REAL HYPERSURFACES WITH KILLING SHAPE OPERATOR IN THE COMPLEX QUADRIC

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Abstract. We introduce the notion of Killing shape operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The Killing shape operator implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with Killing shape operator.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [15], [16], and [17]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications of real hypersurfaces related to commuting Ricci tensor were investigated by Kimura [9], and Pérez and Suh [11], [12] respectively. The classification problems of real hypersurfaces of the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ with certain geometric conditions were mainly discussed in Jeong, Kim and Suh [2], Jeong, Machado, Pérez and Suh [3], [4], Suh [15], [16], [17], where the classification of *contact hypersurfaces*, *parallel Ricci tensor*, *harmonic curvature* and *Jacobi operator* of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [17] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can consider the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Klein [5], [6], [8] and Smyth [14]). The complex quadric can also be regarded as a

2010 *Mathematics Subject Classification*: Primary 53C40. Secondary 53C55.
Key words: Killing shape operator, $\mathfrak{A}$-isotropic, $\mathfrak{A}$-principal, Kähler structure, complex conjugation, complex quadric.
This work was supported by grant Proj. No. NRF-2015-R1A2A1A-01002459 from National Research Foundation of Korea. The first author was supported by MCT-FEDER project MTM-2013-47828-C2-1-P, the second and third authors were supported by NRF-2017-R1A2B4005317, and the fourth by Bokhyun Research 2017.

kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [10]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m{\geq}2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [5], [7] and Reckziegel [13]).

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m.

Moreover, the derivative of the complex conjugation A on Q^m is defined by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M and q denotes a certain 1-form defined on M.

When the shape operator S of M in Q^m satisies $(\nabla_X S)Y = (\nabla_Y S)X$ for any X, Y on M in Q^m, we say that the shape operator is of *Codazzi type.* In [18] we gave a non-existence property of real hypersurfaces of Codazzi type in the complex quadric Q^m with parallel shape operator as follows:

Theorem A. *There do not exist any real hypersurfaces in complex quadric Q^m, $m{\geq}3$, with shape operator of Codazzi type.*

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

When we consider a hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties the unit normal vector field N of M in Q^m can be divided into two classes if either N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [18] and [19]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in Suh [18] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [19]).

The shape operator S of M in Q^m is said to be *Killing* if the operator S satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0$$

for any $X, Y{\in}T_zM$, $z{\in}M$. The equation is equivalent to $(\nabla_X S)X = 0$ for any $X{\in}T_zM$, $z{\in}M$, because of linearization. Moreover, we can give the geometric meaning of Killing Jacobi tensor as follows:

When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot\gamma(0) = X$. Then the transformed vector field $S\dot\gamma$ is Levi-Civita *parallel* along the geodesic γ of the vector field X (see Blair [1] and Tachibana [23]).

In the study of real hypersurfaces in the complex quadric Q^m we considered the notion of parallel Ricci tensor, that is, $\nabla \mathrm{Ric} = 0$ (see Suh [19]). But from the assumption of Ricci being parallel, it was difficult for us to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [19] we gave a classification with the further assumption of $\mathfrak{A}$-isotropic. But fortunately, when we consider Killing shape operator, first we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface in Q^m, $m{\geq}3$, with Killing shape operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Then motivated by such a result, next we give a complete classification for real hypersurfaces in the complex quadric Q^m with Killing shape operator as follows:

Main Theorem 2. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m{\geq}4$, with Killing shape operator. Then M has 4 distinct constant principal curvatures given by*

$$\alpha{\neq}0,\ \beta=\gamma=0,\ \lambda=\frac{(\alpha^2+1)+\sqrt{(\alpha^2+1)^2+2\alpha^2}}{2\alpha},\ \textit{and}$$
$$\mu=\frac{(\alpha^2+1)-\sqrt{(\alpha^2+1)^2+2\alpha^2}}{2\alpha}$$

with corresponding principal curvature spaces respectively

$$T_\alpha=[\xi], T_\beta=[AN], T_\gamma=[A\xi], \phi(T_\lambda)=T_\mu,\ \textit{and dim}\ T_\lambda=\textit{dim}\ T_\mu=m-2.$$

Usually, Killing shape operator is a generalization of parallel shape operator S of M in Q^m , that is, $\nabla_X S = 0$ for any tangent vector field X on M. The parallelism of shape operator has a geometric meaning that every eigen spaces of the shape operator S are parallel along any direction on M in Q^m. Then naturally, by Theorem 2 above we give the following

Corollary. *There do not exist any Hopf real hypersurfaces in Q^m, $m{\geq}3$, with parallel shape operator.*

2. The complex quadric

For more background to this section we refer to [5], [10], [13], [18], [19] and [20]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini-Study metric $\bar g$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini-Study metric $\bar g$ is defined by $\bar g(X,Y) = \Phi(JX,Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\bar\partial \log f_j$ on an open set $U_j = \{[z_0,\ldots,z_j,\ldots,z_{m+1}]{\in}\mathbb{C}P^{m+1}|z_j{\neq}0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar t_j^k$, and $t_j^k = \frac{z_k}{z_j}$ for $j, k = 0, \cdots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

In another way, the complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric Q^m as follows:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\tilde{i}} & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \big\downarrow & & \pi \big\downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2})$ and $y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^\perp$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} | \quad g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z | \quad g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)} \mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)} Q$ respectively. This gives that at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [13]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}}w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}}w &= A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ &= \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X,Y)Z \;=\; & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X \in V(A)$, $t = 0$, there exist many kinds of maximal 2-flats $\mathbb{R}X + \mathbb{R}Z$ for $Z \in V(A)$ orthogonal to $X \in V(A)$. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$, it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by $\mathbb{R}(X+JY)+\mathbb{R}Z$ for any $Z \in V(A)$ orthogonal to $X \in V(A)$ or $\mathbb{R}(X + JY) + \mathbb{R}JZ$ for any $JZ \in JV(A)$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1. *([18]) For each $z \in M$ we have*

- (i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
- (ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies the following

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurface M for a smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transformed JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the equation of Codazzi

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = & \ g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ = & \ (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = & -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & +2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & +\alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 =& 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [13]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN =& \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi =& \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi =& \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 =& 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

4. Killing shape operator and a Key Lemma

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned} R(X,Y)Z \;=\; & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\ & + g(SY,Z)SX - g(SX,Z)SY \end{aligned}$$

for any $X,Y,Z \in T_zM$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m$, $z \in M$, $\rho(X) = g(AX,N)$, where BX and $\rho(X)N$ respectively denote the tangential and normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi,N) = 0$. Then it follows that

$$\begin{aligned} AN &= AJ\xi = -JA\xi = -J(B\xi + \rho(\xi)N) \\ &= -\,(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

The shape operator S of M in Q^m is said to be *Killing* if the operator S satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0. \tag{4.1}$$

for any $X,Y \in T_zM$, $z \in M$.

From (4.1), together with the equation of Codazzi (3.1), it follows that

$$\begin{aligned} 2g((\nabla_X S)Y,Z) = \;& \eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z) - 2\eta(Z)g(\phi X,Y) \\ & + g(X,AN)g(AY,Z) - g(Y,AN)g(AX,Z) \\ & + g(X,A\xi)g(JAY,Z) - g(Y,A\xi)g(JAX,Z). \end{aligned} \tag{4.2}$$

Since we have assumed the real hypersurface M in Q^m is *Hopf*, then $S\xi = \alpha\xi$. This gives

$$(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX.$$

From this, let us put $Y = \xi$ in (4.2) and use $g(A\xi,N) = 0$, we see that

$$\begin{aligned} 2g((X\alpha)\xi + \alpha\phi SX - S\phi SX, Z) = & -g(\phi X,Z) + g(X,AN)g(A\xi,Z) \\ & + g(X,A\xi)g(JA\xi,Z) - g(\xi,A\xi)g(JAX,Z). \end{aligned} \tag{4.3}$$

Here, let us put $X = \xi$ in (4.3) and also use $g(\xi,AN) = 0$, we have

$$2(\xi\alpha)\eta(Z) = g(\xi,A\xi)g(JA\xi,Z) - g(\xi,A\xi)g(JA\xi,Z) = 0.$$

From this we get $\xi\alpha = 0$. Then the derivative $Y\alpha$ in section 3 becomes

$$Y\alpha = 2g(Y,AN)g(\xi,A\xi).$$

From this, together with (4.3), it follows that

$$\begin{aligned} 2g(2g(X,AN)g(\xi,A\xi)\xi + \alpha\phi SX - S\phi SX, Z) = & -g(\phi X,Z) + g(X,AN)g(A\xi,Z) \\ & + g(X,A\xi)g(JA\xi,Z) - g(\xi,A\xi)g(JAX,Z). \end{aligned} \tag{4.4}$$

Then by putting $Z = \xi$ into (4.3), we have

$$\begin{aligned} 4g(X,AN)g(\xi,A\xi) =& g(X,AN)g(A\xi,\xi) + g(X,A\xi)g(JA\xi,\xi) \\ & - g(\xi,A\xi)g(JAX,\xi) \\ =& 2g(X,AN)g(A\xi,\xi). \end{aligned} \tag{4.5}$$

Since $g(A\xi, N) = 0$, (4.5) gives that

$$g(A\xi, \xi)g(AN, X) = 0.$$

Then we have $g(A\xi, \xi) = 0$ or $(AN)^T = 0$, where $(AN)^T$ denotes the tangential part of the vector AN.

Summing up above discussions, we conclude the following

Lemma 4.1. *Let M be a Hopf real hypersurface in Q^m, $m \geq 3$, with Killing shape operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. In above discussion, let us consider the first case $g(A\xi, \xi) = 0$. Then it implies that

$$0 = g(A\xi, \xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N).$$

If we insert $N = \cos t Z_1 + \sin t JZ_2$ for $Z_1, Z_2 \in V(A)$ into the above equation, we have $\cos^2 t - \sin^2 t = 0$. Then by section 2, we have $t = \frac{\pi}{4}$, that is, $N = \frac{1}{\sqrt{2}}(X + JY)$ for some $X, Y \in V(A)$. So the unit normal N is $\mathfrak{A}$-isotropic.

Next we consider the case that $(AN)^T = 0$. Then $AN = (AN)^T + g(AN, N)N = g(AN, N)N$. So it follows that

$$N = A^2 N = g(AN, N)AN = g^2(AN, N)N.$$

So $g(AN, N) = \pm 1$ gives that $AN = \pm N$. That is, the unit normal N is $\mathfrak{A}$-principal. □

Then we are able to consider the classification of Killing shape operator S of M in Q^m into two cases, that the unit normal N is $\mathfrak{A}$-principal or N is $\mathfrak{A}$-isotropic. In section 5 we will discuss a classification of real hypersurfaces in Q^m with Killing shape operator and $\mathfrak{A}$-isotropic unit normal and in section 6 a non-existence of Killing shape operator for hypersurfaces in Q^m when N is $\mathfrak{A}$-principal will be explained in detail.

5. Proof of Main Theorem with $\mathfrak{A}$-isotropic unit normal

In this section let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be written

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \text{and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (3.3) and the anti-commuting $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0,\ g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

Then (4.3) gives the following for any $X, Z \in T_z M$, $z \in M$

$$\begin{aligned} 2g(\alpha\phi SX - S\phi SX, Z) &= -g(\phi X, Z) + g(X, AN)g(A\xi, Z) + g(X, A\xi)g(JA\xi, Z) \\ &= -g(\phi X, Z) + g(X, AN)g(A\xi, Z) - g(X, A\xi)g(AN, Z). \end{aligned} \tag{5.1}$$

Since $A\xi, AN\in T_xM$, $x\in M$, it implies

$$2(\alpha\phi SX - S\phi X) = -\phi X + g(X, AN)A\xi - g(X, A\xi)AN. \tag{5.2}$$

On the other hand, from the formula (5.6) of Suh [19] for a Hopf real hypersurface M with $\mathfrak{A}$-isotropic unit normal N

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \tag{5.3}$$

Then by virtue of (5.2) and (5.3), we have

$$-2S\phi SX = \alpha S\phi X - 3\alpha\phi SX. \tag{5.4}$$

We know that the tangent space T_zM, $z\in M$ is decomposed as follows:

$$T_zM = [\xi]\oplus[A\xi, AN]\oplus\mathcal{Q},$$

where $\mathcal{C}\ominus\mathcal{Q} = \mathcal{Q}^\perp = \mathrm{Span}[A\xi, AN]$.

Lemma 5.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m\geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field. Then*

$$SAN = 0, \quad \text{and} \quad SA\xi = 0.$$

Proof. Let us denote by $\mathcal{C}\ominus\mathcal{Q} = \mathcal{Q}^\perp = \mathrm{Span}[A\xi, AN]$. Since N is isotropic, $g(AN, N) = 0$ and $g(A\xi, \xi) = 0$. By differentiating $g(AN, N) = 0$ and using $(\bar\nabla_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned} 0 =& g(\bar\nabla_X(AN), N) + g(AN, \bar\nabla_X N) \\ =& g(q(X)JAN - ASX, N) - g(AN, SX) \\ =& -2g(ASX, N). \end{aligned}$$

Then $SAN = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have the following formula

$$\begin{aligned} 0 =& g(\bar\nabla_X(A\xi), N) + g(A\xi, \bar\nabla_X N) \\ =& g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ =& -2g(SA\xi, X) \end{aligned}$$

for any $X\in T_zM$, $z\in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

By Lemma 5.1 we know that the distribution $\mathcal{Q}^\perp$ for a Hopf real hypersurface M in Q^m is invariant by the shape operator S, so the distribution $\mathcal{Q}$ is also S-invariant. From this fact we may consider a principal curvature vector $X\in\mathcal{Q}$ such that $SX = \lambda X$, because the distribution $\mathcal{Q}$ can be diagonalized. Then (5.4) gives

$$S\phi X = \frac{3\alpha\lambda}{2\lambda+\alpha}\phi X. \tag{5.5}$$

Here we note that $2\lambda + \alpha\neq 0$. In fact, if $2\lambda + \alpha = 0$, then $\alpha = \lambda = 0$, and from (5.3), it gives us a contradiction. For $X\in\mathcal{Q}$, we know that $g(X, AN) = g(X, A\xi) = 0$. So (5.3) gives the following

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \tag{5.6}$$

Then we consider two cases for $X\in\mathcal{Q}$ or $X\in\mathcal{Q}^{\perp}$.

As a first, for $X\in\mathcal{Q}$ such that $SX = \lambda X$ the formula (5.6) gives

$$2\lambda S\phi X = \alpha S\phi X + (\alpha\lambda + 2)\phi X. \tag{5.7}$$

If $\alpha = 2\lambda$, we should have $2(\lambda^2 + 1)\phi X = 0$, which is impossible. Then we have for $S\phi X = \mu\phi X$

$$S\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X. \tag{5.8}$$

Then (5.5) and (5.8) give

$$\frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X = \frac{3\alpha\lambda}{2\lambda + \alpha}\phi X.$$

From this, any principal curvatures λ and μ of the distribition $\mathcal{Q}$ satisfy the following quadratic equation

$$2\alpha\lambda^2 - 2(\alpha^2 + 1)\lambda - \alpha = 0. \tag{5.9}$$

The solutions become the following constant principal curvatures given by

$$\lambda, \mu = \frac{(\alpha^2 + 1)\pm\sqrt{(\alpha^2 + 1)^2 + 2\alpha^2}}{2\alpha}, \tag{5.10}$$

because the Reeb function α is constant for $\mathfrak{A}$-isotropic unit normal N. Here we note that the Reeb function α can not vanish. If the function α identically vanishes, then (5.9) gives $\lambda = 0$. From this, together with (5.7), we have $\phi X = 0$, which implies a contradiction.

From this, together with Lemma 5.1, the expression of the shape operator becomes the following

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

where the principal curvatures λ and μ are given by (5.10) with multiplicities $m - 2$ respectively.

Summing up the above discussions, we give the following

Theorem 5.2. *Let M be a real hypersurface in the complex quadric Q^m with $\mathfrak{A}$-isotropic unit normal vector field. Then M has 4 distinct constant principal curvatures given by*

$$\alpha\neq 0,\ \beta = \gamma = 0,\ \lambda = \frac{(\alpha^2 + 1) + \sqrt{(\alpha^2 + 1)^2 + 2\alpha^2}}{2\alpha},\ \textit{and}$$

$$\mu = \frac{(\alpha^2 + 1) - \sqrt{(\alpha^2 + 1)^2 + 2\alpha^2}}{2\alpha}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi], T_\beta = [AN], T_\gamma = [A\xi], \phi(T_\lambda) = T_\mu,\ \textit{and dim}\ T_\lambda = \textit{dim}\ T_\mu = m - 2.$$

6. Proof of Main Theorem with $\mathfrak{A}$-principal

In this section let us consider a real hypersurface M in Q^m with Killing shape operator for the case that the unit normal N is $\mathfrak{A}$-principal. In this case the Killing shape operator (4.3) gives that

$$2g(\{\alpha\phi SX - S\phi SX\}, Z) = -g(\phi X, Z) + g(\phi AX, Z),$$

where we have used $g(\xi, A\xi) = -1$ and $JAX = \phi AX + \eta(AX)N$. Then it follows that

$$2(\alpha\phi SX - S\phi SX) = -\phi X + \phi AX. \tag{6.1}$$

Since the unit normal vector field N is $\mathfrak{A}$-principal, $A\xi = -\xi$. Then differentiating this and using Gauss equation give

$$\nabla_X(A\xi) = \bar{\nabla}_X(A\xi) - g(SX, A\xi)N = -q(X)N + \alpha\eta(X)N, \tag{6.2}$$

where q denotes a certain 1-form defined on M as in the introduction. From this, together with $\nabla_X(A\xi) = -\nabla_X\xi = -\phi SX$, we have

$$\phi X = \phi AX.$$

This gives that

$$AX = X - 2\eta(X)\xi.$$

Then we have

$$\begin{aligned} TrA =& g(AN, N) + \sum_{i=1}^{2m-1} g(Ae_i, e_i) \\ =& \sum_{i=1}^{2m-1} g(e_i - 2\eta(e_i)\xi, e_i) \\ =& 2(m-1). \end{aligned} \tag{6.3}$$

But $TrA = 0$, because $T_zQ^m = V(A) \oplus JV(A)$, where $V(A) = \{X \in T_zQ^m | AX = X\}$ and $JV(A) = \{X \in T_zQ^m | AX = -X\}$. This gives us a contradiction. So we assert another theorem as follows:

Theorem 6.1. *There do not exist any real hypersurface in the complex quadric Q^m with Killing shape operator if the unit normal vector field is $\mathfrak{A}$-principal.*

Summing up all of discussions including sections 4 and 5, by Theorems 4.1, 5.2 and 6.1, we give a complete proof of our Main Theorem 2 in the introduction.

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