MEANINGS GIVEN TO ALGEBRAIC SYMBOLISM IN PROBLEM POSING

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ABSTRACT

Some errors in the learning of algebra suggest students have difficulties giving meaning to algebraic symbolism. In this paper, we use problem posing in order to analyze the students' capacity to assign meaning to algebraic symbolism and the difficulties that students encounter in this process depending on the characteristics of the algebraic statements given. We designed a written questionnaire composed of eight closed algebraic statements expressed symbolically, which was administered to 55 students who had finished their compulsory education and that had some previous experience in problem posing. In our analysis of the data, we examine both syntactic and semantic structures of the problem posed. We note that in most cases students posed problems with syntactic structures different to those given. They did not include computations within variables, and changed the kinds of relationships connecting variables. Students easily posed problems for statements with additive structures. Other differences in the type of problems posed depend on the characteristics of the given statements.

Keywords: algebraic symbolism; problem posing; semantic and syntactic structures

INTRODUCTION

The dimension of algebra as language or representation system is one of the most recognized in mathematics and in other disciplines wherein problems can be solved using mathematical models (e.g., physics, engineering, and economics) (Kieran, 2006; Molina, 2009). In this paper, we focus on one of the main components of this dimension: algebraic symbolism.

Many curricula for secondary education tend to prioritize algebraic symbolism over others representations (Bossé, AduGyamfi, & Cheetham, 2011a, b). As a consequence, significant time and effort are devoted to provide meanings and to develop management of algebraic symbolism in compulsory education in many countries (e.g., Ministerio de Educación y Ciencia, 2015; National Council of Teachers of Mathematics, 2000). In most of them, by the end of secondary education students are expected to be able to move between algebraic symbolism and verbal representations (in both directions), among other representation systems. This capacity is linked to a good understanding of the represented concepts (Kaput, 1987; Van Harpen & Presmeg, 2013). From an international perspective, the PISA study (OECD, 2016) highlights the relevance of algebraic symbolism, using it as an indicator of the development of representation competence. Different representations —including algebraic symbolism—, translations between representations, and their connection to real-world situations are descriptors used to describe students' performance through different levels.

Despite of the importance given to algebraic symbolism internationally, many authors stress the limited mastery students' show of this representation system and question the comprehension that students develop (e.g., Fernández-Millán & Molina, 2016; Kieran, 2007). Many studies show frequent and persistent errors made by secondary and even university students when using algebraic symbolism (e.g., Booth, 1982; Cerdán, 2010; Kirshner, 1989; MacGregor & Stacey, 1993; Molina, Rodríguez-Domingo, Cañadas, & Castro, 2017; Ruano,

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Socas, & Paralea, 2008). Some of these errors are related to the visual syntax of algebraic expressions, the understanding of nonstandard algebraic expressions and the use of variables (e.g., meaning given to variables).

One method to analyze the students' content knowledge about different mathematical concepts are translations between representation systems (Rittle-Johnson & Schneider, 2015). One common conclusion of the studies on translations between different representations in algebra is that students have difficulties in maintaining the semantic congruence that characterizes these processes, even when they display an understanding of the initial and final representations (Rodríguez-Domingo & Molina, 2013). Students' difficulties with algebra have been traditionally linked to the translation from verbal to symbolic representation, in the context of problem solving. MacGregor and Stacey (1993) explained that one of the most common errors is the reversal error, that is, to represent the opposite relation to the one indicated. Rodríguez-Domingo, et al (2015) pointed out that other errors are related with the use of variables, for example, as a way of shortening words or with the use of different symbols for the same value; with the completeness of the statement (some expressions are incomplete and others have extra information); with the incorrect interpretation of the operations; with the use of algebraic symbolism, for example, the use of the equal sign to denote that the what is on the left is associated with what is on the right or not interpreting correctly the structure of the algebraic expression. Isik and Kar (2012) assume that "the verbal expressions consist of syntactic and semantic processes as well as other processes, so the errors made cannot be explained merely by syntactic translation" (p. 95).

Focusing on algebraic symbolism and verbal representations as representation systems, translations (in both directions) constitute procedures involved in problem solving and problem posing. Translations from verbal representation to algebraic symbolism are part of the initial steps in problem solving when addressing algebraic problems stated verbally. The other direction of the translation —from the algebraic symbolism to verbal representation— is involved in the process of problem posing. One important aspect in the line of research on problem posing is to explore what problems teachers and students are able to pose (e.g., Cai, 1998; Cai, Hwang, Jiang, & Silver, 2015).

In this paper we focus on problem posing as a process of translation from the symbolic to the verbal representation system. From this viewpoint, problem posing is seen as a useful process for analyzing the meanings given by students to a specific mathematical topic (Castro, 2012; Rittle-Johnson & Schneider, 2015). Through this task, we aim to better understand the richness of the (conceptual) knowledge acquired by students as result of their secondary education.

We start this paper with some key ideas that constitute the conceptual framework and background of our study. These are organized in two blocks: those concerning translations between algebraic symbolism and the verbal representation system, and those concerning problem posing and problems structures. Then, we describe the empirical part of the study here reported.

Translations between Algebraic Symbolism and the Verbal Representation System

As Arcavi (2006) points out, competence in algebra requires to have the capacity of alternating, in a flexible and opportunistic way (to increment efficiency and speed in the execution of procedures), the use of actions without meaning with a search for meaning that is focused on questioning, selecting strategies, pondering, connecting ideas, developing conclusions, or elaborating on new meanings. These components can be classified either as

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procedural or conceptual (Star, 2005). Our approach attends to the conceptual component of knowledge of algebraic symbolism.

In this paper we consider two different external representations systems, used here in Goldin's (1998) terms, verbal language (i.e., the verbal representation system) and algebraic symbolism. The verbal representation system is determined by the use of everyday language, sometimes including specific terminology from academic mathematical language (Cañadas & Figueiras, 2011). Algebraic symbolism is a key formal representation system for topics related to algebra including letters and operational and relational symbols also used in arithmetic. It is used to represent algebraic ideas in a concise and precise way but separated from the initial and specific context in which they arise (Arcavi, 1994). This characteristic provides algebraic symbolism with an important applicability; it makes it possible to transform expressions through known techniques without having to attend to the meanings of symbols involved in the expression. However, at the same time, it makes algebraic symbolism very weak from a semantic viewpoint, leading to students' comprehension difficulties (Wheeler, 1989). Verbal language is less precise than algebraic symbolism, and some factors, such as context and intonation influence the meanings given to algebraic statements[1] (Bossé et al., 2011a, 2011b).

The translation from the symbolic to the verbal representation system has been paid less attention in previous studies (Molina, et al., 2017) and Molina et al. (2006) have observed that it causes fewer difficulties to secondary students than the opposite translation. They suggest using this greater facility as a means to support (a) the development of students' understanding of algebraic symbolism and (b) the improvement of problem-solving skills. Integrated study of posing and solving of problems can potentially help students to become aware of the greater precision and synthetic capability of algebraic symbolism in comparison to verbal language.

PROBLEM POSING AND PROBLEMS STRUCTURES

During the last years, there has been an important change in the way of considering problem posing in mathematics education context, which consist on recognizing "…the awareness that problem posing needs to pervade the education systems around the world" (Singer, Ellerton & Cai, 2013, p. 5). As a consequence, in 2013 a special issue of the journal Educational Studies in Mathematics focused on problem posing attending to the design of problem-posing activities, the nature of problem posing, and the use of problem posing as a research and instruction tool (Singer, et al, 2013; p. 5).

In that issue several authors explore different types of problem posing. For example, Silver (1994) or Brown and Walter (2005) establish problem-posing occurs in three situations: (a) within the process of problem solving (that is, as a reformulation of a problem in order to make the solution more accessible), (b) when the goal is the creation of a new problem from a situation or experience or (c) after solving a particular problem (one might examine some properties or conditions of the problem to generate related problems). Leikin (2015) considers different types of problem posing: (a) problem posing through proving (a strategy of reformulation), (b) problem posing for investigation (transformation of a proof problem into an investigation problem), and (c) problem through investigation (asking new questions due to some properties).

We consider that the goal of problem posing is to create a new problem from a particular situation and that problem posing is a process that requires students to make personal interpretations of specific situations and to

[1] We use the term algebraic statement to refer to general propositions of relations among quantities, some of them unknown, that can be expressed using algebraic symbolism (Rodríguez-Domingo et al., 2015).

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formulate them as mathematical problems (Koichu & Kontorovich, 2012). This process is concluded when the student poses a situation that can be considered a problem for him/herself. Problem posing requires advanced procedures and reflecting on the global structure of the problem and its objective. It implies thinking about the problem elements, the mathematical aspects involved, the relationships between these aspects that allow for posing a problem situation, the ways of solving the problem, and the coherence of the solution (Castro, 2011). In spite of this complexity, previous studies evidence that problem posing is accessible for students of different educative levels, from primary school to prospective secondary school teachers (e.g., Bonotto & Dal Santo, 2015; Ponte & Henriques, 2013; Silver & Cai, 1996; Silver, Mamona-Downs, Leung, & Kenney, 1996).

According to Stoyanova and Ellerton's (1996) a problem-posing situation can be classified as free, semi-structured, or structured. Assuming this framework, a problem-posing situation is free when students are asked to generate a problem from a given situation; semi-structured when students are given an open situation and are invited to explore the structure of that situation, and to complete it by applying knowledge, skills, concepts, and relationships from their previous mathematical experiences; and structured when problem-posing activities are based on a specific problem (Van Harpen & Presmeg, 2013, p. 119). In the context of algebraic statements, we show examples of these three kinds of problem posing situations in Table 1. Within them, free situations have been detected to be the most demanding for Secondary students within the three types (Ngah, Ismail, Tasir, Said, & Haruzuan, 2016).

Table 1. Examples of the kinds of problem posing situations

Problem posing situation	Example
Free	Considering the statement x+3=5. Make up a problem that can be solved using such statement.
Semi-structured	Considering the statement x+3=5. Make up a problem in which ages of different people are related.
Structured	Considering that the ages of two brothers have a difference of three years, this relationship in the case that the older one is five would be x+3=5. Ask as many questions as you can that are in some way related to this problem.

Problem posing may reveal interesting and important aspects of students' mathematical thinking (Cai, 1998; Cai & Hwang, 2002). In the context of school algebra and through the task of problem posing, authors like Resnick, Cauzinille-Marmeche and Mathie (1987) or Dede (2005) pointed out that students should be asked to write stories concerning equations, as a tool to prevent from performing meaningless and senseless operations on equations. Stephens (2003) noticed that students often use techniques to translate between verbal representations to symbolic representation, without a true understanding of the symbolic expressions involved. To deepen in the understanding of symbolic expressions she conducted a study where students have to pose problems based on linear equations. She concluded that problem posing was a difficult task for them because it requires students to think about the meaning of variable and algebraic symbols. Fernández-Millán and Molina (2016) identify characteristics of equations and systems of equations that make the task of problem posing difficult for Secondary students by the end of compulsory education: the inclusion of more than one unknown, the presence of the same unknown in both sides of the equal sign, coefficients higher than two and multiplicative operations among the unknowns. These authors detect a higher ability to give meaning to additive structures than to multiplicative ones.

Isik and Kar (2012) identify that one of the main reasons for the students' difficulties in problem posing given first-degree equations with one-unknown and equation pairs with two unknowns, is the inability to translate the operations and the parentheses in the equations into verbal expressions.

Some of the studies mentioned point out that one of the reasons for the students' difficulties with problem posing might be that they have not been introduced to this process during their education. For this reason, we decided to develop this study with students who had finish their Secondary Education but also had previously studied a mathematical course, which included problem posing tasks.

In this paper, with the aim of investigating students' meanings given to algebraic symbolism, we asked students to pose problems that could be solved through given symbolic statements. We address this study from an approach based on structural variables. Structural variables are characteristics of the problems that assume a specific value among different possible values (Kilpatrick, 1978; Golding & McClintock, 1980). Two kinds of structural variables are syntactic and semantic structures (Castro, Rico y Gil, 1992). Syntactic structure refers to the symbolic representation of the relations between quantities described in the problem. Semantic structure is focused on meanings of the words and mathematical expressions involved in one-step word problems, that is, problems to be solved with only one arithmetic operation (Castro et al., 1992). It captures the meaning of the operations that can be used to solve the problem. Various authors (Greer, 1992; Marshall, 1995; Schmidt & Weiser, 1995) have proposed semantic categories to classify additive and multiplicative problems. We base on Heller and Greeno's (1979) classification of semantic structure, which considers the implication of natural numbers on operations, distinguishing between (a) change, (b) part whole and (c) comparison problems for additive problems; and the multiplicative problems as (a) equal grouping, (b) Cartesian product, and (c) comparison.

RESEARCH OBJECTIVES

In this paper, we use problem posing to analyze: (a) students' capacity to assign meaning to symbolic algebraic statements as well as (b) the difficulties that they encountered in this process according to the characteristics of the algebraic statements given. The selected students had finished their algebraic training in compulsory education and had some experience on problem posing as part of a mathematical course at the university.

To achieve our general objectives and based on the conceptual framework presented, we address the following specific research objectives:

- To describe the semantic structures of the problems posed by students by the end of Secondary Education in free and structured problem posing situations.
- To describe the syntactic structures of the problems posed by students by the end of Secondary Education in free and structured problem posing situations.

The analysis from this structural framework contributes to the literature concerning students' understanding of algebraic statements, and identifies characteristics of algebraic statements that condition students' comprehension.

METHOD

In this section we describe the participants, the data collection design and its implementation and how the data analysis was developed.

Participants

As a condition given by the research objective, participants had finished their mathematics training at Secondary Education. Moreover, we decided that they should have some knowledge about problem posing to avoid that the novelty of the problem posing task would be an issue. Therefore, the sample is intentional, so it is not representative.

Fifty-five students, with a mean age of 21, participated in this study. These students were pre-service primary teachers. Their background met stated the criteria, as we justify in what follows.

a) They had finished secondary compulsory education. So they had studied algebra, at least, during their three years of secondary education, which is compulsory. Therefore we assume that they had experience using letters to symbolize unknowns, using algebraic symbolism to symbolize relations and properties, and translating from verbal to symbolic representations. These are components of the secondary education curriculum.

b) They had previous experience with problem posing. These students have attended the course "Mathematics Foundations for Mathematics Education", a first-year university course on elementary mathematics[2] offered by the School of Education as part of the Primary Teachers Training Degree. This course included problem-posing tasks in an arithmetic setting. In the aforementioned course, problem posing was used as a tool to assess students' understanding of the multiple meanings of arithmetic operations and of semantic categories that allow classifying arithmetic word problems. They were also introduced to the notion of syntactic and semantic structures of the problems, from problem solving and problem posing situations.

Data Collection

We decided to collect the data with a written questionnaire without a time limit. In this way students would be able to individually think on each question and change the order in which to approach each of the questions if they wanted.

Students were asked to pose eight problems that each of them could be solved using a given algebraic statement (see Table 2). For the last three problems we also suggested the students a particular meaning to be assigned to the variables appearing in the given expression. The expressions included were three first-order one-variable equations, two second-order one-variable equations, one first-order two-variable equation, and two systems of

[2] The course aims to help students deepen their understanding of elementary mathematics focusing on the multiples meanings and representations of the mathematics concepts included in the Primary Education curriculum. It is normally attended by students studying the Primary Education Teacher's degree.

first-order two-variable equations (see Table 2). These kinds of algebraic statements were included in the Spanish official curricular documents for compulsory education. To choose the expressions to be considered, we analyzed the algebraic statements appearing in a sample of books from grades 1 to 4 of secondary education (students of ages from 12 to 16) used in Spain considering the following task variables: number of variables, statement order, operational structure included (additive or multiplicative), type of statement (open[3], equation, system of equation). The algebraic expression finally chosen —included in Table 2—, corresponded to the most common algebraic statements found in the books.

Table 2. Characteristics of the symbolic statements included in the questionnaire

Item #	Symbolic statement	Problem posing situation	# of variables	Statement order	Type of statement	Operational structure
1	$x+6=8$	Free	1	1st	Eq	+/-
2	$x^2=16$	Free	1	2nd	Eq	x/:
3	$x+y=7$	Free	2	1st	Eq	+/-
4	$2x-1=5$	Free	1	1st	Eq	+/-; x/:
5	$\left.\begin{array}{r} x+y=7 \\ xy=10 \end{array}\right\}$	Free	2	1st; 2nd	Sys	+/-; x/:
6	$\left.\begin{array}{r} x+3y=3 \\ x-2y=1 \end{array}\right\}$	semi-structured x is the quantity of flour in a cake; y is the quantity of sugar in a cake	2	1st	Sys	+/-; x/:
7	$3x=20$	semi-structured x is the amount of time you require to get from home to the high school	1	1st	Eq	x/:
8	$x(x+1)=18$	semi-structured x is the length of one side of a rectangle	1	2nd	Eq	+/-; x/:

Note. Eq = equation; Sys = system.

The students individually solved the questionnaire during a regular class. One of the authors was the official teacher of the course and was present during data collection. She asked the students to work individually and explained them that they were not required to solve the equations and they could not propose the literal verbal translations of the equations (for example, in Item 3, “ex plus wye equals seven” was not a valid problem).

Categories for the Analysis

The three authors categorized all the students’ answers. Discrepancies were discussed until agreement was met. The categories chosen refer to the syntactic (same, equivalent or different) and semantic structure (additive,

[3] Expressions not containing an equal sign.

multiplicative, both or none) of the problems. We chose them among the various possible classifications of task variables in arithmetic and algebraic problems that we find in the literature (e.g., Puig, 1996).

Syntactic Structure. To analyze the syntactic structure, we translated problems posed to algebraic symbolism by keeping a left-to-right congruence whenever possible, and we compared the obtained translation to the one proposed in the item. If both (translation and initial problem) match, we classified the student's verbal problems as having the *same* syntactic structure (e.g. in item 2 "The square area is 16m^2, which is the length of the sides?"[4]). In cases where the translation was not the same, but was equivalent to the item, we classified it as *equivalent* (e.g., in item 1 "Carlos has 8 candies, but he loses 6, how many candies can Carlos eat?"). In other cases we classified the verbal problem as having a *different* syntactic structure than the given item (e.g., for "There are many toys in my bedroom. Today I have 7 on the floor; if 3 of them belong to my brother, how many toys are mine?" for item 3). Some problems could not be classified in any of these three categories due to being *incomplete* (e.g., "Carlos has x candies. His grandfather gives him the double of candies and his cousin takes one away. Compute the x", for item 4).

Semantic Structure. We first classified the semantic structure attending to the traditional distinction between *additive* (addition and/or subtraction) and *multiplicative* structures (multiplication and/or division) as depending on this distinction we find different semantic categories in the literature. Two additional categories were used to classify problems that involved both or neither of these structures in the results: *both* or *no operation*. The *no operation* category consisted of the statements where a context is described but no operation is expressed.

As we have mentioned previously, different authors have proposed semantic categories to classify additive and multiplicative problems. In Table 3 we include the definition of each of these categories together with some examples. These categories allowed us to capture the meaning given by students to the operations contained in the algebraic statements presented to them in the questionnaire.

We additionally considered another category, *no meaning problems*, to distinguish the problems where no actions were associated to the operations referred in the problem, therefore no meaning was given to them (e.g., "What number must be added to 6 to get an 8?").

Table 3

Types of Additive or Multiplicative Problems

Category	Description	Example

[4] The students presented the problems in Spanish, so all the examples in this paper have been translated into English by the authors.

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Table 3

Types of Additive or Multiplicative Problems

Category	Description	Example
	Additive problems	
Change	Describes an action that increases or decreases an initial quantity.	My sister ate 3 cookies from a 10 cookies package. How many cookies were left?
Part-part-whole	Refers to three quantities; one quantity is a particular whole (set) and the others are two separate quantities (subsets). Part-part-whole problems do not involve action.	In a class of 30 primary students, 20 are girls. How many are boys?
Comparison	Involves a comparison of two distinct, unconnected sets; does not imply action.	I am 5 years older than my sister. If I am 37 years old, how old is my sister?
	Multiplicative problems	
Equal grouping	Proportionality between two factors: one factor indicates the number of things in a group and the other factor indicates the number of equal-sized groups. This second factor acts as a multiplier.	I want to share 100 balloons between the 20 students in my class. How many balloons will each student get?
Comparison	Comparison between two quantities: one identifies the quantity in one group or set while the other number is the comparison factor.	This afternoon I read three times as many pages as my brother read. He read 11 pages. How many pages did I read?
Cartesian	Cartesian problems involve two sets and the pairing of elements between the sets. These problems entail a number of combinations.	How many different ice cream of two flavors can I prepare using chocolate and lemon ice cream and two types of cones?

RESULTS

From the whole group of responses obtained, we discarded 25 responses because they either (a) did not make sense (e.g., in Item 2, "Which is the value of x? By finding it you will be able to check how many dogs I have at home"), (b) asked for a datum contained in the verbal problem, and therefore no algebraic symbolism was needed to solve it (e.g., "A room has an area of 16 m^2, how many square meters does the room have?"), or (c) just called for solving the given symbolic equation/system. As can be seen in Table 4, most of the discarded problems corresponded to Item 4. Items 6 and 8 had the highest numbers of responses left blank. Finally, in all we analyzed 376 responses and 25 were discarded, distributed by item as Table 4 shows.

Table 4

Problems Proposed by the Students

Item	Statement provided	Responses	Discarded responses
1	$x+6=8$	54	1
2	$x^2=16$	52	3
3	$x+y=7$	52	3
4	$2x-1=5$	48	6
5	$x+y=7$	45	4

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Table 4

Problems Proposed by the Students

Item	Statement provided	Responses	Discarded responses
	x*y=10		
6	x-3y=3	37	3
	x-2y=1		
7	3x=20	48	2
8	x(x+1)=18	40	3
Total		376	25

Next, we analyzed the students' verbal proposals according to their syntactic and semantic structures.

Syntactic Structure

Using the categories and subcategories for syntactic structure previously described, we classified the students posed problems as presented in Table 5.

Table 5

Syntactic Structure of the Problems Posed by the Students

Item	Statement provided	Responses	Syntactic structure			
			Same	Equivalent	Different	Incomplete
1	x+6=8	54	43	11	0	0
			(80%)	(20%)	(0%)	(0%)
2	x^2=16	52	26	0	24	2
			(50%)	(0%)	(46%)	(4%)
3	x+y=7	52	39	0	11	2
			(75%)	(0%)	(21%)	(4%)
4	2x-1=5	48	24	0	22	2
			(50%)	(0%)	(45%)	(5%)
5	x+y=7	45	35	0	9	1
	x*y=10		(78%)	(0%)	(20%)	(2%)
6	x-3y=3	37	16	0	14	7
	x-2y=1		(43%)	(0%)	(38%)	(19%)
7	3x=20	48	32	2	11	3
			(67%)	(4%)	(23%)	(6%)
8	x(x+1)=18	40	23	0	16	1
			(58%)	(0%)	(40%)	(2%)

The first item was the only one for which all the students posed a problem whose structure was exactly as given or just equivalent. Among the problems with the same syntactic structure in this item, we included those whose syntactic structure is 6+x=8, x+6=8, 8=x+6, or 8=6+x.

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Items 1 and 7 were the only ones for which some students proposed problems with equivalent syntactic structures. The syntactic structure arose isolating the unknown in one term, and therefore the posed problems did not require the use of algebraic symbolism to solve them, as no operation involving the unknown was described in the verbal statement. Two examples of these problems were: "Juan has eight candies and he ate six. How many candies does he have left?" (Item 1); "The time that one boy needs to get from his house to the high school corresponds to a third of 20. How long does it take him?" (Item 7).

In all but the first item, between 16 (in Item 3) and 39 (in Item 6) out of 55 students showed difficulty in posing a problem. They either: (a) posed an incomplete problem, (b) posed a problem with a different syntactic structure, or (c) gave no answer. Below, we examine the various problems' syntactic structures in order to get more information about these difficulties.

Item 2 had a remarkably high percentage of problems with a syntactic structure different from the original. This was the only item including an exponent, and this characteristic may cause the difficulty. In this item, six students posed problems with a linear equation syntactic structure, such as 2x=16 or 4x=16, either in this form or by isolating the unknown on one side of the equation (e.g., "Sixteen students are going to play Ludo in a class. If four students play in each board, how many groups of four will play?"). Eight students posed problems with no operations involving any unknown. Two of them had the syntactic structure 4*4=x, showing that they had solved the given equation before posing the problem. Another five students included an additional variable, posing problems whose syntactic structures were x+y=16, x*y=16 or 16/x=y (e.g., "Paco has 16 balls in total, but he doesn't know how many he has in each hand. How many does he have so that they add up to 16?"). Three students incorporated some exponent into the syntactic structure of their problem but had difficulty making it correspond to x^2=16 (e.g., "which number multiplied twice by itself gives 16?").

In Item 3, four problems posed had different syntactic structures simply because the students gave values to one of the variables. One student chose particular values for *x* and *y*, and posed a problem where the sum was the unknown ("Maria has five candies and Alberto two lollipops. How many sweets do we both have all together?"). The rest of the students added an extra relationship between *x* and *y* (e.g., "Pedro and Manuel together have seven sweets. If Pedro has two more than Manuel, how many sweets does each one have?"). In this case, the equation posed was included but they added another one, maybe because they needed to have a unique solution of the problem.

Almost half of the problems posed by the students in Item 4 had a syntactic structure different from the original problem. As in other items (2, 5, 6, and 7), six students included an additional variable and five students posed problems where no operation involving the unknown was described in the verbal statement. Two of them had solved the equation and posed a problem with the syntactic structure 2*6-1=x. Regarding this item, we remark on the variety of different syntactic structures given (21 in a total of 48 problems posed). Examples of the symbolic expressions representing the syntactic structures found among these responses include: 2*(2*x*-1)-1; 2*x*-1-5=*y*; *x*-1=5; (3-1)*x*=5; 3*x*-1=5; 2+*x*-1=5; 2*x*=*x*+5; 2*x*+*x*=5; 2+*x*-1=5; *x*+2*x*-1=5. In most of these cases, students tried to involve the coefficient 2 and the term 5, but evidenced many difficulties in relating the quantities in their problems in the way represented by the given expression (e.g., "Javi is playing cards with Antonio, and Javi has double the number of cards. Javi loses one card, and Antonio takes five cards from him. How many cards does Javi have now?"). This item was also the one were more problems were disregarded as previously mentioned.

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In the items involving systems of equations (5 and 6), students showed difficulty in taking both equations into account. In these items, five and seven students, respectively, posed problems that included only one equation in their syntactic structure (e.g., in Item 5, “There are seven boys in a class; how many girls are there if there are ten children total?”), either one of the given ones or a different one. Students’ problems with different syntactic structures from the given system of equations included coefficients and/or independent terms different from the ones given. Three and two students, respectively, also involved extra unknowns (e.g., in Item 5, “The perimeter of a football court is 10 m^2. If you add the length and the width of a tennis court, the total is 7 m. What is the total? What are the different measurements?”).

In Item 7, half of the problems with different syntactic structures did not describe operations involving the unknown. Their syntactic structures are 3*20=x, 3*6.66=x or 20-3=x (e.g., “Juan took 20 minutes in each of the buses to go to the high school. If he took three buses, how much time did he need for the journey?”). The other problems included expressions referring to doubling or tripling those numbers. These students evidenced difficulty in expressing the triple of a quantity. One of the students posed a problem with two unknowns: “From my house to the school there are 20 km; if I walk 3 km in *x* time, how long will it take me from my house to the school?”.

In Item 8, as in Item 4, students’ problems differed widely in their syntactic structures and included more than one variable. All but two of the problems either referred to two unrelated unknowns whose product is 18 (e.g., “How much does the longer side of a rectangle measure if the surface is 18 cm^2?”) or included only additive operations between one to four variables, having as the syntactic structure: x+x+y=18, x+x+(x+1)+(x+1)=18, x+y+z+t=18, x+x+y+y=18, or x+x+1=18 (e.g., “What is the value of x if the sum of two equal sides and one different in a rectangle is 18?”).

Semantic Structure

In this section, we present the analisis of the semantic structure of the problems posed by the students. First, we examined whether they had an additive or multiplicative structure (or both) or did not include an operation (see Table 6). Second, we noted the meaning given to the operation considering the categories previously mentioned (see Table 3).

Table 6

Semantic Structure of the Problems Posed by the Students

			Semantic structure			
Item	Statement provided	Responses	Additive	Multiplicative	Both	Without operation
1	x+6=8	54	54 (100%)	0 (0%)	0 (0%)	0 (0%)
2	x^2=16	52	3 (6%)	48 (92%)	0 (0%)	1 (2%)
3	x+y=7	52	52 (91%)	3 (5%)	0 (0%)	2 (4%)
4	2x-1=5	48	44 (55%)	35 (44%)	31 (64%)	1 (1%)

Table 6

Semantic Structure of the Problems Posed by the Students

Item	Statement provided	Responses	Semantic structure: Additive	Multiplicative	Both	Without operation
5	x+y=7 x*y=10	45	42 (72%)	15 (26%)	14 (31%)	1 (2%)
6	x-3y=3 x-2y=1	37	32 (50%)	26 (41%)	23 (62%)	6 (9%)
7	3x=20	48	3 (6%)	41 (85%)	0 (0%)	4 (8%)
8	x(x+1)=18	40	21 (38%)	34 (62%)	12 (30%)	0 (0%)

Problems posed in Items 1, 2, 3, and 7 involved additive or multiplicative structure (not both) as happened in the statement provided in the questionnaire. As shown in Table 6, it tended to be the same operational structure than the given statement except a few cases. In Item 2, which involves a multiplicative relation, three students posed additive problems. One of these students confused the formulas for perimeter and area. Item 3 does not include any multiplicative relation, but three students posed problems with this structure (e.g., “Raul has seven candies, some of them given by Juan, and Pedro gave him twice as many as Juan and a few others. How many candies did each give to Raul?”). In Item 7, three students posed an additive problem.

In Items 4, 5, 6, and 8, both structures were present in the problems posed by the students, so they captured the kinds of structures in the statements provided in the questionnaire. In items that involve both structures, except in Item 8, the additive structure was more prevalent than the multiplicative one. The number of posed problems without operation in Items 6 and 7 is higher than in other items.

Additive Structure

We now describe the analysis of the meanings given by the students to the additive operations in items where there is an additive structure involved (all but items 2 and 7). Some examples classified in each of the semantic categories are presented in Table 7.

We discarded problems related to number theory (involving numbers and relationships between them) because no meaning is assigned to the operation involved. No meaning problems were distributed among Items 3, 1, 4, 6, 8, and 5 (with frequencies 1, 3, 7, 7, 9, and 10; respectively), having higher frequency in statements with both structures (Items 4, 5, 6, and 8).

Table 7

Examples of Problems Posed by the Students

Category	Example	Item
Change	Peppa had some lipsticks in her bag, Juana added 6 lipsticks and now there are 8 lipsticks. How many were in Peppa's bag?	1

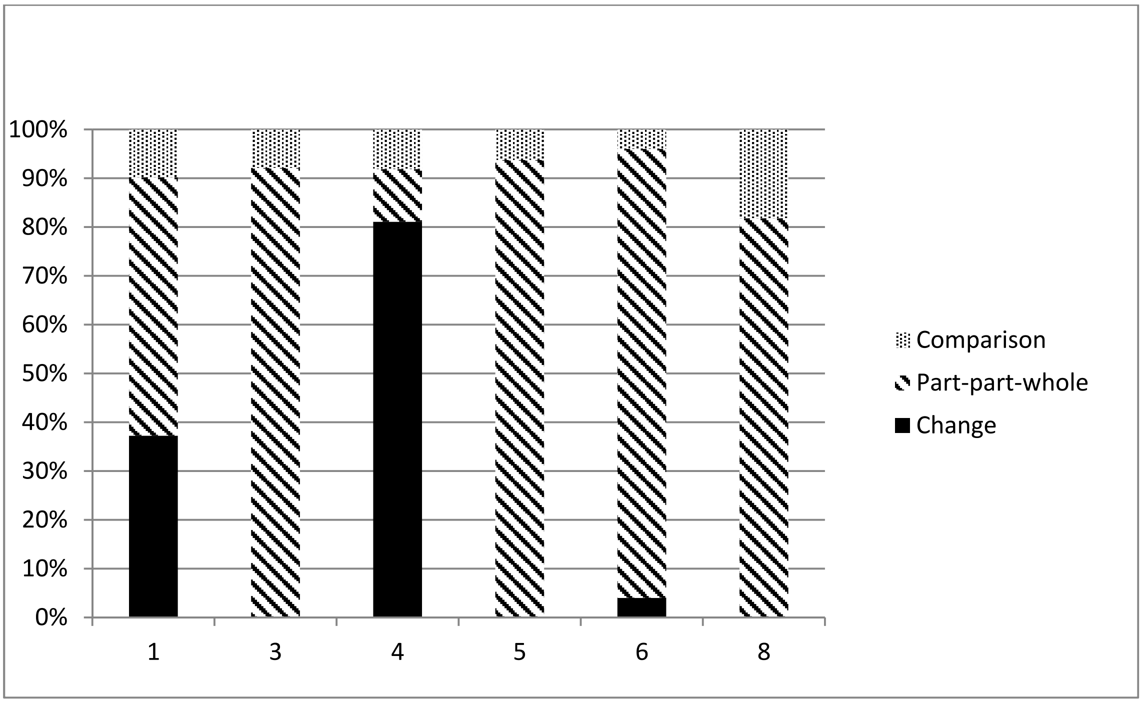
Table 7

Examples of Problems Posed by the Students

Category	Example	Item
Part-part-whole	There are two different kinds of fruits in a basket: apples and pears. There are 7 fruits in total. How many pears and apples are there?	3
Comparison	Discover the value of *x*, assuming that *x* is a girl's age and her brother is 6 years older and he is 8. How old is the girl?	1
No meaning	What number must be added to six to get an eight?	1
	How much flour do we need to sum with the triple amount of sugar to get three as a result? How much flour and sugar will be needed for the subtraction of the flour with the double of sugar to give us 1?	6

We present the frequencies of each kind of additive problem in Figure 1.

Figure 1. *Frequencies of the kinds of additive problems posed by the students according to their semantic structure*



Part-part-whole problems appeared with the highest frequency overall (142 cases) as well as in each item, except for Item 4. Comparison problems had the lowest frequency (14), being present in between 4 and 17% of cases in each item. Change problems were proposed in additive items that include one variable (Items 1 and 4)—constituting the most frequent semantic structure in Item 4 probably as consequence of having a subtraction instead of an addition—and in Item 6, where one student posed a problem with a partial structure of change.

Multiplicative Structure

We now focus on items that involved multiplicative structure (Items 2, 4, 5, 6, 7, and 8). We again discarded the no meaning problems because no meaning is given to the operations involved. There were 53 no meaning

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problems: 19 in Item 2, 10 in Item 5, 7 in Item 4, 7 in Item 8, 6 in Item 6, and 4 in Item 7. In these items, four students posed a no meaning problem without a multiplicative structure (e.g., "Adding the quantity of flour and sugar in a cake, we get 3 g. How much flour and sugar are there? And subtracting the quantity of flour from the quantity of sugar multiplied by 2, we get 1 g.").

In Table 8, we present examples of the students' problems organized by categories.

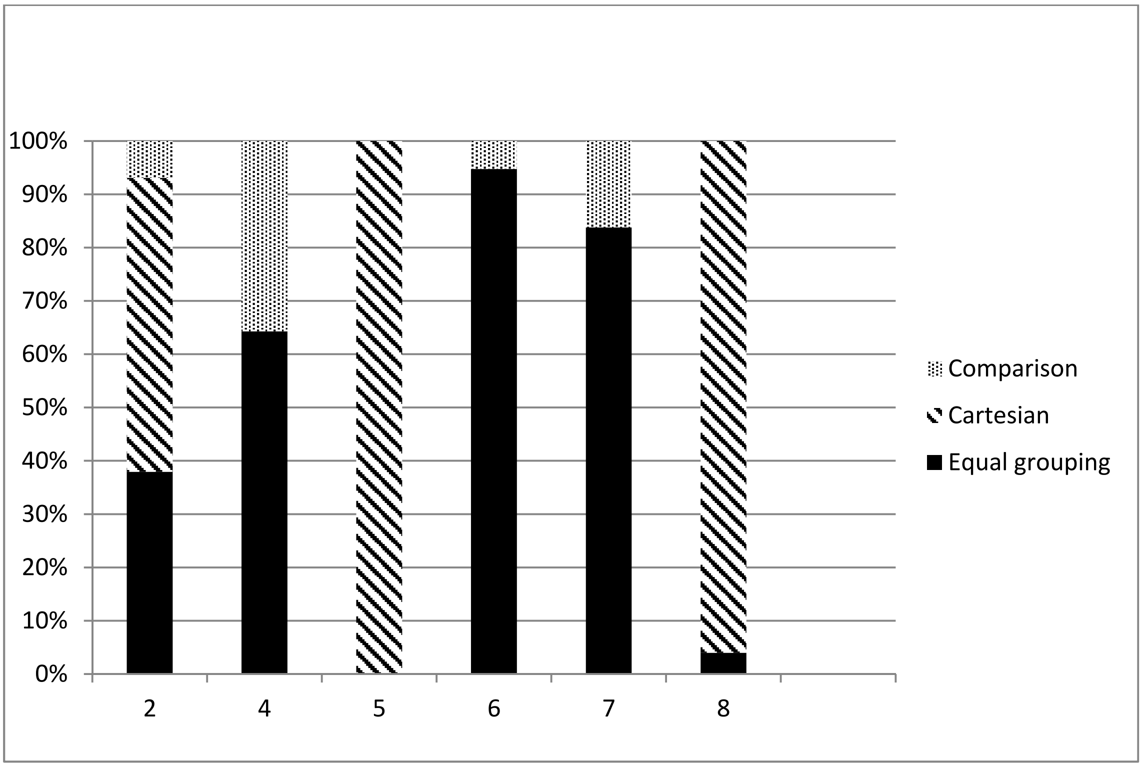
Table 8.

Examples of Problems Posed by the Students

Category	Example	Item
Equal grouping	I have 2 bags with the same amount of candies. If after giving a candy to my brother I still have 5, how many candies are in each bag?	4
Comparison	If going from my house to the park takes me 20 minutes and it is the triple than the time it takes me to go from my house to the school, how much time does it takes me to go from my house to school?	7
Cartesian	My favorite book measures 8 cm in length and 8 cm in width. What is the area of my book?	2
No meaning	Which number multiplied by itself gives 16?	2
	We have 2 pens multiplied by another number; subtracting 1 we get 5 as the result. What is the number that you multiply by 2?"	4

We present the frequencies of the problems posed according to these categories in Figure 2.

Figure 2. *Frequencies of the kinds of multiplicative problems posed by the students according to their semantic structure*



Equal grouping problems were the most frequent (81 cases) and were present in all but one of the items. Comparison problems were the least frequent (20), as also happened in the additive problems, not appearing in Items 5 and 8. Cartesian product problems were posed only in Items 2, 5, and 8, items where the expressions included a product of two variables. The equal-grouping problems included in items 2 and 8, the second with higher frequency, corresponded to cases where students solve the equations before posing the problems and included the solutions in the posed problems (e.g., "I have 16 candies to share with my 4 friends, how many candies will each one of my friends get?" in Item 2).

Discussion

In this paper we use problem posing to analyze: a) the capacity of a group of post-secondary students to assign meaning to symbolic algebraic statements as well as b) the difficulties that they encountered in this process according to the characteristics of the algebraic statements given. These students had completed compulsory education in algebra and had some experience in posing arithmetic problems. In this section, we highlight and comment on the main results to inform research on students' algebraic competence.

We note that in most cases students posed problems with syntactic structures different to the given symbolic expressions. This fact suggests that they encountered difficulty assigning meaning to the given statements. Students did change the kinds of relationships between variables (linear versus quadratic relationships, additive versus multiplicative, and vice versa), incorporated new variables, added new relationship(s) in the statements, included the result of the equation as one of the data points in the problem, did not relate both equations of a system, and did not provide meaning to the operations contained in some of the statement.

Having compared the semantic structures of the problems and the structures of the items included in the questionnaire (Table 2), we conclude that the students easily posed problems with additive structure. In fact, the presence of the additive structure is 25% higher than the multiplicative one (247 to 202). The wider experience with addition than with multiplication problems as well as the ease to express additive relations in comparison with multiplication are two justifications of the differences observed.

Concerning specific items and their characteristics, students did not encounter any difficulties in Item 1. This was the simplest statement in the questionnaire: a first grade equation with one unknown quantity. The variety of syntactic structures in the problems posed by the students for Items 2 and 4 is remarkable. Students' responses suggest that in the case of Item 2 the cause is the presence of the exponent.

We note the difficulty students faced with Item 5 in assigning meaning to the operations. Just five students gave meaning to the multiplicative structure in this item. We observe a similar situation in Item 2, where 19 students posed a no meaning problem, due to the difficulty of giving meaning to the product of two variables. For these two items, as well as for Item 8, Cartesian product problems would be the easiest type of problem to use to give meaning to the product of two variables, but students did not tend to formulate this type of problem. They showed a tendency to interpret multiplicative statements as related to equal-grouping situations. In the case of the additive structure, they showed a tendency to interpret the addition as two separated quantities (part-part whole) or as the modification of an initial quantity (change). Comparison problems have the lowest frequency in both additive and multiplicative structures. This result can be explained by previous studies showing that this type of problem appears with the least frequency in Spanish textbooks (Orrantia, González, & Vicente, 2005).

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Conclusions

In this study we have identified particular difficulties that students encounter when asked to give meaning to symbolic statements. Some students' tendency to solve the given statements in different items or the presence of a coefficient different from 1 can be identified as the causes of these difficulties. Fernández-Millán and Molina (2016) detected secondary students' special difficulties with these types of coefficient, which can also be observed in the result of Item 7, where some students struggled to express the triple of a quantity. These authors refer to the higher precision of algebraic symbolism than verbal language as a reason of the difficulties that students evidence to verbally express this type of multiplicative relationships between quantities where one or two of them is unknown. Our results suggest the same difficulties. Interestingly the experience of our students with problem posing, unlike those in Fernández-Millán and Molina (2016), was not sufficient to make a difference in this regard. Students should be asked to give meaning to symbolism by posing problems or writing narratives concerning algebraic expressions, in line with Dede (2005) suggestion, as a tool to prevent that algebra activities remains meaningless and senseless for students.

The items with higher rates of no response (Items 6 and 8) and with higher numbers of discarded problems (Items 4 and 5) combine additive and multiplicative structures. These items also showed higher rates of no meaning problems, with no meaning given to the operations in the statements. What we call "no meaning" problems, or number theory problems, are a type of problems that can be usually found in textbooks as valid problems. They just express quantitative relations between numbers, therefore they are easier to formulate as no interpretation of the operations is necessary. As a result reinforce a work on symbolism unlinked to its semantic component as happens with procedural components of school algebra. The results show that when students encounter difficulties to assign meaning to the operations they resorted to these type of problems that they know from their experience at school.

The results concerning semantic structures align with previous findings reported by Fernández-Millán and Molina (2016). Among the items that involve multiplicative structure, the difference between posing a problem concerning an equation or an equation system seems not to be very revealing. A smaller number of students proposed problems with different structures for items involving a single equation than for those involving equation systems.

The differences between the different semantic structures in the students' formulations indicate some limitations in students' knowledge related not only to algebra but also to arithmetic. This result is especially remarkable for the students in this study, since they had previous experience posing problems with different semantic structures within the context of arithmetic.

Even after having finished compulsory education, and despite extensive practice with the types of algebraic statements considered in this study, students evidenced difficulties when there was more than one variable in the statement, an exponent, the product of two variables, or a coefficient higher than one, or when having to consider two equations together in a system. The students' formulations indicate a limited comprehension of algebraic symbolism that certainly conditions their capacity to use this representation system. Activities focused on translation including algebraic symbolism where students' attention is directed to the richness of meaning of algebraic statements (and probably also of arithmetic statements) need to be incorporated into the regular practice of secondary mathematics to prevent students' competence in the use of algebraic symbolism from being disconnected from meanings.

Problem posing is an activity that allow revealed the students' meaning of algebraic symbolism. Due to the difficulties found when they finish Secondary education, we think problem posing could be incorporated to this educative level in order to help students to understand the leaning of algebraic symbolism and its meaning. This would be a complementary way to those existing in countries where problem posing are not part of secondary curriculum to work on the different problem structures of algebraic problems.

One of the objectives of the mathematics education course of the teachers education program that the students of this study are studying is to go in depth in problem posing through different semantic and syntactic structures. However, most of the students have shown their debilities in some of these structures. As a consequence, we suggest the teachers' trainers to consider the results for future years.

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