Quadratic estimation problem in discrete-time stochastic systems with random parameter matrices

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Abstract

This paper addresses the least-squares quadratic filtering problem in discrete-time stochastic systems with random parameter matrices in both the state and measurement equations. Defining a suitable augmented system, this problem is reduced to the least-squares linear filtering problem of the augmented state based on the augmented observations. Under the assumption that the moments, up to the fourth-order one, of the original state and measurement vectors are known, a recursive algorithm for the optimal linear filter of the augmented state is designed, from which the optimal quadratic filter of the original state is obtained. As a particular case, the proposed results are applied to multi-sensor systems with state-dependent multiplicative noise and fading measurements and, finally, a numerical simulation example illustrates the performance of the proposed quadratic filter in comparison with the linear one and also with other filters in the existing literature.

Keywords: Random parameter matrices, least-squares quadratic estimation, fading measurements, innovation approach, recursive filter

1. Introduction

A basic assumption in classical estimation theory for linear stochastic systems is the knowledge of the model parameter matrices; also, the addi-

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tive noises and the initial state are assumed to be Gaussian and mutually independent. As it is well known, under these conditions, the systems are Gaussian and the Kalman filter provides the conditional expectation of the state given the observations, that is, the optimal least-squares (LS) estimator. However, there exists a considerable number of situations in which the joint distribution of the state and the observations is not Gaussian and the Kalman filter provides only the linear LS estimator. In these cases, the optimal LS estimator is not a linear function of the observations and, generally, it is not easy to be obtained; this fact has motivated the necessity of looking for suboptimal estimators which are computationally easier, such as linear estimators or, more generally, polynomial estimators.

In systems where the usual assumption of Gaussian noises must be removed in order to obtain a more realistic statistical description of the random processes involved, De Santis et *al.* [1] were the first to obtain a recursive algorithm for the quadratic LS filter, by improving the widely used linear filter. A more general study is carried out in [2], where the arbitrary-order polynomial LS estimation problem is addressed.

Systems with multiplicative noises in the state and/or observation equations constitute another kind of non-Gaussian systems in which the Kalman filter does not provide the optimal LS estimator and, hence, it is necessary to look for suboptimal estimators. This class of systems has been receiving great attention in the last years, mainly due to the fact that this kind of formulation arises in many applications, as image processing problems and communication systems. Therefore, under different hypotheses and performance criterions, the study of the linear LS estimation problem in systems with random multiplicative noises has become an active research area in the last years (see e.g. [3]-[5], and reference therein).

Because of its important applications, it is worth noting especially some classes of systems where the influence of multiplicative noises affects only the measurements of the model; for example, in cases where there are intermittent failures in the observation mechanism, fading phenomena in propagation channels, accidental loss of some measurements, or data inaccessibility during certain times. This kind of systems, named systems with uncertain observations or missing measurements, are modeled including in the observation equation, besides the additive noise, a multiplicative noise component described by a sequence of Bernoulli random variables. Under different hypotheses on the Bernoulli variables and the additive noises involved in the system equations, the linear and polynomial estimation problems have been widely studied in such systems (see e.g. [6]-[12], and references therein). Recently, this missing measurement model, described by Bernoulli variables, has been generalized considering any random variables with arbitrary probability distribution over the interval $[0, 1]$, which allows us to cover some practical applications where only partial information is missing. In this situation, considering also different assumptions on the system noises, the linear LS estimation problem has been treated in [13] and [14].

The above-mentioned systems are a special case of systems with random parameter matrices which clearly are non-Gaussian systems, even under the assumption that the additive noises are Gaussian. Also, systems with random delays and packet dropouts can be transformed into an equivalent stochastic parameterized system (see [15]-[17], among others). Due to the numerous realistic situations and practical applications in which both state transition and measurement are random parameter matrices, such as digital control of chemical processes, systems with human operators, economic systems, and stochastically sampled digital control systems (see e.g. [18]-[21], among others), the linear estimation problem in this type of systems has gained significant research interest in recent years (see e.g. [13], [22]-[24] and references therein). Considering scalar measurements with random observation matrices, the quadratic LS filtering problem has been addressed in [25] by applying the Kalman filter to a suitably augmented system with deterministic observation matrices.

Despite the importance of this kind of systems and the significant improvement that the quadratic LS estimators provide over the linear ones, to the best of the authors knowledge, the quadratic LS estimation problem in systems with both random parameter state transition and measurement matrices has not been investigated. *This paper makes the following contributions: (1) random parameter matrices in both the state and observation equations are considered simultaneously in the system state-space model, thus providing a unified framework to treat some classes of uncertainties, such as multiplicative noises or missing and fading measurements, and, hence, the proposed quadratic LS filter outperforms the linear LS estimators derived in the existing literature for systems with such uncertainties; (2) unlike [25], where deterministic state transition matrices and scalar measurements are assumed, we consider random state transition matrices and multidimensional* *observations; hence, the proposed estimators can be applied to multi-sensor systems and, furthermore, different uncertainty characteristics in the sensors can be considered; specifically, an application to multi-sensor systems with state-dependent multiplicative noise and fading measurements is presented; (3) also, unlike [25], the proposed quadratic filtering algorithm is obtained without requiring the original system transformation into one with deterministic observation matrices.*

The rest of the paper is organized as follows. In Section 2, we present the system model with random parameter matrices to be considered and the assumptions and properties under which the quadratic LS estimation problem is addressed. The augmented system is constructed in Section 3 using the technique proposed by [1], consisting of augmenting the state and observation vectors with their second-order Kronecker powers. Also, in this section, the statistical properties of the augmented processes are analyzed. The proposed methodology reduces the quadratic estimation problem to the linear estimation problem in the augmented system, and the recursive algorithm for the linear LS filter of the augmented state is derived in Section 4. The application to multi-sensor systems with state-dependent multiplicative noise and fading measurements, together with a numerical simulation example which shows the effectiveness of the proposed quadratic estimators in contrast to the linear ones are both presented in Section 5. Finally, some conclusions are drawn in Section 6.

Notation: The notation used throughout this paper is standard. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are the n-dimensional Euclidean space and the set of all $m \times n$ real matrices, respectively. For any matrix A, A^T and A^{-1} denote its transpose and inverse, respectively. The shorthand $Diag(A_1, \dots, A_m)$ denotes a block diagonal matrix whose diagonal blocks are the matrices $A_1, \dots, A_m$, and $[A_1 \cdots \mid A_m]$ represents a partitioned matrix into sub-matrices $A_1, \dots, A_m$. I and K denote the identity and commutation matrices, respectively, of appropriate dimensions. $A^{[2]} = A \otimes A$ where $\otimes$ denotes the Kronecker product. $vec(\cdot)$ stands for the 'vec' or 'stack' operator. $\delta_{k,s}$ is the Kronecker delta function and $\circ$ denotes the Hadamard product.

Moreover, for any random vector or matrix M, we denote $\overline{M} = E[M]$ and $\widetilde{M} = M - \overline{M}$, where $E[\cdot]$ stands for the expectation operator. For arbitrary random vectors β and γ, we denote $Cov[\beta, \gamma] = E[(\beta - E[\beta])\,(\gamma - E[\gamma])^T]$ and $Cov[\beta] = Cov[\beta, \beta]$.

2. Problem formulation

Consider a class of discrete-time linear stochastic systems and denote $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^r$ the state vector and its measurement at time k, respectively. The evolution of the state and its measurements are given by the following equations:

$$x_k = F_{k-1}x_{k-1} + w_{k-1}, \quad k \geq 1, \tag{1}$$
$$y_k = H_k x_k + v_k, \quad k \geq 1, \tag{2}$$

where $\{F_k\}_{k\geq 0}$ and $\{H_k\}_{k\geq 1}$ are sequences of random parameter matrices, $\{w_k\}_{k\geq 0}$ is the process noise and $\{v_k\}_{k\geq 1}$ is the measurement noise.

Our aim is to obtain the least-squares (LS) quadratic estimator of the state x_k based on the measurements $\{y_1, \dots, y_k\}$. As it is known, this estimator is its orthogonal projection onto the space of n-dimensional random variables obtained as linear transformations of $y_1, \dots, y_k$ and their second-order powers, $y_1^{[2]}, \dots, y_k^{[2]}$. To address the LS quadratic estimation problem, it is necessary that $E[y_i^{[2]T} y_i^{[2]}] < \infty$, and therefore, the fourth-order moments of vectors y_i, $i = 1, \dots, k$ must be finite. Specifically, the following assumptions are required:

(A1) The initial state x_0 is a random vector whose moments up to the fourth-order one are known. We will denote $\overline{x}_0 = E[x_0]$, $P_0 = Cov[x_0]$, $P_0^{(3)} = Cov[x_0, x_0^{[2]}]$ and $P_0^{(4)} = Cov[x_0^{[2]}]$.

(A2) $\{F_k\}_{k\geq 0}$ and $\{H_k\}_{k\geq 1}$ are sequences of independent random parameter matrices with known mean matrices $\overline{F}_k$ and $\overline{H}_k$. The covariances and cross-covariances between the entries of the matrices F_k and $F_k^{[2]}$, as well as between the entries of the matrices H_k and $H_k^{[2]}$, are also assumed to be known.

(A3) The noise processes $\{w_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 1}$ are zero-mean white sequences with known moments, up to the fourth-order ones. We will denote

$$Q_k = Cov[w_k], \quad Q_k^{(3)} = Cov[w_k, w_k^{[2]}], \quad Q_k^{(4)} = Cov[w_k^{[2]}],$$
$$R_k = Cov[v_k], \quad R_k^{(3)} = Cov[v_k, v_k^{[2]}], \quad R_k^{(4)} = Cov[v_k^{[2]}].$$

(A4) The initial state x_0, the random parameter matrices $\{F_k\}_{k\geq 0}$, $\{H_k\}_{k\geq 1}$ and the processes $\{w_k\}_{k\geq 0}$, $\{v_k\}_{k\geq 1}$ are mutually independent.

Note that, usually, and without loss of generality, the initial state, x_0, the process noise, $\{w_k\}_{k\geq 0}$, and the observation noise, $\{v_k\}_{k\geq 1}$, are assumed to be white gaussian noises; in such situations, the existence of fourth-order moments required in (A1) and (A3) is clearly satisfied. Also, under the general assumption of non-Gaussian noises, they usually take a finite number of values and their probability distributions are assumed to be known, so that the moments up to the fourth order can be computed, as required by assumptions (A1) and (A3). The conditions established in (A2) are also satisfied in most common applications of systems with random parameter matrices; for example, the multiplicative noises involved in uncertain systems are usually gaussian noises, and missing measurements or random delays and packet dropouts are usually modelled by Bernoulli processes, and hence the means, covariances and cross-covariances required in (A2) can be calculated.

Remark 1. Hereafter, it will be necessary to calculate different expectations associated with the random parameter matrices F_k, $F_k^{[2]}$, H_k and $H_k^{[2]}$. For this purpose, the following property is used:

Let $A = \Big(a_{ij}\Big)_{\substack{i=1,\dots,N_1\\ j=1,\dots,N_2}}$ *and* $B = \Big(b_{ij}\Big)_{\substack{i=1,\dots,M_1\\ j=1,\dots,M_2}}$ *be random parameter matrices, then for any deterministic matrix* $C = \Big(c_{ij}\Big)_{\substack{i=1,\dots,N_2\\ j=1,\dots,M_2}}$, *the* (p,q)*-th entry of the matrix* $E[\widetilde{A}C\widetilde{B}^T]$ *is given by*

$$\Big(E[\widetilde{A}C\widetilde{B}^T]\Big)_{pq} = \sum_{i=1}^{N_2}\sum_{j=1}^{M_2} Cov(a_{pi}, b_{qj})c_{ij}, \quad p = 1,\dots,N_1,\ q = 1,\dots,M_1. \tag{3}$$

3. Quadratic estimation problem statement

Given the system model (1)-(2) under assumptions (A1)-(A4), the problem at hand is to find the LS quadratic estimator, $\widehat{x}^q_{k/k}$, of the state x_k based on the measurements until time k. For this purpose, the following augmented state and measurement vectors are defined by assembling the original vectors and their second-order Kronecker powers:

$$\mathcal{X}_k = \begin{pmatrix} x_k \\ x_k^{[2]} \end{pmatrix}, \quad \mathcal{Y}_k = \begin{pmatrix} y_k \\ y_k^{[2]} \end{pmatrix}.$$

Since the space of linear transformations of $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ is equal to the space of linear transformations of $y_1, \ldots, y_k$ and $y_1^{[2]}, \ldots, y_k^{[2]}$, the LS quadratic filter, $\widehat{x}^q_{k/k}$, is the LS linear estimator of x_k based on $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$. This estimator is obtained by extracting the first n entries of the LS linear estimator of $\mathcal{X}_k$ based on $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$. Therefore, the quadratic estimation problem for the original state is reduced to the linear estimation problem for the augmented state.

In order to address the LS linear estimation problem of the augmented state based on the augmented measurements, the evolution of the vectors $\mathcal{X}_k$ and $\mathcal{Y}_k$ is analyzed. Using the Kronecker product properties [26], the evolution of the second-order powers, $x_k^{[2]}$ and $y_k^{[2]}$, is given by (see [1]):

$$
\begin{aligned}
x_k^{[2]} &= F_{k-1}^{[2]} x_{k-1}^{[2]} + \Phi_{k-1}, \quad k \geq 1, \\
y_k^{[2]} &= H_k^{[2]} x_k^{[2]} + \Psi_k, \quad k \geq 1,
\end{aligned}
$$

where $\Phi_k = (I+K)((F_k x_k) \otimes w_k) + w_k^{[2]}$ and $\Psi_k = (I+K)((H_k x_k) \otimes v_k) + v_k^{[2]}$.

Then, the augmented vectors $\mathcal{X}_k$ and $\mathcal{Y}_k$ satisfy the following equations:

$$
\begin{aligned}
\mathcal{X}_k &= \mathcal{F}_{k-1} \mathcal{X}_{k-1} + \mathcal{W}_{k-1}, \quad k \geq 1, \\
\mathcal{Y}_k &= \mathcal{H}_k \mathcal{X}_k + \mathcal{V}_k, \quad k \geq 1,
\end{aligned}
$$

where

$$
\mathcal{F}_k = Diag(F_k,\, F_k^{[2]}), \quad \mathcal{H}_k = Diag(H_k,\, H_k^{[2]}), \quad \mathcal{W}_k = \begin{pmatrix} w_k \\ \Phi_k \end{pmatrix}, \quad \mathcal{V}_k = \begin{pmatrix} v_k \\ \Psi_k \end{pmatrix}.
$$

For simplicity, since the additive noises of this new model, $\mathcal{W}_k$ and $\mathcal{V}_k$, are non-zero mean vectors, with

$$
\overline{\mathcal{W}}_k = \begin{pmatrix} 0 \\ vec(Q_k) \end{pmatrix}, \quad \overline{\mathcal{V}}_k = \begin{pmatrix} 0 \\ vec(R_k) \end{pmatrix},
$$

the above equations are rewritten in terms of the centered augmented vectors, $X_k = \mathcal{X}_k - \overline{\mathcal{X}}_k$ and $Y_k = \mathcal{Y}_k - \overline{\mathcal{Y}}_k$, obtaining the following augmented system:

$$
X_k = \mathcal{F}_{k-1} X_{k-1} + W_{k-1}, \quad k \geq 1, \tag{4}
$$

$$
Y_k = \mathcal{H}_k X_k + V_k, \quad k \geq 1, \tag{5}
$$

where $W_k = \widetilde{\mathcal{F}}_k \overline{\mathcal{X}}_k + \widetilde{\mathcal{W}}_k$ and $V_k = \widetilde{\mathcal{H}}_k \overline{\mathcal{X}}_k + \widetilde{\mathcal{V}}_k$, being $\overline{\mathcal{X}}_k = \begin{pmatrix} \overline{x}_k \\ vec(D_k) \end{pmatrix}$ with $D_k = E[x_k x_k^T]$.

Taking into account the state equation (1) and under assumptions (A1)-(A4), the mean vector $\overline{x}_k$ and the correlation matrix D_k are recursively calculated by

$$\overline{x}_k = \overline{F}_{k-1}\overline{x}_{k-1}, \quad k \geq 1,$$
$$D_k = \overline{F}_{k-1}D_{k-1}\overline{F}_{k-1}^T + E[\widetilde{F}_{k-1}D_{k-1}\widetilde{F}_{k-1}^T] + Q_{k-1},\ k \geq 1; \quad D_0 = P_0 + \overline{x}_0\overline{x}_0^T,$$

where the (p, q)-th entry of the matrix $E[\widetilde{F}_k D_k \widetilde{F}_k^T]$ is obtained as in (3).

It should be mentioned that the LS linear estimator of X_k based on $Y_1, \ldots, Y_k$ provides the LS linear estimator of $\mathcal{X}_k$ based on $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$, adding the mean vector $\overline{\mathcal{X}}_k$. Therefore, the required quadratic filter $\widehat{x}_{k/k}^q$ is obtained by adding the mean $\overline{x}_k$ to the vector constituted by the first n entries of the LS linear filter of X_k.

In order to obtain the LS linear filter of X_k, the properties of the processes involved in the system (4)-(5) are required.

Clearly, the initial state X_0 is a zero-mean random vector with covariance matrix given by

$$P_0^* = \begin{pmatrix} P_0 & P_0^{(3)} \\ P_0^{(3)T} & P_0^{(4)} \end{pmatrix}.$$

Moreover, it is easy to show that X_0 and $\{W_k\}_{k\geq 0}$, $\{V_k\}_{k\geq 1}$, $\{\mathcal{F}_k\}_{k\geq 0}$, $\{\mathcal{H}_k\}_{k\geq 1}$ are uncorrelated.

Next, the second-order statistical properties of the noise processes $\{W_k\}_{k\geq 0}$ and $\{V_k\}_{k\geq 1}$ are established in propositions 1 and 2, respectively.

Proposition 1. *The noise $\{W_k\}_{k\geq 0}$ is a zero-mean white process with covariance matrix, $E[W_k W_k^T] = Q_k^W$, given by*

$$Q_k^W = \begin{pmatrix} Q_k & Q_k^{12} \\ Q_k^{12T} & Q_k^{22} \end{pmatrix} + E[\widetilde{\mathcal{F}}_k \overline{\mathcal{X}}_k \overline{\mathcal{X}}_k^T \widetilde{\mathcal{F}}_k^T],$$

where

$$Q_k^{12} = ((\overline{F}_k\overline{x}_k)^T \otimes Q_k)(I+K) + Q_k^{(3)},$$
$$Q_k^{22} = (I+K)\left(\left(\overline{F}_k D_k \overline{F}_k^T + E[\widetilde{F}_k D_k \widetilde{F}_k^T]\right) \otimes Q_k\right)(I+K) + Q_k^{(4)}$$
$$+ (I+K)\left((\overline{F}_k\overline{x}_k) \otimes Q_k^{(3)}\right) + \left((\overline{F}_k\overline{x}_k) \otimes Q_k^{(3)}\right)^T (I+K)$$

and

$$E[\widetilde{\mathcal{F}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{F}}_k^T] = \begin{pmatrix} E[\widetilde{F}_k\overline{x}_k\overline{x}_k^T\widetilde{F}_k^T] & E[\widetilde{F}_k\overline{x}_k vec(D_k)^T\widetilde{F}_k^{[2]T}] \\ E[\widetilde{F}_k^{[2]} vec(D_k)\overline{x}_k^T\widetilde{F}_k^T] & E[\widetilde{F}_k^{[2]} vec(D_k) vec(D_k)^T\widetilde{F}_k^{[2]T}] \end{pmatrix},$$

whose blocks are calculated as in (3)*.*

Proof. Clearly, $\forall k \geq 0$, $E[W_k] = 0$. Now, taking into account the mutual independence between $\{w_k\}_{k\geq 0}$, $\{F_k\}_{k\geq 0}$ and the initial state x_0, it is easy to prove that $E[\widetilde{\mathcal{W}}_k\overline{\mathcal{X}}_s^T\widetilde{\mathcal{F}}_s^T] = 0$, $E[\widetilde{\mathcal{F}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_s^T\widetilde{\mathcal{F}}_s^T] = E[\widetilde{\mathcal{F}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{F}}_k^T]\delta_{k,s}$, $\forall k, s \geq 0$, and, hence

$$E[W_k W_s^T] = E[\widetilde{\mathcal{W}}_k\widetilde{\mathcal{W}}_s^T] + E[\widetilde{\mathcal{F}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{F}}_k^T]\delta_{k,s}.$$

Then, we only need to prove that $\forall k, s \geq 0$,

$$E[\widetilde{\mathcal{W}}_k\widetilde{\mathcal{W}}_s^T] := \begin{pmatrix} Q_{k,s}^{11} & Q_{k,s}^{12} \\ Q_{k,s}^{12T} & Q_{k,s}^{22} \end{pmatrix} = \begin{pmatrix} Q_k & Q_k^{12} \\ Q_k^{12T} & Q_k^{22} \end{pmatrix}\delta_{k,s}.$$

- Since $\{w_k\}_{k\geq 0}$ is a zero-mean white sequence with covariances $Q_k, \forall k \geq 0$, it is immediately clear that $Q_{k,s}^{11} = E[w_k w_s^T] = Q_k\delta_{k,s}$.
- Using the Kronecker product properties, Assumption (A3) and since $E[F_k x_k] = \overline{F}_k\overline{x}_k$, it is easy to obtain that $Q_{k,s}^{12} = E[w_k\Phi_s^T] = Q_k^{12}\delta_{k,s}$.
- From the conditional expectation properties we have $E\left[F_k x_k x_k^T F_k^T\right] = \overline{F}_k D_k \overline{F}_k^T + E[\widetilde{F}_k D_k \widetilde{F}_k^T]$, then, using again Assumption (A3) and the Kronecker product properties, we get

$$Q_{k,s}^{22} = E[(\Phi_k - vec(Q_k))(\Phi_s - vec(Q_s))^T] = Q_k^{22}\delta_{k,s}.$$

□

Proposition 2. *The noise* $\{V_k\}_{k\geq 1}$ *is a zero-mean white process with covariance matrix,* $E[V_kV_k^T] = R_k^V$*, given by*

$$R_k^V = \begin{pmatrix} R_k & R_k^{12} \\ R_k^{12T} & R_k^{22} \end{pmatrix} + E[\widetilde{\mathcal{H}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{H}}_k^T],$$

where

$$\begin{aligned} R_k^{12} &= \left((\overline{H}_k\overline{x}_k)^T \otimes R_k\right)(I+K) + R_k^{(3)}, \\ R_k^{22} &= (I+K)\left(\left(\overline{H}_kD_k\overline{H}_k^T + E[\widetilde{H}_kD_k\widetilde{H}_k^T]\right)\otimes R_k\right)(I+K) + R_k^{(4)} \\ &\quad + (I+K)\left((\overline{H}_k\overline{x}_k)\otimes R_k^{(3)}\right) + \left((\overline{H}_k\overline{x}_k)\otimes R_k^{(3)}\right)^T(I+K) \end{aligned}$$

and

$$E[\widetilde{\mathcal{H}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{H}}_k^T] = \begin{pmatrix} E[\widetilde{H}_k\overline{x}_k\overline{x}_k^T\widetilde{H}_k^T] & E[\widetilde{H}_k\overline{x}_k vec(D_k)^T\widetilde{H}_k^{[2]T}] \\ \\ E[\widetilde{H}_k^{[2]}vec(D_k)\overline{x}_k^T\widetilde{H}_k^T] & E[\widetilde{H}_k^{[2]}vec(D_k)vec(D_k)^T\widetilde{H}_k^{[2]T}] \end{pmatrix},$$

whose blocks are calculated as in (3)*.*

Proof. This proof is analogous to that of Proposition 1 and, hence, it is omitted. □

Remark 2. From the augmented state equation (4) and Proposition 1, the following recursive equation for the matrix $\mathcal{D}_k = E[X_kX_k^T]$ holds:

$$\mathcal{D}_k = \overline{\mathcal{F}}_{k-1}\mathcal{D}_{k-1}\overline{\mathcal{F}}_{k-1}^T + E[\widetilde{\mathcal{F}}_{k-1}\mathcal{D}_{k-1}\widetilde{\mathcal{F}}_{k-1}^T] + Q_{k-1}^W, \quad k\geq 1; \quad \mathcal{D}_0 = P_0^*, \tag{6}$$

where

$$E[\widetilde{\mathcal{F}}_k\mathcal{D}_k\widetilde{\mathcal{F}}_k^T] = \begin{pmatrix} E[\widetilde{F}_kD_k\widetilde{F}_k^T] & E[\widetilde{F}_kD_k^{(3)}\widetilde{F}_k^{[2]T}] \\ \\ E[\widetilde{F}_k^{[2]}D_k^{(3)T}\widetilde{F}_k^T] & E[\widetilde{F}_k^{[2]}D_k^{(4)}\widetilde{F}_k^{[2]T}] \end{pmatrix},$$

with $D_k^{(3)} = E[x_kx_k^{[2]T}]$ and $D_k^{(4)} = E[x_k^{[2]}x_k^{[2]T}]$ the blocks of the matrix $\mathcal{D}_k$.

4. LS quadratic estimator

To address the LS linear estimation problem of X_k based on $Y_1,\ldots,Y_k$, an innovation approach is used. Since the measurements are non-orthogonal vectors, this procedure consists of transforming the measurement process $\{Y_k;\ k \geq 1\}$ into an equivalent one of orthogonal vectors $\{\nu_k;\ k \geq 1\}$ called *innovations*. The innovation at time k is defined as $\nu_k = Y_k - \widehat{Y}_{k/k-1}$, where $\widehat{Y}_{k/k-1}$ is the one-stage linear predictor of Y_k. Therefore, the LS linear filter of the augmented state, $\widehat{X}_{k/k}$, can be calculated as a linear combination of the innovations, as follows:

$$\widehat{X}_{k/k} = \sum_{i=1}^{k} E[X_k \nu_i^T] \Pi_i^{-1} \nu_i, \quad k \geq 1, \tag{7}$$

where $\Pi_i = E[\nu_i \nu_i^T]$.

Next, a recursive algorithm for the optimal LS linear filter of the augmented state is derived.

Theorem 1. *The linear filter of the augmented state is recursively obtained by the following relation*

$$\widehat{X}_{k/k} = \widehat{X}_{k/k-1} + \mathcal{G}_k \Pi_k^{-1} \nu_k, \quad k \geq 1; \quad \widehat{X}_{0/0} = 0, \tag{8}$$

where the state predictor, $\widehat{X}_{k/k-1}$, is calculated by

$$\widehat{X}_{k/k-1} = \overline{\mathcal{F}}_{k-1} \widehat{X}_{k-1/k-1}, \quad k \geq 1. \tag{9}$$

The innovation, ν_k, satisfies

$$\nu_k = Y_k - \overline{\mathcal{H}}_k \widehat{X}_{k/k-1}, \quad k \geq 2; \quad \nu_1 = Y_1, \tag{10}$$

The matrix, $\mathcal{G}_k = E[X_k \nu_k^T]$ is determined by

$$\mathcal{G}_k = \Sigma_{k/k-1} \overline{\mathcal{H}}_k^T, \quad k \geq 1, \tag{11}$$

where the prediction error covariance matrix, $\Sigma_{k/k-1}$, is obtained by

$$\Sigma_{k/k-1} = \overline{\mathcal{F}}_{k-1} \Sigma_{k-1/k-1} \overline{\mathcal{F}}_{k-1}^T + E[\widetilde{\mathcal{F}}_{k-1} \mathcal{D}_{k-1} \widetilde{\mathcal{F}}_{k-1}^T] + Q_{k-1}^W, \quad k \geq 1, \tag{12}$$

with $\mathcal{D}_k$ given in (6) *and $\Sigma_{k/k}$, the filtering error covariance matrix, calculated by*

$$\Sigma_{k/k} = \Sigma_{k/k-1} - \mathcal{G}_k \Pi_k^{-1} \mathcal{G}_k^T, \quad k \geq 1; \quad \Sigma_{0/0} = P_0^*. \tag{13}$$

The innovation covariance matrix, Π_k*, satisfies*

$$\Pi_k = E[\widetilde{\mathcal{H}}_k \mathcal{D}_k \widetilde{\mathcal{H}}_k^T] + \overline{\mathcal{H}}_k \mathcal{G}_k + R_k^V, \quad k \geq 1, \tag{14}$$

where

$$E[\widetilde{\mathcal{H}}_k \mathcal{D}_k \widetilde{\mathcal{H}}_k^T] = \begin{pmatrix} E[\widetilde{H}_k D_k \widetilde{H}_k^T] & E[\widetilde{H}_k D_k^{(3)} \widetilde{H}_k^{[2]T}] \\ \\ E[\widetilde{H}_k^{[2]} D_k^{(3)T} \widetilde{H}_k^T] & E[\widetilde{H}_k^{[2]} D_k^{(4)} \widetilde{H}_k^{[2]T}] \end{pmatrix},$$

whose blocks are calculated as in (3)*.*

Proof. From expression (7), relation (8) for the state filter, $\widehat{X}_{k/k}$, in terms of the one-stage predictor, $\widehat{X}_{k/k-1}$, is directly derived.

Expression (9) for the state predictor is immediately obtained from (4) and the Orthogonal Projection Lemma (OPL).

Obtaining an explicit formula for the innovation, $\nu_k = Y_k - \widehat{Y}_{k/k-1}$, is equivalent to calculate $\widehat{Y}_{k/k-1}$, which, from (5) and using again the OPL, can be expressed as

$$\widehat{Y}_{k/k-1} = \overline{\mathcal{H}}_k \widehat{X}_{k/k-1}, \ k \geq 1. \tag{15}$$

Next, identity (11) for $\mathcal{G}_k$ is deduced. Applying the OPL, it is clear that

$$E[X_k \widehat{X}_{k/k-1}^T] = E[\widehat{X}_{k/k-1} \widehat{X}_{k/k-1}^T] = \mathcal{D}_k - \Sigma_{k/k-1}, \quad k \geq 1,$$

therefore,

$$\mathcal{G}_k = E[X_k \nu_k^T] = E[X_k (X_k - \widehat{X}_{k/k-1})^T] \mathcal{H}_k^T = \Sigma_{k/k-1} \overline{\mathcal{H}}_k^T, \quad k \geq 1.$$

Since $\Sigma_{k/k-1} = E[X_k X_k^T] - E[\widehat{X}_{k/k-1} \widehat{X}_{k/k-1}^T]$, using (6) for $E[X_k X_k^T]$ and (9) for $\widehat{X}_{k/k-1}$, expression (12) is easily deduced, taking into account that $E[\widehat{X}_{k-1/k-1} \widehat{X}_{k-1/k-1}^T] = \mathcal{D}_{k-1} - \Sigma_{k-1/k-1}$, $k \geq 1$.

Similarly, $\Sigma_{k/k} = E[X_k X_k^T] - E[\widehat{X}_{k/k} \widehat{X}_{k/k}^T]$ and, therefore, by using (8) for $\widehat{X}_{k/k}$, formula (13) is obtained.

Finally, we prove expression (14) for the innovation covariance matrix $\Pi_k = E[Y_k Y_k^T] - E[\widehat{Y}_{k/k-1} \widehat{Y}_{k/k-1}^T]$. On the one hand, from (5), we have

$$E[Y_k Y_k^T] = E[\mathcal{H}_k X_k X_k^T \mathcal{H}_k^T] + R_k^V$$

where, by considering the conditional expectation properties, it is satisfied that $E[\mathcal{H}_k X_k X_k^T \mathcal{H}_k^T] = \overline{\mathcal{H}}_k \mathcal{D}_k \overline{\mathcal{H}}_k^T + E[\widetilde{\mathcal{H}}_k \mathcal{D}_k \widetilde{\mathcal{H}}_k^T]$. On the other hand, using (15) and the OPL, it is deduced that $E[\widehat{Y}_{k/k-1} \widehat{Y}_{k/k-1}^T] = \overline{\mathcal{H}}_k(\mathcal{D}_k - \Sigma_{k/k-1})\overline{\mathcal{H}}_k^T$, $k \geq 1$. Then, the innovation covariance (14) is proved. □

Remark 3. As mentioned in Section 3, the LS quadratic filter, $\widehat{x}_{k/k}^q$, of the original state x_k is obtained by adding the mean $\overline{x}_k$ to the vector constituted by the first n entries of $\widehat{X}_{k/k}$; specifically,

$$\widehat{x}_{k/k}^q = [I \mid 0]\widehat{X}_{k/k} + \overline{x}_k, \quad k \geq 1.$$

Remark 4. In comparison with the linear filter, the computational cost of the quadratic filter is clearly higher, as the augmented state vector has greater dimension than the original state. Actually, the linear filter of the n-dimensional state vector has the computational order of magnitude $O(n^3)$, while the quadratic filter has the order of magnitude $O((n+n^2)^3)$ as it is obtained from the linear filter of the $(n+n^2)$-dimensional augmented state vector. Hence, the quadratic filter might have an expensive computational cost when the dimension of the original state is very high; nevertheless, this is compensated by the significant improvement that the quadratic estimators generally provide over the linear ones.

5. Application in multi-sensor systems with fading measurements

In this section, the optimal LS quadratic filter obtained in Section 4 is applied to linear discrete-time stochastic systems with fading measurements coming from multiple sensors. The phenomenon of measurement fading occurs in a random way and it is described by different sequences of scalar random variables with a certain probability distribution over the interval $[0, 1]$. Moreover, a simulation example is given to illustrate the effectiveness of the proposed recursive filtering algorithm.

5.1. Multi-sensor model and filtering algorithm

Consider the state equation given in (1) satisfying assumptions (A1)-(A3), and r sensors whose measurements of the state are described by the following observation equations:

$$y_k^i = \theta_k^i C_k^i x_k + v_k^i, \quad k \geq 1, \quad i = 1, 2, \dots, r, \tag{16}$$

where $y_k^i \in \mathbb{R}$, is the measured output provided by sensor i at the sampling time k, $\{C_k^i\}_{k\geq 1}$, are random parameter matrices with compatible dimensions, $\{v_k^i\}_{k\geq 1}$ are the measurement noises, and $\{\theta_k^i\}_{k\geq 1}$ are scalar random variables which model the fading phenomenon of the i-th sensor. In order to apply Theorem 1, the following assumptions of the noise processes and the random parameter matrices are considered:

(*i*) For $i = 1, 2, \dots, r$, the sensor additive noises, $\{v_k^i\}_{k\geq 1}$, are zero-mean white processes. By denoting $v_k = (v_k^1, \dots, v_k^r)^T$, it is supposed that its moments, up to the fourth-order one, are known.

(*ii*) For $i = 1, 2, \dots, r$, $\{C_k^i\}_{k\geq 1}$ are white sequences of random parameter matrices. By denoting $C_k = \left[C_k^{1T} \mid \cdots \mid C_k^{rT} \right]^T$, its mean, $\overline{C}_k$, is known and the covariances and cross-covariances between the entries of the matrices C_k and $C_k^{[2]}$, are also assumed to be known.

(*iii*) For $i = 1, 2, \dots, r$, the noises $\{\theta_k^i\}_{k\geq 1}$ are white sequences of scalar random variables over the interval $[0, 1]$. By denoting $\theta_k = (\theta_k^1, \dots, \theta_k^r)^T$, it is supposed that its moments up to the fourth one are known. We will denote

$$K_k^{\theta} = Cov[\theta_k], \quad K_k^{\theta(3)} = Cov[\theta_k, \theta_k^{[2]}], \quad K_k^{\theta(4)} = Cov[\theta_k^{[2]}].$$

(*iv*) x_0, $\{F_k\}_{k\geq 0}$, $\{\theta_k\}_{k\geq 1}$, $\{C_k\}_{k\geq 1}$, $\{w_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 1}$ are mutually independent.

The observation model (16) can be rewritten in a compact form as follows:

$$y_k = \Theta_k C_k x_k + v_k, \quad k \geq 1,$$

where $y_k = (y_k^1, \dots, y_k^r)^T$ is the measurement vector and $\Theta_k = Diag(\theta_k^1, \dots, \theta_k^r)$. Accordingly, this observation model is a particular case of (2) with $H_k = \Theta_k C_k$, and clearly verifies the assumptions given in Section 2.

The corresponding augmented measurement equation is given by

$$Y_k = \mathcal{T}_k \mathcal{C}_k X_k + V_k, \quad k \geq 1,$$

where $\mathcal{T}_k = Diag(\Theta_k, \Theta_k^{[2]})$, $\mathcal{C}_k = Diag(C_k, C_k^{[2]})$ and $V_k = (\mathcal{T}_k \widetilde{\mathcal{C}}_k + \widetilde{\mathcal{T}}_k \overline{\mathcal{C}}_k)\overline{\mathcal{X}}_k + \widetilde{\mathcal{V}}_k$. This measurement equation is a particular case of (5) with $\mathcal{H}_k = \mathcal{T}_k \mathcal{C}_k$, and it is immediately clear that $\widetilde{\mathcal{H}}_k = \mathcal{T}_k \widetilde{\mathcal{C}}_k + \widetilde{\mathcal{T}}_k \overline{\mathcal{C}}_k$.

By applying the Hadamard product properties, for any deterministic matrix $\mathcal{A} \in \mathbb{R}^{(n+n^2)\times(n+n^2)}$, it is easy to see that

$$E[(\mathcal{T}_k\widetilde{\mathcal{C}}_k + \widetilde{\mathcal{T}}_k\overline{\mathcal{C}}_k)\mathcal{A}(\mathcal{T}_k\widetilde{\mathcal{C}}_k + \widetilde{\mathcal{T}}_k\overline{\mathcal{C}}_k)^T] \\ = E[J_k^\theta J_k^{\theta T}] \circ E[\widetilde{\mathcal{C}}_k\mathcal{A}\widetilde{\mathcal{C}}_k^T] + Cov[J_k^\theta] \circ (\overline{\mathcal{C}}_k\mathcal{A}\overline{\mathcal{C}}_k^T), \quad k \geq 1,$$

where $J_k^\theta = \begin{pmatrix} \theta_k^T \\ \theta_k^{[2]T} \end{pmatrix}$ and $Cov[J_k^\theta] = \begin{pmatrix} K_k^\theta & K_k^{\theta(3)} \\ K_k^{\theta(3)T} & K_k^{\theta(4)} \end{pmatrix}$.

Hence, taking into account this property and Proposition 2, we obtain that the covariance matrix of the noise process $\{V_k\}_{k\geq 1}$ is given by

$$R_k^V = \begin{pmatrix} R_k & R_k^{12} \\ R_k^{12T} & R_k^{22} \end{pmatrix} + E[J_k^\theta J_k^{\theta T}] \circ E[\widetilde{\mathcal{C}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\widetilde{\mathcal{C}}_k^T] + Cov[J_k^\theta] \circ (\overline{\mathcal{C}}_k\overline{\mathcal{X}}_k\overline{\mathcal{X}}_k^T\overline{\mathcal{C}}_k^T)$$

where

$$R_k^{12} = \left((\overline{\Theta}_k\overline{C}_k\overline{x}_k)^T \otimes R_k\right)(I+K) + R_k^{(3)},$$
$$R_k^{22} = (I+K)\left(\left(E[\theta_k\theta_k^T] \circ \left(\overline{C}_kD_k\overline{C}_k^T + E[\widetilde{C}_kD_k\widetilde{C}_k^T]\right)\right) \otimes R_k\right)(I+K) + R_k^{(4)} \\ + (I+K)\left((\overline{\Theta}_k\overline{C}_k\overline{x}_k) \otimes R_k^{(3)}\right) + \left((\overline{\Theta}_k\overline{C}_k\overline{x}_k) \otimes R_k^{(3)}\right)^T (I+K).$$

Thus, starting from the linear filter $\widehat{X}_{k/k}$ given by (8) with $\widehat{X}_{k/k-1}$ the state predictor determined by (9), a recursive optimal linear filtering algorithm is obtained by calculating the innovation ν_k, its covariance matrix Π_k, and the matrix $\mathcal{G}_k$ as follows:

$$\nu_k = Y_k - \overline{\mathcal{T}}_k\overline{\mathcal{C}}_k\widehat{X}_{k/k-1}, \quad k \geq 2; \quad \nu_1 = Y_1,$$
$$\mathcal{G}_k = \Sigma_{k/k-1}\overline{\mathcal{C}}_k^T\overline{\mathcal{T}}_k, \quad k \geq 1,$$
$$\Pi_k = E[J_k^\theta J_k^{\theta T}] \circ E[\widetilde{\mathcal{C}}_k\mathcal{D}_k\widetilde{\mathcal{C}}_k^T] + Cov[J_k^\theta] \circ (\overline{\mathcal{C}}_k\mathcal{D}_k\overline{\mathcal{C}}_k^T) + \overline{\mathcal{T}}_k\overline{\mathcal{C}}_k\mathcal{G}_k + R_k^V, \quad k \geq 1,$$

with $\mathcal{D}_k$ and $\Sigma_{k/k-1}$ given in (6) and (12), respectively, and

$$E[\widetilde{\mathcal{C}}_k\mathcal{D}_k\widetilde{\mathcal{C}}_k^T] = \begin{pmatrix} E[\widetilde{C}_kD_k\widetilde{C}_k^T] & E[\widetilde{C}_kD_k^{(3)}\widetilde{C}_k^{[2]T}] \\ \\ E[\widetilde{C}_k^{[2]}D_k^{(3)T}\widetilde{C}_k^T] & E[\widetilde{C}_k^{[2]}D_k^{(4)}\widetilde{C}_k^{[2]T}] \end{pmatrix},$$

whose blocks are calculated as in (3).

As mentioned in the previous sections, the quadratic filter of the original state is formed by the first n entries of $\widehat{X}_{k/k}$ plus the mean $\overline{x}_k$.

5.2. Numerical simulation example

Consider the following uncertain discrete-time system with fading measurements coming from four sensors:

$$x_k = (0.95 + 0.1\epsilon_{k-1})x_{k-1} + w_{k-1}, \quad k \geq 1,$$
$$y_k^i = \theta_k^i C_k^i x_k + v_k^i, \quad k \geq 1, \quad i = 1, 2, 3, 4$$

where $\{\epsilon_k\}_{k\geq 0}$ is a zero-mean Gaussian white process with unit variance. $\{\theta_k^i\}_{k\geq 1}$, $i = 1, 2, 3, 4$, are independent sequences of discrete-time random variables with the following probability distributions over the interval $[0, 1]$:

- In the first sensor, $\{\theta_k^1\}_{k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) Bernoulli variables with $P[\theta_k^1 = 1] = p^{(1)}$, $\forall k \geq 1$.
- In the second sensor, $\{\theta_k^2\}_{k\geq 1}$ is a sequence of i.i.d. random variables with $P[\theta_k^2 = 0] = 0.2$, $P[\theta_k^2 = 0.5] = 0.6$, $P[\theta_k^2 = 1] = 0.2$, $\forall k \geq 1$.
- In the third sensor, $\{\theta_k^3\}_{k\geq 1}$ is a sequence of i.i.d. random variables uniformly distributed over $[0.3, 0.7]$.
- In the fourth sensor, $\{\theta_k^4\}_{k\geq 1}$ is a sequence of i.i.d. Bernoulli variables with $P[\theta_k^4 = 1] = p^{(4)}$, $\forall k \geq 1$.

The matrices C_k^i, $i = 1, 2, 3, 4$, are defined as $C_k^1 = 0.5 + 0.4\zeta_k^1$, $C_k^2 = 0.6 + 0.4\zeta_k^2$, $C_k^3 = 0.82$ and $C_k^4 = 0.74$, where $\{\zeta_k^i\}_{k\geq 1}$, $i = 1, 2$, are independent zero-mean Gaussian white processes with unit variance. The initial state x_0 is a zero-mean Gaussian variable with $P_0 = 1$. The noise $\{w_k\}_{k\geq 0}$ is a zero-mean Gaussian white process with variance $Q_k = 0.1$, for all k, and $\{v_k^i\}_{k\geq 1}$, $i = 1, 2, 3, 4$, are independent zero-mean white processes with the following probability distributions:

$$P[v_k^1 = -8] = \frac{1}{8}, \quad P[v_k^1 = \frac{8}{7}] = \frac{7}{8}, \quad \forall k \geq 1,$$
$$P[v_k^2 = 1] = \frac{15}{18}, \quad P[v_k^2 = -3] = \frac{2}{18}, \quad P[v_k^2 = -9] = \frac{1}{18}, \quad \forall k \geq 1,$$
$$P[v_k^3 = -1] = \frac{15}{18}, \quad P[v_k^3 = 3] = \frac{2}{18}, \quad P[v_k^3 = 9] = \frac{1}{18}, \quad \forall k \geq 1,$$
$$P[v_k^4 = -\frac{2}{5}] = \frac{9}{10}, \quad P[v_k^4 = \frac{18}{5}] = \frac{1}{10}, \quad \forall k \geq 1.$$

To analyze the performance of the proposed quadratic estimator, we ran a program in MATLAB, in which one hundred iterations of the linear filtering algorithm ([23]) and the proposed quadratic filtering algorithm have been carried out, considering different values of the probabilities that the state x_k is present in the measurements of the first and fourth sensors, $p^{(i)}$, $i = 1, 4$. Linear and quadratic filters of the state are calculated, as well as the corresponding error variances, which provide a measure of the estimation accuracy.

Firstly, for $p^{(1)} = 0.7$ and $p^{(4)} = 0.5$, the performance of the linear and quadratic filtering estimators has been compared in Figure 1 on the basis of the estimates obtained from the corresponding simulated observations of the state. From this figure, it is deduced that the quadratic filter follows the state evolution better than the linear one.

Next, to analyze the effectiveness of the proposed quadratic filter and show the improvement that this estimator provides over the linear one, the linear and quadratic filtering error variances have been calculated for different values of $p^{(1)}$ and $p^{(4)}$. The results are displayed in Figure 2; specifically, the error variances for $p^{(4)} = 0.5$ and different values $p^{(1)} = 0.2$, 0.5, 0.7, 0.9 are shown in Figure 2a, and for $p^{(1)} = 0.5$ and different values $p^{(4)} = 0.2$, 0.5, 0.7, 0.9 are presented in Figure 2b. From this figures it is observed that:

i) For each fixed value of $p^{(1)}$ and $p^{(4)}$, the error variances of the quadratic filter are significantly less than those of each linear filter and, consequently, the quadratic filter outperforms the linear one.

ii) As $p^{(1)}$ or $p^{(4)}$ increases, the filtering error variances are smaller and, therefore, better estimations are obtained.

Analogous results are obtained for other values of the probabilities $p^{(1)}$ and $p^{(4)}$ of the Bernoulli random variables which model uncertainties of the first and fourth sensors, respectively. More generally, we study the linear and quadratic filtering accuracy in function of $p^{(1)}$ and $p^{(4)}$. Specifically, the filter performances are analyzed when $p^{(4)}$ is varied from 0.1 to 0.9 and values of $p^{(1)}$ from 0.1 to 0.9. Taking into account that the filtering error variances have insignificant variation from a certain iteration onwards, Figure 3 shows the linear and quadratic filtering error variances at a fixed iteration (namely, $k = 100$). From this figure it is gathered that the filtering error variances become smaller, and hence better estimations are obtained, as $p^{(1)}$ or $p^{(4)}$ increases. Also, for all the different values of $p^{(1)}$ and $p^{(4)}$, the quadratic filtering error variances decrease more quickly than those of the linear filter and their values are smaller. Actually, the highest error variances of the quadratic filter (obtained for $p^{(i)} = 0.1$, $i = 1, 4$) are smaller than the lowest error variances of the linear filter (obtained for $p^{(i)} = 0.9$, $i = 1, 4$), thus confirming again that the quadratic filtering estimators outperform the linear ones significantly.

Finally, a comparative analysis is presented between the proposed filter and the quadratic filter for multi-sensor systems with uncertain observations [9]. The comparison between these estimators is addressed based on the filtering accumulative mean-square error (AMSE) together with its corresponding filtering mean-square error (MSE) at each time instant k, which are calculated from one thousand independent simulations of the mentioned algorithms considering the probabilities $p^{(1)} = 0.7$ and $p^{(4)} = 0.5$. The AMSE at time k is defined as $\mathrm{AMSE}_k = \sum_{i=1}^{k} \mathrm{MSE}_i$, $k = 1, \ldots, 100$, where the MSE at time k is calculated as $\mathrm{MSE}_k = \frac{1}{1000} \sum_{s=1}^{1000} (x_k^{(s)} - \widehat{x}_{k/k}^{(s)})^2$, with $\{x_k^{(s)}\}_{1 \leq k \leq 100}$ the s-th set of artificially simulated data and $\widehat{x}_{k/k}^{(s)}$ the filter at the sampling time k in the s-th simulation run.

The results of these comparisons are displayed in Figure 4, from which it is observed that both the AMSE and the MSE of the proposed quadratic filters are smaller than those of the quadratic filter in [9]. This shows a better performance of the quadratic filter proposed in this paper over the quadratic filter in [9]; this fact was expected, since the latter ignores the randomness in the parameter matrices.

6. Conclusions

The LS quadratic estimation problem has been investigated for discrete-time linear stochastic systems with random parameter matrices. The main contributions are summarized as follows:

- Using the technique proposed in [1], consisting of augmenting the state and observation vectors with their second-order Kronecker powers, an augmented system with random parameter matrices has been constructed and the quadratic LS filter of the original state is derived from the linear LS filter of the augmented state. The proposed scheme has the following advantages: 1) the filter does not require any transformation of the original system into one with deterministic parameter matrices and can be applied to multi-sensor stochastic uncertain systems considering the possibility of different uncertainties in the observations at each sensor; 2) the filter is globally optimal in the quadratic LS sense and hence outperforms the linear LS estimators for such systems; 3) since the quadratic filter of the original state is derived from the linear filter of the augmented state, its structure is recursive, very simple computationally and suitable for online applications; 4) the linear filtering algorithm has been obtained by an innovation approach, which simplifies substantially the derivation of the algorithm since the innovation process is a white noise.

- The proposed quadratic filter has been applied to systems with fading measurements coming from multiple sensors, when the fading measurement phenomenon in each sensor is described by different sequences of scalar random variables with arbitrary probability distribution over the interval [0,1]. This kind of multi-sensor systems is found in various real-world situations, such as transmission models involving partial loss of measurements.

- The usefulness of the proposed results has been illustrated by a numerical simulation example. Error variance comparison has shown that the quadratic filters outperform the linear ones. Furthermore, a comparative analysis with other linear and quadratic filters that have been reported reveals the superior performance of the proposed quadratic filter. This example has also highlighted the applicability of the proposed algorithm in multi-sensor systems with state-dependent multiplicative noise and fading measurements, which can be addressed by the system model with random parameter matrices considered in this paper.

Noise independence assumptions might be too restrictive in many real-world problems, so they have been relaxed or even removed in several recent studies; for example, the linear estimation problem is addressed in [27] for stochastic uncertain systems with correlated noises and uncertainties caused by correlated multiplicative noises in the state and observation equations.

Hence, a challenging further research topic is to address the quadratic estimation problem for systems with random parameter matrices considering auto-correlation and cross-correlation between the process noise and the measurement noises. Also, an interesting future research topic is to generalize the current results by considering correlation between random state transition and measurement matrices, which would cover the uncertain systems considered in [27], and also systems with randomly delayed measurements or multiple packet dropouts as particular cases.

Acknowledgments

This research is supported by Ministerio de Economía y Competitividad (Grant No. MTM2014-52291-P and FPU programme).

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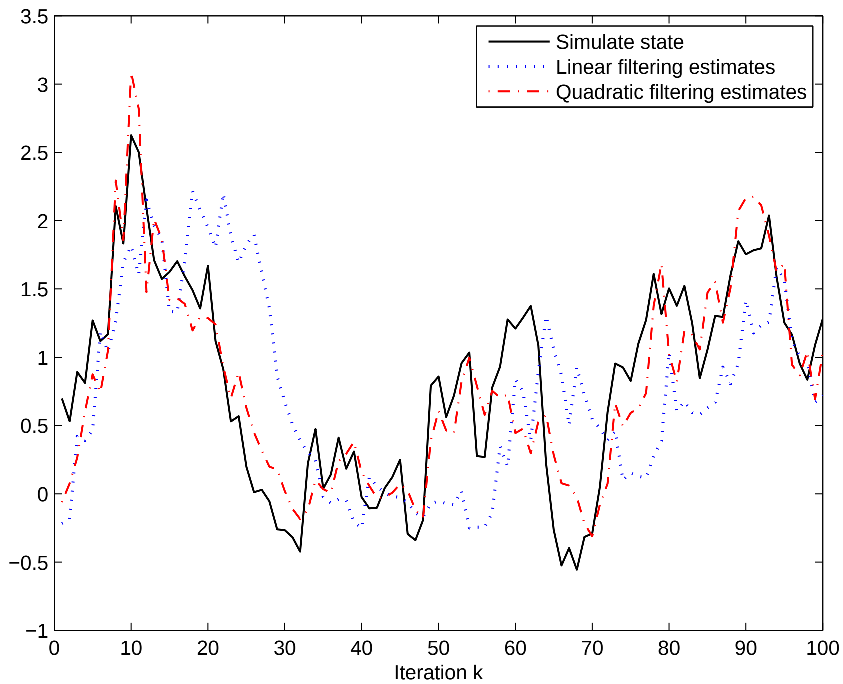
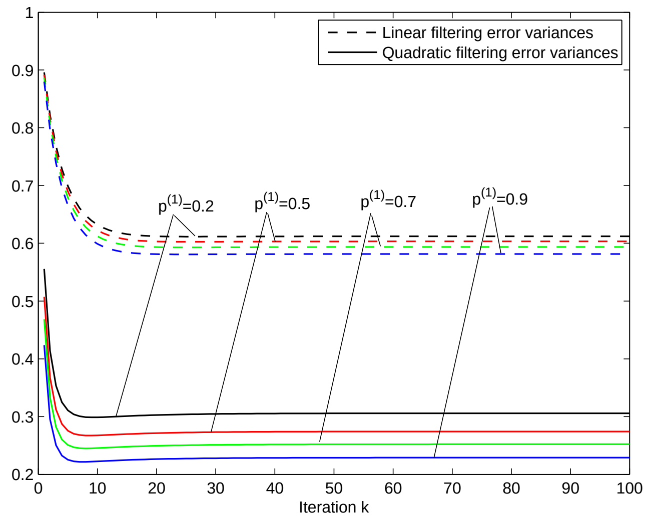
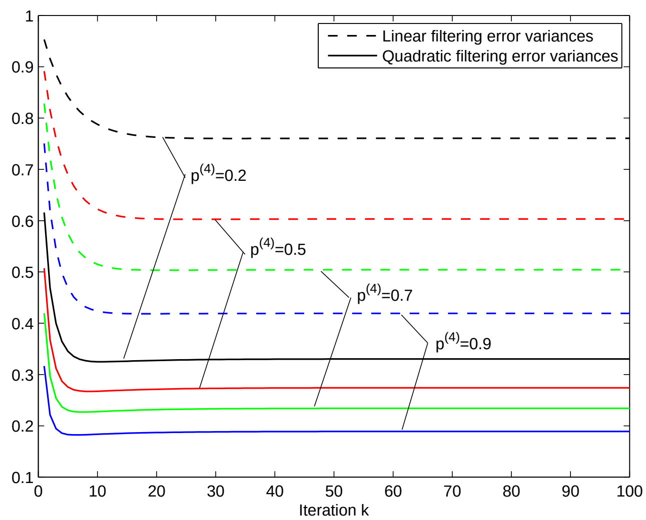


Figure 1: Simulated state and linear and quadratic filtering estimates.



(a)



(b)

Figure 2: (a) Linear and quadratic filtering error variances for $p^{(4)} = 0.5$, $p^{(1)} = 0.2, 0.5, 0.7, 0.9$.
(b) Linear and quadratic filtering error variances for $p^{(1)} = 0.5$, $p^{(4)} = 0.2, 0.5, 0.7, 0.9$.

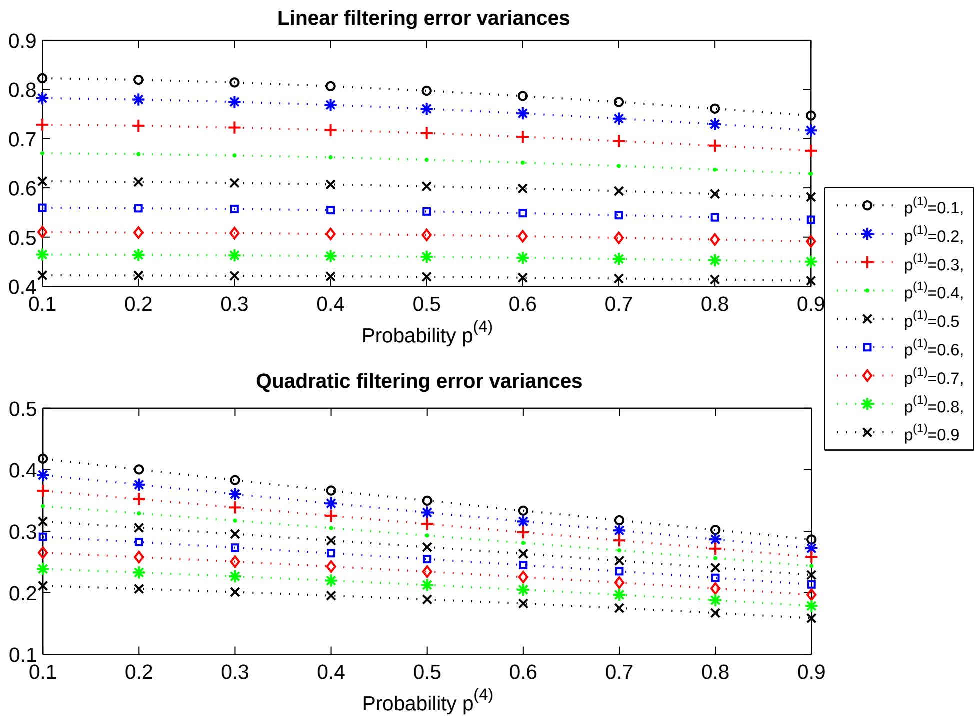
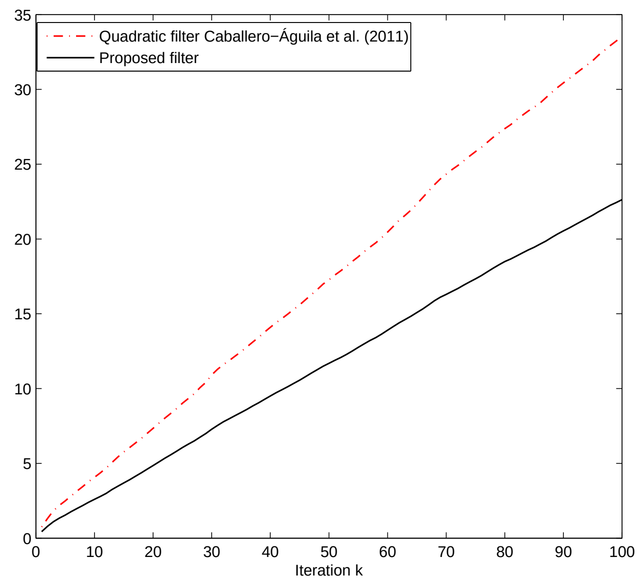
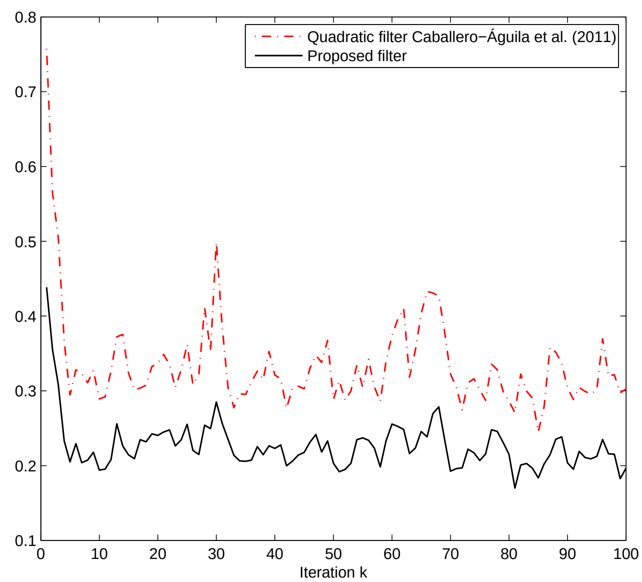


Figure 3: Linear and quadratic filtering error variances at $k = 100$ versus $p^{(4)}$ with $p^{(1)}$ varying from 0.1 to 0.9.



(a)



(b)

Figure 4: (a) Comparison of AMSE_k for two filters. (b) Comparison of MSE_k for two filters.