RULED REAL HYPERSURFACES IN THE COMPLEX QUADRIC

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ABSTRACT. First we introduce the notions of η-parallel and η-commuting shape operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. Next we give a complete classification of real hypersurfaces in the complex quadric Q^m with such kind of shape operators. By virtue of this classification we give a new characterization of ruled real hypersurface foliated by complex totally geodesic hyperplanes Q^{m-1} in Q^m whose unit normal vector field in Q^m is $\mathfrak{A}$-principal.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [15], [16], and [17]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

In the complex projective space $\mathbb{C}P^{m+1}$ some classifications of real hypersurfaces related to η-parallel shape operator were investigated by Kimura [4], Kimura and Maeda [6] respectively. The classification problems of real hypersurfaces of the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ with certain geometric conditions were mainly discussed in Pérez and Suh [10], and Suh [15], [16], [17], where the classification of *contact hypersurfaces*, *parallel Ricci tensor*, *harmonic curvature* and *structure Jacobi operator* of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [17] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can consider the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Kobayashi and Nomizu [8] and Smyth [12], [13] and [14]). The complex quadric can

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also be regarded as a kind of real Grassmann manifold of compact type with rank 2. Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [7] and Reckziegel [11]).

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m.

Moreover, the derivative of the complex conjugation A on Q^m is given by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M, where q denotes a certain 1-form defined on M.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := Eig(A, 1)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

When we consider a hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties the unit normal vector field N of M in Q^m can be considered of two classes if either N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [18] and [19]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in Suh [18] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [19]).

The shape operator S of M in Q^m is said to be η-parallel if it satisfies

$$g((\nabla_X S)Y, Z) = 0$$

for any $X, Y, Z \in \mathcal{C}_z$, $z \in M$, where $\mathcal{C}_z$ denotes the orthogonal complement of the Reeb vector field $\xi_z = JN_z$ of M in T_zM.

Moreover, if the shape operator S of M in Q^m satisfies $g((S\phi - \phi S)X, Y) = 0$ for any $X, Y \in \mathcal{C}$, we say that M is η-commuting.

When the Reeb vector field ξ is a principal vector field of the shape operator of M in Q^m, a real hypersurface M is said to be Hopf. Now let us introduce another kind of real hypersurfaces which is said to be *ruled* real hypersurfaces in the complex quadric Q^m which are not Hopf as follows:

Let γ: $I \to Q^m$ be an integral curve of the Reeb vector field ξ such that $\gamma'(0) = \xi_p$. The distribution $\mathcal{C} = \{X \in TM | X \perp \xi\}$ is said to be integrable if $[X, Y] \in \mathcal{C}$ for any vector fields $X, Y\in\mathcal{C}$. When M is foliated by the integrable totally geodesic complex hyperplane Q^{m-1} in Q^m, then $M = \{x\in Q^{m-1}(t)|t\in I\}$. In such a case we say that M is a *ruled* real hypersurface in Q^m. In such a case, the expression of the shape operator S of the ruled real hypersurface M in Q^m becomes

$$\begin{aligned} S\xi &= \alpha\xi + \beta U \\ SU &= \beta\xi \\ SX &= 0 \end{aligned}$$

for any vector field $X\perp\xi, U$, where U is a unit vector field in $\mathcal{C}$, α and β are functions on M and β does not vanish. Then the above expression holds if and only if $g(SX, Y) = 0$ for any vector fields X and Y in $\mathcal{C}$. By the totally geodesic property of the complex hyperplane Q^{m-1} in Q^m in the construction of the ruled real hypersurface in Q^m, it naturally satisfies the above expression of the shape operator, and conversely if the shape operator satisfies the above formula, we can construct the ruled real hypersurface in Q^m. So as a characterization of ruled real hypersurfaces in Q^m, we summarize this one as follows:

Theorem A. *Let M be a real hypersurface in Q^m, $m\geq 3$. Then M is locally congruent to a ruled real hypersurface foliated by complex totally geodesic Q^{m-1} in Q^m if and only if the shape operator S satisfies $g(SX, Y) = 0$ for any $X, Y\in\mathcal{C}$.*

This Theorem A implies that the shape operator S is η-parallel, that is, $g((\nabla_X S)Y, Z) = 0$ for any $X, Y, Z\in\mathcal{C}$. By linearization, $g((\nabla_X S)X, X) = 0$ for any $X\in\mathcal{C}$. Then this is equivalent to the constancy of $g(S\gamma', \gamma') = \bar{g}(\bar{\nabla}_{\gamma'}\gamma', \bar{\nabla}_{\gamma'}\gamma')$, where $\bar{g}$ and $\bar{\nabla}$ denote respectively the Riemannian metric and the Riemannian connection of the complex quadric Q^m. This means that every geodesic γ: $I\to M$ in Q^m which is orthogonal to the Reeb vector field ξ, that is $\gamma'(0)\perp\xi_p$, and $\gamma(0) = p$, has constant first curvature.

When the stucture tensor ϕ commutes with the shape operator S, that is, $S\phi = \phi S$, we say that M has commuting shape operator. Motivated by this one, Berndt and Suh [2] have proved the following

Theorem B. *Let M be a complete real hypersurface in Q^m, $m\geq 3$, with commuting shape operator. Then M is locally congruent to a tube over $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$.*

Motivated by Theorems A and B, and Theorems 5.3 and 5.4 in section 5, we can assert the following

Main Theorem. *Let M be a real hypersurface in the complex quadric Q^m, $m\geq 4$, with η-parallel and η-commuting shape operator. Then M is locally congruent to a ruled hypersurface foliated by totally geodesic complex hypersurfaces Q^{m-1} in Q^m with $\mathfrak{A}$-principal unit normal vector field.*

If M is Hopf and η-commuting, the shape operator of M commutes with the structure tensor ϕ. Then by a result due to Berndt and Suh [3] M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In such a case the unit normal vector field N is $\mathfrak{A}$-isotropic. In section 5 we prove that the unit normal vector field N of a ruled real hypersurface is $\mathfrak{A}$-principal. But in this case M is non-Hopf.

Remark 1.1. *In Remark 4.4, we have mentioned that the unit normal vector field N of a ruled real hypersurface in Q^m is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Remark 1.2. *In section 6, we construct an example of minimal ruled real hypersurface which is foliated by totally geodesics Q^{m-1} in the complex quadric Q^m from curves in real projective space $\mathbb{R}P^{m+1}$.*

2. The complex quadric

For more background to this section we refer to [7], [8], [11], [18], [19] and [20]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini-Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini-Study metric $\bar{g}$ is defined by $\bar{g}(X,Y) = \Phi(JX,Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\bar{\partial}log f_j$ on an open set $U_j = \{[z_0, \ldots, z_j, \ldots, z_{m+1}]\in\mathbb{C}P^{m+1} | z_j\neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z_k}{z_j}$ for $j, k = 0, \cdots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

In another way, the complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric Q^m as follows:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\tilde{i}} & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \downarrow & & \pi \downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x,x) = g(y,y) = \frac{1}{2} \text{ and } g(x,y) = 0\},$$

where $g(x,y)=\sum_{i=1}^{m+2} x_i y_i$ for any $x=(x_1,\ldots,x_{m+2})$ and $y=(y_1,\ldots,y_{m+2})\in\mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_zS^{2m+3}=H_z\oplus F_z$ and $T_z\tilde{Q}=H_z(Q)\oplus F_z(Q)$ at $z=x+iy\in\tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z=(\mathbb{C}z)^\perp$ and $H_z(Q)=(\mathbb{C}z\oplus\mathbb{C}\bar{z})^\perp$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z=x+iy\in\tilde{Q}$ it can be described as

$$H_z=\{u+iv\in\mathbb{C}^{m+2}|\quad g(x,u)+g(y,v)=0,\quad g(x,v)=g(y,u)\}$$

and

$$H_z(Q)=\{u+iv\in H_z|\quad g(u,x)=g(u,y)=g(v,x)=g(v,y)=0\},$$

where $\mathbb{C}^{m+2}=\mathbb{R}^{m+2}\oplus i\mathbb{R}^{m+2}$, and $g(u,x)=\sum_{i=1}^{m+2}u_ix_i$ for any $u=(u_1,\ldots,u_{m+2})$, $x=(x_1,\ldots,x_{m+2})\in\mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_*H_z=T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_*H_z(Q)=T_{\pi(z)}Q$ respectively. This gives that at the point $\pi(z)=[z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [11]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}}w=\bar{\nabla}_w\bar{z}=\bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w\in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w\in T_{[z]}Q^m$ and any $\lambda\in S^1\subset\mathbb{C}$

$$\begin{aligned}A^2_{\lambda\bar{z}}w=&A_{\lambda\bar{z}}A_{\lambda\bar{z}}w=A_{\lambda\bar{z}}\lambda\bar{w}\\=&\lambda A_{\bar{z}}\lambda\bar{w}=\lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z}=\lambda\bar{\lambda}\bar{\bar{w}}\\=&|\lambda|^2w=w.\end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}}=I$ for any $\lambda\in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}}=I$ and $AJ=-JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m=V(A_{\bar{z}})\oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}})=\mathbb{R}^{m+2}\cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}})=i\mathbb{R}^{m+2}\cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X=X$ and $A_{\bar{z}}JX=-JX$, respectively, for any $X\in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z]\in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X \in V(A)$, $t = 0$, there exist many kinds of maximal 2-flats $\mathbb{R}X + \mathbb{R}Z$ for $Z \in V(A)$ orthogonal to $X \in V(A)$. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$, it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by $\mathbb{R}(X+JY)+\mathbb{R}Z$ for any $Z \in V(A)$ orthogonal to $X \in V(A)$ or $\mathbb{R}(X + JY) + \mathbb{R}JZ$ for any $JZ \in JV(A)$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1. *([18]) For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies the following

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurface M for a smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transformed JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the equation of Codazzi

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = {} & g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ = {} & (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = {} & -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & +2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & +\alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [11]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned}0 =& 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y)\\ &+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi)\\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi)\\ &- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X).\end{aligned} \tag{3.4}$$

4. η-parallel shape operator and a Key Lemma

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}R(X, Y)Z \ =\ & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\\ &+ g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY\\ &+ g(SY, Z)SX - g(SX, Z)SY\end{aligned}$$

for any $X, Y, Z \in T_zM$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m$, $z \in M$, $\rho(X) = g(AX, N)$, where BX and $\rho(X)N$ respectively denote the tangential and normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned}AN =& AJ\xi = -JA\xi = -J(B\xi + \rho(\xi)N)\\ =& -(\phi B\xi + \eta(B\xi)N).\end{aligned}$$

Then we assert the following:

Lemma 4.1. *Let M be a real hypersurface in Q^m, $m \geq 3$, with η-parallel and η-commuting shape operator. Then for any $X, Y, Z \in \mathcal{C}$ we have*

$$\begin{aligned}0 =& g(X, AN)g(AY, Z) + g(Y, A\xi)g(AX, \phi Z) - g(\phi Z, A\xi)g(AX, Y)\\ &- \eta(S\phi Z)g(Y, SX) + g(X, V)g(Y, SZ) + g(Y, V)g(X, SZ).\end{aligned}$$

where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ and V is given by $\phi S\xi$.

Proof. The notion of η-commuting shape operator gives

$$g((S\phi - \phi S)X, Y) = 0$$

for any $X, Y \in \mathcal{C}$. By differentiating this, we have

$$\begin{aligned}g((\nabla_X S)Y, \phi Z) + g((\nabla_X S)Z, \phi Y) =& \eta(SY)g(X, SZ) + \eta(SZ)g(Y, SX)\\ &+ g(X, S\phi Y)g(Z, V) + g(X, S\phi Z)g(Y, V).\end{aligned} \tag{4.1}$$

Then let us consider cyclic formulas with respect X, Y and Z as follows:

$$\begin{aligned}g((\nabla_Y S)Z, \phi X) + g((\nabla_Y S)X, \phi Z) =& \eta(SZ)g(Y, SX) + \eta(SX)g(Z, SY)\\ &+ g(Y, S\phi Z)g(X, V) + g(Y, S\phi X)g(Z, V)\end{aligned} \tag{4.2}$$

and

$$\begin{aligned} g((\nabla_Z S)X,\phi Y) + g((\nabla_Z S)Y,\phi X) &= \eta(SX)g(Z,SY) + \eta(SY)g(X,SZ) \\ &\quad + g(Z,S\phi X)g(Y,V) + g(Z,S\phi Y)g(X,V) \end{aligned} \tag{4.3}$$

Then substract the third one (4.3) from summing up (4.1) and (4.2). From such an obtained equation, and using the equation of Codazzi, it follows that

$$\begin{aligned} g((\nabla_X S)Y,\phi Z) &+ g((\nabla_Y S)X,\phi Z) + g((\nabla_X S)Z - (\nabla_Z S)X,\phi Y) \\ &+ g((\nabla_Y S)Z - (\nabla_Z S)Y,\phi X) \\ =&2\eta(SZ)g(Y,SX) + 2g(X,V)g(Y,S\phi Z) + 2g(Y,V)g(X,S\phi Z) \\ =&2g((\nabla_X S)Y,\phi Z) - \{g(X,AN)g(AY,\phi Z) - g(Y,AN)g(AX,\phi Z) \\ &+ g(X,A\xi)g(JAY,\phi Z) - g(Y,A\xi)g(JAX,\phi Z)\} \\ &+ \{g(X,AN)g(AZ,\phi Y) - g(Z,AN)g(AX,\phi Y) \\ &+ g(X,A\xi)g(JAZ,\phi Y) - g(Z,A\xi)g(JAX,\phi Y)\} \\ &+ \{g(Y,AN)g(AZ,\phi X) - g(Z,AN)g(AY,\phi X) \\ &+ g(X,A\xi)g(JAZ,\phi X) - g(Z,A\xi)g(JAY,\phi X)\}. \end{aligned} \tag{4.4}$$

From this, together with η-commuting property, and using $g(JAY,\phi Z) = -g(AY,J\phi Z) = g(AY,Z)$ for any $Y,Z\in\mathcal{C}$, we have

$$\begin{aligned} g((\nabla_X S)Y,\phi Z) - g(X,AN)g(AY,\phi Z) - g(Y,A\xi)g(AX,Z) - g(Z,A\xi)g(AX,Y) \\ =\eta(SZ)g(Y,SX) + g(X,V)g(Y,S\phi Z) + g(Y,V)g(X,S\phi Z) \end{aligned} \tag{4.5}$$

for any $X,Y,Z\in\mathcal{C}$. Then by replacing Z by ϕZ in (4.5), we have

$$\begin{aligned} g((\nabla_X S)Y,Z) = g(X,AN)g(AY,Z) + g(Y,A\xi)g(AX,\phi Z) - g(\phi Z,A\xi)g(AX,Y) \\ - \eta(S\phi Z)g(Y,SX) + g(X,V)g(Y,SZ) + g(Y,V)g(X,SZ). \end{aligned} \tag{4.6}$$

This gives a complete proof of our Lemma. □

Remark 4.2. *Let M be a tube over a totally complex geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}. Then the unit normal vector field N is $\mathfrak{A}$-isotropic and the shape operator S commutes with the structure tensor ϕ. So the Reeb vector field ξ is principal and the vector field $V = \phi S\xi = 0$. It can be easily seen that the vectors $A\xi$ and AN belong to the distribution $\mathcal{C}$. Then by (4.6) we have $g((\nabla_X S)Y,Z) = 0$ for any $X,Y,Z\in\mathcal{C}$ orthogonal to the vectors $A\xi$ and AN. Moreover, (4.6) gives the following formulas*

$$g((\nabla_{A\xi} S)A\xi,A\xi) = -g(A\xi,A\xi)g(A^2\xi,\phi Z) + g(A\xi,A\xi)g(\xi,\phi Z) = 0,$$

$$g((\nabla_{AN} S)AN,A\xi) = g(AN,AN)g(A^2N,A\xi) - g(AN,AN)g(A^2N,A\xi) = 0,$$

$$g((\nabla_{A\xi} S)AN,AN) = -g(A\xi,A\xi)g(A^2N,\phi AN) + g(AN,A\xi)g(A^2\xi,\phi AN) = 0,$$

and

$$g((\nabla_{A\xi} S)A\xi,AN) = -g(A\xi,A\xi)g(A^2\xi,\phi AN) + g(A\xi,A\xi)g(A^2\xi,\phi AN) = 0.$$

Then all the formulas mentioned above give that the shape operator S is η-parallel.

Now let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be written

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \text{and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (3.3) and the anti-commuting property $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0,\ g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

In Lemma 4.1 let us take skew-symmetric in X and Y, it follows that

$$\begin{aligned} 0 =&\{g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z)\} \\ &+ \{g(Y, A\xi)g(AX, \phi Z) - g(X, A\xi)g(AY, \phi Z)\}. \end{aligned} \tag{4.7}$$

Since we have assumed that the unit normal N is $\mathfrak{A}$-isotropic, we can put $X = AN$ in (4.7). Then it gives that $g(AY, Z) = 0$ for any Y and $Z \in \mathcal{C}$. So Lemma 4.1 gives the following

$$g(X, V)g(Y, SZ) + g(Y, V)g(Z, SX) + g(Z, V)g(X, SY) = 0. \tag{4.8}$$

When the unit normal vector field N is $\mathfrak{A}$-principal, that is, $AN = N$ and $A\xi = -\xi$, then Lemma 4.1 also gives the equation (4.6). Now let us put $S\xi = \alpha\xi + \beta U$ in (4.8). Then we assert the following

Lemma 4.3. *Let M be a complete real hypersurface in Q^m, $m \geq 3$, with η-parallel and η-commuting shape operator. If the unit normal vector field is singular, then*

$$\beta = 0 \quad or \quad g(SY, Z) = 0$$

for any vector fields $Y, Z \in \mathcal{C}$, where $\mathcal{C}$ denotes the orthogonal distribution of the Reeb vector field ξ.

Proof. Now let us put $Z = V = \phi S\xi$ in (4.8) and use $S\xi = \alpha\xi + \beta U$ for some $U \in \mathcal{C}$. Then it follows that

$$\begin{aligned} 0 =& g(SX, Y)\|V\|^2 + g(SY, V)g(X, V) + g(SV, X)g(Y, V) \\ =& g(SX, Y)\|V\|^2 + \beta^2 g(SY, \phi U)g(X, \phi U) + \beta^2 g(S\phi U, X)g(Y, \phi U) \end{aligned} \tag{4.9}$$

for any X, Y and $Z \in \mathcal{C}$. Then for any $X, Y \in \mathcal{C}$ which are orthogonal to ϕU the formula (4.9) gives $g(SX, Y) = 0$. Now we put $X = Y = \phi U$ in (4.9). Then it follows that

$$\begin{aligned} 0 =& g(S\phi U, \phi U)\|V\|^2 + 2\beta^2 g(S\phi U, \phi U) \\ =& 3\beta^2 g(S\phi U, \phi U), \end{aligned} \tag{4.10}$$

where we have used $\|V\|^2 = g(\phi S\xi, \phi S\xi) = \beta^2$. Then (4.10) gives that the function $\beta = 0$ or $g(S\phi U, \phi U) = 0$. Now let us consider the case that $\beta \neq 0$ on the open subset $\mathcal{U}$ in M.

Then $g(S\phi U, \phi U) = 0$ on $\mathcal{U}$. From this, together with putting $Y = \phi U$ in (4.9), we have for any $X \in \mathcal{C}$

$$0 = g(S\phi U, X)\|V\|^2 + \beta^2 g(S\phi U, X) = 2\beta^2 g(S\phi U, X). \tag{4.11}$$

So it follows that $g(S\phi U, X) = 0$ on $\mathcal{U}$ for any $X \in \mathcal{C}$. From this, together with $g(SX, Y) = 0$ for any $X, Y \in \mathcal{C}$ orthogonal to ϕU, we can assert the latter part of Lemma 4.3. From this, we give a complete proof of our Lemma 4.3. □

Remark 4.4. *Let M be a ruled real hypersurface in Q^m foliated by the totally geodesic complex hyperplane Q^{m-1} in section 2. If the Reeb function $\alpha = g(S\xi, \xi) = 0$ and $\beta = g(S\xi, U)$ is constant,and the vector field U is parallel along the integral curve of the Reeb vector field ξ, then the unit normal vector field $N = J\xi$ becomes singular.*

In fact, let us use the equation of Codazzi for $S\xi = \alpha\xi + \beta U$, $SU = \beta\xi$. Then it follows that

$$\begin{aligned} g(\bar{R}(X,Y)\xi, N) =& g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ =& g((\nabla_X S)\xi, Y) - g(((\nabla_Y S)\xi, X) \\ =& d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) \\ & - 2g(S\phi SX, Y) + (X\beta)g(U,Y) - (Y\beta)g(U,X) \\ & + \beta\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}. \end{aligned} \tag{4.12}$$

By putting $X = \xi$ into (4.12) and using the assumption for the ruled hypersurface in Q^m, we have

$$\begin{aligned} g(\bar{R}(\xi, N)N, JY) =& g(\bar{R}(JY, J\xi)N, \xi) = g(\bar{R}(\xi, Y)\xi, N) \\ =& d\alpha(\xi)\eta(Y) - d\alpha(Y) + \alpha\beta g(\phi U, Y) \\ & + (\xi\beta)g(U,Y) + \beta g(\nabla_\xi U, Y) \\ =& 0. \end{aligned} \tag{4.13}$$

This implies $\bar{R}_N\xi = c\xi$ for $c \in \mathbb{R}$, that is, the Reeb vector field ξ is principal for the normal Jacobi operator $\bar{R}_N$. Then by a result due to Berndt and Suh (see Proposition 3.1, [3]*) we know that the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

5. Proof of Main Theorem

In this section we prove our Main Theorem mentioned in the introduction. By the notions of η-parallel and η-commuting shape operator, we give a complete classification of real hypersurfaces in the complex quadric Q^m satisfying these notions. One of the most crucial points of this classification is to give a geometric property that the unit normal vector field of a ruled real hypersurface in Q^m foliated by complex totally geodesic Q^{m-1} is $\mathfrak{A}$-principal. Though in Remark 4.4 we have mentioned the unit normal vector field N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal, but in general N is $\mathfrak{A}$-principal for ruled real hypersurfaces in the complex quadric Q^m.

In order to complete this fact, let us consider a real hypersurface M in Q^m,$m \geq 4$, such that $g((\nabla_X S), Y, Z) = 0$ and $g((S\phi - \phi S)X, Y) = 0$ for any X, Y, $Z \in \mathcal{C}$. We can use the formula (3.3) in section 3. This, together with $g(\xi, AN) = 0$ and Lemma 4.1 yields

$$\begin{aligned} 0 &= g(X, AN)g(AY, Z) + g(Y, A\xi)g(AX, \phi Z) - g(\phi Z, A\xi)g(AX, Y) \\ &\quad - \eta(S\phi Z)g(Y, SX) + g(X, \phi S\xi)g(Y, SZ) + g(\phi S\xi, Y)g(X, SZ) \end{aligned}$$

for any X,Y,$Z \in \mathcal{C}$.

If M is Hopf, that is, the Reeb vector field ξ is a principal vector field of the shape operator S of a real hypersurface M in Q^m, then it follows that $0 = \phi S\xi = S\phi\xi$. From this, together with η-commuting shape operator, $g((S\phi - \phi S)X, Y) = 0$ for any $X, Y \in \mathcal{C}$, it naturally gives that the structure tensor ϕ commutes with the shape operator S, that is, $S\phi = \phi S$. Then by Theorem B we assert the following

Proposition 5.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with η-parallel and η-commuting shape operator. Then M is locally congruent to a tube of radius r over a totally geodesic complex submanifold $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$.*

In a paper due to Berndt and Suh [2] we proved that the unit normal vector field N of M in the complex quadric Q^m is $\mathfrak{A}$-isotropic, that is $g(AN, N) = 0$ for the real hypersurface appearing in Proposition 5.1. Related to this fact, we want to show another proposition as follows:

Proposition 5.2. *There does not exist any real hypersurface in Q^m, $m \geq 3$, with η-parallel shape operator and with $\mathfrak{A}$-isotropic normal vector field N.*

Proof. Let us assume that M is a real hypersurface with η-parallel shape operator in Q^m, $m \geq 3$. That is, the shape operator S of M satisfies the following condition:

$$g((\nabla_X S)Y, Z) = 0 \tag{*}$$

for any tangent vector fields $X, Y, Z \in \mathcal{C}$, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ on M in Q^m. By using the equation of Codazzi, it yields for any X, Y, $Z \in \mathcal{C}$

$$g(g(AX, N)AY - g(AY, N)AX + g(AX, \xi)JAY - g(AY, \xi)JAX, Z) = 0.$$

The vector field $g(AX, N)AY - g(AY, N)AX + g(AX, \xi)JAY - g(AY, \xi)JAX$ in the left-side of the above equation is denoted by $W_{X,Y}$ (simply, W). Then $W_{X,Y} \in T_{[z]}Q^m$ becomes

$$\begin{aligned} W_{X,Y} &= \sum_{i=1}^{2m} g(W_{X,Y}, e_i)e_i = g(W, \xi)\xi + g(W, N)N \\ &= g(W, \xi)\xi, \end{aligned}$$

because $g(W, N) = 0$ and $Z \in \mathcal{C}$. Since N is $\mathfrak{A}$-isotropic, $g(AN, N) = 0$ and $g(AN, \xi) = 0$, we see that $AN \in \mathcal{C} \subset T_{[z]}M$, $[z] \in M$.

Substituting $Y = AN$ in $W_{X,Y}$ and using $A^2 = I$, we have

$$\begin{aligned} & g(AX, N)N - AX - g(AX, \xi)\xi \\ &\qquad = W_{X,AN} = g(W, \xi)\xi = -2g(AX, \xi)\xi. \end{aligned}$$

Then it can be arranged as follows:

$$AX = g(AX, N)N + g(AX, \xi)\xi$$

for any $X \in \mathcal{C}$. From this , applying the real structure A and using the property of $A^2 = I$ again, it follows that

$$X = g(AX, N)AN + g(AX, \xi)A\xi \in \mathcal{C}.$$

This means $\dim_{\mathbb{R}} \mathcal{C} = 2$. But, in fact, any vector $X \in \mathcal{C}$ is expressed by

$$X = \sum_{k=1}^{2m-2} g(X, e_k)e_k$$

with respect to the basis $\{AN, A\xi, e_1, e_2, \cdots, e_{2m-4}\}$ of $\mathcal{C}$. So we get $\dim_{\mathbb{R}} \mathcal{C} = 2m - 2$, which gives a contradiction. From this, we get a complete proof of our proposition. □

Then combining Propositions 5.1 and 5.2, we assert the following

Theorem 5.3. *There do not exist a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with η-parallel and η-commuting shape operator.*

Now let us suppose that M is non-Hopf and write $S\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathcal{C}$ and $\beta \neq 0$. Then the above equation becomes

$$\begin{aligned} 0 = {} & g(X, AN)g(AY, Z) + g(Y, A\xi)g(AX, \phi Z) - g(\phi Z, A\xi)g(AX, Y) \\ & + \beta g(Z, \phi U)g(Y, SX) + \beta g(X, \phi U)g(Y, SZ) + \beta g(Y, \phi U)g(X, SZ) \end{aligned} \tag{5.1}$$

for any $X, Y, Z \in \mathcal{C}$.

Let us take $X, Y, Z \in \mathcal{C}_U = \text{Span}\{\xi, U, \phi U\}^\perp$. From (5.1) we get

$$0 = g(X, AN)g(AY, Z) + g(Y, A\xi)g(AX, \phi Z) - g(Z, AU)g(AX, Y).$$

Taking $X = Z$, we obtain

$$g(Y, A\xi)g(AX, \phi X) = 0 \tag{5.2}$$

for any $X, Y \in \mathcal{C}_U$.

Case 1) Suppose $g(A\xi, Y) = 0$ for any $Y \in \mathcal{C}_U$.

Now we take ϕY instead of Y. Then it follows that

$$g(\phi Y, A\xi) = -g(Y, JA\xi) = g(Y, AJ\xi) = g(Y, AN) = 0 \tag{5.3}$$

for any $Y \in \mathcal{C}_U$.

If we take $X, Y \in \mathcal{C}_U$, $Z = U$ in (5.1), we obtain

$$0 = g(X, AN)g(AY, U) - g(U, AN)g(AZ, Y) \tag{5.4}$$

for any $Z, Y \in \mathcal{C}_U$. From (5.3), (5.4) becomes

$$0 = g(U, AN)g(AX, Y) \tag{5.5}$$

for any $X, Y \in \mathcal{C}_U$. Taking $X = \phi U$, $Y, Z \in \mathcal{C}_U$ in (5.1), we have

$$-g(U, A\xi)g(AY, Z) + \beta g(Y, SZ) = 0 \tag{5.6}$$

for any $Y, Z \in \mathcal{C}_U$ and taking $X, Y \in \mathcal{C}_U$, $Z = \phi U$, we obtain bearing in mind (5.3)

$$g(U, A\xi)g(AX, Y) + \beta g(Y, SZ) = 0.$$

In particular,

$$g(U, A\xi)g(AY, Z) + \beta g(Z, SY) = 0 \tag{5.7}$$

From (5.6) and (5.7) we get

$$g(U, A\xi)g(AY, Z) = 0 \tag{5.8}$$

and

$$g(SY, Z) = 0 \tag{5.9}$$

for any $Y, Z \in \mathcal{C}_U$.

We know $g(AZ, \phi U) = g(\phi Z, AU)$ for any $Z \in \mathcal{C}$. Taking $X = U$, $Y = \phi U$, $Z \in \mathcal{C}_U$ in (5.1), we have

$$-2g(U, AN)g(JAU, Z) + \beta g(U, SZ) = 0$$

and taking $X = \phi U$, $Y = U$, $Z \in \mathcal{C}_U$ in (5.1) it follows

$$-2g(U, A\xi)g(AU, Z) + \beta g(U, SZ) = 0$$

for any $Z \in \mathcal{C}_U$. Therefore

$$g(U, AN)g(JAU, Z) = g(U, A\xi)g(AU, Z) \tag{5.10}$$

for any $Z \in \mathcal{C}_U$. Take $X = U$, $Y = \mathcal{C}_U$, $Z = \phi U$ in (5.1). Then

$$0 = -g(U, AN)g(Y, JAU) + g(U, A\xi)g(AU, Y) + \beta g(Y, SU) \tag{5.11}$$

for any $Y \in \mathcal{C}_U$. From (5.10) and (5.11) we get

$$g(Y, SU) = 0 \tag{5.12}$$

for any $Y \in \mathcal{C}_U$. Therefore we get

$$g(U, AN)g(JAU, Z) = 0 \tag{5.13}$$

$$g(U, A\xi)g(AU, Z) = 0 \tag{5.14}$$

for any $Z \in \mathcal{C}_U$. Taking $X = Z = \phi U$, $Y \in \mathcal{C}_U$ in (5.1) we have

$$\begin{aligned} 0 &= g(\phi U, AN)g(AY, \phi U) + g(U, A\xi)g(A\phi U, Y) + 2\beta g(SY, \phi U) \\ &= 2\beta g(SY, \phi U). \end{aligned}$$

As we suppose $\beta \neq 0$, we get

$$g(SY, \phi U) = 0 \tag{5.15}$$

for any $Y \in \mathcal{C}_U$. From (5.9), (5.12) and (5.15) we have

$$SX = 0 \tag{5.16}$$

for any $X = \mathcal{C}_U$. If we put $X = Y = Z = U$ in (5.1), we get

$$g(U, A\xi)g(AU, \phi U) = 0. \tag{5.17}$$

And taking $X = Y = Z = \phi U$ in (5.1) we obtain

$$g(U,AN)g(A\phi U,U)=3\beta g(\phi U,S\phi U) \tag{5.18}$$

If we put $X=U, Y=\phi U, Z=U$ in (5.1), we have

$$g(U,AN)g(A\phi U,U)+\beta g(U,SU)=0 \tag{5.19}$$

From (5.18) and (5.20) it follows

$$g(U,SU)=-3g(\phi U,S\phi U) \tag{5.20}$$

Now let us put $X=Y=\phi U, Z=U$ in (5.1). Then it follows

$$0=-g(U,A\xi)g(A\phi U,U)+2\beta g(\phi U,SU)$$

and (5.19) yields

$$g(\phi U,SU)=0 \tag{5.21}$$

that is, $S\phi U=\gamma\phi U \quad SU=-3\gamma U+\beta\xi$. By Codazzi equation, bearing in mind that for any $X\in\mathcal{C}_U$

$$g(AX,\xi)=g(AX,N)=0 \text{ and } SX=S\phi X=0.$$

we get

$$g((\nabla_X S)\phi X-(\nabla_{\phi X}S)X,Z)=-2\eta(Z) \tag{5.22}$$

for any Z tangent to M. Taking $Z=\xi$ it follows

$$\beta g([\phi X,X],U)=-2 \tag{5.23}$$

and taking $Z=U$ we get

$$-3\gamma g([\phi X,X],U)=0 \tag{5.24}$$

From (5.23) and (5.24) $\gamma=0$ and we have $S\xi=\alpha\xi+\beta U$, $SU=\beta\xi$, $SX=0$, $X\in\text{Span}\{\xi,U\}^{\perp}$.

Suppose, moreover, that $g(AX,Y)=0$ for any $X,Y\in\mathcal{C}_U$. If $g(AX,U)=0$ for any $X\in\mathcal{C}_U$, then

$$0=g(A\phi X,U)=g(AJX,U)=-g(X,JAU)=g(X,AJU)=g(X,A\phi U).$$

In this case $AX=0$ for any $X\in\mathcal{C}_U$ and this yields $X=0$, therefore $m\leq 2$ and we have a contradiction. Therefore there exists $X\in\mathcal{C}_U$ such that $g(AX,U)\neq 0$. Then for any $X\in\mathcal{C}_U$ we have

$$AX=g(AX,U)U+g(AX,\phi U)\phi U.$$

So by applying complex conjugation A again, it follows $X=g(AX,U)AU+g(AX,\phi U)A\phi U$, which means $\mathcal{C}_U=\text{Span}\{AU,A\phi U\}$ and $m\leq 3$, also a contradiction. Therefore there exist $X,Y\in\mathcal{C}_U$ such that $g(AX,Y)\neq 0$. As $g(X,AN)=g(JX,A\xi)=0$ for any $X\in\mathcal{C}_U$, from (5.4) we have $g(U,AN)=0$ and from (5.6) $g(U,A\xi)=0$. Then these formulas give

$$g(U,AN)=g(U,AJ\xi)=-g(U,JA\xi)=g(\phi U,A\xi)=0,$$

$$g(U,A\xi)=-g(U,AJN)=g(U,JAN)=-g(\phi U,AN)=0.$$

So we have obtained that $AN = g(AN, N)N$ and $N = g(AN, N)AN = g(AN, N)^2 N$. This gives that $g(AN, N)^2 = 1$, which means $\cos^2(2t) = 1$. As $0 \leq t \leq \frac{\pi}{4}$, the unique possibility is $2t = 0$, that is, $t = 0$ and N is $\mathfrak{A}$-principal.

Case 2) Suppose $g(AX, \phi X) = 0$ for any $X \in \mathcal{C}_U$.

This yields $g(AX, Y) = 0$ for any $X, Y \in \mathcal{C}_U$. Take $X, Y = \phi X \in \mathcal{C}_U$, $Z = U$ in (5.1). We have $0 = 2g(X, AN)g(X, A\phi U)$. Therefore we assert

$$g(X, AN)g(X, A\phi U) = 0 \tag{5.25}$$

for any $X \in \mathcal{C}_U$. And taking $X, Y = \phi X \in \mathcal{C}_U$, $Z = \phi U$ in (5.1) we obtain

$$2g(X, AN)g(AX, U) = \beta g(SX, \phi X) \tag{5.26}$$

for any $X \in \mathcal{C}_U$. Taking $X \in \mathcal{C}_U$, $Y = U$, $Z = U$ in (5.1) we get

$$g(X, AN)g(AU, U) - g(U, A\xi)g(X, JAN) - g(U, AN)g(AX, U) = 0 \tag{5.27}$$

for any $X \in \mathcal{C}_U$, and taking $X \in \mathcal{C}_U$, $Y = \phi U$, $Z = \phi U$ in (5.1) we obtain

$$\begin{aligned} 0 = & - g(X, AN)g(AU, U) - g(U, AN)g(AX, U) \\ & - g(U, A\xi)g(X, JAU) + 2\beta g(S\phi U, X). \end{aligned} \tag{5.28}$$

From (5.7) and (5.8) we have

$$g(X, AN)g(AU, U) = 2\beta g(S\phi U, X) \tag{5.29}$$

for any $X \in \mathcal{C}_U$.

Let us suppose that $g(X, AN) = 0$ for any $X \in \mathcal{C}_U$. Then

$$0 = g(\phi X, AN) = -g(X, JAN) = g(X, AJN) = -g(X, A\xi).$$

Therefore $g(X, A\xi) = 0$ for any $X \in \mathcal{C}_U$. As we suppose $g(AX, Y) = 0$ for any $X, Y \in \mathcal{C}_U$, we know

$$AX = g(AX, U)U + g(AX, \phi U)\phi U,$$

for any $X \in \mathcal{C}_U$, that is, $X = g(AX, U)AU + g(AX, \phi U)A\phi U$ for any $X \in \mathcal{C}_U$. Accordingly, it follows that $\mathcal{C}_U = \text{Span}\{AU, A\phi U\}$ and $\dim \mathcal{C}_U \leq 2$. Therefore $\dim M \leq 5$ or $m \leq 3$, which is impossible.

If $g(AX, \phi U) = 0$ for any $X \in \mathcal{C}_U$ we have $g(\phi X, A\phi U) = -g(\phi X, JAU) = -g(X, AU) = 0$, and in this case $AX = g(AX, \xi)\xi + g(AX, N)N$ which yields

$$X = g(AX, \xi)A\xi + g(AX, N)AN$$

for any $X \in \mathcal{C}_U$. We arrive at the same contradiction.

Therefore we must suppose that there exists $X \in \mathcal{C}_U$ such that $g(X, AN) \neq 0$ and from (5.25) $g(X, A\phi U) = 0$. Taking $X, Y \in \mathcal{C}_U$, $Z = U$ in (5.1) we obtain

$$g(X, AN)g(AY, U) + g(Y, A\xi)g(AX, \phi U) = 0 \tag{5.30}$$

for any $X, Y \in \mathcal{C}_U$. Taking, in particular, our previous $X \in \mathcal{C}_U$ in (5.30) we obtain $g(AY, U) = 0$ for any $Y \in \mathcal{C}_U$. As above, this also yields $g(AY, \phi U) = 0$ for any $Y \in \mathcal{C}_U$. Then for any $Y \in \mathcal{C}_U$ we have

$$AY = g(AY, \xi)\xi + g(AY, N)N$$

and, as above, this gives a contradiction. Accordingly, the Case 2) can not appear. So only the Case 1) remains valid. From this, together with Theorem A, we have proved.

Theorem 5.4. *Let M be a non-Hopf real hypersurface in the complex quadric Q^m, $m\geq 4$, with η-parallel and η-commuting shape operator. Then the unit normal vector field N of M is $\mathfrak{A}$-principal and M is locally congruent to a ruled real hypersurface foliated by complex totally geodesic Q^{m-1} in Q^m.*

Summing up above two Theorems 5.3 and 5.4 we give a complete proof of our Main Theorem in the introduction.

6. Examples of ruled real hypersurfaces in complex quadric

In this section, let us construct *ruled real hypersurfaces M^{2m-1} in complex quadric Q^m*, i.e., real hypersurfaces which are foliated by totally geodesic complex hyperquadric Q^{m-1}, from *curves in real projective space* $\mathbb{RP}^{m+1}$.

First we recall *Stiefel manifold* (cf. [5]). Let

$$V_2(\mathbb{R}^{m+2}) = \{(v_1, v_2)|\ v_1, v_2 \in \mathbb{R}^{m+2},\ \|v_1\| = \|v_2\| = 1,\ \langle v_1, v_2\rangle = 0\}$$

be the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$. Then the tangent space $T_{(v_1,v_2)}V_2(\mathbb{R}^{m+2})$ is given as

$$\mathbb{R}(-v_2, v_1) \oplus \{(x_1, x_2) \in \mathbb{R}^{m+2} \times \mathbb{R}^{m+2}|\ x_1, x_2 \perp \mathrm{span}\{v_1, v_2\}\}.$$

Let $\widetilde{\mathbb{G}}_2(\mathbb{R}^{m+2})$ be the *Grassmannian manifolds* of oriented 2-planes in $\mathbb{R}^{m+2}$ and let $\pi^{\mathbb{G}}$: $V_2(\mathbb{R}^{m+2}) \to \widetilde{\mathbb{G}}_2(\mathbb{R}^{m+2})$ be the projection defined by $\pi^{\mathbb{G}}(v_1, v_2) = \mathrm{span}(v_1, v_2)$. Then with respect to the metric on $V_2(\mathbb{R}^{m+2})$ induced from Euclidean space $\mathbb{R}^{m+2} \times \mathbb{R}^{m+2} = \mathbb{C}^{m+2}$ as a submanifold, we can define a Riemannian metric on $\widetilde{\mathbb{G}}_2(\mathbb{R}^{m+2})$ such that π^G is a Riemannian submersion. We consider an embedding:

$$\tilde{i} : V_2(\mathbb{R}^{m+2}) \to S^{2m+3} \subset \mathbb{C}^{m+2}, \quad \tilde{i}(v_1, v_2) = (v_1 + iv_2)/\sqrt{2}.$$

The tangent space $T_{\tilde{i}(v_1,v_2)}\tilde{i}(V_2(\mathbb{R}^{m+1}))$ is given as

$$\mathbb{R}(-v_2 + iv_1) \oplus \{x_1 + ix_2 \in \mathbb{C}^{m+2}|\ x_1, x_2 \perp \mathrm{span}\{v_1, v_2\}\}.$$

Then we have a commutative diagram

$$\begin{array}{ccc} V_2(\mathbb{R}^{m+2}) & \xrightarrow[\tilde{i}]{} & S^{2m+3} \\ {\scriptstyle \pi^{\mathbb{G}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{G}_2(\mathbb{R}^{m+2}) & \xrightarrow[i]{} & \mathbb{CP}^{m+1} \end{array}, \tag{6.1}$$

where π is the Hopf fibration and i is the embedding induced from $\tilde{i}$. Then $i(\mathbb{G}_2(\mathbb{R}^{m+2}))$ is identified with the complex quadric Q^n.

Let I be an interval and let $\gamma : I \to \mathbb{RP}^{m+1}$ be a real 1-dimensional regular curve in real projective space. We denote $\Gamma : I \to SO(n)$ a horizontal lift of γ with respect to the natural projection $SO(m+2) \to \mathbb{RP}^{m+1} = SO(m+2)/S(O(m+1) \times O(1))$. For an expression of the matrix $\Gamma(t) = (e_1(t), \cdots, e_{m+1}(t), e_{m+2}(t))$ by column vectors, we may assume

$$e_j'(t) = \lambda_j(t)e_{m+2}(t) \quad (j = 1, \cdots, m+1), \quad e_{m+2}'(t) = -\sum_{j=1}^{m+1} \lambda_j(t)e_j(t). \tag{6.2}$$

Let $\widetilde{\Phi} : I \times V_2(\mathbb{R}^{m+1}) \to S^{2m+3} \subset \mathbb{C}^{m+2}$ be a map defined by

$$\widetilde{\Phi}(t,(v_1,v_2)) = \Gamma(t)\begin{pmatrix}(v_1+iv_2)/\sqrt{2}\\ 0\end{pmatrix}. \tag{6.3}$$

Then we have the induced map $\Phi : I \times G_2(\mathbb{R}^{m+1}) \to \mathbb{CP}^{m+1}$ defined by

$$\Phi(t,\pi^G((v_1,v_2))) = \pi(\widetilde{\Phi}(t,(v_1,v_2))) \tag{6.4}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} I \times V_2(\mathbb{R}^{m+1}) & \xrightarrow[\tilde{\Phi}]{} & S^{2m+3} \\ {\scriptstyle id\times\pi^{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle \pi} \\ I \times \mathbb{G}_2(\mathbb{R}^{m+1}) & \xrightarrow[\Phi]{} & \mathbb{CP}^{m+1} \end{array}, \tag{6.5}$$

and the image $\Phi(I \times G_2(\mathbb{R}^{m+1}))$ lies in the complex quadric Q^m in $\mathbb{CP}^{m+1}$ and for each $t \in I$, $\Phi(\{t\} \times G_2(\mathbb{R}^{m+1})))$ is a totally geodesic complex hypersurface Q^{m-1} in Q^m.

We compute the differential of $\widetilde{\Phi}$. Using (6.2) we have

$$\begin{aligned} d\widetilde{\Phi}((\partial/\partial t),0) &= \Gamma'(t)\begin{pmatrix}(v_1+iv_2)/\sqrt{2}\\ 0\end{pmatrix} \\ &= \Gamma(t)\begin{pmatrix} O & -\lambda(t) \\ {}^t\lambda(t) & 0\end{pmatrix}\begin{pmatrix}(v_1+iv_2)/\sqrt{2}\\ 0\end{pmatrix} \\ &= \Gamma(t)\begin{pmatrix} 0 \\ (\langle\lambda(t),v_1\rangle + i\langle\lambda(t),v_2\rangle)/\sqrt{2}\end{pmatrix}, \end{aligned} \tag{6.6}$$

where we put $\lambda(t) ={}^t(\lambda_1(t),\cdots,\lambda_{m+1}(t))$. Also we obtain

$$V := d\widetilde{\Phi}(0,(-v_2,v_1)) = \frac{\Gamma(t)}{\sqrt{2}}\begin{pmatrix}-v_2+iv_1\\ 0\end{pmatrix}, \tag{6.7}$$

and

$$d\widetilde{\Phi}(0,(x_1,x_2)) = \frac{\Gamma(t)}{\sqrt{2}}\begin{pmatrix}x_1+ix_2\\ 0\end{pmatrix}, \tag{6.8}$$

where $x_1, x_2 \perp v_1, v_2$. Here V is a vertical vector with respect to the fibration $id \times \pi^{\mathbb{G}} : I \times V_2(\mathbb{R}^{m+1}) \to I \times G_2(\mathbb{C}^{m+1})$. The metric on $I \times V_2(\mathbb{R}^{m+1})$ induced by $\widetilde{\Phi}$ is written as:

$$\|d\widetilde{\Phi}((\partial/\partial t),0)\|^2 = \frac{\langle\lambda(t),v_1\rangle^2 + \langle\lambda(t),v_2\rangle^2}{2},$$

$$\|V\|^2 = 1, \quad \|d\widetilde{\Phi}(0,(x_1,x_2))\|^2 = \frac{\|x_1\|^2+\|x_2\|^2}{2},$$

$$\langle d\widetilde{\Phi}((\partial/\partial t),0), V\rangle = \langle d\widetilde{\Phi}((\partial/\partial t),0), d\widetilde{\Phi}(0,(x_1,x_2))\rangle = \langle V, d\widetilde{\Phi}(0,(x_1,x_2))\rangle = 0.$$

Hence

$$\widetilde{\Phi} \text{ is regular at } (t,(v_1,v_2)) \Leftrightarrow \langle\lambda(t),v_1\rangle^2 + \langle\lambda(t),v_2\rangle^2 \neq 0, \tag{6.9}$$

Proposition 6.1. *Let $\gamma : I \to \mathbb{RP}^{m+1}$ be a real 1-dimensional regular curve in real projective space and let $\Gamma : I \to SO(n)$ be a horizontal lift of γ with respect to the natural projection $SO(m+2) \to \mathbb{RP}^{m+1} = SO(m+2)/S(O(m+1)\times O(1))$. Then the map*

$\Phi : I \times \mathbb{G}_2(\mathbb{R}^{m+1}) \to \mathbb{CP}^{m+1}$ *defined by* (6.4) *is regular at* $(t, (v_1, v_2))$ *if and only if* (6.9) *holds.*

A unit normal vector of $\widetilde{\Phi}$ at $(t, (v_1, v_2))$ is given by

$$\widetilde{N} = \frac{\Gamma(t)}{\sqrt{\langle \lambda(t), v_1\rangle^2 + \langle \lambda(t), v_2\rangle^2}} \begin{pmatrix} 0 \\ -\langle \lambda(t), v_2\rangle + i\langle \lambda(t), v_1\rangle \end{pmatrix}. \tag{6.10}$$

Now we compute the condition for which $\widetilde{\Phi}$ (and Φ) is *minimal.* By (6.8) and (6.10), we see that

$$\langle A^{\widetilde{\Phi}}(0, (x_1, x_2)), (0, (y_1, y_2))\rangle = 0, \quad (x_1, x_2, y_1, y_2 \perp v_1, v_2)$$

where $A^{\widetilde{\Phi}}$ is the shape operator of $\widetilde{\Phi}$. Hence $\widetilde{\Phi}$ (and Φ) is a minimal immersion at the regular points of $\widetilde{\Phi}$ if and only if $\langle A^{\widetilde{\Phi}}(\partial/\partial t, 0), (\partial/\partial t, 0)\rangle = 0$. Using (6.2) and (6.6), we obtain

$$\begin{aligned} D_{d\widetilde{\Phi}((\partial/\partial t),0)} d\widetilde{\Phi}((\partial/\partial t), 0) &= \frac{\Gamma'(t)}{\sqrt{2}} \begin{pmatrix} 0 \\ (\langle \lambda(t), v_1\rangle + i\langle \lambda(t), v_2\rangle) \end{pmatrix} \\ &+ \frac{\Gamma(t)}{\sqrt{2}} \begin{pmatrix} 0 \\ (\langle \lambda'(t), v_1\rangle + i\langle \lambda'(t), v_2\rangle) \end{pmatrix} \\ &= \frac{\Gamma(t)}{\sqrt{2}} \begin{pmatrix} O & -\lambda(t) \\ {}^t\lambda(t) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (\langle \lambda(t), v_1\rangle + i\langle \lambda(t), v_2\rangle) \end{pmatrix} \\ &+ \frac{\Gamma(t)}{\sqrt{2}} \begin{pmatrix} 0 \\ (\langle \lambda'(t), v_1\rangle + i\langle \lambda'(t), v_2\rangle) \end{pmatrix} \\ &= \frac{\Gamma(t)}{\sqrt{2}} \begin{pmatrix} -(\langle \lambda(t), v_1\rangle + i\langle \lambda(t), v_2\rangle)\lambda(t) \\ (\langle \lambda'(t), v_1\rangle + i\langle \lambda'(t), v_2\rangle) \end{pmatrix}. \end{aligned} \tag{6.11}$$

Hence $\widetilde{\Phi}$ (and Φ) is minimal if and only if

$$-\langle \lambda'(t), v_1\rangle \langle \lambda(t), v_2\rangle + \langle \lambda'(t), v_2\rangle\langle \lambda(t), v_1\rangle = 0 \tag{6.12}$$

for any $(v_1, v_2) \in V_2(\mathbb{R}^{m+2})$.

We may assume that $\|\lambda(t)\|^2 = 1$, by changing parameter t if necessarily, so we have $\lambda(t) \perp \lambda'(t)$. On the other hand, (6.12) implies that $\lambda(t) \wedge \lambda'(t) = 0$. Consequently we obtain that $\lambda(t)$ is constant, and $\Gamma(t)$ is a 1-parameter group of $SO(m + 2)$. Hence by the help of [1], minimal ruled hypersurface $\Phi(I \times G_2(\mathbb{R}^{m+1}))$ in Q^m is invariant under a 1-parameter subgroup $\Gamma(t)$ of $SO(m + 2)$.

Theorem 6.2. *Minimal ruled real hypersurface* M^{2m-1} *in complex quadric* Q^m *is invariant under a* 1*-parameter subgroup of* $SO(m + 2)$*.*

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