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Title: Estimating incomplete information in group decision making: A framework of granular computing

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Abstract: A general assumption in group decision making scenarios is that of all individuals possess accurate knowledge of the entire problem under study, including the abilities to make a distinction of the degree up to which an alternative is better than other one. However, in many real world scenarios, this may be unrealistic, particularly those involving numerous individuals and options to choose from conflicting and dynamics information sources. To manage such a situation, estimation methods of incomplete information, which use own assessments provided by the individuals and consistency criteria to avoid discrepancy, have been widely employed under fuzzy preference relations. In this study, we introduce the information granularity concept to estimate missing values supporting the objective of obtaining complete fuzzy preference relations with higher consistency levels. We use the concept of granular preference relations to form each missing value as a granule of information in place of a crisp number. This offers the flexibility that is required to estimate the missing information so that the consistency levels related to the complete fuzzy preference relations are as higher as possible.

Highlights

- A granular procedure for estimating missing information in fuzzy preference relations is proposed.
- The missing values or a fuzzy preference relations are assumed to be granular instead of numeric.
- Information granularity is used here to estimate missing information.
- The consistency levels related to the complete fuzzy preference relations are as higher as possible.

Reply to Reviewers

We would like to express thanks to the reviewers and the Handling Editor for their valuable and constructive comments concerning our submission.

The manuscript has been revised to fully address the issues raised in the reviews. In what follows, we present in detail on how the paper has been modified. To enhance readability of the presentation, the comments of the reviewers and the Handling Editor are shown in italics. Excerpts taken from the revised manuscript are shown in smaller fonts (10 pts).

HANDLING EDITOR

We have collected all reviewers comments. Please, consider comments of Reviewer #1 and reply to his concerns and clearly state the contribution of your work.

We have replied to the concerns of the Reviewer #1 and clarified the contribution of our work. In particular, this is the first "granular" procedure for estimating missing information in group decision making. Until now, all procedures proposed in the literature to deal with missing values in group decision making are "numeric". In addition, this is the first time that the paradigm of granular computing is used to construct a "granular" procedure for estimating missing values in group decision making.

According to InCites Essential Science Indicators from Clarivate Analytics, group decision making is an important topic that has attracted the attention of many researchers. Therefore, in our humble opinion, the "granular" estimation procedure of missing values introduced in this manuscript, which is capable of completing the missing information with higher consistency levels than its "numeric" counterpart as illustrated in the experimental studies, will be of great interest for the readership of this journal and will be surely used in their future research work.

Reviewer 1

I have gone through this revised manuscript, it is well written and I found that it uses existing method [22] and then check with [24] and PSO.

By admitting a certain level of information granularity, “numeric” models can be elevated to an abstract level and the constructs formed there are referred to as “granular” models. Therefore, to build a “granular” model we need a “numeric” counterpart. For example, in:

- X. Zhu, W. Pedrycz, Z. Li. A Design of Granular Takagi-Sugeno Fuzzy Model Through the Synergy of Fuzzy Subspace Clustering and Optimal Allocation of Information Granularity. IEEE Transactions on Fuzzy Systems 26:5 (2018) 2499-2509

the Takagi-Sugeno model was elevated to its “granular” model.

In:

- W. Pedrycz, W. Homenda. From Fuzzy Cognitive Maps to Granular Cognitive Maps. IEEE Transactions on Fuzzy Systems 22:4 (2014) 859-869

the fuzzy cognitive maps were augmented by introducing their generalization coming in the form of “granular” fuzzy cognitive maps.

Similar to these works (but it does not mean that our work is the same that these ones), in our work, we need a “numeric” procedure estimating missing information to build its “granular” model (that is, a “granular” procedure estimating missing information). For this reason, as you say, we use the “numeric” procedure presented in [22], but it does not mean that our “granular” procedure is the same that the “numeric” procedure presented in [22].

On the other hand, in [24], fuzzy neural networks were used as a useful modeling example where the process of building “granular” fuzzy neural networks was discussed. However, in our work, we deal with missing information in group decision making. Therefore, our work is different to the one developed in [24].

Furthermore, Fig. 1 not reflecting new method (Authors already declared that this manuscript not focusing new method).

We apologize for not explaining this well. In this manuscript, we do not introduce a new "numeric" procedure, but we propose the first "granular" approach for estimating missing information in group decision making. That is, this is the first paper in which the paradigm of granular computing is used to elevate a "numeric" procedure estimating missing information to its "granular" form. The "granular" model is more abstract than its "numeric" counterpart (the method presented in [22]) and, in this way, it is more reflective of reality and helps quantify knowledge about the system at hand. As shown in the experimental studies, the "granular" model is capable of completing the missing information with a higher consistency level that the "numeric" model.

We have clarified this in the Introduction Section:

"The objective of this study is to present how to generalize the existing numeric methods dealing with incomplete information to their granular methods. In particular, we present a granular estimation procedure of missing information in group decision making having the procedure proposed in [22] as its numeric counterpart."

On the other hand, we have added Figure 2 to reflect the steps of the new approach in Section 3.

I feel that this manuscript has no good contribution to publish in ASOC.

In Applied Soft Computing journal, we can find published papers in which the framework of granular computing is used to construct a "granular" model from a "numeric" model. For example (these papers are some of them, but you can find more related papers):

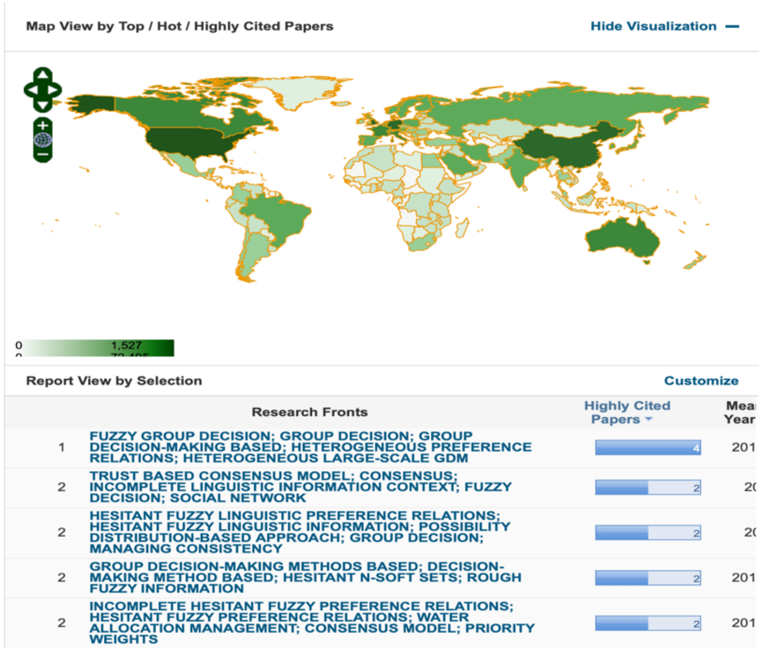
- A.R. Solis, G. Panoutsos. Granular Computing Neural-Fuzzy Modelling: A Neutrosophic Approach. Applied Soft Computing 13:9 (2013) 4010-4021.

- W. Lu, L. Zhang, W. Pedrycz, J. Yang, X. Liu. The Granular Extension of Sugeno-Type Fuzzy Models Based on Optimal Allocation of Information Granularity and its Application to Forecasting of Time Series. Applied Soft Computing 42 (2016) 38-52.

- M. Song, Y. Jing, W. Pedrycz. Granular Neural Networks: A Study of Optimizing Allocation of Information Granularity in Input Space. Applied Soft Computing 77 (2019) 67-75.

Similar to these papers, we present an application of the framework of granular computing but in the field of group decision making and missing information. Therefore, if the aforementioned papers, in which the framework of granular computing is used to construct the "granular" model of a "numeric" model, have been published in Applied Soft Computing, we think that our paper can also be published in this journal.

In addition, as you can see in the following figure, the field of group decision making and incomplete information is a research front according to InCites Essential Science Indicators database from Clarivate Analytics.



	Research Fronts	Highly Cited Papers	Mea Year
1	FUZZY GROUP DECISION; GROUP DECISION; GROUP DECISION-MAKING BASED; HETEROGENEOUS PREFERENCE RELATIONS; HETEROGENEOUS LARGE-SCALE GDM	4	201
2	TRUST BASED CONSENSUS MODEL; CONSENSUS; INCOMPLETE LINGUISTIC INFORMATION CONTEXT; FUZZY DECISION; SOCIAL NETWORK	2	20
2	HESITANT FUZZY LINGUISTIC PREFERENCE RELATIONS; HESITANT FUZZY LINGUISTIC INFORMATION; POSSIBILITY DISTRIBUTION-BASED APPROACH; GROUP DECISION; MANAGING CONSISTENCY	2	20
2	GROUP DECISION-MAKING METHODS BASED; DECISION-MAKING METHOD BASED; HESITANT N-SOFT SETS; ROUGH FUZZY INFORMATION	2	201
2	INCOMPLETE HESITANT FUZZY PREFERENCE RELATIONS; HESITANT FUZZY PREFERENCE RELATIONS; WATER ALLOCATION MANAGEMENT; CONSENSUS MODEL; PRIORITY WEIGHTS	2	201

Given the fact that the research field of group decision making and incomplete information is an important topic and that we propose a "granular" estimation procedure of missing information that is capable of completing the missing information with higher consistency levels than its "numeric" counterpart as illustrated in the experimental studies, we think it is a good contribution to be published in Applied Soft Computing.

There are lot of papers using granularity model. However, it will be publishable if new concept can be proposed for granularity model.

The fundamental principle of granular computing is that of information granularity, which has been explored in system modeling by giving rise to "granular" models. Depending on the context, this principle has been exploited in different ways in order to construct "granular" models. For example, in the aforementioned papers, information granularity has been exploited in different contexts (Takagi-Sugeno model, neural networks, etc.) to construct "granular" models, and these papers have published in top journals like Applied Soft Computing, even though they do not propose a new concept for granularity model. In fact, we cannot propose a new concept of the information granularity, as it is a general principle of granular computing, but we have to exploit it in such a way that the "granular" procedure estimates missing information with the higher consistency level. And this is done here, and this is the first paper in which it is done. That is, information granularity has been exploited in many contexts, but this is the first time that it is exploited to build a "granular" procedure for estimating missing values in group decision making, which achieves good results as it has been shown. Therefore, the idea presented here can be used by other researchers to construct new "granular" estimation procedures from other "numeric" procedures, as it is an important issue in group decision making.

<u>Reviewer 2</u>

The authors have revised the manuscript and added the corresponding remark.

Thank you for considering that our work can be accepted in Applied Soft Computing.

<u>**Reviewer 3**</u>

The paper has been well revised by my suggestions, so I suggest it to be accepted.

Thank you for considering that our work can be accepted in Applied Soft Computing.

Estimating incomplete information in group decision making: A framework of granular computing

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Abstract

A general assumption in group decision making scenarios is that of all individuals possess accurate knowledge of the entire problem under study, including the abilities to make a distinction of the degree up to which an alternative is better than other one. However, in many real world scenarios, this may be unrealistic, particularly those involving numerous individuals and options to choose from conflicting and dynamics information sources. To manage such a situation, estimation methods of incomplete information, which use own assessments provided by the individuals and consistency criteria to avoid discrepancy, have been widely employed under fuzzy preference relations. In this study, we introduce the information granularity concept to estimate missing values supporting the objective of obtaining complete fuzzy preference relations with higher consistency levels. We use the concept of granular preference relations to form each missing value as a granule of information in

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place of a crisp number. This offers the flexibility that is required to estimate the missing information so that the consistency levels related to the complete fuzzy preference relations are as higher as possible.

Keywords: Group decision making, incomplete information, consistency, fuzzy preference relation, information granularity

1. Introduction

Group decision making is characterized as a situation when individuals, from a set of possible options, make a choice collectively [1, 2, 3, 4, 5], which is here no longer attributable to any single individual but the whole group because all of them contribute to the outcome. In such a situation, most of the existing approaches have traditionally supposed that all the individuals have the necessary knowlegde of the problem at hand to make a distinction of the preference degree up to which an option is more suitable than other [6, 7]. However, there exist many problems where this assumption may be idealistic. In [8], it was proved that "increasing the intensity of conflict in a multicriteria comparison increases the likelihood that decision makers consider two alternatives as incomparable", resulting in incomplete information. In particular, in group decision making problems implicating a considerable amount of individuals and options to select from dynamic and contradictory information sources [9], as, for instance, the social network environments [10, 11, 12, 13], it is very common that some of the individuals, even all of them, do not offer all the information required. Therefore, it has been necessary the development of approaches addressing the existence of incomplete information [14, 15].

Given the fact that the attempt to complete assessments between pair of options is easier than providing membership degrees to all the options in an only one step (it means the individuals can evaluate each option in contrast to all the others on the whole), the most usual representation format used by the individuals to provide their assessments is that of preference relations [16]. In addition, among the existing types of preference relations [17, 18], fuzzy preference relations are the most well-known given their ability to model decision processes and their usefulness and capacity to aggregate individuals' assessments into group ones [1, 19]. On the other hand, a drawback of preference relations is that of they generate more information than is actually needed (the individuals must compare every option with all the other ones) and, therefore, the likelihood of obtaining incomplete information is higher than using other representation formats, namely, preference orderings or utility values [20].

Among the existing procedures for dealing with incomplete information in preference relations, those trying to estimate missing values are the most used [15]. On the one hand, we can find methods that estimate missing values in group decision making by using the information given by the rest of individuals along with aggregation procedures [21]. The drawbacks of these approaches is that they require several individuals to estimate the missing information of a particular one, which in conjunction with notable difference between the individuals' preferences could led to the estimation of information not naturally compatible with the rest of the individual's information. On the other hand, we can find methods that estimate missing values using just the own preferences given by the individual. In particular, the methods based on consistency criteria that estimate the individuals' incomplete information using only her/his own evaluations have been satisfactorily employed in group decision making under preference relations [22] (for more details we refer the reader to [15]).

Recently, a promising, innovative, and interesting direction is to pursue building and conceptualizing models formed as granular models [23], which may be realized as generalizations of the existing numeric models. A granular model is constructed at a higher level of abstraction and in this way becomes capable of coping with the essentials of the system modeled.

The objective of this study is to present how to generalize the existing numeric methods dealing with incomplete information to their granular methods. In particular, we present a granular estimation procedure of missing information in group decision making having the procedure proposed in [22] as its numeric counterpart. To do so, we introduce a distribution (allocation) of information granularity [24], which has been already applied successfully to increase both the consensus and the consistency in this kind of problems [25], as an essential factor to complete the missing information when the individuals verbalize their opinions via fuzzy preference relations. Then, distinct from the existing approaches dealing with missing information, we assume the missing values of a fuzzy preference relation are granular instead of numeric. It means that the missing values are considered as information granules [26] as an alternative for numeric values. Therefore, we introduce in the granular preference relation a granularity level that supplies a level of flexibility that is used to complete the missing values. This granular concept is employed to optimize (maximize) an optimization criterion, which is here associated with the individual's consistency, that is, the missing values are estimated with the purpose of increasing the consistency related to the complete fuzzy preference relation.

This study is structured in a bottom-up way and made self-contained. We structure it upon the well-known ideas of group decision making problems and recall a way in which missing information of fuzzy preference relation may be estimated (see Section 2). It uses a consistency criterion to quantify the quality of the estimated missing information. In Section 3, we discuss a way in which missing information of fuzzy preference relations may be estimated through a distribution of information granularity. Strong attention is given to the usage of the component of information granularity in the estimation of the missing values. Three experiments are reported in Section 4. Conclusions and future studies are offered in Section 5.

2. Background

We recall the idea of a fuzzy preference relation and highlight its main characteristics. We center our attention on the consistency related to fuzzy preference relations and look into a way in which missing information may be estimated when they are used.

2.1. Fuzzy preference relations

In the setting of this study, group decision making is a kind of participatory process in which more than one individual, $E = \{e_1, \ldots, e_m\}$, discuss a problem collectively, consider a collection of options, $O = \{o_1, \ldots, o_n\}$, to solve the problem and evaluate them. To do so, two processes are carried out sequentially. The first one, the consensus process [27, 28], is a creative and dynamic manner of achieving agreement among all individuals of the group, which are committed to finding a solution that every individual may actively support, or at least may accept. This guarantees that all concerns, ideas and opinions, are taken into account. The second one, the selection process [22], obtains the final solution in consonance with the evaluations provided. As a result, we arrive at a rank of options from best to worst to solve the problem.

A fundamental issue in that type of problems is the way in which the evaluations provided by the individuals are represented. To do so, as we have already mentioned, fuzzy preference relations have been widely employed.

Definition 1. A fuzzy preference relation P on a set of options O is a fuzzy set on the Cartesian product $O \times O$, that is, it is characterized by a membership function $\mu_P : O \times O \to [0, 1]$.

A fuzzy preference relation P is commonly described by a $n \times n$ matrix $P = (p_{ij})$. In this representation, $p_{ij} = \mu_P(x_i, x_j)$ is the degree in which the option o_i is preferred to the option o_j. In particular, $p_{ij} = 0.5$ means indifference between both options ($o_i \sim o_j$), $p_{ij} = 1$ signifies that the option o_i is entirely preferred to the option o_j, and $p_{ij} > 0.5$ signifies the option o_i is preferred to the option o_j ($o_i \succ o_j$). Furthermore, the elements of the principal diagonal, that is, p_{ii}, are usually written as '$-$' because they are not important here [29].

Many decision making frameworks suppose that individuals can express evaluations between any pair of options. However, it is not all the time possible and, therefore, we have to address the problem of missing information. In a fuzzy preference relation, an entry with a missing value does not mean lack of preference of one option over other one. This can be due to the incapacity of an individual to measure the preference degree of one option over other one. Therefore, if an individual cannot provide the value of p_{ij} due to she/he does not know how better the option o_i is over the option o_j, this does not signify that the agent chooses both options with equal intensity.

These situations are characterized by the concept of an incomplete fuzzy preference relation, which was defined in [22].

Definition 2. A function $f : X \to Y$ is partial when not every element in the set X necessarily maps onto an element in the set Y. However, when every element from the set X maps onto one element of the set Y, then, in this case, we have a total function.

Definition 3. A preference relation P on a set of options O with a partial membership function is an incomplete preference relation.

2.2. Consistency

Undoubtedly decision making is a complex task. It is common that individuals' evaluations do not verify properties that a fuzzy preference relation must satisfy. Consistency, which is related to the transitivity property, is one of them [22]. However, none kind of consistency property is entailed by Definition 1 and, thefeore, a fuzzy preference relations could have entries taking contradictory values, which could lead to wrong decisions [22, 30].

To avoid it, the fuzzy preference relations should satisfy one of the different properties that have been proposed [31]. Given the fact that, for a fuzzy preference relation, the additive transitivity is seen as the parallel concept of the consistency property introduced by Saaty for a multiplicative preference relation [31], a methodology using this property was proposed in [22] for verifying the consistency associated with a fuzzy preference relation. It is founded on the mathematical formulation of the additive transitivity [19]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5), \forall i, j, k \in \{1, \ldots, n\} \tag{1}$$

Additive transitivity entails additive reciprocity, that is, as $p_{ii} = 0.5\ \forall i$, we have that $p_{ij} + p_{ji} = 1$, $\forall i, j \in \{1, \ldots, n\}$, if we make $k = i$ in Eq. (1). As a consequence, we may rewrite Eq. (1) as follows:

$$p_{ik} = p_{ij} + p_{jk} - 0.5, \forall i, j, k \in \{1, \ldots, n\} \tag{2}$$

In [22], the authors used Eq. (1) to estimate the value of an entry via other entries in a fuzzy preference relation. In particular, using an intermediate option o_j, we may estimated the value of p_{ik} $(i \neq k)$ in three different ways [22]:

- We estimate the following value from $p_{ik} = p_{ij} + p_{jk} - 0.5$:

$$(ep_{ik})^{j1} = p_{ij} + p_{jk} - 0.5 \tag{3}$$

- We estimate the following value from $p_{jk} = p_{ji} + p_{ik} - 0.5$:

$$(ep_{ik})^{j2} = p_{jk} - p_{ji} + 0.5 \tag{4}$$

- We estimate the following value from $p_{ij} = p_{ik} + p_{kj} - 0.5$:

$$(ep_{ik})^{j3} = p_{ij} - p_{kj} + 0.5 \tag{5}$$

We then obtain the estimated value of p_{ik} as follows:

$$ep_{ik} = \frac{\sum_{j=1; j \neq i,k}^{n} \left((ep_{ik})^{j1} + (ep_{ik})^{j2} + (ep_{ik})^{j3} \right)}{3(n-2)} \tag{6}$$

In the case that $(ep_{ik})^{jl} = p_{ik}\ \forall j, l$, the given information is completely consistent. However, individuals are not all the time fully consistent. Hence, the evaluation given by an individual may not satisfy Eq. (1). In such a case, some of the estimated values $(ep_{ik})^{jl}$ can not pertain to the range $[0, 1]$. From Eq. (3), Eq. (4) and Eq. (5), we note that the maximum value of any $(ep_{ik})^{jl}$ $(l \in \{1, 2, 3\})$ is equal to 1.5 while the minimum value is equal to -0.5. Therefore, the error between an evaluation and its estimated one in $[0, 1]$ is calculated as follows [22]:

$$\varepsilon p_{ik} = \frac{2}{3} \cdot |ep_{ik} - p_{ik}| \tag{7}$$

The consistency degree cd_{ik} associated with the entry p_{ik} is then obtained as follows:

$$cd_{ik} = 1 - \varepsilon p_{ik} \tag{8}$$

When $\varepsilon p_{ik} = 0$, then $cd_{ik} = 1$, which means there is consistency. The higher the value of εp_{ik} is, the lower the value of cd_{ik} is, and the more inconsistent p_{ik} is concerning the remaining information.

The consistency degrees related to the fuzzy preference relation and the individual options were then defined as follows [22]:

- The consistency degree, cd_i, associated with a given option o_i is calculated as:

$$cd_i = \frac{\sum_{k=1; i \neq k}^{n} (cd_{ik} + cd_{ki})}{2(n-1)} \tag{9}$$

- The consistency degree, cd, associated with a fuzzy preference relation is calculated as:

$$cd = \frac{\sum_{i=1}^{n} cd_i}{n} \tag{10}$$

The higher the value of cd is, the more consistent a fuzzy preference relation is. In particular, when cd is equal to 1, the fuzzy preference relation is fully consistent.

2.3. Estimation procedure of incomplete information

In [22], the authors presented an iterative approach for estimating the incomplete information of a fuzzy preference relation using Eq. (3), Eq. (4) and Eq. (5). Here, we recall its two steps:

1. Missing values to be estimated in each iteration. In the step h, EMV_h denotes the subset of missing values, MV, that we may estimate (by definition, $EMV_0 = \emptyset$). Its definition is:

$$EMV_h = \left\{ (i,k) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \mid i \neq k \wedge \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\} \right\} \tag{11}$$

$$A = \{(i,k) \mid i,k \in \{1,\ldots,n\} \;\wedge\; i \neq k\} \tag{12}$$

$$MV = \{(i,k) \in A \mid p_{ik} \text{ is unknown}\} \tag{13}$$

$$EV = A \setminus MV \tag{14}$$

$$H_{ik}^{h1} = \left\{ j \mid (i,j),(j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \tag{15}$$

$$H_{ik}^{h2} = \left\{ j \mid (j,i),(j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \tag{16}$$

$$H_{ik}^{h3} = \left\{ j \mid (i,j),(k,j) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \tag{17}$$

being A the set of all pair of options, MV the set of pairs of options in which the preference degree of the first option over the second one is unknown or missing, EV the set of pairs of options whose preference degrees are provided by the individual, H_{ik}^{h1}, H_{ik}^{h2}, H_{ik}^{h3}, are the sets of the intermediate option $o_j (j \neq i,k)$ that can be used to estimate the preference degree $p_{ik} (i \neq k)$ in the step h using Eq. (3), Eq. (4) and Eq. (5).

The procedure stops when $EMV_{maxIter} = \emptyset$ ($maxIter > 0$) because we may not estimate more missing values. In addition, in the case that $\bigcup_{l=0}^{maxIter} EMV_l = MV$, all missing values of the incomplete fuzzy preference relation have been estimated and, as a consequence, the procedure has successful estimated all the missing values.

Function *estimate_p(i, k)* **is**
    $cp^1_{ik} = 0$; $cp^2_{ik} = 0$; $cp^3_{ik} = 0$; $\mathcal{K} = 0$;
    $cp^1_{ik} = \left(\left(\sum_{j \in H^{h1}_{ik}} cp^{j1}_{ik}\right) / \# H^{h1}_{ik}\right)$;
    if $H^{h1}_{ik} \neq 0$ **then**
        $\mathcal{K} = \mathcal{K} + 1$;
    end
    $cp^2_{ik} = \left(\left(\sum_{j \in H^{h2}_{ik}} cp^{j2}_{ik}\right) / \# H^{h2}_{ik}\right)$;
    if $H^{h2}_{ik} \neq 0$ **then**
        $\mathcal{K} = \mathcal{K} + 1$;
    end
    $cp^3_{ik} = \left(\left(\sum_{j \in H^{h3}_{ik}} cp^{j3}_{ik}\right) / \# H^{h3}_{ik}\right)$;
    if $H^{h3}_{ik} \neq 0$ **then**
        $\mathcal{K} = \mathcal{K} + 1$;
    end
    $cp_{ik} = (1/\mathcal{K}) \cdot (cp^1_{ik} + cp^2_{ik} + cp^3_{ik})$;
    return cp_{ik}
end

---

2. Estimating a given missing value. In the step h, *estimate_p*(i, k) is applied to estimate a value p_{ik} with $(i, k) \in EMV_h$. It is estimated as the average of all the estimated values obtatined according to all the possible intermediate options o_j by means of Eq. (3), Eq. (4) and Eq. (5).

In summary, given a particular incomplete fuzzy preference relation, we may estimate its missing values using Algorithm 1.

Algorithm 1: Iterative estimation procedure

$EMV_0 = \emptyset$;
$h = 1$;
while $EMV_h \neq \emptyset$ **do**
    foreach $(i, k) \in EMV_h$ **do** `estimate_p(`i,k`)`;
    $h = h + 1$;
end

Remark 1. This procedure estimates the missing values using only the preference values given by the individual. By doing this, the procedure assures that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by the individual [22]. Therefore, if the preference values provided by the individual are inconsistent, the estimated values could also be inconsistent. In such a case, the preference values given by the individual must also be modified if we want to obtain a consistent fuzzy preference relation. To do so, the *estimate_p*(i, k) function should be applied for all the preference values, and not only for the missing values.

3. Estimating missing information through an allocation of information granularity

In this section, we describe how an allocation of information granularity may help to complete a fuzzy preference relation, which has missing values, with the higher possible consistency degree. To address this quest, we introduce an idea of a granular preference relation, which is a generalization of a fuzzy preference relation that is constructed due to a distribution of information granularity [32].

Information granularity [24] is used here as a very important design asset that may be exploited as a means to complete incomplete fuzzy preference relations of higher consistency bringing into a picture the point of values that are non-numeric and quantifying their nature by means of information granules. That is, we give up on the precise numeric values forming the entries of the fuzzy preference relations and make them granular by accepting information granules and allocating a predetermined granularity level to them in order that the granular preference relations constructed in this way "cover" as many values as possible. This position gives rise to the allocation on information granularity [32], which is another essential principle of Granular Computing [33, 34]. In our setting, the allocation of information granularity elevates the fuzzy preference relations to a new level called granular preference relations.

We employ the symbol $\boldsymbol{G}(P)$ to stress that we use granular preference relations, being $\boldsymbol{G}(\cdot)$ a particular formalism of information granules [35]. Note that it is a general expression and that we are not limited to any specific granular formalism used here, namely, probability density functions, fuzzy sets, or intervals, to cite some alternatives that are usually encountered.

Concerning the estimation of missing values via a granular preference relation, there are two crucial aspects to be considered: (i) how to allocate the information granularity to the entries with missing values, and (ii) how to exploit the information granularity to complete incomplete fuzzy preference relations of higher granularity. Both aspects are described in detail in what follows.

3.1. Allocation of information granularity

The information granularity may be distributed in some different ways [24]. For clarity of the presentation, we use here a uniform allocation (distribution), in which all estimated values are treated similarly and become substituted by intervals of the same length. It means that we use intervals as information granules, and, therefore, $\boldsymbol{G}(P) = \boldsymbol{I}(P)$, where $\boldsymbol{I}(\cdot)$ denotes a family of intervals. That is, we take advantage of the estimation procedure described in Section 2.3 and augment it to some extent in order that it becomes adjusted. By these actions, we completely accept that the current knowledge source should be taken with a pinch of salt and the results provided by the estimation procedure should reflect the partial relevance of the procedure in the situation at present. This effect is quantified by making the estimated values granular (namely, more general and abstract) in order that the model may be built around the conceptual framework provided up to now. In addition, we symmetrically distribute the intervals around the estimated values.

3.2. Exploiting information granularity to estimate missing values

In the granular model of fuzzy preference relations, we need to consider that the estimated values are adjusted within the limits offered by the granularity level that is admissible with the purpose of increasing the consistency related to the fuzzy preference relation. Hence, the granularity level is employed to estimate the missing values so that the complete fuzzy preference relation is of higher consistency. We bring about this improvement at the level of each individual. This effect is quantified by the following performance index:

$$Q = \frac{1}{m} \sum_{l=1}^{m} cd^l \tag{18}$$

where m is the number of individuals participating in the decision process and cd^l represents the consistency degree related to the fuzzy preference relation expressed by the individual e_l, which is calculated using Eq. (10).

This optimization problem is willing to maximize the above performance index. It reads as follows:

$$\text{Max}_{P^1,P^2,\ldots,P^m \in \boldsymbol{I}(P)} Q \tag{19}$$

This optimization task is performed for all granular preference relations that are admissible on account of the introduced granularity level. Given the fact that this task is complicated (the search space is quite large as it is composed of $\boldsymbol{I}(P)$), it requires the usage of advanced global optimization techniques. In particular, this optimization task is achieved via the particle swarm optimization [36, 37]. For this problem, this technique is viable since it provides a considerable level of optimization flexibility and is not accompanied by a prohibitive computational overhead level.

The particle swarm optimization is inspired by the foraging behavior of animals. It uses a swarm of particles to model the animals and to search the location of food (optimal solution) in a solution space that is n-dimensional [38] (see Fig. 1). Each particle i is composed of a velocity and a position, which are represented by $\boldsymbol{v}_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ and $\boldsymbol{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$, respectively. In addition, each particle i has the individual memory of its best historical position $\boldsymbol{x}_i^{lbest}$ and its best fitness value y_i^{lbest}. Moreover, the best individual memory $\boldsymbol{x}^{gbest}$ is broadcast across the whole population.

In each generation t, each particle adapts its position and search pattern in the d-th dimension based on its individual memory $\boldsymbol{x}_i^{lbest}$ and the global memory $\boldsymbol{x}^{gbest}$ as follows:

$$v_{id}(t+1) = \omega(t) \cdot v_{id}(t) + c_1 \cdot r_{1d} \cdot (x_{id}^{lbest} - x_{id}(t)) + c_2 \cdot r_{2d} \cdot (x_d^{gbest} - x_{id}(t)) \tag{20}$$

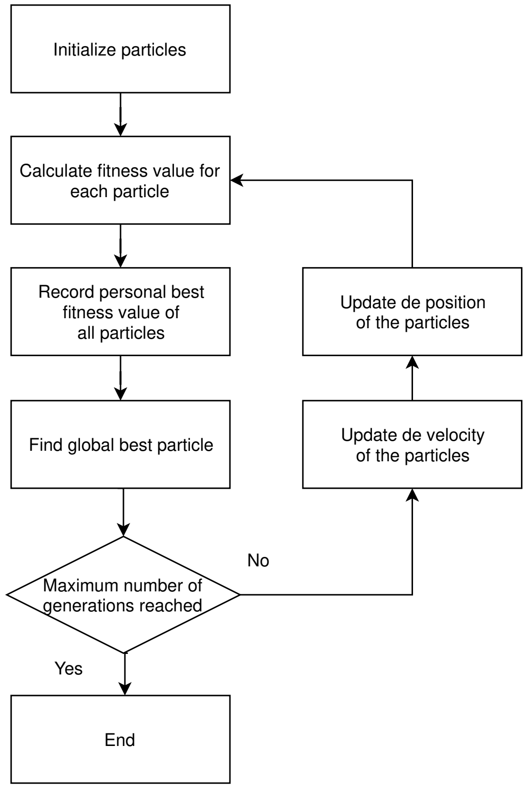


Figure 1: Particle swarm optimization flowchart

$$x_{id}(t+1) = x_{id}(t) + v_{id}(t+1) \tag{21}$$

where c_1 is defined as a cognitive acceleration coefficient and c_2 is defined as a social acceleration coefficient. According to the values of c_1 and c_2, different attention is paid to the global search and the local search. r_{1d} and r_{2d} represent random numbers that are generated in $[0, 1]$. Finally, the local and global search ability of the particles in the generation t is balanced by the inertia weight $\omega(t)$ [39]. For local search, a small value is more suitable while a large value boosts the global search. Its value is usually decreased linearly according to [40]:

$$\omega(t) = (\omega_{start} - \omega_{end}) \cdot \frac{t_{max} - t}{t_{max}} + \omega_{end} \tag{22}$$

where ω_{start} is the initial value of ω and ω_{end} is its final value, the current generation number and the maximum generation number are represented by t and t_{max}, respectively, and $\omega(t)$ is the value of ω in the current generation.

In the particle swarm optimization, a notable aspect is that of establishing an association between the problem's solution and the particle's representation. In our setting, a vector models each particle, assuming each entry of the vector a value between 0 and 1. In essence, if m individuals are part of the group, the vector is composed of $\sum_{l=1}^{m} \#MV^l$ entries, being $\#MV^l$ the number of missing values encountered in the incomplete fuzzy preference relation expressed by the individual e_l.

Let us suppose a granularity level $\alpha \in [0, 1]$, an incomplete fuzzy preference relation P expressed by an individual, and a missing entry p_{ij} of P. Then, the granularity level α implies in this entry of $\boldsymbol{I}(P)$ an interval of admissible values that is calculated as follows:

$$[I_{start}, I_{end}] = [\text{Max}(0, cp_{ij} - \alpha/2), \text{Min}(cp_{ij} + \alpha/2, 1)] \tag{23}$$

As an illustration example, we suppose cp_{ij} is 0.71. In addition, the corresponding component of the particle x is 0.8, and the level of granularity α is 0.4. Using Eq. (23), we get that the corresponding interval to x is equal to $[I_{start}, I_{end}] = [0.51, 0.91]$. Then, using the expression $I_{start} + (I_{end} - I_{start}) \cdot x$ we obtain that the new value of cp_{ij} is equal to 0.83.

The other important aspect in this optimization technique is the definition of the fitness function, which assesses the quality of each particle during the successive generations. In our setting, we aim to maximize the consistency associated with the fuzzy preference relation. Consequently, the fitness function, f, related to the particle is:

$$f = Q \tag{24}$$

where Q is the performance index introduced in Eq. (18). The higher the value returned by the fitness function is, the better the particle is.

The steps of the proposed methodology to estimate missing information in group decision making are illustrated in Fig. 2.

4. Experimental studies

We illustrate the proposal and test its performance in this section by presenting some examples. In all of them, the particle swarm optimization was applied using these values of the parameters:

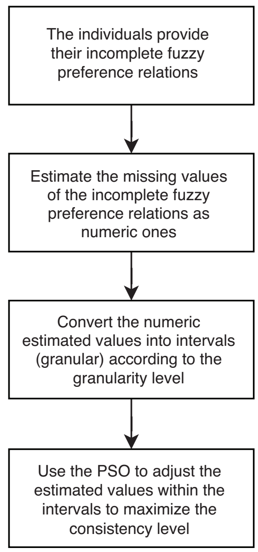


Figure 2: Proposed methodology flowchart

- The swarm was composed of 100 particles. Given the fact that similar outcomes were achieved in different runs of the particle swarm optimization, we found that this size produces "stable" outcomes.

- The number of generations was equal to 1000. This value was chosen because the same values reported by the fitness function were observed after this number of generations.

- c_1 and c_2 were set to 2 as this value is usually used in the existing literature [41, 42, 43].

- ω_{start} was set to 0.9 and ω_{end} was set to 0.4 as we usually encounter these values in the existing literature [40].

4.1. First study

In the first study, a low number of options and individuals is assumed for the sake of simplicity. Four individuals $E = \{e_1, e_2, e_3, e_4\}$ express their evaluations over a collection of five options $O = \{o_1, o_2, o_3, o_4, o_5\}$ by means of these incomplete fuzzy preference relations:

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ x & - & x & x & 0.20 \\ x & x & - & 0.30 & x \\ x & 0.80 & x & - & x \\ x & x & x & x & - \end{pmatrix} \quad P^2 = \begin{pmatrix} - & x & x & x & x \\ 0.20 & - & 0.30 & x & 0.30 \\ 0.70 & x & - & x & x \\ 0.60 & 0.10 & x & - & 0.90 \\ 0.8 & x & 1.00 & 0.30 & x \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & x & 0.70 \\ 0.10 & - & 0.80 & x & 0.30 \\ 0.40 & x & - & x & 0.30 \\ x & 0.10 & x & - & 0.90 \\ 0.90 & x & 0.10 & x & - \end{pmatrix} \quad P^4 = \begin{pmatrix} - & x & x & x & x \\ 0.10 & - & x & 0.90 & x \\ 0.30 & x & - & x & x \\ 0.50 & x & 0.40 & - & x \\ 0.30 & 0.10 & x & x & - \end{pmatrix}$$

On the one hand, if we apply the estimation procedure presented in Section 2.3, the following complete fuzzy preference relations are obtained (the estimated values are in bold):

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.22} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ \mathbf{0.43} & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix} \quad P^2 = \begin{pmatrix} - & \mathbf{0.00} & \mathbf{0.22} & \mathbf{0.20} & \mathbf{0.32} \\ 0.20 & - & 0.30 & \mathbf{0.23} & 0.30 \\ 0.70 & \mathbf{0.49} & - & \mathbf{0.43} & \mathbf{0.53} \\ 0.60 & 0.10 & \mathbf{0.59} & - & 0.90 \\ 0.8 & \mathbf{0.49} & 1.00 & 0.30 & x \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & \mathbf{0.40} & 0.70 \\ 0.10 & - & 0.80 & \mathbf{0.23} & 0.30 \\ 0.40 & \mathbf{0.11} & - & \mathbf{0.08} & 0.30 \\ \mathbf{0.56} & 0.10 & \mathbf{0.80} & - & 0.90 \\ 0.90 & \mathbf{0.21} & 0.10 & \mathbf{0.28}- & \end{pmatrix} \quad P^4 = \begin{pmatrix} - & \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.83} & \mathbf{0.53} \\ 0.10 & - & \mathbf{0.43} & 0.90 & \mathbf{0.30} \\ 0.30 & \mathbf{0.40} & - & \mathbf{0.65} & \mathbf{0.40} \\ 0.50 & \mathbf{0.36} & 0.40 & - & \mathbf{0.35} \\ 0.30 & 0.10 & \mathbf{0.35} & \mathbf{0.60} & - \end{pmatrix}$$

As an example of illustration, the procedure to estimate the missing values in P^1 is as follows:

- The missing values that may be estimated in the initial step are:

$$\begin{aligned} EMV_1 \;=\; & \{(2,1),(2,3),(2,4),(3,1),(3,2),(3,5), \\ & (4,1),(4,3),(4,5),(5,2),(5,3),(5,4)\} \end{aligned}$$

We have the following incomplete fuzzy preference relation once these missing values have been estimated:

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.23} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ x & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix}$$

For instance, the procedure to estimate p_{21}^1 is:

$$\begin{array}{lll} H_{21}^{11} = \emptyset & \Rightarrow & cp_{43}^1 = 0 \\ H_{21}^{12} = \emptyset & \Rightarrow & cp_{43}^2 = 0 \\ H_{21}^{13} = \{5\} & \Rightarrow & cp_{43}^3 = cp_{43}^{53} = p_{25} - p_{15} + 0.50 = 0.10 \\ & & \\ \mathcal{K} = 1 & \Rightarrow & cp_{43} = \dfrac{0 + 0.10 + 0}{1} = 0.10 \end{array}$$

- In the second step, we may estimate the following missing value:

$$EMV_2 = \{(5,1)\}$$

We obtain the following complete fuzzy preference relation once this missing value has been estimated:

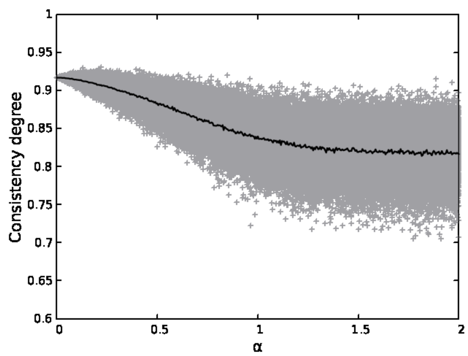
$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.23} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ \mathbf{0.43} & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix}$$

Once all the missing values have been estimated, using the method presented in Section 2.2, we measure the consistency degree related to each fuzzy preference relation:

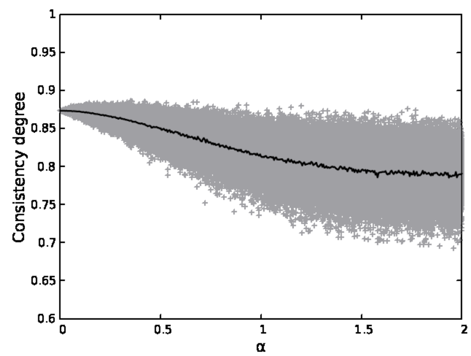
$$cd^1 = 0.917 \quad cd^2 = 0.872 \quad cd^3 = 0.851 \quad cd^4 = 0.920$$

Then, the global consistency is equal to $(0.917+0.872+0.851+0.920)/4) = 0.890$.

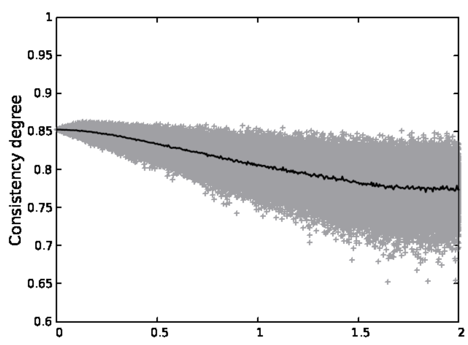
Before applying our approach, it becomes informative to study the effect of the deterioration or improvement of the consistency degree related to the fuzzy preference relations when provided with an imposed level of granularity. A particular value of the level of granularity is allowed to study the impact of the given value for a given fuzzy preference relation P. Then, coming from a granular representation of P, $\boldsymbol{I}(P)$, we generate in a random manner a fuzzy preference relation and compute its corresponding consistency degree. We repeat the calculations 500 times for each value of the level of granularity. In Fig. 3, we show the related plots of the consistency degree in contrast to the given level of granularity. Furthermore, the mean of the consistency degrees are also displayed in these plots.



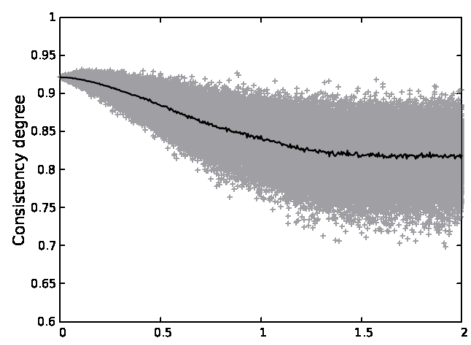
(a) Consistency degree versus α for P^1.



(b) Consistency degree versus α for P^2.



(c) Consistency degree versus α for P^3.



(d) Consistency degree versus α for P^4.

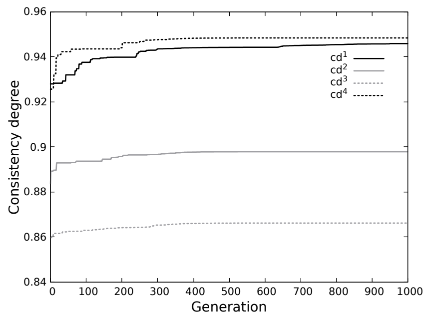
Figure 3: Plots of consistency degrees versus α

Theoretically, when the value of the level of granularity increases, the probability of estimating missing values so that we arrive at a more consistent fuzzy preference relation also increases. It is expected as we intend to exploit the flexibility inserted by the granularity level. However, the likelihood of producing very inconsistent preference relations also increases. Despite this, the average value of the consistency degrees presents some slight downward trend for higher values of the granularity level. Particularly, if the number of missing values is very high, the average consistency degree related to the fuzzy preference relation usually decreases for higher values of the level of granularity.

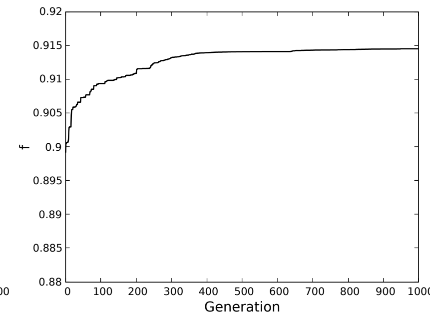
Once studied the effect of the imposed level of granularity in the deterioration or improvement of the consistency degree related to the fuzzy preference relation, we run the approach presented in Section 3.2 to optimize the estimated values assumed by the entries with missing values of the fuzzy preference relations. Taking into consideration different selected values of α, Fig. 4 displays the performance of the particle swarm optimization in relation to the values reported by the fitness function in consecutive generations. At the beginning of the optimization process (first 400 generations) we may observe the most significant improvement. After that, we may observe a slight upward trend until a clearly visible stabilization is reached in the last generations, that is, the values reported by the fitness function are constant.

Comparing with the consistency degrees obtained by the estimation procedure described in Section 2.3 (it is similar to assume a granularity level α equal to 0), our proposal achieves better results (see Fig. 4 and Table 1). As we may observe, a higher imposed level of granularity implies higher values reported by the fitness function and, therefore, the consistency degrees associated with the complete fuzzy preference relations are also higher. It is important to keep in mind that a higher level of granularity implies a higher flexibility introduced in the fuzzy preference relations, which increases the probability of completing incomplete fuzzy preference relations of higher consistency. However, this improvement is not so high as it might be expected. It is due to the fact that the missing values are estimated so that the consistency related to the complete fuzzy preference relation is higher. Anyway, the optimization of the estimated values achieves better consistency degrees than the estimation procedure described in Section 2.3 (when $\alpha = 0$).

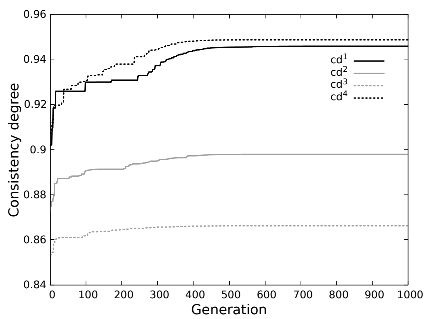
Finally, as illustration example, the following complete fuzzy preference relations are obtained when the estimated values are optimized with $\alpha = 1$



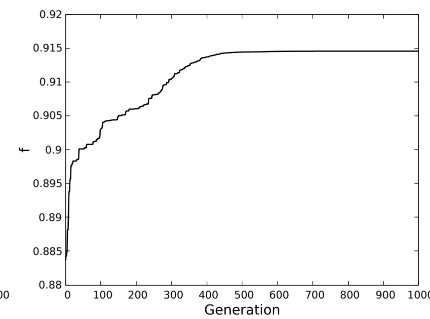
(a) cd with $\alpha = 0.5$.



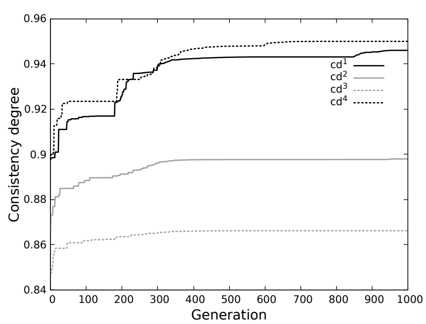
(b) f with $\alpha = 0.5$.



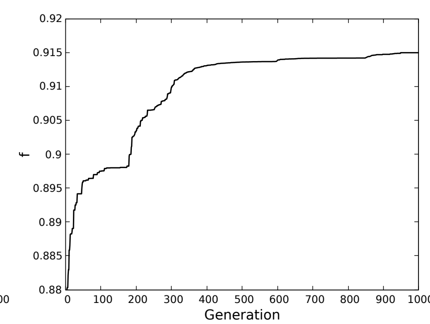
(c) cd with $\alpha = 1$.



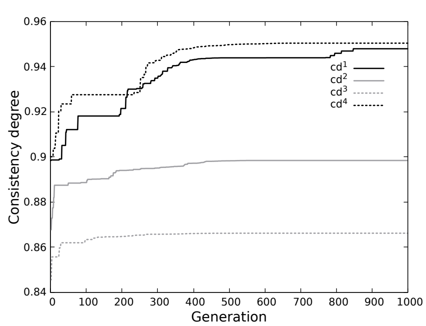
(d) f with $\alpha = 1$.



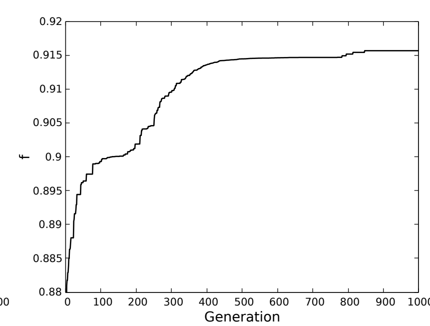
(e) cd with $\alpha = 1.5$.



(f) f with $\alpha = 1.5$.



(g) cd with $\alpha = 2$.



(h) f with $\alpha = 2$.

Figure 4: Plots of the consistency degrees and f in successive generations

Table 1: Results achieved by different values of α.

	cd^1	cd^2	cd^3	cd^4	f
$\alpha = 0$	0.917	0.872	0.851	0.920	0.890
$\alpha = 0.5$	0.946	0.897	0.866	0.948	0.914
$\alpha = 1$	0.946	0.898	0.866	0.949	0.915
$\alpha = 1.5$	0.947	0.898	0.866	0.950	0.915
$\alpha = 2$	0.948	0.899	0.866	0.951	0.916

(the estimated values are in bold):

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.37} & - & \mathbf{0.41} & \mathbf{0.34} & 0.20 \\ \mathbf{0.47} & \mathbf{0.50} & - & 0.30 & \mathbf{0.42} \\ \mathbf{0.71} & 0.80 & \mathbf{0.66} & - & \mathbf{0.59} \\ \mathbf{0.57} & \mathbf{0.54} & \mathbf{0.53} & \mathbf{0.43} & - \end{pmatrix} \quad P^2 = \begin{pmatrix} - & \mathbf{0.34} & \mathbf{0.52} & \mathbf{0.22} & \mathbf{0.41} \\ 0.20 & - & 0.30 & \mathbf{0.10} & 0.30 \\ 0.70 & \mathbf{0.45} & - & \mathbf{0.28} & \mathbf{0.49} \\ 0.60 & 0.10 & \mathbf{0.73} & - & 0.90 \\ 0.8 & \mathbf{0.56} & 1.00 & 0.30 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & \mathbf{0.42} & 0.70 \\ 0.10 & - & 0.80 & \mathbf{0.36} & 0.30 \\ 0.40 & \mathbf{0.19} & - & \mathbf{0.25} & 0.30 \\ \mathbf{0.59} & 0.10 & \mathbf{0.83} & - & 0.90 \\ 0.90 & \mathbf{0.21} & 0.10 & \mathbf{0.37} & - \end{pmatrix} \quad P^4 = \begin{pmatrix} - & \mathbf{0.43} & \mathbf{0.66} & \mathbf{0.82} & \mathbf{0.71} \\ 0.10 & - & \mathbf{0.61} & 0.90 & \mathbf{0.63} \\ 0.30 & \mathbf{0.31} & - & \mathbf{0.60} & \mathbf{0.51} \\ 0.50 & \mathbf{0.27} & 0.40 & - & \mathbf{0.39} \\ 0.30 & 0.10 & \mathbf{0.42} & \mathbf{0.53} & - \end{pmatrix}$$

In summary, it may be concluded that an incomplete fuzzy preference relation may be completed so that the consistency associated with it is higher with the usage of the approach presented in this study. It speaks to the information granularity plays a notable role in the improvement of consistency.

4.2. Second study

In the second study, we suppose the following incomplete fuzzy preference relation:

$$P = \begin{pmatrix} - & 0.30 & 0.60 & 0.70 & x \\ 0.80 & - & x & x & x \\ x & x & - & 0.40 & x \\ 0.20 & 0.60 & x & - & x \\ x & x & x & x & - \end{pmatrix}$$

The estimation procedure presented in Section 2.3 may estimate all the missing values encountered in a fuzzy preference relation if a set of $n-1$ non-leading diagonal preference values is known, where each one of the options

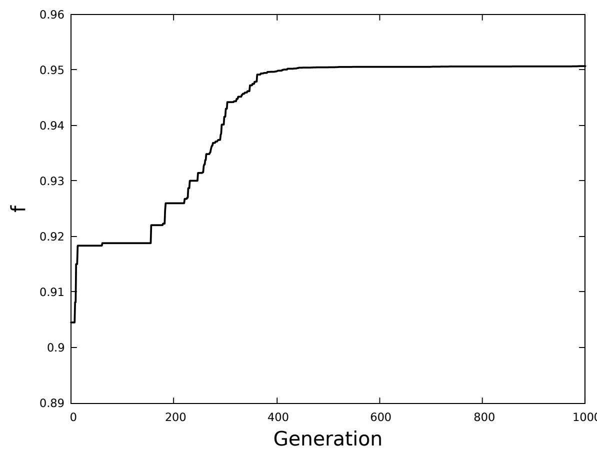


Figure 5: Values reported by f in successive generations.

is compared at least once [22]. Therefore, in this case, it cannot estimate all the missing values as the option o_5 is never compared.

However, we may apply our approach by assuming that the missing values that may be not estimated using the estimation procedure presented in Section 2.3 can assume any value in the unit interval. Therefore, in such a case, the level of granularity assumed is equal to 2. Fig. 5 displays the progression of the values reported by the fitness function.

In this case, the complete fuzzy preference relation obtained is the following (the estimated values are in bold):

$$P = \begin{pmatrix} - & 0.30 & 0.60 & 0.70 & \mathbf{0.49} \\ 0.80 & - & \mathbf{0.87} & \mathbf{0.97} & \mathbf{0.68} \\ \mathbf{0.30} & \mathbf{0.21} & - & \mathbf{0.40} & \mathbf{0.31} \\ 0.20 & 0.60 & \mathbf{0.56} & - & \mathbf{0.39} \\ \mathbf{0.52} & \mathbf{0.41} & \mathbf{0.70} & \mathbf{0.67} & - \end{pmatrix}$$

being the consistency degree related to it equal to 0.951.

In summary, in addition to improve the consistency degree obtained by the estimation procedure presented in Section 2.3, the proposed approach may be also applied in situations in which the above estimation procedure does not work. However, in this case, it estimates preference degree for options that have not been compared at least once and, therefore, even though the preference degrees are estimated so that the consistency level associated with the fuzzy preference relation is as high as possible, the estimated values should be presented to the individual in order that she/he accepts them.

4.3. Third study

In this third study, we test the performance of the proposed approach in different scenarios in which we assume a higher number of indidivuals (m) and options (o). To do so, we randomly generate incomplete fuzzy preference relations and apply the approach presented in [22] and the proposed approach to complete them.

Table 2: Consistency degrees achieved by [22] and the proposed approach.

	$o = 5$	$o = 10$	$o = 15$	$o = 20$
$m = 5$	0.811	0.723	0.827	0.888
	0.853	**0.747**	**0.840**	**0.910**
$m = 10$	0.743	0.655	0.677	0.777
	0.776	**0.682**	**0.698**	**0.801**
$m = 15$	0.645	0.803	0.901	0.798
	0.688	**0.844**	**0.923**	**0.823**
$m = 20$	0.754	0.771	0.697	0.866
	0.788	**0.803**	**0.727**	**0.891**

Table 2 shows the results achieved by the approach presented in [22] (in normal font) and the results achieved by the proposed approach (in bold) in terms of the fitness function f. In the above examples, we have observed that the proposed approach achieves better results when the maximum level of granularity is assumed. Therefore, in this third study, we set $\alpha = 2$. It can be seen that the proposed approach obtains complete fuzzy preference relations with higher consistency degrees.

5. Concluding remarks

This study has formulated, motivated, and solved the problem of estimating missing values of incomplete fuzzy preference relations so that the consistency degree related to the complete fuzzy preference relations obtained are as higher as possible.

This investigation is in line of a general position aligned with the principles of information granularity and the very nature of the resulting information granules. By starting with a collection of incomplete fuzzy preference relations, we have presented a comprehensive algorithm framework that comes up with granular preference relations (in particular, intervals) to estimate the missing values. We have emphasized the motivation and need behind engaging information granules so that the missing values have been estimated to obtain fuzzy preference relations of higher consistency.

We have also shown that the particle swarm optimization algorithm serves as an appropriate optimization framework. However, we should note that while this framework maximizes the values reported by the fitness function, it does not guarantee an optimal result, rather than we may refer to it as the best solution that is produced by the particle swarm optimization framework.

We conclude with some suggestions for future studies:

- In this study, we have shown how to elevate the estimation procedure presented in [22] to its granular form. However, the proposed approach may also be applied to any other numeric estimation procedure [44, 45, 46, 47, 48].

- The allocation of the information granularity was expressed in terms of a uniform distribution, in which all numeric estimated values were treated similarly and became substituted by intervals of the same length that were distributed symmetrically around the estimated values. There is, however, a wealth of possibilites to investigate when it comes to the allocation of the available information granularity: (i) uniform allocation of information granularity with asymmetric position of intervals around the estimated values, (ii) non-uniform allocation of information granularity with symmetrically distributed intervals, and (iii) non-uniform allocation of information granularity with asymmetrically distributed intervals. Furthermore, to assess the relative performance of the above approaches, an interesting reference point is to consider a random allocation of the information granularity. It helps quantify how the optimized and meticulously planned process of allocation of information granularity is better than a simply random allocation process.

- We focused on the formalization of information granules as intervals for the conciseness and clarity of the presentation. However, the underlying conceptual framework is also appropriate to cope with other formal realizations of information granules as, for instance, pythagorean fuzzy sets [49, 50].

Acknowledgments

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References

[1] J. Kacprzyk, Group decision making with a fuzzy linguistic majority, Fuzzy Sets and Systems 18 (1986) 105–118.

[2] F. J. Cabrerizo, F. Chiclana, R. Al-Hmouz, A. Morfez, A. S. Balamash, E. Herrera-Viedma, Fuzzy decision making and consensus: Challenges, Journal of Intelligent & Fuzzy Systems 29 (3) (2015) 1109–1118.

[3] M. J. del Moral, F. Chiclana, J. M. Tapia, E. Herrera-Viedma, A comparative study on consensus measures in group decision making, International Journal of Intelligent Systems 33 (8) (2018) 1624–1638.

[4] I. J. Pérez, F. J. Cabrerizo, E. Herrera-Viedma, A mobile decision support system for dynamic group decision making problems, IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans 40 (6) (2010) 1244–1256.

[5] H. Zhang, Y. Dong, E. Herrera-Viedma, Consensus building for the heterogeneous large-scale gdm with the invidual concerns and satisfactions, IEEE Transactions on Fuzzy Systems 26 (2) (2018) 884–898.

[6] F. J. Cabrerizo, J. M. Moreno, I. J. Pérez, E. Herrera-Viedma, Analyzing consensus approaches in fuzzy group decision making: advantages and drawbacks, Soft Computing 14 (5) (2010) 451–463.

[7] E. Herrera-Viedma, F. J. Cabrerizo, J. Kacprzyk, W. Pedrycz, A review of soft consensus models in a fuzzy environment, Information Fusion 17 (2014) 4–13.

[8] S. Deparis, V. Mousseau, M. Öztürk, C. Pallier, C. Huron, When conflict induces the expression of incomplete preferences, European Journal of Operational Research 221 (3) (2012) 593–602.

[9] I. J. Pérez, F. J. Cabrerizo, S. Alonso, Y. C. Dong, F. Chiclana, E. Herrera-Viedma, On dynamic consensus processes in group decision making problems, Information Sciences 459 (2018) 20–35.

[10] S. Alonso, I. J. Pérez, F. J. Cabrerizo, E. Herrera-Viedma, A linguistic consensus model for web 2.0 communities, Applied Soft Computing 13 (1) (2013) 149–157.

[11] E. Herrera-Viedma, F. J. Cabrerizo, F. Chiclana, J. Wu, M. J. Cobo, K. Samuylov, Consensus in group decision making and social networks, Studies in Informatics and Control 26 (3) (2017) 259–268.

[12] J. A. Morente-Molinera, G. Kou, C. Pang, F. J. Cabrerizo, E. Herrera-Viedma, An automatic procedure to create fuzzy ontologies from users' opinions using sentiment analysis procedures and multi-granular fuzzy linguistic modelling methods, Information Sciences 476 (2019) 222–238.

[13] J. Wu, F. Chiclana, E. Herrera-Viedma, Trust based consensus model for social network in an incomplete linguistic information context, Applied Soft Computing 35 (2015) 827–839.

[14] N. Capuano, F. Chiclana, H. Fujita, E. Herrera-Viedma, V. Loia, Fuzzy group decision making with incomplete information guided by social influence, IEEE Transactions on Fuzzy Systems 26 (3) (2018) 1704–1718.

[15] R. Ureña, F. Chiclana, J. A. Morente-Molinera, E. Herrera-Viedma, Managing incomplete preference relations in decision making: A review and future trends, Information Sciences 302 (2015) 14–32.

[16] I. Millet, The effectiveness of alternative preference elicitation methods in the analytic hierarchy process, Journal of Multi-Criteria Decision Analysis 6 (1) (1997) 41–51.

[17] W. Liu, Y. Dong, F. Chiclana, F. J. Cabrerizo, E. Herrera-Viedma, Group decision-making based on heterogeneous preference relations with self-confidence, Fuzzy Optimziation and Decision Making 16 (4) (2017) 429–447.

[18] Z. S. Xu, A survey of preference relations, International Journal of General Systems 36 (2) (2007) 179–203.

[19] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets and Systems 12 (2) (1984) 117–131.

[20] Y. C. Dong, H. J. Zhang, Multiperson decision making with different preference representation structures: A direct consensus framework and its properties, Knowledge-Based Systems 58 (2014) 45–57.

[21] S.-C. Hsu, T.-C. Wang, Solving multi-criteria decision making with incomplete linguistic preference relations, Expert Systems with Applications 38 (9) (2011) 10882–10888.

[22] E. Herrera-Viedma, F. Chiclana, F. Herrera, S. Alonso, Group decision-making model with incomplete fuzzy preference relations based on additive consistency, IEEE Transactions on Systems, Man, and Cybernetics - Part B: Cybernetics 37 (1) (2007) 176–189.

[23] W. Pedrycz, From numeric models to granular system modeling, Fuzzy Information and Engineering 7 (1) (2015) 1–13.

[24] W. Pedrycz, Allocation of information granularity in optimization and decision-making models: Towards building the foundations of granular computing, European Journal of Operational Research 232 (1) (2014) 137–145.

[25] F. J. Cabrerizo, R. Ureña, W. Pedrycz, E. Herrera-Viedma, Building consensus in group decision making with an allocation of information granularity, Fuzzy Sets and Systems 255 (2014) 115–127.

[26] W. Pedrycz, G. Succi, A. Sillitti, J. Iljazi, Data description: A general framework of information granules, Knowledge-Based Systems 80 (2015) 98–108.

[27] F. J. Cabrerizo, R. Al-Hmouz, A. Morfez, A. S. Balamash, M. A. Martínez, E. Herrera-Viedma, Soft consensus measures in group decision making using unbalanced fuzzy linguistic information, Soft Computing 21 (11) (2017) 3037–3050.

[28] Y. Xu, F. J. Cabrerizo, E. Herrera-Viedma, A consensus model for hesitant fuzzy preference relations and its application in water allocation management, Applied Soft Computing 58 (2017) 265–284.

[29] J. Kacprzyk, M. Fedrizzi, 'soft' consensus measures for monitoring real consensus reaching processes under fuzzy preferences, Control and Cybernetics 15 (1986) 309–323.

[30] Y. Xu, Q. Wang, F. J. Cabrerizo, E. Herrera-Viedma, Methods to improve the ordinal and multiplicative consistency for fuzzy reciprocal preference relations, Applied Soft Computing 67 (2018) 479–493.

[31] E. Herrera-Viedma, F. Chiclana, M. Luque, Some issues on consistency of fuzzy preference relations, European Journal of Operational Research 154 (1) (2004) 98–109.

[32] X. Hu, W. Pedrycz, X. Wang, Optimal allocation of information granularity in system modeling through the maximization of information specificity: A development of granular input space, Applied Soft Computing 42 (2016) 410–422.

[33] A. Bargiela, W. Pedrycz, Granular Computing: An Introduction, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.

[34] F. J. Cabrerizo, E. Herrera-Viedma, W. Pedrycz, A method based on pso and granular computing of linguistic information to solve group decision making problems defined in heterogeneous contexts, European Journal of Operational Research 230 (3) (2013) 624–633.

[35] M. M. Ahmed, N. A. M. Isa, Knowledge base to fuzzy information granule: A review from the interpretability-accuracy perspective, Applied Soft Computing 54 (2018) 121–140.

[36] J. Kennedy, R. C. Eberhart, Particle swarm optimization, in: Proceedings – IEEE International Conference on Neural Networks, Vol. 4, IEEE Press, NJ, 1995, pp. 1942–1948.

[37] Y. Prasad, K. K. Biswas, M. Hanmandlu, A recursive pso scheme for gene selection in microarray data, Applied Soft Computing 71 (2018) 213–225.

[38] P. Liu, J. Liu, Multi-leader PSO (MLPSO): A new PSO variant for solving global optimization problems, Applied Soft Computing 61 (2017) 256–263.

[39] Y. Shi, R. C. Eberhart, A modified particle swarm optimizer, in: Proceedings – IEEE International Conference on Evolutionary Conference. IEEE World Congress on Computational Intelligence, 1998, pp. 69–73.

[40] A. Nickabadi, M. M. Ebadzadeh, R. Safabakhsh, A novel particle swarm optimization algorithm with adaptive inertia weight, Applied Soft Computing 11 (4) (2011) 3658–3670.

[41] B. Mohammadi-Ivatloo, A. Rabiee, A. Soroudi, M. Ehsan, Iteration PSO with time varying acceleration coeffficients for solving non-convex economic dispatch problems, Electrical Power and Energy Systems 42 (1) (2012) 508–516.

[42] R. Poli, J. Kennedy, T. Blackwell, Particle swarm optimization, Swarm Intelligence 1 (1) (2007) 33–57.

[43] F. van den Bergh, P. Engelbrecht, A study of particle swarm optimization particle trajectories, Information Sciences 176 (8) (2006) 937–971.

[44] S. Alonso, F. J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision making with incomplete fuzzy linguistic preference relations, International Journal of Intelligent Systems 24 (2) (2009) 201–222.

[45] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, A web based consensus support system for group decision making problems and incomplete preferences, Information Sciences 180 (23) (2010) 4477–4495.

[46] M. Fedrizzi, S. Giove, Incomplete pairwise comparison and consistency optimization, European Journal of Operational Research 183 (1) (2007) 303–313.

[47] X. Liu, Y. Pan, Y. Xu, S. Yu, Least square completion and inconsistency repair methods for additively consistent fuzzy preference relations, Fuzzy Sets and Systems 198 (2012) 1–19.

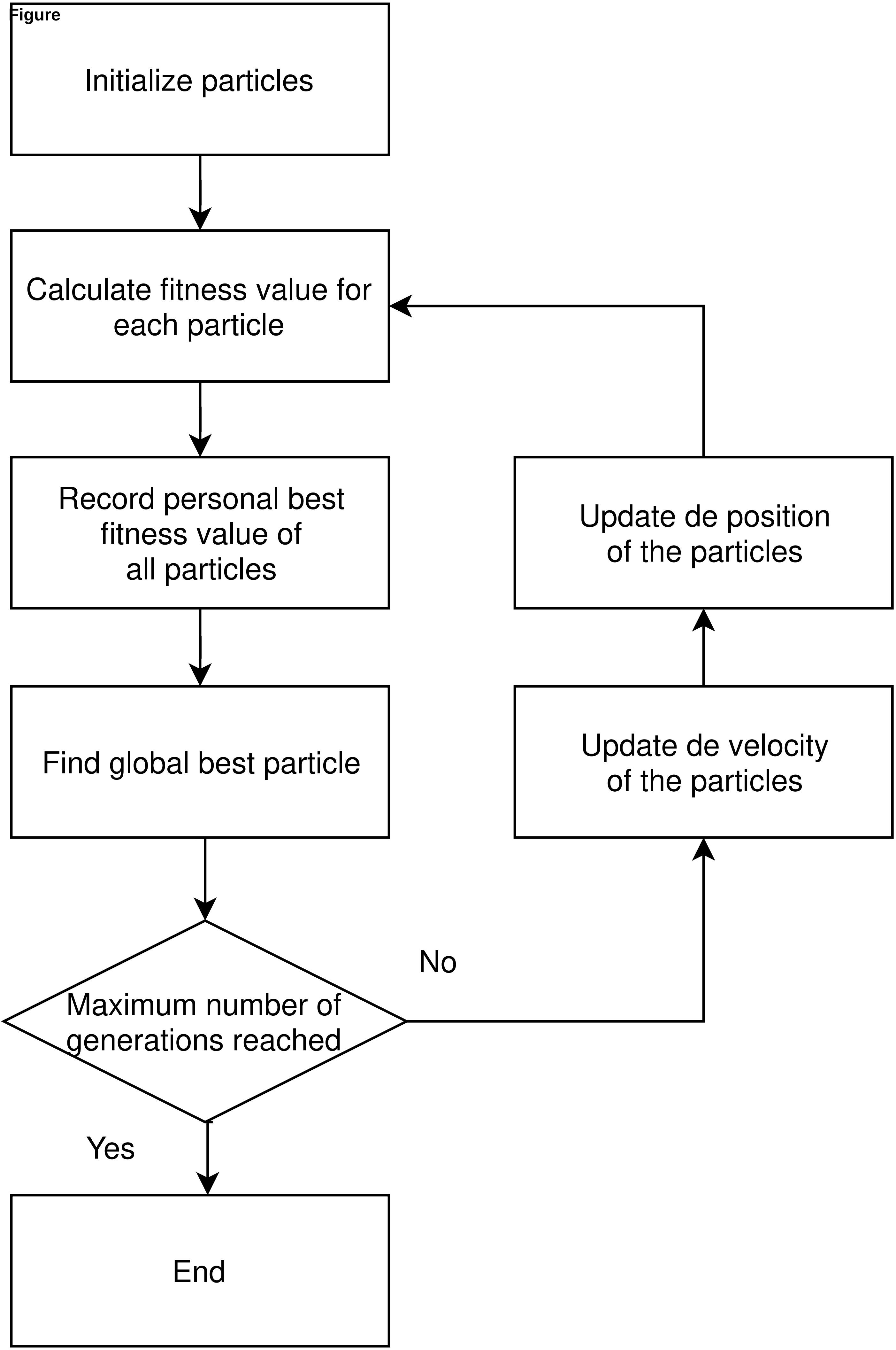
[48] J. Wu, F. Chiclana, Multiplicative consistency of intuitionistic reciprocal preference relations and its application to missing values estimation and consensus building, Knowledge-Based Systems 71 (2014) 187–200.

[49] X. Peng, J. Dai, Approaches to pythagorean fuzzy stochastic multi-criteria decision making based on prospect theory and regret theory

with new distancie measure and score function, International Journal of Intelligent Systems 32 (11) (2017) 1187–1214.

[50] X. Peng, G. Selvachandran, Pythagorean fuzzy set: state of the art and future directions, Artificial Intelligence Review In press.

Figure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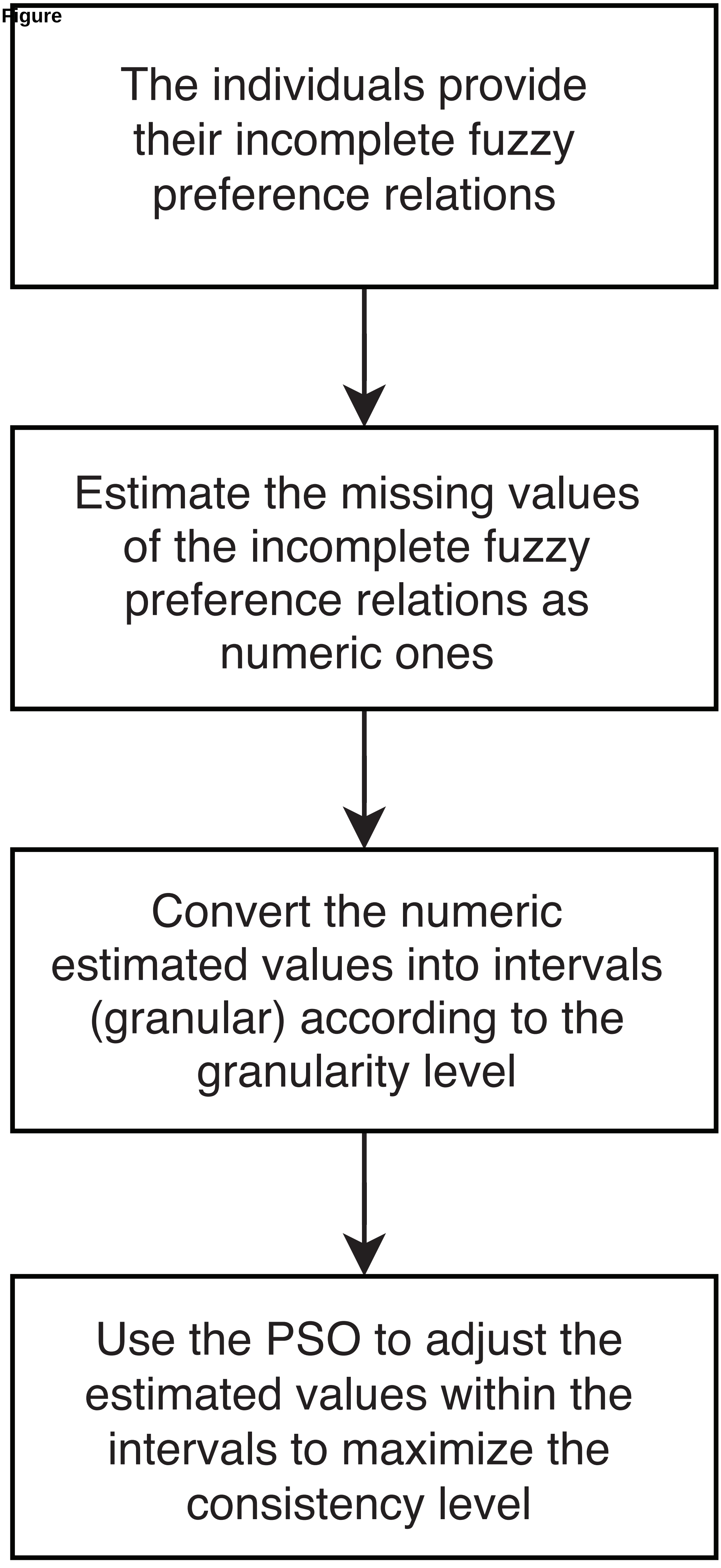
The individuals provide their incomplete fuzzy preference relations
Estimate the missing values of the incomplete fuzzy preference relations as numeric ones
Convert the numeric estimated values into intervals (granular) according to the granularity level
Use the PSO to adjust the estimated values within the intervals to maximize the consistency level

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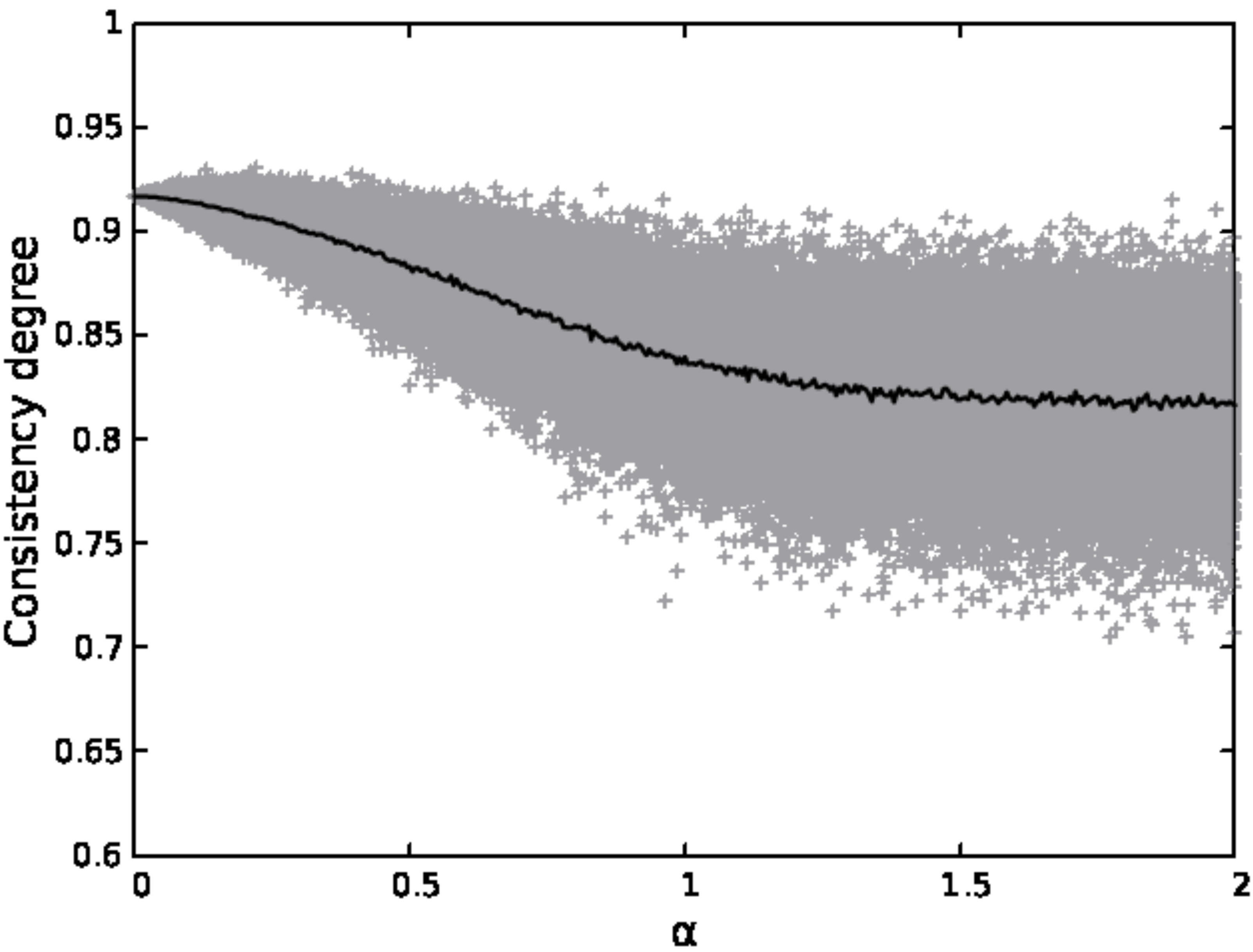

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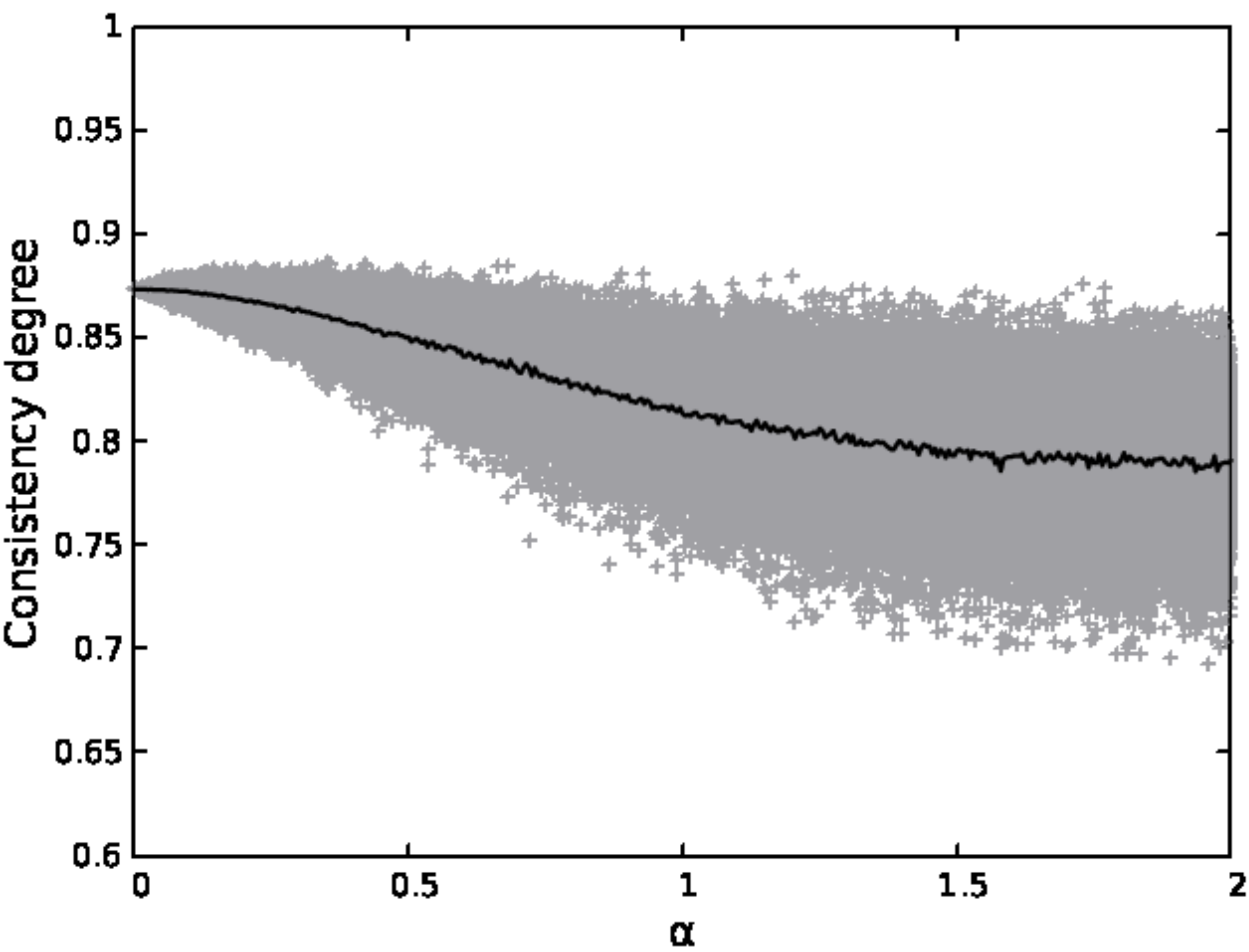

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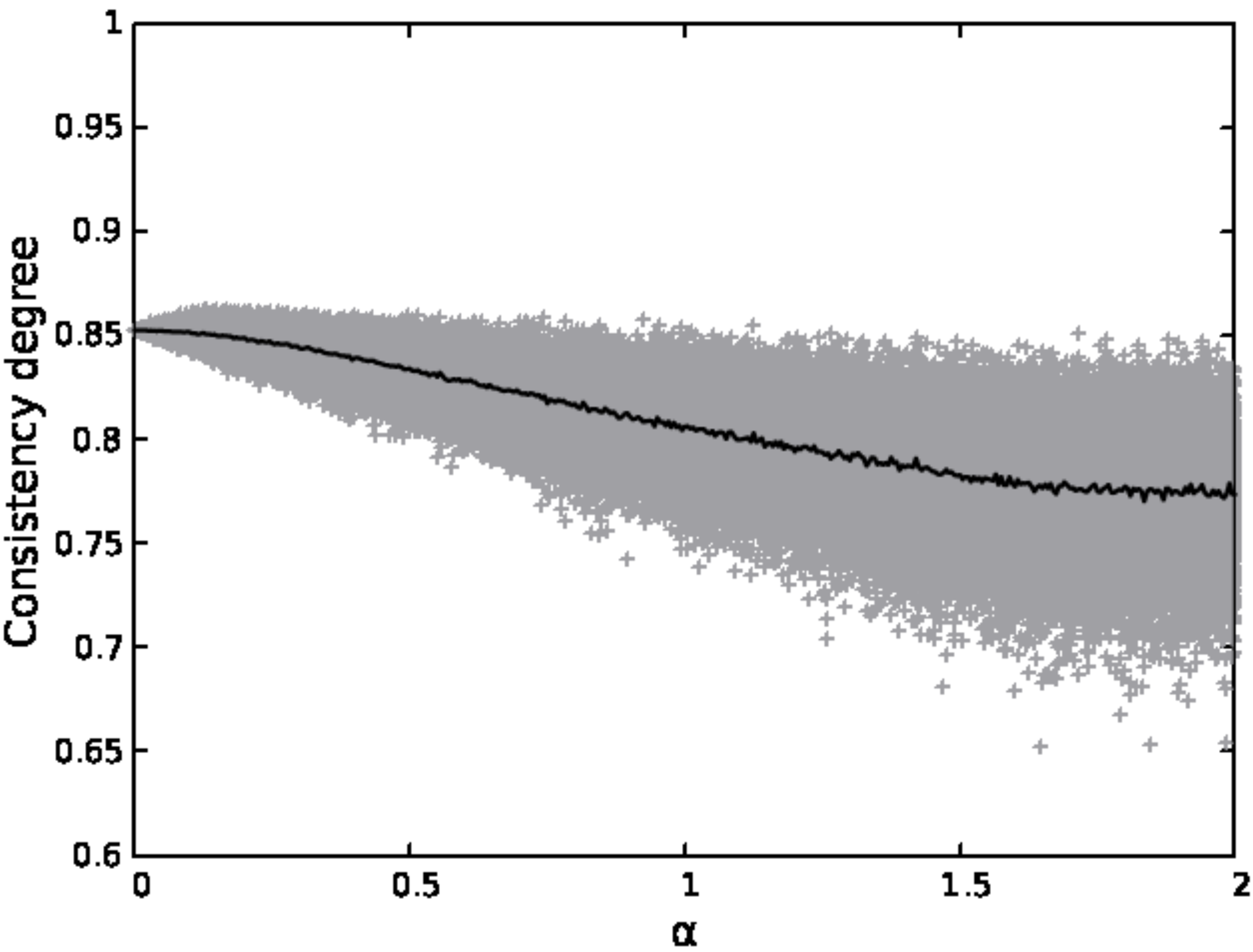

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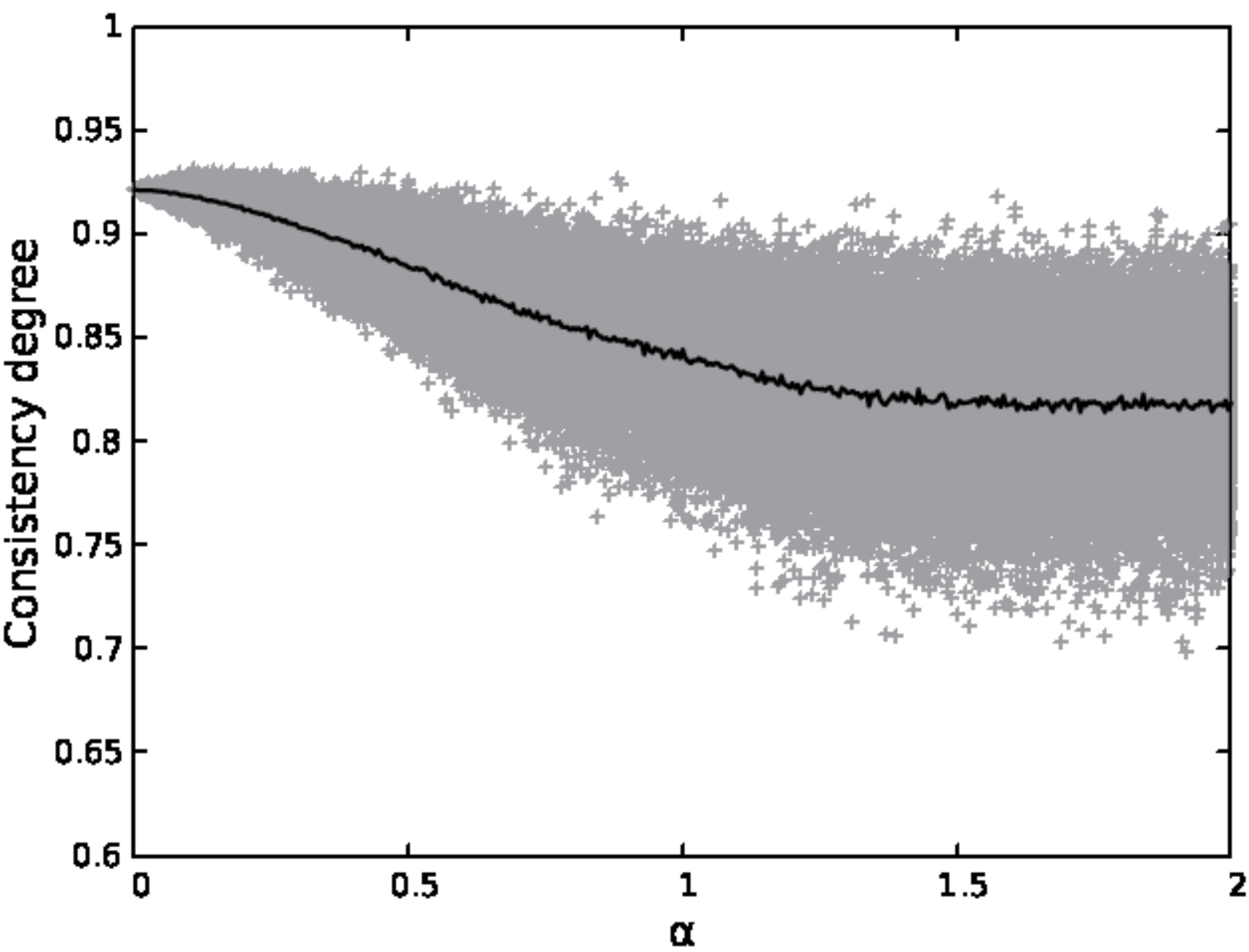

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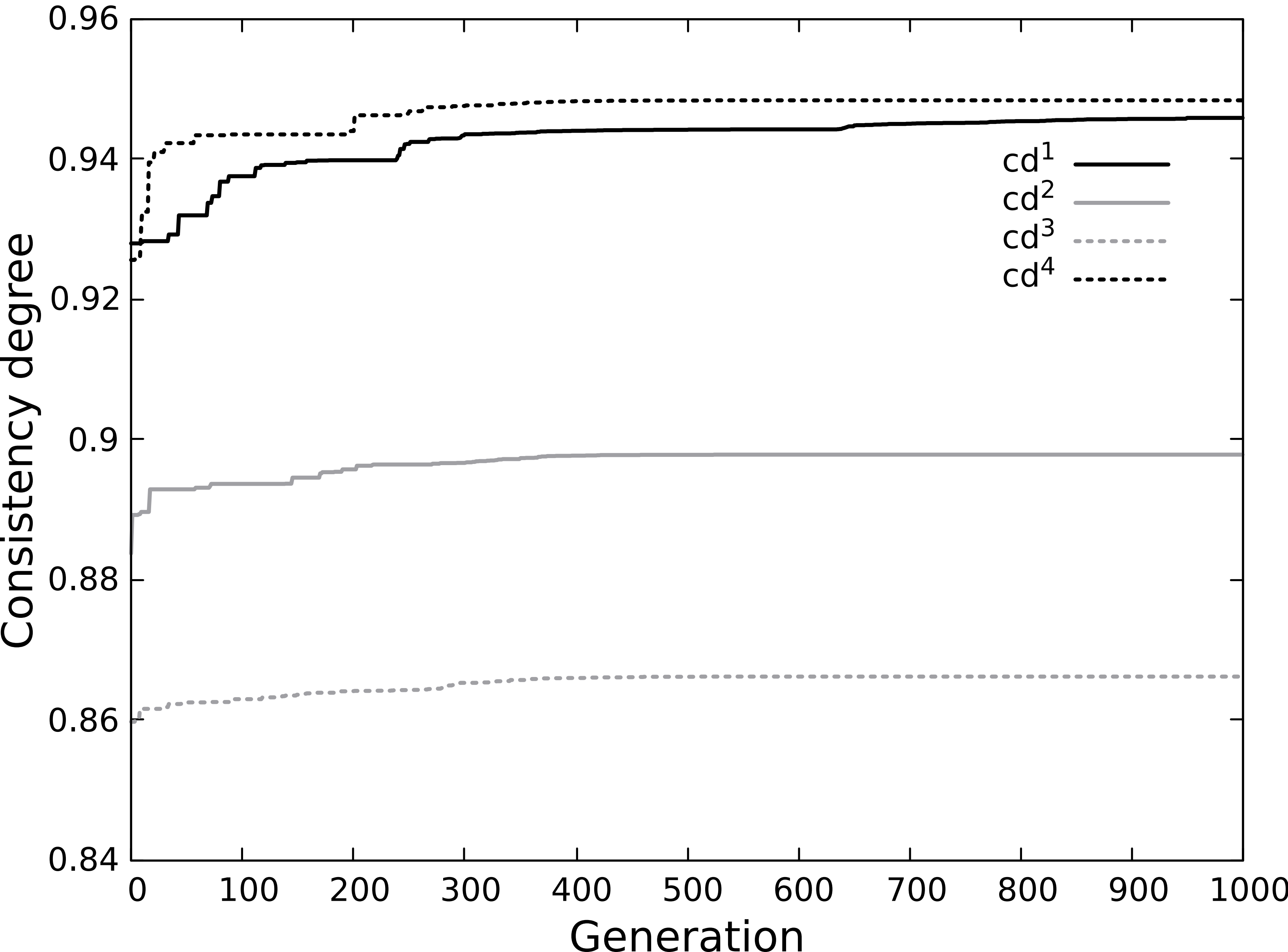
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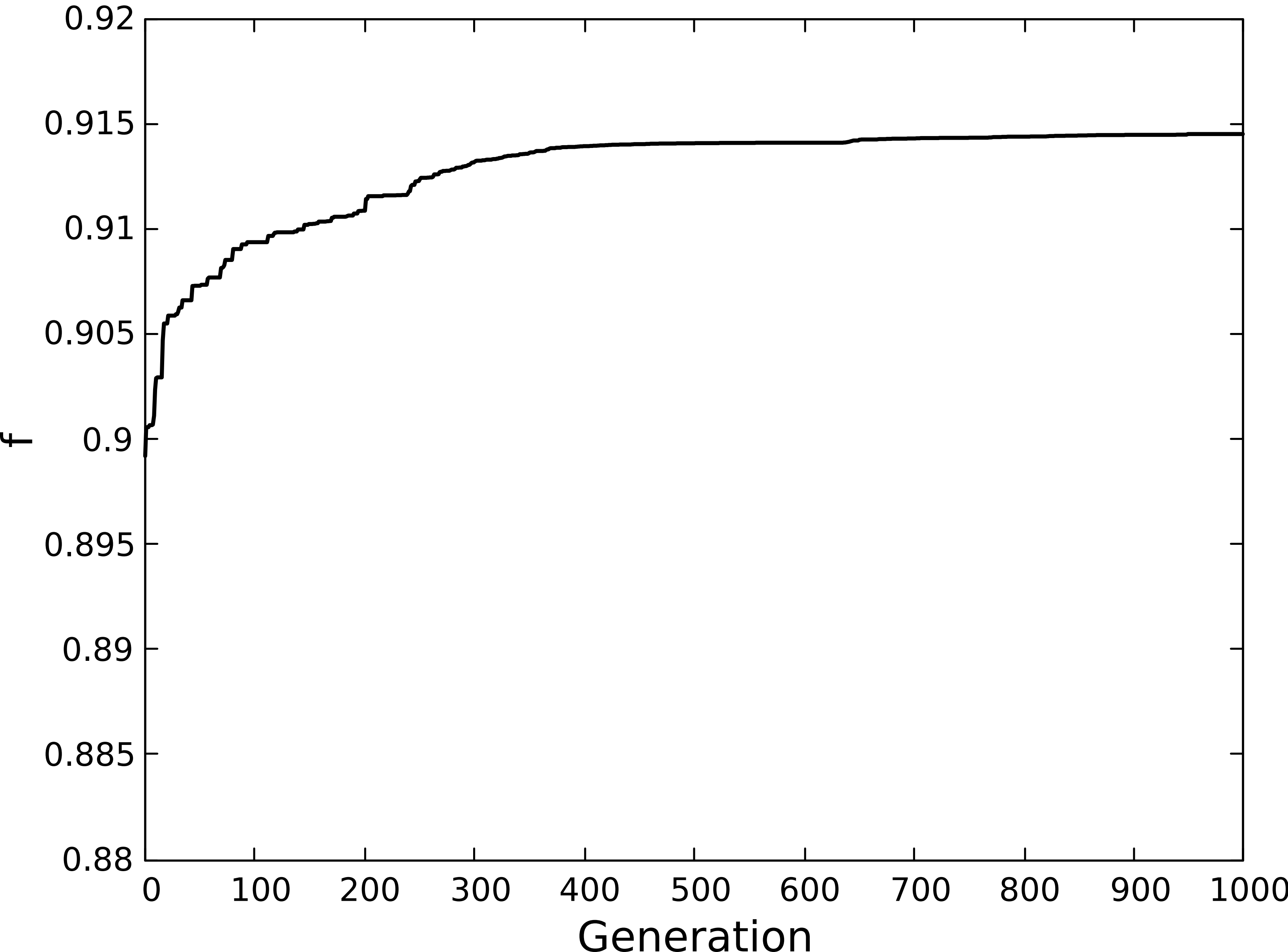

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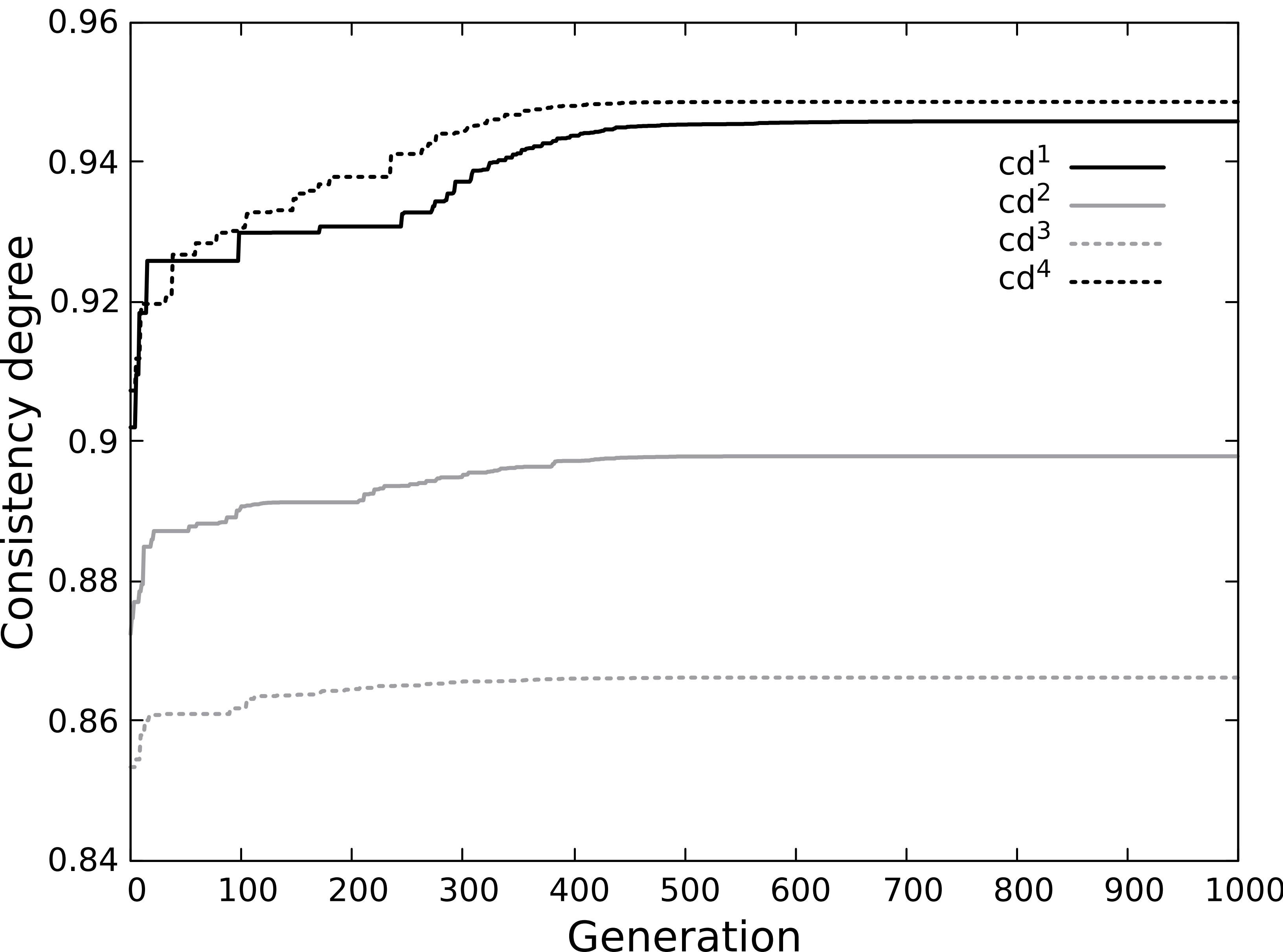

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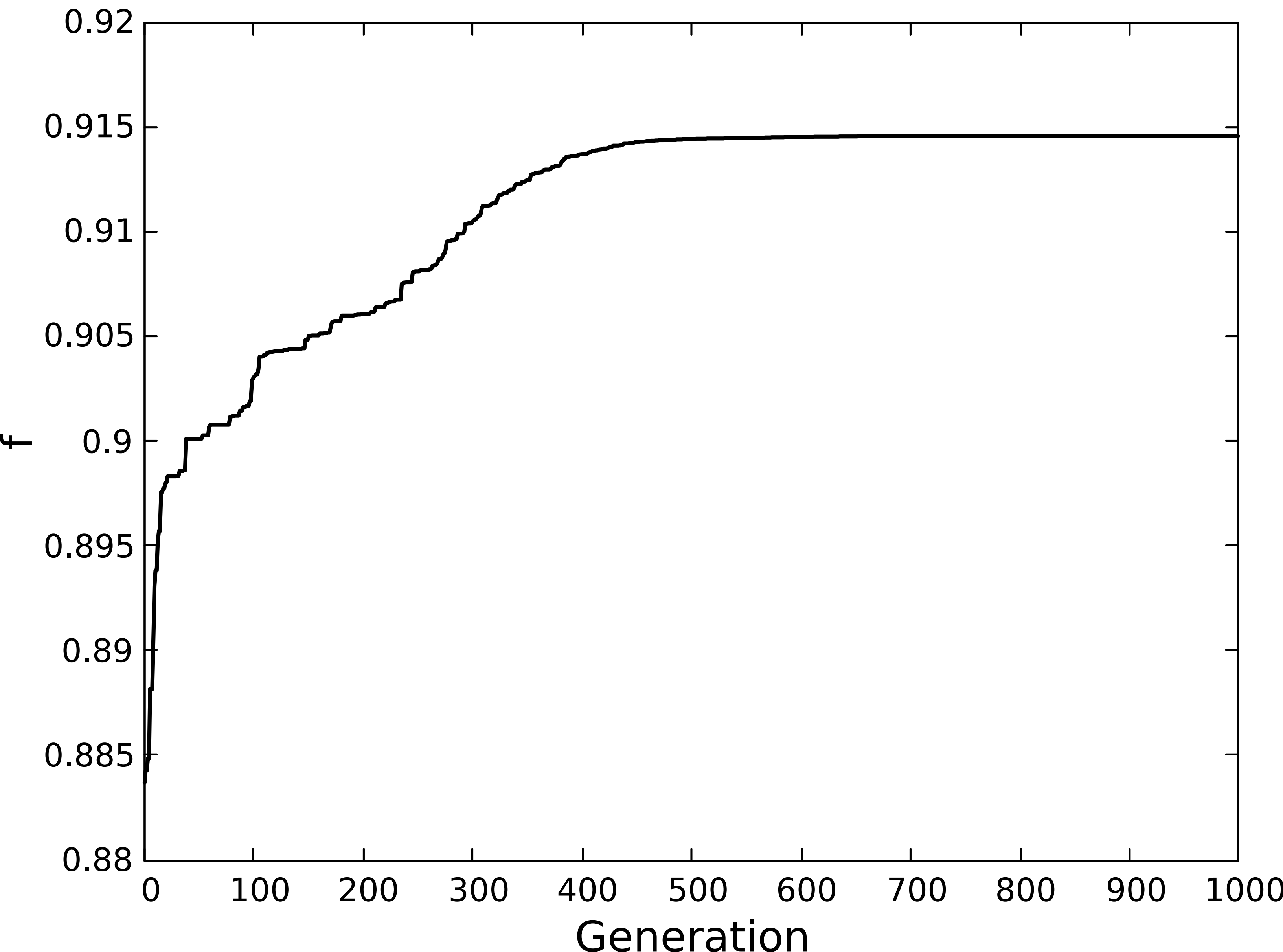
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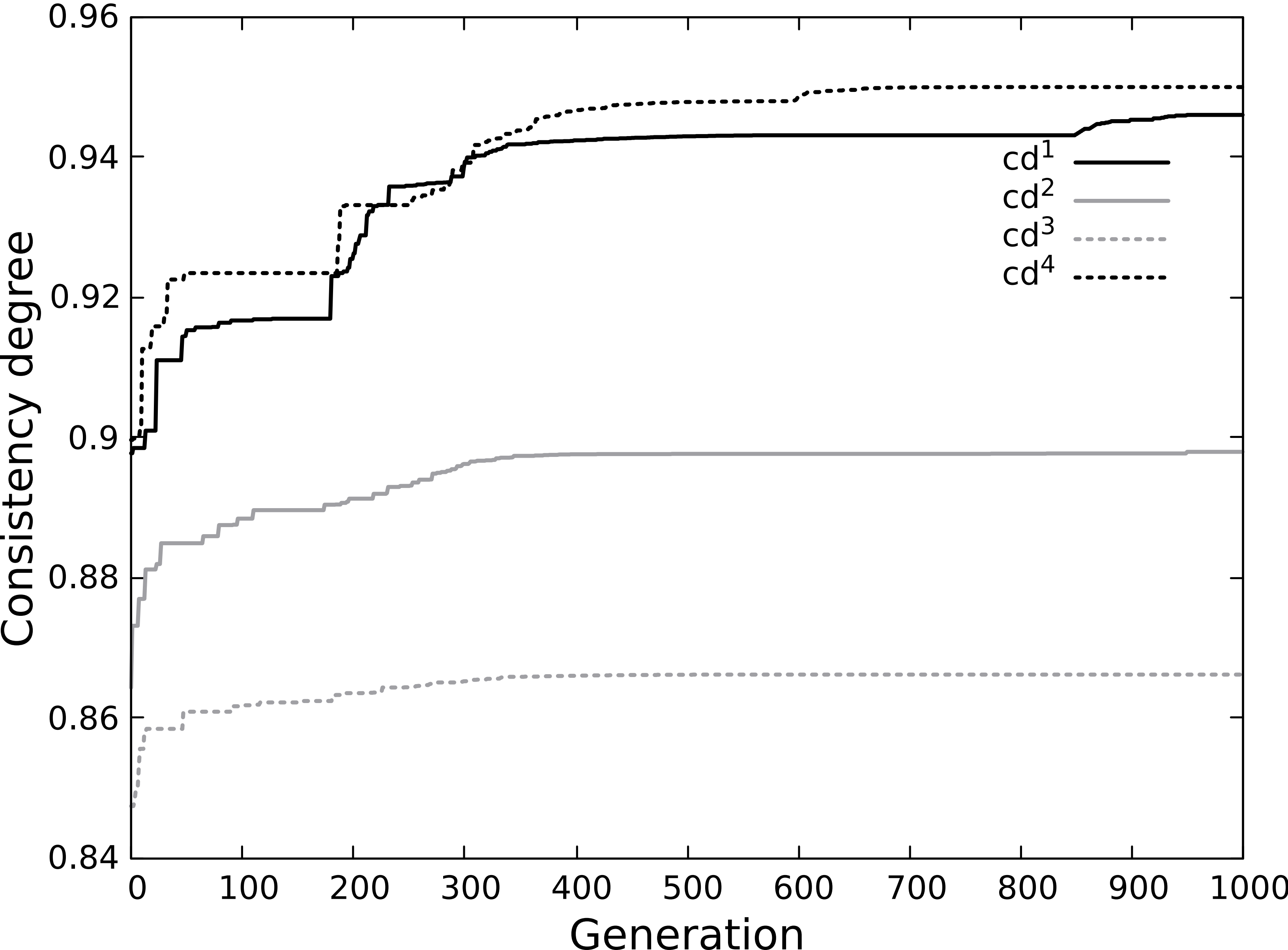

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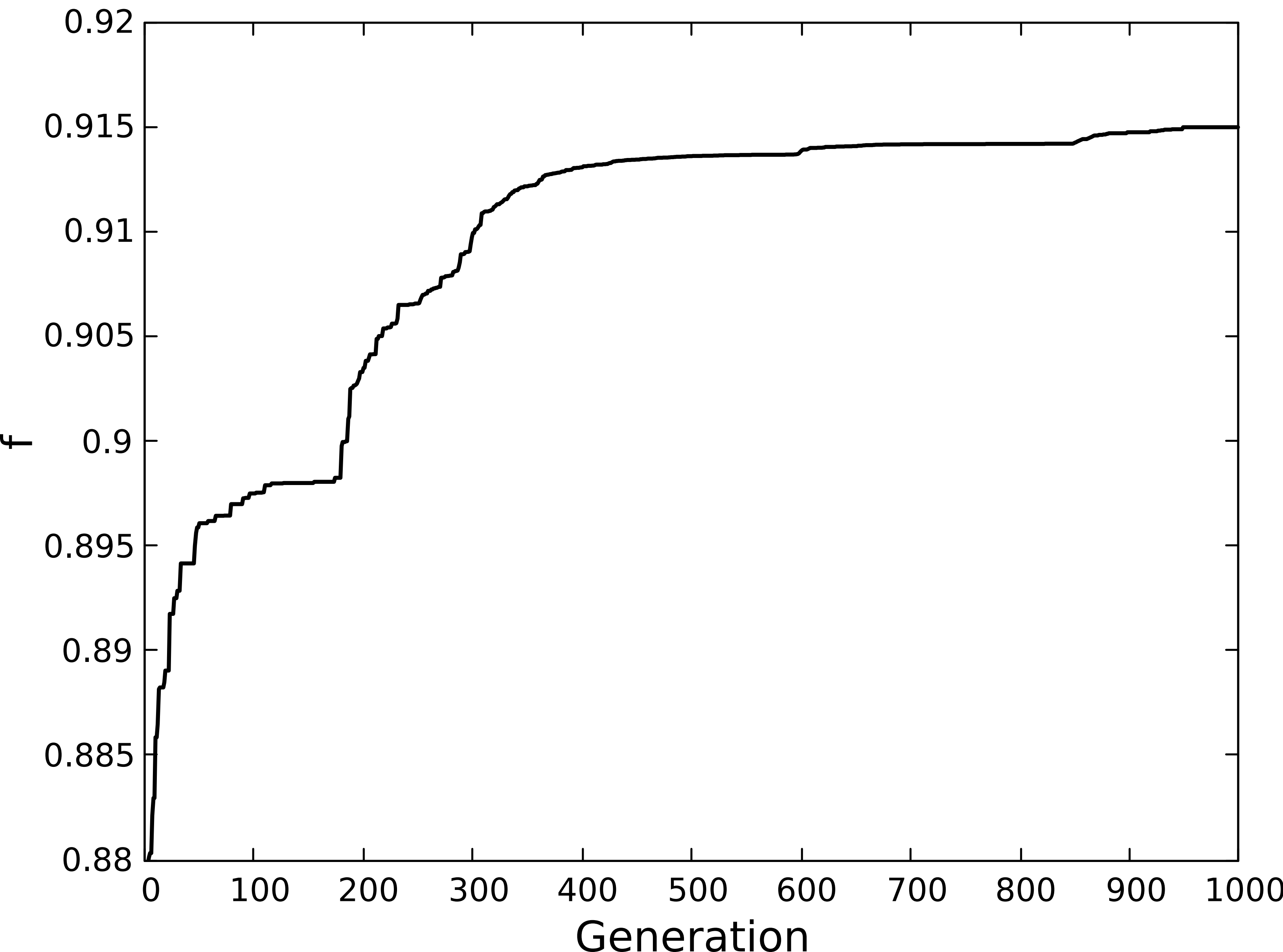
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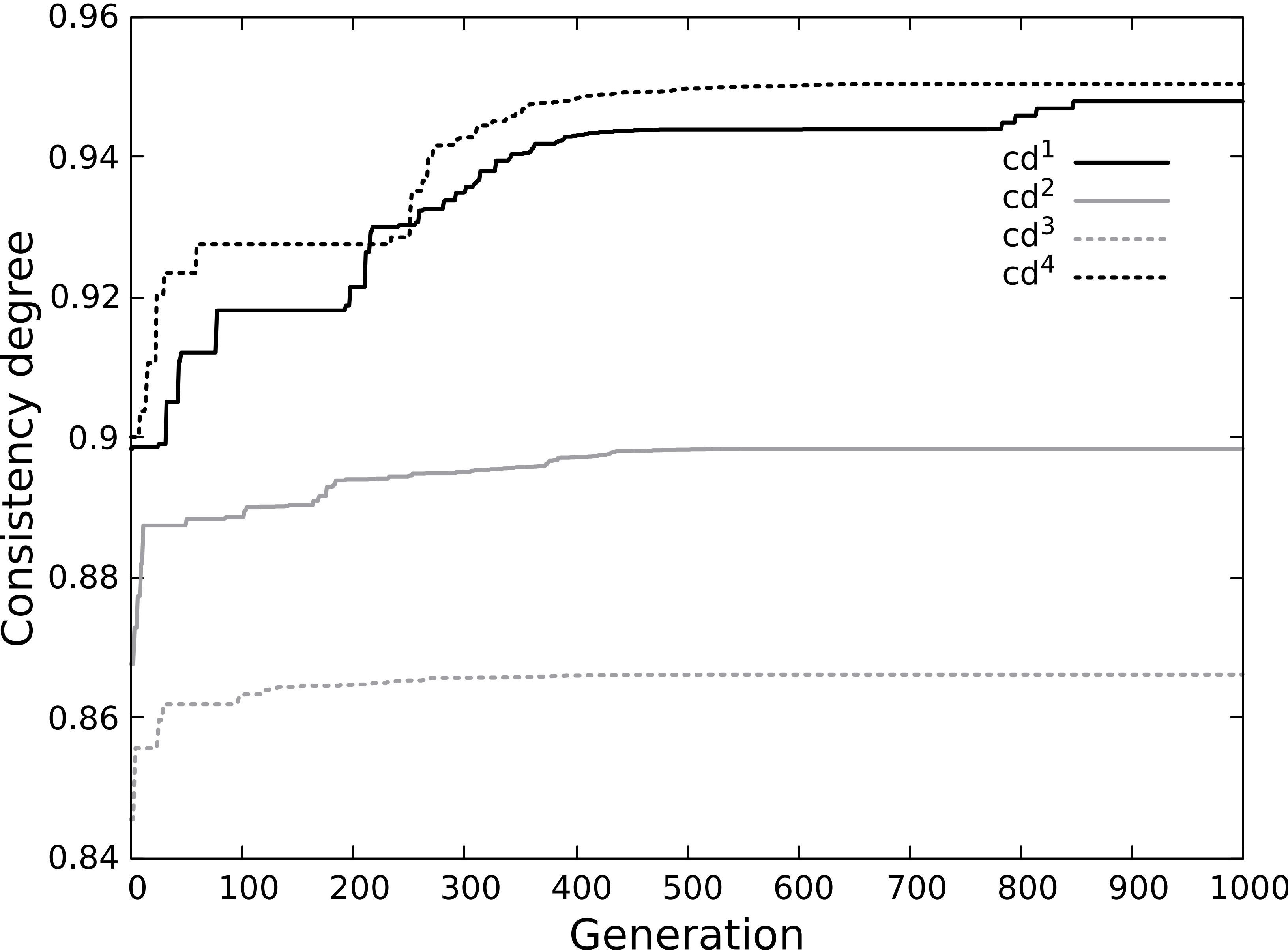
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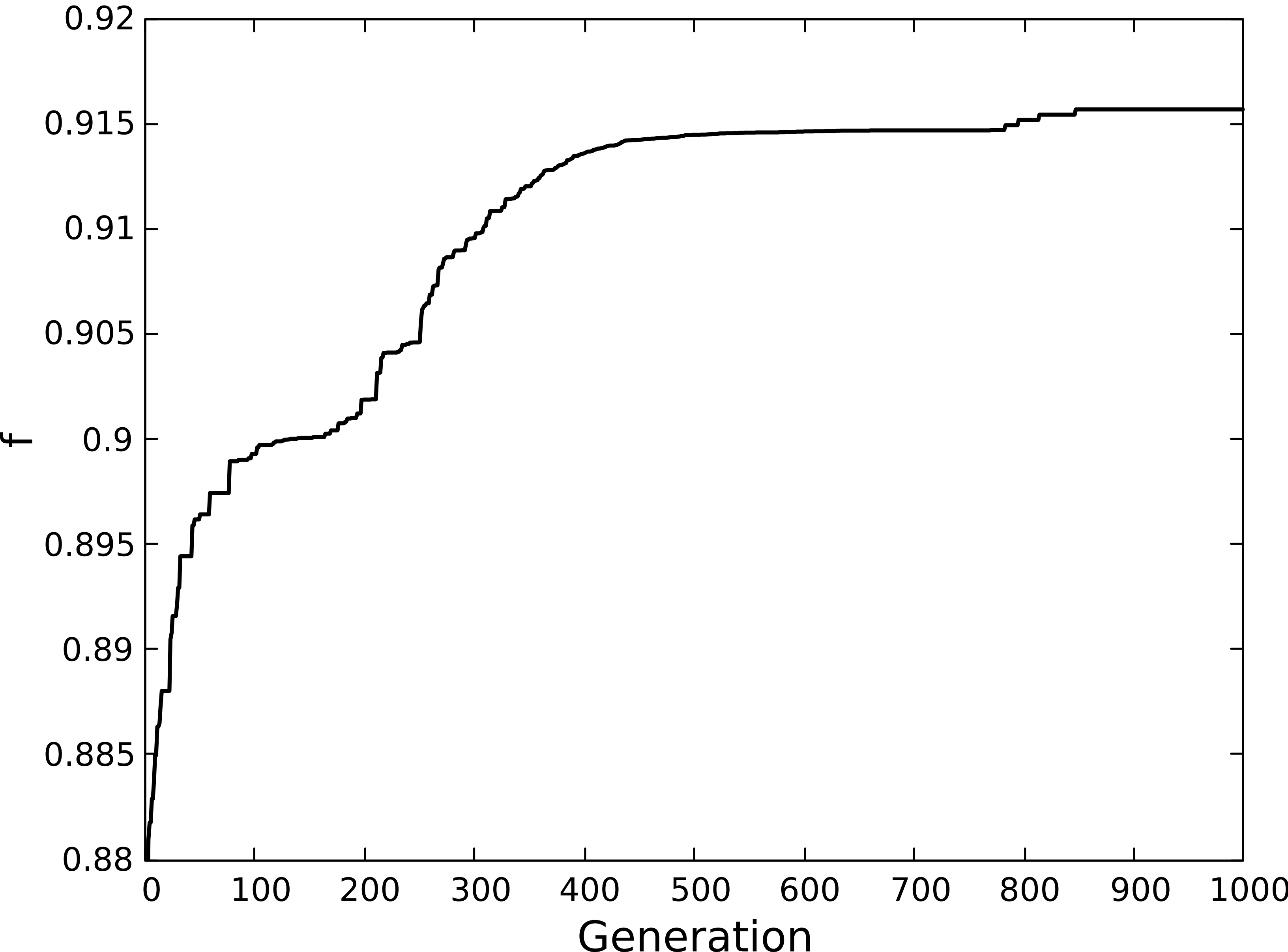

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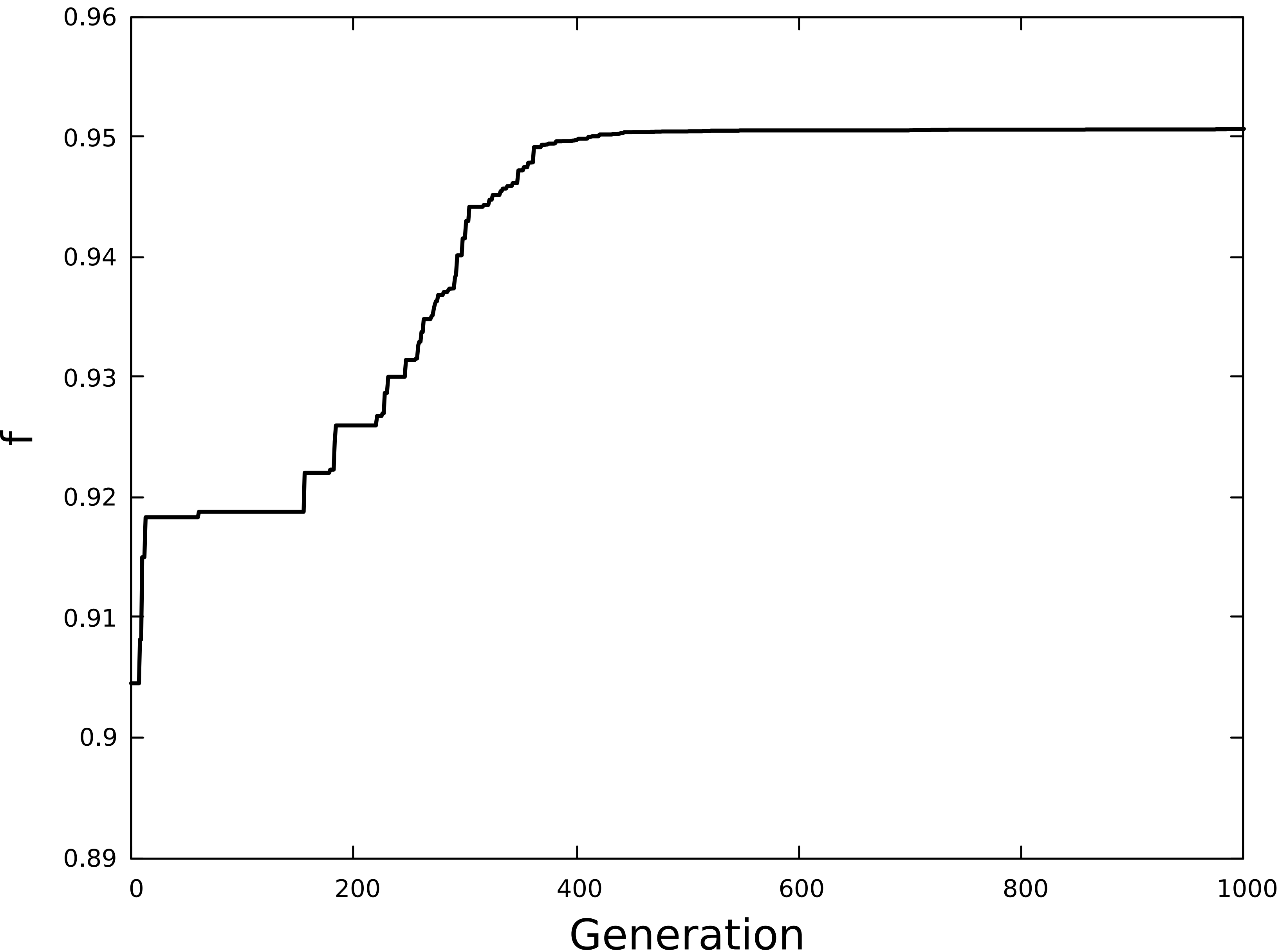
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***Declaration of Interest Statement**

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: