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Quantum black holes in Loop Quantum Gravity

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We study the quantization of spherically symmetric vacuum spacetimes within loop quantum gravity. In particular, we give additional details about our previous work in which we showed that one could complete the quantization of the model and that the singularity inside black holes is resolved. Moreover, we consider an alternative quantization based on a slightly different kinematical Hilbert space. The ambiguity in kinematical spaces stems from how one treats the periodicity of one of the classical variables in these models. The corresponding physical Hilbert spaces solve the diffeomorphism and Hamiltonian constraint but their intrinsic structure is radically different depending on the kinematical Hilbert space one started from. In both cases there are quantum observables that do not have a classical counterpart. However, one can show that at the end of the day, by examining Dirac observables, both quantizations lead to the same physical predictions.

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I. INTRODUCTION

One of the problems that blocks the completion of the quantization program of loop quantum gravity [1] is the suitable implementation of the dynamics. As is usual in canonical approaches to gravity, the dynamics is codified in a set of first class constraints with an algebra involving structure functions, which prevents the direct application of Dirac quantization approach [2]. This problem is also inherited by some of the symmetry reduced models of the theory. Although it is absent in the most studied symmetry reduced models corresponding to homogeneous spacetimes, the so-called loop quantum cosmology [3], the problem reappears as soon as one considers models with spatial dependence in the variables. Some of the most relevant reduced models to be studied, due to their physical implications, are the spherically symmetric spacetimes. They can for instance describe the final stage of the gravitational collapse of astrophysical objects. However, the classical description turns out to be incomplete since this kind of geometries is characterized by the presence of a classical singularity. These reduced models are also interesting from the technical point of view, since they have a Hamiltonian constraint representing invariance under time reparametrizations and a diffeomorphism constraint associated with the symmetry under redefinitions of the radial coordinate. It is one of the simplest inhomogeneous reduced model of general relativity.

The first attempts to quantize spherically symmetric vacuum models [4, 5] applied standard quantization techniques to the "old" (complex) Ashtekar variables and the traditional metric variables respectively, performed a series of canonical transformations and gauge fixings and yielded a quantum theory where the physical states of the spacetime correspond to superpositions of Schwarzschild geometries peaked at a given mass. There is no sense in which the singularity of the spacetime is eliminated by the quantization, as it is in loop quantum cosmology. A reanalysis of these approaches using the "modern" (real) Ashtekar variables and performing a loop quantization yielded essentially the same result [6]. It appears that the unexpected presence of the singularity within these novelty quantization approaches is due to the "severe gauge fixing" adopted for the model before attempting the quantization and there was too little left in terms of dynamics for quantization to be able to do anything about the singularity. On the other hand, several studies of the interior [7] of the Schwarzschild spacetime, exploiting its isometry to the Kantowski–Sachs metric and treating it with loop quantum cosmology techniques, have indicated that the singularity is resolved, therefore creating a tension with the previous studies based on gauge fixing. These are the main motivations to explore alternative quantization approaches where one quantizes before completely gauge fixing the model.

We base our work on previous papers that developed the spherically symmetric framework. The latter was already studied adopting the complex Ashtekar variables [5, 8] and the Ashtekar-Barbero connection [9]. A more complete

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description carried out in Refs. [10, 11] allowed to establish the kinematical framework together with a prescription for the quantum constraints. However, the physical states were not provided. In more recent work [12] it was shown that a redefinition of the constraint algebra of the diffeomorphism and Hamiltonian constraint can be carried out that turns it into a Lie algebra. One can then apply the Dirac quantization approach and solve the model quantum mechanically. The physical space of states was identified and the metric operator realized as an evolving constant of motion and shown to be singularity free. Moreover, new quantum Dirac observables without classical counterpart were identified.

In this paper we give more details about the study of Ref. [12]. We will also analyze two different quantization prescriptions stemming from different choices of kinematical Hilbert space. It turns out that the introduction of a diffeomorphism invariant inner product in one plus one dimensions admits more than one kinematical implementation. The reason for the ambiguity is that one of the variables of the problem (associated with the direction transversal to the radial one) is a scalar. As is usual in loop quantum gravity, when one has scalar variables, one represents them using point holonomies [13]. That can be done in two different ways. One could consider the exponentiation of the variable and therefore the resulting functions are periodic in the variable. An alternative, which is what is normally done in loop quantum cosmology, is to consider a Bohr compactification. In Ref. [12] we proceeded in the first way, without introducing a Bohr compactification. On the other hand, using the Bohr compactification corresponds better to what is done in the full theory. One discovers in this case that there exist superselection sectors associated with the periodicity of the variable. The choice of different kinematical Hilbert spaces leads to quite different implementations of the Hamiltonian constraint and to very different looking spaces of physical states that satisfy all the constraints. So apparently one is faced with two inequivalent quantizations. However, a more careful study of the Dirac observables shows that the physical content of both quantizations is the same.

The paper is organized as follows. In Sec. II we describe the classical system and we establish the new constraint algebra. The kinematics quantum description is provided in Sec. III together with the physical solutions in Sec. IV and Sec. V. A brief description about semiclassical states is included in Sec. VI and the final conclusions in VII. Appendix A includes the relation with the metric variables, the falloff conditions and the boundary terms.

II. CLASSICAL SYSTEM

The reduction of the full theory to spherically symmetric vacuum geometries was considered in Ref. [13], based on the original ideas of Ref. [14]. There, one introduces three Killing vectors compatible with the spherical symmetry and demands that the Lie derivatives of the triad and connection along their orbits is compensated by an internal O(3) transformation. These conditions, as it was shown explicitly in Ref. [13], provide the reduced theory for complex Ashtekar variables. Likewise, the reduction for real Ashtekar variables can be carried out in a similar fashion (see for instance Refs. [9, 11]). Moreover, the same results are obtained if one starts with the metric variables, reduces the theory to spherical symmetry and then introduces Ashtekar-like variables for the reduced theory, as was done for instance in [15].

We will then follow the treatment suggested by Bojowald and Swiderski [11] for spherically symmetric spacetimes within loop quantum gravity. The Ashtekar variables adapted to a spherically symmetric spacetime, are given by

$$\begin{aligned} A = A_a^i \tau_i dx^a = A_x(x)\tau_3 dx + [A_1(x)\tau_1 + A_2(x)\tau_2]d\theta \\ + [A_1(x)\tau_2 - A_2(x)\tau_1]\sin\theta d\phi + \tau_3\cos\theta d\phi, \end{aligned} \tag{1}$$

$$\begin{aligned} E = E_i^a \tau^i \partial_a = \sin\theta\Big(E^x(x)\tau_3\partial_x + [E^1(x)\tau_1 + E^2(x)\tau_2]\partial_\theta\Big) \\ + [E^1(x)\tau_2 - E^2(x)\tau_1]\partial_\phi, \end{aligned} \tag{2}$$

where τ_i are the generators of $SU(2)$ (i.e. $[\tau_i, \tau_j] = \epsilon_{ij}{}^k\tau_k$ with ϵ_{ijk} the totally antisymmetric tensor), x is a radial coordinate, and $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ the angular coordinates. The reduced Poisson algebra is given by

$$\begin{aligned} \{A_x(x), E^x(x')\} = 2G\gamma\delta(x - x'), \\ \{A_i(x), E^j(x')\} = G\gamma\delta_i^j\delta(x - x'), \quad i, j = 1, 2, \end{aligned} \tag{3}$$

where G is the Newton constant and γ is the Immirzi parameter. The spacetime metric is

$$ds^2 = -(Ndt)^2 + q_{xx}(dx + N_r dt)^2 + q_{\theta\theta}d\Omega^2, \tag{4}$$

where N and N_r are the lapse and the shift functions, respectively, t is time coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the induced metric in the unit sphere. In terms of the triad variables, the metric components read $q_{xx} = ((E^1)^2 + (E^2)^2)/E^x$ and $q_{\theta\theta} = E^x$.

In this situation, we are left with three first class constraints: a scalar constraint, a diffeomorphism constraint in the radial direction and a remnant Gauss constraint generating $U(1)$ gauge transformations (the $SU(2)$ symmetry is broken after the reduction). Moreover, we will adopt the usual description, in order to identify the gauge invariant variables of the model, obtained after several canonical transformations. One first introduces polar coordinates, i.e.,

$$E^1 = E^\varphi \cos(\alpha+\beta), \qquad E^2 = E^\varphi \sin(\alpha+\beta),$$
$$A_1 = A_\varphi \cos\beta, \qquad A_2 = A_\varphi \sin\beta, \tag{5}$$

and completes the canonical transformation defining

$$\eta = \alpha+\beta, \quad P^\eta = A_\varphi E^\varphi \sin\alpha = 2A_1E^2 - 2A_2E^1,$$
$$\bar{A}_\varphi = 2A_\varphi \cos\alpha. \tag{6}$$

This transformation together with the redefinition of the $su(2)$ algebra

$$\tilde{\tau}_1(x) = \tau_1\cos\eta + \tau_2\sin\eta,$$
$$\tilde{\tau}_2(x) = -\tau_1\sin\eta + \tau_2\cos\eta, \tag{7}$$

diagonalizes the densitized triad in the form

$$E = E^x(x)\tau_3\sin\theta\partial_x + E^\varphi(x)\tilde{\tau}_1(x)\sin\theta\partial_\theta$$
$$+ E^\varphi(x)\tilde{\tau}_2(x)]\partial_\phi. \tag{8}$$

The canonical Poisson brackets are now given by

$$\{A_x(x), E^x(x')\} = 2G\gamma\delta(x-x'),$$
$$\{\bar{A}_\varphi(x), E^\varphi(x')\} = 2G\gamma\delta(x-x'),$$
$$\{\eta(x), P^\eta(x')\} = 2G\gamma\delta(x-x'). \tag{9}$$

The last canonical transformation

$$\bar{A}_x = A_x + \eta', \quad \bar{P}^\eta = P^\eta + (E^x)', \tag{10}$$

allows one to simplify the treatment of the pure gauge canonical pair η and $\bar{P}^\eta$. The connection is finally given by

$$A = (\bar{A}_x - \eta')\tau_3 dx + \bar{A}_\varphi[\cos\alpha\tilde{\tau}_1 - \sin\alpha\tilde{\tau}_2]d\theta$$
$$+ \bar{A}_\varphi[\sin\alpha\tilde{\tau}_1 + \cos\alpha\tilde{\tau}_2]\sin\theta d\phi + \tau_3\cos\theta d\phi. \tag{11}$$

The spin connection in these variables can be computed straightforwardly, yielding

$$\Gamma = \Gamma^i_a\tau_i dx^a = -\eta'\tau_3 dx + \frac{(E^x)'}{2E^\varphi}\tilde{\tau}_2 d\theta - \frac{(E^x)'}{2E^\varphi}\tilde{\tau}_1\sin\theta d\phi + \tau_3\cos\theta d\phi.$$

Together with the connection A given in Eq. (11), we can compute the components of the curvature $\gamma K = A - \Gamma$

$$2\gamma K_x = \bar{A}_x, \quad 2\gamma K_\varphi = \bar{A}_\varphi, \tag{12}$$

(in the following, we will assume $\gamma = 1$). Therefore, thanks to Eq. (12), one is left at the end of the day with three pairs of canonical variables (E^x, K_x), (E^φ, K_φ) and $(\eta, \bar{P}^\eta)$, whose geometrical interpretation is simple: the triad components E^x and E^φ are related to the metric components —see Eqs. (A1)— and the variables K_x and K_φ are the components of the extrinsic curvature.

The total Hamiltonian for the theory is a linear combination of the Gauss, diffeomorphism and scalar constraints. The contribution of the former to the Hamiltonian is given by $\int dx\lambda_\phi G_\phi$ where $G_\phi := \bar{P}^\eta$ is the Gauss constraint and λ_ϕ is the corresponding Lagrange multiplier. In the following, we will consider the gauge fixing $\eta = 0$ in order to eliminate the gauge freedom corresponding to this constraint. One can in fact implement consistently this gauge symmetry at the quantum level [16], however, our purpose is just simplify the study as much as possible, keeping special attention to other more interesting aspects. Therefore, we will work with a total reduced Hamiltonian

$$H_T = G^{-1}\int dx(NH + N_rH_r), \tag{13}$$

that is a linear combination of the diffeomorphism and scalar constraints

$$H_r := E^\varphi K_\varphi' - (E^x)' K_x\,, \tag{14a}$$

$$H := \frac{((E^x)')^2}{8\sqrt{E^x}E^\varphi} - \frac{E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''}{2E^\varphi}\,, \tag{14b}$$

respectively. In addition, we will not include here contributions on the boundaries for simplicity (see App. A).

One can easily check that the constraint algebra is

$$\{H_r(N_r), H_r(\tilde{N}_r)\} = H_r(N_r\tilde{N}_r' - N_r'\tilde{N}_r), \tag{15a}$$

$$\{H(N), H_r(N_r)\} = H(N_r N'), \tag{15b}$$

$$\{H(N), H(\tilde{N})\} = H_r\left(\frac{E^x}{(E^\varphi)^2}\left[N\tilde{N}' - N'\tilde{N}\right]\right), \tag{15c}$$

and is equipped with structure functions (like in the general theory), with the ensuing difficulties for achieving a consistent quantization [17].

Let us emphasize that, in order to recover a Schwarzschild spacetime, one should consider suitable fall-off conditions for the fields of the previous reduced theory. They were already studied in Ref. [4], and we summarize the main aspects in App. A. In this case, the total action of the model corresponds to the previous reduced canonical theory plus a global degree of freedom emerging out of boundary terms, whose contribution can be identified with the mass of the black hole M, and its canonically conjugated momentum τ, which is the proper time of an observer at infinity.

A. New constraint algebra

Following previous ideas (see Refs. [6, 12]), we will modify the constraint algebra in a suitable way by introducing the new lapse $\bar{N}$ and shift $\bar{N}_r$ functions, such that

$$N_r = \bar{N}_r - 2N\frac{K_\varphi\sqrt{E^x}}{(E^x)'}, \quad N = \bar{N}\frac{(E^x)'}{E^\varphi}. \tag{16}$$

After this transformation, the diffeomorphism constraint (14a) remains unaltered, however, the scalar one (14b) is now

$$H = -\int dx\bar{N}\left[\sqrt{E^x}\left(1 - \frac{[(E^x)']^2}{4(E^\varphi)^2} + K_\varphi^2\right)\right]'. \tag{17}$$

We can integrate the scalar constraint $H(\bar{N})$ by parts, which yields

$$H(\tilde{N}) = -\int dx\tilde{N}\bigg(- \sqrt{E^x}\left(1 + K_\varphi^2\right) + 2GM + \frac{[(E^x)']^2\sqrt{E^x}}{4\left(E^\varphi\right)^2}\bigg), \tag{18}$$

where the new lapse function is now $\tilde{N} := \bar{N}'$. The term $2GM$ emerges after imposing the boundary conditions for the lapse [4, 6, 12] (see App. A), and it ensures, as we will see below, the existence of Schwarzschild-like solutions. With this rescaling the Hamiltonian constraint turns out to have an Abelian algebra with itself, and the usual algebra with the diffeomorphism constraint

$$H_r(\bar{N}_r) = \int dx\bar{N}_r\left[-(E^x)'K_x + E^\varphi K_\varphi'\right]. \tag{19}$$

More explicitly,

$$\{H_r(\bar{N}_r), H_r(\bar{M}_r)\} = H_r(\bar{N}_r\bar{M}_r' - \bar{N}_r'\bar{M}_r), \tag{20a}$$

$$\{H(\tilde{N}), H_r(\bar{N}_r)\} = H(\bar{N}_r\tilde{N}'), \tag{20b}$$

$$\{H(\tilde{N}), H(\tilde{M})\} = 0, \tag{20c}$$

which turns out to be a Lie algebra, allowing us to apply standard quantization techniques.

We might notice that the Hamiltonian constraint (18) can be written at the classical level as

$$H(\tilde{N}) = \int dx \tilde{N} H_- H_+, \tag{21}$$

with

$$H_\pm = \sqrt{\sqrt{E^x}\left(1+K_\varphi^2\right) - 2GM} \pm \frac{(E^x)'\,(E^x)^{1/4}}{2E^\varphi}. \tag{22}$$

Since the interesting physical sectors will be those with $E^x > 0$, $(E^x)' > 0$ and $E^\varphi > 0$, the vanishing of $H(N)$ corresponds in fact to $H_- = 0$. Therefore, after the redefinition of the lapse function $\underline{N} = \tilde{N} H_+/(2E^\varphi)$, we get the constraint

$$H(\underline{N}) = \int dx \underline{N}\left(2E^\varphi\sqrt{\sqrt{E^x}\left(1+K_\varphi^2\right) - 2GM} - (E^x)'\,(E^x)^{1/4}\right). \tag{23}$$

that is classically equivalent to (18). This last form of the Hamiltonian constraint will be more suitable for dealing with the quantization in which only periodic functions of the point holonomies are considered.

Let us remark that the sets of constraints (18) and (19), and (14) lead to the same metric solution: the Schwarzschild metric. This is a direct consequence of the fact that the new scalar constraint is actually a linear combination (with coefficients dependent on the dynamical variables) of the old scalar and diffeomorphism constraints. Concretely, for the exterior, the gauge conditions $E^x = x^2$ and $K_\varphi = 0$, their conservation upon evolution and the restriction to the constraint surface yields $N_r = 0$ and

$$E^\varphi = \frac{x}{\sqrt{1-\frac{2GM}{x}}}, \quad N = 1 - \frac{2GM}{x}. \tag{24}$$

One can straightforwardly realize that the metric components $q_{xx} = (E^\varphi)^2/E^x$ and $q_{\theta\theta} = E^x$ in Eq. (4), together with these results, allow one to recover the Schwarzschild metric.

III. QUANTIZATION: KINEMATICAL STRUCTURE

In order to start with the quantization, we will adopt a standard description $\mathcal{H}^m_{\rm kin} = L^2(\mathbb{R}, dM)$ for the global degree of freedom corresponding to the mass of the black hole. For the remaining ones, we will consider two kinematical Hilbert spaces whose structure is inherited from loop quantum gravity [11, 16]: periodic and quasiperiodic functions of the point holonomies. In particular, one starts with the space of linear combinations of holonomies of $su(2)$-connections along edges e, or in other words, cylindrical functions of the connections through holonomies along the mentioned edges. We can then introduce the notion of graph g for this reduced model, which consists of a collection of edges e_j connecting the vertices v_j. It is natural to associate the variable K_x with non-overlapping edges along the radial direction in the graph and the scalar K_φ with vertices on it (point holonomies). We will also consider that each edge is connected with another by means of the corresponding vertex. In general, a given graph can be written as a linear combination of products of cylindrical functions of the form

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \prod_{e_j\in g} \exp\left(\frac{i}{2}k_j \int_{e_j} dx\, K_x(x)\right) \prod_{v_j\in g} \exp\left(\frac{i}{2}\mu_j K_\varphi(v_j)\right), \tag{25}$$

where the label $k_j \in \mathbb{Z}$ is the valence associated with the edge e_j, and $\mu_j \in \mathbb{R}$ the valence associated with the vertex v_j (usually called "coloring").

A. Prescription A: periodic functions of point holonomies

If one adopts a representation for the point holonomies as periodic functions, the labels μ_j are real but must belong to a countable subset of the real line with equally displaced points. In this case, the labels $\vec{\mu}$ can be relabeled by an integer, for instance, $\vec{n}$. The kinematical Hilbert space turns out to be

$$\mathcal{H}^A_{\rm kin} = \mathcal{H}^m_{\rm kin} \otimes \left[\bigotimes_{j=1}^{V} \ell^2_j \otimes \ell^2_j\right], \tag{26}$$

where ℓ^2 denotes the space of square summable functions. It is equipped with the inner product

$$\langle g,\vec{k},\vec{n},M|g',\vec{k}',\vec{n}',M'\rangle = \delta(M-M')\delta_{\vec{k},\vec{k}'}\delta_{\vec{n},\vec{n}'}\delta_{g,g'}\ , \tag{27}$$

where $\delta_{g,g'}$ is equal to the unit if $g=g'$ or zero otherwise, and with $n_j\in\mathbb{Z}$. This kinematical Hilbert space is then separable.

B. Prescription B: quasiperiodic functions

An alternative choice for the kinematical structure closer to the full theory is for point holonomies represented as quasiperiodic functions of the connection, as is usually done in loop quantum cosmology, whose kinematical Hilbert space is $L^2(\mathbb{R}_{\rm Bohr},d\mu_{\rm Bohr})$, with $\mathbb{R}_{\rm Bohr}$ the Bohr compactification of the real line and $d\mu_{\rm Bohr}$ the natural translationally invariant measure on that set. In this situation, the kinematical Hilbert space $\mathcal{H}^B_{\rm kin}$ turns out to be the tensor product

$$\mathcal{H}^B_{\rm kin} = \mathcal{H}^m_{\rm kin}\otimes\left[\bigotimes_{j=1}^{V}\ell^2_j\otimes L^2_j(\mathbb{R}_{\rm Bohr},d\mu_{\rm Bohr})\right], \tag{28}$$

which is endowed with the inner product

$$\langle g,\vec{k},\vec{\mu},M|g',\vec{k}',\vec{\mu}',M'\rangle = \delta(M-M')\delta_{\vec{k},\vec{k}'}\delta_{\vec{\mu},\vec{\mu}'}\delta_{g,g'}\ . \tag{29}$$

Since the labels $\vec{\mu}$ can take any real value, this kinematical Hilbert space $\mathcal{H}^B_{\rm kin}$ is nonseparable, which is the main difference with respect to the previous Hilbert space $\mathcal{H}^A_{\rm kin}$.

C. Operator representation

The representation of the basic operators is essentially the same in both quantizations. In particular, the mass and the triads act as multiplicative operators on these states

$$\hat{M}|g,\vec{k},\vec{\mu},M\rangle = M|g,\vec{k},\vec{\mu},M\rangle, \tag{30}$$
$$\hat{E}^x(x)|g,\vec{k},\vec{\mu},M\rangle = \ell^2_{\rm Pl}k_j|g,\vec{k},\vec{\mu},M\rangle, \tag{31}$$
$$\hat{E}^\varphi(x)|g,\vec{k},\vec{\mu},M\rangle = \ell^2_{\rm Pl}\sum_{v_j\in g}\delta\big(x-x_j\big)\mu_j|g,\vec{k},\vec{\mu},M\rangle, \tag{32}$$

where k_j is either the color of the edge including the point $x\in e_j$ or, if x is at a vertex, the color of the edge to the right of the vertex. Besides, x_j is the position of the vertex v_j, with $j=1,2,\ldots$ We also must understand $\vec{\mu}$ as a general label which will take the corresponding values in either $\mathcal{H}^A_{\rm kin}$ or $\mathcal{H}^B_{\rm kin}$.

Regarding the only connection component $A^\varphi(x)$ that is present in the scalar constraint, the representation adopted will be in terms of point holonomies of length ρ. The basic operators associated with are

$$N^\varphi_{\pm n\rho}(x)|g,\vec{k},\vec{\mu},M\rangle = |g,\vec{k},\vec{\mu}'_{\pm n\rho},M\rangle,\quad n\in\mathbb{N}, \tag{33}$$

where the new vector $\vec{\mu}'_{\pm n\rho}$ either has just the same components than $\vec{\mu}$ up to $\mu_j\to\mu_j\pm n\rho$ if x coincides with a vertex of the graph located at x_j, or $\vec{\mu}'_{\pm n\rho}$ will be $\vec{\mu}$ with a new component $\{\ldots,\mu_j,\pm n\rho,\mu_{j+1},\ldots\}$ with $x_j<x<x_{j+1}$.

We can also construct at the kinematical level the volume operator, given by

$$\hat{\mathcal{V}}|g,\vec{k},\vec{\mu},M\rangle = 4\pi\ell^3_{\rm Pl}\sum_{v_j\in g}\mu_j\sqrt{k_j}|g,\vec{k},\vec{\mu},M\rangle. \tag{34}$$

IV. REPRESENTATION OF THE HAMILTONIAN CONSTRAINT

The complete physical Hilbert space can be determined after identifying the solutions to both the scalar and diffeomorphism constraints. In this section we will study the solutions of the scalar constraint. We will represent it as a quantum operator and we will find its solutions together with a suitable inner product for them. We will follow two different quantization prescriptions based on the previous kinematical Hilbert spaces.

A. Prescription A

1. The physical Hilbert space

We will start representing the Hamiltonian constraint given by (23) in the kinematical Hilbert space and determining its solutions, from which the physical Hilbert space will be constructed. We could have considered (18) instead of (23). At the end of the day they would have given the same physical results, but the latter is more easily solvable.

In particular, we polymerize the connection components and then the scalar constraint is promoted to a quantum operator with a suitable factor ordering. Let us restrict the study to a particular state Ψ_g and positive masses, i.e.

$$(\Psi_g| = \int_0^\infty dM \prod_{v_j\in g}\int_0^{\pi/\rho} dK_\varphi(v_j)\sum_{\vec{k}}\langle\vec{k},\vec{K}_\varphi,M|\psi(M)\chi(\vec{k})\phi(\vec{k};\vec{K}_\varphi;M), \tag{35}$$

and impose that it be a solution to the scalar constraint

$$(\Psi_g|\hat{H}(N)^\dagger = (\Psi_g|\sum_{v_j\in g}N(v_j)\hat{H}_j^\dagger = 0. \tag{36}$$

This last condition is equivalent —up to a global factor $(-1)(\ell_{\rm Pl}^2 k_j)^{1/4}$— to a set of partial differential equations for each vertex v_j, with $j = 1,\cdots,V$, of the form

$$4i\ell_{\rm Pl}^2\frac{\sqrt{1+m_j^2\sin^2 y_j}}{m_j}\partial_{y_j}\phi_j + \ell_{\rm Pl}^2(k_j-k_{j-1})\phi_j = 0, \tag{37}$$

with $y_j := \rho K_\varphi(v_j)$, ϕ has been decomposed as the product over the vertices of factors of the form $\phi_j = \phi_j(k_j,k_{j-1},y_j,M)$ and

$$m_j^2 = \rho^{-2}\left(1-\frac{2GM}{\sqrt{\ell_{\rm Pl}^2 k_j}}\right)^{-1}. \tag{38}$$

The solutions to this set of differential equations are of the form

$$\phi_j = \exp\left\{\frac{i}{4}m_j(k_j-k_{j-1})F(y_j,im_j)\right\}, \tag{39}$$

with

$$F(\phi,k) = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2 t}}dt, \tag{40}$$

the Jacobi elliptic integral of the first kind. We then conclude that the solutions to the scalar constraint must be

$$\Psi(\vec{k};\vec{K}_\varphi;M) = \psi(M)\prod_{v_j\in g}\chi(k_j)\phi_j\Big(k_j,k_{j-1},K_\varphi(v_j),M\Big) \tag{41}$$

where

$$\phi(k_j,k_{j-1},K_\varphi(v_j),M) = \exp\left\{\frac{i}{4}m_j(k_j-k_{j-1})F\Big(\rho K_\varphi(v_j),im_j\Big)\right\}, \tag{42}$$

$\psi(M)$ is the analog wave function of Kuchař's proposal [4] and $\chi(k_j)$ are arbitrary functions of finite norm on the kinematical Hilbert space.

We may notice that the sign of m_j can change depending on the values of the quantum numbers M and k_j. In the following, we will identify the exterior region with real m_j and the interior with pure imaginary m_j. Now, for the exterior of the black hole, i.e, whenever m_j are real, the functions ϕ_j are pure phases. Therefore the states (41) belong to the kinematical Hilbert space $\mathcal{H}_{\rm kin}^A$ instead of being distributions, which is the usual situation. However, in the interior m_j becomes a pure imaginary number. Let us consider $m_j^2 < 0$ and finite (in particular ρ must be a non-vanishing, positive real number). The Jacobi elliptic integral is now

$$F\Big(y_j, |m_j|\Big) = \int_0^{y_j} \frac{1}{\sqrt{1 - |m_j^2| \sin^2 t}} dt. \tag{43}$$

with the argument $y_j \in (0, \pi]$ (or equivalently $K_\varphi(v_j) \in (0, \pi/\rho]$). If $|m_j| < 1$, this integral is real, and therefore we are in a similar situation as before: the functions ϕ_j are just phases. Otherwise, i.e. $|m_j| > 1$, we have to analyze the problem carefully. In this case one can split the Jacobi elliptic integral (43) in the sum of three contributions:

i) For $y_j \in (0, \arcsin m_j^{-1})$, the corresponding contribution will be called $F_1\Big(y_j, |m_j|\Big)$ and remains real and finite. Then the functions ϕ_j associated to them are pure phases.

ii) When $y_j \in (\arcsin m_j^{-1}, \pi - \arcsin m_j^{-1})$ the Jacobi elliptic integral is

$$F\big(y_j, |m_j|\big) = F_1\big(\arcsin m_j^{-1}, |m_j|\big) + F_2\big(y_j, |m_j|\big).$$

It has a constant real contribution F_1 and an imaginary counterpart $F_2\Big(y_j, |m_j|\Big)$ since in the integrand the square root becomes negative, so that the corresponding ϕ_j are not pure phases. However, they are bounded, and therefore the functions ϕ_j are finite.

iii) Finally, if $y_j \in (\pi - \arcsin m_j^{-1}, \pi]$ the Jacobi elliptic integral is now

$$\begin{aligned} F\big(y_j, |m_j|\big) &= F_1\big(\arcsin m_j^{-1}, i m_j\big) \\ &+ F_2\big(\pi - \arcsin m_j^{-1}, |m_j|\big) + F_3\big(y_j, |m_j|\big). \end{aligned}$$

In this interval the argument of the square root in $F_3\big(y_j, |m_j|\big)$. is positive, and ϕ_j becomes a phase that varies with y_j.

In summary, we conclude that the solutions to the constraint of the form (41) are well defined for both the exterior and the interior of the black hole. In particular, they belong to the kinematical Hilbert space $\mathcal{H}^A_{\rm kin}$ in the sense that they have finite norm with respect to the inner product (27). We may notice that we have not required any self-adjointness condition to the constraint (see App. B for additional details). Nevertheless, it is not a serious obstacle in the sense that we can still find the physical states which codifies the dynamics of the quantum system, in agreement with Ref. [17]. These solutions (and the ones provided in Sec. IV B) generically are not associated with a semiclassical geometry. However, either making special choices of the values of the labels of the states, or considering superpositions of states, one can approximate semiclassical geometries well. We will discuss this in section VI.

Furthermore, as we will see in Sec. V, if one considers all possible superpositions of these solutions Ψ_g within a diffeomorphism class of graphs $[g]$, the resulting state $\Psi_{\rm phys}$ will be an element of $\mathcal{H}_{\rm Diff}$. We then conclude that the physical Hilbert space is a subspace of $\mathcal{H}_{\rm Diff}$ (see Sec. V for additional details) whose inner product is given in Eq. (86). Therefore, this quantization prescription does not require the introduction of additional structures (like a different inner product) with respect to $\mathcal{H}_{\rm Diff}$ in order to reach a consistent quantum description.

2. Observables

In agreement with previous quantizations [4, 6, 12], there exists a Dirac observable corresponding to the mass of the black hole, i.e. $\hat{M}$, which can be identified as an observable on the boundary. Therefore, any physical state will be a linear superposition of black holes with well defined masses. However, the quantization we present here provides a genuine observable owing to quantum geometry effects, as was noticed in Ref. [12]. Let us recall that one can select a basis of physical states with a fixed number V of vertices located at x_j, with $j = 0, 1, \ldots, V$. This number V is preserved under the action of the constraints, then we can construct the corresponding Dirac observable $\hat{V}$ that acting on the states Ψ_g gives the integer number V. This observable has no classical analog, and it can be considered as an observable in the bulk.

Moreover, the restriction to invertible diffeomorphisms in the classical theory leads to identify the triad component E^x with monotonically growing functions. This condition suggests the consideration of states with

$$\chi(k_{j'}) = \delta_{j'j}, \tag{44}$$

at each vertex v_j, respectively, and with monotonically growing integers k_j at the quantum level, which at the end of the day characterize the sequence of radii of the geometry. If we also take into account that the order of the vertices is also preserved for diffeomorphism invariant states, we can identify another observable $\hat{O}(z)$ with $z \in [0,1]$ in the bulk, such that

$$\hat{O}(z)\Psi_{\rm phys} = \ell_{\rm Pl}^2 k_{{\rm Int}(Vz)}\Psi_{\rm phys}, \tag{45}$$

where ${\rm Int}(Vz)$ is the integer part of Vz.

Surprisingly, this observable allows us to construct an evolving constant associated to E^x. Given an arbitrary monotonic function $z(x) : [0,x] \to [0,1]$, the mentioned evolving constant can be constructed as

$$\hat{E}^x(x)\Psi_{\rm phys} = \hat{O}\big(z(x)\big)\Psi_{\rm phys}. \tag{46}$$

It is clear that freedom in the choice of the function $z(x)$ codifies the gauge freedom in E^x.

3. Singularity resolution

One of the questions that one can ask is whether the classical singularity can be avoided within this model. The answer is in the affirmative, and in fact, one can follow two strategies. The first one concerns the requirement of selfadjointness to the metric components. For instance, the classical quantity

$$g_{tx} = -\frac{(E^x)' K_\varphi}{2\sqrt{E^x}} \frac{1}{\sqrt{1 + K_\varphi^2 - \frac{2GM}{\sqrt{E^x}}}}, \tag{47}$$

defined as an evolving constant (i.e. a Dirac observable), must correspond to a selfadjoint operator at the quantum level. Classically, K_φ and E^x are pure gauge, and g_{tx} is just a function of the observable M. The exterior of the black hole can be covered by the conditions $K_\varphi = 0$ and $z(x) = x^2$ (together with an appropriate choice of the lapse and the shift), leading to the usual form of the Schwarzchild metric. Quantum-mechanically, the gauge freedom is encoded in $z(x)$ and the periodic gauge parameter $\mathcal{K}_\varphi$ (we introduce here this calligraphic symbol in order to distinguish between the quantum gauge parameter and the classical one K_φ); its periodicity prevents coordinate singularities. The physical information is codified by $\hat{M}$ and $\hat{O}\big(z\big)$. In the interior of the horizon, if $\hat{g}_{tx}$ is a selfadjoint operator, a necessary condition will be

$$1 + \mathcal{K}_\varphi^2 - \frac{2GM}{\sqrt{\ell_{\rm Pl}^2 k_j}} \geq 0. \tag{48}$$

At the singularity, i.e. $j = 1$, and owing to the bounded nature of $\mathcal{K}_\varphi^2 < \infty$,

$$\sqrt{k_1} \geq \frac{2GM}{\ell_{\rm Pl}(1 + \mathcal{K}_\varphi^2)} > 0. \tag{49}$$

Therefore, this argument strongly suggests that the classical singularity will be resolved at the quantum level since k_1 must be a non-vanishing integer. This truncation of the Hilbert space is consistent because the action of the constraints does not lead outside the space of non-vanishing k's. One can therefore analytically continue the solution to negative values of x and one will have a region of spacetime isometric to the exterior of the black hole beyond where the classical singularity used to be.

We will now proceed to present a different quantization prescription, where in particular the classical singularity can be resolved following alternative reasonings already employed in the literature [18, 19].

B. Prescription B

1. Hamiltonian constraint

We will now deal with the solutions to the scalar constraint for the alternate kinematical Hilbert space of quasi-periodic functions of the point holonomies. For it we will adopt an alternate prescription for promoting the Hamiltonian constraint to a quantum operator. Following the usual strategy in loop quantum cosmology, the scalar constraint corresponds essentially to a difference operator that only mixes states with support in lattices of constant step (i.e., separable subspaces of the kinematical one). However, in order to simplify the analysis, it is more convenient to adopt a prescription as simple as possible while it captures all the relevant physical information. On the one hand, we start by polymerizing the connection $K_\varphi \to \sin(\rho K_\varphi)/\rho$ contained in Eq. (18). On the other hand, the factor ordering ambiguity introduces a freedom in the choice quantum scalar constraint, that we will take advantage of, picking out a factor ordering such that: i) the scalar constraint allows us to decouple the zero volume states, and ii) the different orientations of the triad E^φ are decoupled. It is well known that such features turn out to simplify the subsequent treatment of the scalar constraint [19]. Finally, since there are inverse triad contributions in Eq. (18), we will adopt the standard treatment for them by means of the so-called Thiemann's trick [20], which basically consists in defining them at the classical level by means of Poisson brackets of certain power of the triad with its canonically conjugated momentum, and then promote them to quantum commutators (with the addition that one of the variables may be conveniently polymerized). A factor ordering that fulfills all the previous requirements is

$$\hat{H}(N) = \int dx N(x)\sqrt{\hat{E}^x}\left(\hat{\Theta}\sqrt{\hat{E}^x} + \hat{E}^\varphi\sqrt{\hat{E}^x} - \frac{1}{4}\widehat{\left[\frac{1}{\hat{E}^\varphi}\right]}\left[(\hat{E}^x)'\right]^2\sqrt{\hat{E}^x} - 2G\hat{M}\hat{E}^\varphi\right), \tag{50}$$

where the operator $\hat{\Theta}(x)$ acting on the kinematical states

$$\hat{\Theta}(x)|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}\delta(x - x(v_j))\hat{\Omega}^2_\varphi(v_j)|g,\vec{k},\vec{\mu},M\rangle, \tag{51}$$

is defined by means of the non-diagonal operator

$$\hat{\Omega}_\varphi(v_j) = \frac{1}{4i\rho}|\hat{E}^\varphi|^{1/4}\left[\widehat{\mathrm{sgn}(E^\varphi)}\left(\hat{N}^\varphi_{2\rho} - \hat{N}^\varphi_{-2\rho}\right) + \left(\hat{N}^\varphi_{2\rho} - \hat{N}^\varphi_{-2\rho}\right)\widehat{\mathrm{sgn}(E^\varphi)}\right]|\hat{E}^\varphi|^{1/4}\Big|_{v_j}, \tag{52}$$

where

$$|\hat{E}^\varphi|^{1/4}(v_j)|g,\vec{k},\vec{\mu},M\rangle = \ell_{\rm Pl}^{1/2}|\mu_j|^{1/4}|g,\vec{k},\vec{\mu},M\rangle, \tag{53}$$

$$\widehat{\mathrm{sgn}\big(E^\varphi(v_j)\big)}|g,\vec{k},\vec{\mu},M\rangle = \mathrm{sgn}(\mu_j)|g,\vec{k},\vec{\mu},M\rangle, \tag{54}$$

have been constructed by means of the spectral decomposition of $\hat{E}^\varphi$ on $\mathcal{H}^B_{\rm kin}$. This choice of the scalar constraint, concretely the operator $\hat{\Omega}_\varphi$, is well motivated by previous studies of different cosmological scenarios [18, 19], owing to the particular structure of the subspaces invariant under its action, which will be classified below. Regarding the operator $\widehat{\left[1/\hat{E}^\varphi\right]}$, it will be defined following the mentioned Thiemann's ideas [20], yielding well defined operators on the kinematical Hilbert space. More precisely, the classical identity

$$\frac{\mathrm{sgn}(E^\varphi)}{\sqrt{|E^\varphi|}} = \frac{2}{G}\{K_\varphi, \sqrt{E^\varphi}\}, \tag{55}$$

with $\{\cdot,\cdot\}$ the classical Poisson brackets, is promoted to a quantum operator (recalling that the connection must be conveniently polymerized). Its square allows us to define

$$\widehat{\left[\frac{1}{\hat{E}^\varphi}\right]}|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}\delta(x - x(v_j))\frac{\mathrm{sgn}(\mu_j)}{\ell^2_{\rm Pl}\rho^2}(|\mu_j+\rho|^{1/2} - |\mu_j-\rho|^{1/2})^2|g,\vec{k},\vec{\mu},M\rangle. \tag{56}$$

With all the previous definitions, the action of the Hamiltonian constraint can be computed, yielding

$$\begin{aligned}\hat{H}(N)|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}N(x_j)(\ell^3_{\rm Pl}k_j)\Big[&f_0(\mu_j,k_j,M)|g,\vec{k},\vec{\mu},M\rangle \\ &-f_+(\mu_j)|g,\vec{k},\vec{\mu}_{+4\rho_j},M\rangle - f_-(\mu_j)|g,\vec{k},\vec{\mu}_{-4\rho_j},M\rangle\Big],\end{aligned} \tag{57}$$

with the functions

$$f_{\pm}(\mu_j) = \frac{1}{16\rho^2}|\mu_j|^{1/4}|\mu_j \pm 2\rho|^{1/2}|\mu_j \pm 4\rho|^{1/4} s_{\pm}(\mu_j) s_{\pm}(\mu_j \pm 2\rho), \tag{58}$$

$$f_0(\mu_j, k_j, k_{j-1}, M) = \mu_j \left(1 - \frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right) + \frac{1}{16\rho^2}\Big[(|\mu_j||\mu_j + 2\rho|)^{1/2} s_+(\mu_j) s_-(\mu_j + 2\rho)$$
$$+|\mu_j||\mu_j - 2\rho|)^{1/2} s_-(\mu_j) s_+(\mu_j - 2\rho)\Big] - \frac{\text{sgn}(\mu_j)}{\rho^2}(k_j - k_{j-1})^2 (|\mu_j + \rho|^{1/2} - |\mu_j - \rho|^{1/2})^2, \tag{59}$$

and where the factors

$$s_{\pm}(\mu_j) = \text{sgn}(\mu_j) + \text{sgn}(\mu_j \pm 2\rho), \tag{60}$$

come from the sign functions incorporated in the definition of the quantum operator $\hat{\Omega}_{\varphi}$.

2. *Singularity resolution*

The operator corresponding to the Hamiltonian constraint has been chosen such that it allows for a singularity resolution. Let us consider the set of spin networks with $\mu_j > 0$ and $k_j > 0$, and with an arbitrary number of vertices. One can easily check that this subspace is preserved under the action of the scalar constraint (it is, in consequence, an invariant domain). Concretely, at a given vertex, it preserves k_j and mixes different values of μ_j without approaching $\mu_j = 0$. The immediate consequence is that one can construct nontrivial solutions to the Hamiltonian constraint such that they do not have contributions on states with either $k_j = 0$ and/or $\mu_j = 0$. Therefore, the triad components cannot vanish on the space of solutions, and the classical singularity will not be present in the quantum theory. However, one could even think about extending the previous invariant domain to a bigger one just by adding spin networks with, e.g., $k_j = 0$. In this situation, one can still invoke the arguments shown in Sec. IV A 3 about the selfadjointness of some quantum evolving constants, which will allow one to recover the original invariant domain. Therefore, one can combine these two procedures in order to achieve a resolution of the singularity.

In general, one would expect that for the physical states there will be no well defined notion of horizon or black hole interior, since there is no semiclassical geometry associated with them. In those cases it will be difficult to provide a clear notion of singularity either. However, one of the most interesting situations are the "most disfavorable" cases (from the point of view of the existence of singularities) in which one indeed can have a good approximation of a classical geometry of a black hole throughout most of the spacetime, and therefore one can check what happens in the region close to where the classical singularity would have been. In that case we showed above that the singularity is eliminated, and the effective geometry becomes regular (and discrete) throughout the spacetime.

3. *Discrete geometry*

The action of the constraint on this orthogonal complement does not mix different graphs g. In other words, the subspace associated with a given graph g is preserved by the action of the scalar constraint in the sense that no new vertices are created. In turn, a given graph is partially characterized by the number of vertices and the set $\{k_j\}$, which is preserved by $\hat{H}(N)$. Regarding the color of the vertices, the action of the constraint mixes them by means of a difference operator of step 4ρ in the labels μ_j.

We will assume that any state annihilated by the constraint will belong to the algebraic dual of the dense subspace Cyl on the kinematical Hilbert space. For a generic graph g, it will be of the form

$$(\Psi_g| = \int_0^{\infty} dM \sum_{\vec{k}} \sum_{\vec{\mu}} \langle g, \vec{k}, \vec{\mu}, M| \psi(M) \chi(\vec{k}) \phi(\vec{k}; \vec{\mu}; M). \tag{61}$$

It satisfies the constraint equation

$$\sum_{v_j \in g} (\Psi_g | \hat{H}(N_j)^{\dagger} = 0, \tag{62}$$

where $\hat{H}(N_j) = N_j \hat{C}_j$ is defined in terms of the difference operators $\hat{C}_j$. Since each term in the previous expression is multiplied by $N(x_j)$, which can be any general function, the only possibility is that each element of the previous series vanishes independently. This leads to a set of difference equations, one per each v_j. Up to an irrelevant non-vanishing conformal factor ($\ell_{\rm Pl}^3 k_j$), each difference equation reads

$$
\begin{aligned}
&- f_+(\mu_j - 4\rho)\phi_j(\mu_j - 4) - f_-(\mu_j + 4\rho)\phi_j(\mu_j + 4)\\
&+ f_0(k_j, k_{j-1}, \mu_j, M)\phi_j(\mu_j) = 0.
\end{aligned}
\tag{63}
$$

where the function $\phi(\vec{k}, \vec{\mu}, M)$ admits a natural decomposition in factors of the form

$$
\phi(\vec{k}, \vec{\mu}, M) = \prod_{j=1}^{V} \phi_j(\mu_j), \tag{64}
$$

with $\phi_j(\mu_j) = \phi_j(k_j, k_{j-1}, \mu_j, M)$. Then, the Hamiltonian constraint only mixes states with support in lattices of the labels μ_j, with $j = 1, 2, \dots, V$, of step 4ρ. In addition, since the functions $f_\pm(\mu_j)$ vanish in the intervals $[0, \mp 2\rho]$, respectively, different orientations of the labels μ_j are decoupled. Then, the solution states belong to the subspaces with support on the semilattices $\mu_j = \epsilon_j \pm 4\rho {\rm n}_j$, with ${\rm n}_j \in \mathbb{N}$ and $\epsilon_j \in (0, 4\rho]$. In consequence, the constraint only relates states belonging to separable subspaces of the kinematical one, that we will call $\mathcal{H}^B_{\vec{\epsilon}} = \bigotimes_{j=1}^V \mathcal{H}^B_{\epsilon_j}$ in the following. Analogously to what happens in loop quantum cosmology [19, 21], it suffices to provide the value of $\phi_j(\mu_j = \epsilon_j)$ in order to obtain the function $\phi_j(\mu_j)$ at any other triad section employing the previous difference equation. Moreover, in the limit $\mu_j \to \infty$ the solutions satisfy the differential equation

$$
- 4\mu_j \partial^2_{\mu_j}\phi - 4\partial_{\mu_j}\phi - \frac{\tilde{\lambda} - 1}{\mu_j}\phi + \tilde{\omega}\mu_j\phi = 0 \tag{65}
$$

with

$$
\begin{aligned}
\tilde{\lambda} &= \frac{3}{4} + (k_j - k_{j-1})^2,\\
\tilde{\omega} &= \left(1 - \frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right)
\end{aligned}
\tag{66}
$$

which corresponds to either a Bessel or a modified Bessel differential equation depending if $\tilde{\omega}$ is negative or positive (as we already indicated in Sec. IV A we will refer to these two different situations as the interior or exterior of the black hole, respectively). The immediate consequence is that the solutions to the constraint are different depending on whether we are inside or outside the horizon, and they have to be analyzed separately.

This behavior of the difference equation, together with the numerical studies carried out on flat and closed Friedmann-Robertson-Walker spacetimes in loop quantum cosmology (see Ref.[19, 22]) will allow us to anticipate several aspects of the solutions to the difference equation (63), without solving it explicitly. In the next two subsections and in App. C we provide a discussion about this point, but let us summarize the main results: i) whenever $\ell_{\rm Pl}|k_j|^{1/2} > 2GM$ (exterior of the black hole) we find that the constraint equation can be diagonalized as $\lambda_n(\epsilon_j) - \Delta k_j^2 = 0$, where $\{\lambda_n\}$ is a countable set of positive real numbers that depends on $\epsilon_j \in (0, 4\rho]$ associated with the different discretizations, and for a given ϵ_j, it is expected that the sequence $\{\lambda_n\}$ will depend as well on M and k_j. The solutions, as functions of μ_j, emerge out of the minimum triad section $\mu_j = \epsilon_j$, oscillate several times, and decay exponentially in the limit $\mu_j \to \infty$. This situation can be identified with the one already found in closed homogeneous and isotropic spacetimes [22]. ii) On the other hand, if $\ell_{\rm Pl}|k_j|^{1/2} < 2GM$ (interior of the black hole), the constraint equation in its diagonal form is $\omega_j - (1 - 2GM/\ell_{\rm Pl}|k_j|^{1/2}) = 0$, where $\omega_j \in \mathbb{R}^+$. Again, the solutions emerge out of a minimum triad section, and behave in the limit $\mu_j \to \infty$ as an exact standing wave, i.e., a linear combination of two in and out plane waves in μ_j of frequency $\omega_j^{1/2}/2$. This situation is analogous to the one found in Ref. [19] for flat, homogeneous and isotropic cosmologies.

In both cases, for a given value of either $\lambda_n(\epsilon_j)$ or ω_j, the corresponding solutions are non-degenerated. Let us see all this in more detail.

4. The exterior of the black hole: $\ell_{\rm Pl}|k_j|^{1/2} > 2GM$

Let us recall that, for any choice of ϵ_j, the solutions ϕ_j are completely determined by their initial data $\phi_j(\mu_j = \epsilon_j)$. If we fix it to be real, together with the fact that the coefficients of the corresponding difference equation are also real, we conclude that at any other triad section the corresponding solution $\phi_j(\mu_j)$ will remain a real function. We will also introduce a bijection on the space of solutions that will allow us to achieve a suitable separable form of the constraint equation. If we recall that the zero volume states have been decoupled, the bijection is defined by the scaling of the solutions

$$\phi_j^{\text{out}}(\mu_j) = \hat{b}(\mu_j)\phi_j(\mu_j), \tag{67}$$

with

$$\hat{b}(\mu_j) = \frac{1}{\rho}(|\hat{\mu}_j + \rho|^{1/2} - |\hat{\mu}_j - \rho|^{1/2}), \tag{68}$$

the square root of the inverse triad operator $\widehat{[1/\mu_j]}$ (up to a factor ℓ_{Pl}^{-1}), which has an empty kernel. The new functions $\phi_j^{\text{out}}(\mu_j)$ are the coefficients of the solutions to the difference operators

$$\hat{C}_j^{\text{out}} = \widehat{\left[\frac{1}{\mu_j}\right]}^{-1/2} \hat{C}_j \widehat{\left[\frac{1}{\mu_j}\right]}^{-1/2}. \tag{69}$$

In this situation, the constraint equation admits a separation of the form

$$\hat{C}_j^{\text{out}} = \hat{\mathcal{C}}_j^{\text{out}} - (k_j - k_{j-1})^2, \tag{70}$$

where $\hat{\mathcal{C}}_j^{\text{out}}$ is a difference operator, for each j, whose spectral decomposition can be carried out. For consistency, we will study its positive spectrum by solving the eigenvalue problem

$$\hat{\mathcal{C}}_j^{\text{out}}|\phi_{\lambda_j}^{\text{out}}\rangle = \lambda_j|\phi_{\lambda_j}^{\text{out}}\rangle. \tag{71}$$

In App. C we have included a detailed discussion on what we expect about the spectrum and the eigenfunctions of these operators. In particular, we conclude that the eigenvalues belong to a countable set that will depend on the particular semilattice where the eigenfunction has support, i.e., on $\epsilon_j \in (0, 4\rho]$ (usually called superselection sector in the loop quantum cosmology literature), and for a given ϵ_j the eigenvalues are expected to depend on k_j and M. Therefore, the difference operators $\hat{C}_j^{\text{out}}$, for a given integer n, can be diagonalized as

$$\lambda_n(M, k_j, \epsilon_j) - \Delta k_j^2 = 0. \tag{72}$$

This condition appears to considerably restrict the possible values of Δk_j in the exterior of the black hole. Even one can think seriously in possible inconsistencies owing to the fact that, for a given choice of M, Eq. (72) cannot be satisfied exactly for all j, since both addends take discrete values without any a priori relation. However, we can take advantage of the dependence of $\lambda_n(\epsilon_j)$ on the parameter ϵ_j. The results showed in Ref. [22] indicate that the dependence is continuous and in such a way that one can cover the whole positive real line (up to some interval $[0, \lambda_0]$) with the set $\{\lambda_n(\epsilon_j)\}$ for each j. Therefore, we expect that, for any given choice of M and Δk_j, we will be able to find ϵ_j and $\lambda_n(\epsilon_j)$ fulfilling Eq. (72), up to a region $[0, \lambda_0]$ that must be analyzed carefully. Since the prescription we are adopting for the difference operator is different from the one chosen in Ref. [22], we expect that our choice will successfully provide a satisfactory description also in that region. An interesting question would be whether this dependence on ϵ_j, apart from continuous, is also monotonous. If it is discontinuous, there could be values of Δk_j that could not be properly covered. If there is a non-monotonous dependence of $\lambda_n(\epsilon_j)$ on ϵ_j it would be possible to find several values of ϵ_j that would be associated to the same eigenvalue $\lambda_n(\epsilon_j)$, which would allow us to identify new superselection sectors in the theory (as the usual ones found in loop quantum cosmology) and consequently additional genuine quantum observables would emerge. This is a question that will be studied in a future publication.

Let us remark that the corresponding eigenfunctions are normalized to

$$\langle\phi_{\lambda_n(\epsilon_j)}^{\text{out}}|\phi_{\lambda_{n'}(\epsilon_j')}^{\text{out}}\rangle = \delta_{nn'}\delta_{jj'}, \tag{73}$$

on the corresponding $\mathcal{H}_{\epsilon_j}^B$.

Finally, the role played by ϵ_j is analogous to the one of the parameter α_j introduced in App. B, where the latter allows one to cover the whole real line with the spectrum of the momentum operator on a box $(0, L_j]$.

5. The interior of the black hole $\ell_{\rm Pl}|k_j|^{1/2} < 2GM$

In the interior of the black hole, i.e. $\ell_{\rm Pl}|k_j|^{1/2} < 2GM$, we can carry out a similar analysis. Let us introduce this alternative invertible scaling

$$\phi_j^{\rm in}(\mu_j) = \hat{\mu}_j^{1/2}\phi_j(\mu_j), \tag{74}$$

where these new coefficients correspond to the solutions of the difference operators $\hat{C}_j^{\rm in} = \hat{\mu}_j^{-1/2}\hat{C}_j\hat{\mu}_j^{-1/2}$. This time, the constraint equations now read

$$\hat{\mathcal{C}}_j^{\rm in} + \left(1 - \frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right) = 0, \tag{75}$$

where $\hat{\mathcal{C}}_j^{\rm in}$ is also a difference operator for each j that we will be able to diagonalize after solving the eigenvalue problem

$$\hat{C}_j^{\rm in}|\phi_{\omega_j}^{\rm in}\rangle = \omega_j|\phi_{\omega_j}^{\rm out}\rangle, \tag{76}$$

for $\omega_j \geq 0$. In App. C we find that this counterpart of the spectrum is continuous and non-degenerated. The eigenfunctions are normalized on $\mathcal{H}_{\epsilon_j}^B$ to

$$\langle\phi_{\omega_j}^{\rm in}|\phi_{\omega_j'}^{\rm in}\rangle = \delta\left(\sqrt{\omega_j} - \sqrt{\omega_j'}\right). \tag{77}$$

In this case, the constraint equations $\hat{C}_j^{\rm in}$ acquire the diagonal form

$$\omega_j + \left(1 - \frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right) = 0. \tag{78}$$

which can be satisfied for any M and k_j (compatible with the interior of the black hole), owing to the continuity of the eigenvalues ω_j.

6. Physical Hilbert space

The solutions to the constraint can be computed applying group averaging. In particular, since the vanishing eigenvalue of the constraint belongs to the continuous spectrum (in the constraint equation M is continuous), the group averaging is

$$(\Psi_g^C| = \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_V\ e^{i\sum_j \lambda_j \hat{C}_j^\dagger}\int_0^\infty dM \sum_{\vec{k}}\sum_{\vec{\mu}}\langle g,\vec{k},\vec{\mu},M|\psi(M)\chi(\vec{k})\phi(\vec{k};\vec{\mu};M), \tag{79}$$

assuming the selfadjointness of the constraints $\hat{C}_j$. Besides, if the corresponding solutions $\{\phi_{\vec{\lambda}_n}^{\rm out}\}$ and $\{\phi_{\vec{\omega}}^{\rm in}\}$ provide a basis (possibly generalized) of the kinematical Hilbert space, the previous group averaging yields

$$(\Psi_g^C| = \int_0^\infty dM \sum_{\vec{k}<(2GM/\ell_{\rm Pl})^2}\psi(M)\chi(\vec{k})\langle M|\langle\vec{k}|\langle\phi_{\vec{\omega}(\vec{k},M)}^{\rm in}| + \int_0^\infty dM \sum_{\vec{k}>(2GM/\ell_{\rm Pl})^2}\psi(M)\chi(\vec{k})\langle M|\langle\vec{k}|\langle\phi_{\vec{\lambda}_n(\vec{k},M)}^{\rm out}|, \tag{80}$$

This formal expression of the solutions provides the Kuchař mass function together with the functions that solve the scalar constraint. In order to identify a suitable inner product, let us introduce the representation associated with the canonically conjugate variable to M, which we will call τ, and identify it with a relational time. We then pick out any arbitrary vertex v_j, for instance on the exterior and we solve M in favor of the corresponding eigenvalue ω_j. The solutions are then

$$\Psi_g^C(\vec{k},\vec{\mu};\tau) = \frac{2G}{\ell_{\rm Pl}\sqrt{k_j}}\int_0^\infty d\omega_j\psi(\omega_j)\chi(\vec{k})\phi_{\vec{\omega}(\omega_j)}^{\rm in}(\vec{\mu})e^{iM(\omega_j)\tau} + \sum_{\vec{\lambda}_n(\omega_j)}\psi(\vec{\lambda}_n(\omega_j))\chi(\vec{k})\phi_{\vec{\lambda}_n(\omega_j)}^{\rm out}(\vec{\mu})e^{iM(\vec{\lambda}_n(\omega_j))\tau}, \tag{81}$$

recalling that for the exterior, the parameters ϵ_j cannot be freely chosen, while in the interior they are unconstrained. Besides, the remaining eigenvalues ω'_j and $\lambda_n(\epsilon'_j)$, with $j' \neq j$, are all determined for a given choice of M, and consequently of ω_j. Let us also remark that a similar construction can be provided by selecting any vertex on the interior. The solutions have finite norm

$$\|\Psi_g^C(\tau_0)\|^2 = \sum_{\vec{k}} \sum_{\vec{\mu}} |\Psi_g^C(\vec{k}, \vec{\mu}; \tau_0)|^2 < \infty. \tag{82}$$

Therefore, the space of solutions of the scalar constraint is equipped with a suitable time-independent inner product

$$\langle g, \vec{k}, \vec{\mu} | g', \vec{k}', \vec{\mu}' \rangle = \delta_{\vec{k},\vec{k}'} \delta_{\vec{\mu},\vec{\mu}'} \delta_{g,g'} \ , \tag{83}$$

which coincides with the one of the spin networks with each μ_j restricted to a suitable semilattice ϵ_j. This solution space can be completed with the previous inner product, which turns out to be

$$\mathcal{H}_{\rm C}^B = \bigotimes_{j=1}^{V} \ell_j^2 \otimes \mathcal{H}_{\epsilon_j}^B, \tag{84}$$

the kinematical Hilbert space of the spin networks sector but restricted to the separable subspaces labeled by $\vec{\epsilon}$. Finally, after group averaging these states with the diffeomorphism constraint (see Sec. V), we recover the physical Hilbert space, whose elements are a superposition of the former with arranged vertices in all possible positions.

Regarding the observables of the model, it is worth commenting that we can identify the very same ones found before, that is, the constant of the motion associated with the mass $\hat{M} = -i\partial_\tau$ on the boundary and the new observables $\hat{V}$ and $\hat{O}(z)$ on the bulk defined in Eq. (45).

V. DIFFEOMORPHISM INVARIANT STATES

The physical sector of the system is codified in those states that are invariant under the symmetries of the model: diffeomorphisms in the radial direction and time-reparametrizations. These transformations are classically generated by the constraints (18) and (19). In this section we will deal with the diffeomorphism constraint (for the scalar one see Sec. IV).

The usual strategy followed in loop quantum gravity is to apply the so-called group averaging technique, which picks out the diffeomorphism invariant states as well as it induce a natural inner product on the corresponding complex vector space. One starts with a particular graph g and any element $\Psi_g \in$ Cyl, where Cyl is the space of cylindrical functions for all graphs g. One then considers all the diffeomorphisms, and averages each element Ψ_g with respect to them. The result is all the infinite linear combinations of diffeomorphism invariant functionals belonging to the algebraic dual of Cyl, i.e. $\text{Cyl}^*_{\text{Diff}}$. This averaging is made with a suitable rigging map

$$\eta : \text{Cyl} \to \text{Cyl}^*_{\text{Diff}}, \tag{85}$$

that allows one to identify a natural inner product

$$\langle \eta(\Psi) | \eta(\Phi) \rangle = \langle \eta(\Psi) | \Phi \rangle, \tag{86}$$

which is independent of Ψ and Φ. The completion of $\text{Cyl}^*_{\text{Diff}}$ with respect to the inner product (86) allows the construction of the Hilbert space of diffeomorphism invariant states of the model, i.e. $\mathcal{H}_{\text{Diff}}$. This Hilbert space admits a natural decomposition

$$\mathcal{H}_{\text{Diff}} = \oplus_{[g]} \mathcal{H}_{[g],\text{Diff}}, \tag{87}$$

where $[g]$ runs over the diffeomorphism classes of graphs.

In this symmetry reduced model, the diffeomorphism constraint averages states on the radial direction. Therefore, the resulting space of diffeomorphism invariant states is given by linear combinations of spin networks with vertices in all possible positions along the radial line. The corresponding $\mathcal{H}_{\text{Diff}}$ is endowed with a basis of states that is characterized by the diffeomorphism class of graphs $[g]$, and each state in a given class by colorings of edges and vertices. These states are commonly called (symmetry-reduced) spin-knot states (or s-knot states). Besides, owing to the graph symmetry group, the order of the position of the vertices of the diffeomorphism invariant states must be preserved, i.e., they can be regarded as an ensemble of chunks of volumes arranged in the radial coordinate. Therefore, the states which solve the scalar constraint, for both prescriptions of periodic and quasiperiodic functions, have essentially the same Hilbert space structure that the one of spin network states. On this space the group averaging can be carried out in a standard way, yielding the mentioned diffeomorphism invariant states. Then, since the coloring of the edges k_j is preserved in both quantization prescriptions, we can identify the observables $\hat{V}$, the number of vertex of the graph, and $\hat{O}(z)$, defined in Eq. (45), with no classical analog.

VI. SEMICLASSICAL STATES

An interesting question is whether we can recover a semiclassical description out of this quantum theory. The standard procedure consists in looking for those states where the expectation values of physical observables are peaked on classical trajectories. For example, a good candidate would be

$$\psi(M) = \frac{1}{(2\pi\Delta M^2)^{1/4}} e^{-(M-M_0)^2/4(\Delta M_0)^2}, \quad \chi(k_{j'}) = \delta_{j'j}, \quad k_{j'} > k_j \text{ if } j' > j. \tag{88}$$

such that $M_0 \gg m_{\rm Pl}$, with $m_{\rm Pl} = \hbar/G$ being the Planck mass, and $\Delta M_0/M_0 \ll 1$, where ΔM_0 is the uncertainly on the mass. These states may be associated with a semiclassical description since they provide geometries peaked at a given mass and around a geometry of a given spin network of coloring $\vec{k}$. The consideration of sequences of growing quantum numbers k_j is required since they make the radial variable monotonically growing and avoid double coverings. In this situation, the areas of spheres of symmetry are quantized such that the difference of the areas of two spheres on two arbitrary vertices v_j and $v_{j'}$ would be an integer times a fundamental quanta of area of the order of the square of the Planck length. Besides, it would be interesting to consider more general states $\chi(k_j)$. In this case, the fundamental (but state-dependent) discretization of the geometry might not correspond to spheres of well-defined, quantized area, but determined by the specific expectation values $\langle\vec{k}\rangle_\chi$. Therefore, the intrinsic discretization of the spacetime can be modeled, keeping the semiclassical character of the effective spacetime. In addition, the study alternate semiclassical conditions like, e.g., geometries with a low dispersion on geometrical objects such as the volume operator

$$\frac{\Delta_\Psi \hat{\mathcal{V}}}{\langle\hat{\mathcal{V}}\rangle_\Psi} \ll 1. \tag{89}$$

or even the metric components which would depend as well on the connection variables. These requirements could induce additional restrictions on the physical states. This will be a matter of research for a future publication.

We would like to remark that on the basis of states labeled by M and the eigenvalues of $\hat{O}(z)$ (not necessarily related with semiclassical states) there exist two regions where the corresponding states show (well defined but) different behaviors. One of them corresponding to (the square root of the) eigenvalues of $\hat{O}(z)$ bigger than $2GM$ and the other one in the opposite situation. In the cases in which one approximates semiclassical geometries, for instance when a suitable superposition of masses and k_j's (or certain special choice for the labels of the states) is considered, these regions would correspond to the usual notion of exterior and interior of the black hole, respectively. In these situations, one can study geodesics in the effective metric in the exterior (defining discrete approximations if necessary) and one would see that null geodesics end on the asymptotic boundary. This method would provide an additional notion of what is the exterior region of the (semiclassical) black hole.

VII. CONCLUSIONS

In this manuscript we have analyzed the quantization of spherically symmetric spacetimes adopting loop quantization techniques. In particular, we consider a canonical description in terms of the real Ashtekar-Barbero connection and its conjugate variable. After several canonical transformations and an innocuous gauge fixing, the resulting model is characterized by two local first class constraints codifying the invariance under diffeomorphisms in the radial direction and time reparametrizations. The constraint algebra shows structure functions that obstruct a subsequent quantization. Fortunately, we can avoid this obstacle by means of a redefinition of the constraint algebra after modifying the lapse and the shift functions conveniently, such that the new constraint algebra is a true Lie algebra. We then apply the Dirac approach combined with a quantization of the geometry à la loop. Let us recall that the basic bricks of the quantum theory are holonomies of the Ashtekar-Barbero connection along piecewise-continuous edges and fluxes of densitized triads through surfaces. We study two quantization prescriptions where the corresponding kinematical Hilbert spaces consist in the standard Kuchař mass states tensor product with the Hilbert space of spin networks formed by edges along the radial line joined by vertices (transverse direction), but for each prescription the point holonomies are represented as periodic or quasiperiodic functions of the connection, respectively. In order to solve the dynamics, we start looking for the solutions to the scalar constraint on each prescription. In the case of periodic functions, we study the quantization already proposed in Ref. [12], where the solutions can be explicitly obtained. Besides, they belong to a subspace of the kinematical Hilbert space, and no additional considerations must be taken into account in order to endow them with Hilbert space structure. This is no longer the case when we adopt the alternate prescription for quasiperiodic functions. There, we represent the scalar constraint on the kinematical

Hilbert space adopting a convenient factor ordering. Its solutions have support on semilattices of constant step of the triad associated to the transverse direction. They can be computed out of their initial data in the minimum volume section. Besides, applying group averaging, the resulting Hilbert space differs from the kinematical one. In both cases, we can complete the quantization by implementing the diffeomorphism constraint after applying group averaging to their respective solution spaces, achieving different results for each prescription. However, we were able to identify the very same observables in the two prescriptions: the traditional Dirac observable of the model associated to the mass of the black hole, together with a new one emerging out of both the implementation of the diffeomorphism constraint and the special properties of the scalar one. The latter preserves the number of vertices of the states and the former respects their order. These two facts are the responsible of the emergence of this new observable, with no classical analog. Therefore, both descriptions are equivalent at the level of Dirac observables. Regarding the singularity resolution, we have argued in two different ways how it can be avoided in the quantum theory. One of them consists in looking for observables, like the metric components, suitably defined as evolving constants [6, 12], and require selfadjointness of them. In particular, we found strong arguments that such a requirement forces the absence of singular geometries in the quantum theory. The other procedure to reveal the resolution of singularities, based on ideas of Refs. [18, 19], consists in the selection of a suitable representation for the scalar constraint such that it has an invariant domain free of possible problematic states, and consequently restrict the study to this subspace. In this case the physical states are linear superpositions of spin networks where the triad never vanishes. It in this sense that we claim that the classical singularity is not present in the quantum theory.

The obtained results open new possibilities to address fundamental problems in black hole physics. For instance, the discrete geometry associated to the radial direction as well as the singularity resolution could have enlightening consequences in the evaporation process of a black hole and the information loss paradox [23]. Regarding the semiclassical kinematics [24], an infalling observer will cross into the interior of the black hole in a finite time with respect to an observer at the spatial infinite. Since the radial coordinate takes discrete values and their separation is limited to a minimum value due to the quantization of the area in loop quantum gravity, the blueshift factor for an infalling observer viewed from infinity never diverges. Moreover, the infalling observer will reach the region where the singularity was expected to be. Nevertheless, the singularity resolution would allow to continue the geodesics at another spacetime region. In addition, when coupling a test scalar field with a semiclassical black hole, and assuming that a quantum field theory on curved spacetimes will be able to capture to certain extent most of the relevant physical phenomena of the model, the discretization of the geometry modifies the predictions with respect to the standard continuous description. This may help solving several problems of the latter like the problematic trans-Planckian modes close to the horizon, which could affect the black hole information paradox [23], and the subsequent approaches like the membrane paradigm [25], black hole complementarity [26, 27] or the firewall phenomenon [28, 29]. All these aspects will require detailed studies that go beyond the scope of this paper. Besides, we also expect in a first approximation that following a similar treatment like the one provided in Ref. [30] for cosmological perturbations, a test quantum field theory on this quantum spacetime will in fact experience some and not all the quantum geometry degrees of freedom, as well as the possible generalizations when the backreaction of a perturbed model is incorporated, as was done in Ref. [21] for cosmological settings.

Another interesting extension of all the previous studies concerns the gravitational collapse [31] within loop quantum gravity. In particular, understanding the complete quantum dynamics of the spherically collapse of a scalar field in loop quantum gravity would provide the missing ingredients that would allow us to verify the true nature of black hole evaporation and the black hole information paradox [32].

Finally, we would like to remark that we have carried out a quantization of a symmetry reduced model. In this sense, as in any other similar setting, one has to be careful about interpreting its physical predictions, since their validity must be trusted only after confrontation with the full theory.

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Appendix A: Metric variables, falloff and boundary terms

Let us comment on the fact that the phase space variables introduced in Sec. II are related to the metric ones introduced by Kuchař [4] $ds^2 = \Lambda^2 dx^2 + R^2 d\Omega^2$ by means of the transformation

$$\Lambda = \frac{E^\varphi}{\sqrt{|E^x|}}, \qquad P_\Lambda = -\sqrt{|E^x|}K_\varphi, \tag{A1a}$$

$$R = \sqrt{|E^x|}, \qquad P_R = -2\sqrt{|E^x|}K_x - \frac{E^\varphi K_\varphi}{\sqrt{|E^x|}}, \tag{A1b}$$

where P_Λ, P_R are the momenta canonically conjugate to Λ and R, respectively.

Let us also recall that the falloff conditions of the metric variables were studied in Ref. [4]. The maximally extended Schwarzschild spacetime has two infinities $r \to \pm\infty$, with r the radial coordinate in Kruskal variables. Considering also the Kruskal time t, we then have

$$\begin{aligned} &\Lambda(t,r) = 1 + \frac{2GM_\pm(t)}{|r|} + \mathcal{O}(|r|^{-(1+\varepsilon)}), \\ &R(t,r) = |r| + \mathcal{O}(|r|^{-\varepsilon}), \\ &P_\Lambda(t,r) = \mathcal{O}(|r|^{-\varepsilon}), \\ &P_R(t,r) = \mathcal{O}(|r|^{-(1+\varepsilon)}), \end{aligned} \tag{A2}$$

together with lapse and shift functions

$$\begin{aligned} &N(t,r) = N_\pm(t) + \mathcal{O}(|r|^{-\varepsilon}), \\ &N_x(t,r) = \mathcal{O}(|r|^{-\varepsilon}). \end{aligned} \tag{A3}$$

Finally, one might realize that the previous falloff conditions together with the variations of the action associated to the Hamiltonian defined in Eq. (13) at infinity would give the inconsistent result $N_\pm(t) = 0$. In Ref. [4] was suggested to include a boundary contribution in the action

$$N_+(t)M_+(t) + N_-(t)M_-(t), \tag{A4}$$

where the functions $N_\pm(t) = \pm\dot{\tau}_\pm(t)$ might be treated as prescribed functions, with $\tau_\pm(t)$ the proper time associated to observers moving along worldlines of constant radii at both infinities, respectively. Finally, on the solution space one can check that $M_\pm(t) = M$, with M a constant of the motion corresponding to the mass of the black hole. From this point of view, this contribution can be interpreted as a global degree of freedom, since its contribution to the action is of the form

$$S_\infty = \int dt\, \dot{\tau} M, \tag{A5}$$

whose physical interpretation is an additional term to the symplectic structure.

Appendix B: Selfadjointness of the scalar constraint

Let us comment on the fact that, within the quantization prescription compatible with the kinematical Hilbert space $\mathcal{H}^A_{\rm kin}$, we can require selfadjointness to the scalar constraint whenever we restrict the study to the exterior of the black hole, i.e., $m_j^2 > 0$. After the change of variable

$$dx_j = \frac{m_j dy_j}{4\sqrt{1 + m_j^2 \sin^2 y_j}}, \tag{B1}$$

which essentially gives

$$x_j(y_j, m_j) = \frac{1}{4} m_j F(y_j, i m_j), \tag{B2}$$

with

$$F(\phi,k)=\int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2 t}}dt, \tag{B3}$$

the Jacobi elliptic integral of the first kind, and the new coordinate $x_j\in\big[0,x_j(2\pi,m_j)\big)$, the equations in (37) can be written as

$$\partial_{x_j}\phi_j=i\omega\phi_j, \tag{B4}$$

which is basically the eigenvalue problem of the momentum operator of a free particle in box of length $L(m_j)=x_j(2\pi,m_j)$. It is well known that this operator has an infinite number of self-adjoint extensions. In fact, for each j, such an extension is characterized by a parameter $\alpha_j\in[0,2\pi)$ whenever one restrict the study to the monoparametric family of dense domains

$$D_{\alpha_j}(\partial_{x_j})=\Big\{\psi\in\mathcal{H}:\phi_j\big(L(m_j)\big)=e^{i\alpha_j}\phi_j(0)\quad\text{and}\quad\langle\phi_j|\partial_{x_j}|\phi_j\rangle<\infty\Big\}. \tag{B5}$$

The spectrum of this operator on each dense domain α_j is equal to $\omega_n(\alpha_j,m_j)=(2\pi n-\alpha_j)/L(m_j)$, with eigenfunctions

$$\phi_{\alpha_j}(x_j,m_j)=\sqrt{\frac{1}{L(m_j)}}\exp\left\{i\frac{(2\pi n-\alpha_j)x_j}{L(m_j)}\right\}. \tag{B6}$$

The constraint equation now reads

$$\omega_n(\alpha_j,m_j)-(k_j-k_{j-1})=0. \tag{B7}$$

We may notice that the parameters m_j, k_j and n do not vary continuously. In consequence, in order to the equation (B7) be consistent, one is forced to select a different self-adjoint extension α_j such that (B7) be satisfied $\forall j$. We then cannot restrict the study to any arbitrary domain α_j on each j. As was noticed in the alternative quantization prescription, the role of the parameter α_j can be interpreted with a family of parameters that label the possible discretizations of the triad $\hat{E}^\varphi(v_j)$.

Appendix C: Spectral decomposition of difference operators

In this appendix we include additional details about the spectral properties of the difference operators studied in Sec. IV B.

1. Difference operator: the exterior

Let us focus on the difference operator $\hat{\mathcal{C}}_j^{\rm out}$ introduced in Sec. IV B 4, which has an action of the form

$$\hat{\mathcal{C}}_j^{\rm out}|\mu_j\rangle=f_0^{\rm out}(\mu_j,k_j,M)|\mu_j\rangle-f_+^{\rm out}(\mu_j)|\mu_j+4\rho_j\rangle-f_-^{\rm out}(\mu_j)|\mu_j-4\rho_j\rangle, \tag{C1}$$

with

$$f_\pm^{\rm out}(\mu_j)=\frac{1}{16\rho^2 b(\mu_j)b(\mu_j\pm 4\rho)}|\mu_j|^{1/4}|\mu_j\pm 2\rho|^{1/2}|\mu_j\pm 4\rho|^{1/4}s_\pm(\mu_j)s_\pm(\mu_j\pm 2\rho), \tag{C2}$$

$$f_0^{\rm out}(\mu_j,k_j,k_{j-1},M)=\frac{\mu_j}{b(\mu_j)^2}\left(1-\frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right)+\frac{1}{16\rho^2 b(\mu_j)^2}\Big[(|\mu_j||\mu_j+2\rho|)^{1/2}s_+(\mu_j)s_-(\mu_j+2\rho)$$
$$+(|\mu_j||\mu_j-2\rho|)^{1/2}s_-(\mu_j)s_+(\mu_j-2\rho)\Big], \tag{C3}$$

where $b(\mu_j)$ where defined in Eq. (68). We are interested in its positive spectrum. Therefore, we will study the solutions to the eigenvalue problem

$$\hat{\mathcal{C}}_j^{\rm out}|\phi_{\lambda_j}^{\rm out}\rangle=\lambda_j|\phi_{\lambda_j}^{\rm out}\rangle, \tag{C4}$$

for $\lambda_j \geq 0$. In order to deal with this question, let us start studying the limit $\mu_j \to \infty$. There, the corresponding difference equations become the differential ones

$$-4\mu_j^2\partial_{\mu_j}^2\phi - 8\mu_j\partial_{\mu_j}\phi + \tilde{\omega}\mu_j^2\phi = \gamma_j^2\phi, \tag{C5}$$

with $\gamma_j^2 = \lambda_j + 1$ and $\tilde{\omega}$ given in Eq. (66). These equations correspond to modified Bessel equations where their solutions are combinations of modified Bessel functions, i.e.,

$$\phi = Ax_j^{-1/2}\mathcal{K}_{i\gamma_j}(x_j) + Bx_j^{-1/2}\mathcal{I}_{i\gamma_j}(x_j), \tag{C6}$$

with $x_j = \mu_j\sqrt{\tilde{\omega}}/2$. In the limit $\mu_j \to \infty$, $\mathcal{I}$ grows exponentially and $\mathcal{K}$ decays exponentially. Therefore, the latter is the only contribution to the spectral decomposition. In consequence, this counterpart of the spectrum of the difference operator (C5) is non-degenerate. Moreover, the functions $\mathcal{K}_{i\gamma_j}(x)$ are normalized to

$$\langle\mathcal{K}_{i\gamma_j}|\mathcal{K}_{i\gamma_j'}\rangle = \delta(\gamma_j - \gamma_j'), \tag{C7}$$

in $L^2(\mathbb{R}, x_j^{-1}dx_j)$, since the normalization in this case is ruled by the behavior of $\mathcal{K}_{i\gamma}(x)$ in the limit $x \to 0$, which corresponds to

$$\lim_{x\to 0}\mathcal{K}_{i\gamma_j}(x) \to A\cos\left(\gamma_j \ln|x|\right). \tag{C8}$$

For additional details, see also Ref. [33]

Now we appeal to the results found in Ref. [22], where the homogeneous constraint equation is analogous to ours at each vertex v_j. There it was found that the eigenfunctions of such a difference operator have a similar asymptotic behavior for $\mu_j \to \infty$. However, the spectrum of the corresponding difference operator turns out to be discrete (instead of continuous like the corresponding differential operator) owing to the behavior of its eigenfunctions at $\mu_j \to 0$. Therefore, we expect that λ_n belong to a countable set (for each vertex v_j), which must be determined numerically. We expect that the possible positive values of λ_n will depend on $\epsilon_j \in (0, 4\rho]$, and for a given ϵ_j, they will also depend on $\tilde{\omega}$, i.e., on k_j and M by means of Eq. (66).

Therefore, the corresponding eigenfunctions will be normalized on $\mathcal{H}^B_{\epsilon_j}$ to

$$\langle\phi^{\rm out}_{\lambda_n(\epsilon_j)}|\phi^{\rm out}_{\lambda_{n'}(\epsilon_j)}\rangle = \delta_{nn'}. \tag{C9}$$

2. Difference operator: the interior

In addition, we will consider the difference operator $\hat{\mathcal{C}}_j^{\rm in}$ that was also introduced in Sec. IV B 4, whose action on a state $|\mu_j\rangle$ is of the form

$$\hat{\mathcal{C}}_j^{\rm in}|\mu_j\rangle = f_0^{\rm in}(\mu_j, k_j, M)|\mu_j\rangle - f_+^{\rm in}(\mu_j)|\mu_j + 4\rho_j\rangle - f_-^{\rm in}(\mu_j)|\mu_j - 4\rho_j\rangle,$$

with

$$f_\pm^{\rm in}(\mu_j) = \frac{1}{16\rho^2}|\mu_j|^{-1/4}|\mu_j \pm 2\rho|^{1/2}|\mu_j \pm 4\rho|^{-1/4}s_\pm(\mu_j)s_\pm(\mu_j \pm 2\rho), \tag{C10}$$

$$f_0^{\rm in}(\mu_j, k_j, k_{j-1}, M) = \frac{1}{16\rho^2|\mu_j|}\Big[(|\mu_j||\mu_j + 2\rho|)^{1/2}s_+(\mu_j)s_-(\mu_j + 2\rho)$$
$$+(|\mu_j||\mu_j - 2\rho|)^{1/2}s_-(\mu_j)s_+(\mu_j - 2\rho)\Big] - \frac{{\rm sgn}(\mu_j)}{|\mu_j|\rho^2}(k_j - k_{j-1})^2(|\mu_j + \rho|^{1/2} - |\mu_j - \rho|^{1/2})^2. \tag{C11}$$

We will study then the solutions to

$$\hat{\mathcal{C}}_j^{\rm in}|\phi^{\rm in}_{\omega_j}\rangle = \omega_j|\phi^{\rm out}_{\omega_j}\rangle, \tag{C12}$$

with $\omega_j \in \mathbb{R}^+$, i.e., we will consider only the positive counterpart of its spectrum, since it is the only consistent contribution to Eq. (75). Let us recall that the coefficients $\phi_j(\mu_j)$ of these difference equations are determined by their initial data $\phi_j(\mu_j = \epsilon_j)$ and are real if $\phi_j(\mu_j = \epsilon_j) \in \mathbb{R}$, since the previous functions $f_0^{\rm in}$ and $f_\pm^{\rm in}$ are also real.

We conclude that the eigenfunctions must be real and the spectrum non-degenerated. Besides, in the limit $\mu_j \to \infty$, the solutions satisfy the Bessel equations

$$- 4\partial_{\mu_j}^2 \phi - \frac{\gamma^2}{\mu_j^2}\phi - \omega_j \phi = 0, \tag{C13}$$

whose solutions can be split in linear combinations of Hankel functions of first $H_{i\gamma}^{(1)}(x_j)$ and second kind $H_{i\gamma}^{(2)}(x_j)$, multiplied by a factor $x_j^{1/2}$, that is

$$\phi = A x^{1/2} H_{i\gamma}^{(1)}(x_j) + B x^{1/2} H_{i\gamma}^{(2)}(x_j), \tag{C14}$$

where

$$x_j = \frac{\mu_j \sqrt{\omega_j}}{2}, \tag{C15}$$

and $\gamma^2 = \tilde{\lambda}$, with $\tilde{\lambda}$ given in Eq. (66). The asymptotic limit of these functions is

$$H_{i\gamma}^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4+\gamma\pi/2)}, \quad H_{i\gamma}^{(2)}(x) = \left(H_{i\gamma}^{(1)}(x)\right)^*. \tag{C16}$$

This allows us to conclude that, since $\phi_j^{\rm in}(\mu_j) \in \mathbb{R}$, the most general solution at $\mu_j \to \infty$ must be of the form

$$\lim_{\mu_j\to\infty} \phi_{\omega_j}^{\rm in}(\mu_j) = A \cos\left[\frac{\sqrt{\omega_j}}{2}\mu_j + \beta\right], \tag{C17}$$

with A certain amplitude and β a phase that can depend on k_j, k_{j-1} and ϵ_j. We then conclude that the solutions, as functions of μ_j, are normalizable (in the generalized sense) to

$$\langle \phi_{\omega_j}^{\rm in} | \phi_{\omega_j'}^{\rm in} \rangle = \delta\left(\sqrt{\omega_j} - \sqrt{\omega_j'}\right), \tag{C18}$$

on $\mathcal{H}_{\epsilon_j}^B$.

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