-	-	-
x_5	2.429	1.857	2.071	2.000

Suppose that the risk user perceived is the potential harassment from platform managers when giving negative ratings, and members in subgroups have the same risk attitude, subgroups show different degrees of risk pursuit. The more users prefer to pursue risks (with a more optimistic attitude towards risks), the greater the reference point K will be, such phenomenon of $K_{1l}^{r}\left(r=1,2,3,4,l=1,2,3,4\right)$ is illustrated in **Figure 7**.

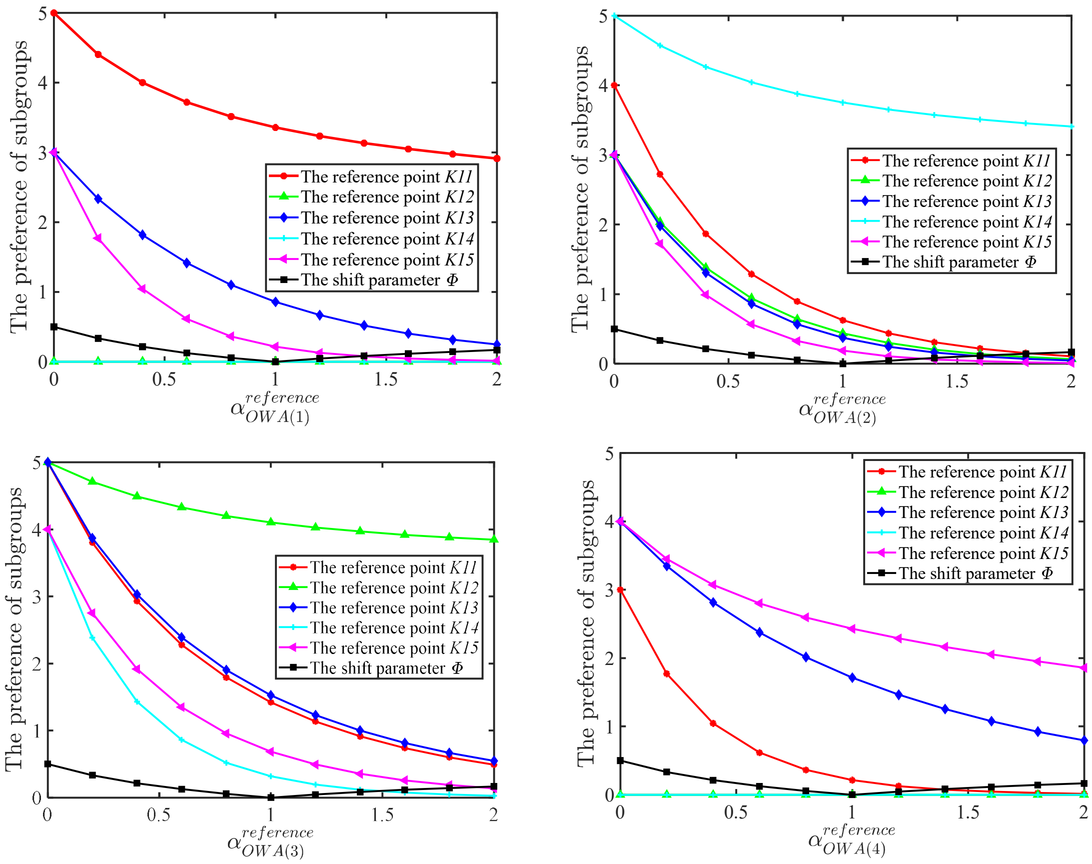


Figure 7 The relationship between the reference point and platform manager attitude

Let $\alpha_{OWA(1)}^{reference}=\alpha_{OWA(2)}^{reference}=\alpha_{OWA(3)}^{reference}=\alpha_{OWA(4)}^{reference}=0.75$, **Table 6** shows the collective preference of each subgroup determined using Eqs. (16)-(19).

Table 6 The polarized collective preferences of subgroups

SG_1	f_1	f_2	f_3	f_4
x_1	3.343	2.410	2.696	2.627
x_2	-	-	-	-
x_3	0.835	0.485	0.556	0.623
x_4	-	-	-	-
x_5	0.200	0.133	0.133	0.133

SG_2	f_1	f_2	f_3	f_4
x_1	0.563	0.394	0.394	0.448
x_2	0.393	0.261	0.505	0.281
x_3	0.335	0.335	0.278	0.278
x_4	3.767	2.763	3.171	2.755
x_5	0.163	0.109	0.109	0.109

SG_3	f_1	f_2	f_3	f_4
x_1	1.466	0.980	1.397	1.085
x_2	4.030	2.699	3.368	2.814
x_3	1.576	1.090	1.357	1.033
x_4	0.075	0.083	0.083	0.092
x_5	0.458	0.311	0.357	0.311

SG_4	f_1	f_2	f_3	f_4
x_1	0.200	0.133	0.200	0.133
x_2	-	-	-	-
x_3	1.687	1.474	1.546	1.331
x_4	-	-	-	-
x_5	2.211	1.843	2.058	1.985

Step 5: Identify clusters with minority opinions

According to the judgment conditions of minority opinions shown in Eq. (21), we can obtain $I_1 = 2.079$, $I_2 = 2.015$, $I_3 = 3.173$, $I_4 = 1.504$, and $\overline{I} = 2.161$, so the subgroup SG_1, SG_2 and SG_4 has minority opinions, which suggests that minority opinions cannot be discovered based solely on the size of subgroups. Besides, the weights of subgroups are obtained as: $\mu_1 = 0.237$, $\mu_2 = 0.230$, $\mu_3 = 0.362$, and $\mu_4 = 0.171$, respectively.

Step 6: Manage minority opinions and rank restaurants

The more pessimistic platform managers are about minority opinions (they think that the influence of minority opinions on the overall users needs to be considered), the greater the weight of subgroups with minority opinions is, such situation is shown in **Figure 8**.

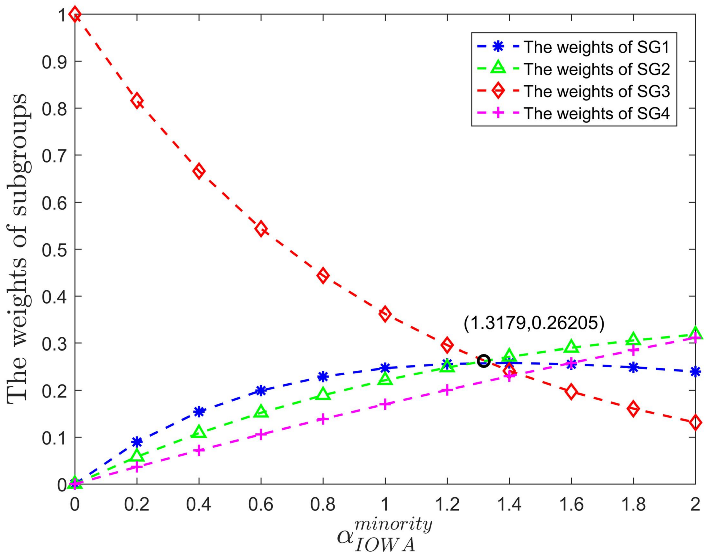


Figure 8 The relationship between the importance of minority opinions and platform manager attitude

According to **Figure 8**, the restaurant ranking results considering the platform manager's optimistic, pessimistic and neutral attitude are introduced as below.

(i) When platform managers would like to consider minority opinions, the parameter $\alpha_{I-IOWA}^{minority}$ in the RIM quantifier Q should satisfy $1 < \alpha_{I-IOWA}^{minority} \le 1.318$ based on Eq. (24). **Figure 8** shows that when $\alpha_{I-IOWA}^{minority} = 1.318$, we have $w_2 = w_3$. When $\alpha_{I-IOWA}^{minority}$ increases more, SG_2 with minority opinions will be given more importance than SG_3 with majority opinions.

Let $\alpha_{I-IOWA}^{minority} = 1.2$, the weights w_r associated with the I-IOWA operator is computed to be $w_1 = 0.258$, $w_2 = 0.260$, $w_3 = 0.267$, and $w_4 = 0.215$. Based on Eqs. (23) and (26), five restaurants are sorted with the final comprehensive evaluation as: $x_1 \succ x_4 \succ x_2 \succ x_3 \succ x_5$.

(ii) When platform managers think that there is no need to consider minority opinions, we let $\alpha_{I-IOWA}^{minority} = 0.3$ and obtain $w_1 = 0.124$, $w_2 = 0.084$, $w_3 = 0.737$, and $w_4 = 0.054$, then five restaurants are sorted with the final comprehensive evaluation as: $x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$.

(iii) When platform managers are indifferent to minority opinions, $\alpha_{I-IOWA}^{minority} = 1$ and $w_1 = 0.247$, $w_2 = 0.221$, $w_3 = 0.362$, and $w_4 = 0.170$, then five restaurants are sorted with the final comprehensive evaluation as: $x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$.

We can find that the group tends to follow the majority's choice when they have a neutral risk attitude. Therefore, the ranking of restaurants in the latter two cases is consistent. The restaurant x_1 transcends the restaurant x_4 in the first case because we consider minority opinions. According to the sorting results, we can combine the item-based collaborative filtering algorithm to provide a recommendation list under different risk attitudes.

5.2 Sensitivity analysis

To visualize the influence of risk attitude of users and platform managers on alternative rankings, we provide the sensitivity analysis in this part. The attitudes of users and platform managers represented by $\alpha_{WOWA}^{criteria}$, $\alpha_{WOWA}^{alternative}$, $\alpha_{OWA}^{similarity}$, $\alpha_{OWA}^{reference}$, and $\alpha_{I-IOWA}^{minority}$ are reflected in four phases (Section 3.1-3.4).

Phase 1: The relationship between the similarity degree and platform manager attitude towards criteria $\alpha_{WOWA}^{criteria}$ and alternative $\alpha_{WOWA}^{alternative}$ is shown in **Figure 9**. We can find that different parameters of risk attitude can lead to different similarity matrix and network structures, so the clustering results and alternative ranking results are also different. Thus, the ranking of alternatives is sensitive to the parameter $\alpha_{WOWA}^{criteria}$ and $\alpha_{WOWA}^{alternative}$.

Phase 2: The trend of the similarity threshold θ and the density of the similarity network G changing with the value of $\alpha_{OWA}^{similarity}$ under a positive and negative attitude towards incomplete evaluation information is shown in **Figure 10**, respectively.

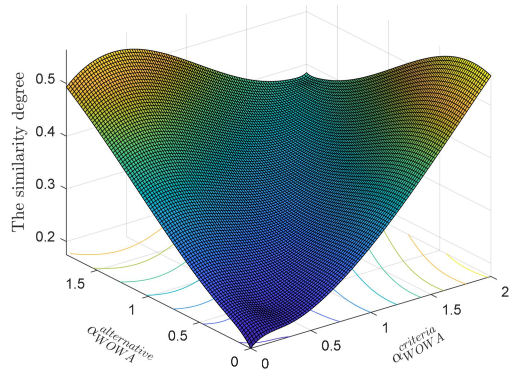


Figure 9 The relationship between the similarity degree and platform manager attitude

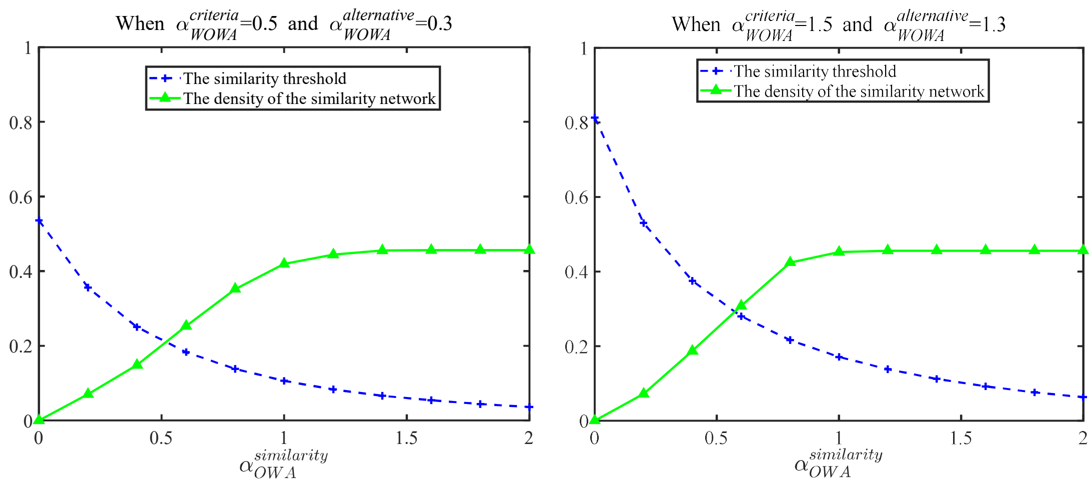
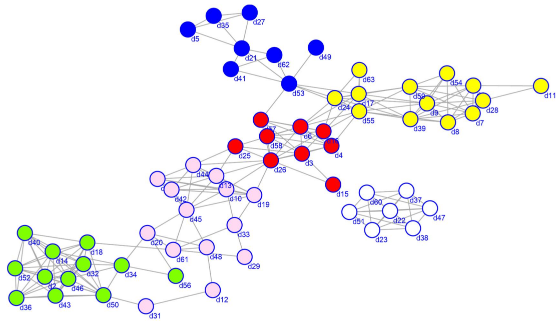


Figure 10 The trend of similarity threshold and the density of the similarity network

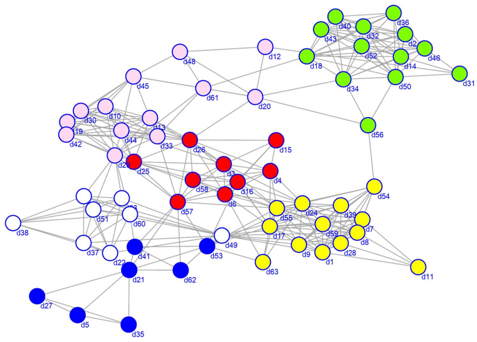
Figure 10 shows that the similarity threshold θ and the density of user similarity network G are both closely related to the risk attitude of the platform managers, but the density of G is only sensitive to the positive attitude, not to the negative attitude. That is, the inflection point of G appears when platform managers are neutral, mainly because when the similarity threshold is lowered to a certain point, most elements of the similarity matrix remain unchanged and the structure of G begins to remain stable.

Naturally, the clustering results of G is not unique. For instance, **Figure 11** shows the different clustering results with different $\alpha_{OWA}^{similarity}$ in the optimistic case when $\alpha_{WOWA}^{criteria}=0.5$ and $\alpha_{WOWA}^{alternative}=0.3$. **Figure 11** also shows how the sparsity of G changes with $\alpha_{OWA}^{similarity}$.

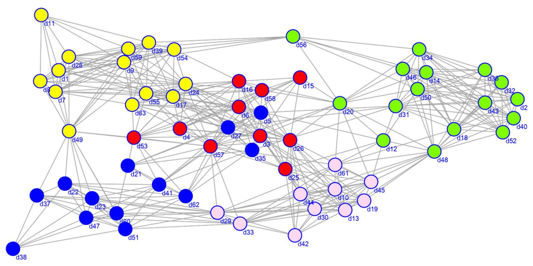
Phase 3 and 4: In **Table 7**, we summarize the changes of the parameter $\alpha_{OWA}^{similarity}$, $\alpha_{OWA}^{reference}$, and $\alpha_{I-IOWA}^{minority}$ in other stages and their impacts on clustering analysis, minority opinion management and alternative ranking when $\alpha_{WOWA}^{criteria}=0.5$ and $\alpha_{WOWA}^{alternative}=0.3$. When $\alpha_{OWA}^{similarity}=0.3$ and $\alpha_{OWA}^{similarity}=0.4$, the cluster number is 6. When $\alpha_{OWA}^{similarity}=0.5$ and $\alpha_{OWA}^{similarity}=0.6$, the cluster number is 5. When $\alpha_{OWA}^{similarity}=0.7$ and $\alpha_{OWA}^{similarity}=0.8$, the cluster number is 4. It is worth noting that although the above clustering numbers are the same, the clustering effect is not the same.



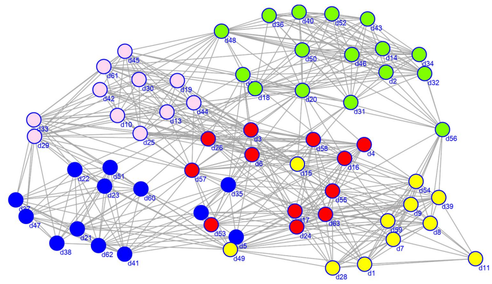
(a) $\alpha_{OWA}^{similarity} = 0.3$



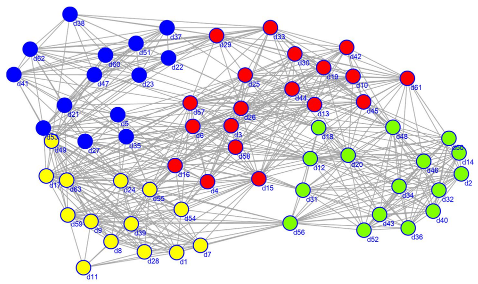
(b) $\alpha_{OWA}^{similarity} = 0.4$



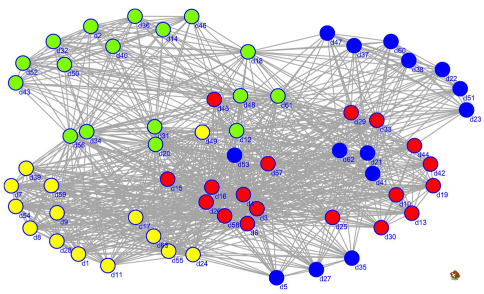
(c) $\alpha_{OWA}^{similarity} = 0.5$



(d) $\alpha_{OWA}^{similarity} = 0.6$



(e) $\alpha_{OWA}^{similarity} = 0.7$



(f) $\alpha_{OWA}^{similarity} = 2$

Figure 11 The clustering results with different $\alpha_{OWA}^{similarity}$ in the optimistic case

From **Table 7**, we can find that (1) Different similarity thresholds lead to different clustering effects, and groups identified as having minority opinions are also different, i.e., risk attitude also has an impact on the judgment of minority opinions. (2) In the case that minority opinions are considered important or indifferent, the difference of clustering effect has a great influence on the alternative ranking except x_5. Meanwhile, when the parameter $\alpha_{OWA}^{similarity}$ is greater than or equal to 0.5, the alternative ranking results tend to be stable. (3) Under the same clustering effect, the consideration of minority opinions has a certain influence on the alternative ranking, except for x_5. (4) The smaller the similarity threshold is, the more stable the community structure is, and the group members are less affected by the risk attitude, which is also consistent with the reality of group thinking, where the group bears more risk than the individual.

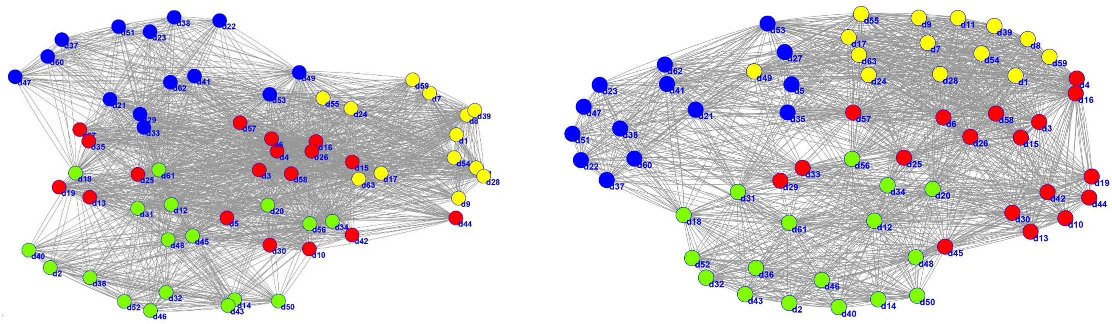
Table 7 The sensitivity analysis when $\alpha_{WOWA}^{criteria}=0.5$ and $\alpha_{WOWA}^{alternative}=0.3$

$\alpha_{OWA}^{similarity}$	0.3			0.4			0.5			0.6			0.7			2		
θ	0.297			0.250			0.213			0.183			0.147			0.106		
t	6			6			5			5			4			4		
The attitude of users toward risks $\alpha_{OWA}^{reference}=0.45$																		
The minority	SG_1, SG_2			SG_1,SG_6			SG_2, SG_4			SG_1, SG_2, SG_4			SG_2, SG_4			SG_2, SG_4		
$\max\alpha_{I-IOWA}^{minority}$	1.527			1.532			1.600			1.430			1.338			1.278		
$\alpha_{I-IOWA}^{minority}$	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1
x_1	4	3	4	1	4	5	3	3	2	1	3	2	1	2	2	1	2	2
x_2	1	2	2	2	2	2	1	2	3	2	2	3	3	3	3	2	3	3
x_3	3	4	3	1	3	3	4	4	4	4	4	4	4	4	4	4	4	4
x_4	2	1	1	3	1	1	2	1	1	3	1	1	2	1	1	3	1	1
x_5	5	5	5	4	5	4	5	5	5	5	5	5	5	5	5	5	5	5
The attitude of users toward risks $\alpha_{OWA}^{reference}=0.75$																		
The minority	SG_6			SG_6			SG_2, SG_4			SG_4			SG_1, SG_2, SG_4			SG_1, SG_2, SG_4		
$\max\alpha_{I-IOWA}^{minority}$	1.729			1.555			1.508			1.198			1.292			1.255		
$\alpha_{I-IOWA}^{minority}$	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1
x_1	4	2	4	3	2	4	3	2	2	1	3	2	1	2	2	1	2	1
x_2	1	3	2	2	4	3	1	3	4	2	2	4	3	3	3	2	3	3
x_3	3	5	3	1	3	2	4	4	3	4	4	3	4	4	4	4	4	4
x_4	2	1	1	4	1	1	2	1	1	3	1	1	2	1	1	3	1	2
x_5	5	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
The attitude of users toward risks $\alpha_{OWA}^{reference}=1.35$																		
The minority	SG_6			SG_6			SG_2, SG_4			SG_1, SG_4			SG_1, SG_2, SG_4			SG_1, SG_2, SG_4		
$\max\alpha_{I-IOWA}^{minority}$	1.721			1.555			1.464			1.232			1.282			1.257		
$\alpha_{I-IOWA}^{minority}$	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1	1.2	0.3	1
x_1	4	2	4	3	2	4	3	2	2	1	3	2	1	2	2	1	2	1
x_2	1	3	2	2	4	3	1	3	5	2	2	4	3	3	3	2	3	3
x_3	2	5	3	1	3	1	4	4	3	4	4	3	4	4	4	4	4	4
x_4	3	1	1	4	1	2	2	1	1	3	1	1	2	1	1	3	1	2
x_5	5	4	5	5	5	5	5	5	4	5	5	5	5	5	5	5	5	5

5.3 Comparison and discussions

At present, there are few studies considering the risk attitudes of various subjects from the whole decision-making process, so it is difficult to provide a complete comparative analysis of the existing literature. We discuss the advantages of this study through the comparative analysis of each phase.

(1) The comparative analysis of Phase 1: Cosine similarity measure (Chao et al., 2018) and Pearson correlation coefficient (Li and Wei, 2019) are often used to compute the similarity between users based on restaurant ratings. **Figure 12** shows the clustering results of similarity networks obtained using the above two methods. We can find that there are some differences between the clustering results shown in **Figure 12** and the results obtained by the proposed method (**Figure 6**). Because of such differences, the results of restaurant ranking are also different. According to the Cosine similarity method, we obtain x_1 f x_2 f x_4 f x_3 f x_5 when the minority opinions are considered, x_4 f x_1 f x_2 f x_3 f x_5 when the minority opinions are not considered, and x_1 f x_4 f x_2 f x_3 f x_5 when the minority opinions are considered indifferent. According to the Pearson correlation coefficient, we obtain x_1 f x_4 f x_3 f x_2 f x_5 when the minority opinions are considered or considered indifferent, and x_4 f x_1 f x_3 f x_2 f x_5 when the minority opinions are not considered.



(a) Cosine similarity method (b) Pearson correlation coefficient

Figure 12 The clustering results of similarity networks obtained using other methods

The similarity network G obtained by the above two methods is much denser because the similarity degree is computed as 1 when users only have one co-commented restaurant which is obviously unreasonable. Besides, the similarity matrix obtained by the above two methods also has some other unreasonable results. For instance, based on incomplete preference information in **Table 3**, we obtain $S_{13} = 0.216 > S_{14} = 0.191$ using the proposed method, but we get $S_{13} = 0.510 < S_{14} = 0.616$ and $S_{13} = 0.291 < S_{14} = 0.476$ using the cosine similarity and Pearson correlation coefficient similarity method. However, from **Table 3**, we can intuitively find that the scores of d_1 and d_3 are closer than d_1 and d_4.

In short, the above two methods have some limitations in calculating similarity based on the incomplete preference information. The proposed method of this paper considers the preference attitude and uses the similarity threshold to regulate the sparse property of the similarity network, so as to make it more applicable in the complicated and variable decision environment.

(2) The comparative analysis of Phase 2: The traditional method (Wu et al., 2019) that constructs the similarity network G based on the original similarity matrix is a special case of the proposed method. When the parameter $\alpha_{OWA}^{similarity}$ is greater than or equal to 2, the similarity threshold $\theta = 0.106 = \min\left(S_{ij}\right)\left(i, j = 1,2,...,63, i \neq j\right)$ with the increase of parameter $\alpha_{OWA}^{similarity}$ no longer affects the structure of G, namely G is constructed directly based on the initial similarity matrix, so the last column of **Table 7** can be obtained using the traditional method that does not consider similarity thresholds. Thus, the proposed method is more flexible than traditional methods in the ever-changing decision-making environment.

(3) The comparative analysis of Phase 3: There are many ways to obtain the collective preference in traditional researches, the most common one is the average method. In the case of $\alpha_{WOWA}^{criteria}=0.5$, $\alpha_{WOWA}^{alternative}=0.3$, and $\alpha_{OWA}^{similarity}=0.75$, we compute the collective preference $U_{kl}^{r}\left(r=1,2,3,4;k=1,2,3,4;l=1,2,3,4,5\right)$ (which is shown in **Table 5**) using the average method to obtain $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$ when we consider minority opinions with $\alpha_{I-IOWA}^{minority}=1.2$ and $x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$ when we do not consider minority opinion or consider it indifferent. In the first case, it is different from the ranking results obtained by the proposed method, indicating that the difference in group preference calculation has an impact on the alternative ranking. The results obtained in the last two cases are the same as those obtained by the proposed method, which means that majority opinions have an overwhelming influence on the ranking of alternatives when no action is taken for minority opinions. Besides, the group preferences in **Table 5** consider the behavioral factors of individuals in a group society, which is conducive to the implementation of different market strategies during the ever-changing decision-making environment.

(4) The comparative analysis of Phase 4: According to the minority opinions identification method proposed by Xu et al. (Xu et al., 2015), only the subgroup SG_4 shown in **Figure 11** (b) is regarded as a minority opinion subgroup when $\alpha_{WOWA}^{criteria}=0.5$, $\alpha_{WOWA}^{alternative}=0.3$, $\alpha_{OWA}^{similarity}=0.75$, and $\alpha_{OWA}^{reference}=0.75$. Actually, according to **Figure 11** (b) and **Figure 8**, it is easy to find that the subgroup SG_1 and SG_2 has a similar minority opinion phenomenon to the subgroup SG_4. We cannot provide a comparative analysis of the management process of minority opinions, because the importance of minority opinion in the literature (Xu et al., 2015) is evaluated by the majority based on the empirical and subjective discussions. In this process, it is difficult to guarantee that all majority group experts are objective and impartial. In this paper, the importance of minority opinions is determined by adjusting risk attitude parameters according to the decision-making environment from an objective perspective.

From the above analysis, the proposed LSGDM method has much more flexibility to deal with incomplete information, detect communities, and manage minority opinions. Besides, we can find that x_1 and x_2 are the top two alternatives and there is controversy over whether to consider minority opinions, while x_5 is the worst with no controversy.

6 Conclusions

To deal with the increasingly complex decision-making environment, we propose an LSGDM method to detect communities based on incomplete preference information and manage the minority opinions considering the group polarization behavior.

According to the attitude of DMs, the proposed LSGDM method adjusts the influence of incomplete information on decision-making results, and obtains the similarity matrix and dynamic clustering results based on the proposed similarity calculation method. Based on the clustering results, we improve the traditional group polarization model to explain the intra-cluster consensus reaching process. Furthermore, we identify and manage the minority opinions objectively and dynamically according to decision contexts. Finally, we determine the recommendation lists of restaurants on Dianping.com using the proposed LSGDM method. The results show that the LSGDM method enables platform managers to respond flexibly to users' diversified demands and to provide suitable recommendation lists considering users' behavior.

This study still has some limitations in theory and applications. In theory, the consensus reaching process among subgroups is rarely explored. In terms of application, only four given criteria are considered and the linguistic comments are ignored. Besides, we do not deeply explore the group recommendation mechanism, but only rank the recommendation lists based on the LSGDM method. In the future, we will deeply study the consensus evolution among subgroups, summarize more criteria that users care about by semantic analysis and extract more user preference information from linguistic comments. We also need to combine LSGDM methods and recommendation problems to improve the accuracy of group recommendations.

Appendix

Proof of property 1. If $w_j = 1/n$ for all $j = 1,2,...,n$, we have that $w*$ interpolates (0, 0) and $\left(j/n, \sum_{h\le j} w_h\right) = \left(j/n, j/n\right)$. Therefore, $w*\left(x\right) = x$ and $\omega_j = \sum_{h\le j} p_{\sigma(h)} - \sum_{h<j} p_{\sigma(h)}$, it is clear that $\omega_j = p_{\sigma(j)}$. Similarly, we can obtain $\omega_l' = q_{\sigma'(l)}$. Thus, $S_{ik} = WA\left(SC_{ik}^{j}, SA_{ik}^{l}\right)$.

Proof of property 2. If $p_j = 1/n$ for all $j = 1,2,...,n$, then $\omega_j = w*\left(\sum_{h\le j} p_{\sigma(h)}\right)$ $-w*\left(\sum_{h<j} p_{\sigma(h)}\right) = w*\left(i/n\right) - w*\left(\left(i-1\right)/n\right)$. Since $w*$ interpolates (0, 0) and $\left(j/n, \sum_{h\le j} w_h\right)$, $w*\left(i/n\right) = \sum_{h\le j} w_h$ and $w*\left(0\right) = 0$. Therefore, $\omega_j = w_j$. Similarly, we can obtain $\omega_l' = w_l'$. Thus, $S_{ik} = OWA\left(SC_{ik}^{j}, SA_{ik}^{l}\right)$.

Proof of property 3. (a) $S_{ik} \ge \frac{1}{2}\left(\min_j\left[SC_{ik}^{\sigma(j)}\right] + \min_l\left[SA_{ik}^{\sigma'(z)}\right]\right)$. According to (12), $\min_j\left[SC_{ik}^{\sigma(j)}\right] = SC_{ik}^{\sigma(n)}$ and $\min_l\left[SA_{ik}^{\sigma'(l)}\right] = SA_{ik}^{\sigma'(z)}$, $S_{ik} = \frac{1}{2}\left(\omega_n SC_{ik}^{\sigma(n)} + \sum_{j=1}^{n-1}\omega_j SC_{ik}^{\sigma(j)} + \omega_z' SA_{ik}^{\sigma'(z)} + \sum_{l=1}^{z-1}\omega_l' SA_{ik}^{\sigma'(l)}\right)$, where $b_j\left(j = 1,...,n-1\right) \ge b_n$, $b_l'\left(l = 1,...,z-1\right) \ge b_z'$, $\sum_{j=1}^{n-1}\omega_j = 1 - \omega_n$, and $\sum_{l=1}^{z-1}\omega_l' = 1 - \omega_n'$, we can obtain $S_{ik} \ge \frac{1}{2}\left(\omega_n SC_{ik}^{\sigma(n)} + SC_{ik}^{\sigma(n)}\sum_{j=1}^{n-1}\omega_j +\right.$

$\omega'_z SA_{ik}^{\sigma'(z)} + SA_{ik}^{\sigma'(z)} \sum_{l=1}^{z-1} \omega'_l \Big)$, that is, $S_{ik} \geq \frac{1}{2}\Big(\omega_n SC_{ik}^{\sigma(n)} + SC_{ik}^{\sigma(n)} \left(1-\omega_n\right) + \omega'_z SA_{ik}^{\sigma'(z)}$ $+ SA_{ik}^{\sigma'(z)} \left(1-\omega'_n\right)\Big) \Rightarrow s_{ik} \geq \frac{1}{2}\left(SC_{ik}^{\sigma(n)} + SA_{ik}^{\sigma'(z)} \right)$, i.e., $S_{ik} \geq \frac{1}{2}\left(\min_j \left[SC_{ik}^{\sigma(j)} \right] + \min_l \left[SA_{ik}^{\sigma'(z)} \right] \right)$. (b) $S_{ik} \leq \frac{1}{2}\left(\max_j \left[SC_{ik}^{\sigma(j)} \right] + \max_l \left[SA_{ik}^{\sigma'(l)} \right] \right)$. According to (12), $S_{ik} = \frac{1}{2}\Big(\omega_1 SC_{ik}^{\sigma(1)} +$ $\sum_{j=2}^{n} \omega_j SC_{ik}^{\sigma(j)} + \omega'_1 SA_{ik}^{\sigma'(1)} + \sum_{l=2}^{z} \omega'_l SA_{ik}^{\sigma'(l)}$, $\max_j \left[SC_{ik}^{\sigma(j)} \right] = SC_{ik}^{\sigma(1)}$ and $\max_l \left[SA_{ik}^{\sigma'(l)} \right] = SA_{ik}^{\sigma'(1)}$, where $SC_{ik}^{\sigma(1)} \geq SC_{ik}^{\sigma(j)} \left(j \geq 2 \right)$, $SA_{ik}^{\sigma'(1)} \geq SA_{ik}^{\sigma'(l)} \left(l \geq 2 \right)$, $\sum_{j=2}^{n} \omega_j = 1-\omega_1$ and $\sum_{j=2}^{n} \omega'_j = 1-\omega'_1$, we can obtain $S_{ik} \leq \frac{1}{2}\Big(\omega_1 SC_{ik}^{\sigma(1)} + SC_{ik}^{\sigma(1)} \sum_{j=2}^{n} \omega_j + \omega'_1 SA_{ik}^{\sigma'(1)} \quad + SA_{ik}^{\sigma'(1)} \sum_{l=2}^{z} \omega'_l \Big)$, that is, $S_{ik} \leq \frac{1}{2}\left(\omega_1 SC_{ik}^{\sigma(1)} + SC_{ik}^{\sigma(1)} \left(1-\omega_1\right) + \omega'_1 SA_{ik}^{\sigma'(1)} + SA_{ik}^{\sigma'(1)} \left(1-\omega'_1\right) \right) \Rightarrow s_{ik} \leq \frac{1}{2}\left(SC_{ik}^{\sigma(1)} + SA_{ik}^{\sigma'(1)} \right)$, i.e., $S_{ik} \leq \frac{1}{2}\left(\max_j \left[SC_{ik}^{\sigma(j)} \right] + \max_l \left[SA_{ik}^{\sigma'(l)} \right] \right)$.

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