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Noetherian and totally noetherian rings and modules

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Doctoral Thesis

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This dissertation has been elaborated in the Department of Algebra of the University of Granada, under the direction of Professor Dr. Pascual Jara Martínez, for the obtaining of the title of Doctor in the Mathematics Doctorate Program of the Universities of Almería, Cádiz, Granada, Jaén and Málaga.

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Contents

Abstract

Given a commutative ring A there are different approaches to understand its structure; one is consider ideals and their arithmetic (multiplicative theory), and another one is to consider modules over A (module theory); in this work we shall mix both; on one hand we shall study ideals; in particular prime ideals, and on the other we shall use categories of modules and functors between them. Recall that the spectrum of A, endowed with Zariski topology, is a bridge between Algebra and Geometry. In this approach we shall consider subsets of the spectrum of A and chain conditions and presheaf constructions on them.

Indeed, given a ring A we shall consider a subset $\mathscr{K} \subseteq \mathrm{Spec}(A)$ closed under generalizations, and the associated hereditary torsion theory $\sigma_{\mathscr{K}}$, or, more generally we shall consider a hereditary torsion theory σ on **Mod**–A, and define chain conditions relative to σ such that we extend the range of examples we may study. The behaviour of these constructions is acceptable from a categorical point of view, so we can construct new categories and functors and so on. The simplest example is provided by a multiplicative set $S \subseteq A$ for which we have the fraction ring $S^{-1}A$, and the category of $S^{-1}A$ modules. A general hereditary torsion theory σ has a similar description whenever A is a σ–noetherian ring; in fact, it is determined by a multiplicative set of finitely generated ideals rather than principal ones.

In both cases we obtain a categorical framework which is useful for some developments; however, a more arithmetic approach might be of interest. For instance, an A–module M is σ–torsion when for each element $m \in M$ there exists an ideal $\mathfrak{h}_m \in \mathscr{L}(\sigma)$ such that $m\mathfrak{h}_m = 0$; a common ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ should be the best choice to work more effectively; this occurs when M is finitely generated; that is, if M is σ–torsion and finitely generated, there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $M\mathfrak{h} = 0$.

With this new approach we have three different notions for noetherian module:

- M is noetherian whenever the lattice of all submodules of M is noetherian.
- M is σ–noetherian whenever the lattice of all σ–closed submodules is noetherian, and the third one for which we have no categorical description is:
- M is totally σ–noetherian whenever for any ascending chain of submodules $\{N_i \mid i \in I\}$ there is an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$, and an element $j \in I$ such that $N_i\mathfrak{h} \subseteq N_j$.

This notion of totally σ–noetherian was introduced by Anderson and Dumitrescu as S–finite

for a multiplicative set $S \subseteq A$, which coincides with our definition of totally σ_S–noetherian.

This more arithmetical approach to chain conditions has the advantage of allowing effective computation, and the disadvantage of losing several categorical and functorial constructions.

The following is a brief description of the content of this dissertation. Chapters one and two are introductory to the theory. In chapter three, we study σ–finitely generated modules, consider the spectrum of a hereditary torsion theory and define the σ–radical. As we noted earlier, if A is a totally σ–noetherian ring, then σ is a finite type hereditary torsion theory, which shows that we are extending the theory of multiplicative sets to multiplicative sets of finitely generated ideals. In particular, we study noetherian spaces and its relationship with chains of totally radical ideals. Finally we include some examples to illustrate the theory.

In chapter four, we consider totally prime submodules as a generalization of prime ideals; our first aim is to show that for every σ–closed prime ideal $\mathfrak{p} \subseteq A$ there exists a plethora of totally σ–prime ideals defined by $\mathfrak{p}$, and that not all of them have the form $\mathfrak{p}\mathfrak{h}$ for some $\mathfrak{h} \in \mathscr{L}(\sigma)$. In particular we show that under the totally σ–noetherian condition the ideals of A are closely related to totally σ–prime ideals. The behaviour with respect to a ring map is also considered, and finally we show that for totally σ–finitely generated modules the prime spectrum contains relevant information about the modules.

In chapter five, we study ring extensions, focussing on idealization, pullback constructions, amalgamated extensions, and how we can induce hereditary torsion theories between them, and what properties are preserved. A problem appears when studying idealization; we know that if A is a PIR, in general the idealization is not a PIR, except in very few cases. To show that this also happens if we consider this notion relative to a hereditary torsion theory σ we develop the structure theory of totally σ–PIRs, showing that every totally σ–PIR is up to the torsion submodule a direct product of indecomposable totally σ–PIRs; this will be of application in the case of S–principal ideal rings. In particular, we show that only in very few cases an idealization is totally PIR. To complete the presentation, we collect some collections of examples to clarify the theory.

1 Introduction

In Commutative Algebra, the study of chain conditions has been a useful tool for classifying rings and establishing properties of rings through their modules, as has been in the develop of their applications. In this theory a prominent role has been played first by prime ideal and later by prime submodules; in fact, they are the link between algebraic and geometric properties of the rings.

We focus our attention on chain conditions: noetherian, artinian, or PIR, and show how prime ideals contain the necessary information to characterize rings and modules satisfying these properties.

There are several level in this study. On the one hand, we can study lattices of ideals, $\mathcal{L}(A)$, or submodules, $\mathcal{L}(M)$; they are modular, bounded and upper–continuous lattices; other properties on these lattices give different classes of ring and modules. For instance, if $\mathcal{L}(A)$ is distributive, the ring A is arithmetic (if, in addition, A is an integral domain, then it is a Prüfer domain). If $\mathcal{L}(A) = \{0, A\}$, then A is a field, and so on.

In this process, the relationships between rings is the cornerstone of the theory. Thus, if $f : A \longrightarrow B$ is a ring map, then we are interested when we can obtain information of B from A. If f is a surjective map, there exists an ideal $\operatorname{Ker}(f) \subseteq A$, and an isomorphism $B \cong A/\operatorname{Ker}(f)$ that allows to identify $\mathcal{L}(B)$ with the sublattice $\{\mathfrak{a} \subseteq A \mid \mathfrak{a} \supseteq \operatorname{Ker}(f)\}$. If B is finitely generated as A–module many properties of A pass to properties of B. If B is finitely generated as A–algebra, we can also study properties of B from the same properties of A; this is the case of the polynomial ring in finitely many indeterminates. If every element of B is a fraction of two elements of A, we also have that many properties of B are determined by properties of A; for any multiplicative set $\Sigma \subseteq A$, being $B = \Sigma^{-1}A$, there exists a lattice isomorphism between $\operatorname{Spec}(B)$ and $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap \Sigma = \varnothing\}$. Further examples of this situation can be shown as the formally power series ring $A[\![X]\!]$, which is the completion of $A[X]$ with respect to a linear topology. The same situation applies for modules.

To organize all this information we shall use categories and functors associated to rings and modules. Thus, for any ring map $f : A \longrightarrow B$ we have a functor $\mathcal{U}_f : \textbf{Mod}\text{–}B \longrightarrow \textbf{Mod}\text{–}A$ which is right adjoint to $- \otimes_A B : \textbf{Mod}\text{–}A \longrightarrow \textbf{Mod}\text{–}B$; or for every prime ideal $\mathfrak{p} \subseteq A$ the ring map

$\lambda_{\mathfrak{p}} : A \longrightarrow A_{\mathfrak{p}}$ defines a functor $L_{\mathfrak{p}} : \mathbf{Mod}\text{–}A \longrightarrow \mathbf{Mod}\text{–}A_{\mathfrak{p}}$ as $L_{\mathfrak{p}}(M) = M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$.

As we point out before, there exists a bridge between algebra and geometry. To any commutative ring A, a geometrical object appears: the prime spectrum Spec(A), with the Zariski topology. The closed subsets are $\{V(\mathfrak{a}) \mid \mathfrak{a} \in \mathscr{L}(A)\}$, where $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

For each prime ideal $\mathfrak{p} \in \text{Spec}(A)$ we have a ring $A_{\mathfrak{p}}$, which has only a maximal ideal. All these local rings acting together give us a great deal of information on A and its modules. For instance, a module map $g : M \longrightarrow N$ is surjective (resp. injective) if, and only if, $g_{\mathfrak{p}} : M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ is surjective (resp. injective) for every prime ideal $\mathfrak{p} \in \text{Spec}(A)$; a local property. On the other hand, for any basic open subset $X(a) \subseteq \text{Spec}(A)$, being $X(a) = \{\mathfrak{p} \in \text{Spec}(A) \mid a \notin \mathfrak{p}\}$ for any $a \in A$, we have a ring $A_a = \Sigma_a^{-1}A$, where $\Sigma_a = \{a^n \in A \mid n \in \mathbb{N}\}$, and a presheaf of rings on the topological space Spec(A), whose fiber at the point $\mathfrak{p} \in \text{Spec}(A)$ is exactly $A_{\mathfrak{p}}$.

In this process some parts of the spectrum become more and more relevant; for this reason, it is necessary to study the localization not in single prime ideal, but in a set of prime ideals: this is the case of A_a mentioned above. Thus appears the notion of hereditary torsion theory; for any hereditary torsion theory σ in $\mathbf{Mod}$–A we build a ring A_σ, a module category $\mathbf{Mod}\text{–}A_\sigma$, and functors. In the development a problem appears: in general, for any A-module M we have two A_σ–modules: M_σ, and $M \otimes_A A_\sigma$. As we saw before, in the case of the localization at $\mathfrak{p}$ they are isomorphic, but in the general case they are not. Therefore, we have three categories: $\mathbf{Mod}$–A, $\mathbf{Mod}\text{–}A_\sigma$, and the category of all modules M_σ, where $M \in \mathbf{Mod}\text{–}A$, and we denote by $\mathbf{Mod}\text{–}(A, \sigma)$. Since for any prime ideal $\mathfrak{p} \in \text{Spec}(A)$ we have a hereditary torsion theory $\sigma_{A\setminus\mathfrak{p}}$, and $M_{\sigma_{A\setminus\mathfrak{p}}} \cong M_{\mathfrak{p}}$, then $\mathbf{Mod}\text{–}(A, \sigma_{A\setminus\mathfrak{p}}) \simeq \mathbf{Mod}\text{–}A_{\mathfrak{p}}$.

To continue this theory, it is necessary to develop an arithmetic of the hereditary torsion theories, as well as a lattice theory.

Once hereditary torsion theories are well understood, we model their applications to rings and modules with applications to finitely generated and noetherian modules; obtaining that if A is a noetherian ring relative to a hereditary torsion theory, then σ can be described through prime ideals; indeed, $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$. As we see, we are always very close to the localization in prime ideals; moreover, the Gabriel filter $\mathscr{L}(\sigma)$ has a cofinal set of finitely generated ideals

(In this sense remember that each filter $\mathscr{L}(\sigma_{A\setminus\mathfrak{p}})$ has a cofinal set of principal ideals. With this background other chain conditions have been explored: artinian, PIR, and so on).

For any module M we have $\mathscr{L}(M)$, the lattice of all submodules of M; if we take a hereditary torsion theory σ, we have another lattice: $C(M,\sigma)$, which is defined by a closure operator: $\mathrm{Cl}_\sigma^M(-)$; in the sense that M is σ–noetherian (resp. σ–artinian) whenever $C(M,\sigma)$ is a noetherian (resp. artinian) lattice, and characterizes when M is σ–noetherian in terms of the localization at prime ideals belonging to $\mathscr{K}(\sigma)$.

The relationship between the lattices $\mathscr{L}(M)$ and $C(M,\sigma)$ yields new examples of ring and modules satisfying chain conditions (remember that any noetherian module is σ–noetherian for any hereditary torsion theory σ). With the drawback that we cannot control efficiently the elements: for example, if M is σ–torsion, for any element $m \in M$ there exists an ideal $\mathfrak{h}_m \in \mathscr{L}(\sigma)$ such that $m\mathfrak{h}_m = 0$. But we can simply consider an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $M\mathfrak{h} = 0$ whenever M is finitely generated, and extend this to a general definition: a module M is totally σ–torsion whenever there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $M\mathfrak{h} = 0$. We gain in description and lose in categorical properties, since now we do not have an associated lattice to study chain conditions.

In this dissertation we study totally σ–torsion modules and properties derived from them. The origin of the theory we shall develop lies in the notion of almost principal ideal domain, introduced by Hamann–Houston–Johnson in [26]. The authors study the problem of determining the structure of the polynomial ring $D[X]$, over an integral domain D with field of fractions K, looking for the structure of the Euclidean domain $K[X]$. In particular, an ideal $\mathfrak{a} \subseteq D[X]$ is said to be **almost principal** whenever there exist a polynomial $F \in \mathfrak{a}$, of positive degree, and an element $0 \neq s \in D$ such that $\mathfrak{a}s \subseteq FD[X] \subseteq \mathfrak{a}$. The integral domain D is an **almost principal domain** whenever every ideal $\mathfrak{a} \subseteq D[X]$, which extends properly to $K[X]$, is almost principal. Noetherian and integrally closed domains are examples of almost principal domains.

This was later extended by Anderson–Kwak–Zafrullah in [6] to consider almost noetherian rings, showing that for an integral domain D the polynomial ring $D[X]$ is almost noetherian if, and only if, $D[X]$ is almost PID.

It was in 2002 when Anderson and Dumitrescu, in [5] abstracted this notion to introduce S–noetherian rings, for a given multiplicative set $S \subseteq A$.

The authors extend this notion to non–necessarily integral domains by defining, for a given multiplicatively closed subset $S \subseteq A$ of a ring A, an ideal $\mathfrak{a} \subseteq A$ to be **S–finite** if there exist a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$, and an element $s \in S$, such that $\mathfrak{a}s \subseteq \mathfrak{a}'$. They define a ring A to be **S–noetherian** whenever every ideal $\mathfrak{a} \subseteq A$ is S–finite. Many authors have worked on S–noetherian rings and related notions, and shown relevant results on its structure. For instance, see [12, 19, 27, 31, 40, 41, 51, 55].

Continuing the abstraction process, in 2021, Jara, in [31], introduced totally σ–noetherian rings. One may think that S–noetherian rings are related to a multiplication set $S \subseteq A$, while totally σ–noetherian ring are related to a multiplicative set $\mathscr{S} \subseteq \mathscr{L}(A)$ of finitely generated ideals covering a wider range of examples.

Allow us to give a brief description of the contents of this dissertation.

This thesis is organized in chapters. This is chapter (1); in chapter (2), we start with recalling the main concepts and facts that will be frequently used in our study. In it A will be a commutative ring and σ be a hereditary torsion theory in **Mod**–A. We introduce some notions of rings and modules using the hereditary torsion theory.

In chapter (3), we introduce the extension of S–noetherian spectrum property. That is; for a multiplicatively closed subset S of a commutative ring A, there were several S–noetherian spectrum properties. In our study, for any commutative ring A, we introduce generalizations of them using a hereditary torsion theory σ instead of a multiplicative closed subset S.

Our interest shall be to study the prime spectrum, Spec(A), of a ring A through S–finite ideals, taking into account that for any multiplicative subset $S \subseteq A$, the prime spectrum has a partition in two subsets: the prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \cap S \neq \varnothing$, and the prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \cap S = \varnothing$.

In [27], the author study the **S–noetherian spectrum property** of a ring A, this means that every ideal $\mathfrak{a} \subseteq A$ is **radically S–finite**, i.e., there exist a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$, and $s \in S$ such that $\mathfrak{a}s \subseteq \operatorname{rad}(\mathfrak{a}')$, and study when this property is inherited by the polynomial ring. A different approach appears in [19], where the author consider rings with a new notion of S**–noetherian spectrum**, i.e., the ascending chain condition on S–radical ideals holds, defines the **prime** S**–radical** of an ideal $\mathfrak{a} \subseteq A$ as the intersections of the prime ideals in the following set: $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p} \text{ and } \mathfrak{p} \cap S = \varnothing\}$, and gives several characterization of rings with S–noetherian spectrum. In this respect, our aim is to show that these two theories are part of a more general theory involving hereditary torsion theories. In particular, we show that totally noetherian σ–radical and Spec–noetherian are characterized through prime ideals.

Hence, in section (3.1) we introduce the main subject: totally σ–finitely generated modules and the lattice of σ–closed submodules. After that, in section (3.2), following [19], we consider the spectrum of a hereditary torsion theory $\mathscr{K}(\sigma)$, and define the prime σ–radical. The main aim is to characterize when it is a noetherian space. In particular, we show that if the ring A is noetherian σ–radical, then we can consider that σ is of finite type. A different approach to noetherian spaces associated to a ring is developed, following [27], in section (3.6). This is not a lattice approach, but we show that a ring A is totally noetherian σ–radical if, and only if, every increasing chain of radical ideals is totally σ–stable. Both approaches to noetherian spaces have in common that they are preserved by polynomial ring constructions, see Theorems (3.25.) and (3.43.), and they can be characterized by prime ideals. Most of these results appear in [34].

The main aim of chapter (4) is to provide new notions of prime ideals and submodules relative to totally torsion with respect to a hereditary torsion theory. This theory extends the developments made in [5], [28] and [50]; also [34], and since these prime objects are established with respect to a hereditary torsion theory, hence it will be of general application.

We begin introducing the basic notions relative to torsion theories, and we address to [25] and [52] as main references for that.

Our main object of studying will be prime ideals and submodules. We shall use three different notions of prime: prime ideal, totally σ–prime ideal and σ–prime ideal, and similarly for submodules.

Which is of interest, with respect to prime submodules, is that if $N \subseteq M$ is a prime submodule, then M/N is a either σ–torsion or σ–torsionfree. Therefore, every hereditary torsion theory produces a partition of the prime spectrum (the set of all prime submodules) of any A–module

M. We have: Spec$(M) = \mathscr{X}(M,\sigma) \cup \mathscr{K}(M,\sigma)$, being $\mathscr{X}(M,\sigma)$ (resp. $\mathscr{K}(M,\sigma)$) the set of all σ–dense (resp. σ–closed) prime submodules. In the particular case in which $M = A$, i.e., it is the ring, we shall represent $\mathscr{X}(A,\sigma)$ (resp. $\mathscr{K}(A,\sigma)$), simply as $\mathscr{X}(\sigma)$ (resp. $\mathscr{K}(\sigma)$).

There is a natural way of extending prime ideals to prime ideals relative to a hereditary torsion theory: σ–prime ideals, i.e., ideals $\mathfrak{a} \subseteq A$ such that $\mathrm{Cl}_\sigma^A(\mathfrak{a}) \neq \mathfrak{a}$ and $\overline{\mathfrak{a}} = \mathrm{Cl}_\sigma^A(\mathfrak{a}) \subseteq A$ is a prime ideal. The theory of σ–prime submodules is useful, but we have no control on the extension $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$. In addition, if $N \subseteq M$ is a σ–prime submodule, not necessarily the ideal $(N : M) \subseteq A$ is σ–prime, contrary to what happens in the prime submodule case.

To fix this malfunction, we introduce totally σ–prime ideals and submodules: an ideal $\mathfrak{a} \subseteq A$ is totally σ-prime if $\mathfrak{a} \notin \mathscr{L}(\sigma)$, and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\overline{\mathfrak{a}} = (\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime; or equivalently, see Proposition (4.5.), if $\mathfrak{a} \notin \mathscr{L}(\sigma)$ and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for every $a, b \in A$, if $ab \in \mathfrak{a}$, then either $a\mathfrak{h} \subseteq \mathfrak{a}$ or $b\mathfrak{h} \subseteq \mathfrak{a}$, and, in a similar way, we define totally σ–prime submodules. Which is important with this notion is that (1) every totally σ–prime submodule is a σ–prime submodule, and (2) if $N \subseteq M$ is totally σ–prime, then $(N : M) \subseteq A$ is totally σ–prime.

As happens with σ–prime, the behaviour of totally σ–prime ideals and submodules is closely linked to prime ideals; hence, for every σ–closed prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$, and every ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$, if $\mathfrak{a}$ is an ideal such that $\mathfrak{p}\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a} \subseteq A$ is totally σ–prime. Totally σ–prime ideals satisfy the usual theorems for prime ideals: avoidance theorem, Cohen's theorem, etc. Thus, a totally σ–finitely generated A–module is totally σ–noetherian, see [31] and [32], if, and only if, every prime σ–closed submodule is totally σ–finitely generated, or equivalently, if, and only if, every totally σ–prime submodule is totally σ–finitely generated.

We want to point out that, since σ–prime ideals and submodules have a better behaviour with respect to categorical properties, totally σ–prime ideals and submodules are closer to the arithmetic of rings and modules than σ–primes.

The contents of this chapter are distributed in sections. In section (4.1) we introduce the different kinds of prime ideals, and its general properties, and show that for every σ–closed prime ideal $\mathfrak{p} \subseteq A$ there exists a plethora of totally σ–prime ideals defined by $\mathfrak{p}$, and that not all of them have the shape $\mathfrak{p}\mathfrak{h}$ for some $\mathfrak{h} \in \mathscr{L}(\sigma)$. In section (4.2) we deal with the existence of totally

σ–prime ideals, and show that they exist whenever A is σ–noetherian; in particular, whenever it is totally σ–noetherian. Ring extensions are studied in section (4.3); given two rings A and B, and a ring map $f : A \longrightarrow B$, we know that every hereditary torsion theory σ in **Mod**–A induces a hereditary torsion theory $\overline{\sigma}$ in **Mod**–B, and show that there exists a correspondence between totally $\overline{\sigma}$–prime ideals of B and totally σ–prime ideals of A, which is a bijection, whenever f is surjective, between totally σ–prime ideals of B and totally $\overline{\sigma}$–prime ideals of A containing Ker(f). Avoidance theorem is proved. In section (4.4) we study totally σ–prime submodules $N \subseteq M$, and characterize them through the totally σ–prime ideal $(N : M) \subseteq A$. In particular, for a general hereditary torsion theory, Cohen's theorem is proved when M is totally σ–finitely generated; and a new version of this theorem is established when σ is of finite type, in this case relative to totally σ–prime ideals instead of submodules. In section (4.5) we study the relationship with the localization at prime ideals. Most of these results appear in [33].

Finally, in chapter (5), we study the behaviour of hereditary torsion theories in ring extensions and ring maps, focussing on idealization, pullback constructions, amalgamated extensions, etc.

For a commutative ring A with unity, and an A–module M. The **idealization** of M with respect to A is $B = M \rtimes A = \{(m, a) \mid m \in M, a \in A\}$ with sum component-wise, and multiplication given by $(m_1, a_1)(m_2, a_2) = (m_1a_2 + m_2a_1, a_1a_2)$. Also, it is called the trivial extension of A by M. Note that A naturally embeds into $M \rtimes A$ via $a \mapsto (0, a)$. If N is a submodule of M, then $N \rtimes \{0\}$ is an ideal of $M \rtimes A$. (This is why the construction is called idealization).$(M \rtimes \{0\})^2 = (0, 0)$ and hence $M \rtimes A$ is non-reduced for any ring A and any non-zero A-module M. While we do not know who first constructed an example using idealization, the idea of using idealization to extend results concerning ideals to modules is due to Nagata [46]. Nagata in his famous book, Local Rings [46], presented a principle, called the principle of idealization. By this principle, modules become ideals. Idealization is advantageous for working on ideal instead of submodules, generalizing results from rings to modules, and illustrate examples on commutative rings with zero divisors. For more about Nagata idealization, we refer the reader to [1, 2, 7, 37, 38, 46].

In section (5.1) we identify the hereditary torsion theory of the idealization depending on the hereditary torsion theory σ in **Mod**–A, which is $f(\sigma)$ in **Mod**–$M \rtimes A$, whose Gabriel filter is:

$\mathscr{L}(f(\sigma)) = \{\mathfrak{b} \subseteq M \rtimes A \mid f^{-1}(\mathfrak{b}) \in \mathscr{L}(\sigma)\}$. We continue with considering σ is of finite type and provide examples. Idealization and principal ideal rings are studied in section (5.1), beginning with the definition of principal ideal rings and the related notions of it, then, constructing theorems according to the hereditary torsion theory, and determine when the idealization is either a Principle ideal ring or a Euclidean ring.

Section (5.2) is devoted to principal ideal domains (PIDs) which constitute an important class of rings, mainly because abelian groups have $\mathbb{Z}$, the ring of integer numbers, as ground ring, and because many other examples are of common use: the polynomial ring in one indeterminate with coefficients in a field, several rings of algebraic numbers, or serial rings. In addition, the structure of finitely generated modules over a PID is also well known, and an important tool in other theories.

PIDs have many generalizations, on the one hand: Bezout domains in which finitely generated ideals are principal, on the other hand: principal ideal rings (PIRs), in which the integral domain condition was removed, and so on. Other generalizations of PIDs concerning various objects related to the ring such as, for instance, a multiplicative set $S \subseteq A$. Thus, in 1988, Hamann, Houston and Johnson, in [26], introduce the notion of almost principal ideal and almost PID. If D is an integral domain with field of fractions K, an ideal $\mathfrak{a} \subseteq A[X]$ is *almost principal* if there exist $F(X) \in \mathfrak{a}$, of positive degree, and $s \in D \setminus \{0\}$ such that $\mathfrak{a}s \subseteq F(X)D[X]$; and the integral domain D is *almost PID* whenever every ideal $\mathfrak{a} \subseteq D[X]$, with proper extension to $K[X]$, is almost principal.

Later, in 2002, Anderson and Dumitrescu, in [5], introduce the notion of S–PIR. In this more abstract framework they perform a study, parallel to the classical one, of PIDs, but relative to a multiplicative set $S \subseteq A$. Many others works on S–PID and S–PIR are realized in the past years, see [5, 8, 10, 11, 13, 32, 39].

Because of the definition of S–PIR involving single elements of A, their theory focuses on arithmetics properties; i.e., an ideal $\mathfrak{a} \subseteq A$ is *S–principal* whenever there exist elements $a \in \mathfrak{a}$ and $s \in S$ such that $\mathfrak{a}s \subseteq aA$, and A is an *S–PIR* whenever every ideal $\mathfrak{a} \subseteq A$, such that $\mathfrak{a} \cap S = \varnothing$, is S–principal. From the very beginning, classical results, such as Kaplansky's lemma, as well as the the spectrum of A relative to S were considered in this new framework.

Recently, in 2023, Jara, in [32], introduced an extension in the notion of PIR, proving some of the classical results mentioned above. This new view is based on considering hereditary torsion theories, σ, instead of multiplicative sets, becoming the S–PIRs examples by considering the hereditary torsion theory σ_S defined by S.

With this new tool, and point of view, some results are seen from a different perspective, and the use of categories and functor and other categorical tools help in obtaining new results. Thus, in this work, using the torsion submodule, we realize an even closer approximation to prime ideals, by determining the exact elements that appear in their definition and description, see [31] and [32].

Having stated the necessary and appropriate background, we address with the characterization of PIR relative to a hereditary torsion theory σ: totally σ–PIR, and its relationship with chain conditions. It is clear that every totally σ–PIR is totally σ–noetherian, but not necessarily totally σ–artinian; nevertheless, *non-trivial* quotients of totally σ–PIR are always totally σ–artinian, Theorem (5.30.).

Our main aim in this section is to establish a structure theorem for totally σ–PIRs, and hence of S–PIRs. To do so we first study those totally σ–PIRs that have only one prime ideal minimal in $\mathcal{K}(\sigma)$, and those who have only a prime ideal maximal in $\mathcal{K}(\sigma)$ (*indecomposable* and *local* totally σ–PIR, respectively); the latter as a tool and the former as a brick to build totally σ–PIRs. In the absence of a primary decomposition theory we deal with the set $\mathcal{K}(\sigma)$ of prime ideals and its lattice structure, and decompose it in subsets which are closed under generalizations and specializations, yielding a partition of $\mathcal{K}(\sigma)$, and finally a lattice decomposition of $A/\sigma A$, following the development done in [23] and [24]. Most of these results will appear in [35].

Section (5.3) deal with another important example of extension are Dorroh extensions. In order to define Dorroh extension, we first define algebras. Given a ring A, an A**–algebra** is an abelian group B satisfying: (1) B is an A–module, (2) B is a ring, not necessarily with unity nor commutative, and (3) the action of A satisfies: $a(b_1b_2) = (ab_1)b_2 = b_1(ab_2)$, for any $a \in A$ and $b_1, b_2 \in B$. See [3].

Hence, if A is a ring and B is A–algebra; a new A–algebra can be build as follows: $B \rtimes A = \{(b,a) \mid\ b \in B \text{ and } a \in A\}$ being the multiplication $(b_1,a_1)(b_2,a_2) = (b_1b_2 + b_1a_2 + b_2a_1, a_1a_2)$. The A–algebra $B \rtimes A$ is called the **Dorroh extension** of B by A, and it may be denoted also by

B_1. Dorroh [18] first used this construction, with $A = \mathbb{Z}$, (the ring of integers), as a means of embedding a (non–unitary) ring A without unity into a ring with unity. In ring theory, Dorroh extension has become an important method for constructing new rings and investing properties of rings. Many ring constructions can be regarded as Dorroh extensions of rings, for instance, the trivial extension of a ring and the amalgamated algebras along an ideal, being the homomorphic image of the Dorroh extension. Furthermore, properties of Dorroh extensions of rings are referred to [3, 14, 17, 21, 44, 53].

Then, in section (5.3) in order to give the structure of B–module, we introduce the **Universal property of the Dorroh extension**, Lemma (5.45.). Using an A–algebra map, we see that to give a structure of B–module on M is equivalent to give a structure of $B \rtimes A$–module. In consequence, we define the B–submodule and then cyclic submodule. That is, given a B–module M, for any element $x \in M$ there is a smallest B–submodule $\langle x \rangle$ containing x: the intersection of all B–submodules containing x, and it is called the **cyclic submodule generated by** x. So, for any subset $S \subseteq A$, the B–submodule generated by S is the smallest B–submodule containing S; it is denoted as $\langle S \rangle$.

We continue by considering the **finiteness conditions** on algebras, and see when the A–algebra $B \rtimes A$ is noetherian. The result is in Theorem (5.47.); Let A be a ring and B an A–algebra. If $B \rtimes A$ is noetherian, then A is a noetherian ring and B is a noetherian A–algebra. Also, if A is a noetherian ring and B a noetherian A–algebra, then $B \rtimes A$ is noetherian. Using the same procedure, we obtain that $B \rtimes A$ is artinian if, and only if, A is artinian and B is an artinian B–algebra. As a generalization of that, we define the hereditary torsion theory on Dorroh extension by setting the Gabriel filters then study the noetherian and artinian notions according to hereditary torsion theory. See Theorems (5.50.) and (5.51.). Besides that, we see the behaviour of chain conditions relative to totally torsion of the A–algebra extension, Theorems (5.52.) and (5.53.).

In category theory, pullbacks play an important tool in the commutative algebra because of their use in producing many examples. One of these examples is Nagata's idealization. Therefore, in section (5.4), we mention the definition of the Pullback construction and some properties of it, That is, if A, B and C are commutative rings with unities, if $\alpha : A \to C$ and $\beta : B \to C$ are ring homomorphisms, the set $D := \{(a, b) \in A \times B | \ \alpha(a) = \beta(b)\}$ of $A \times B$ is called the **pullback** of α and β, see section (5.4). There are many types of pullbacks according to type of maps be used. We are interested in the type of pullbacks as in Proposition (5.55.). See the proposition below. After that, we construct theorems about noetherian and totally noetherian of the pullbacks, using the hereditary torsion theory defined on them in Remark (5.57.). Finally, in this section, we explain the pullbacks properties. The first one is related to the kernel of the opposite maps. A second property is related with that factorization of homomorphisms and the third one says that two pullback squares produces a pullback square.

In order to set up a more general setting of the idealization, D'Anna and Fontana in 2007 introduced and studied the amalgamated duplication using Dorroh extension [17]. Then in 2009, D'Anna, Finocchiaro and Fontana provided a generalization of the amalgamated duplication which depends on $H \rtimes A$ as follows.

Let $f : A \to S$ be a ring homomorphism and $\mathfrak{b}$ an ideal of S. Then in view of the equation $a.j = f(a)j$ for any $a \in A$ and $j \in \mathfrak{b}$, then $\mathfrak{b}$ has the A-module structure. Now, let $\mathfrak{b} \rtimes A$ be the Dorroh extension considered above. Then the **amalgamation of the A-algebra S along $\mathfrak{b}$ with respect to f** is defined to be: $A \bowtie^f \mathfrak{b} := f^{\bowtie}(\mathfrak{b} \rtimes A) = \{(a, f(a)+j) \mid a \in A, j \in \mathfrak{b}\} \subseteq A \times S$, where the multiplication is componentwise. See [16]. They also showed that the amalgamation algebra can be realized as a pullback [16, Proposition 4.7]. Indeed, the amalgamation construction takes its importance from several aspects, namely:

1. It generalizes the Nagata idealization. Let $S = M \rtimes A$ and $f : A \hookrightarrow M \rtimes A$ be the canonical ring embedding that assigns to a the element $(a, 0)$. Then viewing M ($M \cong M \rtimes 0$) as an ideal of $M \rtimes A$, the idealization of M in A is canonically isomorphic to $A \bowtie^f M$. (i.e. $M \rtimes A \cong A \bowtie^f M$).
2. It generalizes the $D + M$ construction. Let M be a maximal ideal of a ring T and let D be a subring of T such that $M \cap D = (0)$. The ring $D + M := \{x + m \mid x \in D, m \in M\}$ is canonically isomorphic to $D \bowtie^f M$, where $f : D \hookrightarrow T$ is the natural embedding.
3. The amalgamation like any construction has several applications in solving open questions; see for example the main results of [16, 17], and also is useful in providing new counter examples.

In consequence, in section (5.5) we define the amalgamated algebra, build the diagram of pullback of the amalgamated algebras, and construct the hereditary torsion theories through the Gabriel filter. Continuing with noetherian and totally noetherian properties on this construction and study the noetherian notion through the prime ideals.

2 Preliminaries

Hereditary torsion theories

In this chapter we recall some basic concepts and facts that will be frequently used throughout this thesis.

Let A be a commutative ring with unity $1 \in A$; an A**–module** M is an abelian group together a right action $\beta : A \longrightarrow \mathrm{End}(M)$, denoted $\beta(a)(m) = ma$, for any $m \in M$ and $a \in A$.

The **category of all modules Mod**–A has as objects the A–modules and morphisms the module maps; i.e., maps $f : M_1 \longrightarrow M_2$ such that $f(x+y) = f(x) + f(y)$ and $f(xa) = f(x)a$, for any $x, y \in M_1$ and $a \in A$.

A **hereditary torsion theory** σ is **Mod**–A is given by one of the following objects:

- A class $\mathcal{T}$ of A–modules which is closed under submodules, quotient modules, direct sums and group–extensions.
- A class $\mathcal{F}$ of A–modules closed under submodules, essential extensions, direct products and group–extensions.
- A left exact subfunctor of the identity σ.
- A filter of ideals $\mathcal{L}$ satisfying: if for any $\mathfrak{a} \subseteq A$ there exists an ideal $\mathfrak{b} \in \mathcal{L}$ such that $(\mathfrak{a} : b) \in \mathcal{L}$, for any $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathcal{L}$.

The relationships between these objects are the following:

(1) Given σ, the **radical torsion**, we have:

- $\mathcal{T} = \{M \in \mathbf{Mod}\text{–}A \mid \sigma M = M\}$;
- $\mathcal{F} = \{M \in \mathbf{Mod}\text{–}A \mid \sigma M = 0\}$;
- $\mathcal{L} = \{\mathfrak{a} \subseteq A \mid \sigma(A/\mathfrak{a}) = A/\mathfrak{a}\}$

In this case we denote $\mathcal{T}$ by $\mathcal{T}_\sigma$; $\mathcal{F}$ by $\mathcal{F}_\sigma$, and $\mathcal{L}$ by $\mathcal{L}(\sigma)$.

(2) Given $\mathcal{T}$, the **torsion class**, we have:

- $\mathcal{F} = \{X \in \mathbf{Mod}\text{–}A \mid \mathrm{Hom}_A(T, X) = 0 \text{ for any } T \in \mathcal{T}\}$;
- $\sigma(M) = \sum\{N \subseteq M \mid N \in \mathcal{T}\}$, for any A–module M;

- $\mathscr{L} = \{\mathfrak{a} \subseteq A \mid A/\mathfrak{a} \in \mathscr{T}\}$.

(3) Given $\mathscr{F}$, the **torsionfree class**, we have:

- $\mathscr{T} = \{X \in \mathbf{Mod}\text{–}A \mid \operatorname{Hom}_A(X, F) = 0$ for any $F \in \mathscr{F}\}$;
- $\sigma(M) = \cap\{N \subseteq M \mid M/N \in \mathscr{F}\}$, for any $M \in \mathbf{Mod}\text{–}A$.

(4) Given $\mathscr{L}$, the **Gabriel filter**, we have:

- $\sigma(M) = \{x \in M \mid \operatorname{Ann}(x) \in \mathscr{L}\}$, for any $M \in \mathbf{Mod}\text{–}A$ and, in the lattice $\mathscr{L}(M)$ of all submodules of M, there is a **closure operator**, $\mathrm{Cl}^M_\sigma(-)$, defined, for any submodule $N \subseteq M$, by the equation: $\mathrm{Cl}^M_\sigma(N)/N = \sigma(M/N)$.

For any hereditary torsion theory σ, an elementary property of $\mathscr{L}(\sigma)$ is that it is closed under multiplication. Indeed, if $\mathfrak{a}, \mathfrak{b} \in \mathscr{L}(\sigma)$ and we consider the short exact sequence

$$0 \longrightarrow \frac{\mathfrak{a}}{\mathfrak{ab}} \longrightarrow \frac{A}{\mathfrak{ab}} \longrightarrow \frac{A}{\mathfrak{a}} \longrightarrow 0$$

Since $\mathfrak{a}/\mathfrak{ab}, A/\mathfrak{a} \in \mathscr{T}_\sigma$, then $A/\mathfrak{ab} \in \mathscr{T}_\sigma$, and $\mathfrak{ab} \in \mathscr{L}(\sigma)$.

Given a hereditary torsion theory σ in **Mod**–A several results and definitions are necessary to develop the theory.

(1) Let $N \subseteq M$ be a submodule, we say:

- $N \subseteq M$ is **σ–dense**, and write $N \subseteq_\sigma M$ whenever M/N is σ–torsion. We denote by $\mathscr{L}(M, \sigma)$ the set of all σ–dense submodules of M.
- $N \subseteq M$ is **σ–closed**, whenever M/N is σ–torsionfree. We denote by $C(M, \sigma)$ the set of all σ–closed submodules of M.

(2) For every submodule N of an A–module M the **σ–closure** of N in M is $\mathrm{Cl}^M_\sigma(N)$.

(3) For any prime ideal $\mathfrak{p} \subseteq A$ we have either $A/\mathfrak{p}$ is σ–torsion or $A/\mathfrak{p}$ is σ–torsionfree. We denote by

- $\mathscr{K}(\sigma) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid A/\mathfrak{p}$ is σ–torsionfree$\}$,
- $\mathscr{Z}(\sigma) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid A/\mathfrak{p}$ is σ–torsion$\}$, and
- $\mathscr{C}(\sigma)$ the set of all maximal elements in $\mathscr{K}(\sigma)$.

(4) $\mathcal{K}(\sigma)$ is **closed under generalizations**; i.e., for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$, if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \in \mathcal{K}(\sigma)$, then $\mathfrak{p} \in \mathcal{K}(\sigma)$, and $\mathcal{Z}(\sigma)$ is **closed under specializations**; i.e., for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$, if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \mathcal{Z}(\sigma)$, then $\mathfrak{q} \in \mathcal{Z}(\sigma)$.

In the set of all hereditary torsion theories on **Mod**–A we consider a partial order as follows:

(1) If σ_1, σ_2 are hereditary torsion theories in **Mod**–A, we say $\sigma_1 \leq \sigma_2$ whenever $\mathcal{T}_{\sigma_1} \subseteq \mathcal{T}_{\sigma_2}$, or equivalently if $\mathcal{F}_{\sigma_2} \subseteq \mathcal{F}_{\sigma_1}$.
(2) For any family $\{\sigma_i \mid i \in I\}$ of hereditary torsion theories in **Mod**–A the **infimum**, $\wedge_i \sigma_1$, is defined as

$$\mathcal{T}_{\wedge_i \sigma_i} = \cap_i \mathcal{T}_{\sigma_i} \text{ and } \mathcal{L}(\wedge_i \sigma_i) = \cap_i \mathcal{L}(\sigma_i).$$

(3) The **supremum** of the family $\{\sigma_i \mid i \in I\}$ is the hereditary torsion theory $\vee_i \sigma_i$, defined

$$\mathcal{F}_{\vee_i \sigma_i} = \cap_i \mathcal{F}_{\sigma_i}.$$

Let us to introduce some examples of hereditary torsion theories.

Examples. 2.1.

(1) For any ring A the hereditary torsion theory $\sigma_1 = \mathrm{id}$ such that $\mathcal{T}_{\sigma_1} =$ **Mod**–A is known as the **total** hereditary torsion theory, and the hereditary torsion theory $\sigma_0 = 0$ such that $\mathcal{T}_{\sigma_0} = \{0\}$ is the **trivial** hereditary torsion theory, and satisfies $\mathcal{L}(\sigma_0) = \{A\}$.
(2) For any multiplicatively closed subset $\Sigma \subseteq A$ the filter

$$\mathcal{L}(\sigma_\Sigma) = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \cap \Sigma \neq \varnothing\}$$

is a Gabriel filter for a principal hereditary torsion theory σ_Σ. Conversely, if σ is a **principal**, hereditary torsion theory, see below, and we define

$$\Sigma_\sigma = \{s \in A \mid sA \in \mathcal{L}(\sigma)\},$$

then it is multiplicative and $\sigma_{\Sigma_\sigma} = \sigma$. Besides, $\Sigma = \Sigma_{\sigma_\Sigma}$ for any multiplicative subset $\Sigma \subseteq A$.

(3) For any prime ideal $\mathfrak{p} \subseteq A$ we have that $A \setminus \mathfrak{p}$ is a multiplicative subset, hence associated to $\mathfrak{p}$ we have a hereditary torsion theory $\sigma_{A\setminus\mathfrak{p}}$.

If $\{\mathfrak{p}_i \mid i \in I\}$ is a family of prime ideals then $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}_i} \mid i \in I\}$ is a hereditary torsion theory

(4) For any hereditary torsion theory σ and any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$, we have $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$. In particular, we have: $\sigma \leq \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$.

Observe that if $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals, then $\sigma_{A\setminus\mathfrak{p}_1} \leq \sigma_{A\setminus\mathfrak{p}_2}$.

A hereditary torsion theory σ is called **half–centered** whenever we have the equality; i.e., $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$.

For any multiplicatively closed subset $\Sigma \subseteq A$, the hereditary torsion theory σ_Σ is half–centered.

Let us now study some particular kind of hereditary torsion theories. A hereditary torsion theory σ is

- of **finite type** whenever $\mathscr{L}(\sigma)$ has a cofinal subset of finitely generated ideals, and
- **principal** whenever it has a cofinal subset of principal ideals.

Principal hereditary torsion theories

The principal hereditary torsion theories are of interest because they are parameterized by multiplicative subsets.

For any multiplicative subset Σ we obtain $\sigma_\Sigma = \sigma_{\overline{\Sigma}}$, being $\overline{\Sigma}$ the **saturation** of Σ. In consequence, we may assume $\Sigma = \overline{\Sigma}$ is a **saturated multiplicative subset** (for any $s, t \in A$ if $st \in \Sigma$, then either $s \in \Sigma$ or $t \in \Sigma$).

Therefore, there is a family of prime ideals, say $\{\mathfrak{p}_i \mid i \in I\}$ such that $A \setminus \Sigma = \cup_i \mathfrak{p}_i$. The set of ideals $\{\mathfrak{p}_i \mid i \in I\}$ can be taken closed in the following sense: *if* $\mathfrak{p} \subseteq \cup_i \mathfrak{p}_i$, *hence* $\mathfrak{p} \in \{\mathfrak{p}_i \mid i \in I\}$, *for any prime ideal* $\mathfrak{p}$, a kind of **avoidance property**. In this way we may associate to any saturated multiplicative subset Σ the set of all prime ideal

$$\mathscr{K}(\Sigma) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap \Sigma = \varnothing\} = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \subseteq A \setminus \Sigma\}.$$

We have that $\mathcal{K}(\Sigma)$ satisfies the **avoidance property**.

It is obvious that if a set of prime ideals $\mathscr{P}$ satisfies the avoidance property, it is generically closed.

As we mention before, if $\Sigma \subseteq A$ is a multiplicative subset, then σ_Σ is a principal hereditary torsion theory. The converse also holds. In this sense, for any ring A there is a bijective correspondence between principal hereditary torsion theories and saturated multiplicative subsets; this bijective correspondence can be extended to sets of prime ideals satisfying the avoidance property by the correspondence $\sigma \mapsto \mathcal{K}(\sigma)$, because every principal hereditary torsion theory is half–centered.

Lemma. 2.2.
Let $\Sigma \subseteq A$ be a multiplicative subset, then $\sigma_\Sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \cap \Sigma = \varnothing\} = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma_\Sigma)\}$. Therefore, σ_Σ is half–centered.

Sometimes, if $\Sigma \subseteq A$ is a multiplicatively closed subset the σ_Σ–closure of N in M was also called the **Σ–saturation** of N in M and was denoted by $\operatorname{Sat}_\Sigma^M(N)$; if $\mathfrak{p} \subseteq A$ is a prime ideal and $\Sigma = A \setminus \mathfrak{p}$, we write $\operatorname{Sat}_\mathfrak{p}^M(N)$.

Finite type hereditary torsion theories

If σ is a finite type hereditary torsion theory and we define

$$\mathscr{G}_\sigma = \mathscr{L}_f(\sigma) = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \in \mathscr{L}(\sigma) \text{ is finitely generated}\},$$

then $\mathscr{G}_\sigma$ is a multiplicative set of finitely generated ideals.

For any multiplicative set of finitely generated ideals $\mathscr{G}$, the filter

$$\mathscr{L}(\sigma_\mathscr{G}) = \{\mathfrak{a} \subseteq A \mid \text{there exists } \mathfrak{b} \in \mathscr{G} \text{ such that } \mathfrak{b} \subseteq \mathfrak{a}\}$$

is a Gabriel filter for a finite type hereditary torsion theory. Furthermore, $\mathscr{G} \subseteq \mathscr{G}_{\sigma_\mathscr{G}}$ for any multiplicative set $\mathscr{G}$ of finitely generated ideals, and $\sigma_{\mathscr{G}_\sigma} = \sigma$.

We may extend this to any set, $\mathscr{S}$, of finitely generated ideals. We may consider

$$\mathscr{L}(\mathscr{S}) = \mathscr{L}(\sigma_\mathscr{S}) = \{\mathfrak{a} \subseteq A \mid \text{there exist } \mathfrak{s}_1, \ldots, \mathfrak{s}_t \in \mathscr{S} \text{ such that } \mathfrak{s}_1 \cdots \mathfrak{s}_t \subseteq \mathfrak{a}\}.$$

Lemma. 2.3.

With the above notation $\mathscr{L}(\mathscr{S})$ is a Gabriel filter.

Let us denote by $\sigma_{\mathscr{S}}$ the hereditary torsion theory it defines.

Proof. Let $\mathfrak{a} \subseteq A$ be an ideal such that there exists $\mathfrak{h} \in \mathscr{L}(\mathscr{S})$ satisfying $(\mathfrak{a} : h) \in \mathscr{L}(\mathscr{S})$, for every $h \in \mathfrak{h}$. We may consider $\mathfrak{h}$ finitely generated with generators $h_1, \ldots, h_s$. For any index i there exist $\mathfrak{s}_{i,1}, \ldots, \mathfrak{s}_{i,t_i} \in \mathscr{S}$ such that $h_i \mathfrak{s}_{i,1} \cdots \mathfrak{s}_{i,t_i} \subseteq \mathfrak{a}$; hence we obtain $\mathfrak{h} \prod_i (\mathfrak{s}_{i,1} \cdots \mathfrak{s}_{i,t_i}) \subseteq \mathfrak{a}$, and $\mathfrak{a} \in \mathscr{L}(\mathscr{S})$. □

There is a correspondence between finite type hereditary torsion theories and multiplicative sets of finitely generated ideals. To any hereditary torsion theory σ we associate $\mathscr{L}_f(\sigma)$, and to any set of finitely generated ideals $\mathscr{S}$ the hereditary torsion theory $\sigma_{\mathscr{S}}$. It will be a bijection if we consider *saturated* sets of finitely generated ideals.

The set $\mathscr{L}_f(\sigma)$ is a filter in the poset $\mathscr{L}_f(A)$, of all finitely generated ideals of A and in addition it satisfies the Gabriel condition: for any $\mathfrak{a} \in \mathscr{L}_f(A)$ such that there exists $\mathfrak{h} \in \mathscr{L}_f(\sigma)$ satisfying for any $h \in \mathfrak{b}$ there exists $\mathfrak{b}_h \in \mathscr{L}_f(\sigma)$ such that $h\mathfrak{b}_h \subseteq \mathfrak{a}$, we have $\mathfrak{a} \in \mathscr{L}_f(\sigma)$. Let us name a **Gabriel sets** those sets of finitely generated ideals satisfying this property. On the other hand, for any multiplicative set of finitely generated ideals $\mathscr{S}$ we have $\mathscr{L}_f(\sigma_{\mathscr{S}}) \supseteq \mathscr{S}$ is a Gabriel set. In consequence, the above correspondence is bijective between finite type hereditary torsion theories and Gabriel sets of finitely generated ideals.

The following results are presented without proof.

Proposition. 2.4.

Every finite type hereditary torsion theory is half–centered.

It is possible to characterize finite type hereditary torsion theories analysing the set $\mathscr{K}(\sigma) \subseteq \operatorname{Spec}(A)$.

Proposition. 2.5.

Let σ be a hereditary torsion theory in **Mod**–*A, the following statements are equivalent:*

(a) σ is of finite type.

(b) σ *is half–centered and* $\mathscr{K}(\sigma) \subseteq \operatorname{Spec}(A)$ *is quasi–compact.*

References for unexplained terms should be [25] and [52].

In the following, we assume A will be a commutative ring, **Mod**–A be the category of A–modules and σ be a hereditary torsion theory on **Mod**–A. Modules are represented by Latin letters: M, N, N_1, $\ldots$, and ideals by Gothics letters: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{b}_1$, $\ldots$ Different hereditary torsion theories will be represented by Greek letters: σ, τ, σ_1, $\ldots$, and induced hereditary torsion theories by adorned Greek letters: σ', $\overline{\tau}$, $\ldots$

3 Totally noetherian spectrum property

3.1 Totally finitely generated modules

For any σ–torsion finitely generated A–module M, if $M = m_1A + \cdots + m_tA$, since $(0 : m_i) \in \mathscr{L}(\sigma)$, for any $i = 1, \ldots, t$, then $\mathfrak{h} := \cap_{i=1}^{t}(0 : m_i) \in \mathscr{L}(\sigma)$, and it satisfies $M\mathfrak{h} = 0$. In general, this result does not hold for σ–torsion non–finitely generated A–modules. Therefore, we shall define an A–module M to be **totally σ–torsion** whenever there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $M\mathfrak{h} = 0$. The notion of totally torsion appears, for instance, in [36, page 462].

For any ideal $\mathfrak{a} \subseteq A$ there are two different notions of finitely generated ideals relative to σ:

- $\mathfrak{a} \subseteq A$ is **σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq A$ such that $\mathrm{Cl}_\sigma^A(\mathfrak{a}) = \mathrm{Cl}_\sigma^A(\mathfrak{a}')$. In the case in which $\mathfrak{a}' \subseteq \mathfrak{a}$, we have $\mathfrak{a}'/\mathfrak{a}$ is σ–torsion.
- $\mathfrak{a} \subseteq A$ is **totally σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ such that $\mathfrak{a}/\mathfrak{a}'$ is totally σ–torsion.

In the same way, see [32], for any ring A there are two different notions of noetherian ring relative to σ:

- A is **σ–noetherian** if every ideal is σ–finitely generated.
- A is **totally σ–noetherian** whenever every ideal is totally σ–finitely generated.

Examples. 3.1.

(1) Every finitely generated ideal is totally σ–finitely generated and every totally σ–finitely generated ideal is σ–finitely generated.

(2) Let $S \subseteq A$ be a multiplicatively closed subset, an ideal $\mathfrak{a} \subseteq A$ is **S–finite** if, and only if, it is totally σ_S–finitely generated. The ring A is **S–noetherian**, see [5], if, and only if, A is totally σ_S–noetherian.

The notions of σ–noetherian and totally σ–noetherian ring can be extended to A–modules in an easy way.

Totally σ–torsion modules are the trivial examples of totally σ–noetherian modules. Also every noetherian module is totally σ–noetherian for every hereditary torsion theory σ.

Observe that these two notions of torsion, and the derived notions from them, are completely different in its behaviour and categorical properties. For instance, due to the definition, for any A–module M there exists a maximum submodule belonging to $\mathscr{T}_\sigma$, the submodule: σM, and it satisfies $M/\sigma M \in \mathscr{F}_\sigma$. On the contrary, in the totally σ–torsion case we can not assure the existence of a maximal totally σ–torsion submodule. The existence of a maximum σ–torsion submodule allows us to build new concepts relative to σ as lattices, closure operators and localization; concepts that we have not in the totally σ–torsion case; for instance, the ring A is σ–noetherian if, and only if, the lattice $C(A,\sigma) = \{\mathfrak{a} \mid A/\mathfrak{a} \in \mathscr{F}_\sigma\}$ is noetherian. Nevertheless, the totally σ–torsion case allows us study arithmetic properties of rings and modules which are hidden with the use of σ–torsion, and these properties are those which we are interested in.

As we point out before, the σ–torsion allows, for any A–module M, to define a lattice

$$C(M,\sigma) = \{N \subseteq M \mid M/N \in \mathscr{F}_\sigma\},$$

and in $\mathscr{L}(M)$, the lattice of all submodules of M, a closure operator

$$\mathrm{Cl}^M_\sigma(-) : \mathscr{L}(M) \longrightarrow C(M,\sigma) \subseteq \mathscr{L}(M).$$

The lattice operations in $C(M,\sigma)$, for any $N_1, N_2 \in C(M,\sigma)$, are defined by

$$\begin{aligned} N_1 \wedge N_2 &= N_1 \cap N_2, \\ N_1 \vee N_2 &= \mathrm{Cl}^M_\sigma(N_1 + N_2). \end{aligned}$$

Using the lattice $C(M,\sigma)$, and, in a parallel way to noetherian modules, for an A–module M we may define:

- M is **σ–artinian** whenever $C(M,\sigma)$ is an artinian lattice; that is, it satisfies the decreasing chain condition.
- M is **totally σ–artinian** if every decreasing chain $\{N_i \mid i \in I\}$ is **totally σ–stable**; that is, there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$, and an index $j \in I$ such that for any $i \in I$ we have $N_j\mathfrak{h} \subseteq N_i$; i.e., $N_j\mathfrak{h} \subseteq (\cap_i N_i)$.

The ring A is **σ–artinian** (resp. **totally σ–artinian**) if it is as module.

3.2 The spectrum of a hereditary torsion theory

A σ–closed submodule $N \subseteq M$ is

- σ**–minimal** whenever N is minimal in $C(M,\sigma) \setminus \{\sigma M\}$, and
- σ**–maximal** or σ**–critical** whenever N is maximal in $C(M,\sigma) \setminus \{M\}$.

One may extend these notions to non–necessarily σ–closed submodules: A submodule $N \subseteq M$ is

- σ**–minimal** whenever $\mathrm{Cl}_\sigma^M(N)$ is σ–minimal.
- σ**–maximal** whenever $\mathrm{Cl}_\sigma^M(N)$ is σ–maximal.

An A–module M is σ**–simple** if $C(M,\sigma) = \{\sigma M, M\}$, this means $\sigma M \neq M$; and σ**–cocritical** whenever it is σ–simple and σ–torsionfree.

The following results are easy consequences of the definitions.

Lemma. 3.2.

Let $N \subseteq M$ be a submodule, the following statements are equivalent:

(a) $N \subseteq M$ is σ–maximal.

(b) M/N is σ–simple.

In particular, $N \subseteq M$ is σ–critical if, and only if, M/N is σ–cocritical.

Lemma. 3.3.

Let $f : M_1 \longrightarrow M_2$ be a module map between σ–cocritical A–modules, then either $f = 0$ or f is a monomorphism.

Lemma. 3.4.

Every non–zero submodule of a σ–cocritical A–module is σ–cocritical.

Let M be an A–module, a submodule $N \subseteq M$ is **prime** whenever $N \neq M$, and for any $m \in M \setminus N$ and $a \in A$, if $ma \in N$, then $Ma \subseteq N$. The set of all prime submodules of M will be denoted as $\mathrm{Spec}(M)$, and called the **spectrum** of M.

Let $\mathscr{K}(M,\sigma)$ be the class of all prime σ–closed submodules of M, and $\mathscr{Z}(M,\sigma)$ be the class of all prime σ–dense submodules of M. In particular, $\{\mathscr{K}(M,\sigma), \mathscr{Z}(M,\sigma)\}$ is a partition of

Spec(M), the prime spectrum of M. Of particular interest is the case in which $M = A$ because we have that Spec(A) $\neq \emptyset$. In this case we write $\mathcal{K}(\sigma)$ and $\mathcal{Z}(\sigma)$ instead of $\mathcal{K}(A,\sigma)$ and $\mathcal{Z}(A,\sigma)$, respectively; and denote by $\mathcal{C}(\sigma)$ the set of all maximal elements in $\mathcal{K}(\sigma)$. By similarity, we denote by $\mathcal{C}(M,\sigma)$ the set of all maximal elements in $C(M,\sigma)$; it satisfies $\mathcal{C}(M,\sigma) \subseteq \mathcal{K}(M,\sigma)$. It is worth noting the difference between $C(M,\sigma)$ and $\mathcal{C}(M,\sigma)$, see next proposition.

Proposition. 3.5.

Let M be an A–module, the following statements hold:

(1) If $N \subseteq M$ is a σ–critical submodule, then $N \subseteq M$ is a prime submodule.

(2) If $N \subseteq M$ is a prime submodule, then either $\mathrm{Cl}^M_\sigma(N) = M$, i.e., $N \subseteq_\sigma M$, or $\mathrm{Cl}^M_\sigma(N) = N$, i.e., $N \subseteq M$ is σ–closed.

(3) A submodule $N \subseteq M$ is σ–critical if, and only if, $N \in \mathcal{C}(M,\sigma)$.

(4) A prime submodule $N \subseteq M$ is σ–critical if, and only if, $N \subseteq M$ is irreducible and $(N : M) \in \mathcal{C}(\sigma)$ whenever σ is half—centered.

Proof. (1). Let $ma \in N$. If $m \notin N$ then $N \subsetneq N + mA \subseteq M$, and $N + mA \subseteq_\sigma M$, and the only homomorphism from $M/(N + mA)$ to M/N is the zero one. Since $h : M/(N + mA) \longrightarrow M/N$, defined $f(x + (N + mA)) = xa$ is a module map, it is zero, hence $Ma \subseteq N$.

(2). Let us assume $\mathrm{Cl}^M_\sigma(N) \neq N$, and let $m \in \mathrm{Cl}^M_\sigma(N) \setminus N$. There exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $m\mathfrak{h} \subseteq N$; hence $M\mathfrak{h} \subseteq N$, and $\mathrm{Cl}^M_\sigma(N) = M$.

(3). It is already clear that N is σ-critical if and only if the lattice of closed submodules of M/N is $\{0, M/N\}$, which is equivalent to N belonging to $\mathcal{C}(M,\sigma)$.

(4). The necessary condition is clear. If a prime irreducible submodule $N \subseteq M$ that satisfies $(N : M) \in \mathcal{C}(\sigma)$, and we call $\mathfrak{p} = (N : M)$, then $N \subseteq M$ is $\sigma_{A\setminus\mathfrak{p}}$–critical. Otherwise, for every $\mathfrak{q} \in \mathcal{C}(\sigma)$ such that $\mathfrak{q} \neq \mathfrak{p}$ we have that M/N is $\sigma_{A\setminus\mathfrak{q}}$–dense. In consequence, for every $N \subsetneq H \subseteq M$ we have: M/H is $\sigma_{A\setminus\mathfrak{p}}$–torsion because $N \subseteq M$ is $\sigma_{A\setminus\mathfrak{p}}$–critical, and $\sigma_{A\setminus\mathfrak{q}}$-torsion, because M/N is. Therefore, since $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{q}} \mid \mathfrak{q} \in \mathcal{C}(\sigma)\}$, then $N \subseteq M$ is a σ–critical submodule. □

In this case, for every prime submodule $N \subseteq M$, we also have $\chi(M/N) = \chi(A/(N : M))$. Where, for any A-module X we denote by $\chi(X)$ is the largest hereditary torsion theory such that the A–module X is torsionfree.

Lemma. 3.6.

Let $\sigma \leq \tau$ be hereditary torsion theories. If M is a σ–cocritical A–module, and τ–torsionfree A–module then M is τ–cocritical.

Lemma. 3.7.

If $\sigma \leq \tau$ are hereditary torsion theories, M is a σ–simple A–module, then M is τ–simple.

3.3 The prime radical

If M is an A–module, if $N \subseteq M$ is a submodule, the **prime σ–radical** of N in M is

$$\mathrm{rad}_\sigma(N) = \cap\{H \mid N \subseteq H \subseteq M, H \in \mathscr{K}(M,\sigma)\}.$$

Lemma. 3.8.

Let σ be a hereditary torsion theory, M an A–module, $N, N_1, N_2, N_3 \subseteq M$ submodules, and $\mathfrak{h} \in \mathscr{L}(\sigma)$, the following statements hold.

(1) If $N_1 \subseteq_\sigma N_2$, then $\mathrm{Cl}^M_\sigma(N_1) = \mathrm{Cl}^M_\sigma(N_2)$.

(2) If $N_1 \subseteq_\sigma N_2 \subseteq_\sigma N_3$, then $N_1 \subseteq_\sigma N_3$.

(3) $\mathrm{Cl}^M_\sigma(N\mathfrak{h}) = \mathrm{Cl}^M_\sigma(N)$.

Proof. (1). If $N_1 \subseteq_\sigma N_2$, then $\mathrm{Cl}^M_\sigma(N_1) \subseteq \mathrm{Cl}^M_\sigma(N_2)$. Moreover, N_2/N_1 is σ–torsion, hence $N_2/N_1 \subseteq \sigma(M/N_1) = \mathrm{Cl}^M_\sigma(N_1)/N_1$, and $N_2 \subseteq \mathrm{Cl}^M_\sigma(N_1)$, hence and we have $\mathrm{Cl}^M_\sigma(N_1) = \mathrm{Cl}^M_\sigma(N_2)$.

(2) and (3) are straightforward. □

Proposition. 3.9.

Let σ be a finite type hereditary torsion theory and M be a σ–finitely generated A–module. For any submodule $N \subseteq M$ which is not σ–dense, there exists a σ–critical submodule $H \subseteq M$ such that $N \subseteq H$.

Proof. Let $\Gamma = \{H \mid H \subseteq M$ such that $N \subseteq H$ and $H \subseteq M$ is not σ–dense$\}$. If $\{H_i\}_i$ is a chain in Γ, then $H = \cup_i H_i$ belongs to Γ. Indeed, if $H \subseteq_\sigma M$, since M is σ–finitely generated, there exists $F \subseteq_\sigma M$, finitely generated. Since $H, F \subseteq M$ are σ–dense, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$, finitely generated, such that $F\mathfrak{h} \subseteq H$, then there exists an index i such that $F\mathfrak{h} \subseteq H_i$, which is a contradiction. □

A σ–closed submodule $N \subseteq M$ is called **σ–radical** whenever $N = \operatorname{rad}_\sigma(N)$, and **finite σ–radical** if there exists a finitely generated submodule $F \subseteq N$ such that $\operatorname{rad}_\sigma(N) = \operatorname{rad}_\sigma(F)$. In this sense, for every σ–dense submodule $N \subseteq M$ we have $\operatorname{rad}_\sigma(N) = M$, and every σ–dense ideal $\mathfrak{a} \subseteq A$ is finite σ–radical.

The σ–radical and the closure operator, defined by σ, are compatible. This and other results on the σ–radical are collected in the following, whose proofs are omitted.

Lemma. 3.10.

(1) For every submodule $N \subseteq M$ we have $N \subseteq \operatorname{rad}_\sigma(N) = \operatorname{rad}_\sigma(\operatorname{Cl}_\sigma^M(N))$.

(2) For every submodule $N \subseteq M$ we have $\operatorname{rad}_\sigma \operatorname{rad}_\sigma(N) = \operatorname{rad}_\sigma(N)$.

(3) For every submodules $N_1, N_2 \subseteq M$ such that $N_1 \subseteq N_2$ we have: $\operatorname{rad}_\sigma(N_1) \subseteq \operatorname{rad}_\sigma(N_2)$.

(4) For every submodules $N_1, N_2 \subseteq M$ such that $N_1 \subseteq_\sigma N_2$ we have: $\operatorname{rad}_\sigma(N_1) = \operatorname{rad}_\sigma(N_2)$.

Corollary. 3.11.

For every submodule $N \subseteq M$ we have $\operatorname{rad}_\sigma(N) = M$ if, and only if, either $N \subseteq_\sigma M$ or $\{H \in \mathscr{K}(M,\sigma) \mid N \subseteq H\}$ is empty.

Lemma. 3.12.

For every submodule $N \subseteq M$ we have $\operatorname{rad}(N) \subseteq \operatorname{rad}_\sigma(N) = \operatorname{Cl}_\sigma^M(\operatorname{rad}(N))$.

Lemma. 3.13.

If $\sigma \leq \tau$ are hereditary torsion theories, for every submodule $N \subseteq M$ we have $\operatorname{rad}_\sigma(N) \subseteq \operatorname{rad}_\tau(N)$. In particular, we have: $\operatorname{rad}_\sigma(\operatorname{rad}_\tau(N)) = \operatorname{rad}_\tau(N)$.

Proof. Since $\operatorname{rad}_\sigma(N) \subseteq \operatorname{rad}_\tau(N)$, taking $\operatorname{rad}_\tau(N)$ instead of N, we have

$$\operatorname{rad}_\sigma \operatorname{rad}_\tau(N) \subseteq \operatorname{rad}_\tau \operatorname{rad}_\tau(N) = \operatorname{rad}_\tau(N) \subseteq \operatorname{rad}_\sigma \operatorname{rad}_\tau(N).$$

Therefore, $\operatorname{rad}_\sigma \operatorname{rad}_\tau(N) = \operatorname{rad}_\tau(N)$, for any submodule $N \subseteq M$. □

Lemma. 3.14.

Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ be ideals, we have: $\operatorname{rad}_\sigma(\mathfrak{a}\mathfrak{b}) = \operatorname{rad}_\sigma(\mathfrak{a} \cap \mathfrak{b}) = \operatorname{rad}_\sigma(\mathfrak{a}) \cap \operatorname{rad}_\sigma(\mathfrak{b})$.

In consequence, $\operatorname{rad}_\sigma(\mathfrak{a}^n) = \operatorname{rad}_\sigma(\mathfrak{a})$.

3.4 Noetherian spaces

Let us recall, from [9], the definition and elementary properties of noetherian topological spaces.

A topological space X is **noetherian** whenever every chain of open subsets is stationary (ACC on open subsets).

Proposition. 3.15. ([9])

Let X be a topological space. The following statements are equivalent:

(a) X is noetherian

(b) Every subspace is noetherian.

(c) X satisfies the maximal condition (every non–empty family of open subsets has maximal elements).

(d) Every open subset is quasi–compact.

(e) Every subset is quasi–compact.

(f) X satisfies the descending chain condition on closed subsets.

(g) X satisfies the minimal condition on closed subsets.

A closed subset $Z \subseteq X$ is **irreducible** whenever $Z = Z_1 \cup Z_2$, for closed subsets $Z_1, Z_2 \subseteq Z$, we have either $Z = Z_1$ or $Z = Z_2$.

Proposition. 3.16. ([9])

Let X be a noetherian topological space, then X is a finite union of irreducible closed subsets.

Corollary. 3.17. ([9])

Every noetherian space is a finite union of irreducible closed subsets.

3.5 The spectrum of a hereditary torsion theory

The subset $\mathscr{K}(\sigma) \subseteq \mathrm{Spec}(A)$, it is **closed under generalizations**, i.e., for every $\mathfrak{q}, \mathfrak{p} \subseteq \mathrm{Spec}(A)$, if $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathscr{K}(\sigma)$, then $\mathfrak{q} \in \mathscr{K}(\sigma)$.

A hereditary torsion theory σ is Spec–**noetherian** if $\mathscr{K}(\sigma)$ is a noetherian space. Also we can say that A is **noetherian σ–radical**.

An open subset $Y \subseteq \mathscr{K}(\sigma)$ is the trace of an open subset in Spec(A), hence there exists $\mathfrak{a} \subseteq A$ such that $Y' = X(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \nsubseteq \mathfrak{p}\}$, and $Y = Y' \cap \mathscr{K}(\sigma)$. Also we have $\mathrm{rad}(\mathfrak{a}) = \cap\{\mathfrak{p} \mid \mathfrak{p} \in V(\mathfrak{a})\}$, and $Y' = X(\mathrm{rad}(\mathfrak{a}))$. It is well known that there is a correspondence between open subsets in Spec(A) and radical ideals.

$$Y' \quad \leftrightarrow \quad \cap\{\mathfrak{p} \mid \mathfrak{p} \notin Y'\}.$$

If $Y \subseteq \mathscr{K}(\sigma)$ is an open subset and consider $\mathfrak{a}_Y = \cap\{\mathfrak{p} \in \mathscr{K}(\sigma) \mid \mathfrak{p} \notin Y\}$, then $X(\mathfrak{a}_Y) \cap \mathscr{K}(\sigma) = Y$, and $\mathfrak{a}_Y$ is named a **σ–radical ideal**. The map $Y \mapsto \mathfrak{a}_Y$ establishes a correspondence between open subsets of $\mathscr{K}(\sigma)$ and σ–radical ideals.

Thus we have:

Proposition. 3.18.

*Let σ be a hereditary torsion theory in **Mod**–A, the following statements are equivalent:*

(a) $\mathscr{K}(\sigma)$ is a topological noetherian space.

(b) A satisfies the ACC on σ–radical ideals.

Proposition. 3.19.

*Let σ be a half–centered hereditary torsion theory in **Mod**–A, the following statements are equivalent:*

(a) $\mathscr{K}(\sigma)$ is a topological noetherian space.

(b) Every half–centered hereditary torsion theory $\tau \geq \sigma$ is of finite type.

This is a direct consequence of the well known fact σ is of finite type if, and only if, σ is half–centered and $\mathscr{K}(\sigma)$ is quasi–compact, and Proposition (3.15.).

Corollary. 3.20.

If σ is half–centered and $\mathscr{K}(\sigma)$ is noetherian, then σ is of finite type.

The converse does not necessarily hold.

Remark. 3.21.

Let σ be a hereditary torsion theory, then $\mathscr{K}(\sigma)$ is generically closed, and defines a hereditary torsion theory $\sigma' = \cap\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$ satisfying $\mathscr{K}(\sigma) = \mathscr{K}(\sigma')$, $\mathrm{rad}_\sigma(\mathfrak{a}) = \mathrm{rad}_{\sigma'}(\mathfrak{a})$ for every ideal $\mathfrak{a} \subseteq A$, etc. In addition, σ' is of finite type if, and only if, $\mathscr{K}(\sigma) \subseteq$ Spec(A) is quasi–compact.

Hence σ' is the smallest finite type hereditary torsion theory bigger than σ. In the same way, σ is Spec–noetherian if, and only if, σ' is.

For that reason, in the study of noetherian spaces, we may restrict ourselves to consider only finite type hereditary torsion theories.

Lemma. 3.22.

If $\sigma_1 \leq \sigma_2$ are hereditary torsion theories and σ_1 is Spec*–noetherian, then σ_2 is* Spec*–noetherian.*

Remember, an ideal $\mathfrak{a} \subseteq A$ is **finite σ–radical** if there are finitely many elements, $a_1, \ldots, a_t \in \mathrm{Cl}_\sigma^A(\mathfrak{a})$, such that $\mathrm{rad}_\sigma(\mathfrak{a}) = \mathrm{rad}_\sigma(a_1, \ldots, a_t)$.

Proposition. 3.23.

Let σ be a hereditary torsion theory, the following statements are equivalent:

(a) A has noetherian σ–radical (σ is Spec*–noetherian).*

(b) A satisfies the ACC for σ–radical ideals.

(c) Every ideal is finite σ–radical.

(d) Every prime ideal in $\mathcal{K}(\sigma)$ is finite σ–radical.

(e) $\mathcal{K}(\sigma)$ satisfies the ACC for prime ideals and $\min(V(\mathfrak{a}) \cap \mathcal{K}(\sigma))$ is finite for every ideal $\mathfrak{a} \subseteq A$.

(f) $\mathcal{K}(\sigma)$ satisfies the ACC for prime ideals and $\min(V(\mathfrak{a}) \cap \mathcal{K}(\sigma))$ is finite for every finitely generated ideal $\mathfrak{a} \subseteq A$.

Proof. (a) $\Leftrightarrow$ (b). It is Proposition (3.18.)

(c) $\Rightarrow$ (a). Let $\{\mathfrak{a}_i \mid i \in I\}$ be a chain of σ–radical ideals. If $\mathfrak{a} = \cup_i \mathfrak{a}_i$, then $\mathfrak{a}$ is radical ideal, hence σ–radical. By the hypothesis, $\mathfrak{a} = \mathrm{rad}_\sigma(\mathfrak{b})$ for some $\mathfrak{b} \subseteq \mathfrak{a}$, finitely generated. Therefore, there exists an index $i \in I$ such that $\mathfrak{b} \subseteq \mathfrak{a}_i$, and the chain stabilizes.

(d) $\Rightarrow$ (c). We define $\Gamma = \{\mathfrak{a} \subseteq A \mid \mathfrak{a}$ is σ–radical and is not finite σ–radical$\}$. If $\Gamma \neq \varnothing$, since it is inductive, by Zorn's lemma, there exists $\mathfrak{a} \in \Gamma$, maximal. We claim $\mathfrak{a}$ is prime. In the contrary, there are $\mathfrak{a}_1, \mathfrak{a}_2 \supsetneqq \mathfrak{a}$ such that $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$. Since

$$\mathrm{rad}_\sigma(\mathfrak{a}) \subseteq \mathrm{rad}_\sigma(\mathfrak{a}_1) \cap \mathrm{rad}_\sigma(\mathfrak{a}_2) = \mathrm{rad}_\sigma(\mathfrak{a}_1\mathfrak{a}_2) \subseteq \mathrm{rad}_\sigma(\mathfrak{a}) = \mathfrak{a},$$

then $\mathrm{rad}_\sigma(\mathfrak{a}_1), \mathrm{rad}_\sigma(\mathfrak{a}_2) \neq A$. If we take $\mathfrak{a}_1 = \mathrm{rad}_\sigma(\mathfrak{a}_1)$, then $\mathfrak{a}_1, \mathfrak{a}_2$ are finite σ–radical. Hence $\mathfrak{a} = \mathrm{rad}_\sigma(\mathfrak{a}) = \mathrm{rad}_\sigma(\mathfrak{a}_1\mathfrak{a}_2)$ is finite σ–radical, which is a contradiction. In consequence, $\Gamma = \varnothing$, and every σ–radical ideal is finite σ–radical.

(f) ⇒ (d). Let $0 \neq \mathfrak{p} \in \mathscr{K}(\sigma)$. If $\mathfrak{p}$ is not finite σ–radical, let $0 \neq a \in \mathfrak{p}$, and $\mathfrak{a}_1 = aA$, then $\min(V(\mathfrak{a}) \cap \mathscr{K}(\sigma))$ is finite. If every $\mathfrak{q} \in \min(V(\mathfrak{a}_1) \cap \mathscr{K}(\sigma))$ satisfies $\mathfrak{p} \subseteq \mathfrak{q}$, then

$$\mathfrak{p} \subseteq \cap \mathfrak{q} = \mathrm{rad}_\sigma(aA) \subseteq \mathrm{rad}_\sigma(\mathfrak{p}) = \mathfrak{p},$$

and $\mathfrak{p}$ is finite σ–radical, which is a contradiction. Hence there exists $\mathfrak{q} \in \min(V(aA) \cap \mathscr{K}(\sigma))$ such that $\mathfrak{p} \nsubseteq \mathfrak{q}$. We write $\min(V(\mathfrak{a}_1) \cap \mathscr{K}(\sigma)) = \{\mathfrak{p}_{1,1}, \ldots, \mathfrak{p}_{1,r_1}\} \cup \{\mathfrak{q}_{1,1}, \ldots, \mathfrak{q}_{1,s_1}\}$, being $\mathfrak{p} \subseteq \mathfrak{p}_{1,*}$ and $\mathfrak{p} \nsubseteq \mathfrak{q}_{1,*}$.

For any ideal $\mathfrak{q}_{1,j}$ we take $b_{1,j} \in \mathfrak{p} \setminus \mathfrak{q}_{1,j}$. Thus, we have a finite set of elements $\{b_{1,1}, \ldots, b_{1,s_1}\}$.

We define $\mathfrak{a}_2 = \mathfrak{a}_1 + \langle b_{1,1} \ldots, b_{1,s_1} \rangle$. Since $\min(V(\mathfrak{a}_2) \cap \mathscr{K}(\sigma))$ is finite, and not every $\mathfrak{q}$ in $\min(V(\mathfrak{a}_2) \cap \mathscr{K}(\sigma))$ satisfies the condition $\mathfrak{p} \subseteq \mathfrak{q}$. We may write

$$\min(V(\mathfrak{a}_2) \cap \mathscr{K}(\sigma)) = \{\mathfrak{p}_{2,1}, \ldots, \mathfrak{p}_{2,r_2}\} \cup \{\mathfrak{q}_{2,1}, \ldots, \mathfrak{q}_{2,s_2}\},$$

being $\mathfrak{p} \subseteq \mathfrak{p}_{2,*}$ and $\mathfrak{p} \nsubseteq \mathfrak{q}_{2,*}$.

For every such ideal $\mathfrak{q}_{2,j}$, we take $b_{2,j} \in \mathfrak{p} \setminus \mathfrak{q}_{2,j}$. Hence, there exists a finite set $\{b_{2,1}, \ldots, b_{2,s_2}\}$. Since

$$\left(\cap_i \mathfrak{p}_{1,i}\right) \cap \left(\cap_j \mathfrak{q}_{1,j}\right) = \mathrm{rad}_\sigma(\mathfrak{a}_1) \subseteq \mathrm{rad}_\sigma(\mathfrak{a}_2) = \left(\cap_i \mathfrak{p}_{2,i}\right) \cap \left(\cap_j \mathfrak{q}_{2,j}\right) \subseteq \mathfrak{q}_{2,j},$$

there exists $\mathfrak{q}_{1,h}$ such that $\mathfrak{q}_{1,h} \subseteq \mathfrak{q}_{2,j}$.

In this way we build finitely generated ideals $\mathfrak{a}_n \subseteq \mathfrak{p}$, for $n \geq 1$, and prime ideals $\mathfrak{q}_{n,*}$. For any $\mathfrak{q}_{1,j}$ there exists a chain starting at $\mathfrak{q}_{1,j}$. Since there are infinitely many ideals, there is a infinite chain starting at one of the $\mathfrak{q}_{1,j}$, which is a contradiction. See [49].

(e) ⇒ (f). It is immediate.

(a) ⇒ (e). First we have that every chain of prime ideals in $\mathscr{K}(\sigma)$ is a chain of σ–radical ideals, hence stationary. On the other hand, if $\min(V(\mathfrak{a}) \cap \mathscr{K}(\sigma))$ is infinite, we may assume $\mathfrak{a}$ is σ–radical, and consider the family $\Gamma = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is } \sigma\text{–radical and } \min(V(\mathfrak{a}) \cap \mathscr{K}(\sigma)) \text{ is infinite}\}$.

By (1), Γ has maximal elements. If $\mathfrak{a} \in \Gamma$ is maximal, then it is not prime, and there are ideals $\mathfrak{a}_1, \mathfrak{a}_2 \supsetneqq \mathfrak{a}$ such that $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$. Since

$$\operatorname{rad}_\sigma(\mathfrak{a}) \subseteq \operatorname{rad}_\sigma(\mathfrak{a}_1) \cap \operatorname{rad}_\sigma(\mathfrak{a}_2) = \operatorname{rad}_\sigma(\mathfrak{a}_1\mathfrak{a}_2) \subseteq \operatorname{rad}_\sigma(\mathfrak{a}) = \mathfrak{a},$$

and $\operatorname{rad}_\sigma(\mathfrak{a}_i)$ has finitely many minimal ideal in $V(\mathfrak{a}_i) \cap \mathscr{K}(\sigma)$, for $i = 1, 2$. Then $V(\mathfrak{a}) \cap \mathscr{K}(\sigma)$ has finitely many minimal ideals, which is a contradiction. □

If $\sigma = \sigma_\Sigma$, the hereditary torsion theory defined by a multiplicative subset $\Sigma \subseteq A$, then we have [19, Theorem 3.6].

Corollary. 3.24.

*Let σ be a hereditary torsion theory in **Mod**–A, $\mathfrak{b} \subseteq A$ an ideal and $\overline{\sigma}$ the hereditary torsion theory induced by σ in **Mod**–$(A/\mathfrak{b})$. If A is noetherian σ–radical, then $A/\mathfrak{b}$ is noetherian $\overline{\sigma}$–radical.*

Proof. First we observe that for any ideal $\mathfrak{b} \subseteq \mathfrak{a} \subseteq A$ we have: $\operatorname{Cl}^A_\sigma(\mathfrak{a})/\mathfrak{b} = \operatorname{Cl}^{A/\mathfrak{b}}_{\overline{\sigma}}(\mathfrak{a}/\mathfrak{b})$, and second that $\mathscr{K}(\overline{\sigma}) = \{\mathfrak{p}/\mathfrak{b} \mid \ \mathfrak{b} \subseteq \mathfrak{p}, \mathfrak{p} \in \mathscr{K}(\sigma)\}$. Therefore, for any ideal $\mathfrak{a}/\mathfrak{b} \subseteq A/\mathfrak{b}$, we may assume $\mathfrak{a} = \operatorname{Cl}^A_\sigma(\mathfrak{a})$ there exists $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, such that $\operatorname{rad}_\sigma(\mathfrak{a}) = \operatorname{rad}_\sigma(\mathfrak{a}')$, hence $\operatorname{rad}_{\overline{\sigma}}(\mathfrak{a}/\mathfrak{b}) = \operatorname{rad}_{\overline{\sigma}}((\mathfrak{a}' + \mathfrak{b})/\mathfrak{b})$, and $\mathfrak{a}/\mathfrak{b}$ has a finite $\overline{\sigma}$–radical. □

The following result holds precisely because σ can be taken of finite type whenever A is noetherian σ–radical.

Theorem. 3.25.

*Let A be a ring, σ be a hereditary torsion theory in **Mod**–A, and $\overline{\sigma}$ be the hereditary torsion theory induced by σ in **Mod**–$A[X]$, the following statements are equivalent:*

(a) A is noetherian σ–radical.

(b) $A[X]$ is noetherian $\overline{\sigma}$–radical.

Proof. (a) ⇒ (b). Let us assume there are σ–closed ideals which are not finite σ–radical, hence in the family of them there are maximal elements, and each maximal element is a prime ideal. For any prime ideal $\mathfrak{q} \subseteq A[X]$ which is maximal in the set of σ–closed and non finite σ–radical ideals, the contraction $\mathfrak{q} \cap A \subseteq A$ is prime and finite radical. Then we consider $A/(\mathfrak{q} \cap A)$

and $A[X]/\mathfrak{q} = (A/(\mathfrak{q}\cap A))[X]$. Therefore, we may assume A is an integral domain and $\mathfrak{q} \subseteq A[X]$ satisfies $\mathfrak{q}\cap A = 0$.

Let K be the field of fractions of A. In $K[X]$ the ideal $\mathfrak{q}K[X]$ is generated by a polynomial, say $F \in \mathfrak{q}$, of degree m, and let a be the leader coefficient of F. The ideal $\mathfrak{q}+(a)$ contains properly $\mathfrak{q}$, hence it has finite σ–radical, because of the maximality of $\mathfrak{q}$. Say $\mathrm{rad}(\mathfrak{q}+(a)) = \mathrm{rad}(F_1,\ldots,F_s)$, for some $F_1,\ldots,F_s \in \mathrm{Cl}^A_\sigma(\mathfrak{q}+(a))$. Since σ is of finite type, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$, finitely generated, such that $(F_1,\ldots,F_s)\mathfrak{h} \subseteq \mathfrak{q}+(a)$, and there are $\mathfrak{a}' \subseteq \mathfrak{q}$, $\mathfrak{a}'' \subseteq A[X]$, finitely generated, such that $(F_1,\ldots,F_s)\mathfrak{h} \subseteq \mathfrak{a}'+a\mathfrak{a}'' \subseteq \mathfrak{q}+(a)$. Therefore,

$$\mathfrak{q} \subseteq \mathrm{rad}_\sigma(\mathfrak{q}+(a)) = \mathrm{rad}_\sigma(F_1,\ldots,F_s) = \mathrm{rad}_\sigma((F_1,\ldots,F_s)\mathfrak{h}) \subseteq \mathrm{rad}_\sigma(\mathfrak{a}'+a\mathfrak{a}'') \subseteq \mathrm{rad}_\sigma(\mathfrak{q}+(a)).$$

Consider the localized A_a, of A at $\{a^n \mid n \in \mathbb{N}\}$. We claim, $\mathfrak{q}A_a = FA_a$, hence it is finite radical. Indeed, if $G \in \mathfrak{q}A_a \subseteq \mathfrak{q}K[X]$, there exists $H \in K[X]$ such that $G = FH$, hence $H \in A_a[X]$, and we have the equality.

We have $\mathfrak{a}+(F) \subseteq \mathfrak{q}$, hence $\mathrm{rad}_\sigma(\mathfrak{a}'+(F)) \subseteq \mathfrak{q}$. Otherwise, if $\mathfrak{q}' \supseteq \mathfrak{a}+(F)$, there are two possibilities:

(1). $a \in \mathfrak{q}'$, hence $\mathfrak{q} \subseteq \mathrm{rad}_\sigma(\mathfrak{q}+(a)) = \mathrm{rad}_\sigma(\mathfrak{a}'+a\mathfrak{a}'') \subseteq \mathfrak{q}'$.

(2). $a \notin \mathfrak{q}'$, hence $\mathfrak{q}'A_a[X]$ is prime and $F \in \mathfrak{q}'A_a[X]$, hence $\mathfrak{q}A_a[X] = FA_a[X] \subseteq \mathfrak{q}A_a[X]$, and $\mathfrak{q} \subseteq \mathfrak{q}'$.

In both cases we obtain $\mathfrak{q} \subseteq \mathfrak{q}'$, and $\mathfrak{q} \subseteq \mathrm{rad}_\sigma(\mathfrak{a}'+(F))$.

(b) ⇒ (a). We shall use Corollary (3.24.) applied to the map $A[X] \longrightarrow A[X]/(X)$. Observe that we have three hereditary torsion theories: σ in **Mod**–A, $\overline{\sigma}$ in **Mod**–$A[X]$, and $\overline{\overline{\sigma}}$ in **Mod**–$A[X]/(X) =$ **Mod**–A. We only need to show that $\sigma = \overline{\overline{\sigma}}$. Indeed, we have: $\mathscr{L}(\overline{\overline{\sigma}}) = \{\mathfrak{h}'/(X) \mid \mathfrak{h}' \in \mathscr{L}(\overline{\sigma})\} = \{\mathfrak{h}'/(X) \mid \mathfrak{h}'\cap A \in \mathscr{L}(\sigma)\} = \mathscr{L}(\sigma)$. □

If $\sigma = 0$, i.e., if $\mathscr{L}(\sigma) = \{A\}$, then we obtain [48, Theorem 2.5].

3.6 Totally noetherian radical rings

There is another approach to associate a like noetherian space to a hereditary torsion theory. This approach follows the work of A. Hamed in [27]. Our aim in this section is to show that hereditary torsion theories provide a useful tool which allows to improve some of the results in [27].

Let $\mathfrak{a} \subseteq A$ be an ideal and σ a hereditary torsion theory in **Mod**–A, the **σ–radical** of $\mathfrak{a}$ is $\text{rad}_\sigma(\mathfrak{a}) = \cap\{\mathfrak{p} \in \mathscr{K}(\sigma) \mid \ \mathfrak{a} \subseteq \mathfrak{p}\}$. An ideal $\mathfrak{a} \subseteq A$ is **finite σ–radical** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq \text{Cl}^A_\sigma(\mathfrak{a})$ such that $\text{rad}_\sigma(\mathfrak{a}') = \text{rad}_\sigma(\mathfrak{a})$. We may assume $\mathfrak{a}' \subseteq \mathfrak{a}$ whenever σ is of finite type.

We say $\mathfrak{a} \subseteq A$ is **totally finite σ–radical** if there exists a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \text{rad}(\mathfrak{a}') \subseteq \text{rad}(\mathfrak{a})$.

Lemma. 3.26.

Let $\mathfrak{a} \subseteq A$ be a totally finite σ–radical, then it is finite σ–radical.

Proof. If $\mathfrak{a}$ is totally finite σ–radical, there exist $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \text{rad}(\mathfrak{a}') \subseteq \text{rad}(\mathfrak{a})$, then $\text{rad}_\sigma(\mathfrak{a}\mathfrak{h}) \subseteq \text{rad}_\sigma(\text{rad}(\mathfrak{a}')) \subseteq \text{rad}_\sigma(\text{rad}(\mathfrak{a}))$, and we have $\text{rad}_\sigma(\mathfrak{a}) = \text{rad}_\sigma(\mathfrak{a}\mathfrak{h}) \subseteq \text{rad}_\sigma(\mathfrak{a}') \subseteq \text{rad}_\sigma(\mathfrak{a})$, hence $\text{rad}_\sigma(\mathfrak{a}') = \text{rad}_\sigma(\mathfrak{a})$. □

A ring A is **totally noetherian σ–radical** whenever every ideal $\mathfrak{a} \subseteq A$ is totally finite σ–radical.

Example. 3.27.

Let $S \subseteq A$ be a multiplicatively closed subset of A, an ideal $\mathfrak{a} \subseteq A$ is **radically S–finite**, see [27], if, and only if, $\mathfrak{a}$ is totally finite σ_S–radical, and the ring A satisfies the **S–noetherian spectrum property** if, and only if, it is totally noetherian σ_S–radical.

In consequence, we have:

Proposition. 3.28.

Let σ be a hereditary torsion theory in **Mod**–*A. If A is totally noetherian σ–radical then it is noetherian σ–radical (σ is* Spec*–noetherian), i.e., $\mathscr{K}(\sigma) \subseteq \text{Spec}(A)$ is a noetherian space with the induced topology.*

Contrary to noetherian σ–radical rings, which can be characterized by the lattice $\mathscr{K}(\sigma)$, totally noetherian σ–radical is not directly a lattice property.

Lemma. 3.29.
*Let σ be a perfect hereditary torsion theory in **Mod**–A. If A is totally noetherian σ–radical, then A_σ, the localization ring of A with respect to σ, has noetherian spectrum.*

Proof. Let $\mathfrak{b} \subseteq A_\sigma$, there exists $\mathfrak{a} \subseteq A$ such that $\mathfrak{b} = \mathfrak{a}_\sigma = \mathfrak{a}A_\sigma$, and a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}') \subseteq \mathrm{rad}(\mathfrak{a})$, then

$$\mathfrak{b} = \mathfrak{a}_\sigma = (\mathfrak{a}\mathfrak{h})_\sigma \subseteq \mathrm{rad}(\mathfrak{a}')_\sigma = \mathrm{rad}_\sigma(\mathfrak{a}') = \mathrm{rad}(\mathfrak{a}'_\sigma) \subseteq \mathrm{rad}(\mathfrak{a})_\sigma = \mathrm{rad}_\sigma(\mathfrak{a}) = \mathrm{rad}(\mathfrak{a}_\sigma),$$

and A_σ has noetherian spectrum. □

The converse does not necessarily hold.

Example. 3.30.
Consider a field K and the polynomial ring $A = K[X_n \mid n \in \mathbb{N}]$, and $\sigma = \sigma_{A\setminus\{0\}}$ the usual hereditary torsion theory in an integral domain. If we take $\mathfrak{a} = (X_n^2 \mid n \in \mathbb{N})$, then $\mathrm{rad}(\mathfrak{a}) = (X_n \mid n \in \mathbb{N})$, and for any $\mathfrak{h} \in \mathscr{L}(\sigma)$, we have $\mathfrak{a}\mathfrak{h}$ is not contained in the radical of a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$.

Lemma. 3.31.
Let $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$ be totally finite σ–radical ideals, then $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_1 \cap \mathfrak{a}_2$ are totally σ–radical ideals.

Proof. We have $\mathrm{rad}(\mathfrak{a}_1\mathfrak{a}_2) = \mathrm{rad}(\mathfrak{a}_1 \cap \mathfrak{a}_2) = \mathrm{rad}(\mathfrak{a}_1) \cap \mathrm{rad}(\mathfrak{a}_2)$. By the hypothesis there exists $\mathfrak{a}'_i \subseteq \mathfrak{a}_1$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_i\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}'_i)$, hence

$$\mathfrak{a}_1\mathfrak{a}_2\mathfrak{h}^2 \subseteq \mathrm{rad}(\mathfrak{a}'_1)\,\mathrm{rad}(\mathfrak{a}'_2) \subseteq \mathrm{rad}(\mathfrak{a}'_1\mathfrak{a}'_2) \subseteq \mathrm{rad}(\mathfrak{a}_1\mathfrak{a}_2).$$

Otherwise,

$$(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}'_1) \cap \mathrm{rad}(\mathfrak{a}'_2) = \mathrm{rad}(\mathfrak{a}'_1 \cap \mathfrak{a}'_2) \subseteq \mathrm{rad}(\mathfrak{a}_1 \cap \mathfrak{a}_2).$$

□

An increasing chain $\{\mathfrak{a}_n \mid n \in \mathbb{N}\}$ is **totally σ–stable** whenever there exist $\mathfrak{b} \in \mathscr{L}(\sigma)$ and an index m such that $\mathfrak{a}_k\mathfrak{b} \subseteq \mathfrak{a}_m$ for every $k \geq m$.

Let $\Sigma = \{s_1, \ldots, s_t\}$ be a finite multiplicative subset, and $s = s_1 \cdots s_t$. We have $\mathscr{L}(\sigma_\Sigma) = \{\mathfrak{a} \subseteq A \mid sA \subseteq \mathfrak{a}\}$. In addition, sA is idempotent, hence, it is generated by an idempotent element $e \in sA$, because it is finitely generated, and $\sigma = \sigma_{\{1,e\}}$. The localization of A at σ is $A_\sigma = eA$. This is the case of jansian finite type hereditary torsion theories. In particular we have:

Lemma. 3.32.

Let σ be a jansian finite type hereditary torsion theory generated by a finitely generated idempotent ideal $\mathfrak{h}$, then

(1) $\mathfrak{a} \subseteq A$ is totally finite σ–radical if, and only if, there exists $\mathfrak{a}' \subseteq \mathfrak{a}$ such that $\mathfrak{a}\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{a}')$.

(2) An increasing chain $\{\mathfrak{a}_n \mid n \in \mathbb{N}\}$ is totally σ–stable if, and only is, there exists an index m such that $\mathfrak{a}_k\mathfrak{h} \subseteq \mathfrak{a}_m$ for every $k \geq m$.

Theorem. 3.33.

*Let σ be a hereditary torsion theory in **Mod**–A. Consider the following statements:*

(1) A is a totally noetherian σ–radical.

(2) Every increasing chain of radical ideals is totally σ–stable.

***(A).** Then (1) $\Rightarrow$ (2).*

***(B).** Let us assume $\mathscr{L}(\sigma)$ satisfies the property:*

(†) There exists a strict decreasing chain with infinitely many elements in $\mathscr{L}(\sigma)$.

In this case, if every increasing chain of radical ideals is totally σ–stable, then A is a totally noetherian σ–radical ring.

***(C).** Since if a hereditary torsion theory σ not satisfying (†) is jansian, and for every jansian hereditary torsion theory, σ statements (1) and (2) are equivalent, then (1) and (2) are equivalent for every hereditary torsion theory σ.*

Proof. **(A).** (1) $\Rightarrow$ (2). Let $\{\mathfrak{a}_n \mid n \in \mathbb{N}\}$ be an increasing chain of radical ideals, then $\mathfrak{a} = \cup_n \mathfrak{a}_n$ is radical, and there exist $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{a}') \subseteq \operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$. There exists an index m such that $\mathfrak{a}' \subseteq \operatorname{rad}(\mathfrak{a}_m)$, hence $\mathfrak{a}\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{a}') \subseteq \operatorname{rad}(\mathfrak{a}_m) = \mathfrak{a}_m$. Hence, for every $k \geq m$ we have $\mathfrak{a}_k\mathfrak{h} \subseteq \mathfrak{a}_m$.

(B). Let us assume that every increasing chain of radical ideals is totally σ–stable and A is not totally noetherian σ–radical, hence there exists an ideal $\mathfrak{a}$ such that for every non–zero finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ and if we take every $\mathfrak{h} \in \mathscr{L}(\sigma)$ we have $\mathfrak{a}\mathfrak{h} \nsubseteq \operatorname{rad}(\mathfrak{a}')$.

If $\{\mathfrak{h}_n \mid n \in \mathbb{N}\}$ is a strict decreasing chain of ideals in $\mathscr{L}(\sigma)$, and $0 \neq a_0 \in \mathfrak{a}$, since $\mathfrak{a}A\mathfrak{h}_0 \nsubseteq \operatorname{rad}(a_0A)$, and $\mathfrak{a}A\mathfrak{h}_1 \nsubseteq \operatorname{rad}(a_0A)$, there exist $a_{1,0}, a_{1,1} \in \mathfrak{a}$ such that $a_{1,0}\mathfrak{h}_0, a_{1,1}\mathfrak{h}_1 \nsubseteq \operatorname{rad}(a_0A)$.

Define $\mathfrak{b}_0 = \operatorname{rad}(a_0A)$, and $\mathfrak{b}_1 = \operatorname{rad}(a_0, a_{1,0}, a_{1,1})$. Then we have: $\mathfrak{a}\mathfrak{h}_0, \mathfrak{a}\mathfrak{h}_1 \nsubseteq \mathfrak{b}_1$, and, by the hypothesis, $\mathfrak{a}\mathfrak{h}_2 \nsubseteq \operatorname{rad}(\mathfrak{b}_1)$. As before, there exists $a_{2,i} \in \mathfrak{a}$ such that $a_{2,i}\mathfrak{h}_i \nsubseteq \operatorname{rad}(\mathfrak{b}_2)$, for any $i = 0, 1, 2$. Define $\mathfrak{b}_2 = \operatorname{rad}(a_0, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}, a_{2,2})$. We can continue in this way to build a strict increasing chain of radical ideals: $\{\mathfrak{b}_n \mid n \in \mathbb{N}\}$, which is a contradiction.

(C). If σ does not satisfy condition (†), then $\mathscr{L}(\sigma)$ has a minimal element $\mathfrak{j}$ which is an idempotent ideal. Let $\mathscr{L}(\sigma) = \{\mathfrak{h} \mid \mathfrak{h} \supseteq \mathfrak{j}\}$. We claim (2) $\Rightarrow$ (1). Indeed, for any ideal $\mathfrak{a} \subseteq A$ we define

$$\Gamma = \{\operatorname{rad}(\mathfrak{a}') \mid \mathfrak{a}' \subseteq \mathfrak{a} \text{ is finitely generated}\}.$$

By the hypothesis, any increasing chain $\{\mathfrak{b}_n \mid n \in \mathbb{N}\}$, of elements in Γ, if σ–stable. i.e., there exists an index m such that $\mathfrak{b}_k\mathfrak{j} \subseteq \mathfrak{b}_m$, for every $k \geq m$. This implies that, Γ contains σ–maximal elements. An element $\mathfrak{b} \in \Gamma$ is σ–maximal whenever for any $\mathfrak{b}' \in \Gamma$ such that $\mathfrak{b} \subseteq \mathfrak{b}'$ we have $\mathfrak{b}'\mathfrak{j} \subseteq \mathfrak{b}$. Let $\mathfrak{b} = \operatorname{rad}(\mathfrak{a}') \in \Gamma$ be a σ–maximal element in Γ. For any $x \in \mathfrak{a} \setminus \mathfrak{b}$ we have $\mathfrak{b} = \operatorname{rad}(\mathfrak{a}') \subsetneqq \operatorname{rad}(\mathfrak{a}' + xA) \in \Gamma$, hence we have $\operatorname{rad}(\mathfrak{a}' + xA)\mathfrak{j} \subseteq \mathfrak{b}$, i.e., $x\mathfrak{j} \subseteq \mathfrak{b}$. In conclusion, $\mathfrak{a}\mathfrak{j} \subseteq \mathfrak{b} = \operatorname{rad}(\mathfrak{a}')$, and $\mathfrak{a}$ is totally finite σ–radical. □

Compare with Theorem 2.1 in [27].

We can show a Cohen's theorem like for totally noetherian σ–radical rings. First we prove a technical lemma.

Lemma. 3.34.
Let A be a ring and $\Gamma = \{\mathfrak{b} \subseteq A \mid \mathfrak{b}$ is radical and is non totally finite σ–radical$\}$. If $\Gamma \neq \varnothing$, then it has maximal elements, and every maximal element of Γ is a prime ideal.

Proof. By Zorn's lemma, to show that Γ has maximal elements we only need to show that every ascending chain in Γ is bounded. Let $\{\mathfrak{b}_n \mid n \in \mathbb{N}\}$ be an increasing chain in Γ, and define $\mathfrak{b} = \cup_n \mathfrak{b}_n$, then $\mathfrak{b}$ is radical and if $\mathfrak{b}$ is totally finite σ–radical there exist $\mathfrak{b}' \subseteq \mathfrak{b}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{b}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{b}')$. There exists an index m such that $\mathfrak{b}' \subseteq \mathfrak{b}_m$, hence $\mathfrak{b}_m\mathfrak{h} \subseteq \mathfrak{b}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{b}') \subseteq \mathrm{rad}(\mathfrak{b}_m)$, and $\mathfrak{b}_m \notin \Gamma$, which is a contradiction. In consequence, Γ has maximal elements.

Let $\mathfrak{b} \in \Gamma$ be a maximal element, and $\mathfrak{a}_1, \mathfrak{a}_2$ be ideals such that $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{b}$ and $\mathfrak{a}_1, \mathfrak{a}_2 \supsetneqq \mathfrak{b}$. We may assume $\mathfrak{a}_1, \mathfrak{a}_2$ are radical ideal, hence they are not totally finite σ–radical, and there are, finitely generated ideals $\mathfrak{a}'_i \subseteq \mathfrak{a}_i$, $i = 1, 2$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_i\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}'_i)$. Therefore, $\mathfrak{a}'_1\mathfrak{a}'_2 \subseteq \mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{b}$ is finitely generated, and $\mathfrak{b}\mathfrak{h} \subseteq \mathfrak{a}_i\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}'_i)$, hence $\mathfrak{b}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}'_1) \cap \mathrm{rad}(\mathfrak{a}'_2) = \mathrm{rad}(\mathfrak{a}'_1\mathfrak{a}'_2)$, and $\mathfrak{b}$ is totally finite σ–radical, which is a contradiction. □

Now we have the theorem that characterizes totally noetherian σ–radical rings in terms of prime ideals.

Theorem. 3.35.

Let A be a ring, the following statements are equivalent:
(a) A is totally noetherian σ–radical.
(b) Every ideal is totally finite σ–radical.
(c) Every prime ideal is totally finite σ–radical.

Proof. We know that (a) ⇒ (b) ⇒ (c). To show (c) ⇒ (b), if there exists a radical ideal which is not totally finite σ–radical, the set Γ in Lemma (3.34.) is non–empty, hence it contains a maximal element which is a prime ideal, and this is a contradiction.

(b) ⇒ (a). Let $\mathfrak{a} \subseteq A$ be an ideal, by the hypothesis $\mathfrak{b} = \mathrm{rad}(\mathfrak{a})$ is totally finite σ–radical, hence there exist $\mathfrak{b}' \subseteq \mathfrak{b}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{b}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{b})$. Since $\mathfrak{b}' \subseteq \mathfrak{b} = \mathrm{rad}(\mathfrak{a})$ is finitely generated, there exists $n \in \mathbb{N}$ such that $(\mathfrak{b}')^n \subseteq \mathfrak{a}$, hence we have: $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{b}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{b}') = \mathrm{rad}((\mathfrak{b}')^n) \subseteq \mathrm{rad}(\mathfrak{a})$, and $\mathfrak{a}$ is totally finite σ–radical. □

Our aim now is to show, similarly to Theorem (3.25.), that if A is totally noetherian σ–radical, then $A[X]$ is totally noetherian $\overline{\sigma}$–radical, being $\overline{\sigma}$ the induced hereditary torsion theory in **Mod**–$A[X]$.

Lemma. 3.36.

Let $\mathfrak{a} \subseteq \mathfrak{b} \subseteq A$ be ideals. If $\mathfrak{b} \subseteq A$ is totally finite σ–radical, then $\mathfrak{b}/\mathfrak{a} \subseteq A/\mathfrak{a}$ is totally finite $\overline{\sigma}$–radical.

Proof. By the hypothesis, there exist a finitely generated ideal $\mathfrak{b}' \subseteq \mathfrak{b}$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{b}\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{b}')$. Hence,

$$\frac{\mathfrak{b}+\mathfrak{a}}{\mathfrak{a}} \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} = \frac{\mathfrak{b}\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} \subseteq \frac{\mathfrak{b}'+\mathfrak{a}}{\mathfrak{a}},$$

and $\mathfrak{b}/\mathfrak{a}$ is totally finite $\overline{\sigma}$–radical. □

Corollary. 3.37.

Let A be a totally noetherian σ–radical ring, for any ideal $\mathfrak{a} \subseteq A$ we have that $A/\mathfrak{a}$ is totally noetherian $\overline{\sigma}$–radical.

Lemma. 3.38.

Let $\mathfrak{a} \subseteq \mathfrak{b} \subseteq A$ be ideals. If $\mathfrak{a} \subseteq A$ is totally finite σ–radical and $\mathfrak{b}/\mathfrak{a} \subseteq A/\mathfrak{a}$ is totally finite $\overline{\sigma}$–radical, then $\mathfrak{b} \subseteq A$ is totally finite σ–radical.

Proof. There exist $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h}_1 \subseteq \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h}_1 \subseteq \operatorname{rad}(\mathfrak{a}')$, and there exist $\mathfrak{b}' \subseteq \mathfrak{b}$, finitely generated, and $\mathfrak{h}_2 \in \mathscr{L}(\sigma)$ such that $\frac{\mathfrak{b}}{\mathfrak{a}} \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} \subseteq \operatorname{rad}\left(\frac{\mathfrak{b}'}{\mathfrak{a}}\right)$. We can take $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$, and the finitely generated ideal $\mathfrak{a}' + \mathfrak{b}' \subseteq \mathfrak{b}$, then we have: $\frac{\mathfrak{b}}{\mathfrak{a}} \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} \subseteq \operatorname{rad}\left(\frac{\mathfrak{b}'}{\mathfrak{a}}\right)$, hence $\mathfrak{b}\mathfrak{h} + \mathfrak{a} \subseteq \mathfrak{b}' + \mathfrak{a}$, and $\mathfrak{b}\mathfrak{h} \subseteq \mathfrak{b}' + \mathfrak{a}$. Therefore, $\mathfrak{b}\mathfrak{h}\mathfrak{h} \subseteq (\mathfrak{b}' + \mathfrak{a})\mathfrak{h} \subseteq \mathfrak{b}' + \operatorname{rad}(\mathfrak{a}') \subseteq \operatorname{rad}(\mathfrak{b}' + \mathfrak{a}')$. □

Lemma. 3.39.

Let $\mathfrak{a} \subseteq A$ be an ideal and $\sigma_1 \leq \sigma_2$ hereditary torsion theories. If $\mathfrak{a}$ is totally finite σ_1–radical, then it is totally finite σ_2–radical.

Lemma. 3.40.

Let $\mathfrak{a} \subseteq A$ be an ideal and $a \in A$ be a regular element. If $\mathfrak{a} + aA \subseteq A$ is totally finite σ–radical and $\mathfrak{a}A_a \subseteq A_a$ is totally σ–radical, then $\mathfrak{a} \subseteq A$ is totally σ–radical.

We denote by A_a the ring of fractions of A with respect to the multiplicative subset $\{a^n \mid n \in \mathbb{N}\}$.

Proof. Since $\mathfrak{a} + aA \subseteq A$ is totally finite σ–radical, there exist $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\mathfrak{a}'' \subseteq A$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{a}' + a\mathfrak{a}'')$. Otherwise, since $\mathfrak{a}A_a \subseteq A_a$ is totally finite $\overline{\sigma}$–radical, there exists $\mathfrak{b}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h}A_0 \subseteq \operatorname{rad}(\mathfrak{a}'A_0)$. In consequence, $\mathfrak{a}' + \mathfrak{b}' \subseteq \mathfrak{a}$ is finitely generated. Otherwise, for every prime ideal $\mathfrak{p} \supseteq \mathfrak{a}' + \mathfrak{b}'$, If $a \in \mathfrak{p}$, then $\mathfrak{a}\mathfrak{h}\,\mathrm{rad}(\mathfrak{a}' + a\mathfrak{a}'') \subseteq \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$. If $a \notin \mathfrak{p}$, then $\mathfrak{p} \cap \{a^n \mid n \in \mathbb{N}\} = \varnothing$, and $\mathfrak{a}\mathfrak{h}A_a \subseteq \mathrm{rad}(\mathfrak{b}'A_a) \subseteq \mathfrak{p}A_a$. In particular, $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{p}A_a \cap A = \mathfrak{p}$. Therefore, $\mathfrak{a}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}' + \mathfrak{b}')$. □

Lemma. 3.41.

Let $\mathfrak{a} \subseteq A$ be a totally finite σ–radical ideal, then $\mathfrak{a}[X] \subseteq A[X]$ is totally finite $\overline{\sigma}$–radical.

Proof. Let $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \mathrm{rad}(\mathfrak{a}')$, hence $\mathfrak{a}[X]\mathfrak{h}[X] \subseteq \mathfrak{a}\mathfrak{h}[X] \subseteq \mathrm{rad}(\mathfrak{a}')[X] \subseteq \mathrm{rad}(\mathfrak{a}'[X])$, and $\mathfrak{a}[X]$ is totally finite $\overline{\sigma}$–radical, because $\mathfrak{a}'[X] \subseteq A[X]$ is finitely generated. □

Lemma. 3.42.

Let D be an integral domain with field of fractions K and $F \in D[X]$, with leader coefficient $a \neq 0$. If $G \in K[X]$ satisfies $FG \in D[X]$, then $G \in D_a[X]$.

Proof. Let $F = \sum_{i=0}^{n} a_i X^i$, $G = \sum_{i=0}^{m} b_i X^i$, and $a_n = a \neq 0$. Since $a_n b_m \in D$, then $b_m \in D_a$. If we assume $b_m, b_{m-1}, \ldots, b_{t-1} \in D_a$, since $a_n b_t + \cdots + a_{n-m+t} b_m \in D$, then $b_t \in D_a$. □

Theorem. 3.43.

*Let A be a ring, σ be a hereditary torsion theory in **Mod**–A, and $\overline{\sigma}$ be the hereditary torsion theory induced by σ in **Mod**–$A[X]$, the following statements are equivalent:*

(a) A is totally noetherian σ–radical.

(b) $A[X]$ is totally noetherian $\overline{\sigma}$–radical.

Proof. (a) ⇒ (b). Consider that the family $\Gamma = \{\mathfrak{a} \mid \mathfrak{a} \text{ is non totally } \sigma\text{–radical}\}$. If $\Gamma \neq \varnothing$, there exists an ideal $\mathfrak{q}_0$, maximal in Γ, hence a prime ideal. If $\mathfrak{p} = \mathfrak{q}_0 \cap A$, then $(A/\mathfrak{p})[X] = A[X]/\mathfrak{p}[X]$. Thus, we have a totally noetherian σ–radical integral domain $D = A/\mathfrak{p}$, a polynomial ring $D[X]$, and a prime ideal $\mathfrak{q} = \mathfrak{q}_0/\mathfrak{p}[X]$ such that $\mathfrak{q} \cap D = 0$. We assume $\mathfrak{q} \neq 0$.

Let K be the field of fractions of D, there exists $F \in D[X]$ such that $\mathfrak{q}K[X] = FK[X]$. If a is the leader coefficient of F, then $a \notin \mathfrak{q}$, hence $\mathfrak{q} \subsetneqq (\mathfrak{q} + (a))$, and in $(\mathfrak{q} + (a))$ is totally finite $\overline{\sigma}$–radical. Otherwise, we claim, $\mathfrak{q}D_a[X] = FD_a[X]$. Since $\mathfrak{q}K[X] = FK[X]$, for any $G \in \mathfrak{q}D_a[X]$, there exists $H \in K[X]$ such that $G = FH$, hence $H \in D_a[X]$. Since $\mathfrak{q}D_a[X]$ is cyclic, it is totally finite σ–radical.

Now we apply Lemma (3.40.) to get that $\mathfrak{q} \subseteq D[X]$ is totally finite $\overline{\sigma}$–radical, which is a contradiction.

(b) ⇒ (a). It is a consequence of Lemma (3.37.) and the second part of the proof of Theorem (3.25.).

□

In the study of noetherian σ–radical rings we found that σ can be considered of finite type. However, in the case of totally noetherian σ–radical we have no criteria to make the analogous statement. Indeed, we know that if A is totally noetherian σ–radical, and $\sigma' = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$, then A is totally noetherian σ'–radical, and $\mathscr{K}(\sigma) = \mathscr{K}(\sigma')$. But, since totally noetherian radical is not a lattice property, we can not affirm the converse.

4 Totally prime ideals and submodules

4.1 Totally prime ideals

Let A be a commutative ring, and σ be a hereditary torsion theory in **Mod**–A with Gabriel filter $\mathscr{L}(\sigma)$. An ideal $\mathfrak{a} \subseteq A$ is

- **prime** if $\mathfrak{a} \neq A$ and for any $a, b \in A$ if $ab \in \mathfrak{a}$, either $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$.
- **totally** σ**–prime** if $\mathfrak{a} \notin \mathscr{L}(\sigma)$ and there exists an ideal $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \in \mathscr{L}(\sigma)$ such that for any $a, b \in A$ if $ab \in \mathfrak{a}$, either $a\mathfrak{h} \subseteq \mathfrak{a}$ or $b\mathfrak{h} \subseteq \mathfrak{a}$.
- σ**–prime** if $\mathfrak{a} \notin \mathscr{L}(\sigma)$ and for any $a, b \in A$ if $ab \in \mathfrak{a}$, either $a \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$ or $b \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$.

The ring A is **prime** (resp. **totally** σ**–prime**, σ**–prime**) whenever $0 \subseteq A$ is a prime (resp. totally σ–prime, σ–prime) ideal.

For any ideal $\mathfrak{a} \subseteq A$, such that $\mathfrak{a} \notin \mathscr{L}(\sigma)$, we have:

$$\mathfrak{a} \subseteq A \text{ is prime } \Rightarrow \mathfrak{a} \subseteq A \text{ is totally } \sigma\text{–prime} \Rightarrow \mathfrak{a} \subseteq A \text{ is } \sigma\text{–prime.}$$

σ–prime ideals

The σ–prime ideals can be characterized through their σ–closure as follows:

Proposition. 4.1.
Let $\mathfrak{a} \subseteq A$ be an ideal such that $\mathfrak{a} \notin \mathscr{L}(\sigma)$, the following statements are equivalent:

(a) $\mathfrak{a} \subseteq A$ is σ–prime.
(b) $\mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq A$ is prime.
(c) For any ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$, if $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$, either $\mathfrak{a}_1 \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$ or $\mathfrak{a}_2 \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$.

Proof. (a) $\Rightarrow$ (b). Let $a, b \in A$ such that $ab \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$, there exists $\mathfrak{c} \in \mathscr{L}(\sigma)$ such that $ab\mathfrak{c} \subseteq \mathfrak{a}$. If $a \notin \mathrm{Cl}^A_\sigma(\mathfrak{a})$, then $(\mathfrak{a} : a) \notin \mathscr{L}(\sigma)$. For any $c \in \mathfrak{c}$ we have $abc \in \mathfrak{a}$; if for any $c \in \mathfrak{c}$ we have $ac \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$, then $((\mathfrak{a} : a) : c) = (\mathfrak{a} : ac) \in \mathscr{L}(\sigma)$, hence $(\mathfrak{a} : a) \in \mathscr{L}(\sigma)$, which is a contradiction. Therefore, there exists $c \in \mathfrak{c}$ such that $ac \notin \mathrm{Cl}^A_\sigma(\mathfrak{a})$, so $b \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$.

(a) $\Rightarrow$ (c). Let $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$, if $\mathfrak{a}_1, \mathfrak{a}_2 \nsubseteq \mathrm{Cl}_\sigma^A(\mathfrak{a})$, there exist $a_i \in \mathfrak{a}_i \setminus \mathrm{Cl}_\sigma^A(\mathfrak{a})$, for $i = 1, 2$, such that $a_1a_2 \in \mathfrak{a}$, hence either $a_1 \in \mathrm{Cl}_\sigma^A(\mathfrak{a})$ or $a_2 \in \mathrm{Cl}_\sigma^A(\mathfrak{a})$, which is a contradiction.

(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are immediate. □

In particular, σ–closed σ–prime ideals are prime.

Corollary. 4.2.

Let $\mathfrak{a} \subseteq A$ be a σ–closed ideal; i. e., $\mathfrak{a} = \mathrm{Cl}_\sigma^A(\mathfrak{a})$, the following statements are equivalent:

(a) $\mathfrak{a} \subseteq A$ is σ–prime.

(b) $\mathfrak{a} \subseteq A$ is prime.

Totally σ–prime ideals

In a parallel way, we may characterize totally σ–prime ideals as follows. The proof of this proposition is similar to the proof of Proposition (4.1.).

Proposition. 4.3.

Let $\mathfrak{a} \subseteq A$ be an ideal such that $\mathfrak{a} \notin \mathscr{L}(\sigma)$, the following statements are equivalent:

(a) $\mathfrak{a} \subseteq A$ is totally σ–prime.

(b) There exists an ideal $\mathfrak{h} = \mathfrak{h}_\mathfrak{a} \in \mathscr{L}(\sigma)$ such that for any ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$, if $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$, either $\mathfrak{a}_1\mathfrak{h} \subseteq \mathfrak{a}$ or $\mathfrak{a}_2\mathfrak{h} \subseteq \mathfrak{a}$.

Totally σ–prime ideals can be characterized through prime ideals of A as follows. First let us point out that if $\mathfrak{a} \subseteq A$ is a totally σ–prime ideal, the associated ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ can be chosen as the annihilator of $\sigma(A/\mathfrak{a})$.

Lemma. 4.4.

Let $\mathfrak{a} \subseteq A$ be a totally σ–prime ideal, and $\mathfrak{k} = \mathrm{Ann}(\sigma(A/\mathfrak{a})) = (\mathfrak{a} : \mathrm{Cl}_\sigma^A(\mathfrak{a}))$, the following statements hold:

(1) $\mathfrak{k} \in \mathscr{L}(\sigma)$;

(2) for any $a, b \in a$ such that $ab \in \mathfrak{a}$ we have that either $a\mathfrak{k} \subseteq \mathfrak{a}$ or $b\mathfrak{k} \subseteq \mathfrak{a}$;

(3) $\mathrm{Cl}_\sigma^A(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{k}) = (\mathfrak{a} : \mathfrak{t})$, for any ideal $\mathfrak{t} \in \mathscr{L}(\sigma)$ such that $\mathfrak{t} \subseteq \mathfrak{k}$;

(4) $\mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq A$ *is prime.*

Proof. By hypothesis $A/\mathfrak{a}$ is not totally σ–torsion, or equivalently, $\mathfrak{a} \notin \mathcal{L}(\sigma)$.

(1). Let $\mathfrak{h} \in \mathcal{L}(\sigma)$, the ideal associated to $\mathfrak{a} \subseteq A$; for any $x \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$ there exists $\mathfrak{t} \in \mathcal{L}(\sigma)$ such that $x\mathfrak{t} \subseteq \mathfrak{a}$; hence either $x\mathfrak{h} \subseteq \mathfrak{a}$ or $A\mathfrak{t}\mathfrak{h} \subseteq \mathfrak{a}$. Therefore, $\mathrm{Cl}^A_\sigma(\mathfrak{a})\mathfrak{h} \subseteq \mathfrak{a}$; i.e., $\mathfrak{h} \subseteq (\mathfrak{a} : \mathrm{Cl}^A_\sigma(\mathfrak{a})) = \mathfrak{k}$.

(2). If we take $\mathfrak{k} = (\mathfrak{a} : \mathrm{Cl}^A_\sigma(\mathfrak{a}))$, for any $x \in A$ if $x\mathfrak{h} \subseteq \mathfrak{a}$, then $x \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$, hence $x\mathfrak{k} \subseteq \mathfrak{a}$. In conclusion, we can take $\mathfrak{h} = \mathfrak{k}$.

(3). It is clear that $(\mathfrak{a} : \mathfrak{k}) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq (\mathfrak{a} : \mathfrak{k})$. Otherwise, if $\mathfrak{t} \in \mathcal{L}(\sigma)$ and $\mathfrak{t} \subseteq \mathfrak{k}$, then $(\mathfrak{a} : \mathfrak{k}) \subseteq (\mathfrak{a} : \mathfrak{t}) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{k})$.

(4) is immediate since $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h})$. □

The relationship between totally σ–prime ideals and prime ideals appears in the next proposition.

Proposition. 4.5.

Let $\mathfrak{a} \subseteq A$ *be an ideal such that* $\mathfrak{a} \notin \mathcal{L}(\sigma)$, *and* $\mathfrak{k} = (\mathfrak{a} : \mathrm{Cl}^A_\sigma(\mathfrak{a}))$, *the following statements are equivalent:*

(a) $\mathfrak{a} \subseteq A$ *is totally* σ*–prime.*

(b) *There exists an ideal* $\mathfrak{h} \in \mathcal{L}(\sigma)$ *such that* $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ *is prime.*

(c) *There exists an ideal* $\mathfrak{h} \in \mathcal{L}(\sigma)$ *such that* $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h})$, *and* $\mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq A$ *is prime.*

Proof. (a) $\Rightarrow$ (b) and (c) are consequence of Lemma (4.4.).

(b) $\Rightarrow$ (c). If we take $\mathfrak{h}$ as in (b), we only need to show that $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h})$. Always we have $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{h}) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$ because $\mathfrak{h} \in \mathcal{L}(\sigma)$. Since $(\mathfrak{a} : \mathfrak{h})$ is prime, then either $(\mathfrak{a} : \mathfrak{h}) \in \mathcal{Z}(\sigma)$ or $(\mathfrak{a} : \mathfrak{h}) \in \mathcal{K}(\sigma)$. If $(\mathfrak{a} : \mathfrak{h}) \in \mathcal{Z}(\sigma)$, then $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = A$, and $\mathfrak{a} \in \mathcal{L}(\sigma)$, which is a contradiction. Therefore, $(\mathfrak{a} : \mathfrak{h}) \in \mathcal{K}(\sigma)$, and $(\mathfrak{a} : \mathfrak{h}) = \mathrm{Cl}^A_\sigma(\mathfrak{a})$.

(c) $\Rightarrow$ (a) is obvious. □

As a consequence, as before, σ–closed totally σ–prime ideals are prime.

Corollary. 4.6.

Let $\mathfrak{a} \subseteq A$ *be a* σ*–closed ideal, the following statements are equivalent:*

(a) $\mathfrak{a} \subseteq A$ *is totally* σ*–prime.*

(b) $\mathfrak{a} \subseteq A$ *is prime.*

Compare with Example 2 in [28], and Corollary(4.2.) above.

Proof. (a) $\Rightarrow$ (b). There exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime. Since $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{h})$, if $\mathfrak{a} \neq (\mathfrak{a} : \mathfrak{h})$, there exists $x \in (\mathfrak{a} : \mathfrak{h}) \setminus \mathfrak{a}$ satisfying $x\mathfrak{h} \subseteq \mathfrak{a}$, hence $x + \mathfrak{a} \in \sigma(A/\mathfrak{a}) = 0$, which is a contradiction.

(b) $\Rightarrow$ (a) is immediate. □

Once we have a totally σ–prime ideal, we can build many others.

Proposition. 4.7.

Let $\mathfrak{a}, \mathfrak{k} \subseteq A$ *be ideals such that* $\mathfrak{k} \in \mathscr{L}(\sigma)$*, the following statements are equivalent:*

(a) $\mathfrak{a} \subseteq A$ *is totally* σ*–prime.*

(b) $\mathfrak{a}\mathfrak{k} \subseteq A$ *is totally* σ*–prime.*

Proof. (a) $\Rightarrow$ (b). Since $\mathfrak{a} \notin \mathscr{L}(\sigma)$, then $\mathfrak{a}\mathfrak{k} \notin \mathscr{L}(\sigma)$ because, we have $\mathfrak{a}\mathfrak{k} \subseteq \mathfrak{a}$. Let $\mathfrak{h} \in \mathscr{L}(\sigma)$ be an ideal associated to $\mathfrak{a}$, and $a, b \in A$ such that $ab \in \mathfrak{a}\mathfrak{k}$. Since $ab \in \mathfrak{a}$, either $a\mathfrak{h} \subseteq \mathfrak{a}$ or $b\mathfrak{h} \subseteq \mathfrak{a}$. Therefore, either $a\mathfrak{h}\mathfrak{k} \subseteq \mathfrak{a}\mathfrak{k}$ or $b\mathfrak{h}\mathfrak{k} \subseteq \mathfrak{a}\mathfrak{k}$.

(b) $\Rightarrow$ (a). If $\mathfrak{a} \in \mathscr{L}(\sigma)$, since $(\mathfrak{a}\mathfrak{k} : k) \supseteq \mathfrak{a}$, for any $k \in \mathfrak{k}$, then $\mathfrak{a}\mathfrak{k} \in \mathscr{L}(\sigma)$, which is a contradiction; in conclusion, $\mathfrak{a} \notin \mathscr{L}(\sigma)$. Otherwise, if $a, b \in A$ satisfy $ab \in \mathfrak{a}$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ satisfies $(\mathfrak{a}\mathfrak{k} : \mathfrak{h}) \subseteq A$ is prime, then $ab\mathfrak{k} \subseteq \mathfrak{a}\mathfrak{a}$, and either $a\mathfrak{h} \subseteq \mathfrak{a}\mathfrak{k} \subseteq \mathfrak{a}$ or $b\mathfrak{k}\mathfrak{h} \subseteq \mathfrak{a}\mathfrak{k} \subseteq \mathfrak{a}$. In conclusion, $\mathfrak{a} \subseteq A$ is totally σ–prime. □

Note that, if $\mathfrak{a} \subseteq A$ is totally σ–prime, there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathrm{Cl}_\sigma^A(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime; therefore, for every ideal $\mathfrak{h}' \in \mathscr{L}(\sigma)$ we have $(\mathfrak{a} : \mathfrak{h}') \subseteq (\mathfrak{a} : \mathfrak{h})$. But this condition is not enough to get that $\mathfrak{a} \subseteq A$ is totally σ–prime; it is further necessary that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ be prime; and, in consequence, that it be the σ–closure of $\mathfrak{a}$ in A.

Theorem. 4.8.

Let σ *be a hereditary torsion theory on* ***Mod***–*A, the following statements hold:*

(1) For any ideals $\mathfrak{h} \in \mathscr{L}(\sigma)$ *and* $\mathfrak{p} \in \mathscr{K}(\sigma)$ *the ideal* $\mathfrak{p}\mathfrak{h} \subseteq A$ *is totally* σ*–prime.*

(2) *For any totally σ–prime ideal $\mathfrak{a} \subseteq A$ there exist ideals $\mathfrak{h} \in \mathscr{L}(\sigma)$ and $\mathfrak{p} \in \mathscr{K}(\sigma)$ such that $\mathfrak{p}\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{p}$.*

(3) *Let $\mathfrak{a} \subseteq A$ be an ideal, if there exist ideals $\mathfrak{h} \in \mathscr{L}(\sigma)$ and $\mathfrak{p} \in \mathscr{K}(\sigma)$ such that $\mathfrak{p}\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, then $(\mathfrak{a} : \mathfrak{h}) = \mathfrak{p}$, and $\mathfrak{a} \subseteq A$ is totally σ–prime.*

Proof. (1) is a consequence of Proposition (4.7.).

(2). If $\mathfrak{a} \subseteq A$ is totally σ–prime, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime, and $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h})$ belongs to $\mathscr{K}(\sigma)$, by Proposition (4.5.); obviously, $\mathrm{Cl}^A_\sigma(\mathfrak{a})\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$.

(3). Otherwise, let $\mathfrak{p} \in \mathscr{K}(\sigma)$, $\mathfrak{h} \in \mathscr{L}(\sigma)$, and $\mathfrak{a} \subseteq A$ an ideal satisfying $\mathfrak{p}\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{p}$; we claim $\mathfrak{a} \subseteq A$ is totally σ–prime. First we note that $\mathrm{Cl}^A_\sigma(\mathfrak{p}\mathfrak{h}) = \mathrm{Cl}^A_\sigma(\mathfrak{a}) = \mathrm{Cl}^A_\sigma(\mathfrak{p}) = \mathfrak{p}$, and also that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is σ–closed. Indeed, if $x \in \mathrm{Cl}^A_\sigma((\mathfrak{a} : \mathfrak{h}))$, there exists $\mathfrak{h}' \in \mathscr{L}(\sigma)$ such that $x\mathfrak{h}' \subseteq (\mathfrak{a} : \mathfrak{h})$, hence $x\mathfrak{h}'\mathfrak{h} \subseteq \mathfrak{a}$, and $x \in \mathrm{Cl}^A_\sigma(\mathfrak{a}) = \mathfrak{p}$. Therefore, $x\mathfrak{h} \subseteq \mathfrak{p}\mathfrak{h} \subseteq \mathfrak{a}$, and $x \in (\mathfrak{a} : \mathfrak{h})$. On the other hand, $(\mathfrak{a} : \mathfrak{h})\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, so $(\mathfrak{a} : \mathfrak{h}) \subseteq \mathfrak{p}$, and since $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{h})$, then $(\mathfrak{a} : \mathfrak{h}) = \mathfrak{p}$. In conclusion, $\mathfrak{a} \subseteq A$ is totally σ–prime. □

Remark. 4.9.

Let $\mathfrak{a} \subseteq A$ be a totally σ–prime ideal and $\mathfrak{h}_\mathfrak{a} = (\mathfrak{a} : \mathrm{Cl}^A_\sigma(\mathfrak{a}))$, the following statements hold:

(1) For any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ the ideal $\mathfrak{h}_\mathfrak{p}$ is exactly the whole ring A.

In particular, $\mathfrak{p} = (\mathfrak{p} : \mathfrak{h})$ for any ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$.

(2) For any ideals $\mathfrak{p} \in \mathscr{K}(\sigma)$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ we have $\mathfrak{p}\mathfrak{h} = \mathfrak{p}\mathfrak{h}_{\mathfrak{p}\mathfrak{h}}$. Indeed, let $\mathfrak{a} = \mathfrak{p}\mathfrak{h}$, then $\mathfrak{p} = \mathrm{Cl}^A_\sigma(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{h}) = (\mathfrak{a} : \mathfrak{h}_\mathfrak{a})$, hence $\mathfrak{h} \subseteq \mathfrak{h}_\mathfrak{a}$. Otherwise, $\mathfrak{p}\mathfrak{h}_\mathfrak{a} \subseteq \mathfrak{a} = \mathfrak{p}\mathfrak{h} \subseteq \mathfrak{p}\mathfrak{h}_\mathfrak{a}$. Therefore, $\mathfrak{p}\mathfrak{h} = \mathfrak{p}\mathfrak{h}_\mathfrak{a}$.

(3) For any totally σ–prime ideal $\mathfrak{a} \subseteq A$ we have $(\mathfrak{a} : \mathfrak{h}) = \mathrm{Cl}^A_\sigma(\mathfrak{a})$, for any ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{h} \subseteq \mathfrak{h}_\mathfrak{a}$. In general we have $\mathrm{Cl}^A_\sigma(\mathfrak{a})\mathfrak{h} \subseteq \mathfrak{a}$, and the equality is not always satisfied.

Let us show an example in which the different notions of prime ideal are independent.

Example. 4.10.

Let $M = \mathbb{Z}_{2^\infty} = \{\frac{a}{2^s} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid a \in \mathbb{Z}, s \in \mathbb{N}\}$, the Prüfer group, and $A = M \rtimes \mathbb{Z}$ its idealization. Let $\Sigma = \{(0, 2^t) \mid t \in \mathbb{N}\}$, and $\sigma = \sigma_\Sigma$, the hereditary torsion theory whose Gabriel filter is $\mathscr{L}(\sigma) = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \cap \Sigma \neq \varnothing\}$.

(1) The ideal $\mathfrak{a} = M \rtimes 2\mathbb{Z} \subseteq A$ is prime but it is not totally σ-prime because $\mathfrak{a} \in \mathscr{L}(\sigma)$.

(2) The ideal $\mathfrak{a} = M \rtimes 12\mathbb{Z} \subseteq A$ is not prime and it is totally σ–prime. Indeed, it we consider the ideal $\mathfrak{h} = M \rtimes 4\mathbb{Z}$, it satisfies $\mathfrak{h} \in \mathscr{L}(\sigma)$, and $(\mathfrak{a} : \mathfrak{h}) = M \rtimes 3\mathbb{Z}$ is a prime ideal.

(3) The ideal $\mathfrak{a} = 0 \subseteq A$ is σ–prime but it is not totally σ–prime. First observe that $\mathrm{Cl}^A_\sigma(0) = M \rtimes 0$; indeed, for any $x = \frac{a}{2^s} + \mathbb{Z} \in M \rtimes 0$ we have $x(0, 2^s) = 0$; therefore, $\mathfrak{a} \subseteq A$ is σ–prime. Second, if there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(0 : \mathfrak{h}) \subseteq A$ is prime, then either there exists a prime integer number p such that $(0 : \mathfrak{h}) = M \rtimes p\mathbb{Z}$, i.e., $\mathfrak{h}(M \rtimes p\mathbb{Z}) = 0$, and we have $\mathfrak{h} \subseteq M \rtimes 0$, which is a contradiction, or $(0 : \mathfrak{h}) = M \rtimes 0$, and $\mathfrak{h}(M \rtimes 0) = 0$, so $M \rtimes 0$ is totally σ–torsion, which is a contradiction.

4.2 Existence of totally prime ideals

One problem we are faced with is the existence of totally σ–prime ideals. Let us consider the following example.

Example. 4.11.
Let K be a field, $\{X_n \mid n \in \mathbb{N}\}$ a set of numerably many indeterminates over K, and A the quotient ring

$$A = \frac{K[X_n \mid n \in \mathbb{N}]}{(X_0, X_1^2 - X_0, X_2^2 - X_1, \ldots)}.$$

We know that A is a zero–dimensional ring with maximal ideal $\mathfrak{m} = (X_n \mid n \in \mathbb{N})$ satisfying $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathfrak{m} \subseteq \mathrm{Nil}(A)$. Under these conditions the hereditary torsion theory σ, whose Gabriel filter is $\{\mathfrak{m}, A\}$, satisfies $\mathscr{K}(\sigma) = \varnothing$. Consequently, there are not totally σ–prime ideals in A.

The existence of totally σ–prime ideals can be ensured by imposing some extra conditions; for example, imposing that A is a σ–noetherian ring. In this case it is sufficient to show that $\mathscr{K}(\sigma)$ is non–empty.

Following [32], and section (3.1), an ideal $\mathfrak{a} \subseteq A$ is **totally σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ and an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}'\mathfrak{h} \subseteq \mathfrak{a}' \subseteq \mathfrak{a}$. The ring A is **totally σ–noetherian** whenever every ideal is totally σ–finitely generated. In the same way an ideal $\mathfrak{a} \subseteq A$ is **σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq A$

such that $\mathrm{Cl}_\sigma^A(\mathfrak{a}) = \mathrm{Cl}_\sigma^A(\mathfrak{a}')$, and the ring A is σ**–noetherian** whenever every ideal is σ–finitely generated. Observe that every totally σ–noetherian ring is σ–noetherian; therefore, σ is of **finite type**; i.e., the Gabriel filter $\mathscr{L}(\sigma)$ has a cofinal set of finitely generated ideals. The converse does not necessarily hold.

The following result is well known.

Lemma. 4.12.

If A is a σ–noetherian ring, then $\mathscr{K}(\sigma) \neq \varnothing$.

Now we show that totally σ–noetherian ring can be characterized through prime and totally σ–prime ideals.

Theorem. 4.13. (Cohen–like theorem)

Let σ be a hereditary torsion theory in ***Mod****–A, the following statements are equivalent:*

(a) A is totally σ–noetherian.

(b) Every totally σ–prime ideal is totally σ–finitely generated.

(c) Every prime ideal is totally σ–finitely generated.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Let $\mathfrak{p} \subseteq A$ be a prime ideal, either $\mathfrak{p} \in \mathscr{L}(\sigma)$ or $\mathfrak{p} \in \mathscr{K}(\sigma)$. In the first case $\mathfrak{p}$ is totally σ–finitely generated, and in the second one also because it is totally σ–prime.

(c) $\Rightarrow$ (a). See Corollary (3.5) in [32]. □

As it was shown in Theorem (4.8.), each totally σ–prime ideal defines a prime ideal in $\mathscr{K}(\sigma)$, and there exists a plethora of totally σ–prime ideals defined by any prime ideal in $\mathscr{K}(\sigma)$. All these ideals constitute an equivalence class for a convenient equivalence relation. Each class has a maximum element; the existence of minimal elements in these equivalence classes depends of the properties that $\mathscr{L}(\sigma)$ satisfies. For instance, if there exists a minimal element in $\mathscr{L}(\sigma)$, there is a minimal element in each of these classes.

A hereditary torsion theory σ is **jansian** whenever $\mathscr{L}(\sigma)$ has a minimal element, say $\mathfrak{h}$. In this case $\mathfrak{h} \subseteq A$ must be an idempotent ideal.

Proposition. 4.14.

Let σ be a jansian hereditary torsion theory in ***Mod***–*A, and let $\mathfrak{h} \in \mathscr{L}(\sigma)$ be minimal element, the following statements holds:*

(1) For any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ the class

$$\{\mathfrak{a} \subseteq A \mid \ \mathrm{Cl}^A_\sigma(\mathfrak{a}) = \mathfrak{p} \text{ and } \mathfrak{a} \subseteq A \text{ is totally } \sigma\text{–prime}\}$$

has a minimal element.

(2) For any ideal $\mathfrak{c} \subseteq A$ the class

$$\mathscr{S} = \{\mathfrak{a} \subseteq A \mid \ \mathfrak{a} \supseteq \mathfrak{c} \text{ and } \mathfrak{a} \subseteq A \text{ is totally } \sigma\text{–prime}\}$$

has minimal elements whenever it is non–empty.

Proof. (1) is obvious from Theorem (4.8.).

(2). Let $\{\mathfrak{a}_i \mid \ i \in I\}$ be a chain in $\mathscr{S}$. For any $\mathfrak{a} \in \mathscr{S}$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime. In consequence, $(\cap_i \mathfrak{a}_i : \mathfrak{h}) = \cap_i(\mathfrak{a}_i : \mathfrak{h})$ is the intersection of a chain of prime ideals, hence it is a prime ideal, and $\cap_i \mathfrak{a}_i \subseteq A$ is totally σ–prime. By Zorn's lemma, if $\mathscr{S}$ is non–empty, it has minimal elements. □

A minimal element in the former class $\mathscr{S}$ is called a **minimal totally σ–prime ideal over** $\mathfrak{c}$.

Theorem. 4.15.

Let A be a totally σ–noetherian ring, and $\mathfrak{c} \notin \mathscr{L}(\sigma)$, there exist finitely many totally σ–prime ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ and an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{c} \subseteq \mathfrak{a}_i$, for any index $i = 1, \ldots, t$, and $\mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \subseteq \mathfrak{c}$.

We may take $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \in \mathscr{K}(\sigma)$.

This is a generalization of [28, Theorem 5]

Proof. Let us define a class of ideals:

$$\mathscr{F} = \{\mathfrak{c} \subseteq A \mid \text{ for any family of totally } \sigma\text{–prime submodules } \mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A \text{ and any } \mathfrak{h} \in \mathscr{L}(\sigma) \text{ we have } \mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \nsubseteq \mathfrak{c}\}.$$

If $\mathscr{F} \neq \emptyset$, since for any chain $\{\mathfrak{c}_i \mid i \in I\}$ in $\mathscr{F}$ we have $\cup_i \mathfrak{c}_i \in \mathscr{F}$, by Zorn's lemma there are maximal elements in $\mathscr{F}$. Indeed, if $\cup_i \mathfrak{c}_i \notin \mathscr{F}$, there are $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A$ totally σ–prime ideals and $\mathfrak{h} \in \mathscr{L}(\sigma)$ (finitely generated) such that $\mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \subseteq \mathfrak{c}$. There are finitely generated ideals $\mathfrak{f}_j$ and ideals $\mathfrak{h}_j \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_j \mathfrak{h}_j \subseteq \mathfrak{f}_j \subseteq \mathfrak{a}_j$, for any index $j = 1, \ldots, t$; hence $\mathfrak{a}_1 \mathfrak{h}_1 \cdots \mathfrak{a}_t \mathfrak{h}_t \mathfrak{h} \subseteq \mathfrak{f}_1 \cdots \mathfrak{f}_t \mathfrak{h} \subseteq \mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \subseteq \mathfrak{c}$. Since $\mathfrak{f}_1 \cdots \mathfrak{f}_t \mathfrak{h}$ is finitely generated, there is an index $i \in I$ such that $\mathfrak{a}_1 \mathfrak{h}_1 \cdots \mathfrak{a}_t \mathfrak{h}_t \mathfrak{h} \subseteq \mathfrak{f}_1 \cdots \mathfrak{f}_t \mathfrak{h} \subseteq \mathfrak{c}_i$, which is a contradiction.

Let $\mathfrak{c} \in \mathscr{F}$ be a maximal element. If $\mathfrak{c} \in \mathscr{L}(\sigma)$, for every totally σ–prime ideal $\mathfrak{a} \subseteq A$ we have $\mathfrak{a}\mathfrak{c} \subseteq \mathfrak{c}$, which is a contradiction; hence, $\mathfrak{c} \notin \mathscr{L}(\sigma)$, and it is not totally σ–prime. Therefore, there are elements $a_1, a_2 \in A$ such that $a_1 a_2 \in \mathfrak{c}$, and for every ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ we have $a_1\mathfrak{h}, a_2\mathfrak{h} \nsubseteq \mathfrak{c}$. Hence $\mathfrak{c} + a_1\mathfrak{h}, \mathfrak{c} + a_2\mathfrak{h} \nsupseteq \mathfrak{c}$, and they do not belong to $\mathscr{F}$. There exist totally prime ideals $\mathfrak{a}_{1,1}, \ldots, \mathfrak{a}_{1,t_1}, \mathfrak{a}_{2,1}, \ldots, \mathfrak{a}_{2,t_2}$, and ideals $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_{k,1} \cdots \mathfrak{a}_{k,t_k} \mathfrak{h}_k \subseteq \mathfrak{c} + a_k\mathfrak{h}$, and their product satisfies:

$$\mathfrak{a}_{1,1} \cdots \mathfrak{a}_{1,t_1} \mathfrak{h}_1 \mathfrak{a}_{2,1} \cdots \mathfrak{a}_{2,t_2} \mathfrak{h}_2 \subseteq (\mathfrak{c} + a_1\mathfrak{h})(\mathfrak{c} + a_2\mathfrak{h}) \subseteq \mathfrak{c},$$

which is a contradiction. In consequence, every maximal element in $\mathscr{F}$ is totally σ–prime, which is also a contradiction. In conclusion, $\mathscr{F}$ must be empty and the result holds. □

We may enhanced this result to show that there are only finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathscr{K}(\sigma)$, minimal prime over $\mathfrak{c}$, satisfying this result.

Corollary. 4.16.
Let A be a totally σ–noetherian ring, and $\mathfrak{c} \notin \mathscr{L}(\sigma)$, there exist finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathscr{K}(\sigma)$, minimal prime over $\mathfrak{c}$, and an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{c} \subseteq \mathfrak{p}_i$, for any index $i = 1, \ldots, t$, and $\mathfrak{p}_1 \cdots \mathfrak{p}_t \mathfrak{h} \subseteq \mathfrak{c}$.

In addition, every minimal prime ideal over $\mathfrak{c}$ belongs to $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$.

Proof. We may assume each ideal $\mathfrak{a}_i = \mathfrak{p}_i$, in the proposition is prime, hence $\mathfrak{p}_i \in \mathscr{K}(\sigma)$. For any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$, minimal over $\mathfrak{c}$ we have $\mathfrak{p}_1 \cdots \mathfrak{p}_t \mathfrak{h} \subseteq \mathfrak{c} \subseteq \mathfrak{p}$, hence there exists an index i such that $\mathfrak{p}_i \subseteq \mathfrak{p}$; therefore $\mathfrak{p} = \mathfrak{p}_i$.

We may assume the elements in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are incomparable; given $\mathfrak{p}_i$ there exists a prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ minimal prime over $\mathfrak{c}$ satisfying $\mathfrak{c} \supseteq \mathfrak{p} \subseteq \mathfrak{p}_i$, hence $\mathfrak{p}_i \subseteq \mathfrak{p}$, and $\mathfrak{p}_i$ is minimal prime over $\mathfrak{c}$. $\square$

Corollary. 4.17.

Let A be a totally σ–noetherian ring, for any $\mathfrak{c} \notin \mathscr{L}(\sigma)$ there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathrm{Cl}_{\sigma}^{A}(\mathfrak{c})^{t}\mathfrak{h} \subseteq \mathfrak{c}$, for some $t \in \mathbb{N}$.

Proof. By the theorem there exist totally σ–prime ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \supseteq \mathfrak{c}$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \subseteq \mathfrak{c}$. For any index $i = 1, \ldots, t$ there exists $\mathfrak{h}_i \in \mathscr{L}(\sigma)$ such that $\mathrm{Cl}_{\sigma}^{A}(\mathfrak{a}_i)\mathfrak{h}_i \subseteq \mathfrak{a}_i$, and $\mathrm{Cl}_{\sigma}^{A}(\mathfrak{c}) \subseteq \mathrm{Cl}_{\sigma}^{A}(\mathfrak{a}_i)$, hence we have

$$\mathrm{Cl}_{\sigma}^{A}(\mathfrak{c})^{t}\left(\prod_{i}\mathfrak{h}_i\right)\mathfrak{h} = \left(\prod_{i=1}^{t}(\mathrm{Cl}_{\sigma}^{A}(\mathfrak{c})\mathfrak{h}_i)\right)\mathfrak{h} \subseteq \left(\prod_{i=1}^{t}(\mathrm{Cl}_{\sigma}^{A}(\mathfrak{a}_i)\mathfrak{h}_i)\right)\mathfrak{h} \subseteq \left(\prod_{i=1}^{t}\mathfrak{a}_i\right)\mathfrak{h} \subseteq \mathfrak{c}.$$

$\square$

Corollary. 4.18.

Let A be a totally σ–noetherian ring; for any ideal $\mathfrak{c} \subseteq A$ such that $\mathfrak{c} \notin \mathscr{L}(\sigma)$ if $\mathfrak{a} \supseteq \mathfrak{c}$ is a minimal totally σ–prime ideal over $\mathfrak{c}$, there exist ideals $\mathfrak{a}' \subseteq A$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a} = \mathfrak{c} + \mathfrak{a}'\mathfrak{h}$.

Proof. There are totally σ–prime ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A$ and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}_1 \cdots \mathfrak{a}_t \mathfrak{h} \subseteq \mathfrak{a}$, and there exists $\mathfrak{h}' \in \mathscr{L}(\sigma)$ and an index i such that $\mathfrak{a}_i\mathfrak{h}' \subseteq \mathfrak{a}$. Since $\mathfrak{c} + \mathfrak{a}_i\mathfrak{h}' \subseteq A$ is totally σ–prime (it is enough to take quotient by $\mathfrak{c}$, and use that $\frac{\mathfrak{a}_i}{\mathfrak{c}}\mathfrak{h} \subseteq \frac{A}{\mathfrak{c}}$ is totally σ–prime). Now, since $\mathfrak{a}$ is minimal over $\mathfrak{c}$, then $\mathfrak{c} = \mathfrak{c} + \mathfrak{a}_i\mathfrak{h}'$. $\square$

The existence of only finitely many totally σ–prime which are minimal over an ideal $\mathfrak{c} \subseteq A$ satisfying $\mathfrak{c} \notin \mathscr{L}(\sigma)$ can be proved whenever we impose together the conditions that appear in Proposition (4.14.) and in Theorem (4.15.).

The existence of minimal totally σ–prime ideals is assured whenever σ is jansian; but in this case if A totally σ–noetherian and σ jansian then that $\mathscr{L}(\sigma) = \{\mathfrak{h}' \subseteq A \mid \mathfrak{h} \subseteq \mathfrak{h}'\}$ for some finitely generated idempotent ideal $\mathfrak{h} \subseteq A$. Since $\mathfrak{h}$ is generated by an idempotent element, there exists a decomposition $A = eA \oplus (1-e)A$. In this case, an A–module M is σ–torsionfree whenever M is a $(1-e)A$–module, and σ–torsion whenever it is an eA–module. Therefore, if $\mathfrak{a} \subseteq A$ is totally σ–prime, then $\mathfrak{a} = e\mathfrak{a} \oplus (1-e)\mathfrak{a}$, $\mathrm{Cl}_{\sigma}^{A}(\mathfrak{a}) = eA \oplus (1-e)\mathfrak{a}$, and $\dfrac{A}{\mathrm{Cl}_{\sigma}^{A}(\mathfrak{a})} = 0 \oplus \dfrac{(1-e)A}{(1-e)\mathfrak{a}}$.

4.3 Ring extensions

For any ring map $f : A \longrightarrow B$ and any hereditary torsion theory σ in **Mod**–A there exists a hereditary torsion theory $\overline{\sigma}$ in **Mod**–B whose Gabriel filter is $\mathscr{L}(\overline{\sigma}) = \{\mathfrak{h} \subseteq B \mid f^{-1}(\mathfrak{h}) \in \mathscr{L}(\sigma)\}$. If there is no confusion we can represent $\overline{\sigma}$ simply as σ.

Let us consider the particular case of the ring extension $A \longrightarrow A[X]$, the polynomial ring extension.

Lemma. 4.19.
*Let σ be a hereditary torsion theory in **Mod**–A, and $\overline{\sigma}$ be the induced hereditary torsion theory in **Mod**–$A[X]$. For any ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a} \notin \mathscr{L}(\sigma)$, the following statements are equivalent:*

(a) $\mathfrak{a} \subseteq A$ is totally σ–prime.

(b) $\mathfrak{a}[X] \subseteq A[X]$ is totally $\overline{\sigma}$–prime.

This is an extension of [28, Example 4].

Proof. Since $\mathfrak{a} \notin \mathscr{L}(\sigma)$, then $\mathfrak{a}[X] \notin \mathscr{L}(\overline{\sigma})$; in particular, $\overline{\sigma}$ is non–trivial in **Mod**–$A[X]$.

(a) $\Rightarrow$ (b). Let $\mathfrak{h} \in \mathscr{L}(\sigma)$ be an ideal such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime. Let $F, G \in A[X]$ such that $FG \in \mathfrak{a}[X]$, since $(\mathfrak{a} : \mathfrak{h})[X] \subseteq A[X]$ is prime, either $F \in (\mathfrak{a} : \mathfrak{h})[X]$ or $G \in (\mathfrak{a} : \mathfrak{h})[X]$, and either $F\mathfrak{h} \subseteq \mathfrak{a}[X]$, hence $F\mathfrak{h}[X] \subseteq \mathfrak{a}[X]$, or $G\mathfrak{h} \subseteq \mathfrak{a}[X]$, hence $F\mathfrak{h}[X] \subseteq \mathfrak{a}[X]$. In consequence, $\mathfrak{a}[X] \subseteq A[X]$ is totally $\overline{\sigma}$–prime.

(b) $\Rightarrow$ (a). Let $\mathfrak{h}' \in \mathscr{L}(\sigma)$ be an ideal associated to $\mathfrak{a}[X]$. Let $a, b \in A$ such that $ab \in \mathfrak{a}$, then $ab \in \mathfrak{a}[X]$, and either $a\mathfrak{h}' \subseteq \mathfrak{a}[X]$ or $b\mathfrak{h}' \subseteq \mathfrak{a}[X]$, hence either $a(\mathfrak{h}' \cap A) \subseteq \mathfrak{a}$ or $b(\mathfrak{h}' \cap A) \subseteq \mathfrak{a}$. □

This result holds, more in general, in one direction, if we consider an arbitrary ring extension.

Proposition. 4.20.
*Let $A \subseteq B$ be a ring extension, and σ be a hereditary torsion theory in **Mod**–A. For any totally $\overline{\sigma}$–prime ideal $\mathfrak{c} \subseteq B$ the ideal $\mathfrak{c} \cap A \subseteq A$ is totally σ–prime.*

Proof. First we observe that $\mathfrak{c} \cap A \notin \mathscr{L}(\sigma)$. On the other hand, let $\mathfrak{h} \in \mathscr{L}(\overline{\sigma})$ the associated ideal to $\mathfrak{c}$. Let $a, b \in A$ such that $ab \in \mathfrak{c} \cap A$; since $ab \in \mathfrak{c}$, we have either $a\mathfrak{h} \subseteq \mathfrak{c}$ or $b\mathfrak{h} \subseteq \mathfrak{c}$, hence either $a(\mathfrak{h} \cap A) \subseteq \mathfrak{c} \cap A$ or $b(\mathfrak{h} \cap A) \subseteq \mathfrak{c} \cap A$. □

For any ring map $f : A \longrightarrow B$ the result also holds. To prove it, we only need to show that it holds whenever f is surjective.

Proposition. 4.21.

*Let $f : A \longrightarrow B$ be a surjective map and σ be a hereditary torsion theory in **Mod**–A. For any ideal totally $\overline{\sigma}$–prime ideal $\mathfrak{c} \subseteq B$ then $f^{-1}(\mathfrak{c}) \subseteq A$ is totally σ–prime.*

Proof. First observe that $f^{-1}(\mathfrak{c}) \subseteq A$ does not belong to $\mathscr{L}(\sigma)$. Let $\mathfrak{h} \in \mathscr{L}(\overline{\sigma})$, the associated ideal to $\mathfrak{c} \subseteq B$. For any $a, b \in A$ such that $ab \in f^{-1}(\mathfrak{c})$, since $f(a)f(b) \in \mathfrak{c}$, then either $f(a)\mathfrak{h} \subseteq \mathfrak{c}$ or $f(b)\mathfrak{h} \subseteq \mathfrak{c}$. Therefore, either $f(af^{-1}(\mathfrak{h})) \subseteq f(a)\mathfrak{h} \subseteq \mathfrak{c}$, and $af^{-1}(\mathfrak{h}) \subseteq f^{-1}(\mathfrak{c})$, and, in the same way, we could obtain $bf^{-1}(\mathfrak{h}) \subseteq f^{-1}(\mathfrak{c})$. □

Corollary. 4.22.

*Let σ be a hereditary torsion theory in **Mod**–A, $\mathfrak{c} \subseteq A$ be an ideal and $\overline{\sigma}$ the induced hereditary torsion theory in **Mod**–$A/\mathfrak{c}$. The following statements hold:*

(1) For any totally $\overline{\sigma}$–prime ideal $\mathfrak{a}/\mathfrak{c} \subseteq A/\mathfrak{c}$ the ideal $\mathfrak{a} \subseteq A$ is totally σ–prime.

(2) For any totally σ–prime ideal $\mathfrak{a}$ such that $\mathfrak{c} \subseteq \mathfrak{a} \subseteq A$ we have $\mathfrak{a}/\mathfrak{c} \subseteq A/\mathfrak{c}$ is totally $\overline{\sigma}$–prime.

(3) There exists a bijective correspondence between totally $\overline{\sigma}$–prime ideals of $A/\mathfrak{c}$ and totally σ–prime ideals $\mathfrak{a} \subseteq A$ such that $\mathfrak{c} \subseteq \mathfrak{a} \subseteq A$.

Proof. Let us show a proof for (2). If $\mathfrak{c} \subseteq \mathfrak{a} \subseteq A$ is totally σ–prime then $\mathfrak{a} \notin \mathscr{L}(\sigma)$, hence $\mathfrak{a}/\mathfrak{c} \notin \mathscr{L}(\overline{\sigma})$. Let $\mathfrak{h} \in \mathscr{L}(\sigma)$ be an ideal such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime. We have $\mathfrak{c} \subseteq \mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{h})$, and $(\mathfrak{a} : \mathfrak{h})/\mathfrak{c} = (\mathfrak{a}/\mathfrak{c} : (\mathfrak{h} + \mathfrak{c})/\mathfrak{c}) \subseteq A/\mathfrak{c}$ is prime. Otherwise, $(\mathfrak{h} + \mathfrak{c})/\mathfrak{c} \in \mathscr{L}(\overline{\sigma})$ In conclusion, $\mathfrak{a}/\mathfrak{c} \subseteq A$ is totally $\overline{\sigma}$–prime. □

Lemma. 4.23. (Prime avoidance lemma)

*Let σ be a hereditary torsion theory in **Mod**–A and $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A$ be totally σ–prime ideals. For any ideal $\mathfrak{a} \subseteq \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_t$ there exist an index $i \in \{1, \ldots, t\}$ and an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{a}_i$.*

Proof. For any index i there exists an ideal $\mathfrak{h}_i \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a}_i : \mathfrak{h}_i) \subseteq A$ is prime. We have the inclusion $\mathfrak{a} \subseteq \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_t \subseteq (\mathfrak{a}_1 : \mathfrak{h}_1) \cup \cdots \cup (\mathfrak{a}_t : \mathfrak{h}_t)$, hence there is an index i such that $\mathfrak{a} \subseteq (\mathfrak{a}_i : \mathfrak{h}_i)$, and $\mathfrak{a}\mathfrak{h}_i \subseteq \mathfrak{a}_i$. □

The **radical** of an ideal $\mathfrak{c} \subseteq A$ is defined as the intersection of all prime ideals containing $\mathfrak{c}$. The behaviour of totally σ-prime ideals with respect to the radical is studied in the following results.

Lemma. 4.24.

Let $\mathfrak{a}, \mathfrak{c} \subseteq A$ *be ideals such that* $\mathfrak{a} \subseteq A$ *is totally* σ*–prime and* $\mathfrak{c} \subseteq \mathfrak{a}$*, there exists an ideal* $\mathfrak{h} \in \mathscr{L}(\sigma)$ *such that* $\operatorname{rad}(\mathfrak{c})\mathfrak{h} \subseteq \mathfrak{a}$.

Proof. Since $\mathfrak{a} \subseteq A$ is totally σ–prime, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime. Since $\mathfrak{c} \subseteq \mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{h})$, then $\operatorname{rad}(\mathfrak{c}) \subseteq (\mathfrak{a} : \mathfrak{h})$, and $\operatorname{rad}(\mathfrak{c})\mathfrak{h} \subseteq \mathfrak{a}$. □

The result can be enhanced as follows.

Proposition. 4.25.

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A$ *be totally* σ*–prime ideals, there exists an ideal* $\mathfrak{h} \in \mathscr{L}(\sigma)$ *such that* $\operatorname{rad}(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t)\mathfrak{h} \subseteq \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t$.

Proof. For any index i, since $\mathfrak{a}_i \subseteq A$ is totally σ–prime, there exists an ideal $\mathfrak{h}_i \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a}_i : \mathfrak{h}_i) \subseteq A$ is prime. If we define $\mathfrak{h} = \mathfrak{h}_1 \cap \cdots \cap \mathfrak{h}_t$, then

$$\begin{aligned}\operatorname{rad}(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t) &\subseteq (\mathfrak{a}_1 : \mathfrak{h}_1) \cap \cdots \cap (\mathfrak{a}_t : \mathfrak{h}_t) \\ &\subseteq (\mathfrak{a}_1 : \mathfrak{h}) \cap \cdots (\mathfrak{a}_t : \mathfrak{h}) = (\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t : \mathfrak{h}).\end{aligned}$$

Therefore, $\operatorname{rad}(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t)\mathfrak{h} \subseteq \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_t$. □

Let σ be a hereditary torsion theory in **Mod**–A, let $\Sigma \subseteq A$ be a multiplicatively closed subset, and $\lambda : A \longrightarrow \Sigma^{-1}A$ the canonical map. For any ideal $\mathfrak{a} \subseteq A$ let $\Sigma^{-1}\mathfrak{a} = \lambda(\mathfrak{a})\Sigma^{-1}A$, and $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) = \{a \in A \mid \text{ there exists } t \in \Sigma \text{ such that } at \in \mathfrak{a}\} = \lambda^{-1}(\Sigma^{-1}\mathfrak{a})$. There exists a hereditary torsion theory $\overline{\sigma}$ in **Mod**–$\Sigma^{-1}A$ whose Gabriel filter is $\mathscr{L}(\overline{\sigma}) = \{\mathfrak{c} \subseteq \Sigma^{-1}A \mid \lambda^{-1}(\mathfrak{c}) \in \mathscr{L}(\sigma)\}$.

Lemma. 4.26.

If $\mathfrak{a} \subseteq A$ *is a totally* σ*–prime ideal such that* $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) \notin \mathscr{L}(\sigma)$*, then* $\Sigma^{-1}\mathfrak{a} \subseteq \Sigma^{-1}A$ *is totally* $\overline{\sigma}$ *prime.*

Proof. Since $\mathfrak{a} \subseteq A$ is totally σ–prime, then $\Sigma^{-1}\mathfrak{a} \notin \mathscr{L}(\overline{\sigma})$ because $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) \notin \mathscr{L}(\sigma)$. There exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(\mathfrak{a} : \mathfrak{h}) \subseteq a$ is prime, and we have $\Sigma^{-1}\mathfrak{h} \in \mathscr{L}(\overline{\sigma})$. For any $a, b \in A$ if

$(a/1)(b/1) \in \Sigma^{-1}\mathfrak{a}$, there exists $t \in \Sigma$ such that $abt \in \mathfrak{a}$; hence either $a\mathfrak{h} \subseteq \mathfrak{a}$ or $bt\mathfrak{h} \subseteq \mathfrak{a}$. In consequence, either $\lambda(a)\Sigma^{-1}\mathfrak{h} \subseteq \Sigma^{-1}\mathfrak{a}$ or $\lambda(b)\Sigma^{-1}\mathfrak{h} \subseteq \mathfrak{a}$, and $\Sigma^{-1}\mathfrak{a} \subseteq \Sigma^{-1}A$ is totally $\overline{\sigma}$–prime.

□

Proposition. 4.27.

*Let σ be a hereditary torsion theory in **Mod**–A, and $\Sigma \subseteq A$ be a multiplicatively closed subset. There exists a bijective correspondence between totally $\overline{\sigma}$–prime ideals in $\Sigma^{-1}A$ and totally σ–prime ideals $\mathfrak{a} \subseteq A$ such that $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) \notin \mathscr{L}(\sigma)$.*

Proof. Let $\mathfrak{c} \subseteq \Sigma^{-1}A$ be a totally $\overline{\sigma}$–prime ideal, and $\mathfrak{h} \in \mathscr{L}(\overline{\sigma})$ such that $(\mathfrak{c} : \mathfrak{h}) \subseteq \Sigma^{-1}A$ is prime. Let $\mathfrak{p} \subseteq A$ be a prime ideal such that $\Sigma^{-1}\mathfrak{p} = (\mathfrak{c} : \mathfrak{h})$. Hence $\Sigma^{-1}\mathfrak{p}\mathfrak{h} \subseteq \mathfrak{c}$, and $\mathfrak{p}\lambda^{-1}(\mathfrak{h}) = \lambda^{-1}(\Sigma^{-1}\mathfrak{p}\mathfrak{h}) \subseteq \lambda^{-1}(\mathfrak{c})$. Therefore, $\mathfrak{p} \subseteq (\lambda^{-1}(\mathfrak{c}) : \lambda^{-1}(\mathfrak{h}))$. Otherwise, if $x \in (\lambda^{-1}(\mathfrak{c}) : \lambda^{-1}(\mathfrak{h}))$, then $x\lambda^{-1}(\mathfrak{h}) \subseteq \lambda^{-1}(\mathfrak{c})$, and $\lambda(x)\mathfrak{h} \subseteq \mathfrak{c}$. Hence $\lambda(x) \in (\mathfrak{c} : \mathfrak{h})$, and $x \in \lambda^{-1}(\mathfrak{c} : \mathfrak{h}) = \mathfrak{p}$. In conclusion, $\lambda^{-1}(\mathfrak{c}) \subseteq A$ is totally σ–prime and $\mathrm{Cl}^A_\Sigma(\lambda^{-1}(\mathfrak{c})) \notin \mathscr{L}(\sigma)$.

The bijective correspondence is obvious. □

Remark. 4.28.

To get the condition $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) \notin \mathscr{L}(\sigma)$, it is sufficient to show that $\mathrm{Cl}^A_\Sigma(\mathfrak{a}) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$; i.e., that the module $\mathrm{Cl}^A_\Sigma(\mathfrak{a})/\mathfrak{a}$ is σ–torsion.

Proposition. 4.29.

Let $\mathfrak{a} \subseteq A$ be a totally σ–prime ideal. If $\mathfrak{p} = \mathrm{Cl}^A_\sigma(\mathfrak{a})$, then $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$. In this situation, $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}}$–prime.

Proof. Since $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$, we only need to show that $\mathfrak{a} \notin \mathscr{L}(\sigma_{A\setminus\mathfrak{p}})$. In the contrary, if $\mathfrak{a} \in \mathscr{L}(\sigma_{A\setminus\mathfrak{p}})$, then $\mathfrak{a} \nsubseteq \mathfrak{p}$, which is a contradiction. □

Compare the next result with Theorem (4.15.).

Proposition. 4.30.

Let A be a σ–noetherian ring and $\mathfrak{a} \subseteq A$ be a totally σ–prime ideal, then there exist only finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathscr{K}(\sigma)$ which are minimal over $\mathfrak{a}$. In addition, $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}_i}$–prime, for any index $i = 1, \ldots, t$.

Proof. For any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ containing $\mathfrak{a}$ we have $\mathfrak{a} \subseteq \mathrm{Cl}_{\sigma}^{A}(\mathfrak{a}) \subseteq \mathfrak{p}$, and there are only finitely many prime ideals in $\mathscr{K}(\sigma)$ containing $\mathrm{Cl}_{\sigma}^{A}(\mathfrak{a})$ since R is σ–noetherian, hence the first part of this result holds. The second part is a consequence of the inclusions $\mathfrak{a} \subseteq \mathfrak{p}_i$ for any index $i = 1, \ldots, t$, and Proposition (4.27.). □

The converse of the last result does not hold; i.e., if A is σ–noetherian, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathscr{K}(\sigma)$ are minimal ideal over and ideal $\mathfrak{a} \subseteq A$ such that $\sigma = \sigma_{A\setminus\mathfrak{p}_1} \wedge \cdots \wedge \sigma_{A\setminus\mathfrak{p}_t}$, if $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}_i}$–prime for any index $i = 1, \ldots, t$, not necessarily $\mathfrak{a} \subseteq A$ is totally σ–prime.

Example. 4.31.
Let $A = \mathbb{Z}_6$, $\mathfrak{p}_1 = 2A$, $\mathfrak{p}_2 = 3A$, whose Gabriel filters are: $\mathscr{L}(\sigma_{A\setminus\mathfrak{p}_1}) = \{3A, A\}$ and $\mathscr{L}(\sigma_{A\setminus\mathfrak{p}_2}) = \{2A, A\}$. First we have $\mathscr{L}(\sigma) = \{A\}$, whenever $\sigma = \sigma_{A\setminus\mathfrak{p}_1} \wedge \sigma_{A\setminus\mathfrak{p}_2}$. The zero ideal $0 \subseteq A$ is not totally σ–prime, but it is totally $\sigma_{A\setminus\mathfrak{p}_i}$–prime for any index $i = 1, 2$.

In a natural way the following problem arises: How one can characterize ideals $\mathfrak{a} \subseteq A$ such that $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}}$–prime?

Observe that

(1) $\mathfrak{a} \subseteq A$ is not $\sigma_{A\setminus\mathfrak{p}}$–dense, hence $\mathfrak{a} \subseteq \mathfrak{p}$.

(2) Since $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}}$–prime, then $\mathfrak{q} := \mathrm{Cl}_{\sigma}^{A}(a) = (\mathfrak{a} : \mathfrak{h}) \subseteq A$ is prime, hence $\mathfrak{q} \subseteq \mathfrak{p}$, and $\mathfrak{h} \nsubseteq \mathfrak{p}$; in consequence, $(\mathfrak{a} : \mathfrak{q}) \nsubseteq \mathfrak{p}$.

(3) The following relationship holds: $(\mathfrak{a} : (\mathfrak{a} : \mathfrak{q})) = \mathfrak{q}$.

The converse also holds, i.e., given an ideal $\mathfrak{a} \subseteq A$ and a prime ideal $\mathfrak{p} \subseteq A$, if $\mathfrak{a} \nsubseteq \mathfrak{p}$, and $\mathfrak{q} := (\mathfrak{a} : (\mathfrak{a} : \mathfrak{q}))$ for some prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, and $(\mathfrak{a} : \mathfrak{q}) \nsubseteq \mathfrak{p}$, then $\mathfrak{a} \subseteq A$ is totally $\sigma_{A\setminus\mathfrak{p}}$–prime.

4.4 Prime submodules

For any A–module M and any submodule, we have the following definitions:

- $N \subseteq M$ is **prime** if $N \neq M$ and for any $m \in M$ and $a \in A$, if $ma \in N$, then either $m \in N$ or $Ma \subseteq N$, i.e., $a \in (N : M)$.

- $N \subseteq M$ is **σ–prime** if $\overline{N} = \mathrm{Cl}_{\sigma}^{M}(N) \neq M$ and for any $m \in M$ and $a \in A$, if $ma \in N$, then either $m \in \overline{N}$ or $Ma \subseteq \overline{N}$, i.e., $a \in (\overline{N} : M)$.
- $N \subseteq M$ is **totally σ–prime** if for every $\mathfrak{k} \in \mathscr{L}(\sigma)$ we have $M\mathfrak{k} \nsubseteq N$, i.e., M/N is not totally σ–torsion, and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for any $m \in M$ and $a \in A$, if $ma \in N$, then either $m \in (N : \mathfrak{h})$ or $Ma \subseteq (N : \mathfrak{h})$, i.e., $a \in (\overline{N} : M)$.

The module M is **prime** (resp. **totally σ–prime**, **σ–prime**) whenever $0 \subseteq A$ is a prime (resp. totally σ–prime, σ–prime) submodule.

Note the difference between totally σ–prime ideal and totally σ–prime submodule, and that in both cases we have imposed the condition that the quotient be totally σ–torsion.

Let us analyse the different notions of prime submodule.

σ–prime submodules

Lemma. 4.32.
If $N \subseteq M$ is a submodule such that $\overline{N} \neq M$, the following statements are equivalent:
(a) $N \subseteq M$ is a σ–prime submodule.
(b) $\frac{N}{\sigma N} \cong \frac{\sigma M + N}{\sigma M} \subseteq \frac{M}{\sigma M}$ is a σ–prime submodule.
(c) $\overline{N} \subseteq M$ is prime.

Proof. For any submodule $N \subseteq M$ let $\widetilde{N} = \frac{\sigma M + N}{\sigma M}$, and $\widetilde{M} = M/\sigma M$.

(a) $\Rightarrow$ (b). Let $\widetilde{m} \in \widetilde{M}$ and $a \in A$; if $\widetilde{m}a \in \widetilde{N}$, there exist $t \in \sigma M$ and $n \in N$ such that $\widetilde{m}a = \widetilde{t+n}$. Let $\mathfrak{h} = \mathrm{Ann}(t) \in \mathscr{L}(\sigma)$, then $ma\mathfrak{h} = n\mathfrak{h} \in N$, hence either $m \in \overline{N}$ or $Ma\mathfrak{h} \subseteq \overline{N}$, and $Ma \in \overline{N}$. Therefore either $\widetilde{m} \in \frac{\overline{N}}{\sigma M}$ or $\widetilde{M}a \in \frac{\overline{N}}{\sigma M}$.

(b) $\Rightarrow$ (a). Let $ma \in N$, then $\widetilde{m}a \in \widetilde{N}$, and either $\widetilde{m} \in \overline{\widetilde{N}} = \frac{\overline{N}}{\sigma M}$ or $\widetilde{M}a \subseteq \overline{\widetilde{N}}$. Therefore, either $m \in \overline{N}$ or $Ma \subseteq \overline{N}$.

(a) $\Rightarrow$ (c). Let $m \in M$ and $a \in A$ such that $ma \in \overline{N}$, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $ma\mathfrak{h} \subseteq N$, then either $m \in \overline{N}$ or $Ma\mathfrak{h} \subseteq \overline{N}$, hence $Ma \subseteq \overline{N}$.

(c) $\Rightarrow$ (a). Let $m \in M$ and $a \in A$ such that $ma \in N \subseteq \overline{N}$, hence either $m \in \overline{N}$ or $Ma \subseteq \overline{N}$, and $N \subseteq M$ is σ–prime. □

Corollary. 4.33.

Let $N \subseteq M$ be a σ–closed submodule, the following statements are equivalent:

(a) $N \subseteq M$ is σ–prime.

(b) $N \subseteq M$ is prime.

Lemma. 4.34.

Let $N \subseteq M$ be a submodule; always we have $\overline{(N:M)} \subseteq (\overline{N}:M)$, and if M is finitely generated, then we have the equality.

Proof. If $a \in \overline{(N:M)}$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $a\mathfrak{h} \subseteq (N:M)$, hence $Ma\mathfrak{h} \subseteq N$, and $a \in (\overline{N}:M)$.

Otherwise, if $a \in (\overline{N}:M)$, then $Ma \subseteq \overline{N}$, and since M is finitely generated, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $Ma\mathfrak{h} \subseteq N$, hence $a \in \overline{(N:M)}$. □

Proposition. 4.35.

If $N \subseteq M$ is a σ–prime submodule, and M is finitely generated, then $(N:M) \subseteq A$ is σ–prime.

Proof. Let $a, b \in A$ such that $ab \in (N:M)$. If $a \notin \overline{(N:M)} = (\overline{N}:M)$, for every $\mathfrak{h} \in \mathscr{L}(\sigma)$ we have $a\mathfrak{h} \nsubseteq (N:M)$, hence $Ma\mathfrak{h} \nsubseteq N$. Since M is finitely generated, then $Ma \nsubseteq \overline{N}$. Since $Mab \subseteq N$, and $Ma \nsubseteq \overline{N}$, then $Mb \subseteq \overline{N}$, hence $b \in (\overline{N}:M) = \overline{(N:M)}$. □

Totally σ–prime submodules

First we show that totally σ–prime submodules can be characterized through prime submodules. Indeed, we have:

Proposition. 4.36.

Let $N \subseteq M$ be a submodule such that M/N is not totally σ–torsion, the following statements are equivalent:

(a) $N \subseteq M$ is totally σ–prime.

(b) There exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(N:\mathfrak{h}) \subseteq M$ is prime.

(c) There exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\overline{N} = (N:\mathfrak{h})$, and $\overline{N} \subseteq M$ is prime.

In this case, $(N:\mathfrak{h}')\subseteq(N:\mathfrak{h})$, for any $\mathfrak{h}'\in\mathscr{L}(\sigma)$, and $(N:\mathfrak{h}')=(N:\mathfrak{h})$ whenever $\mathfrak{h}'\subseteq\mathfrak{h}$.

Proof. (a) ⇒ (b). Let $\mathfrak{h}\in\mathscr{L}$ satisfying that for any $m\in M$ and $a\in A$ such that $ma\in N$ then either $m\mathfrak{h}\subseteq N$ or $Ma\mathfrak{h}\subseteq N$. If $ma\in(N:\mathfrak{h})$, then $ma\mathfrak{h}\subseteq N$, and either $m\mathfrak{h}\subseteq N$ or $Ma\mathfrak{h}\subseteq N$. Therefore, either $m\in(N:\mathfrak{h})$ or $Ma\subseteq(N:\mathfrak{h})$.

(b) ⇒ (a). Let $ma\in N\subseteq(N:\mathfrak{h})$, then either $m\in(N:\mathfrak{h})$ or $Ma\subseteq(N:\mathfrak{h})$, and $N\subseteq M$ is totally σ–prime.

Now we claim that $(N:\mathfrak{h})=\overline{N}$. Indeed, we only need to prove the inclusion $\overline{N}\subseteq(N:\mathfrak{h})$. If $m\in\overline{N}$, there exists $\mathfrak{d}\in\mathscr{L}(\sigma)$ such that $m\mathfrak{d}\subseteq N$, hence either $m\mathfrak{h}\subseteq N$ or $M\mathfrak{d}\mathfrak{h}\subseteq N$; in the second case M/N is totally σ–torsion; therefore, $m\in(N:\mathfrak{h})$, and the claim holds. In consequence, for any $\mathfrak{h}''\in\mathscr{L}(\sigma)$ we have $(N:\mathfrak{h}'')\subseteq(N:\mathfrak{h})$; and if $\mathfrak{h}'\in\mathscr{L}(\sigma)$ satisfies $\mathfrak{h}'\subseteq\mathfrak{h}$, then $(N:\mathfrak{h}')=(N:\mathfrak{h})=\overline{N}$. □

Observe that the condition $\overline{N}=(N:\mathfrak{h})$ for some $\mathfrak{h}\in\mathscr{L}(\sigma)$ is not enough to get that $N\subseteq M$ is totally σ–prime; it is necessary that, in addition, $\overline{N}\subseteq M$ is prime.

Remark. 4.37.

Similarly to the case of totaly σ–prime ideals, we may establish the result of Lemma (4.4.) for modules. Therefore, if $N\subseteq M$ is a totally σ–prime submodule, we can take $\mathfrak{h}=\mathrm{Ann}(\sigma(M/N))=(N:\overline{N})$, and it satisfies $\overline{N}=(N:\mathfrak{h})\subseteq M$ is a prime submodule.

Corollary. 4.38.

Let $N\subseteq M$ be a σ–closed submodule, the following statements are equivalent:

(a) $N\subseteq M$ is totally σ–prime.

(b) $N\subseteq M$ is prime.

The following results are immediate from the definition.

Lemma. 4.39.

*Let $\sigma_1\leq\sigma_2$ hereditary torsion theories in **Mod**–A. If $N\subseteq M$ is a totally σ_1–prime submodule, and $\mathrm{Cl}^M_{\sigma_2}(N)\neq M$, then $N\subseteq M$ is totally σ_2–prime.*

Lemma. 4.40.

Let $N\subseteq M$ be a submodule such that M/N is not totally σ–torsion, the following statements are equivalent:

(a) $N \subseteq M$ *is totally* σ*–prime.*

(b) *There exists* $\mathfrak{h} \in \mathscr{L}(\sigma)$ *such that for every* $L \subseteq M$ *and* $\mathfrak{a} \subseteq A$*, if* $L\mathfrak{a} \subseteq N$*, then either* $L\mathfrak{h} \subseteq N$ *or* $M\mathfrak{a}\mathfrak{h} \subseteq N$*.*

There is also a relationship between totally σ–prime submodules and totally σ–prime ideals. Remember that such a relationship does not exist in the case of σ–prime submodule, where we needed to impose the extra condition that the ambient module was finitely generated.

Lemma. 4.41.

Let $N \subseteq M$ *be a totally* σ*-prime submodule, then* $(N : M) \subseteq A$ *is a totally* σ*–prime ideal.*

Proof. Let $a, b \in A$ such that $ab \in (N : M)$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(N : \mathfrak{h}) \subseteq M$ is prime. If $a\mathfrak{h} \nsubseteq (N : M)$, then $Ma\mathfrak{h} \nsubseteq N$. Let $m \in M$ such that $ma\mathfrak{h} \nsubseteq N$. Since $mab \in M$, then either $ma\mathfrak{h} \subseteq N$ or $Mb\mathfrak{h} \subseteq N$. In consequence, $Mb\mathfrak{h} \subseteq N$, and $b\mathfrak{h} \subseteq (N : M)$. □

There a method to build totally σ–prime submodules from a given prime σ–closed submodule as the following proposition shows.

Proposition. 4.42.

Let $N \subseteq M$ *be a totally* σ*–prime submodule, then for any* $\mathfrak{h}' \in \mathscr{L}(\sigma)$ *we have that* $N\mathfrak{h}' \subseteq M$ *is totally* σ*-prime.*

Proof. Let $ma \in N\mathfrak{h}' \subseteq N$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(N : \mathfrak{h}) \subseteq M$ is prime, hence either $m\mathfrak{h} \subseteq N$ or $Ma\mathfrak{h} \subseteq N$. In consequence, either $m\mathfrak{h}\mathfrak{h}' \subseteq N$ or $Ma\mathfrak{h}\mathfrak{h}' \subseteq N$. □

Proposition. 4.43.

Let $N \subseteq M$ *be a submodule such that* M/N *is not totally* σ*–torsion, the following statements are equivalent:*

(a) $N \subseteq M$ *is totally* σ*–prime.*

(b) $\overline{N} \subseteq M$ *is prime and there exists* $\mathfrak{h} \in \mathscr{L}(\sigma)$ *such that* $(N : \mathfrak{h}') \subseteq (N : \mathfrak{h})$ *for any* $\mathfrak{h}' \in \mathscr{L}(\sigma)$*.*

Proof. We only need to prove that (b) ⇒ (a). Let $ma \in N \subseteq \overline{N}$, hence either $m \in \overline{N}$ or $Ma \subseteq \overline{N}$. If $m \in \overline{N}$, there exists $\mathfrak{h}' \in \mathscr{L}$ such that $a\mathfrak{h}' \subseteq N$, hence $m \in (N : \mathfrak{h}') \subseteq (N : \mathfrak{h})$. If $Ma \in \overline{N}$, for every $x \in M$ there exists $\mathfrak{h}_x \in \mathscr{L}(\sigma)$ such that $xa\mathfrak{h}_x \subseteq N$, hence $xa \in (N : \mathfrak{h}_x) \subseteq (N_{\mathfrak{h}})$. In conclusion, $N \subseteq M$ is totally σ–prime. □

This is a generalization of [50, Proposition 2.17].

In addition we have the following result that characterizes totally σ–prime submodules.

Proposition. 4.44.

Let $N \subseteq M$ be a totally σ–prime submodule, and $\mathfrak{h}' \in \mathscr{L}(\sigma)$. Then any submodule $N' \subseteq M$ such that $\overline{N}\mathfrak{h}' \subseteq N' \subseteq \overline{N}$ is totally σ–prime in M.

Proof. Let $m \in M$ and $a \in A$ such that $ma \in N' \subseteq \overline{N}$, and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(N : \mathfrak{h}) \subseteq M$ is prime. Then either $m\mathfrak{h} \subseteq N$ or $Ma\mathfrak{h} \subseteq N$. In consequence either $m\mathfrak{h}\mathfrak{h}' \subseteq N\mathfrak{h}' \subseteq N'$ or $Ma\mathfrak{h}\mathfrak{h}' \subseteq N\mathfrak{h} \subseteq N'$. □

In consequence, for any A–module M if there exists a totally σ–prime submodule $N \subseteq M$, then $\overline{N} \subseteq M$ is a prime submodule, and conversely, for any prime σ–closed submodule $\overline{N} \subseteq M$ and any ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$, then $\overline{N}\mathfrak{h} \subseteq M$ is totally σ–prime. We have shown that, essentially, these are the totally σ-prime submodules of M; see Proposition (4.44.).

The totally σ–prime submodules also characterize a particular class of modules: the totally σ–noetherian modules. Let us show an extension of Cohen–like's theorem as appears in [32, Corollary 3.5].

Theorem. 4.45. (Cohen–like theorem)

Let M be a totally σ–finitely generated A–module, the following statements are equivalent:

(a) M is totally σ–noetherian.

(b) Every totally σ–prime submodule is totally σ–finitely generated.

(c) Every σ–prime submodule is totally σ–finitely generated.

(d) Every σ–closed prime submodule $P \subseteq M$ is totally finitely generated.

Proof. (a) ⇒ (b) ⇒ (c) ⇒ (d) are obvious.

(d) ⇒ (a). Let $\Gamma = \{N \subseteq M \mid N \text{ is not totally } \sigma\text{–finitely generated}\}$. This set does not contain M. If it is non–empty, since it is inductive, by Zorn's lemma it has maximal elements. Let $N \in \Gamma$

maximal. We show $N \subseteq M$ is prime. Indeed, if there are $m \in M \setminus N$ and $a \in A$ such that $ma \in N$, and $Ma \nsubseteq N$, we consider the short exact sequence: $0 \to (N : a)a \xrightarrow{\alpha} N \times Ma \xrightarrow{\beta} N + Ma \to 0$, being $\alpha(x) = (x, x)$ and $\beta(y, z) = y - z$, for obvious elements x, y and z. By the maximality of N we obtain that $(N : a) \supsetneq N$, hence it is totally σ–finitely generated, hence $(N : a)a$ is; and similarly $N + Ma$ is totally σ–finitely generated. Therefore, $N \times Ma$ is, and N is totally σ–finitely generated, which is a contradiction.

In addition, every N maximal in Γ is σ–closed. Indeed, if $N \subsetneq \overline{N}$, there exists $m \in \overline{N} \setminus N$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $m\mathfrak{h} \subseteq N$, hence $M\mathfrak{h} \subseteq N$, and N is totally σ–finitely generated as M is, which is a contradiction. In conclusion, every maximal element in Γ is a prime σ–closed submodule of M that, by hypothesis, is totally σ–finitely generated; thus, Γ must be empty and M is totally σ–noetherian. □

Observe that, in this case, we have no any extra condition on the hereditary torsion theory σ. In the case in which $M = A$ we have that σ must of finite type; and under this condition we have more characterizations of totally σ–noetherian modules.

Corollary. 4.46.

*Let σ be a finite type hereditary torsion theory in **Mod**–A. If M is a totally σ–finitely generated A–module, the following statements are equivalent:*

(a) M is totally σ–noetherian.

(e) $M\mathfrak{a} \subseteq M$ is totally σ–finitely generated for every totally σ–prime ideal $\mathfrak{a} \in A$.

(f) $M\mathfrak{p} \subseteq M$ is totally σ–finitely generated for every $\mathfrak{p} \in \mathcal{K}(\sigma)$.

Proof. (a) ⇒ (e), and (e) ⇒ (f) are obvious.

(f) ⇒ (a). Following the proof in the above theorem, we find that if $N \in \Gamma$ is maximal, then $(N : M) \subseteq A$ is prime and σ–closed, hence $(N : M) \in \mathcal{K}(\sigma)$. Therefore, $N(N : M) \subseteq N$ is totally σ–finitely generated.

Let us call $\mathfrak{p} := (N : M)$. By hypothesis M is totally σ–finitely generated, hence there are $m_1, \ldots, m_t \in M$ and $\mathfrak{h}_1 \in \mathcal{L}(\sigma)$ such that $M\mathfrak{h}_1 \subseteq (m_1, \ldots, m_t) \subseteq M$. In consequence, using Remark (4.9.),

$$(N : M) \subseteq (N : (m_1, \ldots, m_t)) \subseteq (N : M\mathfrak{h}_1) = (N : M).$$

Therefore, $\mathfrak{p} = (N : (m_1, \ldots, m_t)) = \cap_{i=1}^{t}(N : m_i)$, and there exists an index $i \in \{1, \ldots, t\}$ such that $\mathfrak{p} = (N : m_i)$. Obviously $m_i \in M \setminus N$, hence $N + m_iA$ is totally σ–finitely generated, and there exist $n_1, \ldots, n_s \in N$ and $\mathfrak{h}_2 \in \mathscr{L}(\sigma)$ such that $(N + m_iA)\mathfrak{h}_2 \subseteq (n_1, \ldots, n_s, m_i) \subseteq N + m_iA$. Therefore, $N\mathfrak{h}_2 \subseteq (n_1, \ldots, n_s) + m_i\mathfrak{p} \subseteq (n_1, \ldots, n_s) + M\mathfrak{p}$. Since $M\mathfrak{p}$ is totally σ–finitely generated, there exist $x_1, \ldots, x_r \in M$ and $\mathfrak{h}_3 \in \mathscr{L}(\sigma)$, that we can take finitely generated, such that $M\mathfrak{p}\mathfrak{h}_3 \subseteq (x_1, \ldots, x_r) \subseteq M\mathfrak{p}$. In consequence, we have:

$$N\mathfrak{h}_2\mathfrak{h}_3 \subseteq (n_1, \ldots, n_s)\mathfrak{h}_3 + M\mathfrak{p}\mathfrak{h}_3 \subseteq (n_1, \ldots, n_s)\mathfrak{h}_3 + (x_1, \ldots, x_r) \subseteq N,$$

and N is totally σ–finitely generated, which is a contradiction. □

This is a generalization of [50, Proposition 2.22], and [5, Proposition 4].

4.5 Modules and ring extensions

First we consider the case of a module map, and the behaviour of totally prime submodules; the proof is straightforward.

Proposition. 4.47.

(1) Let $f : M_1 \longrightarrow M_2$ be a module map. If $N_2 \subseteq M_2$ is a totally σ–prime submodule, and $M_1/f^{-1}(N_2)$ is not totally σ–torsion then $f^{-1}(N_2) \subseteq M_1$ is totally σ–prime.

(2) Let $f : M_1 \longrightarrow M_2$ be an epimorphism, there is a bijective correspondence between totally σ–prime submodules of M_2 and totally σ–prime submodules $N \subseteq M$ such that $\mathrm{Ker}(f) \subseteq N$.

A classical result says that an A–module M is prime if, and only if, $\mathrm{Ann}(M) \subseteq A$ is prime and M is torsionfree over $A/\mathrm{Ann}(M)$. We try to recover a similar result in the case of totally σ–prime modules.

Let M be a totally σ–prime A–module, then $\mathrm{Ann}(M) \subseteq A$ is totally σ–prime, hence $\overline{\mathrm{Ann}(M)} \subseteq A$ is prime, see Lemma (4.4.). In consequence, M, which is an A–module, is naturally an $A/\mathrm{Ann}(M)$–module, and $M/\sigma M$ is an $A/\overline{\mathrm{Ann}(M)}$–module. Since $A/\overline{\mathrm{Ann}(M)}$ is an integral domain, we obtain that M is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module whenever, for any $\overline{m} = m + \sigma M \in M/\sigma M$ and any

$\overline{a} = a + \overline{\mathrm{Ann}(M)} \in A/\overline{\mathrm{Ann}(M)}$, if $\overline{m}\,\overline{a} = 0$, then either $\overline{m} = 0$ or $\overline{a} = 0$. This means that there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $m\mathfrak{h} = 0$ and $a\mathfrak{h} \in \mathrm{Ann}(M)$, i.e., $\overline{a}\mathfrak{h} = 0$ in $A/\mathrm{Ann}(M)$.

Thus, the following definition appears in a natural way: an A–module M is **totally σ–torsionfree** whenever there exists an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for any $m \in M$ and any $a \in A$, if $ma = 0$, then either $m\mathfrak{h} = 0$ or $a\mathfrak{h} = 0$. See [50].

With this background, if $\overline{\sigma}$ is the hereditary torsion theory induced by σ in $A/\mathrm{Ann}(M)$, we have:

Proposition. 4.48.

Let M be an A–module which is not totally σ–torsion. The following statements are equivalent:

(a) M is totally σ–prime.

(b) $\mathrm{Ann}(M) \subseteq A$ is totally σ–prime and M is a totally $\overline{\sigma}$–torsionfree module over $A/\mathrm{Ann}(M)$.

Proof. (a) $\Rightarrow$ (b). Let $\mathfrak{h} = (0 : \sigma M) \in \mathscr{L}(\sigma)$ be an ideal such that $\sigma M = (0 : \mathfrak{h}) \subsetneq M$ is prime, see Proposition (4.36.). Let $m \in M$, $a \in A$, and $\overline{a} = a + \mathrm{Ann}(M)$, such that $m\overline{a} = 0$, then $ma = 0$, and either $m\mathfrak{h} = 0$ or $Ma\mathfrak{h} = 0$, hence $\overline{a}\mathfrak{h} = 0$, and M is totally $\overline{\sigma}$–torsionfree.

(b) $\Rightarrow$ (a). Let $\mathfrak{h} \in \mathscr{L}(\sigma)$ be an ideal such that $(\mathrm{Ann}(M) : \mathfrak{h}) \subseteq A$ is prime and either $m\mathfrak{h} = 0$ or $\overline{a}\mathfrak{h} = 0$ whenever $m\overline{a} = 0$. Let $m \in M$ and $a \in A$ such that $ma = 0$, then $m\overline{a} = 0$, hence either $m\mathfrak{h} = 0$ or $\overline{a}\mathfrak{h} = 0$, and $a\mathfrak{h} \in \mathrm{Ann}(M) = (0 : M)$. In conclusion, M is prime. □

Corollary. 4.49.

If M is totally σ–prime, then $M/\sigma M$ is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module.

Proof. If $\overline{m}\,\overline{a} = 0$, then $ma \in \sigma M$, and there exists $\mathfrak{h}' \in \mathscr{L}(\sigma)$ such that $ma\mathfrak{h}' = 0$. Therefore, since M is totally $\overline{\sigma}$–torsionfree, as $A/\mathrm{Ann}(M)$–module, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that either $m\mathfrak{h} = 0$ or $\overline{a}\mathfrak{h} = 0$. In the first case, $m \in \sigma M$, son $\overline{m} = 0$. In the second case, since $\overline{a}\mathfrak{h} = 0$, then $a\mathfrak{h} \subseteq \mathrm{Ann}(M)$, and $\overline{a} \in \overline{\mathrm{Ann}(M)}$, i.e., $\overline{a} = 0$. □

Observe that $\overline{m}$ represents the class of $m \in M$ in $M/\sigma M$, and if $a \in A$, then $\overline{a}$ represents either that class of a in $A/\mathrm{Ann}(M)$ or the class of A in $A/\overline{\mathrm{Ann}(M)}$ depending of the context.

The converse of this corollary does not necessarily hold as the following example shows, i.e., if M is not totally σ–torsion, $\overline{\mathrm{Ann}(M)} \subseteq A$ is totally σ–prime, and $M/\sigma M$ is torsionfree as $A/\overline{\mathrm{Ann}(M)}$–module, it may be that M is not totally σ–prime.

Example. 4.50.

Consider $A = \mathbb{Z}$, and $M = \mathbb{Z}_{2^\infty}$. Then $\mathrm{Ann}(M) = 0 \subsetneq \mathbb{Z}$ is prime. Take σ the hereditary torsion theory defined by Σ, the multiplicative subset of all integers whose prime factors are different of 2. In this case $\overline{\mathrm{Ann}(M)} = 0$, because $\mathbb{Z}$ is an integral domain. Otherwise, $M = \sigma M$, hence $M/\sigma M = 0$ is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module. But M is not totally σ–prime. Indeed, does not exist an ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for any $m \in M$ and any $a \in \mathbb{Z}$ such that $ma = 0$ then either $m\mathfrak{h} = 0$ or $Ma\mathfrak{h} = 0$. Take, for instance, the family of elements $m_t = \frac{1}{2^t} + \mathbb{Z}$ and $a_t = 2^t$, then there is not such an ideal $\mathfrak{h}$.

At this point we have two notions that are closely related:

(α) M is totally $\overline{\sigma}$–torsionfree over $A/\mathrm{Ann}(M)$.

(β) $M/\sigma M$ is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module.

Remark. 4.51.

First we note that if M is σ–torsionfree over A, then M is totally σ–torsion. If $\mathfrak{h} \in \mathscr{L}(\sigma)$ is the associated ideal to M, for any $x \in \sigma M$ there exists $\mathfrak{k} \in \mathscr{L}(\sigma)$ such that $x\mathfrak{k} = 0$, hence, either $x\mathfrak{h} = 0$ or $\mathfrak{k}\mathfrak{h} = 0$; in the second case we have $0 \in \mathscr{L}(\sigma)$, which is imposible. Therefore, $x \in (0 : \mathfrak{h})$, and $\sigma M \subseteq (0 : \mathfrak{h})$ is totally σ–torsion. In this case, if $\mathfrak{k} = (0 : \sigma M)$, we have $\sigma M = (0 : \mathfrak{k})$.

Now we may establish the announced equivalence.

Proposition. 4.52.

If M is a non totally σ–torsion module, the following statements are equivalent:

(a) M is totally $\overline{\sigma}$–torsionfree over $A/\mathrm{Ann}(M)$.

(b) σM is totally σ–torsion and $M/\sigma M$ is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module

Proof. (a) $\Rightarrow$ (b). Let $(m+\sigma M)(a+\overline{\mathrm{Ann}(M)})=0$; if $m+\sigma M\neq 0$, since $ma\in\sigma M$ we have $ma\mathfrak{h}=0$, hence either $m\mathfrak{h}\mathfrak{h}=0$, which is a contradiction, or $a\mathfrak{h}=0$, and $a\in\sigma A$, which implies $a+\overline{\mathrm{Ann}(M)}=0$.

(b) $\Rightarrow$ (a). Let $\mathfrak{h}=(0:\sigma M)$, for any $m\in M$ and $a\in A$ such that $m(a+\mathrm{Ann}(M))=0$, we have $ma\in\sigma M$; hence, we have $(m+\sigma M)(a+\overline{\mathrm{Ann}(M)})=0$; therefore, either $m+\sigma M=0$, which implies $m\mathfrak{h}=0$, or $a+\overline{\mathrm{Ann}(M)}=0$, and there exists $\mathfrak{k}\in\mathscr{L}(\sigma)$ such that $M\mathfrak{k}a=0$; i.e., $Ma\subseteq\sigma M$; hence $Ma\mathfrak{h}=0$, which implies $(a+\mathrm{Ann}(M))\mathfrak{h}=0$. □

It is possible to give a local characterization of totally σ–torsionfree modules in the particular case in which σ is half–centered. To do this we need that A satisfy an additional property.

Let $\sigma=\wedge\{\sigma_{A\setminus\mathfrak{p}}\mid\ \mathfrak{p}\in\mathscr{K}(\sigma)\}$ be a half–centered hereditary torsion theory. The ring A is **universally σ–integral** whenever A is $\sigma_{A\setminus\mathfrak{p}}$–torsionfree for every $\mathfrak{p}\in\mathscr{K}(\sigma)$. Observe that universally σ–integral rings are the natural generalization of integral domains, and they are characterized by the property: $\mathrm{Ann}(a)\subseteq\cap\{\mathfrak{p}\mid\ \mathfrak{p}\in\mathscr{K}(\sigma)\}$, the **$\sigma$–radical** of A, for any $a\in A$. We shall show that rings of this type satisfy two interesting properties.

Proposition. 4.53.

Let A be a ring, $\sigma\wedge\{\sigma_{A\setminus\mathfrak{p}}\mid\ \mathfrak{p}\in\mathscr{K}(\sigma)\}$ a half–centered hereditary torsion theory such that A is a universally σ–integral ring. For any A–module M, the following statements hold:

(I). If σM is totally σ–torsionfree, then $\overline{\mathrm{Ann}(M)}=\mathrm{Cl}^A_\sigma(\mathrm{Ann}(M))=\cap\{\mathrm{Cl}^A_{\sigma_{A\setminus\mathfrak{p}}}(\mathrm{Ann}(M))\mid\ \mathfrak{p}\in\mathscr{K}(\sigma)\}$.

(II). The following statements are equivalent:

(a) M is totally σ–torsionfree.

(b) M is totally $\sigma_{A\setminus\mathfrak{p}}$–torsionfree for any $\mathfrak{p}\in\mathscr{K}(\sigma)$.

Proof. (I). Obviously the inclusion we $\overline{\mathrm{Ann}(M)}=\wedge\mathrm{Cl}^A_\sigma(\mathrm{Ann}(M))$ holds. Otherwise, for any $a\in\wedge\mathrm{Cl}^A_\sigma(\mathrm{Ann}(M))$ and any $\mathfrak{p}\in\mathscr{K}(\sigma)$, there exists $\mathfrak{k}_{\mathfrak{k}}\not\subseteq\mathfrak{p}$ such that $a\mathfrak{h}_{\mathfrak{p}}\subseteq\mathrm{Ann}(M)$; i.e., $Ma\mathfrak{h}_{\mathfrak{p}}=0$. If $\mathfrak{h}=(0:\sigma M)$, then either $Ma\mathfrak{h}=0$; hence $x\in\overline{\mathrm{Ann}(M)}$, or $\mathfrak{h}_{\mathfrak{p}}\mathfrak{h}=0$, and $\mathfrak{h}\subseteq\sigma_{A\setminus\mathfrak{p}}=0$, which is a contradiction. Therefore, we have an equality.

(II).

(a) $\Rightarrow$ (b) is evident.

(b) $\Rightarrow$ (a). Let $m \in M$ and $a \in A$ such that $ma = 0$. Since M is totally $\sigma_{A\setminus\mathfrak{p}}$–torsionfree for every $\mathfrak{p} \in \mathscr{K}(\sigma)$, there exists $\mathfrak{h}_{\mathfrak{p}} \in \mathscr{L}(\sigma_{A\setminus\mathfrak{p}})$, and either $m\mathfrak{h}_{\mathfrak{p}} = 0$ or $a\mathfrak{h}_{\mathfrak{p}} = 0$. Since A is universally σ–integral then $a\mathfrak{h}_{\mathfrak{p}} = 0$ it is not possible. In consequence, $m\mathfrak{h}_{\mathfrak{p}} = 0$, and $\mathrm{Ann}(m) \nsubseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathscr{C}(\sigma)$, and $\mathrm{Ann}(m) \in \cap\{\mathscr{L}(\sigma_{A\setminus\mathfrak{p}}) \mid \mathfrak{p} \in \mathscr{K}(\sigma)\} = \mathscr{L}(\sigma)$. □

There is another approach to totally σ–prime modules based in the annihilator of the σ–torsion module, see Lemma (4.4.); in this case is not necessary to change the ring. Given a totally σ–prime module M, we have that σM is totally σ–torsion; moreover, if $\mathfrak{h} = (0 : \sigma M)$, then it satisfies:

(1) $\mathfrak{h} \in \mathscr{L}(\sigma)$, and

(2) $\sigma M = (0 : \mathfrak{h}) \subseteq M$ is a prime submodule.

Indeed, these two conditions characterize totally σ–prime modules.

Theorem. 4.54.

Given an A–module M, the following statements are equivalent:

(a) M is totally σ–prime.

(b) $\mathfrak{h} = (0 : \sigma M) \in \mathscr{L}(\sigma)$ (i.e., σM is totally σ–torsion), and $\sigma M = (0 : \mathfrak{h}) \subseteq M$ is a prime submodule (i.e., $M/\sigma M$ is prime).

Proof. We prove that (b) $\Rightarrow$ (a). Since $\sigma M \subseteq M$ is a prime submodules, then M is not totally σ–torsion. Otherwise, for any $m \in M$ and $a \in A$, if $ma = 0$ and $m \notin \sigma M$ (i.e., $m\mathfrak{h} \neq 0$), then $Ma \subseteq \sigma M$, and $M\mathfrak{h}a = 0$. □

There is another condition, consequence of (2) above:

(3) $(0 : M\mathfrak{h}) = ((0 : \mathfrak{h}) : M) \subseteq A$ is a prime ideal.

This condition (3) is not enough to assure that M is totally σ–prime; indeed, it is necessary to add to it the condition that appears in Corollary(4.49.): $M/\sigma M$ is a torsionfree $A/\overline{\mathrm{Ann}(M)}$–module.

5 Extensions

In this chapter we shall study some examples of ring constructions, used mainly to build ring examples to prove or disprove results in Commutative Algebra.

We'll consider in it hereditary torsion theories in the category **Mod**–A, and induce others in the categories of modules over the new rings; just note that the starting hereditary torsion theories, will usually be of finite type.

5.1 Idealization

The first example of construction of ring is the idealization, also called also called the **trivial extension** of A by M, introduced by Nagata in 1955.

Let A be a ring and M be an A–module, the **idealization** of M with respect to A is

$$B = M \rtimes A = \{(m, a) \mid m \in M, a \in A\}$$

with sum componentwise, and multiplication given by

$$(m_1, a_1)(m_2, a_2) = (m_1 a_2 + m_2 a_1, a_1 a_2).$$

In consequence, M can be identify with the ideal $M \rtimes 0 \subseteq M \rtimes A$, and A is isomorphic to $(M \rtimes A)/(M \rtimes 0)$. In the same way we may identify A with the subring $0 \rtimes A$ via the ring map $f : A \longrightarrow M \rtimes A$, defined $f(a) = (0, a)$.

For any hereditary torsion theory σ in **Mod**–A we have a hereditary torsion theory $f(\sigma)$ in **Mod**–$M \rtimes A$, whose Gabriel filter is:

$$\mathscr{L}(f(\sigma)) = \{\mathfrak{b} \subseteq M \rtimes A \mid f^{-1}(\mathfrak{b}) \in \mathscr{L}(\sigma)\}$$

Lemma. 5.1.
A basis of filter form $\mathscr{L}(f(\sigma))$ is $\{M\mathfrak{a} \rtimes \mathfrak{a} \mid \mathfrak{a} \in \mathscr{L}(\sigma)\}$.

Proof. Let $\mathfrak{b} \in \mathscr{L}(f(\sigma))$, then $\mathfrak{a} = f^{-1}(\mathfrak{b}) \in \mathscr{L}(\sigma)$. Otherwise, it is clear that $M\mathfrak{a} \rtimes \mathfrak{a} \in \mathscr{L}(f(\sigma))$, for any $a \in \mathfrak{a}$ since $(0, a) \in \mathfrak{b}$, then $M\mathfrak{a} \rtimes \mathfrak{a} = \langle (0, a) \mid a \in \mathfrak{a} \rangle \subseteq \mathfrak{b}$. □

Remark. 5.2.

Observe that if σ is a finite type hereditary torsion theory, since $\{M\mathfrak{a}\bowtie\mathfrak{a} \mid \mathfrak{a}\in\mathscr{L}(\sigma)\}$ is filter basis, if $\mathfrak{a}=\langle a_1,\ldots,a_t\rangle$, then $\mathfrak{a}(M\bowtie A)=((0,a_1),\ldots,(0,a_t))(M\bowtie A)\subseteq M\mathfrak{a}\bowtie\mathfrak{a}$ is finitely generated, and $f^{-1}(\mathfrak{a}(M\bowtie A))\supseteq\mathfrak{a}$, then $\mathfrak{a}(M\bowtie A)\in\mathscr{L}(f(\sigma))$, and $f(\sigma)$ is of finite type. In the contrary, if $f(\sigma)$ is of finite type, non–necessarily M is finitely generated as the following example shows.

Example. 5.3.

Let $A=\mathbb{Z}$ the ring of integer numbers and $M=\oplus_{\mathbb{N}}\mathbb{Z}_2=(\mathbb{Z}_2)^{(\mathbb{N})}$. Let us consider the multiplicative subset $\Sigma=\{2^t \mid t\in\mathbb{N}\}$, and the hereditary torsion theory σ_Σ. It is clear that σ_Σ is principal, hence of finite type. The induced hereditary torsion theory $f(\sigma_\Sigma)$ is also of finite type, since it is $\sigma_{f(\Sigma)}$, but M is not finitely generated.

Remark. 5.4.

We may answer the following question in positive: If $f(\sigma)$ is of finite type, must be σ of finite type?

We may consider a hereditary torsion theory σ in **Mod**–A such that $f(\sigma)$ is of finite type, then σ is of finite type. For any $\mathfrak{a}\in\mathscr{L}(\sigma)$ we consider $M\mathfrak{a}\bowtie\mathfrak{a}\in\mathscr{L}(f(\sigma))$, there is a finitely generated ideal $\mathfrak{b}\in\mathscr{L}(f(\sigma))$ such that $\mathfrak{b}\subseteq M\mathfrak{a}\bowtie\mathfrak{a}$. Since $\mathfrak{c}=f^{-1}(\mathfrak{b})\in\mathscr{L}(\sigma)$, and it is contained in $\mathfrak{a}$. We consider the following diagram

$$\begin{array}{ccccccc}
(M\mathfrak{c}\bowtie\mathfrak{c})\cap M & \rightarrowtail & \mathfrak{b}\cap M & \rightarrowtail & (M\mathfrak{a}\bowtie\mathfrak{a})\cap M & \rightarrowtail & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M\mathfrak{c}\bowtie\mathfrak{c} & \rightarrowtail & \mathfrak{b} & \rightarrowtail & M\mathfrak{a}\bowtie\mathfrak{a} & \rightarrowtail & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{M\mathfrak{c}\bowtie\mathfrak{c}}{(M\mathfrak{c}\bowtie\mathfrak{c})\cap M} & \rightarrowtail & \frac{\mathfrak{b}}{\mathfrak{b}\cap M} & \rightarrowtail & \frac{M\mathfrak{a}\bowtie\mathfrak{a}}{(M\mathfrak{a}\bowtie\mathfrak{a})\cap M} & \rightarrowtail & \frac{M\bowtie A}{M} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathfrak{c} & \rightarrowtail & \frac{\mathfrak{b}+M}{M} & \rightarrowtail & \mathfrak{a} & \rightarrowtail & A
\end{array}$$

Since $\mathfrak{a},\mathfrak{c}\in\mathscr{L}(\sigma)$, then $\frac{b+M}{M}\in\mathscr{L}(\sigma)$ and it is finitely generated.

Remark. 5.5.

Observe that if $\mathfrak{a}\subseteq A$ is finitely generated, non–necessarily $M\mathfrak{a}\bowtie\mathfrak{a}$ is finitely generated. Indeed, we have a short exact sequence

$$M\mathfrak{a} \rightarrowtail M\mathfrak{a} \rtimes \mathfrak{a} \twoheadrightarrow \mathfrak{a}$$

If M is also finitely generated, then $M\mathfrak{a}$ is finitely generated, and $M\mathfrak{a} \rtimes \mathfrak{a}$ also is.

The next result is related to the noetherian condition.

Proposition. 5.6.

Let A be a ring and M an A–module; the prime ideals of $M \rtimes A$ are of the shape: $M \rtimes \mathfrak{p}$, being $\mathfrak{p} \subseteq A$ a prime ideal.

Proof. Since $(M \rtimes 0)^2 = 0$, then $M \rtimes 0$ is contained in each prime ideal of $M \rtimes A$. Let $\mathfrak{q} \subseteq M \rtimes A$ be a prime ideal, since $M \rtimes 0 \subseteq \mathfrak{q}$, then $\mathfrak{q}$ is an homogeneous ideal, say $\mathfrak{q} = M \rtimes \mathfrak{p}$; we show that $\mathfrak{p} \subseteq A$ must be a prime ideal. Indeed, we have $\dfrac{M \rtimes A}{\mathfrak{q}} = \dfrac{M \rtimes A}{M \rtimes \mathfrak{p}} \cong 0 \rtimes \dfrac{A}{\mathfrak{p}}$; since it is an integral domain, then $\mathfrak{p} \subseteq A$ is prime. □

In consequence, if A is a noetherian ring and M is finitely generated, then $M \rtimes \mathfrak{p}$ is finitely generated for any prime ideal $\mathfrak{p} \subseteq A$, and by Cohen's theorem we have $M \rtimes A$ is a noetherian ring.

We may extend this result to consider a hereditary torsion theory σ in **Mod**–A whenever we consider the Cohen–like's theorem, see Theorem (4.13.).

Proposition. 5.7.

Given a totally σ–noetherian ring A, and a totally σ–finitely generated M we have that $M \rtimes A$ is totally $f(\sigma)$–noetherian.

As a consequence we have the following result, see [43].

Theorem. 5.8.

*Given a ring A, an A–module M and a finite type hereditary torsion theory σ in **Mod**–A, the following statements are equivalent:*

(a) $M \rtimes A$ is totally $f(\sigma)$–noetherian (resp. totally $f(\sigma)$–artinian).

(b) A is totally σ–noetherian (resp. totally σ–artinian) and M is totally σ–finitely generated.

In this way we may build examples of totally σ–noetherian rings; simply we can take M to the totally σ–noetherian; for example a totally σ–torsion module.

For any ideal $\mathfrak{a} \subseteq A$, since $\frac{M \times A}{M \rtimes \mathfrak{a}} \cong \frac{A}{\mathfrak{a}}$, then we have a bijective correspondence between Spec(A) and Spec($M \rtimes A$) induces bijective correspondences $\mathscr{K}(\sigma)$ and $\mathscr{K}(f(\sigma))$, and similarly for $\mathscr{Z}(\sigma)$ and $\mathscr{Z}(f(\sigma))$.

Idealization and principal ideal rings

Let us collect some results on PIRs and the idealization; the next three are well known.

Lemma. 5.9.

If $B = M \rtimes A$ is a principal ideal ring, then:

(1) A is a principal ideal ring.

(2) M is a principal A–module.

Lemma. 5.10.

Let $A = A_1 \times A_2$ be a product of rings, and M be an A–module, then

(1) $MA_1, MA_2 \subseteq M$ are submodules.

(2) MA_1 and MA_2 have no isomorphic simple subfactors.

(3) $M = MA_1 \oplus MA_2$.

(4) $M \rtimes A \cong (MA_1 \rtimes A_1) \times (MA_2 \rtimes A_2)$.

Lemma. 5.11.

If the ring $B = M \rtimes A$ has a decomposition $B = B_1 \times B_2$, then

(1) $MB_1 \oplus MB_2$ is a lattice decomposition of M.

(2) $A = AB_1 \times AB_2$ is a ring decomposition.

(3) $B_i = MB_i \rtimes AB_i$ for $i = 1, 2$.

On the other hand we have a decomposition theorem whenever the ring A is a PIR. If A is a PIR, there exists a decomposition $A = \prod_{i=1}^{t} A_i$ as a product of DIPs and SPIRs. Hence we have an isomorphism

$$M \rtimes A \cong \prod_{i=1}^{t} MA_i \rtimes A_i.$$

Therefore, the problem of studying when $M \ltimes A$ is a PIR is reduced to consider the case in which A is either a DIP or a PIR.

To determine when $M \ltimes A$ is a PIR, first we study when $M \ltimes A$ is **homogenous**, i.e., every ideal of $M \ltimes A$ has the shape $N \ltimes \mathfrak{a}$ for some $N \subseteq M$, submodule, and $\mathfrak{a} \subseteq A$, ideal (it is **homogeneous**).

If $N \ltimes \mathfrak{a} \subseteq M \ltimes A$ is an ideal, then $M\mathfrak{a} \subseteq N$. Since the converse also holds, we have the following Lemma.

Lemma. 5.12.

$N \ltimes \mathfrak{a} \subseteq M \ltimes A$ is an ideal if, and only if, $M\mathfrak{a} \subseteq N$.

The next result is the converse of Lemma (5.9.).

Proposition. 5.13. ([4, Theorem 11])

Let A and M be principal. If $M \ltimes A$ is homogeneous, then $M \ltimes A$ is principal.

Proof. Let $N \ltimes \mathfrak{a} \subseteq M \ltimes A$ be an ideal. If $N = mA$ and $\mathfrak{a} = aA$, then $(m, a) \in N \ltimes \mathfrak{a}$, as it is homogeneous. Otherwise, by Theorem 3.3(3), in [7], we have $N \ltimes \mathfrak{a} = mA \ltimes aA \subseteq (mA + Ma) \ltimes aA = (m, a)M \ltimes A \subseteq N \ltimes \mathfrak{a}$. □

In this situation we can characterize when for any ring A and any A–module M the idealization $M \ltimes A$ is either PIR or Euclidean ring (ER). By Lemma (5.10.) we may reduce to consider the case in which A is either a PID or a SPIR. The simplest case is when $M = 0$; in this case $M \ltimes A$ is PIR if, and only if, A is PIR.

Theorem. 5.14.

Let A be a PID and M a non–zero A–module. The following statements are equivalent:

(a) $M \ltimes A$ is a PIR.

(b) A is a field, say K, and $M = A$.

In this case $K \ltimes K$ is an ER.

Proof. (b) ⇒ (a). Let $A = M = K$ be a field, we define $\delta : K \ltimes K \longrightarrow \mathcal{O}$ as

$$\delta(x) = \begin{cases} 0, & \text{if } x \text{ is invertible,} \\ 1, & \text{if } x \text{ is not invertible and } x \neq 0, \\ 2, & \text{if } x = 0. \end{cases}$$

We claim δ is a Euclidean norm. Indeed, $(m,a) \in K \ltimes K$ is invertible if, and only if, $a \neq 0$; for any $x_1 = (m_1, 0)$ and $x_2 = (m_2, 0)$ belonging to $K \ltimes K$ are not invertible, since $x_1 = x_2(0, m_1 m_2^{-1}) + 0$ is a division in $K \ltimes K$ with respect to the norm δ, then $K \ltimes k$ is an ER. In particular, $K \ltimes K$ is a PIR.

(a) ⇒ (b). Let $A = D$ be a PID, which is not a field, and M be a cyclic D–module. There are two cases:

(1) $M = D$. There exists $0 \neq a \in D$ such that $aD \neq D$. The element $(1,a) \in D \ltimes D$ satisfies

$$(1,a)(D \ltimes D) \subseteq D \ltimes aD.$$

We show that $D \ltimes aD$ is not cyclic. If $(x, ay) \in D \ltimes aD$ is a generator, there exists $(u,v) \in D \ltimes D$ such that $(1,a) = (x,ay)(u,v) = (xv + auy, ayv)$, hence $a = ayv$, and y, v are invertible. Therefore, from $1 = xv + auy$ we obtain $y = x + auy^2$, hence $x = (1 - auy)y$. In consequence, $(x,y) = ((1-auy)y, ay) = (1-auy, a)y$, and since y is invertible, $(1-auy, a)$ is a generator of $D \ltimes aD$. Since $aD \subsetneqq D$, if $z \in D \setminus aD$, there is $(u', v') \in D \ltimes D$ such that $(z, 0) = (1 - auy, a)(u', v') = ((1 - auy)v' + u'a, av')$, hence $0 = av'$, and $v' = 0$, so $z = (1 - auy)v' + u'a = u'a \in aD$, which is a contradiction.

(2) There exists $0 \neq d \in D$ such that $M = D/dD$, then $M \ltimes D = \frac{D}{dD} \ltimes D$. The element $(\overline{1}, d) \in \frac{D}{dD} \ltimes D$ satisfies

$$(\overline{1}, d)\left(\frac{D}{dD} \ltimes D\right) \subseteq \frac{D}{dD} \ltimes dD.$$

We claim $\frac{D}{dD} \ltimes dD$ is not cyclic. If $(\overline{x}, dy)$ is a generator, there exists $(\overline{u}, v) \in \frac{D}{dD} \ltimes D$ such that $(\overline{1}, d) = (\overline{x}, dy)(\overline{u}, v) = (\overline{xv}, dyv)$, hence $d = dyv$, and $1 = yv$, so y and v are invertible. From $\overline{1} = \overline{xv} = \overline{xy^{-1}}$, we obtain $\overline{x} = \overline{y}$ and $(\overline{x}, dy) = (\overline{y}, dy) = (\overline{1}, d)y$; a generator of $\frac{D}{dD} \bowtie dD$ is $(\overline{1}, d)$ because y is invertible. Since $\frac{D}{dD} \neq 0$, if we take $m \in D \setminus dD$, then $(\overline{m}, 0) \in \frac{D}{dD} \bowtie dD$, and there exists $(\overline{u'}, v') \in \frac{D}{dD} \bowtie D$ such that $(\overline{m}, 0) = (\overline{1}, d)(\overline{u'}, v') = (\overline{v'}, dv')$, hence $0 = dv'$, and $v' = 0$, so $\overline{m} = \overline{0}$, which is a contradiction.

□

In the same line we can conclude with the following.

Theorem. 5.15.

Let A be a SPIR and M a non–zero A–module, never $M \bowtie A$ is a PIR.

Proof. Let A be a SPIR, and M be a cyclic A–module. Let us denote by $\mathfrak{m} = tA$ the maximal ideal of A. There are two cases:

(1) $M = A$. We have $tA \subsetneqq A$. The element $(1, t) \in (A \bowtie A)$ satisfies

$$(1, t)(A \bowtie A) \subseteq A \bowtie tA.$$

We show that $A \bowtie tA$ is not cyclic. If $(x, ty) \in A \bowtie aA$ is a generator, there exists $(u, v) \in (A \bowtie A)$ such that $(1, t) = (x, ty)(u, v) = (xv + tuy, tyv)$, hence $t = tyv$, and y, v are invertible. Therefore, from $1 = xv + tuy$ we obtain $y = x + tuy^2$, hence $x = (1 - tuy)y$. In consequence, $(x, y) = ((1 - tuy)y, ay) = (1 - tuy, a)y$, and since y is invertible, $(1 - tuy, a)$ is a generator of $A \bowtie tA$. Since $tA \subsetneqq A$, if $z \in A \setminus tA$, there is $(u', v') \in (A \bowtie A)$ such that $(z, 0) = (1 - tuy, t)(u', v') = ((1 - tuy)v' + u't, tv')$, hence $0 = tv'$, and $v' \in tA = 0$, $z = (1 - yuy)v' + u't \in tA$, which is a contradiction.

(2) There exists $0 \neq d \in A$ such that $M = A/dA$, then $M \bowtie A = \frac{A}{dA} \bowtie A$. In this case $Mt = (\frac{A}{dA})t = \frac{dA+tA}{dA} \neq \frac{A}{dA}$. The element $(\overline{1}, d) \in \frac{A}{dA} \bowtie A$ satisfies

$$(\overline{1}, d)\left(\frac{A}{dA} \bowtie A\right) \subseteq \frac{A}{dA} \bowtie dA.$$

We claim $\frac{A}{dA} \bowtie dA$ is not cyclic. If $(\overline{x}, dy)$ is a generator, there exists $(\overline{u}, v) \in \frac{A}{dA} \bowtie A$ such that $(\overline{1}, d) = (\overline{x}, dy)(\overline{u}, v) = (\overline{xv}, dyv)$, hence $d = dyv$, and $w = yv$, y and v are invertible. From $\overline{1} = \overline{xv} = \overline{xwy^{-1}}$, we obtain $\overline{x} = \overline{w^{-1}y}$ and $(\overline{x}, dy) = (\overline{w^{-1}y}, dy) = (\overline{1}, dw)w^{-1}y$. A generator of $\frac{A}{dA} \bowtie dA$ is $(\overline{1}, dw)$ because $w^{-1}y$ is invertible. Since $(\frac{A}{dA})t \neq \frac{A}{dA}$, we take

$\overline{m} \in \frac{A}{dA} \setminus (\frac{A}{dA})t$, then $(\overline{m}, 0) \in \frac{D}{dD} \rtimes dD$, and there exists $(\overline{u'}, v') \in \frac{A}{dA} \rtimes A$ such that $(\overline{m}, 0) = (\overline{1}, dw)(\overline{u'}, v') = (\overline{v'}, dwv')$, hence $0 = dwv'$, and $v' \in tA$, so $\overline{m} = \overline{v'} \in (\frac{A}{dA})t$, which is a contradiction.

□

Corollary. 5.16.

Let A be a ring and M an A–module. The idealization $M \rtimes A$ is a principal ideal ring (a Euclidean ring) if, and only if, $A = M = K$ is a field K.

Later on, we will get a similar result when we consider PIRs relative to a hereditary torsion theory, see Example (5.41.).

5.2 Totally principal ideal rings

We include here this kind of ring because looking for non–trivial examples of totally σ–PIRs the first we constructed was an example of an idealization. Our aim in including here this type of rings has been to obtain a structure theorem which is a generalization of the classical theorems; we do that after proving some intermediate results.

Let A be a ring, and M an A–module, we say

- A is a **principal ideal ring** whenever every ideal $\mathfrak{a} \subseteq A$ is principal, i.e., there exists an element $a \in A$ such that $\mathfrak{a} = aA$.
- M is called a **principal module** whenever every submodule is cyclic.

These definition, with respect to a hereditary torsion theory σ in **Mod**–A, are expressed as follows:

- a submodule $N \subseteq M$ is a **σ–cyclic submodule** (**σ–principal submodule**) if there exists an element $m \in M$ such that $\mathrm{Cl}_\sigma^M(N) = \mathrm{Cl}_\sigma^M(mA)$.
- A is a **σ–principal ideal ring** if every ideal $\mathfrak{a} \subseteq A$ is a σ–cyclic ideal.
- M is a **σ–principal module** whenever every submodule is σ–cyclic.

- a submodule $N \subseteq M$ is a **totally σ–cyclic submodule** (**totally σ–principal submodule**) whenever there exist $\mathfrak{h} \in \mathscr{L}(\sigma)$ and $n \in N$ such that $M\mathfrak{h} \subseteq nA$.
- A is a **totally σ–principal ideal ring** if for every ideal $\mathfrak{a} \subseteq A$ is totally σ–cyclic.
- M is a **totally σ–principal module** whenever every submodule is totally σ–cyclic.

Remark. 5.17.

For any multiplicative subset $\Sigma \subseteq A$ we consider the hereditary torsion theory σ_Σ defined by $\mathscr{L}(\sigma_\Sigma) = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \cap \Sigma \neq \varnothing\}$.

In this particular case we have that for any ideal $\mathfrak{a} \subseteq A$ we have $\mathfrak{a}$ is **Σ–principal** (= σ_Σ–principal) if, and only if, it is **totally Σ–principal** (= totally σ_Σ–principal). If there exists $a \in A$ such that $\mathrm{Cl}^A_{\sigma_\Sigma}(aA) = \mathrm{Cl}^A_\sigma(\mathfrak{a})$, we only need to show that $\mathrm{Cl}^A_{\sigma_\Sigma}(xA) = \mathrm{Cl}^A_\sigma(\mathfrak{a})$ for some $x \in \mathfrak{a}$. Indeed, since $\mathrm{Cl}^A_{\sigma_\Sigma}(aA) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$, there exists $s \in \Sigma$ such that $asA \subseteq \mathfrak{a}$. Now we show that $\mathrm{Cl}^A_{\sigma_\Sigma}(asA) = \mathrm{Cl}^A_\sigma(aA)$; for any $y \in \mathrm{Cl}^A_\sigma(aA)$ there is $s_y \in \Sigma$ such that $ys_y \in aA$, hence $yss_y \in asA$, hence $y \in \mathrm{Cl}^A_\sigma(asA)$.

It is clear that if σ is not a principal hereditary torsion theory the result may be false.

The follow results shows that there are families of totally σ–principal ideal rings which are not principal ideal rings.

Proposition. 5.18. ([10, Proposition 2.1])

Let A be a ring, and $\Sigma = \mathrm{Reg}(A)$, if there exists an essential ideal $\mathfrak{a} \subseteq A$, then A is a totally Σ–principal ideal ring.

Proof. For any non–zero ideal $\mathfrak{b} \subseteq A$ there is an element $0 \neq b \in \mathfrak{b} \cap \mathfrak{a}$, hence $bA \in \mathscr{L}(\sigma_\Sigma)$, and we have $\mathfrak{b}bA \subseteq bA \subseteq \mathfrak{b}$. □

Corollary. 5.19. ([10, Corollary 2.2])

If D is an integral domain and $\Sigma = D \setminus \{0\}$, then D is totally Σ–principal ideal ring.

A construction that provides new examples of PIRs is the direct product. For any family of rings $\{A_i \mid i \in I\}$ the product ring $A = \prod_i A_i$ satisfies: A is a PIR if, and only if, each A_i is a PIR. Indeed, each ideal of $A = \prod_i A_i$ has the shape $\prod_i \mathfrak{a}_i$, being each $\mathfrak{a}_i \subseteq A_i$ an ideal. If A is a PIR then each A_i is a PIR because homomorphic images of PIR are PIR. If every A_i is a PIR, for every $i \in I$ there exists $a_i \in A_i$ such that $\mathfrak{a}_i = a_i D$, and we have $\prod_i \mathfrak{a}_i = (a_i)_i A$ is a principal ideal.

If we study this example relative to a hereditary torsion theory, we find some problems. If σ is a hereditary torsion theory in **Mod**–A, being $A = \prod_i A_i$, with projections $p_j : \prod_i A_i \longrightarrow A_j$, we have a hereditary torsion theory σ_j in each category **Mod**–A_j, defined

$$\mathscr{L}(\sigma_j) = \{\mathfrak{c} \subseteq A_j \mid p_j^{-1}(\mathfrak{c}) \in \mathscr{L}(\sigma)\}.$$

On the other hand, for any $\mathfrak{h} \in \mathscr{L}(\sigma)$, not necessarily $p_i(\mathfrak{h})$ belongs to $\mathscr{L}(\sigma_i)$, which has broken the possibility of passing properties from A to each A_i. In parallel, if we consider a hereditary torsion theory τ_i is each **Mod**–A_i, the natural way to define a hereditary torsion theory in A is considering in **Mod**–A the induced hereditary torsion theory τ_i', defined

$$\mathscr{L}(\tau_i') = \{\mathfrak{a} \subseteq A \mid q_i^{-1}(\mathfrak{a}) \in \mathscr{L}(\tau_i)\},$$

and then consider the intersection of all τ_i, that produces a hereditary torsion theory in A_i that may have no relationship with the original τ_i.

For that reason we restrict to consider families with finitely many elements.

Proposition. 5.20.
Let $\{A_i \mid i = 1, \ldots, n\}$ be a family of rings, and $A = \prod_{i=1}^n A_i$, the following statements hold:

*(1) If for every index i we have a hereditary torsion theory σ_i in **Mod**–A_i and define*

$$\mathscr{L}(\sigma) = \left\{ \mathfrak{a} = \prod_{i=1}^n \mathfrak{a}_i \mid \mathfrak{a}_i \in \mathscr{L}(\sigma_i) \text{ for every index } i \right\},$$

*then $\mathscr{L}(\sigma)$ is the Gabriel filter for a hereditary torsion theory σ in **Mod**–A.*

*(2) If σ is a hereditary torsion theory in **Mod**–A, and, for every index i, we define*

$$\mathscr{L}(\sigma_i) = \{\mathfrak{a}_i \subseteq A_i \mid A_1 \times \cdots \times A_{i-1} \times \mathfrak{a}_i \times A_{i+1} \times \cdots \times A_n \in \mathscr{L}(\sigma)\},$$

*then $\mathscr{L}(\sigma_i)$ is the Gabriel filter for a hereditary torsion theory σ_i in **Mod**–A_i, for every index i.*

It is clear that every principal (resp. σ–principal, totally σ–principal) ring is noetherian (resp. σ–noetherian, totally σ–noetherian).

Given a ring A we say:

- A is a **σ–domain** if A is not σ–torsion ($\Leftrightarrow 0 \notin \mathscr{L}(\sigma)$ and for any $a, b \in A$ such that $ab = 0$ we have either $a \in \mathrm{Cl}^A_\sigma(0)$ or $b \in \mathrm{Cl}^A_\sigma(0)$, or equivalently $0 \subseteq A$ is a σ–prime ideal.
- A is a **totally σ–domain** if A is not totally σ–torsion ($\Leftrightarrow 0 \notin \mathscr{L}(\sigma)$) and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for any $a, b \in A$ satisfying $ab = 0$ we have either $a\mathfrak{h} = 0$ or $b\mathfrak{h} = 0$, or equivalently $0 \subseteq A$ is a totally σ–prime ideal.

We first give a useful characterization of totally σ–domains, see Theorem (4.54.).

Theorem. 5.21.

Given a ring A such that $0 \notin \mathscr{L}(\sigma)$, the following statements are equivalent:

(a) A is a totally σ–domain.

(b) If $\mathfrak{h} = (0 : \sigma A)$, then $\mathfrak{h} \in \mathscr{L}(\sigma)$ and $(0 : \mathfrak{h}) = \sigma A \subseteq A$ is prime.

Proof. (a) $\Rightarrow$ (b). Let $\mathfrak{k} \in \mathscr{L}(\sigma)$ an ideal associated to $0 \subseteq A$, then $(0 : \mathfrak{k}) \subseteq A$ is prime; otherwise, for any $x \in \sigma A$ there exists $\mathfrak{t} \in \mathscr{L}(\sigma)$ such that $x\mathfrak{t} = 0$, hence either $x\mathfrak{k} = 0$; i.e., $x \in (0 : \mathfrak{k})$, or $\mathfrak{t}\mathfrak{k} = 0$; which is a contradiction. In consequence, $\sigma A \subseteq (0 : \mathfrak{k}) \subseteq \sigma A$, and the equality holds. Otherwise, let $\mathfrak{h} = (0 : \sigma A)$, then $\mathfrak{k} \subseteq \mathfrak{h}$, hence $\mathfrak{h} \in \mathscr{L}(\sigma)$, and we have $(0 : \mathfrak{h}) \subseteq (0 : \mathfrak{k}) = \sigma A \subseteq (0 : \mathfrak{h})$, and the result holds.

(b) $\Rightarrow$ (a). By the hypothesis $\mathfrak{h} \in \mathscr{L}(\sigma)$. Let $a, b \in A$ such that $ab = 0 \in (0 : \mathfrak{h})$, hence either $a \in (0 : \mathfrak{h})$; i.e., $a\mathfrak{h} = 0$, or $b \in (0 : \mathfrak{h})$; i.e., $b\mathfrak{h} = 0$. □

As a consequence, we have:

Corollary. 5.22.

A ring A is a totally σ–domain if, and only if, σA is totally σ–torsion and $\sigma A \subseteq A$ is a prime ideal.

Proof. If σA is totally σ–torsion and take $\mathfrak{h} = (0 : \sigma A)$, then $\sigma A \subseteq (0 : \mathfrak{h}) \subseteq \sigma A$; and the result holds from the theorem. □

These results are of interest because they show that the ideal $\mathfrak{h}$, in the Gabriel filter, of the definition of totally σ–domain is directly fixed by the structure of A.

We can consider an ideal $\mathfrak{a} \subseteq A$ instead of $0 \subseteq A$, and define **totally σ–prime ideal**, and, as a consequence of the theorem we have:

Corollary. 5.23.
An ideal $\mathfrak{a} \subseteq A$ *is a totally* σ*–prime ideal if, and only if, taking* $\mathfrak{h} = (\mathfrak{a} : \mathrm{Cl}^A_\sigma(\mathfrak{a}))$, *then* $\mathfrak{h} \in \mathscr{L}(\sigma)$ *and* $(\mathfrak{a} : \mathfrak{h}) = \mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq A$ *is a prime ideal.*

In particular, if $\mathfrak{a} \subseteq A$ *is a totally* σ*–prime ideal, then* $A/\mathfrak{a}$ *is a totally* σ*–prime domain.*

Proof. It is the translation of the previous results to the ring $A/\mathfrak{a}$ whose σ–torsion submodule is $\mathrm{Cl}^A_\sigma(\mathfrak{a})/\mathfrak{a}$. □

At this point it is convenient to remark that for any ideal $\mathfrak{a} \subseteq A$, it is totally σ–prime if, and only if, $\mathrm{Cl}^A_\sigma(\mathfrak{a}) \subseteq A$ is prime and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathrm{Cl}^A_\sigma(\mathfrak{a})\mathfrak{h} \subseteq \mathfrak{a}$.

Proposition. 5.24.
Given a totally σ*–PIR* A, *and a totally* σ*–prime ideal* $\mathfrak{a} \subseteq A$ *we have that* $A/\mathfrak{a}$ *and* $A/\mathrm{Cl}^A_\sigma(\mathfrak{a})$ *are totally* $\overline{\sigma}$*–PID, being* $\overline{\sigma}$*'s the hereditary torsion theories induced by* σ*; i.e.,* $\mathscr{L}(\overline{\sigma}) = \{\mathfrak{b}/\mathfrak{a} \subseteq A/\mathfrak{a} \mid \mathfrak{b} \in \mathscr{L}(\sigma)\}$.

Observe that, in this case, $A/\mathrm{Cl}^A_\sigma(\mathfrak{a})$ is a domain; therefore, to study totally σ–prime ideals, we can restrict ourselves to considering prime ideals in $\mathscr{K}(\sigma)$.

Proposition. 5.25.
If A *is a totally* σ*–PIR, and* $\Sigma \subseteq A$ *a multiplicative subset, then the ring of fractions* A_Σ *is a totally* $\overline{\sigma}$*–PIR.*

Proof. As usual, we denote $\overline{\sigma}$ the hereditary torsion theory induced by σ in A_Σ. For any ideal $\mathfrak{b} \subseteq A_\Sigma$ there exists an ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{b} = \mathfrak{a}A_\Sigma$, and there exist $\mathfrak{h} \in \mathscr{L}(\sigma)$ and $a \in \mathfrak{a}$ such that $\mathfrak{a}\mathfrak{h} \subseteq aA$. Therefore, we have $aA_\Sigma \supseteq \mathfrak{a}\mathfrak{h}A_\Sigma = \mathfrak{b}\mathfrak{h}A_\Sigma$, and $\mathfrak{b}$ is totally $\overline{\sigma}$–principal. □

The spectrum of a totally principal ideal ring

We again study the structure of prime ideals in $\mathscr{K}(\sigma)$ whenever A is a totally σ–PIR. We shall prove one of the main results in this section.

Theorem. 5.26.
Given a totally σ*–PIR* A, *does not exist a chain* $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2$ *of three prime ideals in* $\mathscr{K}(\sigma)$.

Proof. If there exists such a chain; we consider the quotient ring $A/\mathfrak{p}_0$, which is a domain; if $\overline{\sigma}$ is the induced hereditary torsion theory in $A/\mathfrak{p}_0$ by σ, then $A/\mathfrak{p}_0$ is a totally $\overline{\sigma}$–PIR. Therefore we may assume that $\mathfrak{p}_0 = 0$, in consequence A must be an integral domain.

Consider the hereditary torsion theory $\sigma_i = \sigma_{A\setminus\mathfrak{p}_i}$, for $i = 1,2$, the following relationship: $\sigma \leq \sigma_2 \leq \sigma_1$ holds. When we localize at σ_2 obtain new ring: $A_{\mathfrak{p}_2}$; if we adorne $\overline{\sigma_*}$ the hereditary torsion theory induced by σ_* in $A_{\mathfrak{p}_2}$, then $\overline{\sigma} = \overline{\sigma_2} \leq \overline{\sigma_1}$. Now, since A is totally σ–PID, then $A_{\mathfrak{p}_2}$ is $\overline{\sigma}$–PID, hence a PID. Thus, we have a chain of prime ideals: $0 \subsetneqq \mathfrak{p}_1 A_{\mathfrak{p}_2} \subsetneqq \mathfrak{p}_2 A_{\mathfrak{p}_2}$, which is a contradiction. □

In the general case we have:

Corollary. 5.27.

For any totally σ–PIR A there is no chain of totally σ–prime ideals $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2$ such that $\mathfrak{a}_{i+1}/\mathfrak{a}_i$, $i = 1,2$ is not totally σ–torsion.

Proof. If there exists such a chain, we have a chain of prime ideals in $\mathscr{K}(\sigma)$ which contradices the result in the above theorem. □

Let A be a ring, an ideal $\mathfrak{m}$ is called:

- **σ–maximal** if $\mathfrak{m} \notin \mathscr{L}(\sigma)$, and for any ideal $\mathfrak{a} \supseteq \mathfrak{m}$ we have either $\mathrm{Cl}^A_\sigma(\mathfrak{m}) = \mathrm{Cl}^A_\sigma(\mathfrak{a})$ or $\mathrm{Cl}^A_\sigma(\mathfrak{a}) = A$.
- **totally σ–maximal** if $\mathfrak{m} \notin \mathscr{L}(\sigma)$, and there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that for any ideal $\mathfrak{a} \supseteq \mathfrak{m}$ we have either $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{m}$ or $\mathfrak{a} \in \mathscr{L}(\sigma)$.

It is well established that every totally σ–maximal ideal is totally σ–prime.

In a totally σ–PID we have that σA is a prime ideal and that prime ideals in $\mathscr{K}(\sigma)$ are well–arranged. Indeed, as a consequence of Corollary (5.22.), we have:

Corollary. 5.28.

In a totally σ–PID every non σ–torsion prime ideal is totally σ–maximal.

The following is a technical result that will use later in this work and that allows us an approach to all ideals of A in terms of the elements of $\mathscr{K}(\sigma)$.

Theorem. 5.29.
If A is a totally σ–noetherian ring, then $\mathrm{Min}(\mathcal{K}(\sigma))$ *is finite.*

Proof. Consider the family of ideals:

$$\Gamma = \{\mathfrak{a} \subseteq A \mid \text{ there are infinitely prime ideals in } \mathcal{K}(\sigma) \text{ which are minimal over } \mathfrak{a}\}.$$

The result will be true whenever Γ is empty. Let us assume Γ is non–empty, and we shall arrive to a contradiction. Since A is totally σ–noetherian, there are σ–maximal elements in Γ. If $\mathfrak{a} \in \Gamma$ is σ–maximal, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that if $\mathfrak{b} \in \Gamma$ and $\mathfrak{a} \subsetneq \mathfrak{b}$, then $\mathfrak{b}\mathfrak{h} \subseteq \mathfrak{a}$; we show that α is totally σ–prime. Given elements $a, b \in A$ such that $ab \in \mathfrak{a}$, we consider the ideals $\mathfrak{a} + aA$ and $\mathfrak{a} + bA$. If $\mathfrak{a} + aA \in \Gamma$, then $(\mathfrak{a} + aA)\mathfrak{h} \subseteq \mathfrak{a}$, hence $a\mathfrak{h} \subseteq \mathfrak{a}$; similarly if $\mathfrak{a} + bA \in \Gamma$. It rests the case in which $\mathfrak{a} + aA, \mathfrak{a} + bA \notin \Gamma$; hence there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathcal{K}(\sigma)$ which are the minimal over $\mathfrak{a} + aA$, and similarly for $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ and $\mathfrak{a} + bA$. For any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, minimal over $\mathfrak{a}$, since $(\mathfrak{a} + aA)(\mathfrak{a} + bA) \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, then either $\mathfrak{a} + aA \subseteq \mathfrak{p}$; hence there exists $\mathfrak{p}_i \subseteq \mathfrak{p}$; i.e., $\mathfrak{p} = \mathfrak{p}_i$; or $\mathfrak{a} + bA \subseteq \mathfrak{p}$, and we may deduce that there exists one $\mathfrak{q}_j$ such that $\mathfrak{p} = \mathfrak{q}_j$. this means that there are only finitely many prime ideals in $\mathcal{K}(\sigma)$ which are minimal over $\mathfrak{a}$, which is a contradiction. □

With this background, we may prove other important theorem in this section.

Theorem. 5.30.
Let A be a totally σ–PID, for any ideal $\mathfrak{a} \subseteq A$ such that $\sigma A \subsetneq \mathfrak{a}$ the ring $A/\mathfrak{a}$ is totally σ–artinian.

Proof. Since A is totally σ–PID then $\sigma A \subseteq A$ is a domain, and $A/\mathfrak{a}$ is a totally σ–PIR. For simplicity let $B = A/\mathfrak{a}$, and $\overline{\sigma}$ the induced hereditary torsion theory. The following properties hold for B: B is totally σ–noetherian, every prime ideal in $\mathcal{K}(\overline{\sigma})$ is maximal, and $\overline{\sigma} \leq \overline{\sigma_{A\setminus\mathfrak{p}}}$, for any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$.

In the ring $B_{\mathfrak{p}}$ the torsion theories induced by $\overline{\sigma}$ and $\overline{\sigma_{A\setminus\mathfrak{p}}}$ are the same and they are trivial; therefore $B_{\mathfrak{p}}$ is artinian since it has only one prime ideal.

From [31, Theorem 4.5], and Theorem (5.29.) we have that $B = A/\mathfrak{a}$ is totally σ–artinian.

□

Corollary. 5.31.

Given a totally σ–PID A and a decreasing chain of ideals $\{\mathfrak{a}_i \mid i \in I\}$, with $\sigma A \subsetneq \mathfrak{a}_i$ for any index $i \in I$, we have either the chain is σ–stable or $\cap_i \mathfrak{a}_i = \sigma A$.

Proof. If $\cap_i \mathfrak{a}_i \nsubseteq \sigma A$ let us call $\mathfrak{a} = \cap_i \mathfrak{a}_i$. The quotient ring $A/\mathfrak{a}$ is totally σ–artinian, hence the chain $\{\mathfrak{a}_i/\mathfrak{a}\}_i$ is σ–estable. Therefore $\{\mathfrak{a}_i\}_i$ is σ–estable. □

Let us recall that $\mathscr{K}(\sigma)$ has enough information to characterize totally σ–PIRs.

Theorem. 5.32. (Kaplansky–like theorem, [32, Theorem 7.1])

Let σ be a finite type hereditary torsion theory, the following statements are equivalent:

(a) A is totally σ–PIR.

(b) Every prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ is totally σ–principal.

Corollary. 5.33. ([32, Corollary 7.2])

Let σ be a finite type hereditary torsion theory, the following statements are equivalent:

(a) A is totally σ–PIR.

(b) A is σ–PIR and every prime ideal $\mathfrak{p}$ is totally σ–finitely generated.

(c) A is σ–PIR and totally σ–noetherian.

Structure of totally principal ideal rings

Once we have proved that most of the information on totally σ–PIRs resides in the set $\mathscr{K}(\sigma)$, the next steps to determine the structure of totally σ–PIR are: first, to check that the product of finitely many totally PIRs is as well; and second, to study local and indecomposable totally PIRs.

Compare with Proposition (5.20.).

Proposition. 5.34.

Let $\{A_i \mid i = 1, \ldots, n\}$ be a family of rings, and $A = \prod_{i=1}^n A_i$, the following statements hold:

*(I) If for every index i we have a hereditary torsion theory σ_i in **Mod**–A_i such that A_i is a totally σ_i–PIR, and define*

$$\mathscr{L}(\sigma) = \left\{ \mathfrak{a} = \prod_{i=1}^{n} \mathfrak{a}_i \mid \mathfrak{a}_i \in \mathscr{L}(\sigma_i) \text{ for every index } i \right\}, \text{ then}$$

(1) $\mathscr{L}(\sigma)$ *is the Gabriel filter for a hereditary torsion theory* σ *in* ***Mod**–A.*

(2) The ring A is totally σ*–PIR if, and only if,* A_i *is totally* σ_i*–PIR, for every index* i.

(II) If σ *is a hereditary torsion theory in* ***Mod**–A, and, for every index* i*, we define*

$$\mathscr{L}(\sigma_i) = \{\mathfrak{a}_i \subseteq A_i \mid A_1 \times \cdots \times A_{i-1} \times \mathfrak{a}_i \times A_{i+1} \times \cdots \times A_n \in \mathscr{L}(\sigma)\}\,,\ \textit{then}$$

(1) $\mathscr{L}(\sigma_i)$ *is the Gabriel filter for a hereditary torsion theory* σ_i *in* ***Mod**–*A_i*, for every index* i.

(2) The ring A is totally σ*–PIR if, and only if,* A_i *is totally* σ_i*–PIR, for every index* i.

(III) In the case in which $A = \prod_{i=1}^n A_i$ *is a direct product of totally PIRs, then*

(1) $C(\sigma, A) = \prod_{i=1}^n C(\sigma_i, A_i)$*, is the product of lattices, and*

(2) $\mathscr{K}(\sigma) = \cup_{i=1}^n \mathscr{K}(\sigma_i)$*, being incomparable elements of* $\mathscr{K}(\sigma_i)$ *with elements of* $\mathscr{K}(\sigma_j)$*, whenever* $i \neq j$.

Before continuing with the study of local totally PIRs we shall establish a result of prime ideals in $\mathscr{K}(\sigma)$.

Lemma. 5.35.

Let A be a totally σ*–PIR and a prime ideal* $\mathfrak{p} \in \mathscr{K}(\sigma)$*, for every prime ideals* $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneqq \mathfrak{p}$ *we have* $\mathfrak{p}_1 = \mathfrak{p}_2$.

Proof. Since $\mathfrak{p} \in \mathscr{K}(\sigma)$ then $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$. In $A_\mathfrak{p}$ we have two hereditary torsion theories $\overline{\sigma}$ and $\overline{\sigma_{A\setminus\mathfrak{p}}}$, and both of them are trivial; i.e., $\mathscr{L}(\overline{\sigma}) = \{A_\mathfrak{p}\}$. Since $A_\mathfrak{p}$ is totally $\overline{\sigma}$–PIR, then it is a PIR. For any two prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneqq \mathfrak{p}$, since $\mathfrak{p}_1 A_\mathfrak{p}, \mathfrak{p}_2 A_\mathfrak{p} \subsetneqq \mathfrak{p} A_\mathfrak{p}$, then $\mathfrak{p}_1 A_\mathfrak{p} = \mathfrak{p}_2 A_\mathfrak{p}$, hence $\mathfrak{p}_1 = \mathfrak{p}_2$.

□

If A is a local totally σ–PIR, (**local** means that $\mathscr{K}(\sigma)$ only has a maximal element $\mathfrak{m} \in \mathscr{K}(\sigma)$), then $\sigma = \sigma_{A\setminus\mathfrak{m}}$. There are four possibilities:

(1) $\sigma A \in \mathscr{K}(\sigma)$ and $\dim(\mathscr{K}(\sigma)) = 0$: In this case $A_\mathfrak{m}$ is a field.

(2) $\sigma A \in \mathscr{K}(\sigma)$ and $\dim(\mathscr{K}(\sigma)) = 1$: In this case $A_\mathfrak{m}$ is a PID, not a field.

(3) $\sigma A \notin \mathscr{K}(\sigma)$ and $\dim(\mathscr{K}(\sigma)) = 0$: In this case $A_\mathfrak{m}$ is a special PIR.

(4) $\sigma A \notin \mathscr{K}(\sigma)$ and $\dim(\mathscr{K}(\sigma)) = 1$:

In this case we have that $A_{\mathfrak{m}}$ is a PIR and has two non–zero prime ideals, which contradices [30, Lemma 10]. This case never occurs!

This means that given a prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$, there exists only one minimal prime ideal $\mathfrak{p}_0 \in \mathscr{K}(\sigma)$ such that $\mathfrak{p}_0 \subseteq \mathfrak{p}$. On the contrary, given a prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$, we have no control on the prime ideals $\mathfrak{q} \in \mathscr{K}(\sigma)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$.

Theorem. 5.36.

Let A be a totally σ–PIR, there is a decomposition $A/\sigma A = \prod_{i=1}^{n} A_i$ of $A/\sigma A$ as a direct product of totally PIRs A_i, having each A_i a unique minimal prime ideal.

Proof. We may assume A is σ–torsionfree. Using Theorem (5.29.) we have that $\mathscr{K}(\sigma)$ has only finitely many minimal elements. For any $\mathfrak{p} \in \mathrm{Min}(\mathscr{K}(\sigma))$ consider $V(\mathfrak{p}) = \{\mathfrak{q} \in \mathscr{K}(\sigma) \mid \mathfrak{q} \supseteq \mathfrak{p}\}$. It is clear that, as a consequence of Theorem (5.26.), $\{V(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Min}(\mathscr{K}(\sigma))\}$ is a partition of $\mathscr{K}(\sigma)$.

Following the theory that appears in [23] and [24], we have a lattice decomposition of A, say $A = \prod\{A_{[\mathfrak{p}]} \mid \mathfrak{p} \in \mathrm{Min}(\mathscr{K}(\sigma))\}$. If $\sigma_{[\mathfrak{p}]}$ is the hereditary torsion theory induced by σ in $A_{[\mathfrak{p}]}$ then each factor $A_{[\mathfrak{p}]}$ satisfies $\mathscr{K}(\sigma_{[\mathfrak{p}]}) \subseteq \mathrm{Spec}(A_{[\mathfrak{p}]})$ is homeomorphic with $V(\mathfrak{p})$. □

As a consequence of this decomposition theorem we can distinguish a particular class of totally σ–PIR. Indeed, a totally σ–PIR A is called **indecomposable** whenever $\mathrm{Min}(\mathscr{K}(\sigma))$ is unitary. There are different possibilities for indecomposable totally σ–PIRs:

(1) $\sigma A \in \mathscr{K}(\sigma)$ and $\mathscr{K}(\sigma) = \{\mathfrak{m}\}$ is unitary: In this case $A_{\mathfrak{m}}$ is a field.

(2) $\sigma A \in \mathscr{K}(\sigma)$ and $\mathscr{K}(\sigma)$ is not unitary: In this case $\dim(\mathscr{K}(\sigma)) = 1$, $A/\sigma A$ is a totally PID, and for each maximal ideal $\mathfrak{m}$ in $\mathscr{K}(\sigma)$ the ring of fractions $A_{\mathfrak{m}}$ is a PID, not a field.

(3) $\sigma A \notin \mathscr{K}(\sigma)$ and $\mathscr{K}(\sigma) = \{\mathfrak{m}\}$ is unitary: In this case $A_{\mathfrak{m}}$ is a special PIR.

Let us show some examples to illustrate the theory.

Examples. 5.37.

The classical case is obtained when we take σ such that $\mathscr{L}(\sigma) = \{A\}$. A second case: S–principal ideal rings (for $S \subseteq A$ a multiplicative set) is covered if we take $\sigma = \sigma_S$. Our approximation makes very strong use of the torsion ideal $\sigma_S(A) = \{a \mid$ there exists σ_S such that $as = 0\}$, which contains the redundant information of A, from the point of view of S, and whose annihilator provides a useful tool for studying A; and the prime ideals in $\mathscr{K}(\sigma_S)$, which are a small but the most representative part within the S–prime ideals.

Examples. 5.38.

(1) Let A be a ring and M an A–module, consider the idealization $M \rtimes A$. The prime ideals of $M \rtimes A$ have the shape $M \rtimes \mathfrak{p}$, being $\mathfrak{p} \subseteq A$ a prime ideal.

(2) For any ring A an ideal $\mathfrak{a} \subseteq A$ is **regular** whenever it contains a regular element, and it is **semi–regular** if it contains a finitely generates ideal $\mathfrak{b} \subseteq \mathfrak{a}$ such that $(0 : \mathfrak{b}) = 0$. The filter of ideals $\mathscr{L}(\tau_q) = \{\mathfrak{a} \subseteq A \mid \ \mathfrak{a}$ is semi–regular$\}$ is a Gabriel filter, we denote by τ_q the hereditary torsion theory it defines. In fact, the localization of A with respect to τ_q is the ring of finite fractions; see [42] and [56]. For this hereditary torsion theory we have $\mathscr{Z}(\tau_q)$ is the set of all semi–regular prime ideals, and $\mathscr{K}(\sigma)$ the set of all prime ideals $\mathfrak{p}$ such that for every finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{p}$ we have $(0 : \mathfrak{b}) \neq 0$.

(3) Let $A = D$ be an integral domain and M a torsionfree A–module, then $M \rtimes 0$ is the set of all zero–divisors of $M \rtimes D$. Indeed, if (m, d) is a zero–divisor, there exists $0 \neq (x, y) \in M \rtimes D$ such that $0 = (m, d)(x, y) = (my + xd, dy)$; hence $dy = 0$. If $d \neq 0$, then $y = 0$, hence $xd = 0$, this implies that $x = 0$, which is a contradiction.

    Moreover, we have that $M \rtimes D$ has property (A), see [29]. Indeed, if $m_1, \ldots, m_t \in M$ there are non–zero elements $d_1, \ldots, d_t \in D$ such that $m_i d_i = 0$ for every index i; hence $0 \neq d_1 \cdots d_t \in (0 : \langle m_1, \ldots, m_t\rangle)$. Consequently $M \rtimes 0$ is a prime ideal which is not semi–regular.

    On the other hand, every prime ideal $M \rtimes \mathfrak{p}$, with $\mathfrak{p} \neq 0$, is a semi–regular prime ideal. In consequence: $\mathscr{K}(\tau_q) = \{M \rtimes 0\}$, and $\mathscr{Z}(\sigma) = \{M \rtimes \mathfrak{p} \mid \ \mathfrak{a} \in \operatorname{Spec}(D) \setminus \{0\}\}$.

(4) If D is a non–noetherian integral domain and M is finitely generated torsionfree D–module, then $M \rtimes 0$ is a finitely generated; hence totally τ_q–finitely generated; by Cohen–like theorem, see [32, Corollary 3.5], we have that $M \rtimes D$ is a totally τ_q–noetherian ring.

(5) If we take M a torsionfree principal D–module, then $M \rtimes 0$ is totally τ_q–principal; therefore, by Kaplansky–like theorem, see [32, Proposition 7.1], we have that $M \rtimes D$ is totally τ_q–PIR;

indeed, an indecomposable totally τ_q–PID since $\tau_q(M \rtimes D) = M \rtimes 0$.

Examples. 5.39.

(1) Let $A = \mathbb{Z}[X]$, and $\Sigma = \langle X^{2n} \mid n \in \mathbb{N}\rangle \subseteq \mathbb{Z}[X]$ be a multiplicative set. Let $\mathfrak{p} = (F(X))$, for an irreducible polynomial $F(X) \in \mathbb{Z}[X]$, $F(X) \neq X$. Since $\mathfrak{p} \cap \Sigma = \varnothing$, then $\mathfrak{p} \subseteq \mathbb{Z}[X]$ is totally σ–prime; in addition for any ideal $\mathfrak{h} \in \mathscr{L}(\sigma)$, we have $\mathfrak{p}\mathfrak{h} \subseteq \mathbb{Z}[X]$ is totally Σ–prime; i.e., $(X^2F(X)) \subseteq \mathbb{Z}[X]$ is totally σ–prime and it is not prime, but $(XF(X)) \subseteq \mathbb{Z}[X]$ is neither prime not totally Σ–prime.

(2) The same situation occurs if we take a prime ideal $\mathfrak{q} = (p, F(X))$, being $p \in \mathbb{Z}$ a prime integer number and $F(X) = \sum_{i=0}^{t} a_i X^i \in \mathbb{Z}[X]$, $F(X) \neq X$, $p \nmid a_i$ if $a_i \neq 0$, irreducible and irreducible modulo p, or zero.

(3) Observe that $0 \subseteq \mathbb{Z}[X]$ is a totally Σ–prime ring. This means that $\mathbb{Z}[X]$ is totally σ–noetherian, but not totally Σ–artinian.

(4) On the other hand, if we take the multiplicative set $\Sigma = \langle\{X^{2n} \mid n \in \mathbb{N}\} \cup (\mathbb{Z} \setminus \{0\})\rangle \subseteq \mathbb{Z}[X]$, then prime ideals like $\mathfrak{q} = (p, F(X))$ in (2) satisfy $\mathfrak{q} \cap \Sigma \neq \varnothing$, and they are not totally Σ–prime. Thus the only prime ideals which are totally Σ–prime are 0 and those of the shape $(F(X))$, with $F(X) \neq X$ irreducible in $\mathbb{Z}[X]$.

(5) In this case $\mathbb{Z}[X]$ is a totally Σ–principal ideal domain.

Example. 5.40.

(1) Let $A = \mathbb{Z}_{12}[X]$, and $\Sigma = \mathbb{Z}_{12} \setminus 2\mathbb{Z}_{12}$. the prime ideals of $\mathbb{Z}_{12}[X]$ are (2); (3); $(2, F(X))$, being $F(X) = \sum_{i=1}^{t} a_i X^i \in \mathbb{Z}_{12}[X]$, $2 \nmid a_i$ if $a_i \neq 0$, and $F(X)$ is irreducible modulo 2; $(3, F(X))$, being $F(X) = \sum_{i=1}^{t} a_i X^i \in \mathbb{Z}_{12}[X]$, $3 \nmid a_i$ if $a_i \neq 0$, and $F(X)$ is irreducible modulo 3.

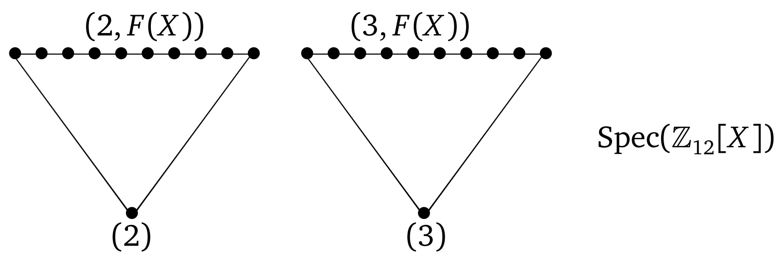
(2) In the case we have:

$$\mathscr{Z}(\sigma_\Sigma) = \{(3)\} \cup \{(3, F(X)) \mid \text{ satisfying the above condition}\};$$
$$\mathscr{K}(\sigma_\Sigma) = \{(2)\} \cup \{(2, F(X)) \mid \text{ satisfying the above condition}\}.$$

Otherwise, $\sigma_\Sigma(\mathbb{Z}_{12}[X]) = 4\mathbb{Z}_{12}[X]$.

(3) In consequence, $\mathbb{Z}_{12}[X]$ is a totally σ_Σ–noetherian rings and it is not a totally σ–artinian ring. In addition, it is not totaly σ–PIR.

(4) A picture of the spectrum of $\mathbb{Z}_{12}[X]$ is the following:



We have that $\mathbb{Z}_{12}[X]$ is totally σ–PIR whenever we take σ such that $\mathcal{K}(\sigma) = \{(2),(3)\}$ or $\mathcal{K}(\sigma) = \{(p)\}$ for $p = 2,3$. In all these cases $\mathbb{Z}_{12}[X]$ has decompositions as product of indecomposables totally σ–PIRs.

Let us consider the particular case of σ with $\mathcal{K}(\sigma) = \{(2),(3)\}$. We have:

- $\sigma = \sigma_{\mathbb{Z}_{12}[X]\setminus(2)} \wedge \sigma_{\mathbb{Z}_{12}[X]\setminus(3)}$,
- $\sigma_{\mathbb{Z}_{12}[X]\setminus(2)}(\mathbb{Z}_{12}[X]) = 3\mathbb{Z}_{12}[X]$, and $\sigma_{\mathbb{Z}_{12}[X]\setminus(3)}(\mathbb{Z}_{12}[X]) = 2\mathbb{Z}_{12}[X]$,
- $\sigma(\mathbb{Z}_{12}[X]) = 6\mathbb{Z}_{12}[X]$.

Therefore, we have the decomposition

$$\frac{\mathbb{Z}_{12}[X]}{6\mathbb{Z}_{12}[X]} \cong \frac{\mathbb{Z}_{12}[X]}{2\mathbb{Z}_{12}[X]} \times \frac{\mathbb{Z}_{12}[X]}{3\mathbb{Z}_{12}[X]} \cong \mathbb{Z}_2[X] \times \mathbb{Z}_3[X].$$

Example. 5.41.
As a byproduct of the theory we may show when an idealization $M \rtimes A$ is a totally σ–PIR, see page 74. Observe that, given a hereditary torsion theory σ, to study when $M \rtimes A$ is a totally σ–PIR we may consider M and A to be σ–torsionfree; indeed, we have $\sigma(M \rtimes A) = \sigma M \rtimes \sigma A$.

In this case, if $M = 0$ we have $M \rtimes A = A$. Otherwise, if $M \neq 0$, then $M \rtimes A$ is never a totally σ–PID, hence $0 \subseteq M \rtimes A$ is not a prime ideal.

If $M \rtimes A$ is a totally σ–PIR, then M and A are totally σ–principal. Conversely, if A is a totally σ–PIR, there is a decomposition $A = A_1 \oplus \cdots \oplus A_t$ being each A_i either an indecomposable totally σ–PIR. We may consider that A is one of the A_i's.

If we take a prime ideal $\mathfrak{p}$, maximal ideal in $\mathcal{K}(\sigma)$, and localize at $\mathfrak{p}$, there are three possibilities:

(1) $A_\mathfrak{p}$ is a field.
(2) $A_\mathfrak{p}$ is a PID, not a field.
(3) $A_\mathfrak{p}$ is a SPIR.

Only in case (1) we have that $M_\mathfrak{p} \rtimes A_\mathfrak{p}$ is a PIR, and for this we need that $M_\mathfrak{p} = A_\mathfrak{p}$. In this case $\mathscr{K}(\sigma) = \{0\}$, hence A must be a field, and $M = A$.

If we call $\mathfrak{P} = M \rtimes \mathfrak{p} \subseteq M \rtimes A$, then $(M \rtimes A)_\mathfrak{P} = M_\mathfrak{p} \rtimes A_\mathfrak{p}$; in consequence in cases (2) and (3) we have that $M \rtimes A$ is not a totally σ–PIR.

One may consult the following references for PIRs: [54], [30].

5.3 Dorroh extension

Algebras

Given a ring A, an A**–algebra** is an abelian group B satisfying:

(i) B is an A–module (in consequence $ab = ba$ for any $a \in A$ and $b \in B$),
(ii) B is a ring, not necessarily with unity nor commutative,
(iii) the action of A satisfies: $a(b_1 b_2) = (ab_1)b_2 = b_1(ab_2)$, for any $a \in A$ and $b_1, b_2 \in B$.

Given an A–algebra B a **left** A**–ideal** $\mathfrak{b}$ of B is an A–submodule closed under the multiplication by elements of B on the left side; i.e., $\mathfrak{b} \subseteq B$ is an A–submodule and a left ideal of B. (**Right** and **two–sided** B–ideals are defined in the same way.)

Given an A–algebra B, an A**–subalgebra** H of B is an A–submodule closed under the multiplication; i.e., $H \subseteq B$ is an A–submodule and $xy \in H$ for any $x, y \in H$.

Given A–algebras B_1 and B_2 and a map $f : B_1 \longrightarrow B_2$, we say that f is an A**–algebra map** whenever f is a module map that preserves the product; i.e., it satisfies:

(i) $f(ax + by) = af(x) + b(f(y)$, for any $a, b \in A$ and $x, y \in B_1$.
(ii) $f(xy) = f(x)f(y)$, for any $x, y \in B_1$.

If $f : B_1 \longrightarrow B_2$ is an A–algebra map, then

- $\mathrm{Ker}(f) = \{x \in B_1 \mid f(x) = 0\}$, the **kernel** of f is a two–sided A–ideal of B_1.
- $\mathrm{Im}(f) = \{f(x) \in B_2 \mid x \in B_1\}$, the **image** of f is an A–subalgebra of B_2.

The operation with A–ideals and A–subalgebras are defined in the natural way.

An A–algebra B is called a **unitary A–algebra** if there exists an element $u \in B$ such that $ux = x = xu$ for every $x \in B$. If for an A–algebra B there exists such element u then it is the only one satisfying this property, and we call it the **unity element** of B. If B_1 and B_2 are unitary A–algebras, with unities, u_1 and u_2 respectively, a **unitary A–algebra map** from B_1 to B_2 is an A–algebra map f satisfying $f(u_1) = u_2$.

Examples. 5.42.

(1) If $A = \mathbb{F}_2$, then $B = \mathbb{F}_2^{\mathbb{N}}$ is a unitary A–algebra.

(2) In the same situation $C = \mathbb{F}_2^{(\mathbb{N})}$ is an Q–algebra which is not unitary.

(3) Also $D = \{(a_n)_n \in B \mid (a_n)_n$ is finally constant$\}$ is a unitary A–algebra.

Examples. 5.43.

(1) For any A–module M the set of all A–endomorphisms of M is a unitary A–algebra.

(2) If we consider $A = \mathbb{R}$ and the A–module $M = A^{\mathbb{N}}$ and $S = \{f \in \mathrm{End}_A(M) \mid \dim_{\mathbb{R}}(\mathrm{Im}(f)) < \infty\}$, then S is an A–algebra which is not unitary.

(3) Let $A = \mathbb{Z}$, and $B = \mathbb{Z} \times \mathbb{Z}$, then B is an unitary A–algebra, and the map $j_1 : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$, defined $j_1(n) = (n, 0)$ is an A–algebra map and it is not a unitary A–algebra map.

Extensions

Let A be a ring and B an A–algebra; a new A–algebra can be build as follows:

$$B \rtimes A = \{(b, a) \in B \times A \mid b \in B \text{ and } a \in A\},$$

being the multiplication

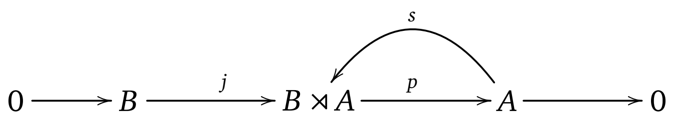
$$(b_1, a_1)(b_2, a_2) = (b_1 b_2 + b_1 a_2 + b_2 a_1, a_1 a_2).$$

The A–algebra $B \rtimes A$ is called the **Dorroh extension** of B by A, and it may be denoted also by B_1.

In this situation we have:

(1) B is identified with $\{(b,0) \mid b \in B\}$, the image of $j : B \longrightarrow B \rtimes A$, is an ideal of $B \rtimes A$.
(2) The image of $p : B \rtimes A \longrightarrow A$, i.e., $(B \rtimes A)/B$, is isomorphic to A as A–algebras.
(3) A is identified with $\{(0,a) \mid a \in A\}$, the image of $i : A \longrightarrow B \rtimes A$, is a subring of $B \rtimes A$. In addition, the map i is an injective A–algebra map.

There is a commutative diagram



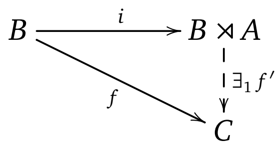
where $s(a) = (0,a)$, for any $a \in A$, hence it satisfies $ps = \mathrm{id}_A$.

Examples. 5.44.

(1) For any non–necessary unitary ring B the ring $B \rtimes \mathbb{Z}$ is called the **Dorroh ring extension** of the ring B.
(2) Let us consider the polynomial ring $\mathbb{Q}[X]$, and let $A = \mathbb{Q}$, $B = X\mathbb{Q}[X]$. In this situation B is an A–algebra, and the Dorroh extension $B \rtimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[X]$, which is a noetherian ring.
(3) The A–algebra D in Example (5.42.) is the Dorroh extension of C and A.

Lemma. 5.45. (Universal property of the Dorroh extension)

Given an A–algebra B, a unitary A–algebra C and an A–algebra map $f : B \longrightarrow C$, there exists a unique unitary A–algebra map $f' : B \rtimes A \longrightarrow C$ such that $f'_{|B} = f$; i.e., the following diagram commutes.



Proof. If f' exists satisfying these properties, then it must be defined $f'(b,a) = b(b) + 1a$, for any $(b,a) \in B \rtimes A$, being $1 \in C$ the unity element. If we define f' is this way, we may check that f' is a unitary A–algebra map, and it is the only one making the diagram commutative. □

Given an A–algebra B and an A–module M, a **structure of B–module** on M is given in defining an A–algebra map $\beta : B \longrightarrow \mathrm{End}_A(M)^{op}$; in consequence, there exists a unique unitary

A–algebra map $\beta' : B \rtimes A \longrightarrow \mathrm{End}_A(M)^{op}$ such that $\beta'_{|B} = \beta$. In conclusion, given an A–algebra map from B to $\mathrm{End}_A(M)^{op}$ is equivalent to give a unitary A–algebra map from $B \rtimes A$ to $\mathrm{End}_A(M)^{op}$, hence to give a structure of B–module on M is equivalent to give a structure of $B \rtimes A$–module.

A pair constituted by an A–module M and an A–algebra map $\beta : B \longrightarrow \mathrm{End}_A(M)^{op}$ is called a **B–module**; hence on an A–module M is equivalent to give either a structure of B–module or a structure of $B \rtimes A$–module. For any $b \in B$ and $m \in M$ we will write $\beta(b)(m) = mb$, so we have $\beta'(b,a) = mb + ma$, for any $(b,a) \in B \rtimes A$, and $m \in M$.

Given an A–module M, an A–submodule $N \subseteq M$ is a **B–submodule** whenever $nb \in N$ for any $n \in N$ and any $b \in B$. The sum and intersection of B–submodules, as the multiplication by left or right B–ideals (i.e., A–ideals of B), are defined in a natural way.

Given a B–module M, for any element $x \in M$ there is a smallest B–submodule $\langle x \rangle$ containing x: the intersection of all B–submodules containing x. This submodule can be also described as

$$x(B \rtimes A) = \{xb + xa \mid \ b \in B, a \in A\},$$

and it is called the **cyclic submodule generated** by x.

Given a B–module M, for any subset $S \subseteq A$, the **B–submodule generated** by S is the smallest B–submodule containing S; it is denoted as $\langle S \rangle$, and its elements are the elements of

$$S(B \rtimes A) = \left\{ \sum_i s_i x_i \mid \ s_i \in S, \ x_i \in B \cup A \right\} = SB + SA.$$

Similar definitions for left B–ideals, right B–ideals and two–sided B–ideals can be performed.

In the following we will work on commutative A–algebras; i.e., A–algebras in which the product or multiplication is commutative.

Finiteness conditions

Given a ring A, an A–algebra B and a B–module M, we say:

- M is **finitely generated** if there is finite subset $S \subseteq M$ such that $M = S(B \rtimes A)$, and **cyclic** if S is an unitary set.

- M is **noetherian** if every submodule is finitely generated, or equivalently if every ascending chain of B–ideals is estable.
- the A–algebra B is **noetherian** whenever it is a noetherian B–module. An A algebra B which is noetherian as A–module is named A–noetherian.

Example. 5.46.

(1) Observe that in the example $(X) \rtimes \mathbb{Q} = \mathbb{Q}[X]$ we have (X) is noetherian, and $\mathbb{Q}[X]$ also is.

(2) We consider $D = \mathbb{F}_2^{(\mathbb{N})}$ and $A = \mathbb{F}_2$, then $D \rtimes A$ is the subring of $\mathbb{F}_2^{\mathbb{N}}$ constituted by all sequences finally constant, neither D nor $D_1 = \mathbb{F}_2^{\mathbb{N}} = B$ are noetherian.

Chain conditions, I

The following result is well known. Let A be a ring and B an A–algebra, we'll study when the A–algebra $B \rtimes A$ is noetherian.

Theorem. 5.47.

Let A be a ring and B an A–algebra.

(1) If $B \rtimes A$ is noetherian, then A is a noetherian ring and B is a noetherian A–algebra.

(2) If A is a noetherian ring and B a noetherian A–algebra, then $B \rtimes A$ is noetherian

Proof. (1). If $B \rtimes A$ is noetherian then A, is noetherian because it is a homomorphic image of $B \rtimes A$. Otherwise, for any ascending chain $\{N_n\}_n$ of B–submodules of B, each N_n is a $B \rtimes A$–submodule of B; indeed, for any $(b,0) \in N_n$ and any $(x,y) \in B \rtimes A$ we have $(b,0)(x,y) = (bx+by,0) \in N_n$, hence N_n is a B–ideal of $B \rtimes A$. In consequence, the ascending chain is stable.

(2). On the other hand, if A is a noetherian ring and B is a noetherian A–algebra, then $B \rtimes A$ is noetherian. For any ascending chain $\{\mathfrak{b}_n\}_n$ of B–ideals of $B \rtimes A$, and any indices $n \leq m$ we have a commutative diagram of A–modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{b}_n \cap B & \longrightarrow & \mathfrak{b}_n & \longrightarrow & \frac{\mathfrak{b}_n}{\mathfrak{b}_n + B} \cong \frac{\mathfrak{b}_n + B}{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{b}_m \cap B & \longrightarrow & \mathfrak{b}_m & \longrightarrow & \frac{\mathfrak{b}_m}{\mathfrak{b}_m + B} \cong \frac{\mathfrak{b}_m + B}{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & B \rtimes A & \longrightarrow & A \cong \frac{B \rtimes A}{B} & \longrightarrow & 0
\end{array}$$

- The first column is a chain of B–ideals of B, hence it is stable; there exists $n_1 \in \mathbb{N}$ such that $\mathfrak{b}_{n_1} \cap B = \mathfrak{b}_m \cap B$ for any $m \geq n_1$.
- The third column is a claim of A–ideals of A, hence there exists an index $n_3 \in \mathbb{N}$ such that $\mathfrak{b}_{n_3} + B = \mathfrak{b}_m + B$ for any $m \geq n_3$.

If we take $n = \max\{n_1, n_3\}$, then we have $\mathfrak{b}_n = \mathfrak{b}_m$ for any $m \geq n$. In consequence, $B \rtimes A$ is noetherian. □

Using the same methodology as before, we obtain that $B \rtimes A$ is artinian if, and only if, A is artinian and B is an artinian B–algebra.

Corollary. 5.48.

Let A be a ring and B an A–algebra.

(1) If $B \rtimes A$ is artinian, then A is a artinian ring and B is an artinian B–algebra.

(2) If A is an artinian ring and B an artinian B–algebra, then $B \rtimes A$ is artinian.

Chain conditions, II

Let σ be a hereditary torsion theory in **Mod**–A, for any A–algebra B we consider $B_1 = B \rtimes A$, and the ring A–algebra map $i : A \longrightarrow B_1$. In this situation there is a hereditary torsion theory τ_1 is B_1, defined by

$$\mathscr{L}(\tau_1) = \{\mathfrak{c} \mid \mathfrak{c} \cap A \in \mathscr{L}(\sigma)\}.$$

In consequence, if $\mathfrak{c} \in \mathscr{L}(\tau_1)$ and $\mathfrak{h} = \mathfrak{c} \cap A \in \mathscr{L}(\sigma)$, then $\mathfrak{h}B_1 = \mathfrak{h}B \rtimes \mathfrak{h} \subseteq \mathfrak{c}$, and the set $\{\mathfrak{h}B \rtimes \mathfrak{h} \mid \mathfrak{h} \in \mathscr{L}(\sigma)\}$ is a filter basis for $\mathscr{L}(\tau_1)$.

On the other hand, using the map $j : B \longrightarrow B_1$, the topology $\mathscr{L}(\tau_1)$ induces a topology $\mathscr{L}'$ in B, defined

$$\mathscr{L}' = \{\mathfrak{b} \cap B \mid \mathfrak{b} \in \mathscr{L}(\tau_1)\},$$

in consequence, a filter basis for $\mathscr{L}'$ is

$$\mathscr{B} = \{\mathfrak{h}B \mid \mathfrak{h} \in \mathscr{L}(\sigma)\}.$$

This filter basis generates a hereditary torsion theory that we'll called τ.

Actually, there exists a natural topology in B defined by the hereditary torsion theory σ, which is

$$\mathscr{L}(B,\sigma) = \{H \subseteq B_A \mid B/H \in \mathscr{T}_\sigma\} = \{H \subseteq B_A \mid \text{there exists } \mathfrak{h} \in \mathscr{L}(\sigma) \text{ such that } \mathfrak{h}B \subseteq H\}.$$

In conclusion, for any hereditary topology σ in **Mod**–A we have topologies τ and τ_1, in B and in B_1, respectively, defined from σ.

Example. 5.49.

(1) Let $\Sigma \subseteq A$ be a multiplicative subset, and $\sigma = \sigma_\Sigma$ the hereditary torsion theory with Gabriel filter $\mathscr{L}(\sigma_\Sigma) = \{\mathfrak{h} \subseteq A \mid \mathfrak{h} \cap \Sigma \neq \varnothing\}$. We may consider τ_1 with Gabriel filter

$$\mathscr{L}(\tau_1) = \{\mathfrak{c} \subseteq B_1 \mid \mathfrak{c} \cap i(\Sigma) \neq \varnothing\},$$
$$\mathscr{L}(\tau) = \{\mathfrak{b} \subseteq B \mid \text{there exists } s \in \Sigma \text{ such that } \mathfrak{b} \cap sB \neq \varnothing\}.$$

(2) Let us consider $A = \mathbb{Z}$, $B = X^2\mathbb{Q}[X]$, for an indeterminate X, and $\Sigma = \{2^t \mid t \in \mathbb{N}\}$. In this situation we have:

$$\mathscr{L}(\sigma) = \{\mathfrak{a} \subseteq \mathbb{Z} \mid \mathfrak{a} \cap \Sigma \neq \varnothing\} = \{2^t\mathbb{Z} \mid t \in \mathbb{N}\}.$$
$$\mathscr{L}(\tau_1) = \{\mathfrak{c} \subseteq B_1 \mid \mathfrak{b} \cap \Sigma \neq \varnothing\} = \langle 2^t\mathbb{Z} + X^2\mathbb{Q}[X] \mid t \in \mathbb{N}\rangle.$$
$$\mathscr{L}(\tau) = \{\mathfrak{b} \subseteq B \mid \mathfrak{b} \cap sB \neq \varnothing \text{ for some } s \in \Sigma\} = \{X^2\mathbb{Q}[X]\}.$$

Theorem. 5.50.

*Given a ring A, an A–algebra B and a hereditary torsion theory σ in **Mod**–A, we have Gabriel filters $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau_1)$ in B and B_1, respectively, and the following statements are equivalent.*

(a) B_1 *is* τ_1*–noetherian.*

(b) A *is* σ*–noetherian and* B *is* τ*–noetherian.*

Proof. (a) $\Rightarrow$ (b). For any ascending chain of ideals $\{\mathfrak{a}_i\}_{i\in I}$ of A we have $\{\mathfrak{a}_i B_1\}_i$ is an ascending chain of ideals of B_1, and there exists $i \in I$ such that $\mathrm{Cl}^{B_1}_{\tau_1}(\mathfrak{a}_i B_1) = \mathrm{Cl}^{B_1}_{\tau_1}(\mathfrak{a}_j B_1)$ for any $j \geq i$.

For any $a \in \mathfrak{a}_j$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $aB_1\mathfrak{h} \subseteq \mathfrak{a}_i B_1$; for any $(x,y) \in B_1$ and any $h \in \mathfrak{h}$ we have the following identity: $(0,a)(x,y)(0,h) = (axh, ayh) \in \mathfrak{a}_i B_1$, hence $ayh \in \mathfrak{a}_i$; therefore, $a\mathfrak{h} \subseteq \mathfrak{a}_i$, and $\mathrm{Cl}^A_\sigma(\mathfrak{a}_j) \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a}_i)$.

Given an ascending chain $\{\mathfrak{b}_i\}_{i\in I}$ of B–ideals of B, each $\mathfrak{b}_i$ is a B_1–ideal, and there exists $i \in I$ such that $\mathrm{Cl}^{B_1}_{\tau_1}(\mathfrak{b}_i) = \mathrm{Cl}^{B_1}_{\tau_1}(\mathfrak{b}_j)$ for any $j \geq i$.

For any $b \in \mathfrak{b}_j$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $bB_1\mathfrak{h} \subseteq \mathfrak{b}_i$, hence, for every $(x,y) \in B_1$ and any $h \in \mathfrak{h}$ we have $(b,0)(x,y)(0,h) = (bxh + byh, 0) \in \mathfrak{b}_i$; in particular, $bB\mathfrak{h} \subseteq \mathfrak{b}_i$, and $b \in \mathrm{Cl}^B_\tau(\mathfrak{b}_i)$.

(b) $\Rightarrow$ (a). For any ascending chain $\{\mathfrak{c}_i\}_{i\in I}$ of ideals of B_1, and for any pair of indices i, j such that $i \leq j$, we consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{c}_i \cap B & \longrightarrow & \mathfrak{c}_i & \longrightarrow & \frac{\mathfrak{c}_i}{\mathfrak{c}_i + B} \cong \frac{\mathfrak{c}_i + B}{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{c}_j \cap B & \longrightarrow & \mathfrak{c}_j & \longrightarrow & \frac{\mathfrak{c}_j}{\mathfrak{c}_j + B} \cong \frac{\mathfrak{c}_j + B}{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & B_1 = B \rtimes A & \xrightarrow{p} & A \cong \frac{B_1}{B} & \longrightarrow & 0
\end{array}$$

We can take each ideal $\mathfrak{c}_i$ an ideal τ_1–closed. We claim $\mathfrak{c}_i \cap B \subseteq B$ is τ–closed since it is an A–submodule of $B_1/\mathfrak{c}_i$; by the hypothesis there exists an index $i_1 \in I$ such that $\mathfrak{c}_i \cap B = \mathfrak{c}_j \cap B$ for any $j \geq i$. Now we claim $\{\mathfrak{c}_i\}_i$ is stable. The chain $\{p(\mathfrak{c}_i)\}_i$ is σ–stable, i.e., there exists an index $i_2 \in I$ such that $\mathrm{Cl}^A_\sigma(p(\mathfrak{a}_i)) = \mathrm{Cl}^A_\sigma(p(\mathfrak{a}_j))$ For any $x \in \mathfrak{j}$. We take $i = \max\{i_1, i_2\}$. For any $j \geq i$ and any $x \in \mathfrak{c}_j$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $p(x)\mathfrak{h} \subseteq p(\mathfrak{c}_i)$. For any $h \in \mathfrak{h}$ there exists $c \in \mathfrak{c}_i$ such that $p(x)h = p(c)$, hence $p(xh - c) = 0$, and $xh - c \in \mathfrak{c}_j \cap B = \mathfrak{c}_i \cap B \subseteq \mathfrak{c}_i$. In conclusion $x\mathfrak{h} \subseteq \mathfrak{c}_i$, and $x \in \mathfrak{c}_i$, which is τ_1–closed.

Be careful because the chain ideals $p(\mathfrak{c}_i) \cong \frac{\mathfrak{c}_i + B}{B}$, $i \in I$, are not σ–closed in A. □

Similar technique could be used to show the next result.

Theorem. 5.51.
*Given a ring A, an A–algebra B and a hereditary torsion theory σ in **Mod**–A, we have Gabriel filters $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau_1)$ in B and B_1, respectively, and the following statements are equivalent.*

(a) B_1 is τ_1–artinian.

(b) A is σ–artinian and B is τ–artinian.

Chain conditions, III

We want to see the behaviour of chain conditions relative to totally torsion of the A–algebra extension. To do that we consider a ring A, an A–algebra B and a hereditary torsion theory σ in **Mod**–A. As we seen before, there are Gabriel filters $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau_1)$ in B and B_1 respectively; hence we may study the relationship between totally noetherian conditions in each of these three A–algebras.

We have a short exact sequence of A–modules:

$$0 \longrightarrow B \longrightarrow B_1 = B \rtimes A \stackrel{p}{\longrightarrow} B_1/B \cong A \longrightarrow 0$$

Theorem. 5.52.
*Given a ring A, an A–algebra B and a hereditary torsion theory σ in **Mod**–A, we have Gabriel filters $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau_1)$ in B and B_1, respectively, and the following statements are equivalent.*

(a) B_1 is totally τ_1–noetherian.

(b) A is totally σ–noetherian and B is totally τ–noetherian.

Proof. (a) $\Rightarrow$ (b). For any ascending chain $\{\mathfrak{a}_i\}_{i\in I}$ of ideals of A we have that $\{\mathfrak{a}_i B_1\}_i$ is an ascending chain of ideals of B_1, hence there exist $\mathfrak{h} \in \mathscr{L}(\sigma)$ and an index $i \in I$ such that $(\cup_j \mathfrak{a}_j B_1)\mathfrak{h} \subseteq \mathfrak{a}_i B_1$, hence $(\cup_j \mathfrak{a}_j)\mathfrak{h} \subseteq \mathfrak{a}_i$, and A is totally σ–noetherian.

For any ascending chain $\{\mathfrak{b}_i\}_{i\in I}$ of B–ideals of B we have that each $\mathfrak{b}_i$ is an ideal of B_1, hence there exist $\mathfrak{h} \in \mathscr{L}(\sigma)$ and an index $i \in I$ such that $(\cup_j \mathfrak{b}_j)\mathfrak{h} \subseteq \mathfrak{b}_i$, i.e., B is totally τ–noetherian.

(b) ⇒ (a). Given an ascending chain $\{\mathfrak{c}_i\}_{i\in I}$ of ideals of B_1 we have each $\mathfrak{c}_i \cap B$ is a B–ideal of B, hence there exist $\mathfrak{h}_1 \in \mathscr{L}(\sigma)$ and an index $i_1 \in I$ such that $(\cup_j(\mathfrak{c}_j \cap B))\mathfrak{h} \subseteq \mathfrak{b}_i \cap B$. On the other hand, $\{p(\mathfrak{c}_i)\}_i$ is an ascending chain of ideals of A, hence there exist $\mathfrak{h}_2 \in \mathscr{L}(\sigma)$ and an index $i_2 \in I$ such that $(\cup_j p(\mathfrak{c}_i))\mathfrak{h} \subseteq p(\mathfrak{c}_i)$. We take $I = \max\{i_1, i_2\}$, and $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{h}_2$. Then for any $j \geq i$ we have $p(\mathfrak{c}_j)\mathfrak{h} \subseteq p(\mathfrak{c}_i)$, so for any $c_j \in \mathfrak{c}_j$ and any $h \in \mathfrak{h}$ there exists $x_{j,h} \in \mathfrak{c}_i$ such that $p(c_j)h = p(x_{j,h})$, hence $c_jh - x_{j,h} \in \mathfrak{c}_j \cap B$. By the hypothesis we have $(\mathfrak{c}_j \cap B)\mathfrak{h} \subseteq \mathfrak{c}_i \cap B$, hence $(c_jh - x_{j,h})\mathfrak{h} \subseteq \mathfrak{c}_i \cap B \subseteq \mathfrak{c}_i$, and $c_jh\mathfrak{h} \subseteq \mathfrak{c}_i$. This means that $\mathfrak{c}_j\mathfrak{h}^2 \subseteq \mathfrak{c}_i$. In consequence, we have $(\cup_j \mathfrak{c}_j)\mathfrak{h}^2 \subseteq \mathfrak{c}_i$; therefore, B_1 is totally τ_1–noetherian. □

Similar technique could be used to show the next result.

Theorem. 5.53.
*Given a ring A, an A–algebra B and a hereditary torsion theory σ in **Mod**–A, we have Gabriel filters $\mathscr{L}(\tau)$ and $\mathscr{L}(\tau_1)$ in B and B_1, respectively, and the following statements are equivalent.*

(a) B_1 is totally τ_1–artinian.

(b) A is totally σ–artinian and B is totally τ–artinian.

References for this results are: [31] and [32].

5.4 Pullback construction

Definition and properties

Let A, B and C be commutative rings with unities, if $\alpha : A \to C$ and $\beta : B \to C$ are ring homomorphisms, the set $D := \{(a, b) \in A \times B|\ \alpha(a) = \beta(b)\}$ of $A \times B$ is called the **pullback** of α and β.

The pullback D can be described using the diagram:

$$\begin{array}{ccc} D & \xrightarrow{\alpha'} & B \\ {\scriptstyle\beta'}\Big\downarrow & & \Big\downarrow{\scriptstyle\beta} \\ A & \xrightarrow[\alpha]{} & C \end{array} \tag{5.1}$$

where α', β' are the restriction to D of the projection of $A \times B$ onto B and A, respectively.

Remark. 5.54.

The pullback D is a subring of $A \times B$.

Proof. Trivially, D is an abelian group, and $(1,1) \in D$. For multiplication, let (a_1, b_1) and $(a_2, b_2) \in D$, then $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$, we want to show that $(a_1a_2, b_1b_2) \in D$. For, $\alpha(a_1a_2) = \alpha(a_1)\alpha(a_2) = \beta(b_1)\beta(b_2) = \beta(b_1b_2)$. Hence, $(a_1a_2, b_1b_2) \in D$. □

Proposition. 5.55.

Let D be the pullback defined in (5.1), if the ring homomorphism α is surjective, then:

(1) α' *is surjective.*

(2) $\text{Ker}(\alpha) \cong \text{Ker}(\alpha')$

Moreover, the following diagram is also a pullback.

$$\begin{array}{ccccc} \text{Ker}(\alpha') & \xrightarrow{\alpha'^{k}} & D & \xrightarrow{\alpha'} & B \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\beta'} & & \Big\downarrow{\scriptstyle\beta} \\ \text{Ker}(\alpha) & \xrightarrow{\alpha^{k}} & A & \xrightarrow{\alpha} & C \end{array}$$

Proof. (1). Let $x \in B$, then $\beta(x) \in C$, since α is surjective then $\exists a \in A$ such that $\alpha(a) = \beta(x)$, hence, $(a, x) \in D$ and $\alpha'(a, x) = x$. Thus, α' is surjective.

(2). Since $\alpha\beta'\alpha'^{k} = \beta\alpha'\alpha'^{k} = 0$, then there exists a unique $\gamma : \text{Ker}(\alpha') \longrightarrow \text{Ker}(\alpha)$ such that $\alpha'^{k} = \alpha^{k}\gamma$. □

Therefore, The D–submodules of B are exactly the ideals of the ring B, and α' induces on B a natural structure of D–module by setting $d \cdot x := \alpha'(d)x$ for all $d \in D$ and $x \in B$.

There are many types of pullbacks according to type of maps to be used. We are interested with the type of pullbacks as in Proposition (5.55.). That is, α and α' are surjective ring homomorphisms, β and β' are injective ring homomorphisms.

Thus, for the following diagram, we have:

$$\begin{array}{ccccc} \mathfrak{k} & \longrightarrow & D & \twoheadrightarrow & B \\ \| & & \downarrow & & \downarrow \\ \mathfrak{k} & \longrightarrow & A & \twoheadrightarrow & C \end{array}$$

(1) $\mathfrak{k} := \mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha')$ is a common ideal of A and D.

(2) If $\mathfrak{a}$ is a common ideal of A and D, then $\mathfrak{a}/\mathfrak{k}$ is a common ideal of B and C. The converse is also true.

(3) For the previous pullback, we can consider the pullbacks in which B and C have no nonzero common ideals and we have: $C \cong \frac{A}{\mathfrak{k}}$.

Proposition. 5.56.

*Let D be the pullback defined in (5.1), such that A and D have $\mathfrak{k} := \mathrm{Ker}(\alpha)$ the maximum common ideal. That is, the **conductor** of A and D is $\mathfrak{k}$.*

Proof. Clearly, $\mathfrak{k}$ is an ideal of A and D. Now, let $\mathfrak{j}$ be any common ideal of A and D. Then, $D\mathfrak{j} \subseteq \mathfrak{j}$ and $\mathfrak{j} \subseteq A$. For any $(a,b) \in D$ and $j \in \mathfrak{j}$ we have: $(a,b)j = (aj, bj) \in \mathfrak{j} \cong \mathfrak{j} \times (0)$. Hence, $(aj, bj) = (aj, 0)$ and $aj \in \mathfrak{j} \subseteq A$. Therefore, $(aj, 0) \in \mathrm{Ker}(\alpha')$, consequently, $\mathfrak{j} \subseteq \mathfrak{k}$. □

Multiplicative subsets and hereditary torsion theory of the pullbacks

We are interested in studying the behaviour of hereditary torsion theories with respect to pullback.

$$\begin{array}{ccc} D & \xrightarrow{\alpha'} & B \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array} \tag{5.2}$$

First let us point out some useful facts.

Remark. 5.57.

(1) Let Σ be a multiplicative subset of the pullback D, then $\Sigma' := \alpha'(\Sigma)$ is a multiplicative subset of B.

(2) Let σ be a hereditary torsion theory in **Mod**–D, then $\alpha'(\sigma)$ is a hereditary torsion theory in **Mod**–B; since $\mathscr{L}(\alpha'(\sigma)) = \{\mathfrak{b} \subseteq B \mid \alpha'^{-1}(\mathfrak{b}) \in \mathscr{L}(\sigma)\}$, then we have: $\mathscr{L}(\alpha'(\sigma)) = \{\mathfrak{b} \subseteq B \mid$ there exists $\mathfrak{d} \in \mathscr{L}(\sigma)$ such that $\alpha'(\mathfrak{d}) \subseteq \mathfrak{b}\}$.

(3) In consequence, if σ is of finite type, then $\alpha'(\sigma)$ is of finite type, and there exists a bijective correspondence between $\mathscr{K}(\alpha'(\sigma))$ and $\{\mathfrak{p} \in \mathscr{K}(\sigma) \mid \operatorname{Ker}(\alpha') \subseteq \mathfrak{p}\}$, and similarly for $\mathscr{Z}(\alpha'(\sigma))$. In particular, if $\sigma = \wedge\{\sigma_{D\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathscr{K}(\sigma)\}$, then $\alpha'(\sigma) = \wedge\{\sigma_{B\setminus\mathfrak{q}} \mid \mathfrak{q} \in \mathscr{K}(\alpha'(\sigma))\}$.

(4) For any ideal $\mathfrak{d} \subseteq D$ we have $\alpha'(\mathrm{Cl}^D_\sigma(\mathfrak{d})) \subseteq \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d}))$; the equality holds WHENEVER $\operatorname{Ker}(\alpha') \subseteq \mathfrak{d}$. Indeed, if $\alpha'(x) \in \mathrm{Cl}^B_{\alpha'(\sigma)}(\mathfrak{d})$, there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\alpha'(x)\mathfrak{h} \subseteq \alpha'(\mathfrak{d})$, hence $x\mathfrak{h} \in \mathfrak{d} + \operatorname{Ker}(\alpha') = \mathfrak{d}$, and $\alpha'(x) \in \alpha'(\mathrm{Cl}^D_\sigma(\mathfrak{d}))$.

(5) For any σ–finitely generated ideal $\mathfrak{d} \subseteq D$ we have $\alpha'(\mathfrak{d})$ is $\alpha'(\sigma)$–finitely generated. Indeed, if $\mathfrak{d}$ is σ–finitely generated there exists $d_1, \ldots, d_s \in D$ such that $\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s)) = \mathrm{Cl}^D_\sigma(\mathfrak{d})$, hence $\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s) + \operatorname{Ker}(\alpha')) = \mathrm{Cl}^D_\sigma(\mathfrak{d} + \operatorname{Ker}(\alpha'))$, and

$$\begin{aligned}
\mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d})) &= \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d} + \operatorname{Ker}(\alpha'))) \\
&= \alpha'(\mathrm{Cl}^D_\sigma(\mathfrak{d} + \operatorname{Ker}(\alpha'))) \\
&= \alpha'(\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s) + \operatorname{Ker}(\alpha'))) \\
&= \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'((d_1, \ldots, d_s) + \operatorname{Ker}(\alpha'))) \\
&= \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(d_1, \ldots, d_s)).
\end{aligned}$$

The converse holds whenever $\operatorname{Ker}(\alpha')$ is σ–finitely generated, $\operatorname{Ker}(\alpha') \subseteq \mathfrak{d}$, and σ is of finite type. Indeed, if $\mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d}))$ is $\alpha'(\sigma)$–finitely generated, there exist $d_1, \ldots, d_s \in D$ such that $\mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(d_1, \ldots, d_s)) = \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d}))$, hence

$$\alpha'(\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s) + \operatorname{Ker}(\alpha')) = \mathrm{Cl}^B_{\alpha'(\sigma)}((d_1, \ldots, d_s)) = \mathrm{Cl}^B_{\alpha'(\sigma)}(\alpha'(\mathfrak{d})) = \alpha'(\mathrm{Cl}^D_\sigma(\mathfrak{d} + \operatorname{Ker}(\alpha'))).$$

Therefore, $\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s) + \operatorname{Ker}(\alpha')) \subseteq \mathrm{Cl}^D_\sigma(\mathfrak{d}) + \operatorname{Ker}(\alpha') = \mathrm{Cl}^D_\sigma(\mathfrak{d})$. On the other hand, for any $x \in \mathrm{Cl}^D_\sigma(\mathfrak{d})$ there exists $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $x\mathfrak{h} \subseteq \mathfrak{d}$, hence $\alpha'(x)\mathfrak{h} \subseteq \mathfrak{d}$, and there exists

$\mathfrak{h}' \in \mathscr{L}(\sigma)$ such that $\alpha' x\mathfrak{h}\mathfrak{h}' \subseteq \alpha'(d_1, \ldots, d_s)$, and $x\mathfrak{h}\mathfrak{h}' \subseteq (d_1, \ldots, d_s) + \mathrm{Ker}(\alpha')$; i.e., $x \in \mathrm{Cl}^D_\sigma((d_1, \ldots, d_s) + \mathrm{Ker}(\alpha'))$, and the equality $\mathrm{Cl}^D_\sigma((d_1, \ldots, d_s)) + \mathrm{Ker}(\alpha') = \mathrm{Cl}^D_\sigma(\mathfrak{d})$ holds.

(6) A similar result holds for totally σ–finitely generated ideals.

Proposition. 5.58.

*Let D be the pullback defined in (5.2), if σ is a finite type hereditary torsion theory in **Mod**–D, then the following are equivalent:*

(a) D is an σ–noetherian ring.

(b) B is an $\alpha'(\sigma)$–noetherian ring and $\mathrm{Ker}(\alpha')$ *is an σ–noetherian D–module.*

Proposition. 5.59.

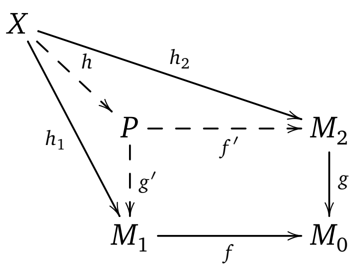
*Let D be the pullback defined in (5.2), if σ is a finite type hereditary torsion theory in **Mod**–D, then the following are equivalent:*

(a) D is a totally σ–noetherian ring.

(b) B is a totally $\alpha'(\sigma)$–noetherian ring and $\mathrm{Ker}(\alpha')$ *is totally σ–noetherian.*

General results

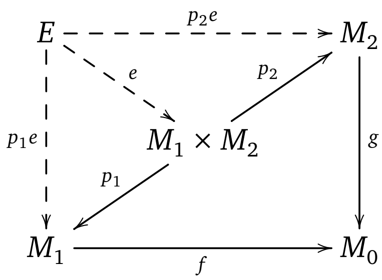
Let M_1, M_2, M_0 be A–modules, and $f : M_1 \longrightarrow M_0$, $g : M_2 \longrightarrow M_0$ be maps, The pullback of f and g is a pair $(P, \{g', f'\})$ such that $f'g = g'f$, and for any pair $(X, \{h_1, h_2\})$ such that $gh_2 = fh_1$ there is a map $h : X \longrightarrow P$ such that $h_1 = f'h$ and $h_2 = g'h$.



(5.3)

The construction of P can be do as follows: build the direct product $M_1 \times M_2$, the projections

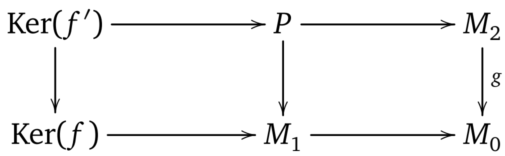
p_1 and p_2, and the equalizer (E, e) of $f p_1$ and $g p_2$:



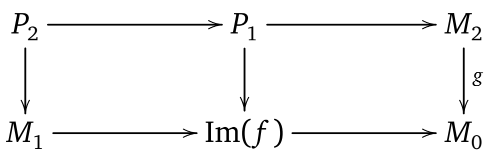
In consequence, the elements of E are:

$$E = \{(m_1, m_2) \in M_1 \times M_2 \mid \; g(m_2) = f(m_1)\}.$$

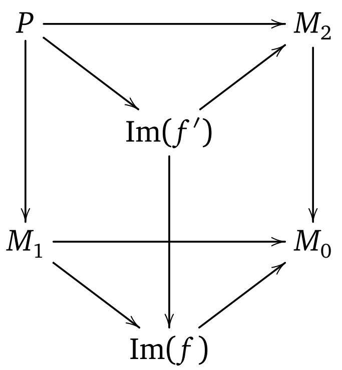
The first property of a pullback diagram, like in (5.3) is related to the kernel of the opposite maps. Indeed, in such a diagram we have an isomorphism



Given a pullback like in (5.3), a second property is related with that factorization of homomorphisms. We may consider the factorization of f in an epimorphism and a monomorphism: $f = f^{ck} f^{kc}$, which produces a two pullback diagrams:

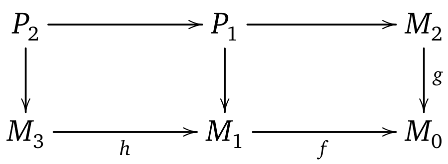


On the other hand we have the natural diagram produced by the factorizations of f and f':



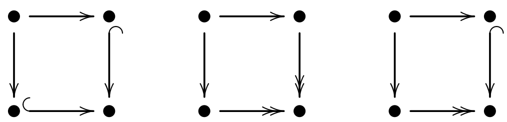
Indeed, there exist isomorphisms $P_1 \cong \mathrm{Im}(f')$ and $P_2 \cong P$ such that the two faces of the last diagram are pullback squares.

The third property on pullback squares says that two pullback squares produces a pullback square.



If P_1 and P_2 are pullback, then P_2 is the pullback of fh and g.

As a consequence, to study pullback squares we may restrict to consider one of the following types:



Taking into account that the opposite map of a monomorphism is a monomorphism, and the opposite map of an epimorphism is an epimorphism.

In the particular case in which M_0, M_1 and M_2 are rings and f, g ring maps, in a pullback square we may give structure of ring to P, and, in addition, the maps f' and g' also are ring maps.

Indeed, the structure is given via the direct product. Let us consider a pullback of rings like

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{5.4}$$

If we consider the case

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & C \end{tikzcd} \tag{5.5}$$

since f and g are surjective maps, there are maps between the spectrum, and we may build a diagram like the following one;

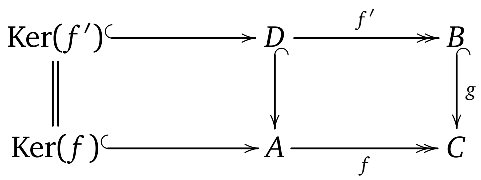
$$\begin{tikzcd} \mathrm{Spec}(C) \arrow[rr, "g^*"] \arrow[dd, "f^*"'] & & \mathrm{Spec}(B) \arrow[dl] \arrow[dd] \arrow[ddr, "(f')^*"] & \\ & \mathrm{Spec}(A) \overset{\bullet}{\cup} \mathrm{Spec}(B) \arrow[dr] & & \\ \mathrm{Spec}(A) \arrow[ur] \arrow[rr] \arrow[drrr, "(g')^*"'] & & Z \arrow[dr, dashed] & \\ & & & \mathrm{Spec}(D) \end{tikzcd} \tag{5.6}$$

In consequence, $\mathrm{Spec}(D)$ is related with the push-out of the set–maps between the spectra. If we impose conditions to A, B, C and f, g, then we may obtain a useful description of $\mathrm{Spec}(D)$, but in general we only will have information on the prime ideals of D and the prime ideals of A and B containing the kernels of f and g, respectively.

The mixed case

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, hook, "g"] \\ A \arrow[r, two heads, "f"'] & C \end{tikzcd} \tag{5.7}$$

corresponds to the case of a ring extension in which the rings sharing an ideal, in this case the kernel of the epimorphisms.



5.5 Amalgamated algebras

Definitions and main results

Let A be a ring and $f : A \longrightarrow B$ be a ring map, hence B is an A–module with action $a \cdot b = f(a)b$, for any $a \in A$ and $b \in B$. For any ideal $H \subseteq B$ we consider

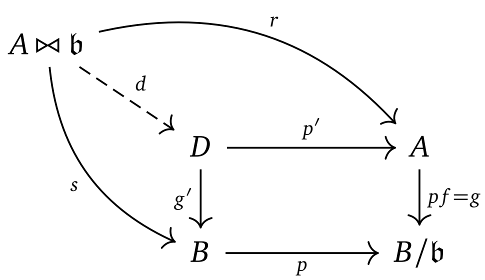
$$A \bowtie H = A \bowtie^f H = \{(a, f(a)+h) \in A \times B \mid a \in A, h \in H\}.$$

It is a subring of B, with componentwise multiplication, and unity $(1,1)$. We call $A \bowtie H$ the **amalgamated algebra** of A and H through f.

There are useful maps related with $A \bowtie H$:

- The map $j : A \longrightarrow A \bowtie H$, defined $j(a) = (a, f(a))$, is a ring map; hence every $A \bowtie H$-module is inherited an A–module structure.
- The map $t : H \longrightarrow A \bowtie H$, defined $t(h) = (0,h)$, is an A–module map.
- There is a ring isomorphism $A \bowtie H/0 \bowtie H \cong A \bowtie 0$.

We have that the amalgamated algebra is an example of a particular pullback construction. Indeed, for any ring map $f : A \longrightarrow B$ and any ideal $\mathfrak{b} \subseteq B$ we consider the pullback diagram



being $p: B \longrightarrow B/\mathfrak{b}$ the projection and $g = pf : A \longrightarrow B/\mathfrak{b}$ the composition. In this situation, the pullback of p and g is $(D, \{g', p'\})$. If we consider the maps $r: A \bowtie \mathfrak{b} \longrightarrow A$, defined $r(x,y) = x$, and $s: A \bowtie \mathfrak{b} \longrightarrow B$, defined $s(x,y) = y$, we have $gr = pfr = ps$, and there exists a unique ring map $d: A \bowtie \mathfrak{b} \longrightarrow D$ such that $r = p'd$ and $s = g'd$. The description of d is just $d(x,y) = (x,y)$; hence it is a bijection, i.e., an isomorphism.

Since p is a surjective map, then p' also is a surjective map, hence we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ker}(p') & \longrightarrow & A \bowtie \mathfrak{b} & \overset{p'}{\twoheadrightarrow} & A \\ \downarrow & & \downarrow g' & \overset{f}{\swarrow} & \downarrow pf=g \\ \mathrm{Ker}(p) = \mathfrak{b} & \longrightarrow & B & \underset{p}{\twoheadrightarrow} & B/\mathfrak{b} \end{array} \tag{5.8}$$

If we start from a hereditary torsion theory σ in **Mod**–A, it defines hereditary torsion theories $\sigma_{A\bowtie\mathfrak{b}}$ in **Mod**–$A \bowtie \mathfrak{b}$ and σ_B in **Mod**–B in the usual way:

$$\begin{aligned} \mathscr{L}(\sigma_{A\bowtie\mathfrak{b}}) &= \{\mathfrak{d} \subseteq A \bowtie \mathfrak{b} \mid j^{-1}(\mathfrak{d}) \in \mathscr{L}(\sigma)\} \\ &= \{\mathfrak{d} \subseteq A \bowtie \mathfrak{b} \mid \text{there exists } \mathfrak{a} \in \mathscr{L}(\sigma) \text{ such that } j(\mathfrak{a}) \subseteq \mathfrak{d}\}. \\ \mathscr{L}(\sigma_B) &= \{\mathfrak{c} \subseteq B \mid f^{-1}(\mathfrak{c}) \in \mathscr{L}(\sigma)\}. \end{aligned} \tag{5.9}$$

In this situation, the ring map $p': A \bowtie \mathfrak{b} \longrightarrow A$ defines a hereditary torsion theory τ in **Mod**–A whose Gabriel filter is

$$\begin{aligned} \mathscr{L}(\tau) &= \{\mathfrak{a} \subseteq A \mid p'^{-1}(\mathfrak{a}) \in \mathscr{L}(\sigma_{A\bowtie\mathfrak{b}})\} \\ &= \{\mathfrak{a} \subseteq A \mid j^{-1}p'^{-1}(\mathfrak{a}) \in \mathscr{L}(\sigma)\} \\ &= \{\mathfrak{a} \subseteq A \mid (p'j)^{-1}(\mathfrak{a}) \in \mathscr{L}(\sigma)\} \\ &= \mathscr{L}(\sigma). \end{aligned}$$

Before a brief remark: if σ is a finite type hereditary torsion theory in **Mod**–A, then $\sigma_{A\bowtie\mathfrak{b}}$ and σ_B are of finite type. In this section we assume, unless otherwise stated, that all hereditary torsion theory is of finite type.

In consequence, given a hereditary torsion theory σ in **Mod**–A, if $A \bowtie \mathfrak{b}$ is $\sigma_{A\bowtie\mathfrak{b}}$–noetherian, then A is σ–noetherian, and $\mathrm{Ker}(p')$ is $\sigma_{A\bowtie\mathfrak{b}}$–noetherian. Since there is an isomorphism of A–modules $\mathrm{Ker}(p') \cong \mathrm{Ker}(p) = \mathfrak{b}$, hence of $A \bowtie \mathfrak{b}$–modules; if $\mathrm{Ker}(p')$ is $\sigma_{A\bowtie\mathfrak{b}}$–noetherian, then $\mathrm{Ker}(p)$ is σ_B–noetherian.

Let us denote by B' the subring $f(A) + \mathfrak{b} \subseteq B$. Observe that, in considering B', we may also obtain $A \bowtie \mathfrak{b}$ as the pullback of A and B'; in this case the maps g and g' are surjective.

If we consider this last situation, B' is $\sigma_{B'}$–noetherian whenever $A \bowtie \mathfrak{b}$ is $\sigma_{A\bowtie\mathfrak{b}}$–noetherian, the converse also holds; hence the following result holds.

Theorem. 5.60.

*Let σ is a finite type hereditary torsion theory in **Mod**–A, the following statements are equivalent:*

(a) $A \bowtie \mathfrak{b}$ *is* $\sigma_{A\bowtie\mathfrak{b}}$*–noetherian.*

(b) A *is* σ*–noetherian and* $B' = f(A) + \mathfrak{b}$ *is* $\sigma_{B'}$*–noetherian (=* $\sigma_{A\bowtie\mathfrak{b}}$*–noetherian).*

(c) A *is* σ*–noetherian and* $\mathrm{Ker}(p) = \mathfrak{b} \subseteq B$ *is* σ_B*–noetherian.*

The same result holds if we consider totally noetherian instead of noetherian. In this case we obtain a generalization to hereditary torsion theories of [41, Theorem 3.2].

In the same situation we have the following consequence:

Corollary. 5.61.

The following statements are equivalent:

(a) A *is* σ*–noetherian.*

(b) $\mathrm{Ker}(f)$ *is* σ*–noetherian and* B *is* $f(\sigma)$*–noetherian.*

And similarly for totally noetherian!

We may apply these results to a particular case of the pullback construction.

Proposition. 5.62.

Let

$$\begin{array}{ccc} D & \xrightarrow{f'} & A \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & C \end{array}$$

*be a pullback of rings such that f is surjective. For any finite type hereditary torsion theory σ in **Mod**–D the following statements are equivalent:*

(a) D *is* σ*–noetherian.*

(b) $\mathrm{Ker}(f)$ *is* σ*–noetherian and* A *is* $f'(\sigma)$*–noetherian.*

Proof. Since f is surjective, f' is also surjective; on the other hand there $\mathrm{Ker}(f) \cong \mathrm{Ker}(f')$, and the sequence $0 \longrightarrow \mathrm{Ker}(f') \xrightarrow{i} D \xrightarrow{f'} A \longrightarrow 0$ is short exact. Hence, D is σ-Noetherian ring if, and only if, A and $\mathrm{Ker}(f)$ are σ-Noetherian D-modules.

The D–submodules of A are exactly the ideals of A because f' is surjective, then A is an σ–Noetherian D-module if, and only if, A is a $f'(\sigma)$-Noetherian ring. Therefore, D is σ-Noetherian ring if, and only if, A is a $f'(\sigma)$-Noetherian ring and $\mathrm{Ker}(f)$ is an σ-Noetherian D-module. □

Now, we study the σ-Artinian property on a pullback. We start by a proposition.

Proposition. 5.63.

Let

$$\begin{array}{ccc} D & \xrightarrow{f'} & A \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & C \end{array}$$

be a pullback of rings such that f *is surjective. For any finite type hereditary torsion theory* σ *in* ***Mod****–*D *the following statements are equivalent:*

(a) D *is* σ*–artinian.*

(b) $\mathrm{Ker}(f)$ *is* σ*–artinian and* A *is* $f'(\sigma)$*–artinian.*

Proof. The proof is similar to the proof of Proposition (5.62.). □

The same results hold if we consider either totally σ–noetherian or totally σ–artinian rings.

Let $j : A \longrightarrow A \bowtie \mathfrak{b}$ be the natural embedding defined by $j(x) = (x, f(x))$ for all $x \in A$. If $\Sigma \subseteq A$ is a multiplicative subset of A, then clearly, $\Sigma' = \{(s, f(s)) \mid s \in \Sigma\}$ and $f(\Sigma)$ are multiplicative subsets of $A \bowtie \mathfrak{b}$ and B, respectively. They meet with the correspondence described in (5.9).

Let us consider the pullback of rings

$$\begin{array}{ccc} A \bowtie \mathfrak{b} & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & \swarrow{\scriptstyle f} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[p]{} & B/\mathfrak{b} \end{array}$$

Theorem. 5.64.

Let $f : A \to B$ be a ring homomorphism, $\mathfrak{b} \subseteq B$ an ideal, and Σ a multiplicative subset of A such that $0 \notin \Sigma$, $\Sigma \cap (\mathfrak{b} :_A A \bowtie \mathfrak{b}) = \varnothing$ and $\mathrm{Ann}(\mathfrak{b}) \cap \Sigma = \varnothing$; then the following statements are equivalent:

(a) $A \bowtie \mathfrak{b}$ is a Σ'–artinian ring.

(b) A is a Σ–artinian ring and $B' = f(A) + \mathfrak{b}$ is an $f(\Sigma)$–artinian ring.

(c) A is a Σ–artinian ring and $\mathfrak{b}$ is Σ'–artinian $A \bowtie \mathfrak{b}$-module (= $f(\Sigma)$–artinian B–module).

Proof. (a) $\Rightarrow$ (b). Since $p_A : A \bowtie \mathfrak{b} \longrightarrow AA$ and $p_B : A \bowtie \mathfrak{b} \longrightarrow B'$ are surjective maps, then A is Σ–artinian and B' is $f(\Sigma)$–artinian.

(b) $\Rightarrow$ (c). Consider $\{\mathfrak{b}_i \mid i \in I\}$ a descending chain of $A \bowtie \mathfrak{b}$–submodules of $\mathfrak{b}$. Since every $A \bowtie \mathfrak{b}$–submodule of $\mathfrak{b}$ is an ideal of $B' = f(A) + \mathfrak{b}$, we have that the chain is of B'–ideals, hence there exists $s \in \Sigma$, and $k \in I$ such that $\mathfrak{b}_k f(s) \subseteq \mathfrak{b}_i$ for any $i \geq k$, $i \in I$; then $j(s)$ satisfies $\mathfrak{b}_k j(s) \subseteq \mathfrak{b}_i$ for any $i \in I$, $i \geq k$, and $\mathfrak{b}$ is a Σ'–artinian $A \bowtie \mathfrak{b}$–module.

(c) $\Rightarrow$ (a). We have short exact sequence of $A \bowtie \mathfrak{b}$–modules:

$$0 \longrightarrow \mathfrak{b} \longrightarrow A \bowtie \mathfrak{b} \longrightarrow A \longrightarrow 0,$$

and $\mathfrak{b}$, A are Σ'–artinian, hence $A \bowtie \mathfrak{b}$ also is. □

We can state and prove the corresponding result relative to a hereditary torsion theory σ in **Mod**–A similarly to Theorem (5.60.). Observe that the main difference is that when we consider multiplicative set in A, we are considering principal hereditary torsion theories, whereas with general hereditary torsion theories we are using finite type hereditary torsion theories; that is, multiplicative sets of finitely generated ideals.

Theorem. 5.65.

In the above situation the following statements are equivalent:

(a) $A \bowtie \mathfrak{b}$ is a $\sigma_{A \bowtie \mathfrak{b}}$–artinian ring.

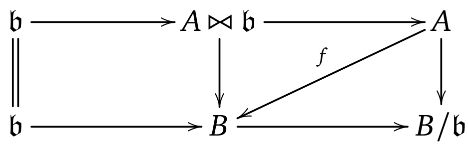
(b) A is a σ–artinian ring and $B' = f(A) + \mathfrak{b}$ is a $\sigma_{B'}$–artinian ring.

(c) A is a σ–artinian ring and $\mathfrak{b}$ is a σ_B–artinian B–module.

Proof. (a) $\Rightarrow$ (b). Consider the surjective map $r : A \bowtie \mathfrak{b} \longrightarrow A$ and the map $s : A \bowtie \mathfrak{b} \longrightarrow B$ whose image is $B' = f(A) \bowtie \mathfrak{b}$, and call $\sigma_{B'}$ the hereditary torsion theory induced from σ through the map f. Therefore, $B' = f(A) \bowtie \mathfrak{b}$ is $\sigma_{B'}$–artinian.

(b) $\Rightarrow$ (c). Given a decreasing chain of A–submodules $\{\mathfrak{b}_i \mid i \in I\}$ of $\mathfrak{b}$, we have a decreasing chain $\{0 \bowtie \mathfrak{b}_i \mid i \in I\}$ of $f(A) \bowtie \mathfrak{b}$, hence it stabilizes, and there exist an index k and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $(0 \bowtie \mathfrak{b}_k)j(\mathfrak{h}) \subseteq \cup_i(0 \bowtie \mathfrak{b}_i)$. In particular, $\mathfrak{b}_k f(\mathfrak{h}) = \mathfrak{b}_k\mathfrak{h} \subseteq \cup_i \mathfrak{b}_i$, and $\mathfrak{b}$ is a σ–artinian A–module.

(c) $\Rightarrow$ (a). We have a pull back diagram



with the top row a short exact sequence. For any ideal $\mathfrak{c} \subseteq A \bowtie \mathfrak{b}$ we have a short exact sequence $0 \to \mathfrak{b} \cap \mathfrak{c} \to \mathfrak{c} \to (\mathfrak{c} + \mathfrak{b})/\mathfrak{b} \to 0$; hence $\mathfrak{c}$ is a σ–artinian module. In particular, for any decreasing chain $\{\mathfrak{c}_i \mid i \in I\}$ of ideals of $A \bowtie \mathfrak{b}$ we have a decreasing chain of σ–artinian A–submodules, hence there exist and index k and $\mathfrak{h} \in \mathscr{L}(\sigma)$ such that $\mathfrak{c}_k j(\mathfrak{h}) \subseteq \cup_i \mathfrak{c}_i$. This means that $A \bowtie \mathfrak{b}$ is a $\sigma_{A \bowtie \mathfrak{b}}$–artinian ring. □

As we realize before, we may assume in this develop of the theory that $f : A \longrightarrow B$ is a surjective map, simple replacing B by $B' = f(A) + \mathfrak{b}$.

Remark. 5.66.

Let us point out that this result solves a question in [43, Question 3.8] when we take $\sigma = \sigma_\Sigma$, for a multiplicative set $\Sigma \subseteq A$.

Study through prime ideals

Based on [20], [22] and [47], M. D'Anna, C. A. Finocchiaro and M. Fontana studied the prime spectrum of the amalgamated algebra along an ideal. We recall their results from [16].

Corollary. 5.67.

Let $X := \operatorname{Spec}(A)$*,* $Y := \operatorname{Spec}(B)$*, and* $W := \operatorname{Spec}(A \bowtie \mathfrak{b})$*,* $\mathfrak{b}_0 := 0 \times \mathfrak{b} \subseteq A \bowtie \mathfrak{b}$*, and* $\mathfrak{b}_1 := f^{-1}(\mathfrak{b}) \times 0 \subseteq A \bowtie \mathfrak{b}$*. For all* $\mathfrak{p} \in X$ *and* $\mathfrak{q} \in Y$*, set*

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie \mathfrak{b} := \{(p, f(p) + b) \mid p \in \mathfrak{p}, b \in \mathfrak{b}\},$$
$$\bar{\mathfrak{q}}^f := \{(a, f(a) + b) \mid a \in A, b \in \mathfrak{b}, f(a) + b \in \mathfrak{q}\}.$$

Then the following statements hold:

(1) The prime ideals of $A \bowtie \mathfrak{b}$ *are of the type* $\mathfrak{p}'^f$ *or* $\bar{\mathfrak{q}}^f$*, for* $\mathfrak{p}$ *varying in* X *and* $\mathfrak{q}$ *in* $Y \setminus V(\mathfrak{b})$ *where* $V(\mathfrak{b}) = \{\mathfrak{q} \in Y \mid \mathfrak{q} \supseteq \mathfrak{b}\}$ *.*

(2) $\mathfrak{p}'^f$ *is a maximal ideal of* $A \bowtie \mathfrak{b}$ *if, and only if,* $\mathfrak{p} \subseteq A$ *is a maximal ideal.*

(3) Let $\mathfrak{q} \subseteq B$ *be a prime ideal not containing* $\mathfrak{b}$*. Then,* $\bar{\mathfrak{q}}^f$ *is a maximal ideal of* $A \bowtie \mathfrak{b}$ *if, and only if,* $\mathfrak{q} \subseteq B$ *is a maximal ideal.*

Proof. See [16]. □

Now we study totally σ–prime ideals and its behaviour with respect to the amalgamation construction.

Un relation with the amalgamated ring extension, let us consider a finite type hereditary torsion theory in **Mod**–A; we may assume that $0 \notin \mathscr{L}(\sigma)$, hence the same holds for $\sigma_{A\bowtie\mathfrak{b}}$ and σ_B.

Proposition. 5.68.

For any ring map $f : A \longrightarrow B$ *and any ideal* $\mathfrak{b} \subseteq B$ *the following statements hold:*

(1) For any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ *we have* $\mathfrak{p}'^f = \mathfrak{p} \bowtie \mathfrak{b} \in \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$*.*

(2) For any prime ideal $\mathfrak{q} \in \mathscr{K}(\sigma_B)$ *we have* $\bar{\mathfrak{q}}^f \in \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$*.*

(3) For any prime ideal $\mathfrak{Q} \in \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$*, if* $\mathfrak{Q} = \mathfrak{p}'^f$*, for some* $\mathfrak{p} \in \operatorname{Spec}(A)$*, then* $\mathfrak{p} \in \mathscr{K}(\sigma)$*; if* $\mathfrak{Q} = \bar{\mathfrak{q}}^f$*, for some* $\mathfrak{q} \in \operatorname{Spec}(B)$*, then* $\mathfrak{q} \in \mathscr{K}(\sigma_B)$*.*

This result can be extended to consider totally prime ideals.

Proof. (1). Since $\mathfrak{p}'^f \subseteq A \bowtie \mathfrak{b}$ is prime, if $\mathfrak{p}'^f \notin \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$, then $\mathfrak{p}'^f \in \mathscr{Z}(\sigma_{A\bowtie\mathfrak{b}})$, hence $\mathfrak{p} \in \mathscr{Z}(\sigma)$, which is a contradiction.

(2). If $\bar{\mathfrak{q}}^f \notin \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$, then $\bar{\mathfrak{q}}^f \in \mathscr{Z}(\sigma_{A\bowtie\mathfrak{b}})$, hence $j^{-1}(\bar{\mathfrak{q}}^f) \in \mathscr{L}(\sigma)$. In this case we have:

$$\begin{aligned} j^{-1}(\bar{\mathfrak{q}}^f) &= \{a \in A \mid j(a) = (a, f(a)) \in \bar{\mathfrak{q}}^f\} \\ &= \{a \in A \mid f(a) \in \mathfrak{q}\} = f^{-1}(\mathfrak{q}). \end{aligned}$$

In consequence, $\mathfrak{q} \in \mathscr{Z}(\sigma_B)$, which is a contradiction.

(3). If $\mathfrak{Q} = \mathfrak{p}'^f$, for some $\mathfrak{p} \in \mathrm{Spec}(A)$, and $\mathfrak{Q} \in \mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$, then $\mathfrak{p} = j^{-1}(\mathfrak{Q}) \in \mathscr{K}(\sigma)$. On the other hand, if $\mathfrak{Q} = \bar{\mathfrak{q}}^f$, for some $\mathfrak{q} \in \mathrm{Spec}(B)$, then $f^{-1}(\mathfrak{q}) = j^{-1}(\mathfrak{Q}) \in \mathscr{K}(\sigma)$; hence $\mathfrak{q} \in \mathscr{K}(\sigma_B)$.

□

Now, if we apply Cohen–like's theorem, see [15] or [32], for the amalgamation, then we have:

Theorem. 5.69.

For any ring map $f : A \longrightarrow B$, any ideal $\mathfrak{b} \subseteq B$, and any finite type hereditary torsion theory in ***Mod****–A the following statements are equivalent:*

1. *$A \bowtie \mathfrak{b}$ is totally σ-Noetherian.*
2. *Every totally σ-prime ideal of $A \bowtie \mathfrak{b}$ is totally σ-finite.*
3. *Every prime ideal in $\mathscr{K}(\sigma_{A\bowtie\mathfrak{b}})$ is totally σ-finite.*

It is easy to prove that, if $\mathfrak{b}$ is totally σ_B–finitely generated, for any prime ideal $\mathfrak{p} \in \mathscr{K}(\sigma)$ we have $\mathfrak{p}$ is totally σ–finitely generated if, and only if, $\mathfrak{p}'^f$ is totally $\sigma_{A\bowtie\mathfrak{b}}$–finitely generated. However we have had difficulties in proving a similar result for the ideals $\mathfrak{q}$ and $\bar{\mathfrak{q}}^f$.

If σ is a hereditary torsion theory in **Mod**–$A \bowtie^f \mathfrak{b}$, it is known that for any prime ideal $\mathfrak{p}$ of $A \bowtie^f \mathfrak{b}$ we have either $\mathfrak{p} \in C(A \bowtie^f \mathfrak{b}, \sigma)$ or $\mathfrak{p} \in \mathscr{L}(\sigma)$, i.e., either $(A \bowtie^f \mathfrak{b})/\mathfrak{p}$ is σ-torsionfree or $(A \bowtie^f \mathfrak{b})/\mathfrak{p}$ is σ-torsion.

Corollary. 5.70.

Let σ be a finite type hereditary torsion theory in ***Mod****–$A \bowtie^f \mathfrak{b}$, the following are equivalent:*

1. *$A \bowtie^f \mathfrak{b}$ is totally σ-noetherian.*
2. *Every prime ideal in $\mathscr{K}(\sigma)$ is totally σ-finitely generated.*

The amalgamated algebras along an ideal generalizes other important constructions, such as the idealization, see section (5.1).

The next proposition proves that the idealization is a special case of the amalgamated algebras.

Proposition. 5.71. ([45])
Let M be an A–module and $f : A \longrightarrow M \rtimes A$ the ring map defined $f(a) = (0,a)$. If take $\mathfrak{b} = M \rtimes 0 \subseteq M \rtimes A$, then $M \rtimes A \cong A \bowtie^f \mathfrak{b}$.

Proof. We'll represent the element of $M \rtimes A$ as $m + a$, instead of (m,a). Let $\lambda : M \rtimes A \longrightarrow A \bowtie^f (M \rtimes 0)$ defined by $\lambda(m,a) = (a, a+m)$; it is well defined, and we can show that it is a ring map isomorphism. Indeed, $\lambda(0,1) = (1,1)$, and for elements $a, a' \in A$ and $m, m' \in M$ we have:

$$\begin{aligned}\lambda((m,a)+(m',a')) &= \lambda(m+m', a+a') = (a+a', a+a'+m+m') \\ &= (a, a+m)+(a', a'+m') = \lambda(m,a)+\lambda(m',a').\end{aligned}$$

$$\begin{aligned}\lambda((m,a)(m'+a')) &= \lambda(ma'+m'a, aa') = (aa', aa'+ma'+m'a) \\ &= (a, a+m)(a', a'+m') = \lambda(m,a)\lambda(m',a').\end{aligned}$$

Trivially, λ is bijective. Hence λ is an isomorphism ring map. □

Therefore, we can easily apply the previous theorems of the amalgamation on the idealization. We finish by observing that the amalgamated algebra along an ideal need not to be an idealization. Since the idealization is always non-reduced when $M \neq 0$ but the amalgamation can be even a domain.

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